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# Modified scattering for odd solutions of cubic nonlinear Schrödinger equations with potential in dimension one

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## Abstract

We show that the global odd solutions of a cubic Schrödinger equation with potential, with small smooth decaying initial data, do not scatter in one space dimension. More precisely, we obtain for the asymptotics of such solutions an explicit expression, involving a logarithmic modulation in the phase of oscillation. This property has been known for long in the potentialless case. In the presence of a (generic) potential, some commutation issues of the Klainerman vector field like operator used in order to exploit dispersion appear.

Our method of proof uses the wave operators of the stationary Schrödinger operator, in order to reduce the problem to an equation without potential, but with a variable coefficients pseudodifferential nonlinearity. Exploiting the fact that we are working only with odd solutions, we may overcome the commutation issues alluded to above, and, using semiclassical analysis, deduce from the PDE an ODE, whose analysis provides the wanted asymptotics of the solution.

## 0 Introduction

In recent years, several works have been devoted to the question of long time asymptotics of solutions of nonlinear hyperbolic equations with data given by a small perturbation of a stationary solution. In higher space dimensions, let us mention the contributions of Soffer and Weinstein [22, 23, 24], and more recently of Cuccagna [4], Bambusi and Cuccagna [2], Cuccagna and Maeda [6], Cuccagna, Maeda and Phan [7]. We do not try to give an exhaustive bibliography, and refer to the most recent papers cited above and their list of references for a more complete description of works in dimension larger or equal to two. Let us mention also, in a more geometric framework, the results of Donninger, Krieger, Szeftel and Wong [14].

We are interested here in *one dimensional* problems for which, in contrast with what happens in higher dimension, and even for small perturbations of the zero state, the dispersion of the

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linear part of the equation is too weak to expect that the solution of the nonlinear problem will have, when time goes to infinity, the same asymptotics as linear solutions. For instance, for one dimensional Klein-Gordon equations with quadratic or cubic nonlinearities, and small and decaying Cauchy data, one proves that the global solutions (when they exist) have asymptotics at infinity that differ, through a logarithmic correction in their phase of oscillation, from the asymptotics of solutions of the linear problem (see [10, 11, 18, 26]). A natural question is thus to ask if one may put into evidence a similar phenomenon when considering initial data that are a small perturbation of a (non zero) stationary solution.

Such a problem has been recently attacked by Kowalczyk, Martel and Muñoz [17] for the so-called “kink problem”. They consider a solution to  $\partial_t^2 \phi - \partial_x^2 \phi = \phi - \phi^3$ , in one space dimension, starting from initial data of the form  $\phi|_{t=0} = H + \varphi_1$ ,  $\partial_t \phi|_{t=0} = \varphi_2$ , where  $H(x) = \tanh(x/\sqrt{2})$  is a stationary solution and  $(\varphi_1, \varphi_2)$  is small in the energy space and odd. They could prove that the local energy decays to zero, as well as the finiteness of some space-time weighted  $L^2$  estimates for the dispersive part of the solution. Their result is probably optimal under the assumptions they are making, but opens new questions. In particular, up to stronger decay assumptions on the initial data, is it possible to uncover the asymptotics of the solution in order to exhibit modified scattering, that is expected from the fact that, in one space dimension, a cubic nonlinearity plays the role of a long range perturbation?

A first step towards such a goal is to study long time asymptotics for small solutions of one dimensional Klein-Gordon or Schrödinger equations in one space dimension, where one adds to the Laplace operator a smooth rapidly decaying potential  $V$ . This has been done by Cuccagna, Georgiev and Visciglia [5], for Schrödinger equations with a nonlinearity vanishing at order  $p > 3$  at zero. In this case, assuming that the operator  $-\Delta + V$  has no eigenvalues, they could show that solutions of the nonlinear problem scatter. One cannot expect the same result if  $p = 3$ : actually, when  $V = 0$ , it has been known since the work of Hayashi and Naumkin [15], Lindblad and Soffer [19] and more recently Ifrim and Tataru [16], that one has only modified scattering when the initial data are small, smooth and decaying. For the defocusing cubic Schrödinger equation, Deift and Zhou [9] showed that the same modified scattering holds for large initial data, using the complete integrability of the equation. In the regime of nonlinearities playing the role of a long range perturbation, we do not know of results showing modified scattering, when one allows variable coefficients, either in the linear part of the equation or in front of the nonlinearity. The only works we are aware of concern time decay of solutions for one dimensional Klein-Gordon equations with variable coefficients nonlinearities by Lindblad and Soffer [20] and Sterbenz [25].

Our goal in this paper is to obtain, for solutions of the cubic Schrödinger equation

$$\left(D_t - \frac{D_x^2}{2} - V(x)\right)u = \kappa(x)|u|^2u,$$

with small smooth initial data  $u_0$  such that  $xu_0$  is in  $L^2$ , a one term asymptotic expansion of the solution displaying the modified scattering phase we expect. We may do that only under convenient assumptions, namely that  $\frac{D_x^2}{2} + V(x)$  has no eigenvalue, that  $V$  is a “generic” even potential belonging to  $\mathcal{S}(\mathbb{R})$ , that  $\kappa$  is smooth, even, with  $\kappa'$  in  $\mathcal{S}(\mathbb{R})$ , and that the initial data  $u_0$  is odd. This last assumption is essential for us, as in the work of Kowalczyk, Martel and Muñoz [17] cited above.

The method we adopt relies, as in many previous works in the subject, on the use of the wave

operators  $W_+$  associated to  $\frac{D_x^2}{2} + V(x)$ . We set  $u = W_+ w$ , for a new unknown  $w$ , that solves an equation of the form

$$(*) \quad \left( D_t - \frac{D_x^2}{2} \right) w = W_+^* [\kappa(x) |W_+ w|^2 W_+ w].$$

In that way, we reduce ourselves to a constant coefficients linear part, but up to a nonlinearity containing the operators  $W_+, W_+^*$ . To analyse this equation, we use a strategy combining ideas from Alazard and Delort [1], Delort [12], Ifrim and Tataru [16] and Stingo [26] in the constant coefficient case: we set  $w(t, x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t})$  for a new unknown  $v$  and obtain from (\*) a semiclassical equation satisfied by  $v$  (with semiclassical parameter  $h = 1/t$ ). We prove then energy estimates for the action of the operator  $(x + tD_x)$  acting on  $w$  (or of the corresponding operator obtained by change of variables acting on  $v$ ). We deduce on the other hand from the PDE an ordinary differential equation satisfied by  $v$ , which is the classical counterpart of the quantum problem (\*). The analysis of that ODE allows one to uncover the asymptotics of  $v$ , and thus of  $w$  and  $u$ , when time goes to infinity.

The new difficulties one has to cope with, in comparison with constant coefficients problems, come from the fact that the operator  $W_+$  in the right hand side of (\*) may be written, under our assumptions, as a *variable coefficients* pseudo-differential operator. Because of that, an operator like  $x + tD_x$  does not commute nicely to it. Nevertheless, using that our unknown  $w$  is odd, one may re-express the results of such a commutator from the action of  $x + tD_x$  itself. This is what allows one to obtain energy inequalities for  $(x + tD_x)w$ . Once such bounds are secured, one may deduce from the PDE satisfied by  $v$  and ODE using symbolic calculus for semiclassical pseudo-differential operators.

## 1 Statement of the theorem and first reductions

### 1.1 Statement of the main theorem

We consider  $V : \mathbb{R} \rightarrow \mathbb{R}$  a potential belonging to  $\mathcal{S}(\mathbb{R})$ . Then the operator  $-\frac{1}{2}\Delta + V = -\frac{1}{2}\frac{d^2}{dx^2} + V$  is a self-adjoint operator whose spectrum is made of an absolutely continuous part, equal to  $[0, +\infty[$ , and of finitely many negative eigenvalues (see Deift-Trubowitz [8]). In this paper, we assume moreover

$$(1.1.1) \quad -\frac{1}{2}\Delta + V \text{ has no eigenvalue}$$

(as for instance if  $V$  is nonnegative). For  $\xi$  in  $\mathbb{R}$ , we define the Jost function  $f_1(x, \xi)$  (resp.  $f_2(x, \xi)$ ) as the unique solution to

$$(1.1.2) \quad -\frac{d^2}{dx^2} f + 2V(x)f = \xi^2 f$$

that satisfies  $f_1(x, \xi) \sim e^{ix\xi}$  when  $x$  goes to  $+\infty$  (resp.  $f_2(x, \xi) \sim e^{-ix\xi}$  when  $x$  goes to  $-\infty$ ). We set

$$(1.1.3) \quad \begin{aligned} m_1(x, \xi) &= e^{-ix\xi} f_1(x, \xi) \\ m_2(x, \xi) &= e^{ix\xi} f_2(x, \xi). \end{aligned}$$

We shall say that the potential  $V$  is generic if

$$(1.1.4) \quad \int_{-\infty}^{+\infty} V(x)m_1(x,0) dx \neq 0.$$

Notice that the above integral is convergent as  $m_1(x, \xi)$  is bounded when  $x$  goes to  $+\infty$  and has at most polynomial growth as  $x$  goes to  $-\infty$  (see [8] Lemma 1 and lemma A.1.1 below). We say that  $V$  is very exceptional if

$$(1.1.5) \quad \int_{-\infty}^{+\infty} V(x)m_1(x,0) dx = 0 \text{ and } \int_{-\infty}^{+\infty} V(x)xm_1(x,0) dx = 0.$$

**Remarks •** There are lots of (even) potentials for which (1.1.1) and (1.1.4) hold true. Actually, it is proved in [8], page 153, that any nonnegative potential in  $\mathcal{S}(\mathbb{R})$  that is not identically zero satisfies (1.1.4).

• We do not know if there are non trivial potentials  $V$ , even, such that (1.1.1) and (1.1.5) hold. If one drops the condition (1.1.1), there are examples of even potentials in  $\mathcal{S}(\mathbb{R})$  that satisfy (1.1.5). If one sets  $V(x) = -3 \cosh^{-2} x$ , it is proved in [3] Lemma 2.1 that the transmission coefficient of this potential satisfies  $T(0) = 1$  (see [8] or Appendix A.1 below for the definition of the transmission coefficient). This implies on the one hand that (1.1.4) does not hold (as (1.1.4) is equivalent to  $T(0) = 0$  – see [8, 27] or (A.1.21) below) and that moreover  $\int xV(x)m_1(x,0) dx = 0$  i.e. that (1.1.5) holds, as follows from (A.1.16) and (A.1.20) in the appendix of the paper.

Our main result is the theorem below, where we denote  $D_t = \frac{1}{i} \frac{\partial}{\partial t}$ ,  $D_x = \frac{1}{i} \frac{\partial}{\partial x}$ . In the rest of the paper, we shall write frequently  $D$  for  $D_x$ , when there is no risk of confusion.

**Theorem 1.1.1** *Let  $V$  be an even potential belonging to  $\mathcal{S}(\mathbb{R})$ , satisfying (1.1.1) and either (1.1.4) or (1.1.5). Let  $\kappa$  be a real valued smooth even function, with  $\partial_x \kappa$  in  $\mathcal{S}(\mathbb{R})$ . For any  $\theta$  in  $]0, \frac{1}{4}[$ , one may find  $s_0$  in  $\mathbb{R}_+$  and for any  $s > s_0$ , some  $\epsilon_0 \in ]0, 1[$ , such that the following statement holds true: For any odd function  $u_0$  in  $H^s(\mathbb{R}; \mathbb{C})$ , satisfying*

$$(1.1.6) \quad \|u_0\|_{H^s} + \|xu_0\|_{L^2} \leq 1,$$

*there is a family of continuous functions  $(\mathcal{A}_\epsilon)_{\epsilon \in ]0, \epsilon_0[}$ , bounded in  $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$  such that, for any  $\epsilon \in ]0, \epsilon_0[$ , the unique global solution of the equation*

$$(1.1.7) \quad \begin{aligned} \left( D_t - \frac{1}{2} D_x^2 - V(x) \right) u &= \kappa(x) |u|^2 u \\ u|_{t=1} &= \epsilon u_0 \end{aligned}$$

*has when  $t$  goes to  $+\infty$  an asymptotic expansion*

$$(1.1.8) \quad u(t, x) = \frac{\epsilon}{\sqrt{t}} \mathcal{A}_\epsilon \left( \frac{x}{t} \right) \exp \left[ -i \frac{x^2}{2t} + i \epsilon^2 L_\kappa \left( t, \frac{x}{t} \right) \left| \mathcal{A}_\epsilon \left( \frac{x}{t} \right) \right|^2 \right] + r(t, x)$$

*where  $L_\kappa(t, x) = \int_1^t \kappa(\tau x) \frac{d\tau}{\tau}$  and  $r$  and  $\mathcal{A}_\epsilon$  satisfy*

$$(1.1.9) \quad \begin{aligned} \|r(t, \cdot)\|_{L^\infty} &= O(\epsilon t^{-\frac{3}{4} + \theta}), & \|r(t, \cdot)\|_{L^2} &= O(\epsilon t^{-\frac{1}{4} + \theta}) \\ \|\mathcal{A}_\epsilon(x) \langle tx \rangle^{-2}\|_{L^\infty} &= O(\epsilon t^{-\frac{1}{4} + \theta}), & \|\mathcal{A}_\epsilon(x) \langle tx \rangle^{-2}\|_{L^2} &= O(\epsilon t^{-\frac{5}{8} + \frac{\theta}{2}}). \end{aligned}$$

**Remarks:** • The index of regularity  $s_0$  in the statement may be estimated from below in function of  $\theta$ . Our proof will give  $s_0 = 1 + \frac{3}{2\theta}$ .

• The global existence of solutions for (1.1.7) with small  $H^1(\mathbb{R})$  data is immediate. The point in the theorem is the asymptotic expansion (1.1.8), which shows that, for the cubic Schrödinger equation with a potential, one gets the same type of asymptotic expansions as in the potentialless case (see Hayashi-Naumkin [15], Lindblad-Soffer [19] and Ifrim-Tataru [16]), at least under our oddness assumption on the initial data.

• The assumptions that  $V$  and  $\kappa$  are even, so that an odd initial data generates an odd solution, will play an essential role in the proof. Actually, already in the case of the linear equation without potential  $(D_t - \frac{D_x^2}{2})u = 0$ , the solution with initial data  $u_0 \in \mathcal{S}(\mathbb{R})$  at  $t = 0$  has principal part when  $t$  goes to  $+\infty$  given by

$$\frac{e^{i\pi/4}}{\sqrt{2\pi t}} \hat{u}_0\left(-\frac{x}{t}\right) e^{-i\frac{x^2}{2t}}.$$

In particular, if  $u_0$  is odd, this may be written as  $\frac{1}{\sqrt{t}} \frac{x}{t} \mathcal{B}\left(\frac{x}{t}\right) e^{-i\frac{x^2}{2t}}$  for some function  $\mathcal{B}$  in  $\mathcal{S}(\mathbb{R})$ , so that  $\|\langle x \rangle^{-1} u(t, x)\|_{L^\infty(dx)}$  will decay like  $t^{-3/2}$  when  $t$  goes to  $+\infty$ , instead of  $t^{-1/2}$  in the general case. Such an enhanced decay will play an essential role below.

• The estimates (1.1.9) for  $\mathcal{A}_\epsilon$  means that  $\mathcal{A}_\epsilon$  has some vanishing property when  $x$  goes to zero. This reflects the fact that we consider only odd solutions.

## 1.2 Reductions

We denote by  $W_+$  the wave operator associated to  $P = -\frac{1}{2}\Delta + V$ , defined as the strong limit

$$(1.2.1) \quad W_+ = s - \lim_{t \rightarrow +\infty} e^{itP} e^{-itP_0}$$

where  $P_0 = -\frac{1}{2}\Delta$ . One knows (see Weder [27] and references therein) that, since we assume that  $V$  has only continuous spectrum according to (1.1.1),

$$(1.2.2) \quad W_+ W_+^* = \text{Id}_{L^2}, \quad W_+^* W_+ = \text{Id}_{L^2}$$

and, more generally, that if  $b$  is any Borel function on  $\mathbb{R}$

$$(1.2.3) \quad b(P) = W_+ b(P_0) W_+^*, \quad b(P_0) = W_+^* b(P) W_+.$$

In particular,  $W_+^* P = P_0 W_+$  so that, if we define

$$(1.2.4) \quad w = W_+^* u$$

the first equation (1.1.7) implies

$$(1.2.5) \quad \left(D_t - \frac{1}{2}D_x^2\right)w = W_+^* [\kappa(x)|W_+ w|^2 W_+ w].$$

Notice that since  $P$  and  $P_0$  preserve the space of odd functions, so do  $W_+$ ,  $W_+^*$ , so that we shall have to study (1.2.5) when  $w$  is odd. For such  $w$ , we shall obtain in appendix A.1 an expression for  $W_+ w$  given by the following proposition.

**Proposition 1.2.1** *Let  $\chi_{\pm}$  be smooth functions, supported for  $\pm x \geq -1$ , with values in  $[0, 1]$ , with  $\chi_{-}(x) = \chi_{+}(-x)$ ,  $\chi_{+}(x) + \chi_{-}(x) \equiv 1$ . There are smooth functions  $e_j(\xi)$ ,  $j = 0, 1$ , satisfying*

$$(1.2.6) \quad |\partial_{\xi}^{\beta} e_j(\xi)| \leq C_{\alpha} \langle \xi \rangle^{-j-\beta}, \quad \forall \beta \in \mathbb{N}, j = 0, 1$$

$$(1.2.7) \quad |\partial_{\xi}^{\beta} [e_0(\xi) + |\xi|e_1(\xi) - 1]| \leq C_{\beta} \langle \xi \rangle^{-1-\beta}, \quad \forall \beta \in \mathbb{N}, \forall \xi \text{ with } |\xi| \geq 1$$

$$(1.2.8) \quad |e_0(\xi) + |\xi|e_1(\xi)| \equiv 1, \text{ for any } \xi \neq 0,$$

there are smooth functions  $m_j(x, \xi)$ ,  $j = 1, 2$ , satisfying for any  $M > 0$ , any  $N$  in  $\mathbb{N}$ , any  $\alpha, \beta$  in  $\mathbb{N}$

$$(1.2.9) \quad \begin{aligned} |\partial_x^{\alpha} \partial_{\xi}^{\beta} [m_1(x, \xi) - 1]| &\leq C_{\alpha\beta N} \langle x \rangle^{-N} \langle \xi \rangle^{-1-\beta}, \quad \forall x > -M, \forall \xi \in \mathbb{R} \\ |\partial_x^{\alpha} \partial_{\xi}^{\beta} [m_2(x, \xi) - 1]| &\leq C_{\alpha\beta N} \langle x \rangle^{-N} \langle \xi \rangle^{-1-\beta}, \quad \forall x < M, \forall \xi \in \mathbb{R} \end{aligned}$$

such that, if we define

$$(1.2.10) \quad \begin{aligned} e_{+}(x, \xi) &= m_1(x, \xi) [e_0(\xi) + |\xi|e_1(\xi)] \\ e_{-}(x, \xi) &= m_2(x, -\xi) [e_0(-\xi) + |\xi|e_1(-\xi)] \end{aligned}$$

and set

$$(1.2.11) \quad a(x, \xi) = \chi_{+}(x)e_{+}(x, \xi) + \chi_{-}(x)e_{-}(x, \xi),$$

then for any odd function  $w$

$$(1.2.12) \quad W_{+}w = a(x, D)w \stackrel{\text{def}}{=} \frac{1}{2\pi} \int e^{ix\xi} a(x, \xi) \hat{w}(\xi) d\xi.$$

**Remark:** We shall see in Appendix A.1 (see (A.1.2)) that since the potential  $V$  is even,  $m_2(x, \xi) = m_1(-x, \xi)$  so that  $a(-x, -\xi) = a(x, \xi)$ . This reflects the fact that  $W_{+}$  preserves the space of odd functions (and the space of even functions).

