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Strong approximations for the p -fold integrated empirical process with applications to statistical tests

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Abstract

The main purpose of this paper is to investigate the strong approximation of the p -fold integrated empirical process, p being a fixed positive integer. More precisely, we obtain the exact rate of the approximations by a sequence of weighted Brownian bridges and a weighted Kiefer process. Our arguments are based in part on the Komlós *et al.* (1975)'s results. Applications include the two-sample testing procedures together with the change-point problems. We also consider the strong approximation of integrated empirical processes when the parameters are estimated. Finally, we study the behavior of the self-intersection local time of the partial sum process representation of integrated empirical processes.

Key words: Integrated empirical process; Brownian bridge; Kiefer process; Rates of convergence; Local time; Two-sample problem; Hypothesis testing; Goodness-of-fit; Change-point.

AMS Classifications: primary: 62G30; 62G20; 60F17; secondary: 62F03; 62F12; 60F15.

1 Introduction

Let $F = \{F(t), t \in \mathbb{R}\}$ be a continuous distribution function [d.f.] and denote by $Q = \{Q(u), u \in [0, 1]\}$ the usual quantile function (generalized inverse) pertaining to F defined as

$$Q(u) := \inf\{t \in \mathbb{R} : F(t) \geq u\} \quad \text{for } u \in (0, 1),$$

$$Q(0) := \lim_{u \downarrow 0} Q(u) \quad \text{and} \quad Q(1) := \lim_{u \uparrow 1} Q(u).$$

The function Q is strictly increasing and we have $F(Q(u)) = u$ for any $u \in [0, 1]$. Consider now a sequence of independent, identically distributed [i.i.d.] random variables [r.v.'s] $\{U_i : i \in \mathbb{N}^*\}$ uniformly distributed on $[0, 1]$ and, for each $i \in \mathbb{N}^*$, set $X_i := Q(U_i)$. The sequence $\{X_i : i \in \mathbb{N}^*\}$ consists of i.i.d. r.v.'s with d.f. F : $F(t) = \mathbb{P}\{X_1 \leq t\}$ for $t \in \mathbb{R}$ (cf., e.g., Shorack and Wellner (1986), p. 3 and the references therein). Moreover, we conversely have $U_i = F(X_i)$.

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For each $n \in \mathbb{N}^*$, let \mathbb{F}_n and \mathbb{U}_n be the empirical d.f.'s based upon the respective samples X_1, \dots, X_n and U_1, \dots, U_n defined by

$$\begin{aligned}\mathbb{F}_n(t) &:= \frac{1}{n} \#\{i \in \{1, \dots, n\} : X_i \leq t\} = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq t\}} \quad \text{for } t \in \mathbb{R}, \\ \mathbb{U}_n(u) &:= \frac{1}{n} \#\{i \in \{1, \dots, n\} : U_i \leq u\} = \frac{1}{n} \sum_{i=1}^n 1_{\{U_i \leq u\}} \quad \text{for } u \in [0, 1],\end{aligned}$$

where $\#$ denotes cardinality. For each $n \in \mathbb{N}^*$, we introduce the *empirical process* α_n and the *uniform empirical process* β_n defined by

$$\alpha_n(t) := \sqrt{n} (\mathbb{F}_n(t) - F(t)) \quad \text{for } t \in \mathbb{R}, \quad (1.1)$$

$$\beta_n(u) := \sqrt{n} (\mathbb{U}_n(u) - u) \quad \text{for } u \in [0, 1]. \quad (1.2)$$

We have of course the usual relations between the empirical process and uniform empirical process:

$$\alpha_n(t) = \beta_n(F(t)) \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*, \quad (1.3)$$

$$\beta_n(u) = \alpha_n(Q(u)) \quad \text{for } u \in [0, 1], n \in \mathbb{N}^*. \quad (1.4)$$

In this paper, we consider integrated empirical d.f.'s based upon the samples X_1, \dots, X_n and U_1, \dots, U_n together with the corresponding integrated empirical processes in the following sense.

Definition 1.1 *We define the families of integrated d.f.'s and integrated empirical d.f.'s associated with the d.f. F , for any $p \in \mathbb{N}$, any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$, as*

$$\begin{aligned}F^{(0)}(t) &:= F(t), \quad \mathbb{F}_n^{(0)}(t) := \mathbb{F}_n(t), \\ F^{(1)}(t) &:= \int_{-\infty}^t F(s) dF(s), \quad \mathbb{F}_n^{(1)}(t) := \int_{-\infty}^t \mathbb{F}_n(s) d\mathbb{F}_n(s), \\ \text{and for } p \geq 2, \\ F^{(p)}(t) &:= \int_{-\infty}^t dF(s_1) \int_{-\infty}^{s_1} dF(s_2) \dots \int_{-\infty}^{s_{p-1}} F(s_p) dF(s_p), \\ \mathbb{F}_n^{(p)}(t) &:= \int_{-\infty}^t d\mathbb{F}_n(s_1) \int_{-\infty}^{s_1} d\mathbb{F}_n(s_2) \dots \int_{-\infty}^{s_{p-1}} \mathbb{F}_n(s_p) d\mathbb{F}_n(s_p),\end{aligned}$$

together with the corresponding family of integrated empirical processes as

$$\alpha_n^{(p)}(t) := \sqrt{n} \left(\mathbb{F}_n^{(p)}(t) - F^{(p)}(t) \right). \quad (1.5)$$

Notice that $F^{(p)}$ (resp. $\mathbb{F}_n^{(p)}$) is a kind of p -fold integral with respect to the measure dF (resp. $d\mathbb{F}_n$). Hence, we will call $F^{(p)}$ (resp. $\mathbb{F}_n^{(p)}$, $\alpha_n^{(p)}$) throughout the paper p -fold integrated d.f. (resp. p -fold integrated empirical d.f., p -fold integrated empirical process). Finally, we define exactly in the same manner the p -fold integrated uniform empirical d.f. $\mathbb{U}_n^{(p)}$ and the p -fold integrated uniform empirical process $\beta_n^{(p)}$.

Below, we provide explicit expressions for $F^{(p)}$ and $\mathbb{F}_n^{(p)}$, the proof of which are postponed to Section 7.

Proposition 1.2 *For each $p \in \mathbb{N}$, we explicitly have, with probability 1,*

$$F^{(p)}(t) = \frac{F(t)^{p+1}}{(p+1)!}, \quad \mathbb{F}_n^{(p)}(t) = \frac{1}{n^{p+1}} \binom{n\mathbb{F}_n(t) + p}{p+1} \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*. \quad (1.6)$$

The particular case where $p = 1$ has often been considered in the literature. [Henze and Nikitin \(2000, 2002\)](#) introduced and deeply investigated the goodness-of-fit testing procedures based on the integrated empirical process. Indeed, the asymptotic properties of their procedures, Kolmogorov-Smirnov, Cramér-von Mises and Watson-type statistics, can be derived from the limiting behavior of the integrated empirical process. [Henze and Nikitin \(2003\)](#) considered a two-sample testing procedure and focused on the approximate local Bahadur efficiencies of their statistical tests. It is noteworthy to point out that tests based on some integrated empirical processes turn out to be more efficient for certain distributions. In [Lachal \(2001\)](#), another version of the p -fold integrated empirical process ($p \in \mathbb{N}^*$) was introduced. For the extension to the multivariate framework, we may refer to [Jing and Wang \(2006\)](#) and [Jing and Yang \(2007\)](#) where some projected integrated empirical processes for testing the equality of two multivariate distributions are considered. Inspired by the work of [Henze and Nikitin \(2003\)](#), [Bouzebda and El Faouzi \(2012\)](#) developed multivariate two-sample testing procedures based on the integrated empirical copula process that are extended to the K -sample problem in [Bouzebda *et al.* \(2011\)](#). Emphasis is placed on the explanation of the strong approximation methodology. The asymptotic behavior of weighted multivariate Cramér-von Mises-type statistics under contiguous alternatives was characterized by [Bouzebda and Zari \(2013\)](#). For more recent references, we refer to [Durio and Nikitin \(2016\)](#) and [Alvarez-Andrade *et al.* \(2017\)](#).

The main purpose of this paper is to investigate the strong approximation of the p -fold integrated empirical process. Next we use the obtained results for studying the asymptotic properties of statistical tests based on this process. We point out that strong approximations are quite useful and have received considerable attention in probability theory. Indeed, many well-known and important probability theorems can be considered as consequences of results about strong approximation of sequences of sums by corresponding Gaussian sequences.

We will first obtain an upper bound in probability for the distance between the p -fold integrated empirical process and a sequence of appropriate Brownian bridges (see [Theorem 2.3](#)). This is the key point of our study. From this, we will deduce a strong approximation of the p -fold integrated empirical process by this sequence of Brownian bridges (see [Corollary 2.5](#)). As an application, we will derive the rates of convergence for the distribution of smooth functionals of each p -fold integrated empirical process (see [Corollary 2.4](#)). Moreover, we will deduce strong approximations for the Kolmogorov-Smirnov and Cramér-von Mises-type statistics associated with the p -fold integrated empirical processes (see [Corollary 2.8](#)).

Second, we will obtain a strong approximation of the p -fold integrated empirical process by a Kiefer processes (see [Theorem 2.6](#)). This latter is of particular interest; indeed, for instance, any kind of law of the iterated logarithm which holds for the partial sums of Gaussian processes may then be transferred to the p -fold integrated empirical processes (see [Corollary 2.7](#)). We may refer to [DasGupta \(2008\)](#) (Chapter 12), [Csörgő and Horváth \(1993\)](#) (Chapter 3), [Csörgő and Révész \(1981\)](#) (Chapters 4-5) and [Shorack and Wellner \(1986\)](#) (Chapter 12) for expositions, details and references about this problem.

We refer to [Csörgő and Hall \(1984\)](#), [Csörgő \(2007\)](#) and [Mason and Zhou \(2012\)](#) for a survey of some applications of the strong approximation and many references. There is a huge literature on the strong approximations and their applications. It is not the purpose of this paper to survey this extensive literature.

The layout of the article is as follows. In [Section 2](#), we first present some strong approximation results for the p -fold integrated empirical process; our main tools are the results of [Komlós *et al.* \(1975\)](#). [Sections 3](#) and [4](#) are devoted to statistical applications, namely the two-sample and change-point problems respectively. In [Section 5](#), we deal with the strong approximation of the p -fold integrated empirical process when parameters are estimated. [Section 6](#) is concerned with the behavior of the self-intersection local time of the partial sum process representation of the p -fold integrated empirical process. To prevent from interrupting the flow of the presentation, all mathematical developments are postponed to [Section 7](#).

2 Strong approximation

2.1 Some processes

First, we introduce some definitions and notations. Let $\mathbb{W} = \{\mathbb{W}(s) : s \geq 0\}$ and $\mathbb{B} = \{\mathbb{B}(u) : u \in [0, 1]\}$ be the standard Wiener process and Brownian bridge, that is, the centered Gaussian processes with continuous sample paths, and covariance functions

$$\mathbb{E}(\mathbb{W}(s)\mathbb{W}(t)) = s \wedge t \quad \text{for } s, t \geq 0$$

and

$$\mathbb{E}(\mathbb{B}(u)\mathbb{B}(v)) = u \wedge v - uv \quad \text{for } u, v \in [0, 1].$$

A Kiefer process $\mathbb{K} = \{\mathbb{K}(s, u) : s \geq 0, u \in [0, 1]\}$ is a two-parameters centered Gaussian process, with continuous sample paths, and covariance function

$$\mathbb{E}(\mathbb{K}(s, u)\mathbb{K}(t, v)) = (s \wedge t)(u \wedge v - uv) \quad \text{for } s, t \geq 0 \quad \text{and } u, v \in [0, 1].$$

It satisfies the following distributional identities:

$$\{\mathbb{K}(s, u) : u \in [0, 1]\} \stackrel{\mathcal{L}}{=} \{\sqrt{s}\mathbb{B}(u) : u \in [0, 1]\} \quad \text{for } s \geq 0$$

and

$$\{\mathbb{K}(s, u) : s \geq 0\} \stackrel{\mathcal{L}}{=} \left\{ \sqrt{u(1-u)}\mathbb{W}(s) : s \geq 0 \right\} \quad \text{for } u \in [0, 1],$$

where $\stackrel{\mathcal{L}}{=}$ stands for the equality in distribution. The interested reader may refer to [Csörgő and Révész \(1981\)](#) for details on the Gaussian processes mentioned above.

2.2 Brownian approximation

It is well-known that the empirical uniform process $\{\beta_n : n \in \mathbb{N}^*\}$ converges to \mathbb{B} in $D[0, 1]$ (the space of all right-continuous real-valued functions defined on $[0, 1]$ which have left-hand limits, equipped with the Skorohod topology; see, for details, [Billingsley \(1968\)](#)). The rate of convergence of this process to \mathbb{B} is an important task in statistics as well as in probability that has been investigated by several authors. We can and will assume without loss of generality that all r.v.'s and processes introduced so far and later on in this paper can be defined on the same probability space (cf. Appendix 2 in [Csörgő and Horváth \(1993\)](#)).

Komlós, Major, and Tusnády [KMT] ([Komlós *et al.* \(1975\)](#), Theorem 3; refer also to [Komlós *et al.* \(1976\)](#)) stated the following Brownian bridge approximation for $\{\beta_n : n \in \mathbb{N}^*\}$ (Formula (2.2)), along with a description of its proof with few details, which has been subsequently refined by [Mason and van Zwet \(1987\)](#) (Formula (2.1)).

Theorem A *On a suitable probability space, we may define the uniform empirical process $\{\beta_n : n \in \mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{\mathbb{B}_n : n \in \mathbb{N}^*\}$, such that, for any $d, n \in \mathbb{N}^*$ satisfying $d \leq n$ and any positive number x ,*

$$\mathbb{P} \left\{ \sup_{u \in [0, d/n]} |\beta_n(u) - \mathbb{B}_n(u)| \geq \frac{1}{\sqrt{n}} (c_1 \log d + x) \right\} \leq c_2 e^{-c_3 x} \quad (2.1)$$

where c_1, c_2 and c_3 are suitable absolute constants. The same inequality holds when replacing the interval $[0, d/n]$ by $[1 - d/n, 1]$. In particular, for $d = n$,

$$\mathbb{P} \left\{ \sup_{u \in [0, 1]} |\beta_n(u) - \mathbb{B}_n(u)| \geq \frac{1}{\sqrt{n}} (c_1 \log n + x) \right\} \leq c_2 e^{-c_3 x}. \quad (2.2)$$

In (2.2), suitable explicit values for c_1, c_2, c_3 were exhibited by [Bretagnolle and Massart \(1989\)](#), Theorem 1: $c_1 = 12, c_2 = 2, c_3 = 1/6$. In his manuscript, [Major \(2000\)](#) details the original proof of (2.2). [Chatterjee \(2012\)](#) provided a new alternative approach for proving the famous KMT theorem.

Remark 2.1 *In the sequel, the precise meaning of “suitable probability space” is that an independent sequence of Wiener processes, which is independent of the originally given sequence of i.i.d. r.v.’s, can be constructed on the assumed probability space. This is a technical requirement which allows the construction of the Gaussian processes displayed in our theorems, and which is not restrictive since one can expand the probability space to make it rich enough (see, e.g., Appendix 2 in [Csörgő and Horváth \(1993\)](#), [de Acosta \(1982\)](#), [Csörgő and Révész \(1981\)](#) and Lemma A1 in [Berkes and Philipp \(1979\)](#)). Throughout this paper, it will be assumed that the underlying probability spaces are suitable in this sense.*

In the following theorem, we state the key point to access the strong Brownian approximation of the p -fold integrated uniform empirical process $\{\beta_n^{(p)} : n \in \mathbb{N}^*\}$.

