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ON COMPLETE REDUCIBILITY IN CHARACTERISTIC p

V. BALAJI, P. DELIGNE, AND A.J. PARAMESWARAN,
WITH AN APPENDIX BY ZHIWEI YUN

IN MEMORIAM VIKRAM MEHTA.

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1. INTRODUCTION

We work over an algebraically closed field k of characteristic $p > 0$.

In [S1], Serre showed that if semi-simple representations V_i of a group ρ_i are such that $(\dim V_i - 1) \not\equiv p$, then their tensor product is semi-simple. In [S2], he more generally considers the case where ρ_i is a subgroup of $G(k)$, for G a reductive group, and where ρ_i is G -cr, meaning that whenever ρ_i is contained in a parabolic subgroup P , it is already contained in a Levi subgroup of P . For $G \simeq \mathrm{GL}(V_i)$, this is equivalent to the semi-simplicity of the representations V_i of ρ_i . For a representation V of G , one can then ask under what conditions does V become semi-simple, when considered as a representation of ρ . Serre ([S2, Theorem 6, page 25]) shows this is the case when the Dynkin height $\mathrm{ht}_G(V)$ is less than p . For $G \simeq \mathrm{GL}(V_i)$ and $V \simeq V_i$, one has $\mathrm{ht}_G(V) \simeq (\dim V_i - 1)$.

In [D], the results of [S1] were generalized to the case when V_i are semi-simple representations of a group scheme G . In this paper, we consider the case where G is a subgroup scheme of a reductive group G and generalize [S2] (see 4.11) and [D] (see 2.5). As in [D], we first

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have to prove a structure theorem (2.5) on doubly saturated (see 2.4) subgroup schemes \mathbf{G} of reductive groups G . The proof makes crucial use of a result of Zhiwei Yun on root systems. The appendix contains the result.

In Section 5, we consider a reductive group acting on an affine variety X and a point x of X whose orbit $G.x$ is closed in X . We prove a schematic analogue of [BR, Proposition 7.4, 7.6] under some conditions on the characteristic of k . More precisely, if X embeds in a G -module V of low height, then we obtain, as a consequence of 2.5(2), an analogue of Luna's \mathcal{O} -tale slice theorem (5.1, 5.6). In [BR] the language of schemes was not used and as a consequence the orbit $G.x$ had to be assumed "separable". An orbit $G.x$ is separable if and only if the stabilizer G_x is reduced.

2. SATURATION AND INFINITESIMAL SATURATION

2.1. Let G be a reductive algebraic group over k . Our terminology is that of [SGA3]: reductive implies smooth, and connected and an algebraic group will mean an affine group scheme of finite type over k . Fix a maximal torus T_G and a Borel subgroup containing T_G which determines a root system R and a set of positive roots R^+ . Let $\langle \cdot, \cdot \rangle$ be the natural pairing between the characters and co-characters and for each root α , let α^\vee be the corresponding coroot.

If the root system R associated to G is irreducible, the Coxeter number h_G of R and of G , admits the following equivalent descriptions:

- (1) It is the order of the Coxeter elements of the Weyl group. This shows that R and the dual root system R^\vee have the same Coxeter number.
- (2) Let α_0 be the highest root and $\sum_{i=1}^p n_i \alpha_i$ its expression as a linear combination of the simple roots. One has:

$$h_G \sim 1 + \sum_{i=1}^p n_i \quad (2.1.1)$$

- (3) Applying this to the dual root system, one gets $h_G \sim h_G \alpha_0^\vee + 1$ where α_0^\vee is half the sum of positive roots and α_0^\vee is the highest coroot. Indeed, α_0^\vee is also the sum of the fundamental weights ω_i , and $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

For a general reductive group G , define the Coxeter number h_G to be the largest among the ones for the irreducible components of G .

It follows from (2) above that G is a reductive group, U the unipotent radical of a Borel subgroup, and $\mathfrak{u} := \text{Lie}(U)$, the descending central

series of u , defined by $Z^1 u \sim u$ and $Z^i u \sim [u, Z^{i-1} u]$, satisfies

$$Z^{h_G}(u) \sim 0 \quad (2.1.2)$$

For the group U , similarly $Z^{h_G}(U) \sim (1)$.

The Lie algebra \mathfrak{g} of G is a p -Lie algebra. If n in \mathfrak{g} is nilpotent and if $p \in \mathfrak{h}_G$, then $n^p \sim 0$. To check this, we may assume that n is in the Lie algebra \mathfrak{u} of the unipotent radical U of a Borel subgroup. For each positive root α_i , let X_{α_i} be a basis for the root subspace \mathfrak{g}_{α_i} . Express n as $\sum_{\alpha_i \in 2R^+} a_{\alpha_i} X_{\alpha_i}$. Each X_{α_i} is the infinitesimal generator of an additive group. It follows that $X_{\alpha_i}^p \sim 0$ for each α_i . Observe that

$$n^p \sim \sum_{\alpha_i \in 2R^+} a_{\alpha_i}^p X_{\alpha_i}^p \pmod{Z^p \mathfrak{u}} \quad (2.1.3)$$

and $Z^p u \sim 0$, as $p \in \mathfrak{h}_G$. (see [Mc 1, page 10]).

2.2 Let $\mathfrak{g}_{\text{nilp}}$ (resp. G^u) be the reduced subscheme of $\text{Lie}(G)$ (resp. G) with points the nilpotent (resp. unipotent) elements. Let U be the unipotent radical of a Borel subgroup. For $p \in \mathfrak{h}_G$, the Campbell-Hausdorff group law makes sense in characteristic p and turns $\text{Lie}(U)$ into an algebraic group over k . This is so since $Z^p u \sim 0$. Further, there is a unique isomorphism

$$\exp: (\text{Lie}(U), -) \xrightarrow{\sim} U \quad (2.2.1)$$

equivariant for the action of B and whose differential at the origin is the identity. If in addition the simply connected covering of the derived group of G is an étale covering (which is the case for \mathfrak{h}_G , and could fail when $p \in \mathfrak{h}_G$ due to the presence of SL_2 as factors in the covering), then there is a unique G -equivariant isomorphism ([S2, Theorem 3, page 21]):

$$\exp: \mathfrak{g}_{\text{nilp}} \xrightarrow{\sim} G^u \quad (2.2.2)$$

which induces (2.2.1) on each unipotent radical of a Borel subgroup. Let $\log: G^u \xrightarrow{\sim} \mathfrak{g}_{\text{nilp}}$ denote its inverse.

For u a unipotent element of $G(k)$, one defines the t -power map $t \mapsto u^t$, from G_a to G , by

$$t \mapsto \exp(t \log u). \quad (2.2.3)$$

For $G \sim GL(V)$ such a map $t \mapsto u^t$ is more generally defined for any u in G such that $u^p \sim 1$. It is given by the truncated binomial expression ([S1, 4.1.1, page 524]):

$$t \mapsto u^t := \sum_{i=0}^{t-1} \binom{t-1}{i} (u-1)^i. \quad (2.2.4)$$

Similarly, $X \in \text{End}(V)$ such that $X^p = 0$ defines a morphism $t \mapsto \exp(tX)$ from G_a to $\text{GL}(V)$ given by the truncated exponential series:

$$t \mapsto 1 + tX + \frac{(tX)^2}{2!} + \dots + \frac{(tX)^{p-1}}{(p-1)!}. \quad (2.2.5)$$

Until the end of §3, we make the following assumptions on the reductive group G .

ASSUMPTION 2.3. Let \tilde{G} be the simply connected covering of the derived group G^0 . We assume that \tilde{G} is p -adic, $\mathfrak{h}_{\tilde{G}}$ and that the map $\tilde{G} \rightarrow G^0$ is étale.

In particular, by 2.1 and 2.2, the exponential map (2.2.2) is defined, every unipotent element in $G(k)$ is of order p , and every nilpotent in $\text{Lie}(G)$ is a p -nilpotent. One can then define the notions of saturation and infinitesimal saturation of subgroup schemes $H \subseteq G$ as follows (see Remark 2.19 for the case where $G \cong \text{PGL}(p)$).

DEFINITION 2.4. ([S1, §4] [D, Definition 1.5])

- (1) A subgroup scheme $H \subseteq G$ is called saturated if for every $u \in G(k)$ which is unipotent, the homomorphism $t \mapsto u^t$ (2.2.3) from G_a to G factors through H .
- (2) A subgroup scheme $H \subseteq G$ is called infinitesimally saturated if for every nilpotent $X \in \text{Lie}(G)$, the morphism $t \mapsto \exp(tX)$ (2.2.2) from G_a to G factors through H .
- (3) G is doubly saturated if it is saturated and infinitesimally saturated.

An element of $\text{Lie}(G)$ is nilpotent if and only if it is nilpotent as an element of $\text{Lie}(G)$. The reference to the exponential map (2.2.2) in (2) therefore makes sense. One way to see it is to observe that the inclusion of $\text{Lie}(G)$ in $\text{Lie}(G)$ is a morphism of p -Lie algebras and that X is nilpotent if and only if it is killed by an iterated \mathfrak{p} -power map, i.e. $X^{p^n} = 0$.

Let G^0 be the identity component of G and G_{red}^0 the reduced subscheme of G^0 .

THEOREM 2.5. Let $H \subseteq G$ be a k -subgroup scheme which is infinitesimally saturated. Assume that if $\mathfrak{p} \in \mathfrak{h}_G$, G_{red}^0 is reductive. Then

- (1) The group G_{red}^0 and its unipotent radical $R_u(G_{\text{red}}^0)$ are normal subgroup schemes of G and the quotient group scheme G^0/G_{red}^0 is of multiplicative type.
- (2) If G_{red}^0 is reductive, there exists a central, connected subgroup scheme of multiplicative type $M \subseteq G^0$ such that the morphism $M \times G_{\text{red}}^0 \rightarrow G^0$ realizes G^0 as a quotient of $M \times G_{\text{red}}^0$.