It follows from (1.2.9), (1.2.10), (1.2.11) that we may write

$$a(x, \xi) = a_0(x, \xi) + a_1(x, \xi)|\xi|$$

where  $a_0$  (resp.  $a_1$ ) is a pseudo-differential symbol of order 0 (resp.  $-1$ ). Consequently, operators  $[x, a(x, D)]$ ,  $[x, a(x, D)^*]$  are bounded on  $L^2$ , so that assumption (1.1.6) implies that  $w_0 = W_{+}^* u_0$  satisfies

$$(1.2.13) \quad \|w_0\|_{H^s} + \|xw_0\|_{L^2} = O(1).$$

The proof of the main theorem is thus reduced to the proof of estimates and asymptotics for the solution  $w$  of (1.2.5) with odd initial condition  $w_0$  satisfying (1.2.13).

## 2 Semiclassical formulation and symbolic calculus

In this section, we shall rewrite equation (1.2.5) as a semiclassical equation, and prove some preliminary results that will be useful to obtain  $L^2$  or  $L^{\infty}$  bounds in the remaining sections.

## 2.1 Semiclassical formulation

Let us introduce some notation following Dimassi-Sjöstrand [13]. An order function on  $\mathbb{R}^2$  is a function  $M : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that there are constants  $C_0 > 0, N_0 > 0$  so that, for any  $(x, \xi, y, \eta)$  in  $\mathbb{R}^2 \times \mathbb{R}^2$ ,

$$M(x, \xi) \leq C_0(1 + |x - y|^2 + |\xi - \eta|^2)^{\frac{N_0}{2}} M(y, \eta).$$

Let  $\delta$  be in  $[0, 1]$ . If  $M$  is an order function, we denote by  $S_\delta(M)$  the space of smooth functions

$$\begin{aligned} (h, x, \xi) &\rightarrow a_h(x, \xi) \\ ]0, 1] \times \mathbb{R}^2 &\rightarrow \mathbb{C} \end{aligned}$$

that satisfy for any  $\alpha, \beta, \gamma$  estimates

$$(2.1.1) \quad |(h\partial_h)^\gamma \partial_x^\alpha \partial_\xi^\beta a_h(x, \xi)| \leq C_{\alpha\beta\gamma} M(x, \xi) h^{-\delta\alpha - (1-\delta)\beta}.$$

Below we shall be mainly interested in the cases  $\delta = 1$  or  $\delta = \frac{1}{2}$ . We shall need also the subclass  $\Sigma_1(\langle \xi \rangle^m)$  of  $S_1(\langle \xi \rangle^m)$ , where  $m$  is in  $\mathbb{R}$ , defined by the condition

$$(2.1.2) \quad |(h\partial_h)^\gamma \partial_x^\alpha \partial_\xi^\beta a_h(x, \xi)| \leq C_{\alpha\beta\gamma} \langle \xi \rangle^{m-\beta} h^{-\alpha}$$

where we gain a negative power of  $\langle \xi \rangle$  for any  $\partial_\xi$  derivative.

Let  $\lambda$  be in  $[0, 1]$ . We associate to a symbol  $(a_h)_h$  in  $S_\delta(M)$  a family of operators acting on (families of) functions in  $\mathcal{S}(\mathbb{R})$  by

$$(2.1.3) \quad \text{Op}_h^\lambda(a_h)v = \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} a_h(\lambda x + (1-\lambda)y, \xi) v(y) dy d\xi,$$

which has a meaning as an oscillatory integral. We shall use actually only the cases  $\lambda = 1$  (usual quantization),  $\lambda = 0$  (dual quantization of the preceding one) and  $\lambda = \frac{1}{2}$  (Weyl quantization), given respectively by

$$\begin{aligned} \text{Op}_h^1(a_h)v &= \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} a_h(x, \xi) v(y) dy d\xi = a(x, hD)v \\ \text{Op}_h^0(a_h)v &= \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} a_h(y, \xi) v(y) dy d\xi \\ &= \mathcal{F}_\xi^{-1} \left[ \int e^{-iy\xi} a_h(y, h\xi) v(y) dy d\xi \right] \\ \text{Op}_h^W(a_h)v &= \text{Op}_h^{\frac{1}{2}}(a_h)v = \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} a_h\left(\frac{x+y}{2}, \xi\right) v(y) dy d\xi. \end{aligned} \tag{2.1.4}$$

Remark that the three definitions coincide if  $a_h$  is independent of  $x$ . We notice also that, if we denote by  $(\text{Op}_h^\lambda(a_h))^*$  the formal adjoint of  $\text{Op}_h^\lambda(a_h)$ , we have the equalities

$$(2.1.5) \quad (\text{Op}_h^1(a_h))^* = \text{Op}_h^0(\bar{a}_h), \quad (\text{Op}_h^0(a_h))^* = \text{Op}_h^1(\bar{a}_h), \quad (\text{Op}_h^W(a_h))^* = \text{Op}_h^W(\bar{a}_h).$$

As (2.1.4) defines operators that are bounded from  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}(\mathbb{R})$ , we may extend them by duality from  $\mathcal{S}'(\mathbb{R})$  to  $\mathcal{S}'(\mathbb{R})$ .

Finally, we shall have to consider as well symbols that are not smooth at  $\xi = 0$ , like  $a_h(x, \xi)|\xi|$ , where  $a_h$  is in  $S_\delta(1)$ . We shall extend the notation  $\text{Op}_h^1(\cdot)$ ,  $\text{Op}_h^0(\cdot)$  to such generalized symbols,



writing  $\text{Op}_h^1(a_h|\xi|)$  (resp.  $\text{Op}_h^0(a_h|\xi|)$ ) for  $\text{Op}_h^1(a_h) \circ |hD|$  (resp.  $|hD| \circ \text{Op}_h^0(a_h)$ ). Of course, we shall use this notation only when we make act these operators on spaces of functions on which  $|hD|$  is well defined, like Sobolev spaces.

Let  $w(t, x)$  be some function in  $C^0([1, +\infty[, H^s(\mathbb{R})) \cap C^1([1, +\infty(, H^{s-2}(\mathbb{R}))$  for some  $s \geq 2$  and define a new function  $v$  by

$$(2.1.6) \quad w(t, x) = \frac{1}{\sqrt{t}} v\left(t, \frac{x}{t}\right).$$

If we set  $h = \frac{1}{t} \in ]0, 1]$ , we get

$$(2.1.7) \quad \begin{aligned} D_t w &= \sqrt{h} \left[ \left( D_t - \frac{1}{2} (xhD_x + hD_x x) \right) v \right] \left( t, \frac{x}{t} \right) \\ &= \sqrt{h} \left[ \left( D_t - \text{Op}_h^W(x\xi) \right) v \right] \left( t, \frac{x}{t} \right) \\ D_x^2 w &= \sqrt{h} \left[ \text{Op}_h^W(\xi^2) v \right] \left( t, \frac{x}{t} \right). \end{aligned}$$

Let us define from (1.2.11)

$$(2.1.8) \quad a_h(x, \xi) = a\left(\frac{x}{h}, \xi\right) = a_{0,h}(x, \xi) + a_{1,h}(x, \xi)|\xi|$$

where, according to (1.2.10),

$$(2.1.9) \quad a_{j,h}(x, \xi) = \chi_+\left(\frac{x}{h}\right) m_1\left(\frac{x}{h}, \xi\right) e_j(\xi) + \chi_-\left(\frac{x}{h}\right) m_2\left(\frac{x}{h}, -\xi\right) e_j(-\xi)$$

for  $j = 0, 1$ . It follows from (1.2.6), (1.2.9) that  $a_{j,h}$  belongs to the subspace  $\Sigma_1(\langle \xi \rangle^{-j})$  of  $S_1(\langle \xi \rangle^{-j})$  defined by (2.1.2), and that for any  $N$  in  $\mathbb{N}$

$$(2.1.10) \quad \left\langle \frac{x}{h} \right\rangle^N \left[ a_{j,h}(x, \xi) - \chi_+\left(\frac{x}{h}\right) e_j(\xi) - \chi_-\left(\frac{x}{h}\right) e_j(-\xi) \right]$$

belongs to  $\Sigma_1(\langle \xi \rangle^{-j-1}) \subset S_1(\langle \xi \rangle^{-j-1})$ . We deduce from (1.2.12) and (2.1.6), (2.1.8)

$$(2.1.11) \quad W_+ w(t, x) = \frac{1}{\sqrt{t}} \left[ \text{Op}_h^1(a_h) v \right] \left( t, \frac{x}{t} \right)$$

so that equation (1.2.5) and (2.1.7), (2.1.5) imply that  $v$  satisfies

$$(2.1.12) \quad \left( D_t - \text{Op}_h^W\left(x\xi + \frac{\xi^2}{2}\right) \right) v = h \text{Op}_h^0(\bar{a}_h) \left[ \kappa\left(\frac{x}{h}\right) |\text{Op}_h^1(a_h) v|^2 \text{Op}_h^1(a_h) v \right].$$

In the rest of the paper, we shall study the solution  $v$  to (2.1.12).

## 2.2 Symbolic calculus

The general formula for the symbol of a composition of operators (in Weyl quantization) is given in Proposition 7.7 of [13]. It turns out that we shall need such a formula only when one of the symbols is a linear form. In this case, the formula just follows from the definition of the quantization in (2.1.4). More precisely, we consider  $M_1$  and  $M_2$  two order functions and symbols

$a_j$  in  $S_\delta(M_j)$ ,  $j = 1, 2$  for some  $\delta \in [0, 1]$ . We assume that  $a_1$  or  $a_2$  is a linear form on  $\mathbb{R}^2$  (in which case the corresponding order function may be taken to be  $(x^2 + \xi^2 + 1)^{\frac{1}{2}}$ ). Then we have the following exact composition formulas:

$$\begin{aligned}
(2.2.1) \quad & \text{Op}_h^1(a_1) \circ \text{Op}_h^1(a_2) = \text{Op}_h^1 \left[ a_1 a_2 + \frac{h}{i} \partial_\xi a_1 \partial_x a_2 \right] \\
& \text{Op}_h^0(a_1) \circ \text{Op}_h^0(a_2) = \text{Op}_h^0 \left[ a_1 a_2 - \frac{h}{i} \partial_x a_1 \partial_\xi a_2 \right] \\
& \text{Op}_h^W(a_1) \circ \text{Op}_h^W(a_2) = \text{Op}_h^W \left[ a_1 a_2 + \frac{h}{2i} \{a_1, a_2\} \right]
\end{aligned}$$

where

$$(2.2.2) \quad \{a_1, a_2\} = \partial_\xi a_1 \partial_x a_2 - \partial_x a_1 \partial_\xi a_2.$$

Let us recall some properties related to the boundedness of our operators on various spaces. We define for  $s$  in  $\mathbb{R}$ ,  $H_{\text{sc}}^s(\mathbb{R}, \mathbb{C})$  as the space of families of functions  $(v_h)_h$  indexed by  $h \in ]0, 1]$ , satisfying

$$(2.2.3) \quad \|(v_h)_h\|_{H_{\text{sc}}^s} \stackrel{\text{def}}{=} \sup_{h \in ]0, 1]} \|v_h\|_{H_h^s} < +\infty$$

where

$$(2.2.4) \quad \|v_h\|_{H_h^s} = \|\text{Op}_h^W(\langle \xi \rangle^s) v_h\|_{L^2}.$$

In the sequel, we shall frequently omit the explicit dependence of  $v_h$  in  $h$  in the notation. We shall use:

**Lemma 2.2.1** *Let  $m, s$  be in  $\mathbb{R}$ ,  $\lambda$  in  $[0, 1]$ .*

(i) *Let  $a$  be in  $S_{\frac{1}{2}}(\langle \xi \rangle^m)$ . Then there is  $C > 0$  such that for any  $(v_h)_h$  in  $H_{\text{sc}}^s(\mathbb{R}, \mathbb{C})$ , any  $h$  in  $]0, 1]$*

$$(2.2.5) \quad \|\text{Op}_h^\lambda(a) v_h\|_{H_h^{s-m}} \leq C \|v_h\|_{H_h^s}.$$

(ii) *Let  $a$  be in  $S_1(\langle \xi \rangle^m)$ . Then there is  $C > 0$  such that for any  $(v_h)_h$  in  $H_{\text{sc}}^s(\mathbb{R}, \mathbb{C})$ , any  $h$  in  $]0, 1]$*

$$(2.2.6) \quad \|\text{Op}_h^0(a) v_h\|_{H_h^{s-m}} + \|\text{Op}_h^1(a) v_h\|_{H_h^{s-m}} \leq C \|v_h\|_{H_h^s}.$$

(iii) *Assume  $m < 0$  and let  $a$  be in  $\Sigma_1(\langle \xi \rangle^m)$ . Then there is  $C > 0$  such that for any  $v$  in  $L^\infty$ , any  $h$  in  $]0, 1]$*

$$(2.2.7) \quad \|\text{Op}_h^1(a) v\|_{L^\infty} \leq C \|v\|_{L^\infty}.$$

The above lemma is proved in Appendix A.2.

We shall use several times the following Sobolev estimates:

**Lemma 2.2.2** (i) *There is  $C > 0$  such that for any  $(v_h)_h$  in  $H_{\text{sc}}^1(\mathbb{R}, \mathbb{C})$ , any  $h$  in  $]0, 1]$*

$$(2.2.8) \quad \|v_h\|_{L^\infty} \leq Ch^{-\frac{1}{2}} \|v_h\|_{L^2}^{\frac{1}{2}} \|hDv_h\|_{L^2}^{\frac{1}{2}}.$$

(ii) *Let  $\chi$  be in  $C_0^\infty(\mathbb{R})$ ,  $\chi \equiv 1$  close to zero,  $s \geq 0$ ,  $\sigma > 0$ . There is a constant  $C > 0$  such that for any  $j$  in  $\mathbb{N}$ , with  $j \leq s - 1$ , any  $(v_h)_h$  in  $H_{\text{sc}}^s(\mathbb{R}, \mathbb{C})$ , any  $h$  in  $]0, 1]$ , any  $\ell$  in  $\mathbb{N}$ ,  $\ell \leq s$ ;*

$$(2.2.9) \quad \|\text{Op}_h^{\text{W}}((1 - \chi)(h^\sigma \xi))v_h\|_{H_h^\ell} \leq Ch^{\sigma(s - \ell)} \|v_h\|_{H_h^s}$$

$$(2.2.10) \quad \| |hD|^j \text{Op}_h^{\text{W}}((1 - \chi)(h^\sigma \xi))v_h \|_{L^\infty} \leq Ch^{\sigma(s - \frac{1}{2} - j) - \frac{1}{2}} \|v_h\|_{H_h^s}.$$

(iii) *Let  $z \rightarrow \gamma(z)$  be in  $L^2$ . There is  $C > 0$  such that for any  $(v_h)_h$  in  $L^2$ , any  $h$  in  $]0, 1]$*

$$(2.2.11) \quad \left\| \text{Op}_h^{\text{W}}\left(\gamma\left(\frac{x + \xi}{\sqrt{h}}\right)\right)v_h \right\|_{L^\infty} \leq Ch^{-\frac{1}{4}} \|v_h\|_{L^2}.$$

*Proof:* (i) Estimate (2.2.8) is just Sobolev embedding.

(ii) Inequality (2.2.9) just follows from the definition of the  $H_h^s$  norm. Estimate (2.2.10) follows from (2.2.8) and (2.2.9) applied with  $\ell = 0$ ,  $s$  replaced by  $s - j$  (resp.  $s - j - 1$ ) and  $v_h$  replaced by  $|hD|^j v_h$  (resp.  $(hD)|hD|^j v_h$ ).

(iii) is an estimate of Ifrim-Tataru [16]. Notice that the distribution kernel of  $\text{Op}_h^{\text{W}}\left(\gamma\left(\frac{x + \xi}{\sqrt{h}}\right)\right)$  is nothing but

$$(2.2.12) \quad \frac{1}{2\pi h} \int e^{i(x-y)\xi/h} \gamma\left(\frac{x+y}{2\sqrt{h}} + \frac{\xi}{\sqrt{h}}\right) d\xi = e^{-i\frac{x^2-y^2}{2h}} \frac{1}{\sqrt{h}} (\mathcal{F}_\xi^{-1}\gamma)\left(\frac{x-y}{\sqrt{h}}\right).$$

As  $\mathcal{F}_\xi^{-1}\gamma$  is in  $L^2$ , inequality (2.2.11) follows by Cauchy-Schwarz.  $\square$

We introduce the following operator

$$(2.2.13) \quad \mathcal{L} = \frac{1}{h} \text{Op}_h^{\text{W}}(x + \xi) = \frac{1}{h} \text{Op}_h^1(x + \xi) = \frac{1}{h} \text{Op}_h^0(x + \xi)$$

that plays an essential role in the long time analysis of solutions to Schrödinger equations. We shall exploit the fact that we work with odd solutions in the proof of the following lemma, that allows to bound a weighted  $L^2$  or  $L^\infty$  norm of the action of a semiclassical operator on an odd function  $v$  in terms of an  $L^2$ -norm of  $\mathcal{L}v$ , with a gain of a positive power of  $h$ .