Theorem 2.2 *Fix $p \in \mathbb{N}^*$. On a suitable probability space, we may define the p -fold integrated uniform empirical process $\{\beta_n^{(p)} : n \in \mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{\mathbb{B}_n : n \in \mathbb{N}^*\}$, such that, for any $d, n \in \mathbb{N}^*$ satisfying $d \leq n$ and large enough x ,*

$$\mathbb{P}\left\{\sup_{u \in [0, d/n]} \left| \beta_n^{(p)}(u) - \mathbb{B}_n^{(p)}(u) \right| \geq \frac{1}{\sqrt{n}} (c_1 \log d + x)\right\} \leq B_p \sum_{k=2}^{p+1} \exp\left(-C_p x^{2/k} n^{1-2/k}\right) \quad (2.3)$$

where B_p and C_p are positive constants depending on p , c_1 is the constant arising in (2.2) and, for each $n \in \mathbb{N}^*$, $\mathbb{B}_n^{(p)}$ is the process defined by

$$\mathbb{B}_n^{(p)}(u) := \frac{1}{p!} u^p \mathbb{B}_n(u) \quad \text{for } u \in [0, 1].$$

The same inequality holds when replacing the interval $[0, d/n]$ by $[1 - d/n, 1]$.

In particular, making $d = n$ in (2.3), we obtain the key estimate for the p -fold integrated empirical process $\{\alpha_n^{(p)} : n \in \mathbb{N}^*\}$ below.

Theorem 2.3 *Fix $p \in \mathbb{N}^*$. On a suitable probability space, we may define the p -fold integrated empirical process $\{\alpha_n^{(p)} : n \in \mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{\mathbb{B}_n : n \in \mathbb{N}^*\}$, such that, for large enough x and all $n \in \mathbb{N}^*$,*

$$\mathbb{P}\left\{\sup_{t \in \mathbb{R}} \left| \alpha_n^{(p)}(t) - \mathbb{B}_n^{(p)}(F(t)) \right| \geq \frac{1}{\sqrt{n}} (c_1 \log n + x)\right\} \leq B_p \sum_{k=2}^{p+1} \exp\left(-C_p x^{2/k} n^{1-2/k}\right). \quad (2.4)$$

An important consequence of Theorem 2.3 is an upper bound for the convergence of distributions of smooth functionals of $\alpha_n^{(p)}$. Indeed, applying (2.4) with $x = c \log n$ for a suitable constant c yields the result below.

Corollary 2.4 *Fix $p \in \mathbb{N}^*$. Let \mathbb{B} be a Brownian bridge and $\mathbb{B}^{(p)}$ the process defined by*

$$\mathbb{B}^{(p)}(u) := \frac{1}{p!} u^p \mathbb{B}(u) \quad \text{for } u \in [0, 1].$$

If $\Phi(\cdot)$ is a Lipschitz functional defined on $D[0, +\infty)$ such that the r.v. $\Phi(\mathbb{B}^{(p)}(F(\cdot)))$ admits a bounded density function, then, as $n \rightarrow \infty$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{\Phi(\alpha_n^{(p)}(\cdot)) \leq x\right\} - \mathbb{P}\left\{\Phi(\mathbb{B}^{(p)}(F(\cdot))) \leq x\right\} \right| = \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right). \quad (2.5)$$

For more comments on this kind of results, we may refer to Csörgő *et al.* (2000), Corollary 1.1 and p. 2459.

By applying (2.4) to $x = c' \log n$ for a suitable constant c' and appealing to Borel-Cantelli lemma, one can obtain the following almost sure approximation of the process $\{\alpha_n^{(p)} : n \in \mathbb{N}^*\}$ based on a sequence of Brownian bridges.

Corollary 2.5 *The following bound holds, with probability 1, as $n \rightarrow \infty$:*

$$\sup_{t \in \mathbb{R}} \left| \alpha_n^{(p)}(t) - \mathbb{B}_n^{(p)}(F(t)) \right| = \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right). \quad (2.6)$$

The next result yields an almost sure approximation for $\{\alpha_n^{(p)} : n \in \mathbb{N}^*\}$ based on a Kiefer process.

Theorem 2.6 *On a suitable probability space, we may define the p -fold integrated empirical process $\{\alpha_n^{(p)} : n \in \mathbb{N}^*\}$, in combination with a Kiefer process $\{\mathbb{K}(s, u) : s \geq 0, u \in [0, 1]\}$, such that, with probability 1, as $n \rightarrow \infty$,*

$$\max_{1 \leq k \leq n} \sup_{t \in \mathbb{R}} \left| \sqrt{k} \alpha_k^{(p)}(t) - \mathbb{K}^{(p)}(k, F(t)) \right| = \mathcal{O}((\log n)^2)$$

where $\mathbb{K}^{(p)}$ is the process defined by

$$\mathbb{K}^{(p)}(s, u) := \frac{1}{p!} u^p \mathbb{K}(s, u) \quad \text{for } s \geq 0, u \in [0, 1].$$

Let us mention that the “extracted” Kiefer process $\{\mathbb{K}(n, u) : n \in \mathbb{N}^*, u \in [0, 1]\}$ may be viewed as the partial sums process of a sequence of independent Brownian bridges $\{\mathbb{B}_i : i \in \mathbb{N}^*\}$:

$$\mathbb{K}(n, u) = \sum_{i=1}^n \mathbb{B}_i(u) \quad \text{for } n \in \mathbb{N}^*, u \in [0, 1].$$

From Theorem 2.6, we deduce the following law of iterated logarithm (“a.s.” stands for “almost surely”).

Corollary 2.7 *We have the following law of iterated logarithm for the p -fold integrated empirical process:*

$$\limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} |\alpha_n^{(p)}(t)|}{\sqrt{\log \log n}} = \frac{(p + 1/2)^{p+1/2}}{p! (p + 1)^{p+1}} \quad \text{a.s.} \quad (2.7)$$

As a direct application of (2.6) and (2.7) to the problem of goodness-of-fit, for testing the null hypothesis

$$\mathcal{H}_0 : F = F_0,$$

we can use the following statistics: the p -fold integrated Kolmogorov-Smirnov statistic

$$\mathbf{S}_n^{(p)} := \sup_{t \in \mathbb{R}} \left| \sqrt{n} \left(\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t) \right) \right|$$

as well as the p -fold integrated Cramér-von Mises statistic

$$\mathbf{T}_n^{(p)} := n \int_{\mathbb{R}} \left(\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t) \right)^2 dF_0(t).$$

Corollary 2.8 Under \mathcal{H}_0 , with probability 1, as $n \rightarrow \infty$, we have

$$\left| \mathbf{S}_n^{(p)} - \sup_{t \in \mathbb{R}} |\mathbb{B}_n^{(p)}(F_0(t))| \right| = \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right), \quad (2.8)$$

$$\left| \mathbf{T}_n^{(p)} - \int_{\mathbb{R}} [\mathbb{B}_n^{(p)}(F_0(t))]^2 dF_0(t) \right| = \mathcal{O}\left(\sqrt{\frac{\log \log n}{n}} \log n\right). \quad (2.9)$$

We finish this part by pointing out the possibility of considering the statistics, for $r > 1$,

$$\sqrt{n} \left(\int_{\mathbb{R}} |\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t)|^r dF_0(t) \right)^{1/r}.$$

It is clear, however, that we have the following convergence in distribution as $n \rightarrow \infty$, under \mathcal{H}_0 :

$$\sqrt{n} \left(\int_{\mathbb{R}} |\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t)|^r dF_0(t) \right)^{1/r} \longrightarrow \left(\int_{\mathbb{R}} |\mathbb{B}^{(p)}(F_0(t))|^r dF_0(t) \right)^{1/r}.$$

In a future research, it would be of interest to deeply investigate such statistics.

3 The two-sample problem

For each $m, n \in \mathbb{N}^*$, let X_1, \dots, X_m and Y_1, \dots, Y_n be independent random samples from continuous d.f.'s F and G , respectively, and let $\mathbb{F}_m^{(p)}$ and $\mathbb{G}_n^{(p)}$ denote their p -fold integrated empirical d.f.'s. Tests for the null hypothesis

$$\mathcal{H}'_0 : F = G,$$

can be based on the *p-fold integrated two-sample empirical process* defined, for each $m, n \in \mathbb{N}^*$, by

$$\boldsymbol{\xi}_{m,n}^{(p)}(t) := \sqrt{\frac{mn}{m+n}} \left(\mathbb{F}_m^{(p)}(t) - \mathbb{G}_n^{(p)}(t) \right) \quad \text{for } t \in \mathbb{R}.$$

Actually, as in [Bouzebda and El Faouzi \(2012\)](#), we will more generally consider the following *modified p-fold integrated two-sample empirical process* (which includes the process $\boldsymbol{\xi}_{m,n}^{(p)}$). Fix a positive integer q which will serve as a power. We define, for each $m, n \in \mathbb{N}^*$,

$$\boldsymbol{\xi}_{m,n}^{(p,q)}(t) := \sqrt{\frac{mn}{m+n}} \left[\left(\mathbb{F}_m^{(p)}(t) \right)^q - \left(\mathbb{G}_n^{(p)}(t) \right)^q \right] \quad \text{for } t \in \mathbb{R}.$$

Set also, for any $m, n \in \mathbb{N}^*$,

$$\varphi(m, n) := \max\left(\frac{\log m}{\sqrt{m}}, \frac{\log n}{\sqrt{n}}\right) \quad \text{and} \quad \phi(m, n) := \max\left(\sqrt{\frac{\log \log m}{m}} \log m, \sqrt{\frac{\log \log n}{n}} \log n\right).$$

Reasonable statistics for testing \mathcal{H}'_0 would be the *modified p-fold integrated Kolmogorov-Smirnov statistic*

$$\mathbf{S}_{m,n}^{(p,q)} := \sup_{t \in \mathbb{R}} |\boldsymbol{\xi}_{m,n}^{(p,q)}(t)|$$

and the *modified p-fold integrated Cramér-von Mises statistic*

$$\mathbf{T}_{m,n}^{(p,q)} := \int_{\mathbb{R}} \boldsymbol{\xi}_{m,n}^{(p,q)}(t)^2 dF(t).$$

The following results are consequences of [Corollary 2.5](#).

Corollary 3.1 *On a suitable probability space, it is possible to define $\{\boldsymbol{\xi}_{m,n}^{(p,q)} : m, n \in \mathbb{N}^*\}$, jointly with two sequences of Brownian bridges $\{\mathbb{B}_m^1 : m \in \mathbb{N}^*\}$ and $\{\mathbb{B}_n^2 : n \in \mathbb{N}^*\}$, such that, under \mathcal{H}'_0 , with probability 1, as $\min(m, n) \rightarrow \infty$,*

$$\sup_{t \in \mathbb{R}} \left| \boldsymbol{\xi}_{m,n}^{(p,q)}(t) - \mathbb{B}_{m,n}^{(p,q)}(t) \right| = \mathcal{O}(\varphi(m, n)),$$

where, for each $m, n \in \mathbb{N}^*$, $\mathbb{B}_{m,n}^{(p,q)}$ is the Gaussian process defined by

$$\mathbb{B}_{m,n}^{(p,q)}(t) := \frac{(p+1)q}{(p+1)!^q} F(t)^{pq+q-1} \left(\sqrt{\frac{n}{m+n}} \mathbb{B}_m^1(F(t)) - \sqrt{\frac{m}{m+n}} \mathbb{B}_n^2(F(t)) \right) \quad \text{for } t \in \mathbb{R}.$$

Corollary 3.2 *Under \mathcal{H}'_0 , with probability 1, as $\min(m, n) \rightarrow \infty$, we have*

$$\left| \mathbf{S}_{m,n}^{(p,q)} - \sup_{t \in \mathbb{R}} \left| \mathbb{B}_{m,n}^{(p,q)}(t) \right| \right| = \mathcal{O}(\varphi(m, n)),$$

$$\left| \mathbf{T}_{m,n}^{(p,q)} - \int_{\mathbb{R}} \mathbb{B}_{m,n}^{(p,q)}(t)^2 dF(t) \right| = \mathcal{O}(\phi(m, n)).$$

Remark 3.3 *The family of statistics indexed by q may be used to maximize the power of the statistical test for a specific alternative hypothesis as argued in [Ahmad and Dorea \(2001\)](#) in the case $p = 1$.*

Now, we fix a positive integer K and we describe the more general K -sample problem. For each $k \in \{1, \dots, K\}$, we consider a setting made of independent observations $\{X_i^k : i \in \{1, \dots, n_k\}\}$ of a real-valued r.v. X^k . The d.f.'s of X_i^k , $i \in \{1, \dots, n_k\}$, are denoted by F^k and they are assumed to be continuous. We would like to test, F_0 being a fixed continuous d.f., the null hypothesis

$$\mathcal{H}_0^K : F^1 = F^2 = \dots = F^K = F_0.$$

For any K -tuple of positive integers $\mathbf{n} = (n_1, \dots, n_K)$, set $|\mathbf{n}| = \sum_{k=1}^K n_k$ and let

$$(Z_1, \dots, Z_{|\mathbf{n}|}) := (X_1^1, \dots, X_{n_1}^1, X_1^2, \dots, X_{n_2}^2, \dots, X_1^K, \dots, X_{n_K}^K)$$

be the pooled sample of total size $|\mathbf{n}|$, $\mathbb{D}_{K,\mathbf{n}}^{(p)}$ be the p -fold integrated empirical d.f. based upon $Z_1, \dots, Z_{|\mathbf{n}|}$, and, for each $k \in \{1, \dots, K\}$, $\mathbb{F}_{n_k}^{(p),k}$ be the p -fold integrated empirical d.f. based upon $X_1^k, \dots, X_{n_k}^k$. Of course, we have the following identity:

$$\mathbb{D}_{K,\mathbf{n}}^{(p)} = \frac{1}{|\mathbf{n}|} \sum_{k=1}^K n_k \mathbb{F}_{n_k}^{(p),k}. \quad (3.1)$$

Next, we define the p -fold integrated K -sample empirical process in the following way: for any K -tuple $\mathbf{n} = (n_1, \dots, n_K) \in (\mathbb{N}^*)^K$,

$$\boldsymbol{\xi}_{K,\mathbf{n}}^{(p)}(t) := \sum_{k=1}^K n_k \left(\mathbb{F}_{n_k}^{(p),k}(t) - \mathbb{D}_{K,\mathbf{n}}^{(p)}(t) \right)^2 \quad \text{for } t \in \mathbb{R}.$$

Obvious candidates for testing Hypothesis \mathcal{H}_0^K are the p -fold integrated K -sample Kolmogorov-Smirnov statistic

$$\mathbf{S}_{K,\mathbf{n}}^{(p)} := \sup_{t \in \mathbb{R}} \boldsymbol{\xi}_{K,\mathbf{n}}^{(p)}(t)$$

and the p -fold integrated K -sample Cramér-von Mises functional (the usual square being included in the definition of $\boldsymbol{\xi}_{K,\mathbf{n}}^{(p)}$)

$$\mathbf{T}_{K,\mathbf{n}}^{(p)} := \int_{\mathbb{R}} \boldsymbol{\xi}_{K,\mathbf{n}}^{(p)}(t) dF_0(t).$$

Set

$$\phi_K(\mathbf{n}) := \max_{1 \leq k \leq K} \left\{ \sqrt{\frac{\log \log n_k}{n_k}} \log n_k \right\}.$$

As a consequence of Corollary 2.5 and by using similar arguments to those used in Bouzebda *et al.* (2011), we obtain the following results.