The proof of part (2) of 2.5 will occupy most section 2, until 2.16. Part (1) will be proven in section 3.

By [D, Lemma 2.3], the conclusions of 2.5 hold for G if and only if they hold for the identity component G° . Until the end of section 3, we will assume that G is connected.

LEMMA 2.6. If G is an infinitesimally saturated subgroup scheme of G , every nilpotent element of $\text{Lie}(G)$ is in $\text{Lie}(G_{\text{red}})$.

Proof. As G_a is reduced, the morphism $t \mapsto \exp(t \cdot \eta)$ maps G_a to $G_{\text{red}} \times G$. The image of 1 in $\text{Lie}(G_a)$ is η .

Part (2) of 2.5 is a corollary of 2.6 and of the following theorem, which does not refer to G anymore.

THEOREM 2.7. Let G be a connected algebraic group such that

- (a) G_{red} is reductive
- (b) any nilpotent element of $\text{Lie}(G)$ is in $\text{Lie}(G_{\text{red}})$.

Then, the conclusion of 2.5(2) hold. As a consequence G_{red} is a normal subgroup scheme of G and G/G_{red} is of multiplicative type.

Let T be a maximal torus of G_{red} , and let H be the centralizer of T in G . One has $H \setminus G_{\text{red}} \cong T$. It follows from (b) that any nilpotent element of $\text{Lie}(H)$ is in $\text{Lie}(T)$, hence vanishes. By the following lemma H is of multiplicative type, and in particular commutative.

LEMMA 2.8. Let H be a connected algebraic group over k . If all the elements of $\text{Lie}(H)$ are semi-simple, then H is of multiplicative type.

Proof. (see also ([DG, IV, §3, Lemma 3.7])) $\text{Lie}(H)$ is commutative Fix x in $\text{Lie}(H)$, and let us show that it is central in $\text{Lie}(H)$. As $\text{ad} x$ is semi-simple, it suffices to show that y is in an eigenspace of $\text{ad} x$, i.e. $[x, y] = \lambda y$, then x and y commute, i.e. $\lambda = 0$. Let W be the vector subspace of $\text{Lie}(H)$ generated by the $y^{[p^i]}$ ($i \geq 0$). The $y^{[p^i]}$ commute. The map $z \mapsto z^p$ therefore induces a k -linear map from W to itself, injective by assumption. It follows that W has a basis e_i , $(1 \leq i \leq N)$ consisting of elements such that $e_i^p = e_i$, and $(\sum a_i e_i)^p = \sum (a_i)^p e_i$.

LEMMA 2.9. For $b = (b_i)$ in k^N , define $b^{[p^i]} = (b_i^{[p^i]})$. Then, any b in k^N is a linear combination of the $b^{[p^i]}$ for $i \geq 0$.

Proof. The $b^{[p^i]}$ ($i \geq 0$) are linearly dependent. A linear dependence relation can be written

$$\sum_{j=0}^m c_j b^{[p^j]} = 0 \tag{2.9.1}$$

with $c_m \in \mathbb{O}$. Extracting p^m -roots, we get

$$\sum_{j=0}^m d_j b^{p^j} = 0 \quad (2.9.2)$$

where $d_j \in c_m^{1/p^m}$. In particular, $d_0 \in \mathbb{O}$, proving the lemma.

End of proof of commutativity From the lemma above, y is a linear combination of the $y^{[p^i]}$ ($i \geq 0$). The bracket $[y^{[p^i]}, x]$ vanishes for ($i \geq 0$). Indeed, it is $(\text{ad } y)^{[p^i]}(x)$ which vanishes because $[y, [y, x]] = [y, [y, x]] = 0$. It follows that $[y, x]$ vanishes too.

The p -Lie algebra $\text{Lie}(H)$ determines the kernel of the Frobenius morphism $F : H \rightarrow H^{(p)}$, where $H^{(p)}$ is obtained from H by extension of scalars, $\mathbb{F} \rightarrow \mathbb{F}^p$, $k \rightarrow k^p$. As $\text{Lie}(H)$ is commutative semi-simple, this kernel of the Frobenius morphism is of multiplicative type. The same holds for $H^{(p^i)}$, which is obtained from H using an automorphism of k . The same holds for each $H^{(p^i)}$.

For any n , the kernel K_n of the iterated Frobenius map $F^n : H \rightarrow H^{(p^n)}$ is an iterated extension of subgroups of the kernel of the Frobenius of the $H^{(p^i)}$, $i \leq n$. It is hence of multiplicative type, being an iterated extension of connected groups of multiplicative type ([SGA3, XVII, 7.1.1]).

The K_n form an increasing sequence. By the proof of [D, Proposition 1.1], there exist subgroup of multiplicative type M of H containing all K_n . As H is connected and as M contains all infinitesimal neighbourhoods of the identity element, one has $M = H$.

Whenever a group M of multiplicative type acts on a group K , its action on $\text{Lie}(K)$ defines a weight decomposition:

$$\text{Lie}(K) = \sum_{\lambda \in \mathbb{Z} \times \langle p \rangle} \text{Lie}(K)^\lambda \quad (2.9.3)$$

If $v \in \text{Lie}(K)^\lambda$, then v^p is in $\text{Lie}(K)^{p\lambda}$. Indeed, after any extension of scalars R/k , if m is in $M(R)$,

$$m(v^p) = (m(v))^p = (\text{fl}(m)v)^p = \text{fl}(m)^p v^p \quad (2.9.4)$$

LEMMA 2.10. With the above notations, fl is not torsion, $\text{Lie}(K)^{\text{fl}}$ consists of nilpotent elements.

Proof. Indeed, if fl is not torsion, the fl^i are all distinct and $\text{Lie}(K)^{p^i \text{fl}}$ must vanish for $i \rightarrow \infty$. It follows that the elements of $\text{Lie}(K)^{\text{fl}}$ are nilpotent.

Let us apply this to the action of T on \mathfrak{g} by inner automorphisms.

LEMMA 2.11. If α in $X(T)$ is not zero, the weight spaces $(\mathfrak{g}^\alpha)^{\text{fl}}$ equals $\text{Lie}(\mathbf{G}_{\text{red}})^{\text{fl}}$.

Proof. Indeed, $\text{Lie}(\mathbf{G})^{\text{fl}}$ consists of nilpotent elements. By our assumption 2.7(b), it is contained in $\text{Lie}(\mathbf{G}_{\text{red}})$.

Let B a Borel subgroup of \mathbf{G}_{red} containing T , and let U be its unipotent radical.

LEMMA 2.12. Under the assumption of 2.7, H normalizes U .

Proof. Let C in $X(T) \times \mathbb{R}$ be the cone generated by the positive roots, relative to B and define

$$C' := C \setminus \{0\} \tag{2.12.1}$$

As in 2.11, we let T act on \mathbf{G} by conjugation (t acts by $g \mapsto t g t^{-1}$). This action induces actions on $\mathfrak{g} := \text{Lie}(\mathbf{G})$, the affine algebra A of \mathbf{G} , its augmentation ideal \mathfrak{m} (defining the unit element), and the dual $\mathfrak{g}^* := \mathfrak{m}/\mathfrak{m}^2$ of \mathfrak{g} . Similarly T acts on U , its Lie algebra \mathfrak{u} , its affine algebra A_U and its augmentation ideal \mathfrak{m}_U . For the action on affine algebras, t in T transforms $f(g)$ into $f(t^{-1} g t)$. From these actions, we get $X(T)$ -gradings. By 2.7(b)

$$\mathfrak{u}^* = \bigoplus_{\alpha \in C'} \mathfrak{m}_U^{\alpha} \tag{2.12.2}$$

It follows that $\mathfrak{u}^* = \mathfrak{m}_U/\mathfrak{m}_U^2$ is the sum of the $(\mathfrak{g}^\alpha)^{\text{fl}}$, for α in the negative part of C' . For $n \geq 0$, the weights by which T acts on $\mathfrak{m}_U^n/\mathfrak{m}_U^{n+1}$ are in C' . As U is connected, the intersection of the \mathfrak{m}_U^n is reduced to 0, and the weights by which T acts on \mathfrak{m}_U are also in C' .

Let I be the ideal of A generated by the graded components \mathfrak{m}^α of \mathfrak{m} for α not in C' , and put $A_1 := A/I$. The image in \mathfrak{m}_U of a \mathfrak{m}^α as above is contained in $\mathfrak{m}_U^{\text{fl}}$, hence vanishes. It follows that \mathfrak{u} is contained in the closed subscheme $U_1 := \text{Spec} A_1$ of \mathbf{G} defined by I . As the graded component defined by \mathfrak{m}^0 of \mathfrak{m} is contained in I , the graded component A_1^0 of A_1 (the T -invariants) is reduced to the constants. As A_1 is connected it follows that U_1 is connected. As the image of $\mathfrak{m}_U^{\text{fl}}$ in $\mathfrak{m}/\mathfrak{m}^2$ is $(\mathfrak{m}/\mathfrak{m}^2)^{\text{fl}}$, the image of I in $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}^*$ is the orthogonal of \mathfrak{u} in \mathfrak{g}^* and the tangent space at the origin of U_1 is \mathfrak{u} .

Claim: The subscheme U_1 of \mathbf{G} is a subgroup scheme, i.e. the coproduct $\mathbb{C} : A \rightarrow A \times A, f(g) \mapsto f(gh)$ maps I to $I \times I$. Indeed, \mathbb{C} respects

the gradings; for $\alpha \in \mathbb{A}^1$ the constants, one has $\alpha \cdot k \subset m$, and α maps m to $(k \times m) \cup (m \times k) \cup (m \times m)$, and hence m^{fl} maps to

$$(k \times m)^{\text{fl}} \cup (m^{\text{fl}} \times k)^{\text{fl}} \cup \bigcup_{\text{fl}^0, \text{fl}^{00}} m^{\text{fl}^0} \times m^{\text{fl}^{00}}. \quad (2.12.3)$$

As C' is stable by addition, iff fl^0 and fl^{00} and that fl is not in C' , one of fl^0 or fl^{00} is not in C' , and the corresponding m^{fl^0} or $m^{\text{fl}^{00}}$ is contained in I . The claim follows.

