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► **To cite this version:**

Matthieu Josuat-Vergès. Derivatives of the tree function. The Ramanujan Journal, 2015, 38 (1), pp.1-15. <10.1007/s11139-015-9684-x>. <hal-01396526>

**HAL Id: hal-01396526**

**<https://hal.science/hal-01396526v1>**

Submitted on 14 Nov 2016

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# DERIVATIVES THE TREE FUNCTION

MATTHIEU JOSUAT-VERGÈS

ABSTRACT. We study some sequences of polynomials that appear when we consider the successive derivatives of the tree function (or Lambert's  $W$  function). We show in particular that they are related with a generalization of Cayley trees, called Greg trees. Besides the combinatorial result in itself, it is interesting to see how this is related with previous work: similar problems were considered first by Ramanujan, and more recently in the theory of completely monotonic functions and its link with probability. Also of great interest is the fact that these Greg trees were introduced as genealogical trees in a problem of textual criticism, where they had a priori no mathematical meaning.

## 1. INTRODUCTION

Lambert's  $W$  function is a special function that has been widely studied and naturally appear in various contexts in pure or applied sciences, see for example the comprehensive article by Corless et al. [4]. It is usually defined by the equation

$$W(z)e^{W(z)} = z.$$

We are interested here in the computation of derivatives of this function. We can derive the first values from the defining equation, and the general pattern that appears is

$$(1) \quad \frac{d^n}{dz^n} W(z) = \frac{\exp(-nW(z))}{(1+W(z))^{2n-1}} P_n(W(z))$$

for some polynomials  $P_n$  (this is done in [4]). Kalugin and Jeffrey [13] proved that the polynomials  $(-1)^{n-1}P_n$  have positive coefficients, settling a conjecture of Sokal. This result was motivated by the fact that we can deduce the sign of all these functions: they are alternately positive and negative, and it follows that  $W(z)$  is a so-called Bernstein function [20]. Various similar results have been obtained in a rather analytical perspective [14, 15], and a probability law related with  $W(z)$  as a Bernstein function is studied in [18].

In fact, the polynomials  $P_n$  are closely related with another sequence that appeared first in Ramanujan's notebook (see [1] and [19]), and inspired quite a few combinatorial works [3, 5, 11, 12, 17, 24]. The goal of this article is to introduce *Greg trees* in this context. They are a generalization of Cayley trees, and were defined by Flight [8] as genealogical trees for manuscripts. (Flight named these trees after Sir Walton

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2000 *Mathematics Subject Classification.* 05C05, 11B83, 26A48.

*Key words and phrases.* Cayley trees, Bernstein function, completely monotonic functions.

This work was supported by the ANR project CARMA.

Wilson Greg, a renowned British scholar who worked in textual criticism, and was in particular an expert of Shakespeare's texts [23].) Similar trees were also considered in [6] as some kind of phylogenetic trees. The connection was noticed by Knuth, in the correction of Exercise 50 of Section 7.2.1.5 in the Prefascicle 3B of the Art of Computer Programming [16]. Here, we explain this connection, by showing that some operations on Greg trees can prove formulas of the kind of Equation (1). In fact, we study here three closely-related sequences of polynomials  $F_n$ ,  $G_n$  and  $H_n$ . The first one is related with Ramanujan's polynomials, and the latter two give the enumeration of Greg trees, respectively in the rooted and unrooted cases. (Note that these sequences can be seen as particular cases of a two-variable polynomials, see Section 7. However, from the point of view of the enumeration of Greg trees, only these cases are considered.)

## 2. DEFINITIONS

Since the defining equation  $W(z)e^{W(z)} = z$  has several solutions,  $W(z)$  can be considered as a multivalued function. Here we only consider the principal branch, and see  $W(z)$  as a holomorphic function on  $\mathbb{C} - (-\infty, -\frac{1}{e}]$ , such that

$$W(z) = \sum_{n \geq 1} (-n)^{n-1} \frac{z^n}{n!}$$

in a neighborhood of 0 (see [4]). From the function  $W(z)$ , we define the tree function:

$$T_1(z) = -W(-z).$$

It is well-known that  $n^{n-1}$  is number of rooted Cayley trees [2], so that  $T_1(z)$  is the exponential generating function of these. More generally, let us define:

$$T_\alpha(z) = \sum_{n \geq 1} n^{n-\alpha} \frac{z^n}{n!}.$$

We have  $T(z) = T_1(z)$  and we will also consider the cases where  $\alpha = 0$  or  $\alpha = 2$ . In particular, the generating function unrooted Cayley trees is:

$$T_2(z) = \sum_{n \geq 1} n^{n-2} \frac{z^n}{n!}.$$

There are several relations linking these functions, for example:

$$T_0(z) = \frac{1}{1 - T(z)}, \quad T_2'(z) = \frac{T(z)}{z}, \quad T_2(z) = T(z) - \frac{1}{2}T(z)^2.$$

We will use these identities in the sequel but leave as an exercise to find either an analytical or a bijective proof for each of them. Whereas  $T_1$  and  $T_2$  are clearly interesting in a combinatorial context, we will see that  $T_0$  is also related with a nice sequence of polynomials, but we do not get into other cases.