Theorem 3.4 *On a suitable probability space, it is possible to define $\{\boldsymbol{\xi}_{K,\mathbf{n}}^{(p)} : \mathbf{n} \in (\mathbb{N}^*)^K\}$, jointly with K sequences of Brownian bridges $\{\mathbb{B}_m^k : m \in \mathbb{N}^*\}$, $k \in \{1, \dots, K\}$, such that, under \mathcal{H}_0^K , with probability 1, for $\mathbf{n} = (n_1, \dots, n_K)$ such that $\min_{1 \leq k \leq K} n_k \rightarrow \infty$,*

$$\sup_{t \in \mathbb{R}} \left| \boldsymbol{\xi}_{K,\mathbf{n}}^{(p)}(t) - \mathbb{B}_{K,\mathbf{n}}^{(p)}(t) \right| = \mathcal{O}(\phi_K(\mathbf{n})),$$

where, for each $\mathbf{n} = (n_1, \dots, n_K) \in (\mathbb{N}^*)^K$, $\mathbb{B}_{K,\mathbf{n}}^{(p)}$ is the process defined by

$$\mathbb{B}_{K,\mathbf{n}}^{(p)}(t) := \frac{F_0(t)^{2p}}{p!^2} \left[\sum_{k=1}^K \mathbb{B}_{n_k}^k(F_0(t))^2 - \left(\sum_{k=1}^K \sqrt{\frac{n_k}{|\mathbf{n}|}} \mathbb{B}_{n_k}^k(F_0(t)) \right)^2 \right] \quad \text{for } t \in \mathbb{R}.$$

In the particular case $K = 2$ (i.e. the two-sample problem), the corresponding settings are related to the previous ones according as

$$\boldsymbol{\xi}_{2,(n_1,n_2)}^{(p)}(t) = \left(\boldsymbol{\xi}_{n_1,n_2}^{(p)}(t) \right)^2, \quad \phi_2((n_1, n_2)) = \phi(n_1, n_2), \quad \mathbf{S}_{2,(n_1,n_2)}^{(p)} = \left(\mathbf{S}_{n_1,n_2}^{(p,1)} \right)^2, \quad \mathbf{T}_{2,(n_1,n_2)}^{(p)} = \mathbf{T}_{n_1,n_2}^{(p,1)}.$$

Notice that $\mathbb{B}_{K,\mathbf{n}}^{(p)}(t) \geq 0$ for any $t \in \mathbb{R}$ and $\mathbf{n} \in (\mathbb{N}^*)^K$ as it is easily seen with the aid of the Cauchy-Schwarz inequality. For $K = 2$, this process writes $\mathbb{B}_{2,(n_1,n_2)}^{(p)} = \left(\mathbb{B}_{n_1,n_2}^{(p,1)} \right)^2$, this is the square of a Gaussian process.

The next result, which is an immediate consequence of the previous theorem (observe that $\mathbf{S}_{K,\mathbf{n}}^{(p)}$ and $\mathbf{T}_{K,\mathbf{n}}^{(p)}$ are bounded linear functionals of the process $\boldsymbol{\xi}_{K,\mathbf{n}}^{(p)}$), gives the limit null distributions of the statistics under consideration.

Corollary 3.5 *Under \mathcal{H}_0^K , with probability 1, for $\mathbf{n} = (n_1, \dots, n_K)$ such that $\min_{1 \leq k \leq K} n_k \rightarrow \infty$, we have*

$$\left| \mathbf{S}_{K,\mathbf{n}}^{(p)} - \sup_{t \in \mathbb{R}} \mathbb{B}_{K,\mathbf{n}}^{(p)}(t) \right| = \mathcal{O}(\phi_K(\mathbf{n})),$$

$$\left| \mathbf{T}_{K,\mathbf{n}}^{(p)} - \int_{\mathbb{R}} \mathbb{B}_{K,\mathbf{n}}^{(p)}(t) dF_0(t) \right| = \mathcal{O}(\phi_K(\mathbf{n})).$$

4 The change-point problem

Here and elsewhere, $[t]$ denotes the largest integer not exceeding t . In many practical applications, we assume the structural stability of statistical models and this fundamental assumption needs to be tested before it can be applied. This is called the analysis of structural breaks, or change-points, which

has led to the development of a variety of theoretical and practical results. For good sources of references to research literature in this area along with statistical applications, the reader may consult Brodsky and Darkhovsky (1993), Csörgő and Horváth (1997) and Chen and Gupta (2000). For recent references on the subject we may refer, among many others, to Bouzebda (2012), Aue and Horváth (2013), Chan *et al.* (2013), Horváth and Rice (2014), Alvarez-Andrade and Bouzebda (2014) and Bouzebda (2014).

In this section, we deal with testing changes in d.f.'s for a sequence of independent real-valued r.v.'s X_1, \dots, X_n . The corresponding null hypothesis that we want to test is

$$\mathcal{H}_0'' : X_1, \dots, X_n \text{ have d.f. } F.$$

As frequently done, the behavior of the derived tests will be investigated under the alternative hypothesis of a single change-point

$$\mathcal{H}_1'' : \exists k^* \in \{1, \dots, n-1\} \text{ such that } X_1, \dots, X_{k^*} \text{ have d.f. } F \text{ and } X_{k^*+1}, \dots, X_n \text{ have d.f. } G.$$

The d.f.'s F and G are assumed to be continuous. The critical integer k^* can be written as $\lfloor ns \rfloor$ for a certain $s \in (0, 1)$. Then, testing the null hypothesis \mathcal{H}_0'' can be based on functionals of the following process: set, for each $n \in \mathbb{N}^*$,

$$\tilde{\alpha}_n^{(p)}(s, t) := \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left(\mathbb{F}_{\lfloor ns \rfloor}^{(p)-}(t) - \mathbb{F}_{n - \lfloor ns \rfloor}^{(p)+}(t) \right) \quad \text{for } s \in (0, 1), t \in \mathbb{R}, \quad (4.1)$$

where $\mathbb{F}_k^{(p)-}$ is the p -fold integrated empirical d.f. based upon the k first observations while $\mathbb{F}_{n-k}^{(p)+}$ is that based upon the $(n-k)$ last ones. In (4.1) we extend the definition of $\mathbb{F}_k^{(p)-}$ and $\mathbb{F}_k^{(p)+}$ to the case where $k = 0$ by setting $\mathbb{F}_0^{(p)-} = \mathbb{F}_0^{(p)+} = 0$, so that $\tilde{\alpha}_n^{(p)}(s, t) = 0$ if $s \in (0, 1/n)$.

We can define the r.v.'s $X_1, \dots, X_{\lfloor ns \rfloor}$ and $X_{\lfloor ns \rfloor+1}, \dots, X_n$ on a probability space on which we can simultaneously construct two Kiefer processes $\{\mathbb{K}_1(s, u) : s \in \mathbb{R}, u \in [0, 1]\}$ and $\{\mathbb{K}_2(s, u) : s \in \mathbb{R}, u \in [0, 1]\}$ such that the ‘‘restricted’’ processes $\{\mathbb{K}_1(s, u) : s \in [1, n/2], u \in [0, 1]\}$ and $\{\mathbb{K}_2(s, u) : s \in [n/2, n], u \in [0, 1]\}$ are independent. It turns out that a natural approximation of $\{\tilde{\alpha}_n^{(p)} : n \in \mathbb{N}^*\}$ is given by the sequence of Gaussian processes $\left\{ \mathring{\mathbb{K}}_n^{(p)}(s, F(t)) : s \in [0, 1], t \in \mathbb{R}, n \in \mathbb{N}^* \right\}$ defined by

$$\mathring{\mathbb{K}}_n^{(p)}(s, u) := \frac{1}{p!} u^p \mathring{\mathbb{K}}_n(s, u) \quad \text{for } s, u \in [0, 1], n \in \mathbb{N}^*, \quad (4.2)$$

where, for each $n \in \mathbb{N}^*$, $\mathring{\mathbb{K}}_n := \{\mathring{\mathbb{K}}_n(s, u) : s, u \in [0, 1]\}$ is the Gaussian process defined by

$$\mathring{\mathbb{K}}_n(s, u) := \begin{cases} \frac{1}{\sqrt{n}} [\mathbb{K}_2(\lfloor ns \rfloor, u) - s(\mathbb{K}_1(\lfloor n/2 \rfloor, u) + \mathbb{K}_2(\lfloor n/2 \rfloor, u))] & \text{for } s \in [0, \frac{1}{2}], u \in [0, 1], \\ \frac{1}{\sqrt{n}} [-\mathbb{K}_1(\lfloor n(1-s) \rfloor, u) + (1-s)(\mathbb{K}_1(\lfloor n/2 \rfloor, u) + \mathbb{K}_2(\lfloor n/2 \rfloor, u))] & \text{for } s \in [\frac{1}{2}, 1], u \in [0, 1]. \end{cases}$$

More precisely, we have the following result.

Theorem 4.1 *On a suitable probability space, it is possible to define $\{\tilde{\alpha}_n^{(p)} : n \in \mathbb{N}^*\}$, jointly with a sequence of Gaussian processes $\{\mathring{\mathbb{K}}_n^{(p)} : n \in \mathbb{N}^*\}$ as above, such that, under \mathcal{H}_0'' , with probability 1, as $n \rightarrow \infty$,*

$$\sup_{s \in (0, 1)} \sup_{t \in \mathbb{R}} \left| \tilde{\alpha}_n^{(p)}(s, t) - \mathring{\mathbb{K}}_n^{(p)}(s, F(t)) \right| = \mathcal{O} \left(\frac{(\log n)^2}{\sqrt{n}} \right).$$

According to Csörgő *et al.* (1997), a way to test change-point is to use the following statistics:

$$\sigma_n^{(p)} := \sup_{s \in (0,1)} \sup_{t \in \mathbb{R}} \left| \tilde{\alpha}_n^{(p)}(s, t) \right|. \quad (4.3)$$

The corollary below is a consequence of Theorem 4.1 which can be proved by following exactly the same lines of Alvarez-Andrade *et al.* (2017).

Corollary 4.2 *If \mathcal{H}_0'' holds true, then we have the convergence in distribution, as $n \rightarrow \infty$,*

$$\sigma_n^{(p)} \xrightarrow{\mathcal{L}} \sup_{s, u \in [0,1]} \left| \overset{\circ}{\mathbb{K}}^{(p)}(s, u) \right|,$$

where $\overset{\circ}{\mathbb{K}}^{(p)} = \{ \overset{\circ}{\mathbb{K}}^{(p)}(s, u) : s, u \in [0, 1] \}$ is a Gaussian process with mean zero and covariance

$$\mathbb{E} \left(\overset{\circ}{\mathbb{K}}^{(p)}(s, u) \overset{\circ}{\mathbb{K}}^{(p)}(s', u') \right) = \frac{1}{p!^2} u^p u'^p (u \wedge u' - uu')(s \wedge s' - ss').$$

One has $\overset{\circ}{\mathbb{K}}^{(p)}(s, u) = \frac{1}{p!} u^p \overset{\circ}{\mathbb{K}}(s, u)$ where $\overset{\circ}{\mathbb{K}}$ is a tied-down Kiefer process. We refer to Csörgő and Horváth (1997) for more details on the process $\overset{\circ}{\mathbb{K}}^{(p)}$ in the case where $p = 1$.

Actually, according to Csörgő *et al.* (1997), the most appropriate way to test change-point is to use the following weighted statistic:

$$\sigma_{n,w}^{(p)} := \sup_{s \in (0,1)} \sup_{t \in \mathbb{R}} \frac{\left| \tilde{\alpha}_n^{(p)}(s, t) \right|}{w(\lfloor ns \rfloor / n)} \quad (4.4)$$

where w is a positive function defined on $(0, 1)$, increasing in a neighborhood of zero and decreasing in a neighborhood of one satisfying the condition

$$I(w, \varepsilon) := \int_0^1 \exp \left(-\frac{\varepsilon w^2(s)}{s(1-s)} \right) \frac{ds}{s(1-s)} < \infty$$

for some constant $\varepsilon > 0$. For a history and further applications of $I(w, \varepsilon)$, we refer to Csörgő and Horváth (1993), Chapter 4. From Szyszkowicz (1992), an example of such function w is given by

$$w(t) := \left(t(1-t) \log \log \frac{1}{t(1-t)} \right)^{1/2} \quad \text{for } t \in (0, 1).$$

By using similar techniques to those which are developed in Csörgő and Horváth (1997), one may show that

$$\sigma_{n,w}^{(p)} \xrightarrow{\mathcal{L}} \sup_{s, u \in [0,1]} \frac{\left| \overset{\circ}{\mathbb{K}}^{(p)}(s, u) \right|}{w(s)}.$$

For more details, we refer to Alvarez-Andrade and Bouzebda (2014).

Remark 4.3 *As in Szyszkowicz (1994), we mention that the statistic given by (4.3) should be more powerful for detecting changes that occur in the middle, i.e., near $n/2$, where $k/n(1 - k/n)$ reaches its maximum, than for the ones occurring near the end points. The advantage of using the weighted statistic defined in (4.4) is the detection of changes that occur near the end points, while retaining the sensitivity to possible changes in the middle as well.*

We hope that the results presented in Sections 3 and 4 will be the prototypes of other various applications.

5 Strong approximation of the integrated empirical process when parameters are estimated

In this section, we are interested in the strong approximation of the integrated empirical process when parameters are estimated. Our approach is in the same spirit of [Burke et al. \(1979\)](#). Let us introduce, for each $n \in \mathbb{N}^*$, the p -fold integrated estimated empirical process $\widehat{\alpha}_n^{(p)}$:

$$\widehat{\alpha}_n^{(p)}(t) := \sqrt{n} \left(\mathbb{F}_n^{(p)}(t) - F^{(p)}(t, \widehat{\boldsymbol{\theta}}_n) \right) \quad \text{for } t \in \mathbb{R}, \quad (5.1)$$

where $\{\widehat{\boldsymbol{\theta}}_n : n \in \mathbb{N}^*\}$ is a sequence of estimators of a parameter $\boldsymbol{\theta}$ from a family of d.f.'s $\{F(t, \boldsymbol{\theta}) : t \in \mathbb{R}, \boldsymbol{\theta} \in \Theta\}$ (Θ being a subset of \mathbb{R}^d and d a fixed positive integer) related to a sequence of i.i.d. r.v.'s $\{X_i : i \in \mathbb{N}^*\}$. Let us mention that a general study of the weak convergence of the estimated empirical process was carried out by [Durbin \(1973\)](#). For a more recent reference, we may refer to [Genz and Haeusler \(2006\)](#) where the authors investigated the empirical processes with estimated parameters under auxiliary information and provided some results regarding the bootstrap in order to evaluate the limiting laws.

Let us introduce some notations.

(5.1) The transpose of a vector V of \mathbb{R}^d will be denoted by V^\top .

(5.2) The norm $\|\cdot\|$ on \mathbb{R}^d is defined by

$$\|(y_1, \dots, y_d)\| := \max_{1 \leq i \leq d} |y_i|.$$

(5.3) For a function $(t, \boldsymbol{\theta}) \mapsto g(t, \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$, $\nabla_{\boldsymbol{\theta}} g(t, \boldsymbol{\theta}_0)$ denotes the vector in \mathbb{R}^d of partial derivatives $((\partial g / \partial \theta_1)(t, \boldsymbol{\theta}), \dots, (\partial g / \partial \theta_d)(t, \boldsymbol{\theta}))$ evaluated at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, and $\nabla_{\boldsymbol{\theta}}^2 g(t, \boldsymbol{\theta})$ denotes the $d \times d$ matrix of second order partial derivatives $((\partial^2 g / \partial \theta_i \partial \theta_j)(t, \boldsymbol{\theta}))_{1 \leq i, j \leq d}$.

(5.4) For a vector $V = (v_1, \dots, v_d) \in \mathbb{R}^d$, $\int V$ denotes the vector $(\int v_1, \dots, \int v_d)$.

Next, we write out the set of all conditions (those of [Burke et al. \(1979\)](#)) which we will use in the sequel.