**Lemma 2.2.3** (i) *Let  $q$  be in  $\mathbb{R}_+$ ,  $c$  an element of  $S_1(\langle \xi \rangle^{-q})$ . There is a constant  $C > 0$  such that for any family of odd functions  $(v_h)_h$  in  $H_{\text{sc}}^{-q}(\mathbb{R}, \mathbb{C})$  such that  $(\mathcal{L}v_h)_h$  is in  $H_{\text{sc}}^{-q}(\mathbb{R}, \mathbb{C})$ , one has for any  $h$  in  $]0, 1]$  the estimate*

$$(2.2.14) \quad \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v_h \right\|_{L^2} \leq Ch \left[ \|\mathcal{L}v_h\|_{H_h^{-q}} + \|v_h\|_{H_h^{-q}} \right].$$

*Moreover, for any  $\sigma \in ]0, 1]$ , any  $s \geq 1/\sigma$ , there is  $C > 0$  and for any  $(v_h)_h$  in  $H_{\text{sc}}^{s-q}(\mathbb{R}, \mathbb{C})$ , odd, with  $(\mathcal{L}v_h)_h$  in  $H_{\text{sc}}^{-q}(\mathbb{R}, \mathbb{C})$ , any  $h$  in  $]0, 1]$*

$$(2.2.15) \quad \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v_h \right\|_{H_h^1} \leq Ch^{1-\sigma} \left[ \|\mathcal{L}v_h\|_{H_h^{-q}} + \|v_h\|_{H_h^{s-q}} \right].$$

(ii) Let  $a_{j,h}$  be in  $S_1(\langle \xi \rangle^{-j})$ ,  $j = 0, 1$  and set

$$a_h(x, \xi) = a_{0,h}(x, \xi) + a_{1,h}(x, \xi)|\xi|.$$

There is  $C > 0$  such that for any odd  $(v_h)_h$  in  $L^2$ , such that  $(\mathcal{L}v_h)_h$  is in  $L^2$ , one has for any  $h$  in  $]0, 1]$

$$(2.2.16) \quad \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(a_h)v_h \right\|_{L^2} \leq Ch \left[ \|\mathcal{L}v_h\|_{L^2} + \|v_h\|_{L^2} \right].$$

Moreover, for any  $\sigma \in ]0, 1]$ , any  $s \geq 1/\sigma$ , there is  $C > 0$  such that for any odd  $(v_h)_h$  in  $H_{sc}^s(\mathbb{R}, \mathbb{C})$  with  $(\mathcal{L}v_h)_h$  in  $L^2$ , one has for any  $h$  in  $]0, 1]$

$$(2.2.17) \quad \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(a_h)v_h \right\|_{L^\infty} \leq Ch^{\frac{1}{2}-\frac{\sigma}{2}} \left[ \|\mathcal{L}v_h\|_{L^2} + \|v_h\|_{H_{sc}^s} \right].$$

*Proof:* (i) We prove first (2.2.14) when  $q = 0$ . We write  $v$  for  $v_h$  to simplify notation. Since  $v$  is odd

$$v(x) = \frac{1}{2}[v(x) - v(-x)] = \frac{x}{2} \int_{-1}^1 (\partial v)(\lambda x) d\lambda$$

so that, since by (2.2.1),  $[\text{Op}_h^1(c), x] = -ih \text{Op}_h^1(\frac{\partial c}{\partial \xi})$ ,

$$(2.2.18) \quad \text{Op}_h^1(c)v = \frac{h}{2} \int_{-1}^1 \text{Op}_h^1\left(\frac{\partial c}{\partial \xi}\right)[(Dv)(\lambda x)] d\lambda + i\frac{x}{2} \int_{-1}^1 \text{Op}_h^1(c)[(Dv)(\lambda x)] d\lambda.$$

We may write by (2.2.13),  $D = \mathcal{L} - \frac{x}{h}$ , so that (2.2.18) is the sum of the following quantities

$$(2.2.19) \quad \frac{h}{2} \int_{-1}^1 \text{Op}_h^1\left(\frac{\partial c}{\partial \xi}\right)[(\mathcal{L}v)(\lambda x)] d\lambda + i\frac{x}{2} \int_{-1}^1 \text{Op}_h^1(c)[(\mathcal{L}v)(\lambda x)] d\lambda$$

and

$$(2.2.20) \quad -\frac{1}{2} \int_{-1}^1 \text{Op}_h^1\left(\frac{\partial c}{\partial \xi}\right)[(\lambda x)v(\lambda x)] d\lambda - i\frac{x}{2h} \int_{-1}^1 \text{Op}_h^1(c)[(\lambda x)v(\lambda x)] d\lambda.$$

Since  $c, \frac{\partial c}{\partial \xi}$  are in  $S_1(1)$ , it follows from (ii) of lemma 2.2.1 that the  $L^2$  norm of the product of  $\langle \frac{x}{h} \rangle^{-2}$  by (2.2.19) is bounded from above by

$$(2.2.21) \quad Ch \int_{-1}^1 \|(\mathcal{L}v)(\lambda x)\|_{L^2(dx)} d\lambda \leq Ch \|\mathcal{L}v\|_{L^2},$$

so by the right hand side of (2.2.14) with  $q = 0$ . Consider next (2.2.20), where we commute the  $x$  against  $v$  to the operators  $\text{Op}_h^1(\frac{\partial c}{\partial \xi}), \text{Op}_h^1(c)$ . We get by (2.2.1) expressions of the form

$$h \int_{-1}^1 \lambda \text{Op}_h^1(c_1)[v(\lambda x)] d\lambda, \quad x \int_{-1}^1 \lambda \text{Op}_h^1(c_2)[v(\lambda x)] d\lambda, \quad \frac{x^2}{h} \int_{-1}^1 \lambda \text{Op}_h^1(c_3)[v(\lambda x)] d\lambda$$

for new symbols  $c_1, c_2, c_3$  in  $S_1(1)$ . The product of these quantities by  $\langle \frac{x}{h} \rangle^{-2}$  has  $L^2$  norm bounded from above by  $Ch\|v\|_{L^2}$ , so by the right hand side of (2.2.14) with  $q = 0$ . When  $q > 0$

we write  $c(x, \xi) = \tilde{c}(x, \xi) \langle \xi \rangle^{-q}$ , with  $\tilde{c}$  in  $S_1(1)$ , so that (2.2.14) with  $q = 0$  applied to the odd function  $\tilde{v} = \langle hD \rangle^{-q} v$  gives

$$\left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v \right\|_{L^2} \leq Ch \left[ \|\mathcal{L} \langle hD \rangle^{-q} v\|_{L^2} + \|v\|_{H_h^{-q}} \right]$$

from which the wanted estimate follows, as  $[\mathcal{L}, \langle hD \rangle^{-q}]$  is bounded from  $H_{\text{sc}}^{-q}$  to  $L^2$ .

To prove (2.2.15), we take  $\chi$  in  $C_0^\infty(\mathbb{R})$ , equal to one close to zero and decompose

$$\left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v = \text{Op}_h^1(\chi(h^\sigma \xi)) \left[ \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v \right] + \text{Op}_h^1((1 - \chi)(h^\sigma \xi)) \left[ \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v \right].$$

By (2.2.9) and (2.2.5)

$$\left\| \text{Op}_h^1((1 - \chi)(h^\sigma \xi)) \left[ \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v \right] \right\|_{H_h^1} \leq Ch^{\sigma(s-1)} \|v\|_{H_h^{s-q}}$$

which is bounded by the right hand side of (2.2.15) if  $s\sigma \geq 1$ . On the other hand

$$\left\| \text{Op}_h^1(\chi(h^\sigma \xi)) \left[ \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v \right] \right\|_{H_h^1} \leq Ch^{-\sigma} \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(c)v \right\|_{L^2}$$

to which we may apply (2.2.14) to finish the proof of (2.2.15).

(ii) Let us deduce (2.2.16) and (2.2.17) from (2.2.14) and (2.2.15). Note that it is enough to show (2.2.16) and

$$(2.2.22) \quad \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(a_h)v \right\|_{H_h^1} \leq Ch^{1-\sigma} \left[ \|\mathcal{L}v\|_{L^2} + \|v\|_{H_h^s} \right],$$

as (2.2.17) will follow from these two inequalities and (2.2.8). If we replace in (2.2.16), (2.2.22),  $a_h$  by  $a_{0,h}$ , these two inequalities follow from (2.2.14) and (2.2.15). Consider now the contribution of  $a_{1,h}(x, \xi)|\xi|$ . Applying (2.2.14), (2.2.15) with  $q = 1$  to  $a_{1,h}$ , we get

$$\begin{aligned} \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(a_{1,h}|\xi|)v \right\|_{L^2} &\leq Ch \left[ \|\mathcal{L}|hD|v\|_{H_h^{-1}} + \||hD|v\|_{H_h^{-1}} \right] \\ \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(a_{1,h}|\xi|)v \right\|_{H_h^1} &\leq Ch^{1-\sigma} \left[ \|\mathcal{L}|hD|v\|_{H_h^{-1}} + \||hD|v\|_{H_h^{s-1}} \right]. \end{aligned}$$

As  $[\mathcal{L}, |hD|] = i \text{sgn}(hD)$  is bounded on  $L^2$ , we bound the above two expressions respectively by the right hand side of (2.2.16) and (2.2.22). This concludes the proof.  $\square$

### 2.3 Reduction to local operators

We want to express the action of a pseudo-differential operator on a function  $f$  from the product of  $f$  and of the restriction of the symbol of the operator to

$$(2.3.1) \quad \Lambda = \{(x, \xi) \in \mathbb{R}^2; x + \xi = 0\}$$

up to a convenient remainder. We shall consider symbols in the subclass of  $S_1(\langle \xi \rangle^{-j})$  that we define now.

**Definition 2.3.1** We denote by  $\tilde{S}_1(\langle \xi \rangle^m)$  the space of symbols  $a$  in  $S_1(\langle \xi \rangle^m)$  such that for any integer  $N$

$$(2.3.2) \quad \left\langle \frac{x}{h} \right\rangle^N h \partial_x a \in S_1(\langle \xi \rangle^m).$$

We shall prove the following result:

**Proposition 2.3.2** Consider a function  $a_h(x, \xi)$  of the form

$$(2.3.3) \quad a_h(x, \xi) = a_{0,h}(x, \xi) + a_{1,h}(x, \xi)|\xi|.$$

(i) Assume that  $a_{j,h}$  belongs to  $S_1(\langle \xi \rangle^{-j})$ ,  $j = 0, 1$ . Define

$$R^1(f) = \text{Op}_h^1(a_h)f - a_h(x, -x)f.$$

Then for any  $\sigma, \delta \in ]0, \frac{1}{2}[$ , any  $s \geq 1 + \frac{1}{\sigma}$ , there is a constant  $C > 0$  such that, for any function  $f$  for which the right hand side of the following inequalities is finite, for any  $h$  in  $]0, 1]$

$$(2.3.4) \quad \|R^1(f)\|_{L^2} \leq C \left[ h^{1-\sigma} \|\mathcal{L}f\|_{L^2} + h^\delta \|f\|_{H_h^s} \right]$$

$$(2.3.5) \quad \|R^1(f)\|_{L^\infty} \leq C \left[ h^{\frac{1}{2} - \frac{3}{2}\sigma} \|\mathcal{L}f\|_{L^2} + h^{\frac{1}{2} - \frac{\sigma}{2} - \delta} \|f\|_{H_h^s} + h^\delta \|f\|_{L^\infty} \right].$$

(ii) Assume that in (2.3.3),  $a_{j,h}$  belongs to  $\tilde{S}_1(\langle \xi \rangle^{-j})$ ,  $j = 0, 1$ . Define

$$R^0(f) = \text{Op}_h^0(\bar{a}_h)f - \bar{a}_h(x, -x)f.$$

Then for any  $\sigma, \delta$  in  $]0, \frac{1}{2}[$ , any  $s \geq 1 + \frac{1}{\sigma}$ , any  $N$  in  $\mathbb{N}$ , there is  $C > 0$  such that, for  $\ell = 0, 1$ , and any function  $f$ , any  $h$  in  $]0, 1]$ ,

$$(2.3.6) \quad \|(hD)^\ell R^0(f)\|_{L^2} \leq C \left[ h^{1-\sigma(\ell+1)} \|\mathcal{L}f\|_{L^2} + h^{\delta-\sigma(\ell+1)} \|f\|_{H_h^s} + h^{-\sigma(\ell+1)} \left\| \left\langle \frac{x}{h} \right\rangle^{-N} f \right\|_{L^2} \right]$$

$$(2.3.7) \quad \|R^0(f)\|_{L^\infty} \leq C \left[ h^{\frac{1}{2} - \frac{3}{2}\sigma} \|\mathcal{L}f\|_{L^2} + h^{\frac{1}{2} - \frac{3}{2}\sigma - \delta} \|f\|_{H_h^s} + h^{-\frac{1}{2} - \frac{3}{2}\sigma} \left\| \left\langle \frac{x}{h} \right\rangle^{-N} f \right\|_{L^2} + h^\delta \|f\|_{L^\infty} \right].$$

We first settle the case of smooth symbols.

**Lemma 2.3.3** Let  $\chi$  in  $C_0^\infty(\mathbb{R})$ ,  $\chi \equiv 1$  close to zero.

(i) Let  $a_{j,h}$  be in  $S_1(\langle \xi \rangle^{-j})$ ,  $j = 0, 1$  and set

$$R_j^1(f) = \text{Op}_h^1(a_{j,h})f - a_{j,h}(x, -x)f.$$

Then for any  $\sigma$  in  $]0, \frac{1}{2}[$ , any  $s \geq \frac{1}{\sigma}$ , any  $\ell$  in  $\mathbb{N}$ , there is  $C > 0$  such that for any function  $f$ , any  $h$  in  $]0, 1]$ ,

$$(2.3.8) \quad \|R_j^1(f)\|_{H_h^\ell} \leq Ch^{1-\sigma\ell} \left[ \|\text{Op}_h^1(\chi(h^\sigma \xi))\mathcal{L}f\|_{L^2} + \|f\|_{H_h^s} \right]$$

$$(2.3.9) \quad \|R_j^1(f)\|_{L^\infty} \leq Ch^{\frac{1}{2}-\frac{\sigma}{2}} \left[ \|\text{Op}_h^1(\chi(h^\sigma \xi))\mathcal{L}f\|_{L^2} + \|f\|_{H_h^s} \right].$$

(ii) Let  $a_{j,h}$  be in  $\tilde{S}_1(\langle \xi \rangle^{-j})$ ,  $j = 0, 1$  and set

$$R_j^0(f) = \text{Op}_h^0(a_{j,h})f - a_{j,h}(x, -x)f.$$

Then for any  $\sigma \in ]0, \frac{1}{2}[$ , any  $s \geq \frac{1}{\sigma}$ , any  $\ell, N$  in  $\mathbb{N}$ , any function  $f$ , any  $h \in ]0, 1]$ ,

$$(2.3.10) \quad \|R_j^0(f)\|_{H_h^\ell} \leq Ch^{1-\sigma\ell} \left[ \|\mathcal{L}f\|_{L^2} + \|f\|_{H_h^s} + \frac{1}{h} \left\| \left\langle \frac{x}{h} \right\rangle^{-N} f \right\|_{L^2} \right]$$

$$(2.3.11) \quad \| |hD|^\ell R_j^0(f) \|_{L^\infty} \leq Ch^{\frac{1}{2}-\sigma(\ell+\frac{1}{2})} \left[ \|\mathcal{L}f\|_{L^2} + \|f\|_{H_h^s} + \frac{1}{h} \left\| \left\langle \frac{x}{h} \right\rangle^{-N} f \right\|_{L^2} \right].$$

*Proof:* We treat (i) and (ii) at the same time. We shall prove (2.3.8) and (2.3.10). Then (2.3.9) (resp. (2.3.11)) follows from (2.2.8) combined with (2.3.8) (resp. (2.3.10)). We write

$$(2.3.12) \quad \begin{aligned} a_{j,h}(x, \xi) - a_{j,h}(x, -x) &= a_{j,h}(x, \xi)(1 - \tilde{\chi})(h^\sigma \xi) \\ &\quad + (a_{j,h}(x, \xi) - a_{j,h}(x, -x))\tilde{\chi}(h^\sigma \xi) \\ &\quad + a_{j,h}(x, -x)(\tilde{\chi}(h^\sigma \xi) - 1) = I + II + III \end{aligned}$$

where  $\tilde{\chi} \in C_0^\infty(\mathbb{R})$  is such that  $\tilde{\chi}\chi = \tilde{\chi}$ ,  $\tilde{\chi} \equiv 1$  close to zero. We have

$$\|\text{Op}_h^1(a_{j,h}(x, \xi)(1 - \tilde{\chi})(h^\sigma \xi))f\|_{H_h^\ell} \leq C \|\text{Op}_h^1((1 - \tilde{\chi})(h^\sigma \xi))f\|_{H_h^{\ell-j}} \leq Ch^{\sigma(s-\ell+j)} \|f\|_{H_h^s}$$

according to lemma 2.2.1 (ii) and (2.2.9). In the same way

$$\|\text{Op}_h^0(a_{j,h}(x, \xi)(1 - \tilde{\chi})(h^\sigma \xi))f\|_{H_h^\ell} = \|\text{Op}_h^0((1 - \tilde{\chi})(h^\sigma \xi))\text{Op}_h^0(a_{j,h})f\|_{H_h^\ell} \leq Ch^{\sigma(s-\ell+j)} \|f\|_{H_h^s}.$$

If  $s\sigma \geq 1$ , we get the terms in  $h^{1-\sigma\ell}\|f\|_{H_h^s}$  in the right hand sides of (2.3.8), (2.3.10), so that term  $I$  in (2.3.12) induces a contribution to  $R_j^1(f)$ ,  $R_j^0(f)$  estimated by the right hand side of (2.3.8), (2.3.10). Term  $III$  is treated in the same way, as  $(hD)^\ell[a_{j,h}(x, -x)]$  is uniformly bounded, for any  $\ell$ . Consider now term  $II$  and write

$$(2.3.13) \quad (a_{j,h}(x, \xi) - a_{j,h}(x, -x))\tilde{\chi}(h^\sigma \xi) = b_{j,h}(x, \xi)\tilde{\chi}(h^\sigma \xi)(x + \xi)$$

with

$$(2.3.14) \quad b_{j,h}(x, \xi) = \int_0^1 \frac{\partial a_{j,h}}{\partial \xi}(x, \lambda\xi - (1 - \lambda)x) d\lambda.$$

This is a symbol in  $S_1(1)$ , and by (2.2.1), we may write

$$\begin{aligned} \text{Op}_h^1((a_{j,h}(x, \xi) - a_{j,h}(x, -x))\tilde{\chi}(h^\sigma \xi))f \\ = h\text{Op}_h^1(b_{j,h}(x, \xi)\tilde{\chi}(h^\sigma \xi))\mathcal{L}f - \frac{h}{i}\text{Op}_h^1\left(\frac{\partial}{\partial \xi}[b_{j,h}(x, \xi)\tilde{\chi}(h^\sigma \xi)]\right)f. \end{aligned}$$

The  $H_h^\ell$ -norm of this quantity is bounded from above by the right hand side of (2.3.8), since  $\tilde{\chi}\chi = \tilde{\chi}$ , since we may apply lemma 2.2.1 (i) to the symbols of  $S_1(1)$   $b_{j,h}\tilde{\chi}(h^\sigma \xi)$ ,  $\partial_\xi[b_{j,h}\tilde{\chi}(h^\sigma \xi)]$ , and since  $\|\text{Op}_h^1(\tilde{\chi}(h^\sigma \xi))g\|_{H_h^\ell} \leq Ch^{-\sigma\ell}\|g\|_{L^2}$ . We have thus proved (2.3.8).

To show (2.3.10), we have to estimate the  $\text{Op}_h^0$ -quantization of (2.3.13), when  $a_{j,h}$  is in  $\tilde{S}_1((\xi)^{-j})$ . By (2.2.1), we have here

$$(2.3.15) \quad \text{Op}_h^0(b_{j,h}\tilde{\chi}(h^\sigma\xi)(x+\xi))f = h\text{Op}_h^0(\tilde{\chi}(h^\sigma\xi))\text{Op}_h^0(b_{j,h})\mathcal{L}f + \frac{h}{i}\text{Op}_h^0(\tilde{\chi}(h^\sigma\xi))\text{Op}_h^0\left(\frac{\partial b_{j,h}}{\partial x}\right)f.$$

By (ii) of lemma 2.2.1, we may still estimate the  $H_h^\ell$  norm of the first term in the right hand side by  $h^{1-\sigma\ell}\|\mathcal{L}f\|_{L^2}$ , and so by the right hand side of (2.3.10). Since  $a_{j,h}$  is assumed to be in  $\tilde{S}_1((\xi)^{-j})$ , it follows from (2.3.14) and (2.3.2) that we may write

$$\frac{\partial b_{j,h}}{\partial x} = \frac{1}{h}\left\langle\frac{x}{h}\right\rangle^{-N}c_1 + c_2$$

where  $c_1$  and  $c_2$  are in  $S_1(1)$ . We bound the  $H_h^\ell$  norm of the last term in (2.3.15) by

$$(2.3.16) \quad Ch^{1-\sigma\ell}\left[\frac{1}{h}\left\|\text{Op}_h^0(c_1)\left[\left\langle\frac{x}{h}\right\rangle^{-N}f\right]\right\|_{L^2} + \left\|\text{Op}_h^0(c_2)f\right\|_{L^2}\right].$$

(Notice that, because we are working with the 0-quantization, the factor  $\langle\frac{x}{h}\rangle^{-N}$  goes against  $f$ ). Using again lemma 2.2.1 (ii), we bound (2.3.16) by the right hand side of (2.3.10). This concludes the proof of the lemma.  $\square$

We need another lemma to study non smooth symbols.