**Proposition 2.1.** *There are polynomials  $F_n(x)$ ,  $G_n(x)$  and  $H_n(x)$  such that for any  $n \geq 1$  we have:*

$$(2) \quad \frac{d^n}{dz^n} T_0(z) = \frac{\exp(nT(z))}{(1-T(z))^{n+2}} F_n \left( \frac{T(z)}{1-T(z)} \right).$$

$$(3) \quad \frac{d^n}{dz^n} T_1(z) = \frac{\exp(nT(z))}{(1-T(z))^n} G_n \left( \frac{T(z)}{1-T(z)} \right).$$

$$(4) \quad \frac{d^n}{dz^n} T_2(z) = \frac{\exp(nT(z))}{(1-T(z))^{n-1}} H_n \left( \frac{T(z)}{1-T(z)} \right).$$

With the initial case  $F_1(x) = G_1(x) = H_1(x) = 1$ , these polynomials satisfy the recursion:

$$(5) \quad \begin{aligned} F_{n+1}(x) &= (2n+2+(n+2)x)F_n(x) + (1+x)^2 F'_n(x), \\ G_{n+1}(x) &= (2n+nx)G_n(x) + (1+x)^2 G'_n(x), \\ H_{n+1}(x) &= (2n-1+(n-1)x)H_n(x) + (1+x)^2 H'_n(x). \end{aligned}$$

*Proof.* We only present the case of  $T_1(z)$  and the other two are similar. By differentiation of  $T_1(z) = ze^{T_1(z)}$ , we get:

$$T_1'(z) = e^{T_1(z)} + zT_1'(z)e^{T_1(z)},$$

hence:

$$T_1'(z) = \frac{e^{T_1(z)}}{1 - ze^{T_1(z)}} = \frac{e^{T_1(z)}}{1 - T_1(z)}.$$

This proves the case  $n = 1$ , and the general case is done inductively. Indeed, suppose that  $G_n$  exists such that Identity (3) is true, by differentiating both sides, we get:

$$\begin{aligned} \frac{d^{n+1}}{dz^{n+1}} T_1(z) &= \frac{nT'(z)e^{nT(z)}}{(1-T(z))^n} G_n \left( \frac{T(z)}{1-T(z)} \right) \\ &\quad + \frac{nT'(z)e^{nT(z)}}{(1-T(z))^{n+1}} G_n \left( \frac{T(z)}{1-T(z)} \right) + \frac{T'(z)e^{nT(z)}}{(1-T(z))^{n+2}} G'_n \left( \frac{T(z)}{1-T(z)} \right), \end{aligned}$$

and then:

$$\begin{aligned} \frac{d^{n+1}}{dz^{n+1}} T_1(z) &= \frac{ne^{(n+1)T(z)}}{(1-T(z))^{n+1}} G_n \left( \frac{T(z)}{1-T(z)} \right) \\ &\quad + \frac{ne^{(n+1)T(z)}}{(1-T(z))^{n+2}} G_n \left( \frac{T(z)}{1-T(z)} \right) + \frac{e^{(n+1)T(z)}}{(1-T(z))^{n+3}} G'_n \left( \frac{T(z)}{1-T(z)} \right), \\ \frac{d^{n+1}}{dz^{n+1}} T_1(z) &= \frac{e^{(n+1)T(z)}}{(1-T(z))^{n+1}} \left( n \frac{2-T(z)}{1-T(z)} G_n \left( \frac{T(z)}{1-T(z)} \right) + \frac{1}{(1-T(z))^2} G'_n \left( \frac{T(z)}{1-T(z)} \right) \right). \end{aligned}$$

This equation shows that  $G_{n+1}$  exists, with:

$$G_{n+1} \left( \frac{T(z)}{1-T(z)} \right) = n \frac{2-T(z)}{1-T(z)} G_n \left( \frac{T(z)}{1-T(z)} \right) + \frac{1}{(1-T(z))^2} G'_n \left( \frac{T(z)}{1-T(z)} \right).$$

This gives the recursion since we have  $G_{n+1}$  in terms of  $G_n$ .

Similar computations give the result for  $F_n$  and  $H_n$ . □

The first values of these polynomials are as follows:

$n$	$F_n(x)$	$G_n(x)$
1	1	1
2	$3x+4$	$x+2$
3	$15x^2+40x+27$	$3x^2+10x+9$
4	$105x^3+420x^2+565x+256$	$15x^3+70x^2+113x+64$
5	$945x^4+5040x^3+10150x^2+9156x+3125$	$105x^4+630x^3+1450x^2+1526x+625$
6	$10395x^5+69300x^4+185850x^3+250768x^2+170359x+46656$	$945x^5+6930x^4+20650x^3+31346x^2+24337x+7776$

$n$	$H_n(x)$
1	1
2	1
3	$x+3$
4	$3x^2+13x+16$
5	$15x^3+85x^2+171x+125$
6	$105x^4+735x^3+2005x^2+2551x+1296$
7	$945x^5+7875x^4+26950x^3+47586x^2+43653x+16807$

The coefficients of  $G_n(x)$  are Sloane's [A048160](#), and those of  $H_n(x)$  are [A048159](#). These two triangles of integers were defined by Flight [8]. The row sums, i.e. the integers  $G_n(1)$ , appear in Felsenstein's article [6].