(i) The estimator $\widehat{\boldsymbol{\theta}}_n$ admits the following form: for each $n \in \mathbb{N}^*$,

$$\sqrt{n} \left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l(X_i, \boldsymbol{\theta}_0) + \varepsilon_n,$$

where $\boldsymbol{\theta}_0$ is the theoretical true value of $\boldsymbol{\theta}$, $l(\cdot, \boldsymbol{\theta}_0)$ is a measurable d -dimensional vector-valued function, and ε_n converges to zero as $n \rightarrow \infty$ in a manner to be specified later on. Notice that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n l(X_i, \boldsymbol{\theta}_0) = \sqrt{n} \int_{-\infty}^t l(s, \boldsymbol{\theta}_0) d\mathbb{F}_n(s).$$

(ii) The mean value of $l(X_i, \boldsymbol{\theta}_0)$ vanishes: $\mathbb{E}(l(X_i, \boldsymbol{\theta}_0)) = 0$.

(iii) The matrix $M(\boldsymbol{\theta}_0) := \mathbb{E}(l(X_i, \boldsymbol{\theta}_0)^\top l(X_i, \boldsymbol{\theta}_0))$ is a finite nonnegative definite $d \times d$ matrix.

(iv) The vector-valued function $(t, \boldsymbol{\theta}) \mapsto \nabla_{\boldsymbol{\theta}} F(t, \boldsymbol{\theta})$ is uniformly continuous in $t \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathbf{V}$, where \mathbf{V} is the closure of a given neighborhood of $\boldsymbol{\theta}_0$.

- (v) Each component of the vector-valued function $t \mapsto l(t, \boldsymbol{\theta}_0)$ is of bounded variation in t on each finite interval of \mathbb{R} .
- (vi) The vector-valued function $t \mapsto \nabla_{\boldsymbol{\theta}} F(t, \boldsymbol{\theta}_0)$ is uniformly bounded in $t \in \mathbb{R}$, and the vector-valued function $(t, \boldsymbol{\theta}) \mapsto \nabla_{\boldsymbol{\theta}}^2 F(t, \boldsymbol{\theta})$ is uniformly bounded in $t \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathbf{V}$.

(vii) Set

$$\ell(s, \boldsymbol{\theta}_0) := l(F^{-1}(s, \boldsymbol{\theta}_0), \boldsymbol{\theta}_0) \quad \text{for } s \in (0, 1)$$

where

$$F^{-1}(s, \boldsymbol{\theta}_0) = \inf\{t \in \mathbb{R} : F(t, \boldsymbol{\theta}_0) \geq s\}.$$

The limiting relations below hold:

$$\lim_{s \searrow 0} \sqrt{s \log \log(1/s)} \|\ell(s, \boldsymbol{\theta}_0)\| = 0 \quad \text{and} \quad \lim_{s \nearrow 1} \sqrt{(1-s) \log \log[1/(1-s)]} \|\ell(s, \boldsymbol{\theta}_0)\| = 0,$$

(viii) Set

$$\ell'_s(s, \boldsymbol{\theta}_0) := \frac{\partial \ell}{\partial s}(s, \boldsymbol{\theta}_0) \quad \text{for } s \in (0, 1).$$

The partial derivative $\ell'_s(s, \boldsymbol{\theta}_0)$ exist for every $s \in (0, 1)$ and the bounds below hold: there is a positive constant C such that

$$s \|\ell'_s(s, \boldsymbol{\theta}_0)\| \leq C \quad \text{for all } s \in (0, \frac{1}{2}) \quad \text{and} \quad (1-s) \|\ell'_s(s, \boldsymbol{\theta}_0)\| \leq C \quad \text{for all } s \in (\frac{1}{2}, 1).$$

Now, we state an analogous result to Theorem 3.1 of [Burke et al. \(1979\)](#). For each $n \in \mathbb{N}^*$, let $\{G_n(t) : t \in \mathbb{R}\}$ be the process defined by

$$\begin{aligned} G_n(t) &:= \frac{1}{\sqrt{n}} \left(\mathbb{K}(n, F(t, \boldsymbol{\theta}_0)) - \left(\int_{\mathbb{R}} l(s, \boldsymbol{\theta}_0) d_s \mathbb{K}(n, F(s, \boldsymbol{\theta}_0)) \right) \nabla_{\boldsymbol{\theta}} F(t, \boldsymbol{\theta}_0)^\top \right) \\ &= \frac{1}{\sqrt{n}} \left(\mathbb{K}(n, F(t, \boldsymbol{\theta}_0)) - \mathbf{W}(n) \nabla_{\boldsymbol{\theta}} F(t, \boldsymbol{\theta}_0)^\top \right) \quad \text{for } t \in \mathbb{R}, \end{aligned}$$

where we set

$$\mathbf{W}(\tau) := \int_{\mathbb{R}} l(s, \boldsymbol{\theta}_0) d_s \mathbb{K}(\tau, F(s, \boldsymbol{\theta}_0)) \quad \text{for } \tau \geq 0.$$

The process $\{\mathbf{W}(\tau) : \tau \geq 0\}$ is a d -dimensional Brownian motion with a covariance matrix of rank that of $M(\boldsymbol{\theta}_0)$. The estimated empirical process given by $\widehat{\alpha}_n^{(p)}(t)$ defined by (5.1) will be approximated by the sequence of processes $\{G_n^{(p)} : n \in \mathbb{N}^*\}$ defined by

$$G_n^{(p)}(t) := \frac{1}{p!} F(t, \boldsymbol{\theta}_0)^p G_n(t) \quad \text{for } t \in \mathbb{R}, \quad (5.2)$$

as described in the next theorem. Set

$$\varepsilon_n^{(p)} := \sup_{t \in \mathbb{R}} \left| \widehat{\alpha}_n^{(p)}(t) - G_n^{(p)}(t) \right|. \quad (5.3)$$

Theorem 5.1 *Suppose that the sequence of estimators $\{\widehat{\boldsymbol{\theta}}_n : n \in \mathbb{N}^*\}$ satisfies Conditions (i), (ii) and (iii). Then, as $n \rightarrow \infty$,*

- (a) $\varepsilon_n^{(p)} \xrightarrow{\mathbb{P}} 0$ if Conditions (iv), (v) hold and $\varepsilon_n \xrightarrow{\mathbb{P}} 0$;

(b) $\varepsilon_n^{(p)} \xrightarrow{a.s.} 0$ if Conditions (vi)–(viii) hold and $\varepsilon_n \xrightarrow{a.s.} 0$;

(c) $\varepsilon_n^{(p)} = \mathcal{O}(\max(h(n), n^{-\epsilon}))$ for some $\epsilon > 0$ if Conditions (vi)–(viii) hold and $\varepsilon_n = \mathcal{O}(h(n))$ for some function h satisfying $h(n) > 0$ and $h(n) \rightarrow 0$.

The limiting Gaussian process $G_n^{(p)}$ of Theorem 5.1 depends crucially on F and also on the true theoretical value θ_0 . In general, Theorem 5.1 cannot be used to test the composite hypothesis :

$$F \in \{F(t, \theta) : t \in \mathbb{R}, \theta \in \Theta\}.$$

In order to circumvent this problem, Burke et al. (1979) proposed an approximate solution, they introduce another process:

$$\widehat{G}_n(t) := \frac{1}{\sqrt{n}} \left(\mathbb{K}(n, F(t, \widehat{\theta}_n)) - \mathbf{W}(n) \nabla_{\theta} F(t, \widehat{\theta}_n)^{\top} \right).$$

Under some regularity conditions, Burke et al. (1979) show that (see Theorem 3.2 therein), as $n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} \left| \widehat{G}_n(t) - G_n(t) \right| \xrightarrow{\mathbb{P}} 0.$$

Setting $\widehat{G}_n^{(p)}(t) := \frac{1}{p!} F(t, \widehat{\theta}_n)^p \widehat{G}_n(t)$, one can show that, as $n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} \left| \widehat{G}_n^{(p)}(t) - G_n^{(p)}(t) \right| \xrightarrow{\mathbb{P}} 0. \quad (5.4)$$

Consequently, we have, as $n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} \left| \widehat{\alpha}_n^{(p)}(t) - \widehat{G}_n^{(p)}(t) \right| \xrightarrow{\mathbb{P}} 0.$$

6 Local time of the integrated empirical process

In this section, we are mainly concerned with the behavior of the local time of the p -fold integrated empirical process. This behavior can be characterized by using a representation that expresses the integrated empirical process in terms of a partial sums process, see (6.1) below. Let us recall the definition of the process β_n given in (1.2) and let us introduce the modified p -fold integrated uniform empirical process $\tilde{\beta}_n^{(p)}$ defined, for each $n \in \mathbb{N}^*$, by

$$\begin{aligned} \tilde{\beta}_n^{(p)}(u) &:= \int_0^u dv_1 \int_0^{v_1} dv_{p-1} \dots \int_0^{v_{p-1}} \beta_n(v_p) dv_p \\ &= \frac{1}{(p-1)!} \int_0^u (u-v)^{p-1} \beta_n(v) dv \quad \text{for } u \in [0, 1]. \end{aligned}$$

In this part, we focus on the particular r.v. $\mathcal{A}_n^{(p)} := \tilde{\beta}_n^{(p)}(1)$. It is easily seen that the representation below holds:

$$\mathcal{A}_n^{(p)} = \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{p!} (1 - U_i)^p - \frac{1}{(p+1)!} \right) = \frac{S_n^{(p)}}{\sqrt{n}}$$

where $\{S_n^{(p)} : n \in \mathbb{N}^*\}$ is the following partial sums process where the summands are i.i.d. r.v.'s with mean zero:

$$S_n^{(p)} := \frac{1}{p!} \sum_{i=1}^n \left((1 - U_i)^p - \frac{1}{p+1} \right). \quad (6.1)$$

This is a random walk with continuously distributed jumps. In the particular case where $p = 1$, we retrieve the representation provided by [Henze and Nikitin \(2002\)](#) p. 185, namely

$$\mathcal{A}_n^{(1)} = \frac{S_n^{(1)}}{\sqrt{n}} \quad \text{with} \quad S_n^{(1)} := \sum_{i=1}^n \left(\frac{1}{2} - U_i \right).$$

Notice that we are dealing with a sum of strongly non-lattice r.v.'s as, i.e., in p. 210 of [Bass and Khoshnevisan \(1993a\)](#). Indeed, we easily check that the characteristic function $\chi^{(p)}$ of the $(1 - U_i)^p - 1/(p+1)$'s, namely

$$\chi^{(p)}(z) := \int_0^1 e^{iz(u^p - 1/(p+1))} du = \frac{e^{-iz/(p+1)}}{pz^{1/p}} \int_0^z e^{iv} \frac{dv}{v^{1-1/p}}$$

satisfies the conditions

$$\forall z \in \mathbb{R}^*, |\chi^{(p)}(z)| < 1 \quad \text{and} \quad \limsup_{|z| \rightarrow \infty} |\chi^{(p)}(z)| < 1.$$

Next, we fix a neighborhood I of 0, e.g., $I = [-1/2, 1/2]$, and we define the local time

$$\lambda^{(p)}(x, n) := \sum_{i=1}^n 1_I(S_i^{(p)} - x) \quad \text{for} \quad x \in \mathbb{R}, n \in \mathbb{N}^*. \quad (6.2)$$

The local time $\lambda^{(p)}(x, n)$ represents the number of visits of the random walk $\{S_n^{(p)} : n \in \mathbb{N}^*\}$ in the neighborhood $x + I$ of x up to discrete time n . Our aim is to obtain the rate of the approximation of the self-intersection local time

$$L_n^{(p)}(t) := \sum_{1 \leq i < j \leq [nt]} \int_{\mathbb{R}} 1_I(S_i^{(p)} - x) 1_I(S_j^{(p)} - x) dx$$

by the integrated local time of some standard Wiener process. The quantity $L_n^{(p)}(t)$ enumerates in a certain manner the couples (i, j) of distinct and ordered indices up to time $[nt]$ such that $S_i - S_j$ is less than the diameter of I .

To this aim, we recall that, if $\{\mathbb{W}(t) : t \geq 0\}$ is the standard Wiener process with $\mathbb{W}(0) = 0$, then its local time process $\{l(x, t) : t \geq 0, x \in \mathbb{R}\}$ is defined as

$$l(x, t) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{x-\varepsilon \leq \mathbb{W}(s) \leq x+\varepsilon\}} ds \quad \text{for} \quad x \in \mathbb{R}, t \geq 0. \quad (6.3)$$

Following exactly the same lines of [Alvarez-Andrade et al. \(2017\)](#), we can prove the two following results.

Theorem 6.1 *We have, with probability 1, as $n \rightarrow \infty$,*

$$\sup_{t \in [0, 1]} \left| L_n^{(p)}(t) - \frac{1}{2} n^{3/2} \int_{\mathbb{R}} l_n(x, t)^2 dx \right| = \mathcal{O}\left(n^{5/4} (\log n)^{1/2} (\log \log n)^{1/4}\right),$$

where l_n is the normalized local time

$$l_n(x, t) := \frac{1}{\sqrt{n}} l(\sqrt{n}x, [nt]).$$

Corollary 6.2 *We have, with probability 1, for any $t \in (0, 1]$, there exist two positive constants κ_1 and κ_2 such that, almost surely, for large enough n ,*

$$\kappa_1 \frac{[nt]^{3/2}}{\sqrt{\log \log n}} \leq L_n^{(p)}(t) \leq \kappa_2 [nt]^{3/2} \sqrt{\log \log n}.$$

In particular, for any $t \in (0, 1]$, almost surely, as $n \rightarrow \infty$,

$$L_n^{(p)}(t) = \frac{1}{2} [nt]^{3/2+o(1)}.$$

7 Mathematical developments

This section is devoted to the proofs of our results. The previously displayed notations continue to be used in the sequel.

7.1 Some bounds for the empirical process, Brownian bridge and the Kiefer process

Let us immediately point out an obvious fact which will be used several times thereafter:

$$0 \leq F(t) \leq 1 \quad \text{and} \quad 0 \leq \mathbb{F}_n(t) \leq 1 \quad \text{for} \quad t \in \mathbb{R}, n \in \mathbb{N}^*$$

which obviously entails that, for any $p \in \mathbb{N}$,

$$0 \leq F^{(p)}(t) \leq 1 \quad \text{and} \quad 0 \leq \mathbb{F}_n^{(p)}(t) \leq 1 \quad \text{for} \quad t \in \mathbb{R}, n \in \mathbb{N}^*. \quad (7.1)$$

Similarly, we have, for any $p \in \mathbb{N}$,

$$0 \leq \mathbb{U}_n^{(p)}(u) \leq 1 \quad \text{for} \quad u \in [0, 1], n \in \mathbb{N}^*. \quad (7.2)$$

We also mention some bounds that we will use further. By appealing to Chung's law of the iterated logarithm for the empirical process, see [Chung \(1949\)](#), which stipulates that

$$\limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} |\alpha_n(t)|}{\sqrt{\log \log n}} = \frac{1}{\sqrt{2}} \quad \text{a.s.},$$

we see that, with probability 1, as $n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} |\alpha_n(t)| = \mathcal{O}\left(\sqrt{\log \log n}\right). \quad (7.3)$$

Moreover, by [Komlós et al. \(1975\)](#), on a suitable probability space, we can define the uniform empirical process $\{\beta_n : n \in \mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{\mathbb{B}_n : n \in \mathbb{N}^*\}$ together with a Kiefer process $\{\mathbb{K}(s, u) : s \geq 0, u \in [0, 1]\}$, such that, with probability 1, as $n \rightarrow \infty$,

$$\sup_{u \in [0, 1]} |\beta_n(u) - \mathbb{B}_n(u)| = \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) \quad (7.4)$$

and

$$\max_{1 \leq k \leq n} \sup_{u \in [0, 1]} \left| \sqrt{k} \beta_k(u) - \mathbb{K}(k, u) \right| = \mathcal{O}((\log n)^2)$$

from which we extract, with probability 1, as $n \rightarrow \infty$,

$$\sup_{u \in [0, 1]} \left| \beta_n(u) - \frac{1}{\sqrt{n}} \mathbb{K}(n, u) \right| = \mathcal{O}\left(\frac{(\log n)^2}{\sqrt{n}}\right). \quad (7.5)$$

As a result, by putting (7.3) into (7.4) and (7.5), one derives the following bounds: with probability 1, as $n \rightarrow \infty$,

$$\sup_{u \in [0, 1]} |\mathbb{B}_n(u)| = \mathcal{O}\left(\sqrt{\log \log n}\right) \quad \text{and} \quad \sup_{u \in [0, 1]} |\mathbb{K}(n, u)| = \mathcal{O}\left(\sqrt{n \log \log n}\right). \quad (7.6)$$

Notice that the second bound in (7.6) comes also from the law of the iterated logarithm for the Kiefer process; see [Csörgő and Révész \(1981\)](#), p. 81.