**Lemma 2.3.4** *Let  $\chi$  be in  $C_0^\infty(\mathbb{R})$ , equal to one close to zero,  $\ell = 0, 1$ . For any  $\sigma \in ]0, \frac{1}{2}[$ , define*

$$(2.3.17) \quad R(f) = |hD|f - |x|\text{Op}_h^1(\chi(h^\sigma\xi))f.$$

*Then for any  $s \geq \frac{1}{\sigma} + 1$ , any  $\delta \in ]0, \frac{1}{2}[$ , there is  $C > 0$  such that for any function  $f$ , any  $h$  in  $]0, 1]$ ,*

$$(2.3.18) \quad \|(hD)^\ell R(f)\|_{L^2} \leq C\left[h^{\delta-\sigma\ell}\|f\|_{L^2} + h^{1-\sigma\ell}\|\mathcal{L}f\|_{L^2} + h^{1-\delta-\sigma\ell}\|f\|_{H_h^s}\right]$$

*and*

$$(2.3.19) \quad \|R(f)\|_{L^\infty} \leq C\left[h^\delta\|f\|_{L^\infty} + h^{\frac{1}{2}-\frac{\sigma}{2}}\|\mathcal{L}f\|_{L^2} + h^{\frac{1}{2}-\frac{\sigma}{2}-\delta}\|f\|_{H_h^s}\right].$$

*Proof:* We write

$$(2.3.20) \quad R(f) = (|hD| - |x|)\text{Op}_h^1(\chi(h^\sigma\xi))f + |hD|\text{Op}_h^1((1-\chi)(h^\sigma\xi))f.$$

The  $H_h^\ell$  norm of the last term is bounded from above, according to (2.2.9) by  $h^{s\sigma-\sigma(\ell+1)}\|f\|_{H_h^s}$ , so by the right hand side of (2.3.18). If we estimate the  $L^\infty$  norm of the last term in (2.3.20) using (2.2.10), we get a quantity in  $h^{-\frac{1}{2}+\sigma(s-\frac{3}{2})}\|f\|_{H_h^s}$ , that is controlled by the right hand side of (2.3.19). Denote  $g = \text{Op}_h^1(\chi(h^\sigma\xi))f$  and let us study the  $H_h^\ell$  or  $L^\infty$  norms of the first term in the right hand side of (2.3.20) i.e. of  $(|hD| - |x|)g$ . We consider a partition of unity

$$1 = \chi_-^1(\xi) + \chi_0(\xi) + \chi_+^1(\xi)$$



where  $\chi_{\pm}^1(\xi)$  is supported for  $\pm\xi \geq 1$ ,  $\chi_-^1(\xi) = \chi_+^1(-\xi)$ , and  $\chi_0$  is compactly supported and even. Then, as the inverse Fourier transform of  $\xi^\ell |\xi| \chi_0(\xi)$  is in  $L^1(\mathbb{R})$  for  $\ell$  in  $\mathbb{N}$ , we obtain for any such  $\ell$ , any positive  $\delta$ , any  $p = 2$  or  $\infty$

$$(2.3.21) \quad \|(hD)^\ell \text{Op}_h^1(\chi_0(h^{-\delta}\xi)|\xi|)g\|_{L^p} \leq Ch^{\delta(1+\ell)} \|g\|_{L^p}.$$

Consider next the function  $\chi_+^1(h^{-\delta}\xi)|\xi| = \chi_+^1(h^{-\delta}\xi)\xi$  and write

$$(2.3.22) \quad \chi_+^1(h^{-\delta}\xi)\xi = \chi_+^1(h^{-\delta}(-x))(-x) + b_+(x, \xi)(x + \xi)$$

with

$$b_+(x, \xi) = \int_{-1}^1 \psi_+((\lambda\xi - (1-\lambda)x)h^{-\delta}) d\lambda$$

where  $\psi_+(\xi) = \partial_\xi[\xi\chi_+(\xi)]$ . We notice that  $b_+$ ,  $h^\delta \frac{\partial b_\pm}{\partial \xi}$  are in  $S_\delta(1) \subset S_{\frac{1}{2}}(1)$ . Moreover, it follows from (2.2.1) that

$$(2.3.23) \quad \text{Op}_h^1(\chi_+(h^{-\delta}\xi)\xi)g = -x\chi_+^1(-h^{-\delta}x)g + \text{Op}_h^1(b_+)h\mathcal{L}g - \frac{h}{i}\text{Op}_h^1\left(\frac{\partial b_+}{\partial \xi}\right)g.$$

We may write in the same way

$$(2.3.24) \quad \text{Op}_h^1(-\chi_-(h^{-\delta}\xi)\xi)g = x\chi_-^1(-h^{-\delta}x)g + \text{Op}_h^1(b_-)h\mathcal{L}g - \frac{h}{i}\text{Op}_h^1\left(\frac{\partial b_-}{\partial \xi}\right)g$$

for another symbol  $b_-$  in  $S_{\frac{1}{2}}(1)$  such that  $h^\delta \frac{\partial b_-}{\partial \xi}$  is in  $S_{\frac{1}{2}}(1)$ . Decompose then

$$(2.3.25) \quad |hD|g = \text{Op}_h^1(\chi_0(h^{-\delta}\xi)|\xi|)g + \left[ \text{Op}_h^1\left(\chi_+^1(h^{-\delta}\xi)\xi\right) + x\chi_+^1(-h^{-\delta}x) \right]g \\ + \left[ \text{Op}_h^1\left(\chi_-^1(h^{-\delta}\xi)(-\xi)\right) - x\chi_-^1(-h^{-\delta}x) \right]g + |x|(1-\chi_0)(h^{-\delta}x)g.$$

We deduce from (2.3.23), (2.3.24) and (2.3.25) that, for  $p = 2$  or  $\infty$

$$(2.3.26) \quad \left\| (hD)^\ell \left[ |hD|g - |x|(1-\chi_0)(h^{-\delta}x)g - \text{Op}_h^1(\chi_0(h^{-\delta}\xi)|\xi|)g \right] \right\|_{L^p} \\ \leq Ch \left[ \|(hD)^\ell \text{Op}_h^1(b_+)h\mathcal{L}g\|_{L^p} + \|(hD)^\ell \text{Op}_h^1(b_-)h\mathcal{L}g\|_{L^p} \right. \\ \left. + \left\| (hD)^\ell \text{Op}_h^1\left(\frac{\partial b_+}{\partial \xi}\right)\mathcal{L}g \right\|_{L^p} + \left\| (hD)^\ell \text{Op}_h^1\left(\frac{\partial b_-}{\partial \xi}\right)\mathcal{L}g \right\|_{L^p} \right].$$

When  $p = 2$ , recalling that  $b_\pm$  is in  $S_{\frac{1}{2}}(1)$  and  $h^\delta \frac{\partial b_\pm}{\partial \xi}$  is in  $S_{\frac{1}{2}}(1)$ , we use (i) of lemma 2.2.1 to bound the right hand side of (2.3.26) by

$$(2.3.27) \quad Ch \left[ \|\mathcal{L}g\|_{H_h^\ell} + h^{-\delta} \|g\|_{H_h^\ell} \right].$$

As  $g = \text{Op}_h^1(\chi(h^\sigma\xi))f$ , we get an estimate in

$$Ch^{1-\sigma\ell} \left[ \|\mathcal{L}g\|_{L^2} + h^{-\delta} \|g\|_{L^2} \right].$$

Finally, since  $[\mathcal{L}, \text{Op}_h^1(\chi(h^\sigma\xi))]$  is bounded on  $L^2$ , we deduce from (2.3.26) that for  $\ell = 0, 1$

$$(2.3.28) \quad \left\| (hD)^\ell \left[ |hD|g - (1-\chi_0)(h^{-\delta}x)|x|g - \text{Op}_h^1(\chi_0(h^{-\delta}\xi)|\xi|)g \right] \right\|_{L^2} \\ \leq Ch^{1-\sigma\ell} \left[ \|\mathcal{L}f\|_{L^2} + h^{-\delta} \|f\|_{L^2} \right].$$

Applying (2.2.8), we deduce from (2.3.28)

$$(2.3.29) \quad \begin{aligned} & \| |hD|g - (1 - \chi_0)(h^{-\delta}x)|x|g - \text{Op}_h^1(\chi_0(h^{-\delta}\xi)|\xi|)g \|_{L^\infty} \\ & \leq Ch^{\frac{1}{2} - \frac{\sigma}{2}} [\| \mathcal{L}f \|_{L^2} + h^{-\delta} \| f \|_{L^2}]. \end{aligned}$$

To prove (2.3.18), we have to estimate in  $L^2$  the action of  $(hD)^\ell$  on the first term in the right hand side of (2.3.20) i.e.  $\|(hD)^\ell[|hD| - |x|]g\|_{L^2}$ . According to (2.3.28) and (2.3.21) with  $p = 2$ , this is smaller than the sum of the right hand side of (2.3.18) and of

$$(2.3.30) \quad h^{\delta(1+\ell)} \|g\|_{L^2} + \|(hD)^\ell \chi_0(h^{-\delta}x)|x|g\|_{L^2} \leq Ch^\delta \|g\|_{H_h^\ell} \leq Ch^{\delta - \sigma\ell} \|f\|_{L^2}$$

for  $\ell = 0, 1$ , using that  $\delta \leq 1$  and, for the last estimate, the definition of  $g = \text{Op}_h^1(\chi(h^\sigma\xi))f$ . As (2.3.30) is also controlled by the right hand side of (2.3.18), this concludes the proof of that inequality.

To get (2.3.19), we use (2.3.29) and (2.3.21) with  $\ell = 0$ ,  $p = \infty$ . We obtain a bound for  $\|(|hD| - |x|)g\|_{L^\infty}$  in terms of the right hand side of (2.3.19) plus

$$h^\delta \|g\|_{L^\infty} + \|\chi_0(h^{-\delta}x)|x|g\|_{L^\infty}$$

that we estimate by  $h^\delta \|f\|_{L^\infty}$  and so by the right hand side of (2.3.19). This concludes the proof.  $\square$

*Proof of Proposition 2.3.2:* (i) We write according to (2.3.3)

$$(2.3.31) \quad \begin{aligned} \text{Op}_h^1(a_h)f - a_h(x, -x)f &= (\text{Op}_h^1(a_{0,h}) - a_{0,h}(x, -x))f \\ &+ (\text{Op}_h^1(a_{1,h}) - a_{1,h}(x, -x))[|hD|f] \\ &+ a_{1,h}(x, -x)[|hD|f - |x|f] = I + II + III. \end{aligned}$$

We write

$$III = a_{1,h}(x, -x)R(f) - a_{1,h}(x, -x)|x|\text{Op}_h^1((1 - \chi)(h^\sigma\xi))f$$

where  $R(f)$  is defined by (2.3.17). Combining (2.3.18) with  $\ell = 0$ , the fact that  $a_{1,h}(x, -x) = O(\langle x \rangle^{-1})$  and (2.2.9), we obtain that the  $L^2$  norm of  $III$  is bounded from above by the right hand side of (2.3.4). If we use instead (2.3.19) and (2.2.10), we obtain for the  $L^\infty$  norm of  $III$  a bound by the right hand side of (2.3.5). By (2.3.8) (resp. (2.3.9)) the  $L^2$  (resp.  $L^\infty$ ) norm of  $I$  is bounded from above by the right hand side of (2.3.4) (resp. (2.3.5)). We are left with studying  $II$ . We apply (2.3.8) (resp. (2.3.9)) with  $j = 1$ ,  $\ell = 0$  and  $s$  replaced by  $s - 1$ . We obtain that the  $L^2$ -norm of  $II$  is bounded from above by

$$Ch \left[ \|\text{Op}_h^1(\chi(h^\sigma\xi))\mathcal{L}|hD|f\|_{L^2} + \| |hD|f \|_{H_h^{s-1}} \right]$$

and its  $L^\infty$  norm by

$$Ch^{\frac{1}{2} - \frac{\sigma}{2}} \left[ \|\text{Op}_h^1(\chi(h^\sigma\xi))\mathcal{L}|hD|f\|_{L^2} + \| |hD|f \|_{H_h^{s-1}} \right]$$

if  $s \geq 1 + \frac{1}{\sigma}$ . Commuting  $\mathcal{L}$  and  $|hD|$ , we get again contributions bounded from above by the right hand side of (2.3.4) and (2.3.5) respectively.

(ii) Under the assumptions of (ii) of the proposition, we write

$$(2.3.32) \quad \begin{aligned} \text{Op}_h^0(\bar{a}_h)f - \bar{a}_h(x, -x)f &= (\text{Op}_h^0(\bar{a}_{0,h}) - \bar{a}_{0,h}(x, -x))f \\ &\quad + |hD|(\text{Op}_h^0(\bar{a}_{1,h}) - \bar{a}_{1,h}(x, -x))f \\ &\quad + (|hD| - |x|)[\bar{a}_{1,h}(x, -x)f] = I + II + III. \end{aligned}$$

By (2.3.10) with  $j = 0$ , (resp. (2.3.11) with  $j = 0$ ,  $\ell = 0$ ), the  $H_h^\ell$  (resp.  $L^\infty$ ) norm of  $I$  is bounded from above by the right hand side of (2.3.6) (resp. (2.3.7)). To study the  $H_h^\ell$  (resp.  $L^\infty$ ) norm of  $II$ , we apply (2.3.10) with  $\ell$  replaced by  $\ell + 1$  (resp. (2.3.11) with  $\ell = 1$ ) in the case  $j = 1$ . We get bounds by the right hand side of (2.3.6), (2.3.7) respectively.

Finally, write  $III$  as

$$(2.3.33) \quad III = (|hD| - |x|\text{Op}_h^1(\chi(h^\sigma\xi)))[\bar{a}_{1,h}(x, -x)f] - |x|\text{Op}_h^1((1 - \chi)(h^\sigma\xi))[\bar{a}_{1,h}(x, -x)f].$$

We get for  $\ell = 0, 1$

$$(2.3.34) \quad \begin{aligned} &\| (hD)^\ell [|x|\text{Op}_h^1((1 - \chi)(h^\sigma\xi))[\bar{a}_{1,h}(x, -x)f]] \|_{L^2} \\ &\leq \| \text{Op}_h^1((1 - \chi)(h^\sigma\xi))[\bar{a}_{1,h}(x, -x)f] \|_{L^2} + \| x\text{Op}_h^1((1 - \chi)(h^\sigma\xi)\xi^\ell)[\bar{a}_{1,h}(x, -x)f] \|_{L^2} \\ &\leq C \left[ h^{\sigma s} \|\bar{a}_{1,h}(x, -x)f\|_{H_h^s} + h^{\sigma(s-\ell)} \|x\bar{a}_{1,h}(x, -x)f\|_{H_h^s} + h^{1+\sigma(s-\ell+1)} \|\bar{a}_{1,h}(x, -x)f\|_{H_h^s} \right] \end{aligned}$$

by (2.2.9). Since  $\bar{a}_{1,h}(x, -x)$ ,  $x\bar{a}_{1,h}(x, -x)$  are in  $L^\infty$  as well as their  $hD$  derivatives, we get a bound in  $h^{1-\sigma\ell}\|f\|_{H_h^s}$  as  $s \geq 1/\sigma$ , so by the right hand side of (2.3.6). Consider next the first term in the right hand side of expression (2.3.33) of  $III$ . By (2.3.18), the  $L^2$  norm of the action of  $(hD)^\ell$  on that term is bounded from above by

$$(2.3.35) \quad C \left[ h^{\delta-\ell\sigma} \|\bar{a}_{1,h}(x, -x)f\|_{L^2} + h^{1-\ell\sigma} \|\mathcal{L}[\bar{a}_{1,h}(x, -x)f]\|_{L^2} + h^{1-\delta-\ell\sigma} \|\bar{a}_{1,h}(x, -x)f\|_{H_h^s} \right].$$

The middle term may be bounded from

$$(2.3.36) \quad h^{1-\sigma\ell} \|\mathcal{L}f\|_{L^2} + h^{-\sigma\ell} \|[hD, \bar{a}_{1,h}(x, -x)]f\|_{L^2}.$$

Since

$$(2.3.37) \quad [hD, \bar{a}_{1,h}(x, -x)] = c_{1,h}(x) + hc_{0,h}(x)$$

with  $c_{1,h} = O(\langle \frac{x}{h} \rangle^{-N})$  for any  $N$ , as  $a_{1,h}$  is in  $\tilde{S}_1(\langle \xi \rangle^{-1})$ ,  $c_{0,h} = O(1)$ , we bound (2.3.36), and then (2.3.35) by the right hand side of (2.3.6). This concludes the proof of that inequality.

To finish the proof of the proposition, we are left with bounding the  $L^\infty$  norm of  $III$  by the right hand side of (2.3.7). By (2.2.8), the  $L^\infty$  norm of the last term in (2.3.33) is bounded from above by

$$(2.3.38) \quad \begin{aligned} &Ch^{-\frac{1}{2}} \| |x|\text{Op}_h^1((1 - \chi)(h^\sigma\xi))[\bar{a}_{1,h}(x, -x)f] \|_{L^2}^{\frac{1}{2}} \\ &\quad \times \left\| (hD) \left[ |x|\text{Op}_h^1((1 - \chi)(h^\sigma\xi))[\bar{a}_{1,h}(x, -x)f] \right] \right\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

These two  $L^2$  norms are the quantities (2.3.34), with  $\ell = 0$  or  $1$ , which have been estimated respectively by  $h\|f\|_{H_h^s}$  and  $h^{1-\sigma}\|f\|_{H_h^s}$ . We thus obtain for (2.3.38) a bound in  $h^{\frac{1}{2}-\frac{\sigma}{2}}\|f\|_{H_h^s}$

which is smaller than the right hand side of (2.3.7). Finally, the  $L^\infty$  norm of the first term in the right hand side of (2.3.33), is controlled according to (2.3.19) by

$$(2.3.39) \quad C \left[ h^\delta \|\bar{a}_{1,h}(x, -x)f\|_{L^\infty} + h^{\frac{1}{2}-\frac{\sigma}{2}} \|\mathcal{L}(\bar{a}_{1,h}(x, -x)f)\|_{L^2} + h^{\frac{1}{2}-\frac{\sigma}{2}-\delta} \|\bar{a}_{1,h}(x, -x)f\|_{H_h^s} \right].$$

The middle term is (2.3.36) (with  $\ell = \frac{1}{2}$ ) multiplied by  $h^{-\frac{1}{2}}$ . Because of (2.3.37), it is bounded by the right hand side of (2.3.7). The same holds true trivially for the other contributions to (2.3.39). This concludes the proof.  $\square$

**Corollary 2.3.5** *With the notations and assumptions of (ii) of Proposition 2.3.2, for any  $\sigma, \delta$  in  $]0, \frac{1}{2}]$ , any  $s \geq 1 + \frac{1}{\sigma}$ , there is  $C > 0$  such that, for any odd function  $f$*

$$(2.3.40) \quad \|R^0(f)\|_{L^2} \leq C [h^{1-\sigma} \|\mathcal{L}f\|_{L^2} + h^{\delta-\sigma} \|f\|_{H_h^s}]$$

$$(2.3.41) \quad \|R^0(f)\|_{L^\infty} \leq C [h^{\frac{1}{2}-\frac{3}{2}\sigma} \|\mathcal{L}f\|_{L^2} + h^{\frac{1}{2}-\frac{3}{2}\sigma-\delta} \|f\|_{H_h^s} + h^\delta \|f\|_{L^\infty}].$$

*Proof:* We apply (2.3.6) with  $\ell = 0$ , (2.3.7), taking  $N = 2$ . Since  $f$  is odd, we may bound  $\|\langle \frac{x}{h} \rangle^{-2} f\|_{L^2}$  in the right hand side of (2.3.6), (2.3.7) from (2.2.14) with  $c \equiv 1$ ,  $q = 0$ . This gives (2.3.40), (2.3.41).  $\square$

### 3 Proof of the main theorem

#### 3.1 Energy estimates

The goal of this subsection is to establish energy estimates for the action of  $\mathcal{L}$  on the solution  $v$  of (2.1.12).