We also have to examine some of the shifted polynomials:

$n$	$F_n(x-1)$	$G_n(x-1)$
1	1	1
2	$3x+1$	$x+1$
3	$15x^2+10x+2$	$3x^2+4x+2$
4	$105x^3+105x^2+40x+6$	$15x^3+25x^2+18x+6$
5	$945x^4+1260x^3+700x^2+196x+24$	$105x^4+210x^3+190x^2+96x+24$
6	$10395x^5+17325x^4+12600x^3+5068x^2+1148x+120$	$945x^5+2205x^4+2380x^3+1526x^2+600x+120$

$n$	$H_n(x-1)$
1	1
2	1
3	$x+2$
4	$3x^2+7x+6$
5	$15x^3+40x^2+46x+24$
6	$105x^4+315x^3+430x^2+326x+120$
7	$945x^5+3150x^4+4900x^3+4536x^2+2556x+720$

The coefficients of  $F_n(x-1)$  are [A075856](#) and appeared in Ramanujan's notebooks [19] (with a definition equivalent to the present one), and several other works [9, 10].

The polynomials  $G_n(x-1)$  appeared in [21] where it is shown that they count rooted Cayley trees according to the number of *improper edges* (see below). The unrooted analog  $H_n(x-1)$  appeared in Zeng's article [24]. The coefficients of these polynomials are respectively [A054589](#) and [A217922](#).

3. GREG TREES

**Definition 3.1** ([8]). Let  $n \geq 0$ . A *Greg tree* of size  $n$  is a tree such that:

- there are  $n$  vertices labeled by integers from 1 to  $n$ , and the other vertices are unlabeled,
- the unlabeled vertices have degree at least 3.

Let  $\mathcal{G}_n$  denote the set of *Greg trees* with  $n$  labeled vertices. Let  $\text{unl}(T)$  denote the number of unlabeled vertices of some  $T \in \mathcal{G}_n$ . Let  $\mathcal{C}_n \subset \mathcal{G}_n$  denote the subset of Cayley trees, i.e. those  $T$  with  $\text{unl}(T) = 0$ .

Figure 1 shows the Greg trees of size 3.

Since we do not specify a bound on the number of unlabeled vertices, it is not *a priori* clear that there is a finite number of Greg trees of size  $n$ . But the condition that unlabeled vertices have degree at least 3 gives such a bound. Indeed, let  $u$  be the number of unlabeled vertices in a Greg tree of size  $n$ , so  $u + n$  is the total number of vertices and there are  $u + n + 1$  edges. The condition on the degree easily gives  $2(u + n + 1) \geq 3u$ , hence  $u \leq 2(n + 1)$ .

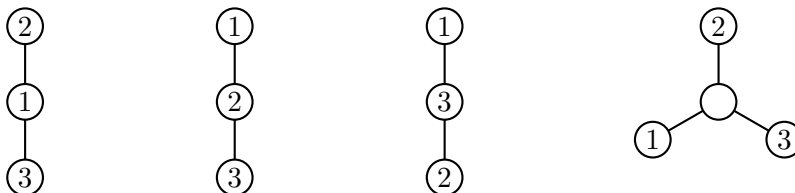


FIGURE 1. Greg trees of size 3.

**Definition 3.2.** Let  $\mathcal{G}_n^\bullet$  denote the set of *rooted Greg trees*, i.e. Greg trees of size  $n$  with a distinguished vertex called the *root*, but with the additional rule that if the root is unlabeled it may have degree 2.

Let  $\mathcal{C}_n^\bullet \subset \mathcal{G}_n^\bullet$  denote the subset of rooted Cayley trees.

See Figure 2 for the rooted Greg trees of size 2. We represent the root as a vertex with a double line circle. Note that since only the root is allowed to have degree 2, the same argument as in the unrooted case gives a bound on the number of unlabeled vertices, so that there is a finite number of rooted Greg trees of size  $n$ .

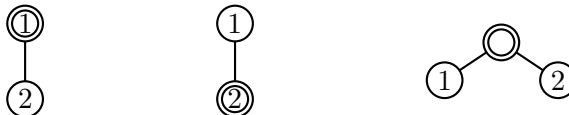


FIGURE 2. Rooted Greg trees of size 2.