7.2 Proof of Proposition 1.2

We begin by making an observation: because of the hypothesis that the df F is continuous, the sampled variables X_1, X_2, \dots, X_n are almost surely all different. Then, we can define with probability 1 the order statistics

$$X_{1,n} < X_{2,n} < \dots < X_{n,n}$$

associated with X_1, X_2, \dots, X_n . Notice that the event $\{X_{i,n} \leq t\}$ is equal to $\{n\mathbb{F}_n(t) \leq i\}$. Hence, we can write that, for any function f , with probability 1,

$$\int_{-\infty}^t f(s) d\mathbb{F}_n(s) = \frac{1}{n} \sum_{i=1}^n f(X_i) 1_{\{X_i \leq t\}} = \frac{1}{n} \sum_{i=1}^{n\mathbb{F}_n(t)} f(X_{i,n}). \quad (7.7)$$

Before proving (1.6), we first show by induction that

$$\mathbb{F}_n^{(p)}(t) = \frac{1}{n^{p+1}} \# \{(i_1, \dots, i_{p+1}) \in \mathbb{N}^{p+1} : 1 \leq i_1 \leq \dots \leq i_{p+1} \leq n\mathbb{F}_n(t)\}. \quad (7.8)$$

Of course, (7.8) holds for $p = 0$. Pick now a positive integer p and suppose that

$$\mathbb{F}_n^{(p-1)}(t) = \frac{1}{n^p} \# \{(i_1, \dots, i_p) \in \mathbb{N}^p : 1 \leq i_1 \leq \dots \leq i_p \leq n\mathbb{F}_n(t)\}.$$

By Definition 1.1, we see that the family of functions $\mathbb{F}_n^{(p)}$ can be recursively defined by $\mathbb{F}_n^{(0)} = \mathbb{F}_n$ and, for any $p \in \mathbb{N}^*$ and any $t \in \mathbb{R}$, by

$$\mathbb{F}_n^{(p)}(t) = \int_{-\infty}^t \mathbb{F}_n^{(p-1)}(s) d\mathbb{F}_n(s).$$

Therefore, by (7.7) and remarking that $\mathbb{F}_n(X_{i,n}) = i/n$, a.s.,

$$\begin{aligned} \mathbb{F}_n^{(p)}(t) &= \frac{1}{n} \sum_{i=1}^n \mathbb{F}_n^{(p-1)}(X_i) 1_{\{X_i \leq t\}} = \frac{1}{n} \sum_{i=1}^{n\mathbb{F}_n(t)} \mathbb{F}_n^{(p-1)}(X_{i,n}) \\ &= \frac{1}{n^{p+1}} \sum_{i=1}^{n\mathbb{F}_n(t)} \# \{(i_1, \dots, i_p) \in \mathbb{N}^p : 1 \leq i_1 \leq \dots \leq i_p \leq n\mathbb{F}_n(X_{i,n})\} \\ &= \frac{1}{n^{p+1}} \sum_{i=1}^{n\mathbb{F}_n(t)} \# \{(i_1, \dots, i_p) \in \mathbb{N}^p : 1 \leq i_1 \leq \dots \leq i_p \leq i\} \\ &= \frac{1}{n^{p+1}} \# \{(i_1, \dots, i_p, i_{p+1}) \in \mathbb{N}^{p+1} : 1 \leq i_1 \leq \dots \leq i_p \leq i_{p+1} \leq n\mathbb{F}_n(t)\}. \end{aligned}$$

Hence, (7.8) is valid for any $p \in \mathbb{N}$.

Now, we observe that the cardinality in (7.8) is nothing but the number of combinations with repetitions of $p+1$ integers lying between 1 and $n\mathbb{F}_n(t)$, which coincides with the number of combinations without repetition of $p+1$ integers lying between 1 and $n\mathbb{F}_n(t) + p$. This is the result concerning $\mathbb{F}_n^{(p)}$ announced in (1.6). Finally, the formula concerning $F^{(p)}$ can be easily obtained by induction too. The proof of Proposition 1.2 is finished. \square

In the proposition below, we provide a representation of $\mathbb{F}_n^{(p)}$ by means of \mathbb{F}_n .

Proposition 7.1 *The integrated empirical d.f. $\mathbb{F}_n^{(p)}$ can be expressed by means of \mathbb{F}_n as follows: with probability 1,*

$$\mathbb{F}_n^{(p)}(t) = \frac{\mathbb{F}_n(t)^{p+1}}{(p+1)!} + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1}} \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*, \quad (7.9)$$

where the coefficients $a_k^{(p)}$, $1 \leq k \leq p$, are positive integers.

Proof. By expanding the combination in (1.6), we get that, a.s.,

$$\mathbb{F}_n^{(p)}(t) = \frac{1}{(p+1)!n^{p+1}} \prod_{i=0}^p (n\mathbb{F}_n(t) + i) = \frac{\mathbb{F}_n(t)}{(p+1)!n^p} \prod_{i=1}^p (n\mathbb{F}_n(t) + i) \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*.$$

Appealing to the classical expansion

$$\prod_{i=1}^p (x + i) = \sum_{k=0}^p \begin{bmatrix} p+1 \\ k+1 \end{bmatrix} x^k,$$

where $[\cdot]$ are the unsigned Stirling numbers of the first kind (see, e.g., http://en.wikipedia.org/wiki/Stirling_number).

We immediately derive (7.9) by setting $a_k^{(p)} := \begin{bmatrix} p+1 \\ k+1 \end{bmatrix}$. \square

In the proposition below, we rely $\alpha_n^{(p)}$ to α_n .

Proposition 7.2 *The p -fold integrated empirical process $\alpha_n^{(p)}$ is related to the empirical process α_n according to, with probability 1,*

$$\alpha_n^{(p)}(t) = \frac{1}{p!} F(t)^p \alpha_n(t) + \sum_{k=2}^{p+1} b_k^{(p)} \frac{F(t)^{p+1-k}}{n^{(k-1)/2}} \alpha_n(t)^k + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1/2}} \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*, \quad (7.10)$$

where the coefficients $a_k^{(p)}$, $1 \leq k \leq p$, are those of Proposition 7.1 and the $b_k^{(p)}$, $2 \leq k \leq p+1$, are positive real numbers less than 1. Similarly,

$$\beta_n^{(p)}(u) = \frac{1}{p!} u^p \beta_n(u) + \sum_{k=2}^{p+1} b_k^{(p)} \frac{u^{p+1-k}}{n^{(k-1)/2}} \beta_n(u)^k + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{U}_n(u)^k}{n^{p-k+1/2}} \quad \text{for } u \in [0, 1], n \in \mathbb{N}^*. \quad (7.11)$$

Proof. By Definition 1.1 and Formulae (1.6) and (7.9), we write that, a.s., for $t \in \mathbb{R}$, $n \in \mathbb{N}^*$,

$$\alpha_n^{(p)}(t) = \frac{1}{(p+1)!} \sqrt{n} (\mathbb{F}_n(t)^{p+1} - F(t)^{p+1}) + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1/2}}.$$

Applying the elementary identity below obtained by writing $a = (a - b) + b$ and using the binomial theorem

$$a^{p+1} - b^{p+1} = (p+1)b^p(a-b) + \sum_{k=2}^{p+1} \binom{p+1}{k} b^{p+1-k}(a-b)^k$$

to $a = \mathbb{F}_n(t)$ and $b = F(t)$ yields, a.s., for $t \in \mathbb{R}$, $n \in \mathbb{N}^*$,

$$\begin{aligned} \alpha_n^{(p)}(t) &= \frac{1}{p!} \sqrt{n} F(t)^p (\mathbb{F}_n(t) - F(t)) \\ &+ \frac{1}{(p+1)!} \sqrt{n} \sum_{k=2}^{p+1} \binom{p+1}{k} F(t)^{p+1-k} (\mathbb{F}_n(t) - F(t))^k + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1/2}}. \end{aligned} \quad (7.12)$$

Substituting $\mathbb{F}_n(t) - F(t) = \alpha_n^{(p)}(t)/\sqrt{n}$ into (7.12) gives (7.10) by setting $b_k^{(p)} := \binom{p+1}{k}/(p+1)!$. \square

7.3 Proof of Theorem 2.2

Set $A_p := \sum_{k=1}^p a_k^{(p)} > 0$ and $I = [0, d/n]$ or $[1 - d/n, 1]$. Making use of (7.11) together with (7.2) and the fact that $0 \leq b_k^{(p)} \leq 1$, it is clear that, a.s., for any $d, n \in \mathbb{N}^*$ such that $d \leq n$,

$$\sup_{u \in I} \left| \beta_n^{(p)}(u) - \mathbb{B}_n^{(p)}(u) \right| \leq \sup_{u \in I} |\beta_n(u) - \mathbb{B}_n(u)| + \frac{1}{\sqrt{n}} \sum_{k=2}^{p+1} \frac{1}{n^{k/2-1}} \sup_{u \in [0,1]} |\beta_n(u)|^k + \frac{A_p}{\sqrt{n}}.$$

Therefore,

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{u \in I} \left| \beta_n^{(p)}(u) - \mathbb{B}_n^{(p)}(u) \right| \geq \frac{1}{\sqrt{n}} (c_1 \log d + x) \right\} \\ & \leq \mathbb{P} \left\{ \sup_{u \in I} |\beta_n(u) - \mathbb{B}_n(u)| + \frac{1}{\sqrt{n}} \sum_{k=2}^{p+1} \frac{1}{n^{k/2-1}} \sup_{u \in [0,1]} |\beta_n(u)|^k \geq \frac{1}{\sqrt{n}} (c_1 \log d + x - A_p) \right\}. \end{aligned}$$

Now, using the elementary inequality

$$\mathbb{P} \left\{ \sum_{k=1}^r \xi_k \geq \sum_{k=1}^r a_k \right\} \leq \sum_{k=1}^r \mathbb{P} \{ \xi_k \geq a_k \},$$

which is valid for any positive integer r , any r.v.'s ξ_1, \dots, ξ_r and any real numbers a_1, \dots, a_r , we obtain

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{u \in I} \left| \beta_n^{(p)}(u) - \mathbb{B}_n^{(p)}(u) \right| \geq \frac{1}{\sqrt{n}} (c_1 \log d + x) \right\} \\ & \leq \mathbb{P} \left\{ \sup_{u \in I} |\beta_n(u) - \mathbb{B}_n(u)| \geq \frac{1}{\sqrt{n}} (c_1 \log d + \frac{x}{p+1} - A_p) \right\} \\ & \quad + \sum_{k=2}^{p+1} \mathbb{P} \left\{ \sup_{u \in [0,1]} |\beta_n(u)|^k \geq \frac{x n^{k/2-1}}{p+1} \right\}. \end{aligned} \tag{7.13}$$

On the other hand, the inequality of [Dvoretzky et al. \(1956\)](#) stipulates that there exists a positive constant c_4 such that, for any $x > 0$ and any $n \in \mathbb{N}^*$,

$$\mathbb{P} \left\{ \sup_{t \in \mathbb{R}} |\mathbb{F}_n(t) - F(t)| \geq \frac{x}{\sqrt{n}} \right\} \leq c_4 e^{-2x^2}. \tag{7.14}$$

Actually (7.14) simply reads, by means of β_n , for any $x > 0$ and any $n \in \mathbb{N}^*$, as

$$\mathbb{P} \left\{ \sup_{u \in [0,1]} |\beta_n(u)| \geq x \right\} \leq c_4 e^{-2x^2}.$$

Then,

$$\begin{aligned} \sum_{k=2}^{p+1} \mathbb{P} \left\{ \sup_{u \in [0,1]} |\beta_n(u)|^k \geq \frac{x}{p+1} \right\} & \leq c_4 \sum_{k=2}^{p+1} \exp \left(-2 \left(\frac{x n^{k/2-1}}{p+1} \right)^{2/k} \right) \\ & \leq c_4 \left(e^{-[2/(p+1)]x} + \sum_{k=3}^{p+1} \exp \left(-\frac{2}{p+1} x^{2/k} n^{1-2/k} \right) \right). \end{aligned} \tag{7.15}$$

Now, by putting (2.1) and (7.15) into (7.13), we immediately complete the proof of Theorem 2.2 with

$$B_p := c_2 e^{c_3 A_p} + c_4 \quad \text{and} \quad C_p := \min(c_3, 2)/(p+1).$$

□

7.4 Proof of Corollary 2.4

The functional Φ being Lipschitz, there exists a positive constant L such that, for any functions v, w ,

$$|\Phi(v) - \Phi(w)| \leq L \sup_{t \in \mathbb{R}} |v(t) - w(t)|,$$

inequality that we will use in the form

$$\Phi(w) - L \sup_{t \in \mathbb{R}} |v(t) - w(t)| \leq \Phi(v) \leq \Phi(w) + L \sup_{t \in \mathbb{R}} |v(t) - w(t)|. \quad (7.16)$$

Let us choose for v, w the processes $V_n := \alpha_n^{(p)}(\cdot)$ and $W_n := \mathbb{B}_n^{(p)}(F(\cdot))$. Applying the elementary inequality $|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A)$ to the events $A = \{\Phi(V_n) \leq x\}$ and $B = \{\Phi(W_n) \leq x\}$ provides, for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}^*$,

$$|\mathbb{P}\{\Phi(V_n) \leq x\} - \mathbb{P}\{\Phi(W_n) \leq x\}| \leq \mathbb{P}\{\Phi(V_n) \leq x \leq \Phi(W_n)\} + \mathbb{P}\{\Phi(W_n) \leq x \leq \Phi(V_n)\}.$$

By (7.16), we see that

$$\begin{aligned} \mathbb{P}\{\Phi(V_n) \leq x \leq \Phi(W_n)\} &\leq \mathbb{P}\left\{\Phi(W_n) - L \sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)| \leq x \leq \Phi(W_n)\right\}, \\ \mathbb{P}\{\Phi(W_n) \leq x \leq \Phi(V_n)\} &\leq \mathbb{P}\left\{\Phi(W_n) \leq x \leq \Phi(W_n) + L \sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)|\right\}, \end{aligned}$$

from which we deduce, by addition, that

$$|\mathbb{P}\{\Phi(V_n) \leq x\} - \mathbb{P}\{\Phi(W_n) \leq x\}| \leq \mathbb{P}\left\{|\Phi(W_n) - x| \leq L \sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)|\right\}. \quad (7.17)$$

On the other hand, by choosing $x = c \log n$ for a large enough constant c in (2.4) and putting $\epsilon_n := (c+c_1) \log n / \sqrt{n}$, we obtain the estimate below valid for large enough n :

$$\mathbb{P}\left\{\sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)| \geq \epsilon_n\right\} \leq \frac{B_p}{n} = o\left(\frac{\log n}{\sqrt{n}}\right). \quad (7.18)$$

Now, by (7.17), we write

$$\begin{aligned} &|\mathbb{P}\{\Phi(V_n) \leq x\} - \mathbb{P}\{\Phi(W_n) \leq x\}| \\ &= \mathbb{P}\left\{\sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)| < \epsilon_n, |\Phi(W_n) - x| \leq L \sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)|\right\} \\ &\quad + \mathbb{P}\left\{\sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)| \geq \epsilon_n, |\Phi(W_n) - x| \leq L \sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)|\right\} \\ &\leq \mathbb{P}\{|\Phi(W_n) - x| \leq L\epsilon_n\} + \mathbb{P}\left\{\sup_{t \in \mathbb{R}} |V_n(t) - W_n(t)| \geq \epsilon_n\right\}. \end{aligned} \quad (7.19)$$

Noticing that the distribution of \mathbb{B}_n does not depend on n , which entails the equality

$$\mathbb{P}\{|\Phi(W_n) - x| \leq L\epsilon_n\} = \mathbb{P}\{|\Phi(W) - x| \leq L\epsilon_n\}$$

where $W := F(\cdot)^p \mathbb{B}(F(\cdot))/p!$, and recalling the assumption that the r.v. $\Phi(W)$ admits a density function bounded by M say, we get that, for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}^*$,

$$\mathbb{P}\{|\Phi(W_n) - x| \leq L\epsilon_n\} \leq 2LM\epsilon_n. \quad (7.20)$$

Finally, putting (7.18) and (7.20) into (7.19) leads to (2.5), which completes the proof of Corollary 2.4.