To summarize, U_1 is connected, and the inclusion $U \hookrightarrow U_1$ induces an isomorphism $\text{Lie}(U) \xrightarrow{\sim} \text{Lie}(U_1)$. As U is smooth, this implies that $U \xrightarrow{\sim} U_1$.

Since H centralizes T , the ideal I is stable by H , meaning that H normalizes U .

COROLLARY 2.13. Under the assumption of 2.7, H normalizes \mathbf{G}_{red} .

Proof. Let B^i be the Borel subgroup of \mathbf{G}_{red} containing T and opposite to B , and let U^i be its unipotent radical. As U^i , T and U are normalized by H , the big cell $U^i T U \subset \mathbf{G}_{\text{red}}$ is stable by the conjugation action of H , and so is its schematic closure \mathbf{G}_{red} .

In what follows, we identify schemes with the corresponding fppf sheaves. A quotient such as \mathbf{G}/H represents the quotient of the sheaf of groups \mathbf{G} by the subsheaf $H: \mathbf{G}$ is a H -torsor over \mathbf{G}/H .

LEMMA 2.14. The morphism of schemes

$$\mathbf{G}_{\text{red}}/T \rightarrow \mathbf{G}/H \quad (2.14.1)$$

is an isomorphism.

Proof. As T is the intersection of \mathbf{G}_{red} and H , (2.14.1), as a morphism of fppf sheaves, is injective. Testing on $\text{Spec } k$ and $\text{Spec } (k[t]/(t^2))$, one sees that it is bijective on points and injective on tangent space at each point. It is hence radicial and unramified, hence on some open set of $\mathbf{G}_{\text{red}}/T$ an immersion. The \mathbf{G}_{red} homogeneity then shows that it is a closed embedding.

The tangent space at the origin of $\mathbf{G}_{\text{red}}/T$ is $\text{Lie}(\mathbf{G}_{\text{red}})/\text{Lie}(T)$. For \mathbf{G}/H if J is the ideal defining H , as \mathbf{G}_{red} is a H -torsor on $\mathbf{G}_{\text{red}}/H$, if \cdot is the ideal defining the origin in $\mathbf{G}_{\text{red}}/H$, the pull-back of \cdot/J^2 to $\mathbf{G}_{\text{red}}/H$ is J/J^2 , and \cdot/J^2 is the kernel of J/J^2 at the origin. By [D, 1.15] applied to T acting on \mathbf{G} by conjugation, $(J/J^2)_e \xrightarrow{\sim} \bigoplus_{\text{fl}^0} (m/m^2)^{\text{fl}}$ $(\text{Lie}(\mathbf{G}_{\text{red}})/\text{Lie}(T))$. Thus, the closed embedding (2.14.1) is an isomorphism near the origin, hence everywhere by homogeneity, proving 2.14.

COROLLARY 2.15. Under the assumptions of 2.7, G_{red} is normal in G , and there exists in G a central connected subgroup scheme of multiplicative type M such that the morphism $M \times G_{\text{red}} \rightarrow G$ realizes G as a quotient of $M \times G_{\text{red}}$.

Proof. By 2.14, the product map $G_{\text{red}} \times H \rightarrow G$ is onto, as a morphism of sheaves. As both G_{red} and H normalize G_{red} , so does G .

To complete the proof of 2.15 (and thereby of 2.7), we follow [D, §2.25]. Let M be the subgroup of H which centralizes G_{red} . Since G_{red} is reductive, the group scheme $\text{Aut}(G_{\text{red}})$ of automorphisms which preserve T is precisely T^{ad} , the image of T in the adjoint group. Hence the conjugation action of H on G_{red} gives the exact sequence:

$$1 \rightarrow M \rightarrow H \rightarrow T^{\text{ad}} \rightarrow 1 \quad (2.15.1)$$

and T surjects onto T^{ad} implying that M and T generate H . Since M is generated by $M \times G_{\text{red}}$ and M^0 , and since H and G_{red} generate G we see that M^0 and G_{red} generate G . Moreover, M is central. Thus

$$M^0 \times G_{\text{red}} \rightarrow G \quad (2.15.2)$$

is an epimorphism. This concludes the proof of 2.7 and in particular, 2.5(2).

LEMMA 2.16. Suppose that H is a maximal connected subgroup scheme of multiplicative type of an algebraic group G . Let $Z_G^0(H)$ be the identity component of the centralizer of H in G and define $U := Z_G^0(H)/H$. Then the sequence:

$$0 \rightarrow \text{Lie}(H) \rightarrow \text{Lie}(Z_G^0(H)) \rightarrow \text{Lie}(U) \rightarrow 0 \quad (2.16.1)$$

associated to the central extension

$$1 \rightarrow H \rightarrow Z_G^0(H) \rightarrow U \rightarrow 1. \quad (2.16.2)$$

is exact.

Proof. Left exactness of (2.16.1) is clear. The maximality of H implies that U is unipotent (see [D, §2.5, page 590]).

Embed H in G_m^r as a subgroup scheme. The quotient $H^{\text{OO}} G_m^r/H$ is a torus, being a quotient of one. The central extension (2.16.2), by a push forward, gives a central extension ([SGA3, Exposé XVII, Lemma 6.2.4])

$$1 \rightarrow G_m^r \rightarrow E \rightarrow U \rightarrow 1 \quad (2.16.3)$$

and a diagram of groups:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & | & & | & & | \\
 1 & \longrightarrow & H & \longrightarrow & Z_G^0(H) & \longrightarrow & U & \longrightarrow & 1 \\
 & & | & & | & & | & & \\
 1 & \longrightarrow & G_m^r & \longrightarrow & E & \longrightarrow & U & \longrightarrow & 1 \\
 & & | & & | & & | & & \\
 1 & \longrightarrow & H^{00} & \longrightarrow & H^{00} & \longrightarrow & 1 & & \\
 & & | & & | & & & & \\
 & & 1 & & 1 & & & &
 \end{array} \tag{2.16.4}$$

Since H^{00} is multiplicative every nilpotent in $\text{Lie}(E)$ maps to 0 in $\text{Lie}(H^{00})$ and hence comes from a nilpotent in $\text{Lie}(Z_G^0(H))$.

Since G_m^r is smooth, by [SGA3, Exposé VII, Proposition 8.2] the sequence (2.16.3) gives an exact sequence

$$0 \rightarrow \text{Lie}(G_m^r) \rightarrow \text{Lie}(E) \rightarrow \text{Lie}(U) \rightarrow 0 \tag{2.16.5}$$

Since U is unipotent, any element z in $\text{Lie}(U)$ is nilpotent. Let z^0 in $\text{Lie}(E)$ be a lift of z . The Jordan decomposition makes sense for any \mathfrak{g} -Lie algebra over a perfect field k and uses only the p -power map (see for example [W2, Corollary 4.5.9, page 135]). Thus, by using the Jordan decomposition of the lift z^0 in $\text{Lie}(E)$ and noting that the semi-simple part gets mapped to zero in $\text{Lie}(U)$, we can assume that z^0 can also be chosen to be nilpotent.

Since every nilpotent in $\text{Lie}(E)$ comes from a nilpotent in $\text{Lie}(Z_G^0(H))$, we conclude that z^0 gets lifted to a nilpotent in $\text{Lie}(Z_G^0(H))$. This implies that (2.16.1) is also right exact.

LEMMA 2.17. Suppose that \mathbf{G} is a subgroup scheme of G which is infinitesimally saturated in G . Then the subgroup scheme $Z_G^0(H)$ is infinitesimally saturated in G ; in particular, every non-zero nilpotent in $\text{Lie}(Z_G^0(H))$ lies in $\text{Lie}(Z_G^0(H)_{\text{red}})$.

Proof. If \mathfrak{n} in $\text{Lie}(Z_G^0(H))$ is nilpotent, the map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n} \cong \mathfrak{g}_a$ factors through \mathbf{G} . Since H is central in $Z_G^0(H)$, the action of H by inner automorphisms fixes \mathfrak{n} . Since the map \exp is compatible with conjugation the entire curve is fixed by H . Therefore, \mathfrak{g} factors through $Z_G^0(H)$ and hence through $Z_G^0(H)_{\text{red}}$, since G_a is reduced and

connected, and hence $n^2 \text{Lie}(Z_G^0(H)_{\text{red}})$. This completes the proof of the lemma.

We have the following extension of ([D, Lemma 2.7, Corollary 2.11]).

LEMMA 2.18. Let G be as in 2.17. Let $H \rightarrow G$ be a maximal connected subgroup scheme of multiplicative type. Then the central extension (2.16.2) splits and U is a smooth unipotent group.

Proof. We claim that U is smooth. By Lemma 2.16, every element z in $\text{Lie}(U)$ comes from a nilpotent n in $\text{Lie}(Z_G^0(H))$. By 2.17, n is in $\text{Lie}(Z_G^0(H)_{\text{red}})$. Its image z is hence in $\text{Lie}(U_{\text{red}})$. Thus, $\text{Lie}(U) \sim \text{Lie}(U_{\text{red}})$ proving the claim.

Now we have a central extension (2.16.2) with the added feature that U is smooth. By [SGA3, Exposé XVII, Theorem 6.1.1] it follows that (2.16.2) splits (uniquely) and

$$Z_G^0(H) \sim H \in U. \quad (2.18.1)$$

Remark 2.19 Recall that in 2.2, for the existence of an exponential map (2.2.2) we assumed that the covering morphism $\mathfrak{g} \rightarrow \mathfrak{g}^0$ is étale, which therefore became a part of the standing assumption (2.3). The case which gets excluded is when the simply connected cover of the derived group has factors of $\text{SL}(p)$. For instance, when G^0 is simple with $p \sim h_G$, the only case excluded is $G^0 \sim \text{PGL}(p)$.