**Proposition 3.1.1** *For any  $s \geq 0$ , there is a constant  $C > 0$  such that, for any odd  $v_0$  in  $H^s(\mathbb{R}, \mathbb{C})$  with  $xv_0$  in  $L^2$ , the solution  $v$  of (2.1.12), with initial data  $v|_{t=1} = v_0$ , satisfies for any  $t \geq 1$  such that the solution exists up to time  $t$ ,*

$$(3.1.1) \quad \|\mathcal{L}v(t, \cdot)\|_{L^2} + \|v(t, \cdot)\|_{H_h^s} \leq \|\mathcal{L}v(1, \cdot)\|_{L^2} + \|v(1, \cdot)\|_{H_h^s} + C \int_1^t \|v(\tau, \cdot)\|_{L^\infty}^2 \left( \|\mathcal{L}v(\tau, \cdot)\|_{L^2} + \|v(\tau, \cdot)\|_{H_{h(\tau)}^s} \right) \frac{d\tau}{\tau}$$

where  $h$  denotes  $\frac{1}{t}$  and  $h(\tau) = \frac{1}{\tau}$ .

**Remark:** The problem (1.1.7) is globally well posed in  $H^1$  and locally well-posed in the space of functions  $u$  in  $H^1$  with  $xu \in L^2$ , so that our solution  $v$  exists for any  $t \geq 1$  and  $\|\mathcal{L}v(t, \cdot)\|_{L^2}$  is finite at any  $t$ .

We shall prove first two lemmas.

**Lemma 3.1.2** *Let  $a_h$  be the symbol defined in (2.1.8). Then  $\text{Op}_h^1(a_h)$ ,  $\text{Op}_h^0(\kappa(x/h)\bar{a}_h)$  are bounded on  $H_{\text{sc}}^s$  for any  $s$ , and  $\text{Op}_h^1(a_h)$  is bounded on  $L^\infty$ , uniformly in  $h$ .*

*Proof:* Since

$$(3.1.2) \quad a_h(x, \xi) = a_{0,h}(x, \xi) + |\xi|a_{1,h}(x, \xi)$$

with  $a_{0,h}$  in  $\Sigma_1(1) \subset S_1(1)$ ,  $a_{1,h}$  in  $\Sigma_1(\langle \xi \rangle^{-1}) \subset S_1(\langle \xi \rangle^{-1})$ , the Sobolev boundedness properties follow from (ii) of lemma 2.2.1 and from the fact that  $|hD|$  is bounded from  $H_{\text{sc}}^s$  to  $H_{\text{sc}}^{s-1}$  for any  $s$ .

To prove the  $L^\infty$  boundedness of  $\text{Op}_h^1(a_h)$ , we use the structure (2.1.9) of  $a_{0,h}$ ,  $a_{1,h}$ . By (1.2.9),  $\chi_+(\frac{x}{h})[m_1(\frac{x}{h}, \xi) - 1]$  and  $\chi_-(\frac{x}{h})[m_2(\frac{x}{h}, \xi) - 1]$  are in  $\Sigma_1(\langle \xi \rangle^{-1})$ . Consequently, we may write

$$(3.1.3) \quad a_h(x, \xi) - \chi_+\left(\frac{x}{h}\right)[e_0(\xi) + e_1(\xi)|\xi|] - \chi_-\left(\frac{x}{h}\right)[e_0(-\xi) + e_1(-\xi)|\xi|] \\ = b_0(x, \xi) + b_1(x, \xi)|\xi|$$

with  $b_0$  in  $\Sigma_1(\langle \xi \rangle^{-1})$ ,  $b_1$  in  $\Sigma_1(\langle \xi \rangle^{-2})$ . We may rewrite this as  $b'_0(x, \xi) + b_1(x, \xi)\chi_0(\xi)|\xi|$ , with  $b'_0$  again in  $\Sigma_1(\langle \xi \rangle^{-1})$ , and  $\chi_0$  in  $C_0^\infty(\mathbb{R})$  equal to one close to zero. By (iii) of lemma 2.2.1,  $\text{Op}_h^1(b'_0)$ ,  $\text{Op}_h^1(b_1)$  are bounded on  $L^\infty$ , and  $\chi_0(hD)|hD|$  is also bounded on that space, as its distribution kernel is bounded by  $\frac{C}{h}\langle \frac{x}{h} \rangle^{-2}$ . Modulo operators bounded on  $L^\infty$ , we may thus study the action on that space of the quantization of

$$(3.1.4) \quad \chi_+\left(\frac{x}{h}\right)[e_0(\xi) + e_1(\xi)|\xi|] + \chi_-\left(\frac{x}{h}\right)[e_0(-\xi) + e_1(-\xi)|\xi|].$$

If we multiply (3.1.4) by  $\chi_0(\xi)$ , we get again a symbol whose quantization is bounded on  $L^\infty$ . On the other hand, by (1.2.7),  $(1 - \chi_0)(\xi)[e_0(\xi) + e_1(\xi)|\xi| - 1]$  is in  $\Sigma_1(\langle \xi \rangle^{-1})$ , so the associated operator is bounded on  $L^\infty$  by lemma 2.2.1. We are thus left with

$$\text{Op}_h^1\left[\left(\chi_+\left(\frac{x}{h}\right) + \chi_-\left(\frac{x}{h}\right)\right)(1 - \chi_0)(\xi)\right]$$

which is trivially bounded on  $L^\infty$ . This concludes the proof.  $\square$

**Lemma 3.1.3** *Let  $a_h$  be the symbol defined in (2.1.8).*

(i) *There is  $C > 0$  such that, for any odd  $v$  in  $L^2 \cap L^\infty$  such that  $\mathcal{L}v$  is in  $L^2$ , we have*

$$(3.1.5) \quad \left\| \mathcal{L}\left[\text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\left[|\text{Op}_h^1(a_h)v|^2|\text{Op}_h^1(a_h)v|\right]\right] \right\|_{L^2} + \left\| \mathcal{L}\left[|\text{Op}_h^1(a_h)v|^2|\text{Op}_h^1(a_h)v|\right] \right\|_{L^2} \\ \leq C\|v\|_{L^\infty}^2[\|\mathcal{L}v\|_{L^2} + \|v\|_{L^2}].$$

(ii) *For any  $s \geq 0$ , there is  $C > 0$  and for any  $v$  in  $L^\infty \cap H_{\text{sc}}^s$ , any  $h \in ]0, 1]$*

$$(3.1.6) \quad \left\| \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\left[|\text{Op}_h^1(a_h)v|^2|\text{Op}_h^1(a_h)v|\right] \right\|_{H_h^s} + \left\| |\text{Op}_h^1(a_h)v|^2|\text{Op}_h^1(a_h)v| \right\|_{H_h^s} \\ \leq C\|v\|_{L^\infty}^2\|v\|_{H_h^s}.$$

*Proof:* (i) Using that

$$\mathcal{L}[|w|^2 w] = 2(\mathcal{L}w)|w|^2 - w^2 \overline{\mathcal{L}w},$$

we may write

$$\mathcal{L}\left[\text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\left[|\text{Op}_h^1(a_h)v|^2|\text{Op}_h^1(a_h)v|\right]\right]$$

as a combination of terms of the form

$$(3.1.7) \quad \left[\mathcal{L}, \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\right]\left[|\text{Op}_h^1(a_h)v|^2|\text{Op}_h^1(a_h)v|\right]$$

$$(3.1.8) \quad \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\left[\left([\mathcal{L}, \text{Op}_h^1(a_h)]v\right)\left(\text{Op}_h^1(a_h)v\right)\left(\overline{\text{Op}_h^1(a_h)v}\right)\right]$$

$$(3.1.9) \quad \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\left[\left(\text{Op}_h^1(a_h)\mathcal{L}v\right)\left(\text{Op}_h^1(a_h)v\right)\left(\overline{\text{Op}_h^1(a_h)v}\right)\right]$$

and of terms similar to (3.1.8), (3.1.9), where the conjugation bar lies on another factor (that does not modify the estimates below). By lemma 3.1.2, the  $L^2$  norm of (3.1.9) is bounded from above by  $C\|v\|_{L^\infty}^2\|\mathcal{L}v\|_{L^2}$ , so by the right hand side of (3.1.5). To bound the  $L^2$  norm of (3.1.8) by the right hand side of (3.1.5), we see again by lemma 3.1.2 that it suffices to prove

$$(3.1.10) \quad \|\left[\mathcal{L}, \text{Op}_h^1(a_h)\right]v\|_{L^2} \leq C\left[\|\mathcal{L}v\|_{L^2} + \|v\|_{L^2}\right].$$

We write, by (3.1.2),

$$(3.1.11) \quad \left[\mathcal{L}, \text{Op}_h^1(a_h)\right]v = \left[\mathcal{L}, \text{Op}_h^1(a_{0,h})\right]v + \left[\mathcal{L}, \text{Op}_h^1(a_{1,h})\right]|hD|v + \text{Op}_h^1(a_{1,h})\left[\mathcal{L}, |hD|\right]v.$$

As  $[\mathcal{L}, |hD|] = i\text{sgn}(hD)$  is bounded on  $L^2$ , (ii) of lemma 2.2.1 implies that the  $L^2$  norm of the last term in (3.1.11) is bounded by the right hand side of (3.1.10). According to (2.2.1) and the definition (2.2.13) of  $\mathcal{L}$ , for any function  $w$ ,

$$(3.1.12) \quad \left[\mathcal{L}, \text{Op}_h^1(a_{j,h})\right]w = i\text{Op}_h^1\left[\frac{\partial a_{j,h}}{\partial \xi} - \frac{\partial a_{j,h}}{\partial x}\right]w, \quad j = 0, 1.$$

By (2.1.1),  $\frac{\partial a_{j,h}}{\partial \xi}$  is in  $S_1(\langle \xi \rangle^{-j})$  and by (2.1.10),  $\langle \frac{x}{i} \rangle^N h \frac{\partial a_{j,h}}{\partial x}$  is in  $S_1(\langle \xi \rangle^{-j})$  for any  $N$ . It follows from (2.2.14) and lemma 2.2.1 (ii) that, if  $w$  is odd,

$$\|\left[\mathcal{L}, \text{Op}_h^1(a_{j,h})\right]w\|_{L^2} \leq C\left[\|w\|_{H_h^{-j}} + \|\mathcal{L}w\|_{H_h^{-j}}\right], \quad j = 0, 1.$$

We apply this inequality to  $w = v$  when  $j = 0$  and to  $w = |hD|v$  when  $j = 1$ . We deduce from the fact that  $[\mathcal{L}, |hD|]$  is bounded on  $L^2$  that the  $L^2$  norm of the first two terms in the right hand side of (3.1.11) is bounded by the right hand side of (3.1.10). Consider finally (3.1.7) and write

$$(3.1.13) \quad \left[\mathcal{L}, \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right)\right]f = \left[\mathcal{L}, \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_{0,h}\right)\right]f \\ + |hD|\left[\mathcal{L}, \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_{1,h}\right)\right]f + \left[\mathcal{L}, |hD|\right]\text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_{1,h}\right)f.$$

By (2.2.1), we have for  $j = 0, 1$

$$(3.1.14) \quad \left[\mathcal{L}, \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_{j,h}\right)\right]f = i\text{Op}_h^0\left[\frac{\partial}{\partial \xi}\left(\bar{a}_{j,h}\kappa\left(\frac{x}{h}\right)\right) - \frac{\partial}{\partial x}\left(\bar{a}_{j,h}\kappa\left(\frac{x}{h}\right)\right)\right].$$

Again, as  $\frac{\partial}{\partial \xi}(\bar{a}_{j,h}\kappa(\frac{x}{h}))$  is in  $S_1(\langle \xi \rangle^{-j-1}) \subset S_1(\langle \xi \rangle^{-j})$  and  $\langle \frac{x}{h} \rangle^N h \frac{\partial}{\partial x}(\bar{a}_{j,h}\kappa(\frac{x}{h}))$  is in  $S_1(\langle \xi \rangle^{-j})$ , we may write the right hand side of (3.1.14) as

$$\text{Op}_h^0 \left[ c_{1,j}(x, \xi) + \frac{1}{h} \left\langle \frac{x}{h} \right\rangle^{-2} c_{2,j}(x, \xi) \right] = \text{Op}_h^0(c_{1,j}) + \frac{1}{h} \text{Op}_h^0(c_{2,j}) \left\langle \frac{x}{h} \right\rangle^{-2}$$

where  $c_{1,j}, c_{2,j}$  are in  $S_1(\langle \xi \rangle^{-j})$ , and where we exploited the definition of the 0-quantification. Consequently

$$\left\| |hD|^j \left[ \mathcal{L}, \text{Op}_h^0 \left( \kappa \left( \frac{x}{h} \right) \bar{a}_{j,h} \right) \right] f \right\|_{L^2} \leq C \left[ \|f\|_{L^2} + \frac{1}{h} \left\| \left\langle \frac{x}{h} \right\rangle^{-2} f \right\|_{L^2} \right].$$

We thus deduce from (3.1.13) that

$$\left\| \left[ \mathcal{L}, \text{Op}_h^0 \left( \kappa \left( \frac{x}{h} \right) \bar{a}_h \right) \right] f \right\|_{L^2} \leq C \left[ \|f\|_{L^2} + \frac{1}{h} \left\| \left\langle \frac{x}{h} \right\rangle^{-2} f \right\|_{L^2} \right].$$

The  $L^2$  norm of (3.1.7) will thus be bounded by

$$(3.1.15) \quad C \left[ \left\| \text{Op}_h^1(a_h)v \right\|_{L^\infty}^2 \left\| \text{Op}_h^1(a_h)v \right\|_{L^2} + \frac{C}{h} \left\| \text{Op}_h^1(a_h)v \right\|_{L^\infty}^2 \left\| \left\langle \frac{x}{h} \right\rangle^{-2} \text{Op}_h^1(a_h)v \right\|_{L^2} \right].$$

As  $v$  is odd, we deduce from (2.2.16) and lemma 3.1.2 that (3.1.15) is bounded from above by the right hand side of (3.1.5).

(ii) Estimate (3.1.6) follows from the boundedness properties of lemma 3.1.2 together with the elementary inequality

$$\|w_1 w_2 w_3\|_{H_h^s} \leq C \sum_{i=1}^3 \|w_i\|_{H_h^s} \prod_{\substack{1 \leq j \leq 3 \\ j \neq i}} \|w_j\|_{L^\infty}$$

that holds for  $s \geq 0$ . □

*Proof of Proposition 3.1.1:* The equation (2.1.12) satisfied by  $v$  may be written as

$$(3.1.16) \quad \left( D_t - \text{Op}_h^W \left( x\xi + \frac{\xi^2}{2} \right) \right) v = hf$$

where according to (3.1.6)

$$\|f\|_{H_h^s} \leq C \|v\|_{L^\infty} \|v\|_{H_h^s}.$$

The Sobolev energy inequality implies

$$(3.1.17) \quad \|v(t, \cdot)\|_{H_h^s} \leq \|v(1, \cdot)\|_{H^s} + C \int_1^t \|v(\tau, \cdot)\|_{L^\infty} \|v(\tau, \cdot)\|_{H_{h(\tau)}^s} \frac{d\tau}{\tau}.$$

(Notice that  $[D_t - \text{Op}_h^W(x\xi), \text{Op}_h^W(\langle \xi \rangle^s)] = 0$ ). Next, we make act  $\mathcal{L}$  on (3.1.16), using that  $[\mathcal{L}, D_t - \text{Op}_h^W(x\xi + \frac{\xi^2}{2})] = 0$ , which just reflects the commutation relation  $[x + tD_x, D_t - \frac{D_x^2}{2}] = 0$ . We obtain

$$(3.1.18) \quad \left( D_t - \text{Op}_h^W \left( x\xi + \frac{\xi^2}{2} \right) \right) \mathcal{L}v = h\mathcal{L}f$$

and  $\|\mathcal{L}f\|_{L^2}$  is bounded from above by the right hand side of (3.1.5). Using the energy inequality for (3.1.18) and combining with (3.1.17), we get (3.1.1). This concludes the proof. □

### 3.2 Reduction to a differential equation

Recall that we have defined in (2.3.1) the set

$$\Lambda = \{(x, \xi) \in \mathbb{R} \times \mathbb{R}; x + \xi = 0\}.$$

Let  $\gamma$  be a  $C_0^\infty(\mathbb{R})$  function equal to one close to zero, and define for any function  $f$

$$(3.2.1) \quad f_\Lambda = \text{Op}_h^W\left(\gamma\left(\frac{x + \xi}{\sqrt{h}}\right)\right)f.$$

We have the following estimates:

**Lemma 3.2.1** *With the above notation*

$$(3.2.2) \quad \|f_\Lambda - f\|_{L^2} \leq Ch^{\frac{1}{2}}\|\mathcal{L}f\|_{L^2}$$

$$(3.2.3) \quad \|f_\Lambda - f\|_{L^\infty} \leq Ch^{\frac{1}{4}}\|\mathcal{L}f\|_{L^2}$$

$$(3.2.4) \quad \|\text{Op}_h^W[(x + \xi)^2]f_\Lambda\|_{L^2} \leq Ch^{\frac{3}{2}}\|\mathcal{L}f\|_{L^2}$$

$$(3.2.5) \quad \|\text{Op}_h^W[(x + \xi)^2]f_\Lambda\|_{L^\infty} \leq Ch^{\frac{5}{4}}\|\mathcal{L}f\|_{L^2}.$$

*Proof:* Set  $\tilde{\gamma}(z) = \frac{1-\gamma(z)}{z}$ . Then using the last equality (2.2.1)

$$f - f_\Lambda = \text{Op}_h^W\left(\tilde{\gamma}\left(\frac{x + \xi}{\sqrt{h}}\right)\left(\frac{x + \xi}{\sqrt{h}}\right)\right)f = \sqrt{h}\text{Op}_h^W\left(\tilde{\gamma}\left(\frac{x + \xi}{\sqrt{h}}\right)\right)\mathcal{L}f.$$

Then (3.2.2) (resp. (3.2.3)) follows from (i) of lemma 2.2.1 (resp. from (iii) of lemma 2.2.2).