□

7.5 Proof of Corollary 2.5

Applying (2.4) to $x = c' \log n$ for a sufficiently large constant c' yields, for large enough n ,

$$\begin{aligned} \pi_n &:= \mathbb{P} \left\{ \sup_{t \in \mathbb{R}} \left| \alpha_n^{(p)}(t) - \mathbb{B}_n^{(p)}(F(t)) \right| \geq c_6 \frac{\log n}{\sqrt{n}} \right\} \\ &\leq B_p \sum_{k=2}^{p+1} \exp \left(-C_p c'^{2/k} (\log n)^{2/k} n^{1-3/k} \right) = \mathcal{O} \left(\frac{1}{n^2} \right) \end{aligned}$$

where c_6 is a positive constant. Hence the series $(\sum \pi_n)$ is convergent and by appealing to Borel-Cantelli lemma, we get that

$$\mathbb{P} \left(\limsup_{n \in \mathbb{N}^*} \left\{ \sup_{t \in \mathbb{R}} \left| \alpha_n^{(p)}(t) - \mathbb{B}_n^{(p)}(F(t)) \right| \geq c_6 \frac{\log n}{\sqrt{n}} \right\} \right) = 0,$$

which clearly implies Corollary 2.5. \square

7.6 Proof of Theorem 2.6

In view of (7.10), we write that, a.s., for any $s \in (0, 1)$, any $t \in \mathbb{R}$ and any $k \in \mathbb{N}^*$,

$$\begin{aligned} \sqrt{k} \alpha_k^{(p)}(t) - \mathbb{K}^{(p)}(k, F(t)) &= \frac{1}{p!} F(t)^p \left[\sqrt{k} \alpha_k(t) - \mathbb{K}(k, F(t)) \right] \\ &\quad + \sum_{i=2}^{p+1} b_i^{(p)} \frac{F(t)^{p+1-i}}{k^{(i-1)/2}} \alpha_k(t)^i + \sum_{i=1}^p a_i^{(p)} \frac{\mathbb{F}_k(t)^i}{k^{p-i+1/2}}. \end{aligned} \quad (7.21)$$

Then, by using (7.1) together with the fact that $0 \leq b_i^{(p)} \leq 1$, we deduce that, a.s., for any $n \in \mathbb{N}^*$,

$$\begin{aligned} &\max_{1 \leq k \leq n} \sup_{t \in \mathbb{R}} \left| \sqrt{k} \alpha_k^{(p)}(t) - \mathbb{K}^{(p)}(k, F(t)) \right| \\ &\leq \frac{1}{p!} \max_{1 \leq k \leq n} \sup_{t \in \mathbb{R}} \left| \sqrt{k} \alpha_k(t) - \mathbb{K}(k, F(t)) \right| + \sum_{i=2}^{p+1} \max_{1 \leq k \leq n} \left(\frac{1}{k^{(i-1)/2}} \sup_{t \in \mathbb{R}} |\alpha_k(t)|^i \right) + \sum_{i=1}^p \max_{1 \leq k \leq n} \frac{a_i^{(p)}}{k^{p-i+1/2}} \\ &\leq \max_{1 \leq k \leq n} \sup_{t \in \mathbb{R}} \left| \sqrt{k} \alpha_k(t) - \mathbb{K}(k, F(t)) \right| + \sum_{i=2}^{p+1} \left(\max_{1 \leq k \leq n} \sup_{t \in \mathbb{R}} |\alpha_k(t)| \right)^i + A_p. \end{aligned} \quad (7.22)$$

Finally, by putting (7.5) and (7.3) into (7.22), we completes the proof of Theorem 2.6. \square

7.7 Proof of Corollary 2.7

From Theorem 2.6, we deduce that, with probability 1, as $n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} \left| \alpha_n^{(p)}(t) \right| = \frac{1}{\sqrt{n}} \sup_{t \in \mathbb{R}} \left| \mathbb{K}^{(p)}(n, F(t)) \right| + \mathcal{O} \left(\frac{(\log n)^2}{\sqrt{n}} \right).$$

Therefore, a.s.,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} \left| \alpha_n^{(p)}(t) \right|}{\sqrt{\log \log n}} &= \limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} \left| \mathbb{K}^{(p)}(n, F(t)) \right|}{\sqrt{n \log \log n}} \\ &= \limsup_{n \rightarrow \infty} \frac{\sup_{u \in [0,1]} |u^p \mathbb{K}(n, u)|}{p! \sqrt{n \log \log n}} \\ &= \frac{\sqrt{2}}{p!} \sup_{u \in [0,1]} \sqrt{\text{Var}(u^p \mathbb{K}(1, u))}. \end{aligned} \quad (7.23)$$

In the last equality, we have used Strassen's law of iterated logarithm for Gaussian processes. Observing that $\text{Var}(u^p \mathbb{K}(1, u)) = u^{2p+1}(1-u)$ and that

$$\sup_{u \in [0,1]} u^{2p+1}(1-u) = \frac{(2p+1)^{2p+1}}{(2p+2)^{2p+2}},$$

(7.23) readily implies (2.7) which proves Corollary 2.7. \square

7.8 Proof of Corollary 2.8

We work under Hypothesis \mathcal{H}_0 . Let us introduce the p -fold integrated empirical process related to the d.f. F_0 :

$$\alpha_{0,n}^{(p)}(t) := \sqrt{n} \left(\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t) \right) \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*.$$

By the triangular inequality, we plainly have

$$\left| \mathbf{S}_n^{(p)} - \sup_{t \in \mathbb{R}} |\mathbb{B}_n^{(p)}(F_0(t))| \right| \leq \sup_{t \in \mathbb{R}} \left| \alpha_{0,n}^{(p)}(t) - \mathbb{B}_n^{(p)}(F_0(t)) \right|$$

from which together with (2.6) we deduce (2.8).

Similarly,

$$\begin{aligned} \left| \mathbf{T}_n^{(p)} - \int_{\mathbb{R}} [\mathbb{B}_n^{(p)}(F_0(t))]^2 dF_0(t) \right| &\leq \int_{\mathbb{R}} \left| \alpha_{0,n}^{(p)}(t)^2 - [\mathbb{B}_n^{(p)}(F_0(t))]^2 \right| dF_0(t) \\ &\leq \sup_{t \in \mathbb{R}} \left| \alpha_{0,n}^{(p)}(t) - \mathbb{B}_n^{(p)}(F_0(t)) \right| \\ &\quad \times \left(\sup_{t \in \mathbb{R}} \left| \alpha_{0,n}^{(p)}(t) \right| + \sup_{t \in \mathbb{R}} |\mathbb{B}_n^{(p)}(F_0(t))| \right). \end{aligned} \quad (7.24)$$

In the last inequality above appears the supremum

$$\sup_{t \in \mathbb{R}} |\mathbb{B}_n^{(p)}(F_0(t))| \leq \sup_{u \in [0,1]} |\mathbb{B}_n(u)|.$$

Then, by putting (2.6), (2.7) and (7.6) into (7.24), we immediately deduce (2.9). The proof of Corollary 2.8 is finished. \square

7.9 Proof of Corollary 3.1

For each $m, n \in \mathbb{N}^*$, let $\alpha_m^{(p),1}$ and $\alpha_n^{(p),2}$ denote the empirical processes respectively associated with the samples X_1, \dots, X_m and Y_1, \dots, Y_n . By replacing $\mathbb{F}_m^{(p)}(t)$ by $\alpha_m^{(p),1}(t)/\sqrt{m} + F^{(p)}(t)$ and $\mathbb{G}_n^{(p)}(t)$ by $\alpha_n^{(p),2}(t)/\sqrt{n} + G^{(p)}(t)$, using the binomial theorem and recalling that, under \mathcal{H}'_0 , $F = G$, we write

$$\begin{aligned} \boldsymbol{\xi}_{m,n}^{(p,q)}(t) &= \sqrt{\frac{mn}{m+n}} \left[\left(\frac{\alpha_m^{(p),1}(t)}{\sqrt{m}} + F^{(p)}(t) \right)^q - \left(\frac{\alpha_n^{(p),2}(t)}{\sqrt{n}} + F^{(p)}(t) \right)^q \right] \\ &= \sqrt{\frac{mn}{m+n}} \sum_{k=1}^q \binom{q}{k} (F^{(p)}(t))^{q-k} \left[\left(\frac{\alpha_m^{(p),1}(t)}{\sqrt{m}} \right)^k - \left(\frac{\alpha_n^{(p),2}(t)}{\sqrt{n}} \right)^k \right] \\ &= q (F^{(p)}(t))^{q-1} \left(\sqrt{\frac{n}{m+n}} \alpha_m^{(p),1}(t) - \sqrt{\frac{m}{m+n}} \alpha_n^{(p),2}(t) \right) + \Delta_{m,n}(t) \end{aligned}$$

where

$$\Delta_{m,n}(t) = \sqrt{\frac{mn}{m+n}} \sum_{k=2}^q \binom{q}{k} (F^{(p)}(t))^{q-k} \left[\left(\frac{\alpha_m^{(p),1}(t)}{\sqrt{m}} \right)^k - \left(\frac{\alpha_n^{(p),2}(t)}{\sqrt{n}} \right)^k \right].$$

By (2.7) and (7.1), it is easily seen that, with probability 1, as $m, n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} |\Delta_{m,n}(t)| = \mathcal{O}\left(\frac{(\log \log m)^{q/2}}{\sqrt{m}}\right) + \mathcal{O}\left(\frac{(\log \log n)^{q/2}}{\sqrt{n}}\right). \quad (7.25)$$

On the other hand, by Corollary 2.5, we can construct two sequences of Brownian bridges $\{\mathbb{B}_m^1 : m \in \mathbb{N}^*\}$ and $\{\mathbb{B}_n^2 : n \in \mathbb{N}^*\}$ such that, with probability 1, as $m, n \rightarrow \infty$,

$$\begin{aligned} \sup_{t \in \mathbb{R}} \left| \alpha_m^{(p),1}(t) - \frac{1}{p!} F(t)^p \mathbb{B}_m^1(F(t)) \right| &= \mathcal{O}\left(\frac{\log m}{\sqrt{m}}\right), \\ \sup_{t \in \mathbb{R}} \left| \alpha_n^{(p),2}(t) - \frac{1}{p!} F(t)^p \mathbb{B}_n^2(F(t)) \right| &= \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right). \end{aligned} \quad (7.26)$$

Setting $\mathbb{B}_{m,n}^{(p,q)}$ as in Corollary 3.1, we have

$$\begin{aligned} \xi_{m,n}^{(p,q)}(t) - \mathbb{B}_{m,n}^{(p,q)}(t) &= \frac{q}{(p+1)!^{q-1}} F(t)^{(p+1)(q-1)} \left[\sqrt{\frac{n}{m+n}} \left(\alpha_m^{(p),1}(t) - \frac{1}{p!} F(t)^p \mathbb{B}_m^1(F(t)) \right) \right. \\ &\quad \left. - \sqrt{\frac{m}{m+n}} \left(\alpha_n^{(p),2}(t) - \frac{1}{p!} F(t)^p \mathbb{B}_n^2(F(t)) \right) \right] + \Delta_{m,n}(t). \end{aligned} \quad (7.27)$$

By putting (7.25) and (7.26) into (7.27), we deduce the result announced in Corollary 3.1. \square

7.10 Proof of Theorem 3.4

Let us introduce, for each $k \in \{1, \dots, K\}$, the p -fold integrated empirical process associated with the d.f. F^k

$$\alpha_n^{(p),k}(t) := \sqrt{n} \left(\mathbb{F}_n^{(p),k}(t) - F_0^{(p),k}(t) \right) \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*.$$

By recalling (3.1) and making use of the most well-known variance formula

$$\sum_{k=1}^K n_k (x_k - \bar{x})^2 = \sum_{k=1}^K n_k (x_k - x_0)^2 - |\mathbf{n}| (\bar{x} - x_0)^2$$

where we have denoted $|\mathbf{n}| = \sum_{k=1}^K n_k$ and $\bar{x} = \frac{1}{|\mathbf{n}|} \sum_{k=1}^K n_k x_k$ for $\mathbf{n} = (n_1, \dots, n_K)$, we rewrite $\xi_{K,\mathbf{n}}^{(p)}(t)$ under Hypothesis \mathcal{H}_0^K as

$$\begin{aligned} \xi_{K,\mathbf{n}}^{(p)}(t) &= \sum_{k=1}^K n_k \left(\mathbb{F}_{n_k}^{(p),k}(t) - F_0^{(p)}(t) \right)^2 - \frac{1}{|\mathbf{n}|} \left(\sum_{k=1}^K n_k \left(\mathbb{F}_{n_k}^{(p),k}(t) - F_0^{(p)}(t) \right) \right)^2 \\ &= \sum_{k=1}^K \alpha_{n_k}^{(p),k}(t)^2 - \left(\sum_{k=1}^K \sqrt{\frac{n_k}{|\mathbf{n}|}} \alpha_{n_k}^{(p),k}(t) \right)^2. \end{aligned}$$

Next, setting $\mathbb{B}_{K,\mathbf{n}}^{(p)}$ as in Theorem 3.4, we have

$$\xi_{K,\mathbf{n}}^{(p)}(t) - \mathbb{B}_{K,\mathbf{n}}^{(p)}(t) = \Delta_{\mathbf{n}}^1(t) - \Delta_{\mathbf{n}}^2(t) \quad (7.28)$$

where we put, for any $t \in \mathbb{R}$ and any $\mathbf{n} = (n_1, \dots, n_K) \in \mathbb{N}^*$,

$$\begin{aligned}\Delta_{\mathbf{n}}^1(t) &= \sum_{k=1}^K \left(\alpha_{n_k}^{(p),k}(t)^2 - \frac{F_0(t)^{2p}}{p!^2} \mathbb{B}_{n_k}^k(F_0(t))^2 \right), \\ \Delta_{\mathbf{n}}^2(t) &= \left(\sum_{k=1}^K \sqrt{\frac{n_k}{|\mathbf{n}|}} \alpha_{n_k}^{(p),k}(t) \right)^2 - \left(\frac{F_0(t)^p}{p!} \sum_{k=1}^K \sqrt{\frac{n_k}{|\mathbf{n}|}} \mathbb{B}_{n_k}^k(F_0(t)) \right)^2.\end{aligned}$$