**Theorem 3.3.** *For any  $n \geq 1$ , we have:*

$$(6) \quad G_n(x) = \sum_{T \in \mathcal{G}_n^\bullet} x^{\text{unl}(T)}, \quad H_n(x) = \sum_{T \in \mathcal{G}_n} x^{\text{unl}(T)}.$$

It is possible to prove this theorem by examining how to build a Greg tree of size  $n + 1$  from a Greg tree of size  $n$ , which leads precisely to the relations in (5) as was done in [8]. But we take a different point of view here. Rather than the recursion in (5), we show that the structure of the right-hand sides of (3) and (4) is explained through some combinatorial operations on Greg trees.

**Definition 3.4.** Let  $T \in \mathcal{C}_m$  and  $n < m$ . We define a Greg tree  $T|_n \in \mathcal{G}_n$  as the result of the following process:

- Remove all labels greater than  $n$  (thus creating new unlabeled vertices).
- Remove each unlabeled vertex of degree 2, and join the two pendant edges into a single edge.
- Remove each unlabeled vertex of degree 1, as well as the incident edge.
- Repeat the last two steps, until the result is a Greg tree.

This process is illustrated in Figure 3.

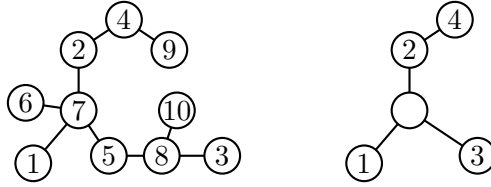


FIGURE 3. The operation  $T \mapsto T|_n$  (here  $n = 4$ ).

**Proposition 3.5.** *Let  $T \in \mathcal{G}_n$ . We have:*

$$(7) \quad \sum_{m \geq n} \sum_{\substack{X \in \mathcal{C}_m \\ X|_n = T}} \frac{z^{m-n}}{(m-n)!} = \frac{e^{nT(z)}}{(1-T(z))^{n-1}} \left( \frac{T(z)}{1-T(z)} \right)^{\text{unl}(T)}.$$

*Proof.* In this proof, we make an extensive use of the symbolic method for labeled combinatorial objects [7], which permits to obtain the result in a purely combinatorial manner. The idea is to find a canonical decomposition for a Cayley tree  $X \in \mathcal{C}_m$  satisfying  $X|_n = T$ , as a collection of rooted Cayley trees indexed in a particular way. Note that this tree  $X$  is considered as having size  $m - n$ , i.e. only the labels greater than  $n$  are part of the labeled combinatorial structure. We rewrite the right hand side of the equation as:

$$(8) \quad e^{nT(z)} \left( \frac{1}{1-T(z)} \right)^{n-1+\text{unl}(T)} T(z)^{\text{unl}(T)}.$$

Then the construction is as follows:

- To each of the  $n$  labeled vertices of  $T$ , we associate an unordered collection of rooted trees, each giving a factor  $e^{T(z)}$ .
- To each of the  $n - 1 + \text{unl}(T)$  edges of  $T$ , we associate an ordered collection of rooted trees, each giving a factor  $\frac{1}{1-T(z)}$ .
- To each of the  $\text{unl}(T)$  unlabeled vertices of  $T$ , we associate a rooted tree, each giving a factor  $T(z)$ .

Here, we understand that all these trees are part of the same combinatorial structure, i.e. each integer between  $n + 1$  and  $m$  appears exactly once as the label of a vertex of some tree. We have thus defined a combinatorial class whose generating function is clearly given by Equation (8), and it remains to check that it maps bijectively to the set of  $X$  such that  $X|_n = T$ . We can build such a Cayley tree  $X$  satisfying  $X|_n = T$  from a collection of trees as above in a straightforward way:

- An unordered collection  $C$  of rooted trees associated to a labeled vertex  $v$  is attached to  $T$  by adding an edge from  $v$  to the root of each tree in  $C$ .
- An ordered collection  $C$  of rooted trees associated to an edge  $e$  (say, from vertices  $v_1$  to  $v_2$ ) is attached to  $T$  by: removing  $e$ , adding an edge from  $v_1$  to the root of the first tree in  $C$ , an edge from the root of a tree in  $C$  to the root of the next tree, and an edge from the root of the last tree in  $C$  to  $v_2$ .
- A rooted tree associated with an unlabeled vertex of  $T$  is attached to  $T$  by putting its root in place of the unlabeled vertex.

The fact that we have a bijection can be proved formally by an induction on  $m$ , for example. We omit details.  $\square$

As a consequence of the previous proposition, we have:

$$\begin{aligned} \frac{d^n}{dz^n} T_2(z) &= \frac{d^n}{dz^n} \sum_{m \geq 0} \sum_{T \in \mathcal{C}_m} \frac{z^m}{m!} = \sum_{m \geq n} \sum_{T \in \mathcal{C}_m} \frac{z^{m-n}}{(m-n)!} = \sum_{U \in \mathcal{G}_n} \sum_{m \geq n} \sum_{\substack{T \in \mathcal{C}_m \\ T|_n = U}} \frac{z^m}{(m-n)!} \\ &= \sum_{U \in \mathcal{G}_n} \frac{e^{nT(z)}}{(1-T(z))^{n-1}} \left( \frac{T(z)}{1-T(z)} \right)^{\text{unl}(T)} = \frac{e^{nT(z)}}{(1-T(z))^{n-1}} H_n \left( \frac{T(z)}{1-T(z)} \right). \end{aligned}$$

So the combinatorics of Greg trees proves the formula in Equation (4). Note in particular that we do not use the recursion for the polynomial  $H_n(x)$  or for the construction of Greg trees.