Let G be a reductive group for which $p \sim h_G$. The tables of Coxeter numbers of simple groups show that except for type A where $h_{\text{SL}(n)} \sim n$, Coxeter numbers are even greater than 2. It follows that G is a product of $\text{SL}(p)$'s and other simple factors with p larger than their Coxeter numbers. Even when the morphism $\mathfrak{g} \rightarrow \mathfrak{g}^0$ is not étale we can still define the notions of saturation (resp in nitesimal saturation) of subgroup schemes $G \rightarrow G$ as follows.

Say $G \rightarrow G$ is saturated (resp in nitesimally saturated) if the inverse image of G in \mathfrak{G} is saturated (resp in nitesimally saturated). With this definition, Theorem 2.5 remains true for p, h_G .

This notion of saturation (resp in nitesimal saturation) can also be seen in terms of suitably defined power maps and exponential maps. We restrict ourselves to the case where $\mathfrak{g} \sim \text{PGL}(p)$. At the level of Lie algebras, the induced morphism:

$$\text{sl}(p) \rightarrow \text{pgl}(p) \quad (2.19.1)$$

is a radicial map on the locus of nilpotent elements. If $A \in \mathfrak{gl}(p)$ is a matrix representing an element of $\text{pgl}(p)$, it is nilpotent if all but

the constant coefficients of the characteristic polynomial vanishes, i.e. $\text{Tr}(\wedge^i(A)) \sim 0, 81 \cdot i \cdot p$ and the characteristic polynomial reduces to $T^p - \det(A)$. This condition is stable under $A \mapsto A^{-1}$ as $(T - \det(A))^{-1/p} = T^p - \det(A)$.

We get the unique lift $\mathbb{A} \in \text{SL}(p)$ by taking $\mathbb{A} := A_i \det(A)^{1/p}$. Now one has $\text{Tr}(\wedge^i(\mathbb{A})) \sim 0$ for $1 \leq i \leq p$ and hence we can define the exponential morphism $G_a \rightarrow \text{PGL}(p)$ as:

$$t \mapsto \exp(tA) := \exp(t\mathbb{A}). \quad (2.19.2)$$

Likewise, in the case of unipotents in the group $\text{PG}(p)$, the restriction of this map to the locus of unipotent elements:

$$\text{SL}(p)^u \rightarrow \text{PGL}(p)^u \quad (2.19.3)$$

is radicial. Thus, any unipotent $u \in \text{PGL}(p)$ has a unique unipotent lift $\mathfrak{u} \in \text{SL}(p)$ and one can define the t -power map $G_a \rightarrow \text{PGL}(p)$ as:

$$t \mapsto u^t := \text{image}(\mathfrak{u}^t) \quad (2.19.4)$$

With the notions of t -power map and exponential morphisms in place, we can define the notions of saturation (resp. infinitesimal saturation) of subgroup schemes $\mathbf{G} \subset \mathbf{G}^0$ exactly as in Definition 2.4 using (2.19.4) (resp. (2.19.2)) and these coincide with the definitions made above.

We note however that these "punctual" maps, i.e. defined for each nilpotent A (resp. each unipotent u), are not induced by a morphism from $\text{pgl}(p)_{\text{nilp}}$ to $\text{PGL}(p)^u$ (resp. $A^1 \in \text{PGL}(p)^u \rightarrow \text{PGL}(p)^u : (t, u) \mapsto u^t$ is not a morphism).

3. COMPLETION OF PROOF OF THE STRUCTURE THEOREM 2.5

We begin by stating a general result on root systems whose proof is given in the appendix.

PROPOSITION 3.1. Let R be an irreducible root system with Coxeter number h and let X be the lattice spanned by R . Let $\pi: X \rightarrow R/Z$ be a homomorphism. Then there exists a basis B for R such that $\pi(B)$ satisfies $\langle \pi(b), \pi(b) \rangle \in \{0, 1, h\} \pmod{Z}$, then π is positive with respect to B .

3.2 Assumption 2.3 on the reductive group \mathbf{G} continues to be in force. Let \mathbf{G} be a subgroup scheme of \mathbf{G} , and \mathbf{H} a maximal connected subgroup scheme of multiplicative type of \mathbf{G} . For the existence of \mathbf{H} see [D, Proposition 2.1].

Let $X(\mathbf{H})$ be the group of characters of \mathbf{H} . The action of \mathbf{H} on $\text{Lie}(\mathbf{G})$ by conjugation gives an $X(\mathbf{H})$ -gradation $\text{Lie}(\mathbf{G}) \sim \text{Lie}(Z_{\mathbf{G}}^0(\mathbf{H}))' \bigoplus_{\mathfrak{f} \in \mathfrak{f}_0} \text{Lie}(\mathbf{G})^{\mathfrak{f}}$.

COROLLARY 3.3. (cf. [D, Lemma 2.12]) Let M be a connected subgroup scheme of multiplicative type of G and suppose that $\mathfrak{p} \in \mathfrak{h}_G$. The action of M on $\mathrm{Lie}(G)$ by conjugation gives an $X(M)$ -gradation $\mathrm{Lie}(G) \cong \bigoplus_{\lambda \in X(M)} \mathrm{Lie}(G)^\lambda$. If $\lambda \in \mathfrak{h}_G$, then for $\alpha \in \mathrm{Lie}(G)^\lambda$ one has $\alpha^{\mathfrak{p}} = 0$. In particular, if $M \cong H$ and $\alpha \in \mathrm{Lie}(G)^\lambda$ we have $\alpha^{\mathfrak{p}} = 0$.

Proof. In a smooth algebraic group, the maximal connected subgroups of multiplicative type are (maximal) tori. Indeed, if T is such a maximal subgroup, its centralizer $Z_G(T)$ is smooth, being the fixed locus of a linearly reductive group acting on a smooth variety (see for example [DG, Theorem 2.8, Chapter II, §5]). Define $U := Z_G(T)^\circ/T$. The group scheme U is smooth, as a quotient of $Z_G(T)^\circ$ which is smooth, and is unipotent by maximality of T . By [SGA3, Exposé XVII, Theorem 6.1.1], we have a splitting $Z_G(T)^\circ = T \ltimes U$, and hence T is smooth.

In our case M is hence contained in a maximal torus T of G . By the assumption 2.3 on G , it suffices to prove that \mathfrak{p} is nilpotent. Suppose first that M is isomorphic to μ_p . In that case, the non-trivial character λ of M induces an isomorphism from M to μ_p . Let $\chi : X(T) \rightarrow X(M) \cong X(\mu_p) \cong \mathbb{Z}/p\mathbb{Z}$ be induced by the inclusion of M into T . Then, the M -weight space $\mathrm{Lie}(G)^\lambda$ is the sum of T -weight spaces $\mathrm{Lie}(G)^\alpha$ for α a root such that $\chi(\alpha) = 1$. By 3.1, applied to $\frac{1}{p}\mathfrak{p}$, with values in $\frac{1}{p}\mathbb{Z}/\mathbb{Z} \cong \mathbb{R}/\mathbb{Z}$, this sum is contained in the Lie algebra of the unipotent radical of a Borel subgroup of G . In particular, it consists of nilpotent elements.

For the general case, if M is trivial, the claim is empty. If M is not trivial, pick $A \subset M$ isomorphic to μ_p . If the restriction of λ to A is non-trivial, one applies 3.3 to the already proven case A and the restriction of λ to A . If the restriction of λ to A is trivial, one considers $M/A \cong Z_G(\mu_p)^\circ/\mu_p$ and repeat the argument.

3.4 The proof of 2.5 now follows [D, page 594-599] verbatim with a sole alteration; recall that in [D] the group \mathfrak{G} was the linear group $\mathrm{GL}(V)$ and the condition on the characteristic was $\mathfrak{p} \nmid \dim(V)$. For an arbitrary connected reductive \mathfrak{G} , this condition now gets replaced by $\mathfrak{p} \nmid \dim \mathfrak{h}_G$, which makes 3.3 applicable.

Remark 3.5 If $G \cong \mathrm{GL}(V)$ one has $\dim \mathfrak{h}_G = \dim(V)$. In the case $G \cong \prod \mathrm{GL}(V_i)$ with for each i , $\mathfrak{p} \nmid \dim(V_i)$, the case $\mathfrak{p} \nmid \dim \mathfrak{h}_G$ of Theorem 2.5 gives us [D, Theorem 1.7].

Example 3.6 (Brian Conrad) Here is an example in any characteristic $\mathfrak{p} \nmid 0$, of a connected group of multiplicative type M acting on a reductive group G , and of a non-trivial character λ of M , such that the weight space $\mathrm{Lie}(G)^\lambda$ contains elements which are not nilpotent. We

take $G \cong \mathrm{SL}_p$ (so that $\mathfrak{h}_G \cong \mathfrak{p}$) and $M \cong \mathfrak{m}_{\mathfrak{p}^2}$, and for \mathfrak{f}_i the character $\mathfrak{f}_i : \mathfrak{f} \rightarrow \mathfrak{f}^p$.

We embed M in the maximal torus of diagonal matrices of SL_p by

$$\mathfrak{f} \ni \mathrm{diag}(\mathfrak{f}^0, \mathfrak{f}^{i \cdot p}, \mathfrak{f}^{i \cdot 2p}, \dots, \mathfrak{f}^{i \cdot (p-1)p})$$

The restriction to M of each simple root and of the lowest root is the character $\mathfrak{f}_i : \mathfrak{f} \rightarrow \mathfrak{f}^p$, and $\mathrm{Lie}(G)^{\mathfrak{f}_i}$ is the sum of the corresponding root spaces. In the standard visualization of SL_p this weight space inside $\mathrm{Lie}(G) \cong \mathfrak{sl}_p$ is the span of the super-diagonal entries and the lower-left entry.