By setting, for any $k \in \{1, \dots, K\}$, any $t \in \mathbb{R}$ and any $\mathbf{n} = (n_1, \dots, n_K) \in \mathbb{N}^*$,

$$\delta_{k,\mathbf{n}}(t) = \alpha_{n_k}^{(p),k}(t) - \frac{F_0(t)^p}{p!} \mathbb{B}_{n_k}^k(F_0(t)) \quad \text{and} \quad \epsilon_{k,\mathbf{n}}(t) = \alpha_{n_k}^{(p),k}(t) + \frac{F_0(t)^p}{p!} \mathbb{B}_{n_k}^k(F_0(t))$$

and writing $\Delta_{\mathbf{n}}^1(t)$ and $\Delta_{\mathbf{n}}^2(t)$ as

$$\Delta_{\mathbf{n}}^1(t) = \sum_{k=1}^K \delta_{k,\mathbf{n}}(t) \epsilon_{k,\mathbf{n}}(t) \quad \text{and} \quad \Delta_{\mathbf{n}}^2(t) = \sum_{k=1}^K \sqrt{\frac{n_k}{|\mathbf{n}|}} \delta_{k,\mathbf{n}}(t) \sum_{k=1}^K \sqrt{\frac{n_k}{|\mathbf{n}|}} \epsilon_{k,\mathbf{n}}(t),$$

we derive the following inequalities:

$$\begin{aligned}\sup_{t \in \mathbb{R}} |\Delta_{\mathbf{n}}^1(t)| &\leq \sum_{k=1}^K \left(\sup_{t \in \mathbb{R}} |\delta_{k,\mathbf{n}}(t)| \right) \left(\sup_{t \in \mathbb{R}} |\epsilon_{k,\mathbf{n}}(t)| \right), \\ \sup_{t \in \mathbb{R}} |\Delta_{\mathbf{n}}^2(t)| &\leq \left(\sum_{k=1}^K \sup_{t \in \mathbb{R}} |\delta_{k,\mathbf{n}}(t)| \right) \left(\sum_{k=1}^K \sup_{t \in \mathbb{R}} |\epsilon_{k,\mathbf{n}}(t)| \right).\end{aligned}\tag{7.29}$$

By (2.5), (2.7) and (7.6), we get the bounds, a.s., for each $k \in \{1, \dots, K\}$, as $n_k \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} |\delta_{k,\mathbf{n}}(t)| = \mathcal{O}\left(\frac{\log n_k}{\sqrt{n_k}}\right) \quad \text{and} \quad \sup_{t \in \mathbb{R}} |\epsilon_{k,\mathbf{n}}(t)| = \mathcal{O}\left(\sqrt{\log \log n_k}\right).\tag{7.30}$$

Finally, by putting (7.30) into (7.29), and next into (7.28), we finish the proof of Theorem 3.4. \square

7.11 Proof of Theorem 4.1

In the computations below, the superscript “ $-$ ” in the quantities \mathbb{F} , $\mathbb{F}^{(p)}$, α and β refers to the k first observations while the superscript “ $+$ ” refers to the $(n - k)$ last ones. By (1.5) and (4.1), we write that, for $n \in \mathbb{N}^*$, $s \in (0, 1)$ and $t \in \mathbb{R}$,

$$\begin{aligned}\tilde{\alpha}_n^{(p)}(s, t) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left[\left(\mathbb{F}_{\lfloor ns \rfloor}^{(p)-}(t) - F^{(p)}(t) \right) - \left(\mathbb{F}_{n - \lfloor ns \rfloor}^{(p)+}(t) - F^{(p)}(t) \right) \right] \\ &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left(\frac{\alpha_{\lfloor ns \rfloor}^{(p)-}(t)}{\sqrt{\lfloor ns \rfloor}} - \frac{\alpha_{n - \lfloor ns \rfloor}^{(p)+}(t)}{\sqrt{n - \lfloor ns \rfloor}} \right).\end{aligned}$$

Using (7.10), we derive, a.s., for any $n \in \mathbb{N}^*$, any $s \in (0, 1)$ and any $t \in \mathbb{R}$, the form

$$\tilde{\alpha}_n^{(p)}(s, t) = \text{I}_n(s, t) - \text{II}_n(s, t) + \text{III}_n(s, t) + \text{IV}_n(s, t)\tag{7.31}$$

where

$$\begin{aligned}
\text{I}_n(s, t) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{p! n^{3/2}} \frac{\alpha_{\lfloor ns \rfloor}^-(t)}{\sqrt{\lfloor ns \rfloor}} F(t)^p, \\
\text{II}_n(s, t) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{p! n^{3/2}} \frac{\alpha_{n - \lfloor ns \rfloor}^+(t)}{\sqrt{n - \lfloor ns \rfloor}} F(t)^p, \\
\text{III}_n(s, t) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \sum_{k=2}^{p+1} b_k^{(p)} F(t)^{p+1-k} \left(\frac{\alpha_{\lfloor ns \rfloor}^-(t)^k}{\lfloor ns \rfloor^{k/2}} - \frac{\alpha_{n - \lfloor ns \rfloor}^+(t)^k}{(n - \lfloor ns \rfloor)^{k/2}} \right), \\
\text{IV}_n(s, t) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \sum_{k=1}^p a_k^{(p)} \left(\frac{\mathbb{F}_{\lfloor ns \rfloor}^-(t)^k}{\lfloor ns \rfloor^{p-k+1}} - \frac{\mathbb{F}_{n - \lfloor ns \rfloor}^+(t)^k}{(n - \lfloor ns \rfloor)^{p-k+1}} \right).
\end{aligned}$$

Concerning III_n , we have the estimate below:

$$\begin{aligned}
|\text{III}_n(s, t)| &\leq \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \sum_{k=2}^{p+1} \left[\left(\frac{|\alpha_{\lfloor ns \rfloor}^-(t)|}{\sqrt{\lfloor ns \rfloor}} \right)^k + \left(\frac{|\alpha_{n - \lfloor ns \rfloor}^+(t)|}{\sqrt{n - \lfloor ns \rfloor}} \right)^k \right] \\
&\leq \frac{1}{\sqrt{n}} \alpha_{\lfloor ns \rfloor}^-(t)^2 \sum_{k=0}^{p-1} \left(\frac{|\alpha_{\lfloor ns \rfloor}^-(t)|}{\sqrt{\lfloor ns \rfloor}} \right)^k + \frac{1}{\sqrt{n}} \alpha_{n - \lfloor ns \rfloor}^+(t)^2 \sum_{k=0}^{p-1} \left(\frac{|\alpha_{n - \lfloor ns \rfloor}^+(t)|}{\sqrt{n - \lfloor ns \rfloor}} \right)^k
\end{aligned} \tag{7.32}$$

We learn from (7.1) that $|\alpha_n(t)/\sqrt{n}| = |\mathbb{F}_n(t) - F(t)| \leq 1$ for any $t \in \mathbb{R}$ and any $n \in \mathbb{N}^*$ and, of course, similar inequalities hold for $\alpha_{\lfloor ns \rfloor}^-$ and $\alpha_{n - \lfloor ns \rfloor}^+$. We deduce that both sums displayed in (7.32) are not greater than p and by (7.3), with probability 1, as $n \rightarrow \infty$, uniformly in s and t ,

$$|\text{III}_n(s, t)| \leq \frac{p}{\sqrt{n}} \left(\alpha_{\lfloor ns \rfloor}^-(t)^2 + \alpha_{n - \lfloor ns \rfloor}^+(t)^2 \right) = \mathcal{O}\left(\frac{\log \log n}{\sqrt{n}}\right). \tag{7.33}$$

Concerning IV_n , we have the estimate below:

$$|\text{IV}_n(s, t)| \leq A'_p \left(\frac{n - \lfloor ns \rfloor}{n^{3/2}} \sum_{k=1}^p \frac{\mathbb{F}_{\lfloor ns \rfloor}^-(t)^k}{\lfloor ns \rfloor^{p-k}} + \frac{\lfloor ns \rfloor}{n^{3/2}} \sum_{k=1}^p \frac{\mathbb{F}_{n - \lfloor ns \rfloor}^+(t)^k}{(n - \lfloor ns \rfloor)^{p-k}} \right) \tag{7.34}$$

where we set $A'_p := \max_{1 \leq i \leq p} a_i^{(p)} > 0$. Because of (7.1) and the convention that $\mathbb{F}_{\lfloor ns \rfloor}^- = 0$ if $s \in (0, 1/n)$, we see that both sums displayed in (7.34) are not greater than p and, as $n \rightarrow \infty$, uniformly in s and t ,

$$\text{IV}_n(s, t) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \tag{7.35}$$

As a byproduct, we get from (7.31), (7.33) and (7.35) that, with probability 1, as $n \rightarrow \infty$, uniformly in s and t ,

$$\tilde{\alpha}_n^{(p)}(s, t) = \text{I}_n(s, t) - \text{II}_n(s, t) + \mathcal{O}\left(\frac{\log \log n}{\sqrt{n}}\right). \tag{7.36}$$

Next, it is convenient to introduce, for $n \in \mathbb{N}^*$ and $s, u \in (0, 1)$,

$$\begin{aligned}\gamma_n(u) &= \sqrt{n} \beta_n(u) = \sum_{i=1}^n (1_{\{U_i \leq u\}} - u), \\ \gamma_{\lfloor ns \rfloor}^-(u) &= \sqrt{\lfloor ns \rfloor} \beta_{\lfloor ns \rfloor}^-(u) = \sum_{i=1}^{\lfloor ns \rfloor} (1_{\{U_i \leq u\}} - u), \\ \gamma_{n-\lfloor ns \rfloor}^+(u) &= \sqrt{n - \lfloor ns \rfloor} \beta_{n-\lfloor ns \rfloor}^+(u) = \sum_{i=\lfloor ns \rfloor+1}^n (1_{\{U_i \leq u\}} - u), \\ \mathbf{I}'_n(s, u) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{p! n^{3/2}} \frac{\beta_{\lfloor ns \rfloor}^-(u)}{\sqrt{\lfloor ns \rfloor}} u^p = \frac{n - \lfloor ns \rfloor}{p! n^{3/2}} u^p \gamma_{\lfloor ns \rfloor}^-(u), \\ \mathbf{II}'_n(s, u) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{p! n^{3/2}} \frac{\beta_{n-\lfloor ns \rfloor}^+(u)}{\sqrt{n - \lfloor ns \rfloor}} u^p = \frac{\lfloor ns \rfloor}{p! n^{3/2}} u^p \gamma_{n-\lfloor ns \rfloor}^+(u),\end{aligned}$$

and

$$\delta_n(s, u) = \mathbf{I}'_n(s, u) - \mathbf{II}'_n(s, u).$$

Then, by (1.3), we plainly have the following equalities:

$$\mathbf{I}'_n(s, F(t)) = \mathbf{I}_n(s, t), \quad \mathbf{II}'_n(s, F(t)) = \mathbf{II}_n(s, t), \quad \gamma_n(u) = \gamma_{\lfloor ns \rfloor}^-(u) + \gamma_{n-\lfloor ns \rfloor}^+(u)$$

and we rewrite (7.36), with probability 1, as $n \rightarrow \infty$, uniformly in s and t , as

$$\tilde{\alpha}_n^{(p)}(s, t) = \delta_n(s, F(t)) + \mathcal{O}\left(\frac{\log \log n}{\sqrt{n}}\right). \quad (7.37)$$

Now, let us rewrite $\delta_n(s, u)$ as

$$\delta_n(s, u) = \frac{u^p}{p! \sqrt{n}} \left(\gamma_{\lfloor ns \rfloor}^-(u) - \frac{\lfloor ns \rfloor}{n} \gamma_n(u) \right) = \frac{u^p}{p! \sqrt{n}} \left(\frac{n - \lfloor ns \rfloor}{n} \gamma_n(u) - \gamma_{n-\lfloor ns \rfloor}^+(u) \right). \quad (7.38)$$

We know from Komlós *et al.* (1975) and Csörgő and Horváth (1997) that, with probability 1, as $n \rightarrow \infty$,

$$\begin{aligned}\sup_{s \in [0, 1/2]} \sup_{u \in [0, 1]} \left| \gamma_{\lfloor ns \rfloor}^-(u) - \mathbb{K}_2(\lfloor ns \rfloor, u) \right| &= \mathcal{O}((\log n)^2), \\ \sup_{s \in [1/2, 1]} \sup_{u \in [0, 1]} \left| \gamma_{n-\lfloor ns \rfloor}^+(u) - \mathbb{K}_1(\lfloor ns \rfloor, u) \right| &= \mathcal{O}((\log n)^2).\end{aligned} \quad (7.39)$$

In particular, we have, with probability 1, as $n \rightarrow \infty$,

$$\sup_{u \in [0, 1]} \left| \gamma_{\lfloor n/2 \rfloor}^-(u) - \mathbb{K}_2(\lfloor n/2 \rfloor, u) \right| = \mathcal{O}((\log n)^2), \quad (7.40)$$

$$\sup_{u \in [0, 1]} \left| \gamma_{n-\lfloor n/2 \rfloor}^+(u) - \mathbb{K}_1(\lfloor n/2 \rfloor, u) \right| = \mathcal{O}((\log n)^2). \quad (7.41)$$

As a byproduct, by adding (7.40) and (7.41), we readily infer that, with probability 1, as $n \rightarrow \infty$,

$$\sup_{u \in [0, 1]} \left| \gamma_n(u) - (\mathbb{K}_1(\lfloor n/2 \rfloor, u) + \mathbb{K}_2(\lfloor n/2 \rfloor, u)) \right| = \mathcal{O}((\log n)^2). \quad (7.42)$$

Recall the definition of the process $\overset{\circ}{\mathbb{K}}_n^{(p)}$ given by (4.2). From (7.38), (7.39), and (7.42), we deduce that, with probability 1, as $n \rightarrow \infty$,

$$\sup_{s,u \in [0,1]} |\delta_n(s,u) - \overset{\circ}{\mathbb{K}}_n^{(p)}(s,u)| = \mathcal{O}\left(\frac{(\log n)^2}{\sqrt{n}}\right). \quad (7.43)$$

Finally, we conclude by using the triangle inequality

$$\begin{aligned} \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n^{(p)}(s,t) - \overset{\circ}{\mathbb{K}}_n^{(p)}(s,F(t))| &\leq \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n^{(p)}(s,t) - \delta_n(s,F(t))| \\ &\quad + \sup_{s,u \in [0,1]} |\delta_n(s,u) - \overset{\circ}{\mathbb{K}}_n^{(p)}(s,u)| \end{aligned} \quad (7.44)$$

and next by putting (7.37) and (7.43) into (7.44). The proof of Theorem 4.1 is completed. \square

7.12 Proof of Theorem 5.1

Recall the definition of $\hat{\alpha}_n^{(p)}(t)$ given by (5.1) and write it as follows: by using (7.10) *mutatis mutandis*, a.s., for any $t \in \mathbb{R}$ and any $n \in \mathbb{N}^*$,

$$\hat{\alpha}_n^{(p)}(t) = \frac{1}{p!} F(t, \hat{\boldsymbol{\theta}}_n)^p \hat{\alpha}_n(t) + \sum_{k=2}^{p+1} b_k^{(p)} F(t, \hat{\boldsymbol{\theta}}_n)^{p+1-k} \frac{\hat{\alpha}_n(t)^k}{n^{(k-1)/2}} + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1/2}}. \quad (7.45)$$

Substituting $\hat{\alpha}_n(t) = (\hat{\alpha}_n(t) - G_n(t)) + G_n(t)$ into (7.45) and using the binomial theorem yield, a.s., for any $t \in \mathbb{R}$ and any $n \in \mathbb{N}^*$,

$$\begin{aligned} \hat{\alpha}_n^{(p)}(t) - G_n^{(p)}(t) &= \frac{1}{p!} F(t, \hat{\boldsymbol{\theta}}_n)^p (\hat{\alpha}_n(t) - G_n(t)) + \frac{1}{p!} \left(F(t, \hat{\boldsymbol{\theta}}_n)^p - F(t, \boldsymbol{\theta}_0)^p \right) G_n(t) + \sum_{k=1}^p a_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1/2}} \\ &\quad + \sum_{k=2}^{p+1} b_k^{(p)} \frac{F(t, \hat{\boldsymbol{\theta}}_n)^{p+1-k}}{n^{(k-1)/2}} \sum_{i=0}^k \binom{k}{i} G_n(t)^{k-i} (\hat{\alpha}_n(t) - G_n(t))^i. \end{aligned} \quad (7.46)$$