As for the rooted case, the operation  $T \mapsto T|_n$  can be defined in a similar way. The only difference is that an unlabeled vertex with degree 2 is removed only if it is not the root. Similarly, we have:

**Proposition 3.6.** *Let  $T \in \mathcal{G}_n^\bullet$ . We have:*

$$(9) \quad \sum_{m \geq n} \sum_{\substack{X \in \mathcal{C}_m^\bullet \\ X|_n = T}} \frac{z^{m-n}}{(m-n)!} = \frac{e^{nT(z)}}{(1-T(z))^n} \left( \frac{T(z)}{1-T(z)} \right)^{\text{unl}(T)}.$$

And we can deduce Equation (3) in the same way as we obtained Equation (4).

For the sake of completeness, let us mention that we could have considered a slightly more general definition for rooted Greg trees, by allowing an unlabeled root to have degree 1 or 2. Then the generating function in size  $n$  is the polynomial  $(1+x)G_n(x)$ , and this can be easily proved bijectively (the term  $xG_n(x)$  gives the generating function for the case where the root is unlabeled and has degree 1). The formulas for  $G_n(x)$  can be modified accordingly. In particular, the previous proposition would be perhaps more natural with this alternative definition, as there would be  $(1-T(z))^{n-1}$  in the denominator just as in Proposition 3.5 instead of  $(1-T(z))^n$ .

The reason of our choice is that the present definition of Greg trees agrees with [8], and considering  $G_n(x)$  rather than  $(1+x)G_n(x)$  makes a better connection with previous works mentioned in the introduction. It would also be possible to define bi-rooted Greg trees, where we have two (ordered) roots allowed to have degree 1 or 2, and the generating function of these is  $\sum x^{\text{unl}(T)} = (1+x)^3 F_n(x)$ . The proof is along the same line: they appear by the operation  $T \mapsto T|_n$  defined on bi-rooted Cayley trees (whose exponential generating function is  $T_0(z)$ ).

#### 4. GENERATING FUNCTIONS

It is possible to write down formulas for the generating functions of the polynomials  $F_n(x)$ ,  $G_n(x)$  and  $H_n(x)$ . These essentially follow from Equations (2), (3), (4) and Taylor series expansions.

**Lemma 4.1.** *The compositional inverse of  $\frac{T(z)}{1-T(z)}$  is  $\frac{z}{1+z} \exp\left(-\frac{z}{1+z}\right)$ .*

*Proof.* We see  $\frac{T(z)}{1-T(z)}$  as the composition of  $T(z)$  and  $\frac{z}{1-z}$ . The compositional inverse of these two functions are respectively  $ze^{-z}$  and  $\frac{z}{1+z}$ . The result follows.  $\square$

Let us define  $F_0(x) = \frac{1}{1+x}$ ,  $G_0(x) = \frac{x}{1+x}$ , and  $H_0(x) = \frac{x(x+2)}{2(x+1)}$ . Although these are not polynomials, we can check that Equations (2), (3), (4), and the recursions in Equation (5) are true for  $n = 0$ , so these definitions are rather natural. In particular, the generating functions have a nice form as follows:

**Theorem 4.2.**

$$\begin{aligned} \sum_{n \geq 0} \frac{u^n}{n!} F_n(x) &= \frac{1}{(1+x)^2} T_0 \left( \frac{u+x}{1+x} e^{-\frac{x}{1+x}} \right), \\ \sum_{n \geq 0} \frac{u^n}{n!} G_n(x) &= T \left( \frac{u+x}{1+x} e^{-\frac{x}{1+x}} \right), \\ \sum_{n \geq 0} \frac{u^n}{n!} H_n(x) &= (1+x) T_2 \left( \frac{u+x}{1+x} e^{-\frac{x}{1+x}} \right). \end{aligned}$$

*Proof.* Let us begin with the case of  $G_n(x)$ . By a Taylor series expansion and the definition of  $G_n(x)$  in Equation (3), we have:

$$T(y+z) = \sum_{n \geq 0} \frac{y^n}{n!} \frac{d^n}{dz^n} T(z) = \sum_{n \geq 0} \frac{1}{n!} \left( \frac{e^{T(z)} y}{1-T(z)} \right)^n G_n \left( \frac{T(z)}{1-T(z)} \right).$$