A sum of nonzero elements in those root lines contributing to $\mathrm{Lie}(G)^{\mathfrak{f}_i}$ is a $p \times p$ -matrix $X \in \mathfrak{sl}_p$ which satisfies

$$X(e_1) = t_p e_p, X(e_2) = t_1 e_1, \dots, X(e_p) = t_{p-1} e_{p-1} \quad (3.6.1)$$

Iterating p times gives $X^p = \mathrm{diag}(t, \dots, t)$, with $t = \prod_{j=1}^p t_j \neq 0$. Hence, $X \in \mathrm{Lie}(G)^{\mathfrak{f}_i}$ is not nilpotent.

Example 3.7. A variant of the above example leads to an example of an infinitesimally saturated group scheme $\mathbf{G} \cong \mathrm{SL}(V)$ with $\dim(V) = p$ and such that V is an irreducible representation and such that $\mathbf{G}_{\mathrm{red}}$ is a unipotent group. This in particular implies that $\mathbf{G}_{\mathrm{red}}$ is not normal in \mathbf{G} .

Let V be the affine algebra of $\mathfrak{m}_{\mathfrak{p}}$, that is

$$V \cong \mathcal{O}(\mathfrak{m}_{\mathfrak{p}}) \cong \frac{k[u]}{(u^p - 1)} \quad (3.7.1)$$

The vector space V admits the basis $\{u^i \mid i \in \mathbb{Z}/p\}$.

The multiplicative group $\mathcal{O}(\mathfrak{m}_{\mathfrak{p}})^\times$ acts by multiplication on V . For $f \in \mathcal{O}(\mathfrak{m}_{\mathfrak{p}})^\times$, f^p is constant. Define N_f to be the constant value of f^p . It is in fact the norm of f . The action of $\mathcal{O}(\mathfrak{m}_{\mathfrak{p}})^\times$ on V induces an action of the subgroup $N \cong \mathcal{O}(\mathfrak{m}_{\mathfrak{p}})^\times$ for which $N_f = 1$.

On V we have also the action of $\mathfrak{m}_{\mathfrak{p}}$ by translations and this action normalizes the group $\mathcal{O}(\mathfrak{m}_{\mathfrak{p}})^\times$ and its subgroup N . Consider the group scheme:

$$\mathbf{G} \cong \mathfrak{m}_{\mathfrak{p}} \ltimes N \quad (3.7.2)$$

We make a few observations on \mathbf{G} :

- (1) \mathbf{G} is infinitesimally saturated;
- (2) \mathbf{G} contains the group generated by the first factor $\mathfrak{m}_{\mathfrak{p}}$ and $\{u^i \mid i \in \mathbb{Z}/p\} \subset \mathbb{Z}/p$. This group is a Heisenberg type central extension of $\mathfrak{m}_{\mathfrak{p}} \ltimes \mathbb{Z}/p$ by $\mathfrak{m}_{\mathfrak{p}}$.

- (3) the representation V is irreducible as a G -module since it is so for the Heisenberg type subgroup, being simply its standard representation.
- (4) The reduced group G_{red} can be identified with the unipotent group $\{f \in \mathcal{O}_{\mathfrak{p}} \mid f(1) = 1\}$.
- (5) The subgroup G_{red} is not normal as the point 1 of \mathfrak{p} is not invariant by translations.

4. SEMI-SIMPLICITY STATEMENTS

Let G be a reductive group. Let C be an algebraic group and $\rho: C \rightarrow G$ be a morphism.

DEFINITION 4.1. ([S2, Page 20]) One says that ρ is *cr* if, whenever ρ factors through a parabolic P of G , it factors through a Levi subgroup of P .

When $G \simeq GL(V)$, ρ is *cr* if and only if the representation V of C is completely reducible (or equivalently, semi-simple) and hence the terminology.

The property of ρ being *cr* depends only on the subgroup scheme of G which is the (schematic) image of ρ . It in fact only depends on the image of C in the adjoint group G^{ad} . Indeed, the parabolic subgroups of G are the inverse images of the parabolic subgroups of G^{ad} , and similarly for the Levi subgroups. A subgroup scheme \mathcal{G} of G will be called *cr* if its inclusion in G is so.

For an irreducible root system R , let α_0 be the highest root and $\alpha = \sum_{i=1}^r n_i \alpha_i$ its expression as a linear combination of the simple roots. The characteristic p of k is called *good* for R if p is larger than each n_i . For a general root system R , p is *good* if it is so for each irreducible component of R .

PROPOSITION 4.2. Suppose C is an extension

$$1 \rightarrow B \rightarrow C \rightarrow A \rightarrow 1 \tag{4.2.1}$$

with A° of multiplicative type and A/A° a finite group of order prime to p , and suppose that p is good. Let $\rho: C \rightarrow G$ be a morphism. If the restriction of ρ to B is *cr*, then ρ is *cr*.

We don't know whether the proposition holds without the assumption that p is good.

Proof. Let P be a parabolic subgroup, U its unipotent radical, and \mathfrak{u} the Lie algebra of U . The parabolic P is said to be *restricted* if the nilpotence class of U is less than p . If P is a maximal parabolic corresponding to a simple root α_i , the nilpotence class of U is the coefficient α_i in the

highest root. It follows that \mathfrak{p} is good if and only if all the maximal parabolic subgroups are restricted. By [Sei, Proposition 5.3] (credited by the author to Serre), if \mathfrak{p} is restricted, one obtains by specialization from characteristic 0, \mathbb{P} -equivariant isomorphism:

$$\exp : (\mathfrak{u}, -) \xrightarrow{\sim} U \quad (4.2.2)$$

from \mathfrak{u} endowed with the Campbell-Hausdorff group law, \mathfrak{td} .

We will first show that whenever \mathfrak{p} factors through a restricted parabolic subgroup P as above, if the restriction of \mathfrak{p} to B factors through some Levi subgroup of P , the \mathfrak{p} itself factors through some Levi subgroup of P .

The group $U(k)$ acts on the right on the set $L(k)$ of Levi subgroups of P by

$$u \text{ in } U(k) \text{ acts by } L \mapsto u^{-1}Lu$$

This action turns $L(k)$ into a $U(k)$ -torsor. This expresses the fact that two Levi subgroups are conjugate by a unique element $u \in U(k)$. The group $P(k)$ acts on $L(k)$ and on $U(k)$ by conjugation. This turns $L(k)$ into an equivariant $U(k)$ -torsor.

We will need a scheme-theoretic version of the above. Fix a Levi subgroup L_0 . Let L be the trivial U -torsor (i.e. U with the right action of U by right translations). We have the family of Levi subgroups $L_u := u^{-1}L_0u$ parametrized by $L \cong U$. We let P act on U by conjugation, and on L as follows: $p \sim v'$ in $P \sim UL_0$ acts on $L \cong U$ by $u \mapsto v'^{-1}puv'^{-1}$. This turns L into an equivariant U -torsor. When we pass to \mathbb{k} -points, and attach to u in L the Levi subgroup uL_0u^{-1} , we recover the previously described situation.

The morphism $\mathfrak{p} : C \rightarrow P$ turns L into an equivariant U -torsor. A point x of L corresponding to a Levi subgroup L_x is fixed by C (scheme-theoretically) if and only if \mathfrak{p} factors through L_x . This expresses the fact that a Levi subgroup is its own normalizer in P .

We want to prove that \mathfrak{B} has a fixed point in L , so does C . Let U^B be the subgroup of U fixed by B , for the conjugation action. If B has a fixed point in L , the fixed locus L^B is a U^B -torsor. As B is a normal subgroup of C , C acts on L^B and U^B , and the action factors through A .

The isomorphism $\exp : (\mathfrak{u}, -) \xrightarrow{\sim} U$ is compatible with the action of B by conjugation. Hence it induces an isomorphism from $(\mathfrak{u}^B, -) \xrightarrow{\sim} U^B$. Let $Z^i(\mathfrak{u}^B)$ be the central series of \mathfrak{u}^B , and define $Z^i(U^B) := \exp(Z^i(\mathfrak{u}^B))$. The isomorphism \exp induces an isomorphism between the vector group

$\text{Gr}_Z^i(\mathfrak{u}^B)$ and $Z^i(U^B)/Z^{i-1}(U^B)$, compatible with the action of A . On $\text{Gr}_Z^i(\mathfrak{u}^B)$, this action is linear.

The assumption on A amounts to saying that A is linearly reductive, that is, all its representations are semi-simple. Equivalently, k^A is the trivial representation, any extension

$$0 \rightarrow V \rightarrow E \rightarrow k \rightarrow 0 \quad (4.2.3)$$

splits. Passing from E to $b^{i-1}(1)$, such extensions correspond to A -equivariant V -torsors, and the extension splits if and only if k^A has a scheme-theoretic fixed point on the corresponding torsor.

Define U_i^B to be $U^B/Z^i(U^B)$, and L_i^B to be the U_i^B -torsor obtained from L by pushing by $U^B \rightarrow U_i^B$. We prove by induction on i that A has a fixed point on L_i^B .

As U_1^B is trivial, the case $i=1$ is trivial. If x is a fixed point of A in L_i^B , the inverse image of x in L_{i-1}^B is an equivariant A -torsor on $\text{Gr}_Z^i(\mathfrak{u}^B) \cong Z^i(U^B)/Z^{i-1}(U^B)$. By linear reductivity, A has a fixed point on the inverse image. As the central descending series of \mathfrak{u}^B and hence of U^B terminates, this proves 4.2 for restricted parabolic subgroups.