To prove (3.2.4) and (3.2.5), we remark that repeated applications of the third equality (2.2.1) gives

$$\text{Op}_h^W\left((x + \xi)^2\right)f_\Lambda = h^{\frac{3}{2}}\text{Op}_h^W\left(\gamma\left(\frac{x + \xi}{\sqrt{h}}\right)\left(\frac{x + \xi}{\sqrt{h}}\right)\right)\mathcal{L}f.$$

Applying again (i) of lemma 2.2.1 and (iii) of lemma 2.2.2, we obtain (3.2.4) and (3.2.5).  $\square$

Let us state now the ordinary differential equation satisfied by  $v_\Lambda$  when  $v$  solves (2.1.12).

**Proposition 3.2.2** *Let  $\theta \in ]0, \frac{1}{4}[$  and take  $s \geq 1 + \frac{3}{2\theta}$ . Let  $v$  be an odd solution of (2.1.12). Then  $v_\Lambda$  solves the ODE*

$$(3.2.6) \quad D_t v_\Lambda = -\frac{x^2}{2}v_\Lambda + h\kappa\left(\frac{x}{h}\right)|v_\Lambda|^2 v_\Lambda + hr$$

where  $r$  satisfies

$$(3.2.7) \quad \|r(t, \cdot)\|_{L^2} \leq Ch^{\frac{1}{2}}(1 + \|v(t, \cdot)\|_{L^\infty}^2)\|\mathcal{L}v(t, \cdot)\|_{L^2} + Ch^{\frac{1}{4}-\theta}\|v(t, \cdot)\|_{L^\infty}^2\|v(t, \cdot)\|_{H_h^s}$$

and

$$(3.2.8) \quad \|r(t, \cdot)\|_{L^\infty} \leq Ch^{\frac{1}{4}}(1 + \|v(t, \cdot)\|_{L^\infty}^2)\|\mathcal{L}v(t, \cdot)\|_{L^2} + Ch^{\frac{1}{4}-\theta}\|v(t, \cdot)\|_{L^\infty}^2\|v(t, \cdot)\|_{H_h^s} + Ch^{\frac{1}{4}}\|v(t, \cdot)\|_{L^\infty}^3,$$

$h$  still denoting  $1/t$ .



Let us prove first

**Lemma 3.2.3** *Let  $v$  be a solution to (2.1.12). Then  $v_\Lambda$  satisfies*

$$(3.2.9) \quad \left( D_t - \text{Op}_h^W \left( x\xi + \frac{\xi^2}{2} \right) \right) v_\Lambda \\ = h \text{Op}_h^W \left( \gamma \left( \frac{x + \xi}{\sqrt{h}} \right) \right) \left[ \text{Op}_h^0 \left( \kappa \left( \frac{x}{h} \right) \bar{a}_h \right) [|\text{Op}_h^1(a_h)v|^2 |\text{Op}_h^1(a_h)v|] \right] + hR$$

where  $R$  satisfies

$$(3.2.10) \quad \|R(t, \cdot)\|_{L^2} \leq Ch^{\frac{1}{2}} \|\mathcal{L}v(t, \cdot)\|_{L^2} \\ \|R(t, \cdot)\|_{L^\infty} \leq Ch^{\frac{1}{4}} \|\mathcal{L}v(t, \cdot)\|_{L^2}.$$

*Proof:* Let us compute

$$\left[ D_t - \text{Op}_h^W \left( x\xi + \frac{\xi^2}{2} \right), \text{Op}_h^W \left( \gamma \left( \frac{x + \xi}{\sqrt{h}} \right) \right) \right].$$

If  $a(t, x, \xi)$  is a symbol, it follows from the definition (2.1.4) of

$$\text{Op}_h^W(a)v = \frac{1}{2\pi} \int e^{i(x-y)\xi} a \left( t, \frac{x+y}{2}, h\xi \right) v(y) dy d\xi$$

with  $h = \frac{1}{t}$  that

$$(3.2.11) \quad [D_t, \text{Op}_h^W(a(t, x, \xi))] = \text{Op}_h^W((D_t - h\xi \cdot D_\xi a)(t, x, \xi)).$$

Moreover, it follows from Proposition 7.7 in [13] that, if  $a_1, a_2$  are symbols, where one of them is a polynomial of degree smaller or equal to two,

$$(3.2.12) \quad [\text{Op}_h^W(a_1), \text{Op}_h^W(a_2)] = \frac{h}{i} \text{Op}_h^W(\{a_1, a_2\}).$$

Actually, in the expansion (7.21) of the symbol of a composition in [13], terms of even order are symmetric in  $(a_1, a_2)$ , so that they cancel out in the symbol of the commutator. Only terms of odd order remain and, if  $a_1$  or  $a_2$  is a polynomial of degree at most two in  $(x, \xi)$ , one just gets the right hand side of (3.2.12). It follows from (3.2.11), (3.2.12) that

$$\left[ D_t - \text{Op}_h^W \left( x\xi + \frac{\xi^2}{2} \right), \text{Op}_h^W \left( \gamma \left( \frac{x + \xi}{\sqrt{h}} \right) \right) \right] = i \frac{h^{\frac{3}{2}}}{2} \text{Op}_h^W \left( \gamma' \left( \frac{x + \xi}{\sqrt{h}} \right) \right) \mathcal{L}.$$

This implies (3.2.9), with estimates (3.2.10) for the remainder, using (i) of lemma 2.2.1 and (iii) of lemma 2.2.2.  $\square$

*Proof of Proposition 3.2.2:* We start from equation (3.2.9). As  $R$  in the right hand side of (3.2.9) contributes to  $r$  in the right hand side of (3.2.6), we see by difference of these two equations that we have to show that

$$(3.2.13) \quad h^{-1} \left[ \text{Op}_h^W \left( x\xi + \frac{\xi^2}{2} \right) v_\Lambda + \frac{1}{2} x^2 v_\Lambda \right]$$

and

$$(3.2.14) \quad \text{Op}_h^W\left(\gamma\left(\frac{x+\xi}{\sqrt{h}}\right)\right) \left[ \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right) \left[ |\text{Op}_h^1(a_h)v|^2 \text{Op}_h^1(a_h)v \right] - \kappa\left(\frac{x}{h}\right)|v_\Lambda|^2 v_\Lambda \right]$$

are estimated in  $L^2$  (resp.  $L^\infty$ ) by the right hand side of (3.2.7) (resp. (3.2.8)).

Consider first (3.2.13) and write  $x\xi + \frac{\xi^2}{2} = -\frac{1}{2}x^2 + \frac{1}{2}(x+\xi)^2$ . Then (3.2.13) equals

$$\frac{1}{2}h^{-1}\text{Op}_h^W((x+\xi)^2)v_\Lambda.$$

Its  $L^2$  (resp.  $L^\infty$ ) norm is bounded from above by (3.2.7) (resp. (3.2.8)) according to (3.2.4) (resp. (3.2.5)).

Consider next (3.2.14). Notice first that

$$\text{Op}_h^W\left(\gamma\left(\frac{x+\xi}{\sqrt{h}}\right) - 1\right) \left[ \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right) \left[ |\text{Op}_h^1(a_h)v|^2 \text{Op}_h^1(a_h)v \right] \right]$$

contributes to  $r$  in (3.2.6) as a consequence of (3.2.2), (3.2.3) and of (3.1.5). We are thus reduced to showing that

$$(3.2.15) \quad \text{Op}_h^0\left(\kappa\left(\frac{x}{h}\right)\bar{a}_h\right) \left[ |\text{Op}_h^1(a_h)v|^2 \text{Op}_h^1(a_h)v \right] - \kappa\left(\frac{x}{h}\right)|v_\Lambda|^2 v_\Lambda$$

may be estimated by the right hand side of (3.2.7) in  $L^2$  and (3.2.8) in  $L^\infty$ .

We apply Corollary 2.3.5 with  $\delta = \frac{1}{4}$ ,  $\sigma = \frac{2}{3}\theta$ ,  $s \geq 1 + \frac{3}{2\theta}$ ,  $\theta \in ]0, \frac{1}{4}[$ . We may write (3.2.15) as

$$(3.2.16) \quad \kappa\left(\frac{x}{h}\right) \left[ \bar{a}_h(x, -x) |\text{Op}_h^1(a_h)v|^2 \text{Op}_h^1(a_h)v - |v_\Lambda|^2 v_\Lambda \right]$$

up to terms bounded in  $L^2$  by

$$(3.2.17) \quad C[h^{\frac{1}{2}}\|\mathcal{L}f\|_{L^2} + h^{\frac{1}{4}-\theta}\|f\|_{H_h^s}]$$

and in  $L^\infty$  by

$$(3.2.18) \quad C[h^{\frac{1}{2}-\theta}\|\mathcal{L}f\|_{L^2} + h^{\frac{1}{4}-\theta}\|f\|_{H_h^s} + h^{\frac{1}{4}}\|f\|_{L^\infty}]$$

where  $f = |\text{Op}_h^1(a_h)v|^2 \text{Op}_h^1(a_h)v$  (Notice that  $f$  is an odd function as  $v$  is odd: see the remark after Definition 1.2.1). By lemmas 3.1.3 and 3.1.2, (3.2.17) (resp. (3.2.18)) is smaller than the right hand side of (3.2.7) (resp. (3.2.8)). We have thus reduced to showing that (3.2.16) may be estimated in  $L^2$  (resp.  $L^\infty$ ) by the right hand side of (3.2.7) (resp. (3.2.8)). By Proposition 2.3.2 (i), if we replace in (3.2.16) one factor  $\text{Op}_h^1(a_h)v$  by  $a_h(x, -x)v$ , and use the boundedness of  $\text{Op}_h^1(a_h)$  on  $L^\infty$  established in lemma 3.1.2, we may rewrite (3.2.16) as

$$(3.2.19) \quad \kappa\left(\frac{x}{h}\right) \left[ |a_h(x, -x)|^4 |v|^2 v - |v_\Lambda|^2 v_\Lambda \right]$$

modulo an expression bounded in  $L^2$  by

$$(3.2.20) \quad C(h^{1-\sigma}\|\mathcal{L}v\|_{L^2} + h^\delta\|v\|_{H_h^s})\|v\|_{L^\infty}^2$$

and in  $L^\infty$  by

$$(3.2.21) \quad C\left(h^{\frac{1}{2}-\frac{3}{2}\sigma}\|\mathcal{L}v\|_{L^2} + h^{\frac{1}{2}-\frac{\sigma}{2}-\delta}\|v\|_{H_h^s} + h^\delta\|v\|_{L^\infty}\right)\|v\|_{L^\infty}^2$$

so by the right hand sides of (3.2.7) and (3.2.8) respectively, taking into account the definition of  $\delta, \theta$ .

We are left with showing similar estimates for (3.2.19), that we rewrite as  $I + II$  with

$$(3.2.22) \quad \begin{aligned} I &= \kappa\left(\frac{x}{h}\right)[|a_h(x, -x)|^4 - 1]|v|^2v \\ II &= \kappa\left(\frac{x}{h}\right)[|v|^2v - |v_\Lambda|^2v_\Lambda]. \end{aligned}$$

The fact that  $II$  has  $L^2$  (resp.  $L^\infty$ ) norm bounded from above by the right hand side of (3.2.7) (resp. (3.2.8)) follows from (3.2.2), (3.2.3) and the fact that, by (2.2.12),  $\text{Op}_h^W\left(\gamma\left(\frac{x+\xi}{\sqrt{h}}\right)\right)$  is bounded on any  $L^p$  space.

Let us study  $I$ . Notice that according to (2.2.14), (2.2.17)

$$\begin{aligned} \left\|\left\langle\frac{x}{h}\right\rangle^{-2}|v|^2v\right\|_{L^2} &\leq Ch\|v\|_{L^\infty}^2[\|\mathcal{L}v\|_{L^2} + \|v\|_{L^2}] \\ \left\|\left\langle\frac{x}{h}\right\rangle^{-2}|v|^2v\right\|_{L^\infty} &\leq Ch^{\frac{1}{2}-\frac{\sigma}{2}}\|v\|_{L^\infty}^2[\|\mathcal{L}v\|_{L^2} + \|v\|_{H_h^s}]. \end{aligned}$$

Consequently, to get  $L^2$  (resp.  $L^\infty$ ) bounds of the form (3.2.20) (resp. (3.2.21)) for  $I$ , it suffices to show that

$$(3.2.23) \quad |a_h(x, -x)|^4 - 1 = O\left(\left\langle\frac{x}{h}\right\rangle^{-2}\right).$$

By (2.1.8), (2.1.9),

$$a_h(x, -x) = \chi_+\left(\frac{x}{h}\right)m_1\left(\frac{x}{h}, -x\right)[e_0(-x) + |x|e_1(-x)] + \chi_-\left(\frac{x}{h}\right)m_2\left(\frac{x}{h}, -x\right)[e_0(x) + |x|e_1(x)].$$

By (1.2.9), we may replace above  $m_1$  and  $m_2$  by 1, up to a  $O\left(\left\langle\frac{x}{h}\right\rangle^{-N}\right)$  contribution. We have thus reduced  $a_h(x, -x)$  in (3.2.23) to

$$\chi_+\left(\frac{x}{h}\right)[e_0(-x) + |x|e_1(-x)] + \chi_-\left(\frac{x}{h}\right)[e_0(x) + |x|e_1(x)].$$

Notice that  $\chi_\pm(x/h) - \mathbb{1}_\pm(x) = O\left(\left\langle\frac{x}{h}\right\rangle^{-N}\right)$  for any  $N$ , so that we may reduce the left hand side of (3.2.23), up to an admissible error in  $O\left(\left\langle\frac{x}{h}\right\rangle^{-2}\right)$ , to

$$|\mathbb{1}_+(x)(e_0(-x) + |x|e_1(-x)) + \mathbb{1}_-(x)(e_0(x) + |x|e_1(x))|^4 - 1.$$

Using (1.2.8), we see that this last expression vanishes. This concludes the proof of the proposition.  $\square$

### 3.3 Proof of the main theorem

We prove first  $L^2$  and  $L^\infty$  estimates for the solution of (2.1.12).

**Proposition 3.3.1** *Fix  $s > 7$ ,  $C_0 > 0$ . There are constants  $A, B, K > 0$ ,  $\epsilon_0 \in ]0, 1[$  such that for any  $\epsilon \in ]0, \epsilon_0[$ , any odd function  $v_0$  in  $H^s(\mathbb{R}, \mathbb{C})$  with*

$$(3.3.1) \quad \|v_0\|_{H^s} + \|xv_0\|_{L^2} \leq C_0,$$

*the solution  $v$  to (2.1.12) with initial data  $v|_{t=1} = \epsilon v_0$  exists for  $t \geq 1$  in  $C^0([1, +\infty[, H^s(\mathbb{R}, \mathbb{C}))$  and satisfies the estimates*

$$(3.3.2) \quad \begin{aligned} \|v(t, \cdot)\|_{H_h^s} + \|\mathcal{L}v(t, \cdot)\|_{L^2} &\leq A\epsilon t^{K\epsilon^2} \\ \|v(t, \cdot)\|_{L^\infty} &\leq B\epsilon. \end{aligned}$$

We prove first:

**Lemma 3.3.2** *Let  $\theta \in ]0, \frac{1}{4}[$  and take  $s \geq 1 + \frac{3}{2\theta}$ . There are constants  $A, B, K > 0$ ,  $\epsilon_0 \in ]0, 1[$  with*

$$(3.3.3) \quad \epsilon_0^{-1} \gg K \gg B \gg A$$

*such that, if estimates (3.3.2) hold on some interval  $[1, T]$  when  $\epsilon < \epsilon_0$ , then for any  $1 \leq t' \leq t \leq T$ , one has the bounds*

$$(3.3.4) \quad \left| |v_\Lambda(t, x)|^2 - |v_\Lambda(t', x)|^2 \right| \leq \frac{\epsilon^2 B^2}{32} t'^{-\frac{1}{4} + \theta}.$$

*Proof:* Computing  $\partial_t |v_\Lambda(t, x)|^2$  from (3.2.6), we get by integration

$$|v_\Lambda(t, x)|^2 = |v_\Lambda(t', x)|^2 + 2\operatorname{Re} i \int_{t'}^t r(\tau, x) \bar{v}_\Lambda(\tau, x) \frac{d\tau}{\tau}$$

from which we deduce

$$(3.3.5) \quad \left| |v_\Lambda(t, x)|^2 - |v_\Lambda(t', x)|^2 \right| \leq 2 \int_{t'}^t \|r(\tau, x)\|_{L^\infty} \|\bar{v}_\Lambda(\tau, x)\|_{L^\infty} \frac{d\tau}{\tau}.$$

By (3.2.3) and the first estimate (3.3.2),

$$(3.3.6) \quad \|v(t, \cdot) - v_\Lambda(t, \cdot)\|_{L^\infty} \leq Ct^{-\frac{1}{4}} \|\mathcal{L}v\|_{L^2} \leq CA\epsilon t^{K\epsilon^2 - \frac{1}{4}} \leq CA\epsilon$$

if  $K\epsilon^2$  is small enough. Taking the second estimate (3.3.2) into account, we get

$$(3.3.7) \quad \|v_\Lambda(t, \cdot)\|_{L^\infty} \leq (CA + B)\epsilon.$$

Moreover, by (3.2.8) and (3.3.2)

$$(3.3.8) \quad \|r(t, \cdot)\|_{L^\infty} \leq Ct^{-\frac{1}{4} + \theta} [(1 + B^2\epsilon^2)A\epsilon + B^3\epsilon^3]$$

if  $K\epsilon_0^2 \leq \theta$ . Plugging (3.3.7) and (3.3.8) in the right hand side of (3.3.5), and taking  $A \ll B \ll \epsilon_0^{-1}$ , we get the bound (3.3.4).  $\square$

*Proof of Proposition 3.3.1:* As already remarked after the statement of Theorem 1.1.1 and Proposition 3.3.1, existence of the solution and finiteness of the quantities to be estimated in (3.3.2) is not an issue. We prove (3.3.2) by bootstrap, assuming that these estimates hold on some interval  $[1, T]$ . Let us show that, then, on the same interval, (3.3.2) holds with  $A$  (resp.  $B$ ) replaced by  $\frac{A}{2}$  (resp.  $\frac{B}{2}$ ). Coupled with the time continuity of the left hand side of (3.3.2), this will show that these inequalities hold for any  $t$ .

Plugging our assumption (3.3.2) in the right hand side of (3.1.1), we get

$$\|\mathcal{L}v(t, \cdot)\|_{L^2} + \|v(t, \cdot)\|_{H_h^s} \leq \frac{A}{4}\epsilon + CAB^2\epsilon^3 \int_1^t \tau^{K\epsilon^2} \frac{d\tau}{\tau} \leq A\epsilon \left[ \frac{1}{4} + \frac{CB^2}{K} \right]$$

if  $A$  has been taken large enough relatively to the constant  $C_0$  in (3.3.1). Under condition (3.3.3) on the constants, we bound this by  $\frac{A}{2}\epsilon$  as wanted.