Therefore, to get the result we can make the substitutions:

$$u = \frac{e^{T(z)} y}{1-T(z)}, \quad x = \frac{T(z)}{1-T(z)}.$$

From the lemma above, we obtain that  $z = \frac{x}{1+x} e^{-\frac{x}{1+x}}$ . Since  $T(z) = z e^{T(z)}$ , we can check that  $xy = zu$ . So:

$$y+z = \left( \frac{u}{x} + 1 \right) z = \frac{u+x}{1+x} e^{-\frac{x}{1+x}}$$

and the result follows. The other two generating functions are obtained by expanding  $T_0(y+z)$  and  $T_2(y+z)$  and making the same substitution.  $\square$

Note that we have proved in particular that

$$H(x, u) = (1+x) \left( G(x, u) - \frac{1}{2} G(x, u)^2 \right)$$

where  $G(x, u) = \sum_{n \geq 0} \frac{u^n}{n!} G_n(x)$  and  $H(x, u) = \sum_{n \geq 0} \frac{u^n}{n!} H_n(x)$ . To understand this bijectively, it is better to use the generating functions without constant terms:

$$\tilde{G} = \sum_{n \geq 1} \frac{u^n}{n!} G_n(x), \quad \tilde{H} = \sum_{n \geq 1} \frac{u^n}{n!} H_n(x).$$

Then the relation becomes  $\tilde{H} = \tilde{G} - \frac{1+x}{2} \tilde{G}^2$ , which we can rewrite as  $\tilde{G} = \tilde{H} + \frac{1+x}{2} \tilde{G}^2$ . Then we distinguish three cases for a rooted Greg tree:

- the root is the vertex with label 1,
- the root is unlabeled and has degree 2,
- the other cases.

Clearly, the first two cases give the generating functions  $\tilde{H}$  and  $\frac{x}{2} \tilde{G}^2$ . So it remains to check that the other cases give the generating function  $\frac{1}{2} \tilde{G}^2$ . So let  $T \in \mathcal{G}_n^\bullet$  be among these other cases. Since the vertex with label 1 is not the root, we can define an edge  $e$  as the first one in the shortest path from the root to the vertex with label 1. Then, remove this edge and say that its two endpoints are the roots of the two subtrees thus created. Since the root is not an unlabeled vertex with degree 2, these two subtrees are indeed Greg trees. In this way we obtain the generating function  $\frac{1}{2} \tilde{G}^2$ .

In the other direction, we can have  $G_n$  in terms of  $H_n$ . Indeed we have the relation:

$$(10) \quad G_n(x) = (n + (n-1)x) H_n(x) + (x+x^2) H'_n(x).$$

This also can be proved combinatorially.

We distinguish the following cases for a rooted Greg tree:

- The root is a labeled vertex.
- The root is an unlabeled vertex with degree at least 3.

- The root is an unlabeled vertex with degree equal to 2.

The first two cases give the generating functions  $nH_n(x)$  and  $xH'_n(x)$ . It remains to show that the last case give the generating function  $(n-1)xH_n(x) + x^2H'_n(x)$ . This can be interpreted by the fact that the generating function for Greg trees with a distinguished edge (with no restriction) is  $(n-1)H_n(x) + xH'_n(x)$ . We leave the details as an exercise.

## 5. THE W FUNCTION HAS THE BERNSTEIN PROPERTY

It was briefly explained in the introduction that the positivity of the polynomials  $(-1)^{n+1}P_n(x)$  (where  $P_n(x)$  is defined via Equation (1)) shows that  $W(z)$  is a Bernstein function. In fact Kalugin and Jeffrey [13] show much more than the positivity: the coefficients of  $(-1)^{n+1}P_n(x)$  form a unimodular and log-concave sequence. Another proof of the latter property is given by Lin and Zeng [17, Proposition 5.1].

The goal of this section is to show that the sequences  $F_n$ ,  $G_n$  and  $H_n$  are also related with some Bernstein functions in the same way as  $P_n$ .

We refer to the book [20] for the theory of completely monotonic functions and Bernstein functions. In particular, we do not present here the motivation for studying this kind of functions. Let us just mention that Bernstein functions appear in measure theory, because they are related with convolution semigroups of probability laws over positive reals. See [18] for some probabilistic aspects of the function  $W(z)$  related with the Bernstein property.

**Definition 5.1.** Let  $f : (0, \infty) \rightarrow \mathbb{R}$  of class  $C^\infty$ . We say that

- $f$  is *completely monotonic* if for all  $z > 0$  and  $n \geq 0$ ,  $(-1)^n f^{(n)}(z) \geq 0$ ,
- $f$  is a *Bernstein function* if  $f(z) > 0$  for any  $z > 0$ , and  $f'$  is completely monotonic.