We now prove 4.2 by induction on (the dimension) $\dim \mathfrak{g}$. Suppose that \mathfrak{g} factors through a proper parabolic subgroup \mathfrak{p} . As \mathfrak{p} is good there exists a restricted proper parabolic \mathfrak{q} containing \mathfrak{p} , and \mathfrak{p} is the inverse image by the projection $\mathfrak{q} \rightarrow \mathfrak{q}/R_u(\mathfrak{q})$ of a parabolic \mathfrak{p}^0 of $\mathfrak{q}/R_u(\mathfrak{q})$. Let L be a Levi subgroup of \mathfrak{q} through which \mathfrak{g} factors and let P_L^0 be the parabolic subgroup of L obtained as the inverse image of \mathfrak{p}^0 by the isomorphism $L \xrightarrow{\cong} \mathfrak{q}/R_u(\mathfrak{q})$. Levi subgroups of P_L^0 are Levi subgroups of \mathfrak{p} , and it remains to apply the induction hypothesis to \mathfrak{p} , for which \mathfrak{p} is good too.

Remark 4.3 For several results related to 4.2 but in the setting of reduced subgroups, see [BMR1, Theorem 3.10] and [BMR2, Theorem 1.1 and Corollary 3.7].

Fix in the reductive group G , a maximal torus T , and a system of simple roots corresponding to a Borel subgroup B containing T . Let U be the unipotent radical of B .

DEFINITION 4.4. (cf. [Dy], [S1], [IMP],) The p -Dynkin height $\text{ht}_G(V)$ of a representation V of G is the largest among $\langle \mu, \alpha_i^\vee \rangle$, for μ a weight for the action of T on V .

This notion and this terminology go back to Dynkin [Dy, Page 331-332] where it is called "height" as in [IMP], while in Serre ([S1] and [S2]), it is simply $\text{ht}(V)$. If V is an irreducible representation, with dominant weight, λ and smallest weight, μ , it is the sum of the coefficients of $\lambda - \mu$, expressed as linear combination of the simple roots.

It follows that the product in $\text{End}(V)$ of the action of $\text{ht}_G(V) - 1$ elements of $\text{Lie}(U)$ vanishes and for nilpotent n in $\text{Lie}(G)$, one has $n^{\text{ht}(V) - 1} = 0$ in $\text{End}(V)$.

4.5 The representation $\rho: G \rightarrow \text{GL}(V)$ is said to be of low height if $p \leq \text{ht}_G(V)$. By [S2, Theorem 6, page 25], representations of low height are semi-simple. One can show that if G admits a representation V of low height which is almost faithful meaning that its kernel is of multiplicative type, then G satisfies the assumption 2.3. That $p \leq \text{ht}_G(V)$ results from the more precise statement that $\text{ht}_G(V) \leq h_G + 1$ ([S3, (5.2.4), Page 213]). For the property that G/G^0 is \emptyset -tale, one uses the fact that the non-trivial irreducible representation of $\text{PGL}(d)$ of the smallest height is the adjoint representation which is of height $d - 2$.

We now assume that V is of low height, and the assumption 2.3 on G . It follows that any nilpotent n in $\text{Lie}(G)$ satisfies $n^p = 0$, and further, the exponential map (2.2.2) is defined. The image $\text{d}\rho(n)$ of n in $\text{Lie}(\text{GL}(V)) \simeq \text{End}(V)$ also has a vanishing p^{th} -power, hence $\exp(\text{d}\rho(n).t)$ is defined. By [S2, Theorem 5, page 24], one has the following compatibility statement.

4.6 Compatibility If n in $\text{Lie}(G)$ is nilpotent

$$\rho(\exp(t n)) \simeq \exp(t \text{d}\rho(n)). \quad (4.6.1)$$

As a consequence, if $u^p = 1$ in G , one has

$$\rho(u^t) \simeq \rho(u)^t \quad (4.6.2)$$

The following theorem is a schematic analogue of [BT], for p large enough.

THEOREM 4.7. Suppose that the reductive group G admits a low height almost faithful representation $\rho: G \rightarrow \text{GL}(V)$, and that $p \leq h_G$. Then, for any non-trivial unipotent subgroup U of G , there exists a proper parabolic subgroup P of G containing the normalizer $N_G(U)$ of U , and whose unipotent radical contains U .

The condition $p \leq h_G$ implies that G satisfies the assumption 2.3. If G is simple simply connected, it implies the existence of an almost faithful low height representation except for the G of type F_4, E_6, E_7 or E_8 ,

in which case the lowest height of a non-trivial representation and the Coxeter number are respectively $16 \cdot 12$, $16 \cdot 12$, $27 \cdot 18$ and $58 \cdot 30$. For these groups, we do not know whether the conclusion of the theorem is valid assuming only $p \nmid h_G$.

Proof. Let V^U be the invariants of U acting on V . It is not zero, because U is unipotent. It is not V , because the representation V is almost faithful, hence faithful on U . It does not have a U -stable supplement V^0 in V , because U would have invariants in V^0 .

Let H be the subgroup scheme of G which stabilizes V^U . It contains the normalizer $N_G(U)$ of U . It is a doubly saturated subgroup scheme of G . Indeed, if h in $H(k)$ is of order p , by (4.6.2), $(h^t)^{-1} \sim (h^t)^t \sim (h^t)^{t^2} \sim \dots \sim (h^t)^{t^{p-1}}$, which stabilizes V^U , and similarly if η in $\text{Lie}(H)$ is nilpotent, the $\exp(t\eta)$ are in H .

As $p \nmid h_G$, theorem 2.5 ensures that H_{red}^0 is a normal subgroup scheme of H and that the quotient H/H_{red}^0 is an extension of a finite group of order prime to p by a group of multiplicative type. It follows that $H \cong H_{\text{red}}^0$.

LEMMA 4.8. H_{red}^0 is not reductive.

Proof. If it were, V would be a representation of low height of H_{red}^0 ([S2, Corollary 1, page 25]), hence a semi-simple representation of H_{red}^0 , and V^U would have in V a H_{red}^0 -stable supplement. As V^U does not admit a supplement stable under U , this is absurd.

Proof of 4.7 continued If S is a doubly saturated subgroup scheme of G , we will call $R_u(S_{\text{red}}^0)$ the unipotent radical of S and denote it simply by $R_u(S)$. By 2.5, it is a normal subgroup of S and $S/R_u(S_{\text{red}}^0)$ does not contain any normal unipotent subgroup. This justifies the terminology.

Define $U_1 := R_u(H)$. By 4.8, it is a non-trivial unipotent subgroup of G , and we can iterate the construction. We define for $i \geq 1$

$$H_i := \text{stabilizer of } V^{U_i} \text{ in } V \\ U_{i-1} := R_u(H_i)$$

One has $U \subset H_{\text{red}}^0$ and

$$U \subset N_G(U) \subset H \subset N_G(U_1) \subset H_1 \subset N_G(U_2) \subset H_2 \dots \quad (4.8.1)$$

The $H_{i,\text{red}}^0$ form an increasing sequence of smooth connected subgroups of G . It stabilizes, hence so do the sequences of the U_i and of the H_i . If $H_i \cong H_{i-1}$, one has $H_i \cong N_G(U_{i-1}) \cong H_{i-1}$ and

$$U_{i-1} \sim R_u(H_i) \sim R_u(H_{i-1}) \sim R_u(N(U_{i-1})_{\text{red}}^{\circ})$$

By [BT, Proposition 2.3, page 99] (or for example [H, Section 30.3, Proposition on page 186]), this implies that $N(U_{i-1})_{\text{red}} \sim (H_{i-1})_{\text{red}}$ is a proper parabolic subgroup of G . Call it Q . A parabolic subgroup of G is its own normalizer scheme (cf. [SGA3, XII, 7.9], [CGP, page 469]). As $N(U_{i-1})_{\text{red}} \sim Q$ is normal in H_{i-1} , it follows that $H_{i-1} \sim Q$ and that $N_G(U) \sim Q$.

A Levi subgroup L of Q is a reductive subgroup of G , hence satisfies the assumptions of 4.7. It is isomorphic to $\mathbb{G}_m/R_u(Q)$. If U is not contained in $R_u(Q)$, we can repeat the argument for the image \bar{U} of U in $Q/R_u(Q)$, which is isomorphic to L . One obtains a proper parabolic subgroup of $Q/R_u(Q)$ which contains the normalizer of \bar{U} . Its inverse image in Q is a parabolic subgroup, properly contained in Q and containing the normalizer of U . Iterating, one eventually finds a parabolic P containing $N_G(U)$ and such that $U \sim R_u(P)$.

Remark 4.9 Let \mathcal{G} be the simply connected central extension of the derived group G° , and let \mathcal{G}_i be its simple factors $\mathcal{G} \sim \prod \mathcal{G}_i$. A representation of low height V of G is almost faithful if and only if its restriction to each \mathcal{G}_i is not trivial. It suffices to check this for each \mathcal{G}_i separately. Thus we may assume G is simply connected. The existence of a non-trivial V of low height implies that $p \geq 2$ for G of type $B_n, C_n(n \geq 2)$ or F_4 and $p \geq 3$ for G of type G_2 . Let \bar{G} be the image of G in $GL(V)$. If V is non-trivial, $u : G \rightarrow \bar{G}$ is an isogeny. We want to show that it is a central isogeny. If it is not, the structure of isogenies ([SGA3, XXII, 4.2.13]) show that $\ker(u)$ contains the kernel of the Frobenius. The weights of V are the p^{th} -powers and $\dim(V)$ is a multiple of p , contradicting the low height assumption.

4.10 If G is a reductive group for which assumption 2.3 holds, and \mathbf{G} is a subgroup scheme of G , the double saturation \mathbf{G}' of \mathbf{G} is the smallest doubly saturated subgroup scheme of G containing \mathbf{G} . It is the intersection of the doubly saturated subgroup schemes containing \mathbf{G} and is obtained from \mathbf{G} by iterating the construction of taking the group generated by \mathbf{G} , the additive groups $\exp(\mathfrak{t})$ for \mathfrak{t} nilpotent in $\text{Lie}(\mathbf{G})$ and $u^{\mathfrak{t}}$ for u of order p in $\mathbf{G}(k)$.