To get the  $L^\infty$  estimate, we write by (3.2.3)

$$\|v(t, \cdot) - v_\Lambda(t, \cdot)\|_{L^\infty} \leq Ct^{-\frac{1}{4}} \|\mathcal{L}v(t, \cdot)\|_{L^2} \leq C\epsilon A$$

if  $K\epsilon_0^2 \leq \frac{1}{4}$ , according to assumption (3.3.2). If  $B \gg A$ , we bound this by  $\epsilon\frac{B}{4}$ . To get the second estimate (3.3.2) with  $B$  replaced by  $\frac{B}{2}$ , we are reduced to showing that  $\|v_\Lambda(t, \cdot)\|_{L^\infty} \leq \epsilon\frac{B}{4}$ . But if  $s > 7$ , and if we take in lemma 3.3.2 some  $\theta$  in  $]0, \frac{1}{4}[$ , close enough to  $\frac{1}{4}$  to ensure the assumption of that lemma, we deduce from (3.3.4) applied with  $t' = 1$  that

$$\|v_\Lambda(t, \cdot)\|_{L^\infty}^2 \leq \|v_\Lambda(1, \cdot)\|_{L^\infty}^2 + \frac{\epsilon^2 B^2}{32} < \frac{\epsilon^2 B^2}{16},$$

if  $B$  is taken large enough relatively to the  $L^\infty$  norm of the initial data. This concludes the proof.  $\square$

To finish the proof of Theorem 1.1.1, we study first the asymptotic behaviour of the solution  $v_\Lambda$  to (3.2.6).

**Lemma 3.3.3** *Let  $\theta \in ]0, \frac{1}{4}[$ ,  $s > 1 + \frac{3}{2\theta}$ . Let  $v$  be the solution to (2.1.12) corresponding to an odd initial data  $\epsilon w_0$ , with  $w_0$  satisfying (1.2.13). Then, if  $\epsilon_0$  is small enough, there is a family  $(\mathcal{A}_\epsilon(x))_{\epsilon \in ]0, \epsilon_0[}$  of continuous functions, bounded in  $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ , and a constant  $C > 0$ , such that  $\mathcal{A}_\epsilon(x)$  satisfies for any  $\epsilon \in ]0, \epsilon_0[$ , any  $t \geq 1$  the vanishing property*

$$(3.3.9) \quad \|\mathcal{A}_\epsilon(x) \langle tx \rangle^{-2}\|_{L^2} \leq Ct^{-\frac{5}{8} + \frac{\theta}{2}}$$

$$(3.3.10) \quad \|\mathcal{A}_\epsilon(x) \langle tx \rangle^{-2}\|_{L^\infty} \leq Ct^{-\frac{1}{4} + \theta}$$

and such that, for any  $\epsilon \in ]0, \epsilon_0[$ , any  $t \geq 1$ ,

$$(3.3.11) \quad \left\| v_\Lambda(t, x) - \epsilon \mathcal{A}_\epsilon(x) \exp \left[ -i \frac{tx^2}{2} + i\epsilon^2 L_\kappa(t, x) |\mathcal{A}_\epsilon(x)|^2 \right] \right\|_{L^\infty} \leq C\epsilon t^{-\frac{1}{4} + \theta}$$

and

$$(3.3.12) \quad \left\| v_\Lambda(t, x) - \epsilon \mathcal{A}_\epsilon(x) \exp \left[ -i \frac{tx^2}{2} + i \epsilon^2 L_\kappa(t, x) |\mathcal{A}_\epsilon(x)|^2 \right] \right\|_{L^2} \leq C \epsilon t^{-\frac{1}{4} + \theta},$$

where  $L_\kappa(t, x) = \int_1^t \kappa(\tau x) \frac{d\tau}{\tau}$ .

*Proof:* If we plug (3.3.2) in the right hand side of (3.2.7), (3.2.8) written with  $\theta$  replaced by some  $\theta'$  smaller than the  $\theta$  of the statement, and close enough to it so that  $s \geq 1 + \frac{3}{2\theta'}$ , and if we take  $\epsilon_0$  small enough relatively to  $\theta - \theta'$ , we get estimates

$$(3.3.13) \quad \begin{aligned} \|r(t, \cdot)\|_{L^2} &\leq C \epsilon t^{-\frac{1}{4} + \theta} \\ \|r(t, \cdot)\|_{L^\infty} &\leq C \epsilon t^{-\frac{1}{4} + \theta} \end{aligned}$$

for some  $C > 0$ . By (3.3.4)

$$\mathcal{B}_\epsilon(x) = \lim_{t \rightarrow +\infty} \epsilon^{-2} |v_\Lambda(t, x)|^2$$

exists, the limit being uniform in  $x$  and  $\epsilon$ , and we have

$$(3.3.14) \quad \left\| |v_\Lambda(t, \cdot)|^2 - \epsilon^2 \mathcal{B}_\epsilon(\cdot) \right\|_{L^\infty} = O(\epsilon^2 t^{-\frac{1}{4} + \theta}), \quad t \rightarrow +\infty.$$

Define

$$(3.3.15) \quad g(t, x) = v_\Lambda(t, x) \exp \left[ i \frac{tx^2}{2} - i \epsilon^2 L_\kappa(t, x) \mathcal{B}_\epsilon(x) \right].$$

We deduce from (3.2.6) that

$$(3.3.16) \quad D_t g(t, x) = \frac{\kappa(tx)}{t} [|v_\Lambda(t, x)|^2 - \epsilon^2 \mathcal{B}_\epsilon(x)] g(t, x) + \frac{1}{t} r(t, x).$$

By (3.3.14), (3.3.13), we see that  $\epsilon^{-1} g(t, x)$  converges uniformly when  $t$  goes to  $+\infty$  to some continuous limit  $\mathcal{A}_\epsilon(x)$ , with

$$\|g(t, \cdot) - \epsilon \mathcal{A}_\epsilon(\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{4} + \theta}), \quad t \rightarrow +\infty$$

so that (3.3.11) holds, since necessarily  $|\mathcal{A}_\epsilon(x)|^2 = \mathcal{B}_\epsilon(x)$ . Integrating (3.3.16), using (3.3.14) and applying Gronwall lemma we get

$$|g(t, x)| \leq C \left[ |g(1, x)| + \int_1^t |r(\tau, x)| \frac{d\tau}{\tau} \right].$$

Taking the  $L^2$  norm of this inequality and using the first inequality (3.3.13), we conclude that  $\|g(t, \cdot)\|_{L^2}$  is uniformly  $O(\epsilon)$ , so that  $\mathcal{A}_\epsilon$  is in  $L^2(\mathbb{R})$  uniformly in  $\epsilon$ . Moreover, this uniform bound, (3.3.14) and the first estimate (3.3.13) imply that the right hand side of (3.3.16) is  $O(\epsilon t^{-\frac{5}{4} + \theta})$  in  $L^2$  norm. Integrating from  $t$  to  $+\infty$ , we get (3.3.12).

By (2.2.16) applied with  $a_h \equiv 1$ , in which we plug (3.3.2), we get

$$(3.3.17) \quad \|\langle xt \rangle^{-2} v(t, \cdot)\|_{L^2} \leq C \epsilon t^{K \epsilon^2 - 1}.$$

Let us write

$$\begin{aligned} \int \epsilon^2 |\mathcal{A}_\epsilon(x)|^2 \langle tx \rangle^{-4} dx &\leq \int |\epsilon^2 |\mathcal{A}_\epsilon(x)|^2 - |v_\Lambda(t, x)|^2| \langle tx \rangle^{-4} dx \\ &\quad + \int ||v_\Lambda(t, x)|^2 - |v(t, x)|^2| \langle tx \rangle^{-4} dx + \int \langle tx \rangle^{-4} |v(t, x)|^2 dx. \end{aligned}$$

By (3.3.14), the first term in the right hand side is  $O(\epsilon^2 t^{-\frac{5}{4}+\theta})$ . By (3.3.17), the last term is  $O(\epsilon^2 t^{-2+2K\epsilon^2})$ . By (3.2.3) and (3.3.2), the middle term is  $O(\epsilon^2 t^{-\frac{5}{4}+K\epsilon^2})$ . This gives (3.3.9) if  $\epsilon$  is small enough.

To get (3.3.10), we write in the same way

$$\epsilon |\mathcal{A}_\epsilon(x)| \langle tx \rangle^{-2} \leq |\epsilon |\mathcal{A}_\epsilon(x)| - |v_\Lambda| | \langle tx \rangle^{-2} + ||v_\Lambda| - |v|| \langle tx \rangle^{-2} + \langle tx \rangle^{-2} |v|.$$

The first (resp. second) term in the right hand side is  $O(\epsilon t^{-\frac{1}{4}+\theta})$  by (3.3.11) (resp. by (3.2.3) and (3.3.2)). The last term is controlled by (2.2.17) (with  $a_h = 1$  and  $\sigma$  close to  $\frac{1}{2}$ ) by  $C\epsilon t^{-\frac{1}{4}+\theta}$  as well. This concludes the proof.  $\square$

*Proof of Theorem 1.1.1:* To prove the theorem, we have to deduce (1.1.8) from (3.3.11), (3.3.12). By (3.2.2), (3.2.3) and (3.3.2), we see that (3.3.11) and (3.3.12) remain true if we replace in their left hand side  $v_\Lambda$  by  $v$  and  $\epsilon_0$  is small enough. Recall that by (1.2.2), (1.2.4) and (2.1.6)

$$u(t, x) = W_+ w(t, x) \text{ with } w(t, x) = \frac{1}{\sqrt{t}} v\left(t, \frac{x}{t}\right)$$

so that according to (2.1.11)

$$\begin{aligned} (3.3.18) \quad u(t, x) &= \frac{1}{\sqrt{t}} [\text{Op}_h^1(a_h)v] \left(t, \frac{x}{t}\right) \\ &= \frac{1}{\sqrt{t}} [\text{Op}_h^1(a_h)v_\Lambda] \left(t, \frac{x}{t}\right) + \frac{1}{\sqrt{t}} [\text{Op}_h^1(a_h)(v - v_\Lambda)] \left(t, \frac{x}{t}\right). \end{aligned}$$

By (3.2.2), (3.2.3), (3.3.2) and lemma 3.1.2, the  $L^2$  (resp.  $L^\infty$ ) norm of the last terms in (3.3.18) is  $O(\epsilon t^{-\frac{1}{2}+K\epsilon^2})$  (resp.  $O(\epsilon t^{-\frac{3}{4}+K\epsilon^2})$ ), so may be incorporated to the remainder in (1.1.8). We apply to the first term Proposition 2.3.2 (i) where we take

$$(3.3.19) \quad \delta = \frac{1}{4} - \theta + \gamma, \quad \frac{2\theta}{3} < \sigma < \min\left(4\theta, \frac{1}{6} + \frac{2}{3}\theta\right)$$

for a small positive  $\gamma$ . The assumption  $s > 1 + \frac{3}{2\theta}$  of the theorem implies that the assumptions of the proposition are satisfied and the choices (3.3.19), together with estimates (3.3.2), show that the remainders (2.3.4), (2.3.5) are  $O(\epsilon t^{-\frac{1}{4}+\theta})$  if  $\epsilon$  is small enough. Consequently, modulo again contributions to  $r$  in (1.1.8), we may replace the first term in the right hand side of (3.3.18), by

$$(3.3.20) \quad \frac{1}{\sqrt{t}} a_h\left(\frac{x}{t}, -\frac{x}{t}\right) v_\Lambda\left(t, \frac{x}{t}\right).$$

We replace above  $v_\Lambda$  by its expansion obtained in (3.3.11), (3.3.12), again modulo a contribution to  $r$  in (1.1.8). Moreover, according to the expressions (2.1.8), (1.2.11) of  $a_h$ , we are reduced to

$$\begin{aligned} (3.3.21) \quad \frac{\epsilon}{\sqrt{t}} \mathcal{A}_\epsilon\left(\frac{x}{t}\right) & \left[ \chi_+(x) e_+\left(x, -\frac{x}{t}\right) + \chi_-(x) e_-\left(x, -\frac{x}{t}\right) \right] \\ & \times \exp\left[-i\frac{x^2}{2t} + i\epsilon^2 L_\kappa\left(t, \frac{x}{t}\right) \left| \mathcal{A}_\epsilon\left(\frac{x}{t}\right) \right|^2\right]. \end{aligned}$$

As  $\mathbb{1}_\pm(x) - \chi_\pm(x) = O(\langle x \rangle^{-N})$  for any  $N$ , (3.3.9) and (3.3.10) show that, modulo again a contribution to  $r$  in (1.1.8), we may replace (3.3.21) by

$$(3.3.22) \quad \frac{\epsilon}{\sqrt{t}} \mathcal{A}_\epsilon\left(\frac{x}{t}\right) \left[ \mathbb{1}_+(x) \left( e_0\left(-\frac{x}{t}\right) + \left|\frac{x}{t}\right| e_1\left(-\frac{x}{t}\right) \right) + \mathbb{1}_-(x) \left( e_0\left(\frac{x}{t}\right) + \left|\frac{x}{t}\right| e_1\left(\frac{x}{t}\right) \right) \right] \\ \times \exp\left[-i\frac{x^2}{2t} + i\epsilon^2 L_\kappa\left(t, \frac{x}{t}\right) \left| \mathcal{A}_\epsilon\left(\frac{x}{t}\right) \right|^2\right],$$

where we used the expressions (1.2.10) of  $e_+, e_-$  and the fact that in  $\mathbb{1}_\pm(x)e_\pm(x, \xi)$ , we may replace  $m_1, m_2$  by 1, since the error generated is again  $O(\langle x \rangle^{-N})$  according to (1.2.9). Finally, if we replace in (3.3.22)  $e_0, e_1$  by the expressions in function of the transmission and reflection coefficients computed in (A.1.29) below, we get

$$\frac{\epsilon}{\sqrt{t}} \mathcal{A}_\epsilon\left(\frac{x}{t}\right) \exp\left[-i\frac{x^2}{2t} + i\epsilon^2 L_\kappa\left(t, \frac{x}{t}\right) \left| \mathcal{A}_\epsilon\left(\frac{x}{t}\right) \right|^2\right]$$

which gives the principal part in (1.1.8). □

## A Appendix

### A.1 Proof of Proposition 1.2.1

We shall give here the proof of Proposition 1.2.1, relying on the results of Deift-Trubowitz [8] and Weder [27].

If  $V$  is a real valued potential in  $\mathcal{S}(\mathbb{R})$ , denote for  $\xi$  real (or for  $\xi$  in the upper half-plane  $\text{Im } \xi \geq 0$ ) by  $f_1(x, \xi), f_2(x, \xi)$  the solutions of  $-y'' + 2V(x)y = \xi^2 y$  satisfying respectively  $f_1(x, \xi) \sim e^{ix\xi}$ ,  $x \rightarrow +\infty$ ,  $f_2(x, \xi) \sim e^{-ix\xi}$ ,  $x \rightarrow -\infty$ . If  $V$  is even, we have  $f_1(-x, \xi) = f_2(x, \xi)$  by uniqueness. Set

$$(A.1.1) \quad m_1(x, \xi) = e^{-ix\xi} f_1(x, \xi), \quad m_2(x, \xi) = e^{ix\xi} f_2(x, \xi).$$

Under our evenness assumptions on  $V$

$$(A.1.2) \quad m_1(-x, \xi) = m_2(x, \xi).$$

By lemma 1 of [8],  $m_1$  solves the Volterra equation

$$(A.1.3) \quad m_1(x, \xi) = 1 + \int_x^{+\infty} D_\xi(x' - x) 2V(x') m_1(x', \xi) dx'$$

where

$$(A.1.4) \quad D_\xi(x) = \int_0^x e^{2ix'\xi} dx' = \frac{e^{2ix\xi} - 1}{2i\xi}.$$

If  $V$  is in  $\mathcal{S}(\mathbb{R})$ , (ii) of lemma 1 of [8] shows that (1.2.9) holds for  $m_1$  (and thus also for  $m_2$ ) when  $\alpha = \beta = 0$ . To get also estimates for the derivatives, we need to establish the following lemma, whose proof relies on the same ideas as in [8]:



**Lemma A.1.1** Denote for any  $\beta, N$  in  $\mathbb{N}$  by  $\Omega_N^\beta(x)$  a smooth positive function such that  $\Omega_N^\beta(x) = \langle x \rangle^{-N}$  for  $x \geq 1$  and  $\Omega_N^\beta(x) = \langle x \rangle^\beta$  for  $x \leq -1$ . Then for any  $N, \alpha, \beta$  in  $\mathbb{N}$ , there is  $C > 0$  such that for any  $\xi$  with  $\text{Im } \xi \geq 0$ , any  $x$

$$(A.1.5) \quad |\partial_x^\alpha \partial_\xi^\beta [m_1(x, \xi) - 1]| \leq C \Omega_N^{\beta+1}(x) \langle \xi \rangle^{-1-\beta}.$$

*Proof:* Following the proof of lemma 1 in [8], we write

$$(A.1.6) \quad m_1(x, \xi) = 1 + \sum_{n=1}^{+\infty} g_n(x, \xi)$$

with

$$(A.1.7) \quad g_n(x, \xi) = \int_{x \leq x_1 \leq \dots \leq x_n} \prod_{j=1}^n D_\xi(x_j - x_{j-1}) 2V(x_j) dx_1 \dots dx_n,$$

using the convention  $x_0 = x$ . Set  $\Omega(x) = \Omega_0^1(x)$  and

$$K_\xi(y, y') = D_\xi(y - y') \Omega(y')^{-1} 2V(y) \Omega(y).$$

Then we may rewrite  $g_n$  as

$$g_n(x, \xi) = \Omega(x) \int_{x \leq x_1 \leq \dots \leq x_n} \prod_{j=1}^n K_\xi(x_j, x_{j-1}) \Omega(x_n)^{-1} dx_1 \dots dx_n,$$

or equivalently

$$(A.1.8) \quad g_n(x, \xi) = \Omega(x) \int_{y_1 \geq 0, \dots, y_n \geq 0} \prod_{j=1}^n K_\xi(x + y_1 + \dots + y_j, x + y_1 + \dots + y_{j-1}) \\ \times \Omega(x + y_1 + \dots + y_n)^{-1} dy_1 \dots dy_n.$$

By (A.1.4), we have

$$|\partial_\xi^\beta D_\xi(y)| \leq C_\beta \langle \xi \rangle^{-1} \langle y \rangle^{1+\beta}.$$

Fix some integer  $m$ . The definition of  $K_\xi$  implies that for  $\alpha + \beta \leq m$

$$(A.1.9) \quad |\partial_x^\alpha \partial_\xi^\beta K_\xi(x + y_1 + \dots + y_j, x + y_1 + \dots + y_{j-1})| \\ \leq C \langle \xi \rangle^{-1} \Omega(x + y_1 + \dots + y_{j-1})^{-1} \langle x + y_1 + \dots + y_j \rangle^{-1-\beta} \\ \times W(x + y_1 + \dots + y_j) \langle y_j \rangle^{1+\beta},$$

where  $W$  is some smooth rapidly decaying function. When  $y_1 \geq 0, \dots, y_j \geq 0$ , we may bound

$$\langle y_j \rangle^{1+\beta} \Omega(x + y_1 + \dots + y_{j-1})^{-1} \langle x + y_1 + \dots + y_j \rangle^{-1-\beta} \leq C \Omega(x)^\beta.$$

Consequently, (A.1.8) implies that

$$(A.1.10) \quad |\partial_x^\alpha \partial_\xi^\beta g_n(x, \xi)| \leq C \Omega(x)^{\beta+1} \langle \xi \rangle^{-n} \int_{y_1 \geq 0, \dots, y_n \geq 0} \prod_{j=1}^n W(x + y_1 + \dots + y_j) dy_1 \dots dy_n.$$

Define  $G(x) = \int_x^{+\infty} W(z) dz$ , so that the last integral above may be written

$$\begin{aligned} (-1)^{n-1} \int_{y_1 \geq 0, \dots, y_{n-1} \geq 0} \prod_{j=1}^{n-1} G'(x + y_1 + \dots + y_j) G(x + y_1 + \dots + y_{n-1}) dy_1 \dots dy_{n-1} \\ = \frac{1}{n!} G(x)^n. \end{aligned}$$

As  $|G(x)| \leq C_N \Omega_N^0(x)$  for any  $N$ , it follows from (A.1.10) that, for any  $N$ ,

$$(A.1.11) \quad |\partial_x^\alpha \partial_\xi^\beta g_n(x, \xi)| \leq \frac{C_N^{n+1}}{n!} \langle \xi \rangle^{-n} \Omega_N^{\beta+1}(x).$$

If we sum for  $n \geq \beta + 1$ , we get a bound by the right hand side of (A.1.5).