**Theorem 5.2** (Kalugin and Jeffrey [13]).  $W(z)$  is a Bernstein function.

*Proof.* The regularity and positivity are rather elementary and the only difficulty is to check that  $(-1)^{n-1}W^{(n)}(z) \geq 0$  for  $n \geq 1$ . From  $T(z) = -W(-z)$  and the definition of  $G_n$  in (3), we have

$$(-1)^{n-1} \frac{d^n}{dz^n} W(z) = \frac{\exp(-nW(z))}{(1+W(z))^n} G_n \left( \frac{-W(z)}{1+W(z)} \right).$$

Since  $W(z) > 0$ , we have  $-1 < \frac{-W(z)}{1+W(z)} < 0$ . So it remains to show that  $G_n(x) > 0$  for any  $n \geq 1$  and  $-1 < x < 0$ . As was mentioned above, the polynomials  $G_n(x-1)$  were considered by Shor [21] who showed that they count Cayley trees according to the number of improper edges, and in particular, they have positive coefficients. It follows that  $G_n(x) > 0$  for  $n \geq 1$  and  $-1 < x < 0$ .  $\square$

**Remark 5.3.** The polynomials  $G_n(x-1)$  are the same as those denoted  $R_n(x)$  in [17] where they are called Ramanujan polynomials. Lin and Zeng show in [17, Proposition 5.1] that we have the relation

$$(11) \quad P_n(x) = (-1-x)^{n-1} G_n \left( \frac{-x}{1+x} \right),$$

and also show properties of the coefficients of  $P_n$  such as log-concavity from properties of  $G_n(x)$ . Note that Equation (11) could also be obtained by comparing Equations (1) and (3).

**Theorem 5.4.**  $-T_2(-z) = \frac{1}{2}W(z)^2 + W(z)$  is a Bernstein function.

*Proof.* Whenever  $f(z)$  is a Bernstein function,  $e^{-f(z)}$  is completely monotonic, see [20, Theorem 3.6]. In the case of  $W(z)$ , we have  $e^{-W(z)} = \frac{1}{z}W(z)$ . A primitive of  $W(z)/z$  is

$$\int \frac{W(z)}{z} dz = \frac{1}{2}W(z)^2 + W(z) = -T_2(-z).$$

Since it has nonnegative values, it follows that  $\frac{1}{2}W(z)^2 + W(z)$  is also a Bernstein function.

This result can be also proved by checking the signs of the derivatives, and the fact that the polynomials  $H_n(x)$  defined in Equation (4) are nonnegative on  $(-1, 0)$ .  $\square$

**Theorem 5.5.**  $1 - T_0(-z) = W(z)/(1 + W(z))$  is a Bernstein function.

*Proof.* A direct calculation shows that  $z \mapsto \frac{z}{1+z}$  is a Bernstein function. Since the composition of two Bernstein functions has the same property (see [20, Corollary 3.7]), we get the result.

This can also be proved by checking the sign of the derivatives, and the fact that the polynomials  $F_n(x)$  defined in Equation (2) are nonnegative on  $(-1, 0)$ .  $\square$

In fact, on the analytical level it is rather elementary to prove that  $W(z)$  has stronger a property than being a Bernstein function. We follow the terminology from [20], and call Nevanlinna-Pick function a holomorphic function that preserves the complex upper half-plane  $\mathbb{H}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ . Those functions that are nonnegative on  $(0, \infty)$  are a particular class of Bernstein, called *complete Bernstein functions*. See [20, Chapter 6] for details, in particular Theorems 6.2 and 6.7.

**Definition 5.6.** A Bernstein function  $f(z)$  is called *complete* if it has an analytic continuation on  $\mathbb{H}^+$  such that  $f(\mathbb{H}^+) \subset \mathbb{H}^+$ .

**Theorem 5.7.**  $W(z)$  is a complete Bernstein function.

*Proof.* The analytic continuations of  $W(z)$  were studied in [4, Section 4]. It is possible to extend  $W(z)$  as a holomorphic function on  $\mathbb{C} \setminus (-\infty, -\frac{1}{e}]$ . From  $W(z)e^{W(z)} = z$ , it follows that  $W(z) \in \mathbb{R} \Rightarrow z \in \mathbb{R}$ . The image of  $\mathbb{H}^+$  is a connected set included in  $\mathbb{C} - \mathbb{R}$ , so it is included either in  $\mathbb{H}^+$  or  $\mathbb{H}^-$ . In a neighborhood of 0, we have  $W(z) = z + O(z^2)$ , so we can conclude that  $W$  maps  $\mathbb{H}^+$  into itself. Then, Theorem 6.2 from [20] shows the result.  $\square$

## 6. IMPROPER EDGES IN CAYLEY TREES

The notion of *improper edges* in Cayley trees was introduced by Shor [21], and then used by Chen and Guo [3], Guo and Zeng [11], Zeng [24].

**Definition 6.1.** Let  $T \in \mathcal{C}_n^\bullet$ . A vertex  $v_2$  is called a *descendant* of a vertex  $v_1$  if the shortest path from the root to  $v_2$  goes through  $v_1$ . Let  $e$  be an edge of  $T$ , and let  $u$  and  $v$  denote its two endpoints in such a way that  $v$  is a descendant of  $u$ . The edge  $e$  of  $T$  is called *improper* if the label of  $u$  is greater than that of (at least) one of its descendants. Let  $\text{imp}(T)$  denote the number of improper edges of  $T$ .