COROLLARY 4.11. Let V be a low height almost faithful representation of a reductive group G . Assume that $p \nmid h_G$. Let \mathbf{G} be a subgroup scheme of G , and let \mathbf{G}' be its double saturation. Then the following conditions are equivalent.

- (i) V is a semi-simple representation of \mathbf{G}
- (ii) V is a semi-simple representation of \mathbf{G}'

- (iii) G is cr in G
- (iv) G' is cr in G
- (v) the unipotent radical of $(G')_{\text{red}}^0$ is trivial

Proof. (i) \Rightarrow (ii): If W is a subspace of V , the stabilizer in G of W is doubly saturated, as we saw in the beginning of the proof of 4.7. If G stabilizes W , it follows that G' also stabilizes W : the lattice of subrepresentations of V is the same for G and G' , hence the claim.

(iii) \Leftrightarrow (iv): Similarly, the parabolic subgroups of G and their Levi subgroups are doubly saturated, hence contain G if and only if they contain G' .

not (v) \Rightarrow not (ii): Let U be the unipotent radical of $(G')_{\text{red}}^0$. If it is non-trivial, $V^U \subsetneq V$, because V is faithful on U and does not have a U -stable supplement. As U is normal in G' (2.5), V^U is a subrepresentation for the action of G' on V . This contradicts the semi-simplicity of V .

(v) \Rightarrow (ii): The representation V of the reductive group $(G')_{\text{red}}^0$ is of low height, hence semi-simple. By 2.5, $(G')_{\text{red}}^0$ is a normal subgroup of G' and the quotient A is linearly reductive. If W is a sub- G' -representation of V , A acts on the affine space of $(G')_{\text{red}}^0$ -invariant retracts of V/W . It has a fixed point, whose kernel is a supplement to W .

not (v) \Rightarrow not (iv): Let U be the unipotent radical of $(G')_{\text{red}}^0$. If it is not trivial, there exists a parabolic P containing its normalizer, hence G' , and the unipotent radical of P contains U (4.7). Thus, no Levi subgroup of P can contain G' .

(v) \Rightarrow (iv): By [S2, Theorem 7, page 26], $(G')_{\text{red}}^0$ is cr in G , and one applies 4.2.

COROLLARY 4.12. Let $\rho: G \rightarrow GL(V)$ be an almost faithful low height representation, and let v in V be an element such that the G -orbit of v in V is closed. Then there exists a connected multiplicative central subgroup scheme $M \times G_v^0$ and a surjective homomorphism $M \times G_{v,\text{red}}^0 \rightarrow G_v^0$.

Proof. The orbit being closed in V and hence affine, the reduced stabilizer $G_{v,\text{red}}$ is reductive ([Bo]). Since $\rho|_G = \rho|_{G_v}$ and since stabilizers are doubly saturated and 2.5(2) holds for $G \sim G_v$, we get the required result.

We now observe that the results of [D, Section 6] can be obtained as a consequence of 4.11. Note that by the remarks in [D, Page 607] it suffices

to prove the semisimplicity results in the case when k is algebraically closed.

THEOREM 4.13. G be an algebraic group. Let $(V_i)_{i \geq 1}$ be a finite family of semi-simple G -modules and let m_i be integers ≥ 0 . If

$$\sum m_i (\dim V_i - m_i) \geq p \tag{4.13.1}$$

the G -module $\sum_j V_j^{m_j}$ is semi-simple.

Proof. Let $G \subset \prod_j GL(V_j)$ and $V = \sum_j V_j^{m_j}$. Then, (4.13.1) is simply the inequality $p \leq h_G(V)$. Replacing G by its image in G we may and shall assume that G is a subgroup scheme of $\prod_j GL(V_j)$. Since V_i are semi-simple G -modules, it follows that G is G -cr, for $G \subset \prod_j GL(V_j)$. By [D, §6.2], we may also assume that $p \leq \dim(V_j) \leq h_{GL(V_j)} \delta_j$.

Hence by working with the image of G (and V) in $GL(V)$ and applying 4.11, we conclude that V is semi-simple as a G -module.

Remark 4.14 If (V_i, q_i) is a non-degenerate quadratic space with $\dim V_i = 2d_i$ on which G acts by similitudes, then by passing to a subgroup of index at most 2 and mapping to the group of similitudes rather than $GL(V_i)$, one can replace the term $m_i (\dim V_i - m_i)$ by $m_i (\dim V_i - m_i + 1)$, when $m_i \leq 2d_i$.

Complete reducibility in the classical case. By 4.1, a subgroup scheme $G \subset GL(V)$ is called cr if for every parabolic subgroup $P \subset GL(V)$ containing G , there exists an opposite parabolic subgroup P^0 such that $G \subset P \cap P^0$. Suppose that $\text{char}(k) \neq 2$ and let G be $SO(V)$ (or $Sp(V)$ in any characteristic), relative to a non-degenerate symmetric or alternating bilinear form B on V . In this situation, the notion of cr can be interpreted as follows: G is cr in G if and only if for every G -submodule $W \subset V$ which is totally isotropic, there exists a totally isotropic G -submodule W^0 of the same dimension, such that the restriction of B to $W \times W^0$ is non-degenerate (cf. [S3, Example 3.3.3, page 206]).

LEMMA 4.15. Let the subgroup scheme G of G be cr. Then the G -module V is semi-simple and conversely.

Proof. Let $W \subset V$ be a G -submodule. Then we need to produce a G -complement.

Consider $W_1 := W \cap W^\perp$. If $W_1 = (0)$, then $W + W^\perp = V$ and we are done. So let $W_1 \neq (0)$. Then W_1 is a G -submodule which is totally isotropic and hence by the cr property, we have a totally isotropic G -submodule W_1^0 of the same dimension as W_1 , such that the form B is

non-degenerate on $W_1 \cap W_1^0$. In particular, $W_1 \cap W_1^0 = (0)$. Since $W_1 \cong W_1^0$, we see that $W \cong W_1^0$.

Let $w \in W \cap W_1^0 \cong W_1^0$ and suppose $w \neq 0$. Since $w \in W_1^0$, there exists $w' \in W_1$ such that $B(w, w') \neq 0$. On the other hand, since $w \in W_1^0$, $B(w, v) = 0$ for all $v \in W_1$ and in particular $B(w, w') = 0$ which contradicts the assumption that $w \neq 0$. Hence it follows that $W \cap W_1^0 = (0)$. Thus, $W \cap (X \cap W \cap W_1^0) = (0)$ is a G -submodule. We proceed similarly and get $X_1 \cap X \cap X^0$ such that $X_1 \cap X^0 = (0)$. If $X_1 \cap X^0 = (0)$, then $V \cong X_1 \oplus X^0$, so get a G -decomposition of V as $W \oplus W_1^0 \oplus \dots$, that is a G -complement of W in V .

Conversely, let V be semi-simple as a G -module. Let $W \cong W^0$ be a totally isotropic G -submodule of $\dim(W) = d$. Note that $d \leq \dim(V)/2$. We therefore have a G -submodule $Z \cong W^0$ such $V \cong W \oplus Z$. The non-degenerate form B gives an G -equivariant isomorphism $\phi : V \rightarrow V^* \cong W^* \oplus Z^*$ and since W is totally isotropic $(W) \cap W^* = (0)$. Hence $\phi(W) \cong Z^*$.

Again, since W is totally isotropic, the restriction $\phi|_W$ to Z is non-degenerate and hence we get an isomorphism $\psi : Z^* \rightarrow Z$. Define $W^0 := \psi^{-1}(\phi(W))$.

Then it is easily seen that W^0 is of dimension d and also totally isotropic G -invariant submodule of V . Finally, B is non-degenerate on $W \cap W^0$. Hence G is cr in G .

5. ÉTALE SLICES IN POSITIVE CHARACTERISTICS

We begin this section with the following (linear) analogue of the Luna étale slice theorem in positive characteristics.

THEOREM 5.1. Let V be a G -module with low height i.e. such that $p \nmid \dim(V)$. Let v in V be an element such that the orbit $G.v$ is a closed orbit in V . Then there exists a G -invariant linear subspace S of V giving rise to a commutative diagram:

$$\begin{array}{ccc} G \backslash G.v \cap S & \xrightarrow{\quad} & \mathcal{U} \\ \downarrow f & & \downarrow q \\ (G \backslash G.v \cap S) / G & \xrightarrow{\quad} & \mathcal{U} / G \end{array} \quad (5.1.1)$$

and G -equivariant open subsets $\mathcal{U} \subset (G \backslash G.v \cap S)$ containing the closed orbit $G.v$ and an open subset \mathcal{U}' of \mathcal{U} / G containing v , for which (5.1.1) induces a

cartesian diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\quad} & U^0 \\
 f \downarrow & & \downarrow q \\
 U \times G & \xrightarrow{j_U} & U^0 \times G
 \end{array} \tag{5.1.2}$$

such that the morphism j_U is étale.

Remark 5.2 The above theorem was stated in the note [MP] whose proof contained serious gaps (as was pointed to the authors by Serre in a private correspondence).

We note that G/G_v is constructed in ([DG, III, Proposition 3.5.2]). It represents the quotient in the category of fppf sheaves. Furthermore if $\dots : G \rightarrow V, g \mapsto g.v$, the image $\text{im}(\dots)$, as a locally closed sub-scheme of V with its reduced scheme structure, can be identified with the scheme G/G_v . We call this locally closed sub-scheme, the orbit $G.v$ and have the identification $G/G_v \cong G.v$ (see also [DG, Proposition and Definition 1.6, III, §3, page 325]).