By (7.1) and by appealing to the elementary identity $a^p - b^p = (a - b) \sum_{i=0}^{p-1} a^i b^{p-i-1}$, we extract from (7.46) the following inequality: a.s., for any $n \in \mathbb{N}^*$,

$$\begin{aligned} \sup_{t \in \mathbb{R}} |\hat{\alpha}_n^{(p)}(t) - G_n^{(p)}(t)| &= \sup_{t \in \mathbb{R}} |\hat{\alpha}_n(t) - G_n(t)| + \sup_{t \in \mathbb{R}} \left| F(t, \hat{\boldsymbol{\theta}}_n) - F(t, \boldsymbol{\theta}_0) \right| \sup_{t \in \mathbb{R}} |G_n(t)| + \frac{A_p}{\sqrt{n}} \\ &\quad + \frac{A_p''}{\sqrt{n}} \left(\sum_{k=0}^{p+1} \sup_{t \in \mathbb{R}} |G_n(t)|^k \right) \left(\sum_{k=0}^{p+1} \sup_{t \in \mathbb{R}} |\hat{\alpha}_n(t) - G_n(t)|^k \right) \end{aligned}$$

where we set $A_p'' := \max_{0 \leq i, k \leq p} \binom{k}{i} > 0$. Recall the notation (5.3) of $\varepsilon_n^{(p)}$ and set

$$\boldsymbol{\eta}_n := \sup_{t \in \mathbb{R}} |\hat{\alpha}_n(t) - G_n(t)|.$$

We have thus obtained the inequality

$$\varepsilon_n^{(p)} \leq \boldsymbol{\eta}_n + \sup_{t \in \mathbb{R}} \left| F(t, \hat{\boldsymbol{\theta}}_n) - F(t, \boldsymbol{\theta}_0) \right| \sup_{t \in \mathbb{R}} |G_n(t)| + \frac{A_p''}{\sqrt{n}} \left(\sum_{k=0}^p \boldsymbol{\eta}_n^k \right) \left(\sum_{k=0}^p \sup_{t \in \mathbb{R}} |G_n(t)|^k \right) + \frac{A_p}{\sqrt{n}}. \quad (7.47)$$

We know from Theorem 3.1 of [Burke et al. \(1979\)](#) that $\boldsymbol{\eta}_n$ satisfies the same limiting results than those displayed in Theorem 5.1 for $\varepsilon_n^{(p)}$.

Next, we need to derive some bounds for $\sup_{t \in \mathbb{R}} |G_n(t)|$ and $\sup_{t \in \mathbb{R}} |F(t, \widehat{\boldsymbol{\theta}}_n) - F(t, \boldsymbol{\theta}_0)|$ as $n \rightarrow \infty$. First, by using (7.6) and noticing that the same bound holds true for $\mathbf{W}(n)$, and by Condition (iv) and the definition of $G_n(t)$, we see that, with probability 1, as $n \rightarrow \infty$,

$$\sup_{t \in \mathbb{R}} |G_n(t)| = \mathcal{O}\left(\sqrt{\log \log n}\right). \quad (7.48)$$

On the other hand, using the one-term Taylor expansion of $F(\cdot, \boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}_0$, there exists $\boldsymbol{\theta}_n^*$ lying in the segment $[\boldsymbol{\theta}_0, \widehat{\boldsymbol{\theta}}_n]$ such that

$$F(t, \widehat{\boldsymbol{\theta}}_n) - F(t, \boldsymbol{\theta}_0) = \left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \nabla_{\boldsymbol{\theta}} F(t, \boldsymbol{\theta}_n^*)^\top. \quad (7.49)$$

In case (a) of Theorem 5.1, $\sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)$ is asymptotically normal and then $n^{1/4}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)$ tends to zero as $n \rightarrow \infty$ in probability. Therefore, by (7.48) and (7.49), $\sup_{t \in \mathbb{R}} |F(t, \widehat{\boldsymbol{\theta}}_n) - F(t, \boldsymbol{\theta}_0)| \sup_{t \in \mathbb{R}} |G_n(t)|$ also tends to zero as $n \rightarrow \infty$, in probability. Putting this into (7.47), we easily complete the proof of Theorem 5.1 in this case. In cases (b) and (c) of Theorem 5.1, referring to Burke *et al.* (1979) p. 779, we have the following bound for $\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0$: with probability 1, as $n \rightarrow \infty$,

$$\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 = \mathcal{O}\left(\sqrt{\frac{\log \log n}{n}}\right).$$

By putting this into (7.49) and next in (7.47) with the aid of (7.48), we complete the proof of Theorem 5.1 in these two cases.

Finally, concerning $\widehat{G}_n^{(p)}(t)$, we have

$$\widehat{G}_n^{(p)}(t) - G_n^{(p)}(t) = \frac{1}{p!} F(t, \widehat{\boldsymbol{\theta}}_n)^p \left(\widehat{G}_n(t) - G_n(t)\right) + \frac{1}{p!} \left(F(t, \widehat{\boldsymbol{\theta}}_n)^p - F(t, \boldsymbol{\theta}_0)^p\right) G_n(t),$$

from which we deduce

$$\sup_{t \in \mathbb{R}} \left| \widehat{G}_n^{(p)}(t) - G_n^{(p)}(t) \right| \leq \sup_{t \in \mathbb{R}} \left| \widehat{G}_n(t) - G_n(t) \right| + \sup_{t \in \mathbb{R}} \left| F(t, \widehat{\boldsymbol{\theta}}_n) - F(t, \boldsymbol{\theta}_0) \right| \sup_{t \in \mathbb{R}} |G_n(t)|.$$

Using the same bounds than previously, we immediately derive (5.4). \square

A Appendix : other integrated empirical distribution functions

To end up this article, let us point out that a similar analysis may be carried out with other integrated empirical d.f.s and integrated processes. For instance, we present below two other families of integrated empirical d.f.'s. The underlying d.f. F is still assumed to be continuous.

Definition A.1 *We define the families of integrated d.f.'s and integrated empirical d.f.'s, for any $p \in \mathbb{N}$, any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$, as*

$$\widetilde{F}^{(p)}(t) := \int_{-\infty}^t F(s)^p dF(s), \quad \widetilde{\mathbb{F}}_n^{(p)}(t) := \int_{-\infty}^t \mathbb{F}_n(s)^p d\mathbb{F}_n(s)$$

and

$$\check{F}^{(p)}(t) := \int_{-\infty}^t (F(t) - F(s))^p dF(s), \quad \check{\mathbb{F}}_n^{(p)}(t) := \int_{-\infty}^t (\mathbb{F}_n(t) - \mathbb{F}_n(s))^p d\mathbb{F}_n(s)$$

together with the corresponding family of integrated empirical processes as

$$\widetilde{\alpha}_n^{(p)}(t) := \sqrt{n} \left(\widetilde{\mathbb{F}}_n^{(p)}(t) - \widetilde{F}^{(p)}(t) \right), \quad \check{\alpha}_n^{(p)}(t) := \sqrt{n} \left(\check{\mathbb{F}}_n^{(p)}(t) - \check{F}^{(p)}(t) \right).$$

We have, from (7.7), a.s., for any $p \in \mathbb{N}$, any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$,

$$\tilde{\mathbb{F}}_n^{(p)}(t) = \frac{1}{n} \sum_{i=1}^{n\mathbb{F}_n(t)} \mathbb{F}_n(X_{i,n})^p \quad \text{and} \quad \check{\mathbb{F}}_n^{(p)}(t) = \frac{1}{n} \sum_{i=1}^{n\mathbb{F}_n(t)} (\mathbb{F}_n(t) - \mathbb{F}_n(X_{i,n}))^p.$$

Since $\mathbb{F}_n(X_{i,n}) = i/n$, we obtain the following closed forms.

Proposition A.2 *For each $p \in \mathbb{N}$, we explicitly have, with probability 1,*

$$\tilde{F}^{(p)}(t) = \check{F}^{(p)}(t) = \frac{F(t)^{p+1}}{p+1} \quad \text{for } t \in \mathbb{R},$$

and

$$\tilde{\mathbb{F}}_n^{(p)}(t) = \frac{1}{n^{p+1}} \sum_{i=1}^{n\mathbb{F}_n(t)} i^p, \quad \check{\mathbb{F}}_n^{(p)}(t) = \frac{1}{n^{p+1}} \sum_{i=0}^{n\mathbb{F}_n(t)-1} i^p \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*. \quad (\text{A.1})$$

Observe the relation, a.s. valid, for all $p \in \mathbb{N}^*$, any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$,

$$\check{\mathbb{F}}_n^{(p)}(t) = \tilde{\mathbb{F}}_n^{(p)}(t) - \frac{1}{n} F_n(t)^p.$$

Proposition A.3 *The empirical d.f. $\tilde{\mathbb{F}}_n^{(p)}$ can be expressed by means of \mathbb{F}_n as follows: with probability 1,*

$$\tilde{\mathbb{F}}_n^{(p)}(t) = \frac{\mathbb{F}_n(t)^{p+1}}{p+1} + \sum_{k=1}^p \tilde{a}_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1}} \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*, \quad (\text{A.2})$$

where the coefficients $\tilde{a}_k^{(p)}$, $1 \leq k \leq p$, are rational numbers.

Proof. Appealing to Bernoulli's formula

$$\sum_{i=1}^n i^p = \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \frac{1}{p+1} \sum_{k=1}^{p-1} \binom{p+1}{k} B_{p+1-k} n^k$$

where the B_k 's are the Bernoulli numbers (see, e.g., http://en.wikipedia.org/wiki/Bernoulli_number), (A.1) immediately yields (A.2) with the coefficients $\tilde{a}_k^{(p)} := \binom{p+1}{k} B_{p+1-k} / (p+1)$ for $k \in \{1, \dots, p-1\}$ and $\tilde{a}_p^{(p)} := 1/2$. \square

Below, we state the expression of $\tilde{\alpha}_n^{(p)}$ by means of α_n analogous to (7.10).

Proposition A.4 *The integrated empirical process $\tilde{\alpha}_n^{(p)}$ is related to the empirical process α_n according to, with probability 1,*

$$\tilde{\alpha}_n^{(p)}(t) = F(t)^p \alpha_n(t) + \sum_{k=2}^{p+1} \tilde{b}_k^{(p)} \frac{F(t)^{p+1-k}}{n^{(k-1)/2}} \alpha_n(t)^k + \sum_{k=1}^p \tilde{a}_k^{(p)} \frac{\mathbb{F}_n(t)^k}{n^{p-k+1/2}} \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*,$$

where the coefficients $\tilde{a}_k^{(p)}$, $1 \leq k \leq p$, are those of Proposition A.3 and the $\tilde{b}_k^{(p)}$, $2 \leq k \leq p+1$, are positive real numbers less than $p!$.

The coefficients $b_k^{(p)}$ are given by $b_k^{(p)} := \binom{p+1}{k} / (p+1)$.

More generally, we could define a broader family indexed by polynomials of two variables.

Definition A.5 We define the family of integrated d.f.'s and integrated empirical d.f.'s, for any polynomials \mathbf{P} of two variables, any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$, as

$$F^{(\mathbf{P})}(t) := \int_{-\infty}^t \mathbf{P}(F(s), F(t)) dF(s), \quad \mathbb{F}_n^{(\mathbf{P})}(t) := \int_{-\infty}^t \mathbf{P}(\mathbb{F}_n(s), \mathbb{F}_n(t)) \mathbb{F}_n(s),$$

together with the corresponding family of integrated empirical processes as

$$\alpha_n^{\mathbf{P}}(t) := \sqrt{n} (\mathbb{F}_n^{\mathbf{P}}(t) - F^{\mathbf{P}}(t)).$$

Below, we state the last result of the paper which is a representation of $\alpha_n^{\mathbf{P}}$ by means of α_n analogous to (7.10). This is the key point for deriving bounds similar to those obtained throughout the paper.

Proposition A.6 The empirical process $\alpha_n^{\mathbf{P}}$ can be express as follows: with probability 1,

$$\alpha_n^{\mathbf{P}}(t) = \mathbf{Q}(t) \alpha_n(t) + \mathbf{R}_n(t) \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*, \quad (\text{A.3})$$

where \mathbf{Q} is the polynomial function of F defined by

$$\mathbf{Q}(t) = \mathbf{P}(F(t), F(t)) + \int_0^{F(t)} \frac{\partial \mathbf{P}}{\partial y}(x, F(t)) dx \quad \text{for } t \in \mathbb{R},$$

and \mathbf{R}_n satisfies the inequality

$$|\mathbf{R}_n(t)| \leq \frac{\mathbf{A}}{\sqrt{n}} + \mathbf{B} \sum_{k=2}^{\mathbf{d}} \frac{\alpha_n(t)^k}{n^{(k-1)/2}} \quad \text{for } t \in \mathbb{R}, n \in \mathbb{N}^*,$$

\mathbf{A} , \mathbf{B} , \mathbf{d} being three constants depending on \mathbf{P} .

Proof. Set $\mathbf{P}(x, y) = \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \mathbf{a}_{ij} x^i y^j$ for some integers \mathbf{p}, \mathbf{q} and some coefficients \mathbf{a}_{ij} . Then

$$F^{(\mathbf{P})}(t) = \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \frac{\mathbf{a}_{ij}}{i+1} F(t)^{i+j+1}$$

and, a.s., for any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$,

$$\mathbb{F}_n^{(\mathbf{P})}(t) = \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \mathbf{a}_{ij} \mathbb{F}_n(t)^j \widetilde{\mathbb{F}}_n^{(i)}(t) = \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \frac{\mathbf{a}_{ij}}{i+1} \mathbb{F}_n(t)^{i+j+1} + \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \sum_{k=1}^i \mathbf{a}_{ij} \widetilde{a}_k^{(i)} \frac{\mathbb{F}_n(t)^{j+k}}{n^{i-k+1}}.$$

Consequently, a.s., for any $n \in \mathbb{N}^*$ and any $t \in \mathbb{R}$,

$$\alpha_n^{\mathbf{P}}(t) = \sqrt{n} \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \frac{\mathbf{a}_{ij}}{i+1} (\mathbb{F}_n(t)^{i+j+1} - F(t)^{i+j+1}) + \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \sum_{k=1}^i \mathbf{a}_{ij} \widetilde{a}_k^{(i)} \frac{\mathbb{F}_n(t)^{j+k}}{n^{i-k+1/2}}$$

which, by using the same method than (7.10), we rewrite as (A.3) with

$$\begin{aligned} \mathbf{Q}(t) &= \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \frac{i+j+1}{i+1} \mathbf{a}_{ij} F(t)^{i+j} = \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \mathbf{a}_{ij} F(t)^{i+j} + \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \frac{j}{i+1} \mathbf{a}_{ij} F(t)^{i+j}, \\ \mathbf{R}_n(t) &= \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \sum_{k=1}^i \mathbf{a}_{ij} \widetilde{a}_k^{(i)} \frac{\mathbb{F}_n(t)^{j+k}}{n^{i-k+1/2}} + \sum_{i=0}^{\mathbf{p}} \sum_{j=0}^{\mathbf{q}} \sum_{k=2}^{i+j+1} \binom{i+j+1}{k} \frac{\mathbf{a}_{ij}}{i+1} \frac{\alpha_n(t)^k}{n^{(k-1)/2}} F(t)^{i+j+1-k}. \end{aligned}$$

We easily conclude by using (7.1). □

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