Then Shor's result is:

$$G_n(x-1) = \sum_{T \in \mathcal{C}_n^\bullet} x^{\text{imp}(T)}.$$

His definition of this polynomial is a recursion for the coefficients which is equivalent to (5). The unrooted analog (and in fact a more general statement) is due to Zeng [24]: by considering an unrooted tree as a rooted tree where the root is the vertex labeled 1, we have a notion of improper edge on unrooted tree, and

$$H_n(x-1) = \sum_{T \in \mathcal{C}_n} x^{\text{imp}(T)}.$$

Hence we have:

$$\sum_{T \in \mathcal{C}_n^\bullet} (1+x)^{\text{imp}(T)} = \sum_{T \in \mathcal{G}_n^\bullet} x^{\text{unl}(T)}, \quad \sum_{T \in \mathcal{C}_n} (1+x)^{\text{imp}(T)} = \sum_{T \in \mathcal{G}_n} x^{\text{unl}(T)}.$$

This calls for bijective proofs: we would like to find an explicit map  $\alpha : \mathcal{G}_n^\bullet \rightarrow \mathcal{C}_n^\bullet$  with the property that, for all  $T \in \mathcal{C}_n^\bullet$ ,

$$\sum_{U \in \alpha^{-1}(T)} x^{\text{unl}(U)} = (1+x)^{\text{imp}(T)},$$

and with the analog property for the unrooted case. We leave this as an open problem for interested readers.

In one direction, we have the map  $T \mapsto T|_n$  from Cayley trees to Greg trees (with fewer vertices), and we have the map  $\alpha$  from Greg trees to Cayley trees. It could be interesting to see how these maps are related. For example, we can consider the composition  $\beta_n : T \mapsto \alpha(T|_n)$ , which gives a Cayley tree on  $n$  vertices for each Cayley tree  $T$  on more vertices. The properties of the map  $\alpha$  and  $T \mapsto T|_n$  show that each class  $\beta_n^{-1}(T)$  has a simple generating function:

$$\begin{aligned} \sum_{m \geq n} \sum_{\substack{U \in \mathcal{C}_m \\ \beta_n(U)=T}} \frac{z^{m-n}}{(m-n)!} &= \sum_{V \in \alpha^{-1}(T)} \sum_{m \geq n} \sum_{\substack{U \in \mathcal{C}_m \\ U|_n=V}} \frac{z^{m-n}}{(m-n)!} \\ &= \sum_{V \in \alpha^{-1}(T)} \frac{e^{nT(z)}}{(1-T(z))^{n-1}} \left( \frac{T(z)}{1-T(z)} \right)^{\text{unl}(V)} \\ &= \frac{e^{nT(z)}}{(1-T(z))^{n-1+\text{imp}(T)}}. \end{aligned}$$

In particular, this property of the map  $\beta_n$  gives an alternative way to compute  $\frac{d^n}{dz^n}T(z)$ , since we get:

$$\begin{aligned} \frac{d^n}{dz^n}T(z) &= \sum_{m \geq n} \sum_{T \in \mathcal{C}_m} \frac{z^{m-n}}{(m-n)!} = \sum_{U \in \mathcal{C}_n} \sum_{T \in \beta_n^{-1}(U)} \frac{z^{|T|}}{|T|!} \\ &= \frac{e^{nT(z)}}{(1-T(z))^{n-1}} \sum_{U \in \mathcal{C}_n} \frac{1}{(1-T(z))^{\text{imp}(T)}}. \end{aligned}$$

It could be quite interesting to find an explicit description of such a map  $\beta_n$ , although it is a weaker problem than finding one for the map  $\alpha$ .

### 7. FINAL REMARKS

The three sequence of polynomials studied in this article have a common generalization. Following Zeng [24], we consider the double sequence of polynomials

$$Q_{n,k}(x) = (x+n-1)Q_{n-1,k}(x) + (n+k-2)Q_{n-1,k-1}(x),$$

where  $n \geq 1$ ,  $0 \leq k \leq n-1$ , and  $Q_{1,0}(x) = 1$ . We have  $Q_{n,k}(x) = \psi_{k+1}(n-1, x+n)$  where  $\psi_k(r, x)$  was defined by Ramanujan [19] via the equation

$$\sum_{k \geq 0} \frac{(x+k)^{r+k} e^{-u(x+k)} u^k}{k!} = \sum_{k=1}^{r+1} \frac{\psi_k(r, x)}{(1-u)^{r+k}}.$$

Then one can check that

$$\begin{aligned} \sum_{k=0}^{n-1} Q_{n,k}(-1)x^k &= xF_{n-1}(x-1), \\ \sum_{k=0}^{n-1} Q_{n,k}(0)x^k &= G_n(x-1), \\ \sum_{k=0}^{n-1} Q_{n,k}(1)x^k &= H_n(x-1). \end{aligned}$$

However, it is not clear if Greg trees are in some way related with  $Q_{n,k}(x)$  for other values of  $x$ .

### 8. ACKNOWLEDGEMENT

We thank Sloane's OEIS [22] for bringing the references [6, 8] to our knowledge.

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