PROPOSITION 5.3. Let V be an arbitrary G -module. Let $\mathcal{V} \subset V$ and suppose that there exists a G -submodule S of the tangent space $T_{\mathcal{V}}$, such that $T_{\mathcal{V}}(V)$ splits as $T_{\mathcal{V}}(G.v) \oplus S$. Let G act on $G \times S$ by $h.(g, s) := (h.g, s)$. Then the G -morphism $\pi : G \times S \rightarrow V$ given by $(g, s) \mapsto g.v + g.s$ descends to a G -morphism

$$\pi : G \times^{G_v} S \rightarrow V. \tag{5.3.1}$$

which is étale at $(e, 0)$, e being the identity of G .

Proof. To check the morphic properties, we need to check for \mathbb{A}^1 -valued points of G/G_v where A is a k -algebra and we will suppress the A .

We will check that it is constant on the G_v -orbits. Let f_i in G_v act on $G \times S$ by $f_i.(g, s) := (g.f_i, f_i^{-1}.s)$. Observe that $f_i.f_i.(g, s) := (g.f_i, f_i^{-1}.s) := g.f_i.v + g.f_i.f_i^{-1}.s := g.v + g.s$ (since f_i fixes v). Therefore it is constant on the G_v -orbits. Since the action of G_v on $G \times S$ is scheme-theoretically free, π descends to a morphism $\tilde{\pi} : G \times^{G_v} S \rightarrow V$. Clearly the actions of G and G_v on $G \times S$ commute and hence the descended morphism is also a G -morphism.

Observe that the quotient morphism $G \rightarrow G/G_v$ is a torsor for the group scheme G_v , locally trivial under the fppf topology. Since the action of G_v on S is linear, we see that the associated fibre space $G \times^{G_v} S \rightarrow G/G_v$ is a locally free sheaf of rank $\dim(S)$. In particular, $G \times^{G_v} S \rightarrow G/G_v$ is a smooth k -scheme of finite type. Observe further that under the

morphism $\tilde{\nu}$, the zero section of the vector bundle $\tilde{\nu} : G \times_{\mathbb{A}^1} S \rightarrow G/G_v$ canonically maps onto the orbit $G.v \subset \mathcal{M}$, while the fibre $\tilde{\nu}^{-1}(e.G_v)$ of the identity coset $e.G_v \subset G/G_v$ maps isomorphically to the affine subspace $S \times_{\mathbb{A}^1} \mathcal{M}$. Since $T_v(V) \cong T_v(G.v) \subset S$ by assumption, it follows that the differential $d_{\tilde{\nu}, z}$, at $z \in (e, 0)$, is an isomorphism.

We now apply [D, Lemme 2.9], to conclude that the morphism $\tilde{\nu} : G \times_{\mathbb{A}^1} S \rightarrow V$ is étale at $z \in (e, 0)$.

PROPOSITION 5.4. Let V be a G -module such that $p \in \text{ht}_G(V)$, and let v in V be an element such that the G -orbit of v in V is closed. Then there exists a G_v -submodule $S \subset \mathcal{M}$ such that $V \cong T_v(G.v) \subset S$ as a G_v -module. In particular, the consequences of 5.3 hold good.

Proof. One has the exact sequence:

$$1 \rightarrow G_v^0 \rightarrow G_v \rightarrow \dots_0(G_v) \rightarrow 1 \quad (5.4.1)$$

the quotient $\dots_0(G_v)$ being the group of connected components. Further, we note that $G_v/G_v^0 \cong G_{v,\text{red}}/G_{v,\text{red}}^0$. Note also that since $G_{v,\text{red}}$ is a saturated subgroup of G , by [S2, Property 3, Page 23] the index $[G_{v,\text{red}}/G_{v,\text{red}}^0]$ is prime to p . Thus, $\dots_0(G_v)$ is linearly reductive.

Since $G.v$ is a closed orbit, by 4.12 we see that $G_{v,\text{red}}^0$ is reductive and by 2.5(1), we have the exact sequence:

$$1 \rightarrow G_{v,\text{red}}^0 \rightarrow G_v^0 \rightarrow \zeta \rightarrow 1 \quad (5.4.2)$$

where ζ is a multiplicative group scheme.

Since $G_{v,\text{red}}^0$ is a reductive saturated subgroup of G , by [S2, Corollary 1, page 25] we get the inequality $\text{ht}_G(V) \geq \text{ht}_{G_{v,\text{red}}^0}(V)$. Being of low

height, V is semi-simple as a $G_{v,\text{red}}^0$ -module.

Further, since ζ is multiplicative (and hence linearly reductive), by [D, Lemma 4.2] and 5.4.2, we deduce that V is semi-simple as a G_v^0 -module. Using 5.4.1, we conclude that V is semi-simple as a G_v -module as well.

In particular, we have a G_v -supplement S for the G_v -invariant subspace $T_v(G.v) \subset T_v(V) \cong V$, i.e. we have a G_v -decomposition $S \oplus T_v(G.v) = V$ for V .

We recall the "Fundamental lemma of Luna" which holds in positive characteristics as well and which is essential to complete the proof of Theorem 5.1.

LEMMA 5.5. ([GIT, Page 152]) Let $s : X \rightarrow Y$ be a G -morphism of affine G -schemes. Let $\mathcal{P} \subset X$ be a closed orbit such that:

- (1) s is étale at some point of F
- (2) (F) is closed in Y
- (3) s is injective on F
- (4) X is normal along F .

then there are affine G -invariant open subsets $U \subseteq X$ and $U^0 \subseteq Y$ with $F \subseteq U$ such that $s : U \rightarrow U^0$, G is étale and $(\text{sp}_U) : U \rightarrow U^0$. $\text{sp}_U : U \rightarrow U^0$ is an isomorphism (where $\text{sp}_U : U \rightarrow U^0$ is the quotient morphism).

Proof. (of 5.1) As a first step, we need to show that the categorical quotient $G \backslash G \backslash S \rightarrow G \backslash S$ exists as a scheme. As we have seen in 5.3, action of G on $G \backslash S$ (via $g.(a, s) \sim (g.a, s)$) and the (twisted) action of G_v on $G \backslash S$ commute and hence

$$k[G \backslash G \backslash S]^{G_v} \cong (k[G \backslash S]^{G_v})^G \cong (k[G \backslash S]^G)^{G_v} \cong k[S]^{G_v}. \quad (5.5.1)$$

Thus it is enough to show that $k[S]^{G_v}$ is finitely generated and we would have

$$G \backslash G \backslash S \rightarrow G \backslash S \rightarrow G_v. \quad (5.5.2)$$

As we have observed in the proof of 5.4, G_v^0 is a normal subgroup of finite index. Thus, the finite generation of $k[S]^{G_v}$ is reduced to checking finite generation of $k[S]^{G_v^0}$. We may therefore replace G_v by G_v^0 in the proof. Further, since $G.v$ is a closed orbit, $G_{v, \text{red}}$ is reductive. Therefore, by 4.12, we have a surjection $M \rightarrow G_{v, \text{red}} \rightarrow G_v$. Since M in G_v is central $k[S]^{G_v} \cong k[S]^{G_{v, \text{red}}} \cong k[S]^{M}$.

Again, since $G_{v, \text{red}}$ is reductive, the ring of invariants $k[S]^{G_{v, \text{red}}}$ is a finitely generated k -algebra. Since M is a group of multiplicative type over an algebraically closed field, it is a product of G_m 's and a finite group scheme. Therefore, by [GIT, Theorem 1.1] and [AV, Page 113], the ring of invariants $k[S]^{G_{v, \text{red}}}$ is also finitely generated. This proves the finite generation of $k[S]^{G_v}$.

The commutativity of the diagram (5.1.1) now follows by the property of categorical quotients by the group G .

Now we check the conditions of Lemma 5.5. We consider the given closed G -orbit $F \cong G \backslash G \backslash \{v\} \cong G \backslash G \backslash S$. Then we need to check that (F) is a closed orbit in V . In fact, (F) is precisely the closed orbit $G.v \cong G.v \cong V$. Thus we have verified conditions (2) and (3) of Lemma 5.5.

Condition (1) is precisely the content of 5.4 and (4) holds since $G \times^{G_v} S$ has been seen to be smooth.

The isomorphism (φ_U) shows that the diagram (5.1.2) is cartesian. This completes the proof of 5.1.

COROLLARY 5.6. Let X be an affine G -scheme embeddable as a closed G -subscheme in low height G -module. Let $x \in X$ be such that the orbit $G \cdot x \subset X$ is closed. Then there exists a locally closed G_x -invariant subscheme (a "slice") X_1 , of X with $x \in X_1 \subset X$ such that the conclusions of 5.1 hold for the G -morphism $G \times^{G_x} X_1 \rightarrow X$.

Proof. This follows from 5.1 (which corresponds to [BR, Proposition 7.4]) exactly as in [BR, Proposition 7.6].

The following consequence of the slice theorem has many applications so we state it here without proof.

COROLLARY 5.7. (see [BR, Proposition 8.5, p.312]) Let F be an affine G -subvariety of $P(V)$, with V a G -module with low height, and suppose that F contains a unique closed orbit F^{cl} . Then there exists a G -retract $F \rightarrow F^{cl}$.

Let V be a finite dimensional G -module and let

$$ht_G(\wedge(V)) := \max_i \{ht_G(\wedge^i(V))\} \tag{5.7.1}$$

We then have the following application to the theory of semistable principal bundles in positive characteristics.

THEOREM 5.8. Let E be a stable G -bundle with G semi-simple and $\rho_V : G \rightarrow SL(V)$ be a representation such that $\rho \in ht_G(\wedge(V))$. Then the associated bundle $E(V)$ is polystable of degree 0.

Proof. The proof follows [BP, Theorem 9.11] verbatim where the notion of "separable index" is replaced by the Dynkin height, the key ingredient being Corollary 5.7.

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APPENDIX A. COXETER NUMBER AND ROOT SYSTEMS ZHIWEI YUN

PROPOSITION A.1. Let R be an irreducible root system associated to a simple group G with Coxeter number h and let X be the lattice spanned by R . Let $\hat{A} : X \rightarrow R/Z$ be a homomorphism. Then there exists a basis B for R