We are thus left with studying

$$(A.1.12) \quad \sum_{n=1}^{\beta} \partial_x^\alpha \partial_\xi^\beta g_n(x, \xi).$$

Notice that (A.1.11) summed for  $n = 1, \dots, \beta$  gives, when  $|\xi| \leq 1$ , the estimate (A.1.5) for (A.1.12) as well. Assume from now on that  $|\xi| \geq 1$  and let us prove by induction on  $n = 1, \dots, \beta$  that  $|\partial_x^\alpha \partial_\xi^\beta g_n(x, \xi)|$  is bounded by the right hand side of (A.1.5). We may write from (A.1.7)

$$(A.1.13) \quad \begin{aligned} g_n(x, \xi) &= \int_{x \leq x_1} D_\xi(x_1 - x) 2V(x_1) g_{n-1}(x_1, \xi) dx_1 \\ &= \int_{y_1 \geq 0} D_\xi(y_1) 2V(y_1 + x) g_{n-1}(y_1 + x, \xi) dy_1 \end{aligned}$$

with  $g_0 \equiv 1$ . We use in (A.1.13) the last expression (A.1.4) for  $D_\xi$ . We have then to consider two kind of terms. The first one is

$$\begin{aligned} \int_{y_1 \geq 0} \frac{e^{2iy_1\xi}}{\xi} 2V(y_1 + x) g_{n-1}(y_1 + x, \xi) dy_1 \\ = -\frac{1}{2i\xi^2} 2V(x) g_{n-1}(x, \xi) - \int_{y_1 \geq 0} \frac{e^{2iy_1\xi}}{2i\xi^2} \partial_{y_1} [2V(y_1 + x) g_{n-1}(y_1 + x, \xi)] dy_1. \end{aligned}$$

Repeating the integrations by parts, we end up with contributions that, according to the induction hypothesis (and the fact that  $g_0 \equiv 1$ ), satisfy estimates of the form (A.1.5) (with  $\Omega_N^\beta(x)$  replaced by  $\langle x \rangle^{-N}$ ), and an integral term of the form

$$(A.1.14) \quad \int_{y_1 \geq 0} \frac{e^{2iy_1\xi}}{\xi^{M+1}} \partial_{y_1}^M [2V(y_1 + x) g_{n-1}(y_1 + x, \xi)] dy_1$$

for  $M$  as large as we want. If  $M = \beta$ , we see that (A.1.14) satisfies (A.1.5). The second type of terms coming from (A.1.13) to consider is

$$\frac{1}{\xi} \int_{y_1 \geq 0} 2V(y_1 + x) g_{n-1}(y_1 + x, \xi) dy_1$$

which trivially satisfies (A.1.5) by the induction hypothesis applied to  $g_{n-1}$ . This concludes the proof.  $\square$

In order to obtain the representation (1.2.12) for  $W_+w$ , when  $w$  is odd, we recall first the definition of the transmission and reflection coefficients. The wronskian of  $(f_1(x, \xi), f_1(x, -\xi))$  (resp.  $(f_2(x, \xi), f_2(x, -\xi))$ ) is nonzero for any  $\xi$  in  $\mathbb{R}^*$  (see [8], page 144), so that, for real  $\xi \neq 0$ , we may find unique coefficients  $T_1(\xi), T_2(\xi)$  non zero,  $R_1(\xi), R_2(\xi)$  such that

$$(A.1.15) \quad \begin{aligned} f_2(x, \xi) &= \frac{R_1(\xi)}{T_1(\xi)} f_1(x, \xi) + \frac{1}{T_1(\xi)} f_1(x, -\xi) \\ f_1(x, \xi) &= \frac{R_2(\xi)}{T_2(\xi)} f_2(x, \xi) + \frac{1}{T_2(\xi)} f_2(x, -\xi). \end{aligned}$$

By Theorem I in [8], these functions extend as smooth functions on  $\mathbb{R}$ , and they satisfy the following properties

$$(A.1.16) \quad \begin{aligned} T_1(\xi) &= T_2(\xi) \stackrel{\text{def}}{=} T(\xi) \\ T(\xi)\overline{R_2(\xi)} + R_1(\xi)\overline{T(\xi)} &= 0 \\ |T(\xi)|^2 + |R_j(\xi)|^2 &= 1, \quad j = 1, 2 \\ \overline{T(\xi)} &= T(-\xi), \quad \overline{R_j(\xi)} = R_j(-\xi). \end{aligned}$$

If the potential  $V$  is even, we have seen that  $f_1(-x, \xi) = f_2(x, \xi)$ , so that, plugging this equality in the first relation (A.1.15), comparing to the second one, and using that  $T_1 = T_2$ , we conclude that

$$(A.1.17) \quad R_1(\xi) = R_2(\xi).$$

We denote by  $R(\xi)$  this common value. The integral representations of the scattering coefficients (see [8] page 145)

$$(A.1.18) \quad \begin{aligned} \frac{R(\xi)}{T(\xi)} &= \frac{1}{2i\xi} \int e^{2ix\xi} 2V(x)m_1(x, \xi) dx \\ \frac{1}{T(\xi)} &= 1 - \frac{1}{2i\xi} \int 2V(x)m_1(x, \xi) dx \end{aligned}$$

together with (A.1.5) and the fact that  $V \in \mathcal{S}(\mathbb{R})$ , show that  $\partial_\xi^\beta R(\xi) = O(\langle \xi \rangle^{-N})$  for any  $N, \beta$  and  $\partial_\xi^\beta (T(\xi) - 1) = O(\langle \xi \rangle^{-1-\beta})$  for any  $\beta$ .

We need the following lemma:

**Lemma A.1.2** *The functions  $T, R$  satisfy*

$$(A.1.19) \quad T(0) = 1 + R(0)$$

*in the following two cases:*

- *The generic case  $\int V(x)m_1(x, 0) dx \neq 0$ .*
- *The very exceptional case  $\int V(x)m_1(x, 0) dx = 0$  and  $\int V(x)xm_1(x, 0) dx = 0$ .*

*Proof:* Summing the two equalities (A.1.18) and making an expansion at  $\xi = 0$  using (A.1.5), we get

$$\begin{aligned} R(\xi) + 1 &= T(\xi) \left[ 1 - \frac{1}{i\xi} \int_{-\infty}^{+\infty} V(x)m_1(x, \xi) dx + \frac{1}{i\xi} \int_{-\infty}^{+\infty} e^{2ix\xi} V(x)m_1(x, \xi) dx \right] \\ &= T(\xi) \left[ 1 + 2 \int_{-\infty}^{+\infty} xV(x)m_1(x, 0) dx + O(\xi) \right], \quad \xi \rightarrow 0 \end{aligned}$$

so that

$$(A.1.20) \quad R(0) + 1 - T(0) = 2T(0) \int_{-\infty}^{+\infty} xV(x)m_1(x,0) dx.$$

In the generic case, by (A.1.18)

$$(A.1.21) \quad T(\xi) = i\xi \left[ - \int_{-\infty}^{+\infty} V(x)m_1(x,0) dx + O(\xi) \right]^{-1}, \quad \xi \rightarrow 0$$

so that  $T(0) = 0$ . This shows that (A.1.20) vanishes in the two considered cases.  $\square$

*Proof of Proposition 1.2.1:* We have to prove that  $W_+$  acting on odd functions is given by (1.2.12). Recall (see for instance Weder [27] formula (2.20), Schechter [21]) that  $W_+w$  is given by

$$(A.1.22) \quad W_+w = F_+^* \hat{w}$$

where  $F_+^*$  is the adjoint of the distorted Fourier transform, given by

$$(A.1.23) \quad F_+^* \Phi = \frac{1}{2\pi} \int \psi_+(x, \xi) \Phi(\xi) d\xi$$

where

$$(A.1.24) \quad \psi_+(x, \xi) = \mathbb{1}_{\xi>0} T(\xi) f_1(x, \xi) + \mathbb{1}_{\xi<0} T(-\xi) f_2(x, -\xi).$$

Let  $\chi_{\pm}$  be the functions defined in the statement of Proposition 1.2.1 and write

$$\psi_+(x, \xi) = \chi_+(x) \psi_+(x, \xi) + \chi_-(x) \psi_+(x, \xi).$$

Replace in  $\chi_+ \psi_+$  (resp.  $\chi_- \psi_-$ )  $\psi_+$  by (A.1.24) where we express  $f_2$  from  $f_1$  (resp.  $f_1$  for  $f_2$ ) using the first (resp. second) formula (A.1.15). We get, using notation (A.1.1)

$$(A.1.25) \quad \begin{aligned} \psi_+(x, \xi) = & \chi_+(x) \left[ e^{ix\xi} (T(\xi)m_1(x, \xi)\mathbb{1}_{\xi>0} + m_1(x, \xi)\mathbb{1}_{\xi<0}) + e^{-ix\xi} R(-\xi)m_1(x, -\xi)\mathbb{1}_{\xi<0} \right] \\ & + \chi_-(x) \left[ e^{ix\xi} (m_2(x, -\xi)\mathbb{1}_{\xi>0} + T(-\xi)m_2(x, -\xi)\mathbb{1}_{\xi<0}) + e^{-ix\xi} R(\xi)m_2(x, \xi)\mathbb{1}_{\xi>0} \right]. \end{aligned}$$

Using (A.1.2), we deduce from (A.1.22), (A.1.23) and (A.1.25) that

$$(A.1.26) \quad W_+w = \frac{1}{2\pi} \int e^{ix\xi} e_1(x, \xi) \hat{w}(\xi) d\xi + \frac{1}{2\pi} \int e^{-ix\xi} e_2(x, \xi) \hat{w}(\xi) d\xi$$

with

$$(A.1.27) \quad \begin{aligned} e_1(x, \xi) &= \chi_+(x)m_1(x, \xi)[T(\xi)\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}] + \chi_-(x)m_1(-x, -\xi)[\mathbb{1}_{\xi>0} + T(-\xi)\mathbb{1}_{\xi<0}] \\ e_2(x, \xi) &= \chi_+(x)R(-\xi)m_1(x, -\xi)\mathbb{1}_{\xi<0} + \chi_-(x)R(\xi)m_1(-x, \xi)\mathbb{1}_{\xi>0}. \end{aligned}$$

If  $w$  is odd, we may rewrite (A.1.26) as

$$W_+w = \frac{1}{2\pi} \int e^{ix\xi} a(x, \xi) \hat{w}(\xi) d\xi$$

with

$$(A.1.28) \quad \begin{aligned} a(x, \xi) &= e_1(x, \xi) - e_2(x, -\xi) \\ &= \chi_+(x)m_1(x, \xi)[(T(\xi) - R(\xi))\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}] \\ &\quad + \chi_-(x)m_1(-x, -\xi)[\mathbb{1}_{\xi>0} + (T(-\xi) - R(-\xi))\mathbb{1}_{\xi<0}]. \end{aligned}$$

Setting

$$(A.1.29) \quad e_0(\xi) = \frac{T(\xi) - R(\xi) + 1}{2}, \quad e_1(\xi) = \frac{T(\xi) - R(\xi) - 1}{2\xi}$$

we see that  $e_0, e_1$  satisfy (1.2.6), since  $e_1$  is smooth at  $\xi = 0$  because of (A.1.19). Actually, as  $T(\xi) - 1$  is  $O(\langle \xi \rangle^{-1})$ , we even get that  $e_1 = O(\langle \xi \rangle^{-2})$ . We rewrite thus (A.1.28) as

$$a(x, \xi) = \chi_+(x)m_1(x, \xi)[e_0(\xi) + e_1(\xi)|\xi|] + \chi_-(x)m_1(-x, -\xi)[e_0(-\xi) + e_1(-\xi)|\xi|]$$

which gives (1.2.11) using (A.1.2). The equality

$$e_0(\xi) + e_1(\xi)|\xi| = (T(\xi) - R(\xi))\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}$$

implies (1.2.7), as  $T(\xi) - R(\xi) - 1$  is a symbol of order  $-1$ . Moreover, (1.2.8) follows from the fact that  $|T(\xi) - R(\xi)| = 1$ , which is a consequence of (A.1.16). Since (1.2.9) follows from (A.1.5) and (A.1.2), this concludes the proof of the proposition.  $\square$

## A.2 Proof of lemma 2.2.1

*Proof of lemma 2.2.1:* (i) When  $m = 0$  i.e.  $a \in S_{\frac{1}{2}}(1)$ ,  $s = 0$ ,  $\lambda = \frac{1}{2}$  (i.e. one considers the Weyl quantization), property (2.2.5) is just  $L^2$  boundedness of  $\text{Op}_h^W(a)$ , which is Theorem 7.11 of [13]. Since one may express any quantization from the Weyl one (see (7.16) in [13]), the same property holds for  $\text{Op}_h^\lambda$  for any  $\lambda \in [0, 1]$ . The case of arbitrary  $s, m$  follows from the symbolic calculus of [13] Chapter 7. Actually, (2.2.5) is equivalent, by the same reasoning as above, to the  $L^2$  boundedness of

$$\text{Op}_h^W(\langle \xi \rangle^{s-m}) \circ \text{Op}_h^W(a) \circ \text{Op}_h^W(\langle \xi \rangle^{-s})$$

and that last operator may be written as  $\text{Op}_h^W(b)$  for some symbol  $b$  in  $S_{\frac{1}{2}}(1)$  by Theorem 7.9 and Proposition 7.7 of [13].

(ii) We cannot deduce directly lemma 2.2.1 (ii) from the results of [13], as symbols in  $S_1(\langle \xi \rangle^m)$  defined by inequalities (2.1.1) with  $\delta = 1$  are not covered by the assumptions made on that reference. Though, when  $m = s = 0$ , we may reduce (2.2.6) to the similar inequality for symbols in  $S_{\frac{1}{2}}(1)$ , that are treated in [13]. Actually, define the operator  $\Theta$  by

$$\Theta v(x) = v(x\sqrt{h}).$$

Then

$$\Theta \circ \text{Op}_h^1(a) \circ \Theta^{-1} = \text{Op}_h^1(b), \quad \Theta \circ \text{Op}_h^0(a) \circ \Theta^{-1} = \text{Op}_h^0(b)$$

with  $b(x, \xi) = a(x\sqrt{h}, \xi/\sqrt{h})$ . As  $b$  belongs to  $S_{\frac{1}{2}}(1)$ ,  $\text{Op}_h^1(b)$ ,  $\text{Op}_h^0(b)$  are bounded on  $L^2$ , uniformly in  $h$ , so that  $\text{Op}_h^1(a)$ ,  $\text{Op}_h^0(a)$  are bounded on  $L^2$ , uniformly in  $h$ .

To deduce (2.2.6) for general  $s, m$  from that property, we notice first that by (2.1.5) it suffices to treat the case of  $\text{Op}_h^1(a)$ . Take  $\varphi$  in  $C_0^\infty(\mathbb{R} - \{0\})$ ,  $\psi$  in  $C_0^\infty(\mathbb{R})$  forming a Littlewood-Paley partition of unity, so that  $1 = \psi(\xi) + \sum_1^{+\infty} \varphi(2^{-k}\xi)$ . Define for  $k \geq 1$ ,  $\Delta_k^h = \text{Op}_h^1(\varphi(2^{-k}\xi))$ ,  $\Delta_0^h = \text{Op}_h^1(\psi(\xi))$ . Then the  $\mathcal{L}(H_{\text{sc}}^s, H_{\text{sc}}^{s-m})$  boundedness of  $\text{Op}_h^1(a)$  follows from an estimate of the form

$$(A.2.1) \quad \|\Delta_\ell^h \circ \text{Op}_h^1(a) \circ \Delta_k^h\|_{\mathcal{L}(L^2)} \leq C_N 2^{km-N|k-\ell|}$$

for any  $N, k, \ell$ , with a constant  $C_N$  uniform in  $h \in ]0, 1[$ . Writing  $\varphi(2^{-\ell}hD) = (2^{-\ell}hD)^N \tilde{\varphi}(2^{-\ell}hD)$  or  $\varphi(2^{-k}hD) = (2^{-k}hD)^N \tilde{\varphi}(2^{-k}hD)$ , with  $\tilde{\varphi}(z) = \frac{\varphi(z)}{z^N}$ , and using that if  $a$  is in  $S_1(\langle \xi \rangle^m)$ ,  $hDa$  belongs to the same class by (2.1.1), one deduces immediately (A.2.1) from the  $L^2$  boundedness of the quantization of elements of  $S_1(1)$  seen above.

(iii) We take here  $a$  in the subclass  $\Sigma_1(\langle \xi \rangle^m)$ , and using the same Littlewood-Paley partition of unity as above, we write

$$a(x, \xi) = \sum_{k=0}^{+\infty} a_k(x, \xi)$$

with  $a_0(x, \xi) = a(x, \xi)\psi(\xi)$ ,  $a_k(x, \xi) = a(x, \xi)\varphi(2^{-k}\xi)$  for  $k \geq 1$ . Denote by  $K_k(x, y)$  the distributional kernel of  $\text{Op}_h^1(a_k)$  i.e.

$$(A.2.2) \quad K_k(x, y) = \frac{2^k}{(2\pi h)} \int e^{i\frac{2^k \xi}{h}(x-y)} a(x, 2^k \xi) \varphi(\xi) d\xi$$

when  $k \geq 1$ , and the similar expression with  $\varphi$  replaced by  $\psi$  when  $k = 0$ . By (2.1.2),  $\partial_\xi^\beta [a(x, 2^k \xi) \varphi(\xi)] 2^{-km}$  is bounded uniformly in  $k$ , so that, performing integration by parts in (A.2.2), we get

$$|K_k(x, y)| \leq C_N \frac{2^k}{h} 2^{km} \left(1 + \frac{2^k}{h} |x - y|\right)^{-N}$$

for any  $N$ . This immediately implies that  $\text{Op}_h^1(a_k)$  is bounded from  $L^\infty$  to  $L^\infty$ , uniformly in  $h \in ]0, 1[$ , with  $O(2^{km})$  operator norm. As  $m < 0$ , the sum  $\text{Op}_h^1(a)$  of these operators is also bounded on  $L^\infty$ . This concludes the proof.  $\square$

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