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ON COMPLETE REDUCIBILITY IN CHARACTERISTIC p

V. BALAJI, P. DELIGNE, AND A.J. PARAMESWARAN,
WITH AN APPENDIX BY ZHIWEI YUN

IN MEMORIAM VIKRAM MEHTA.

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1. INTRODUCTION

We work over an algebraically closed field k of characteristic $p > 0$.

In [S1], Serre showed that if semi-simple representations V_i of a group Γ are such that $\sum(\dim V_i - 1) < p$, then their tensor product is semi-simple. In [S2], he more generally considers the case where Γ is a subgroup of $G(k)$, for G a reductive group, and where Γ is G -cr, meaning that whenever Γ is contained in a parabolic subgroup P , it is already contained in a Levi subgroup of P . For $G = \prod \mathrm{GL}(V_i)$, this is equivalent to the semi-simplicity of the representations V_i of Γ . For a representation V of G , one can then ask under what conditions does V become semi-simple, when considered as a representation of Γ . Serre ([S2, Theorem 6, page 25]) shows this is the case when the Dynkin height $ht_G(V)$ is less than p . For $G = \prod \mathrm{GL}(V_i)$ and $V = \otimes V_i$, one has $ht_G(V) = \sum(\dim V_i - 1)$.

In [D], the results of [S1] were generalized to the case when the V_i are semi-simple representations of a group scheme \mathcal{G} . In this paper, we consider the case when \mathcal{G} is a subgroup scheme of a reductive group G and generalize [S2] (see 4.11) and [D] (see 2.5). As in [D], we first

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have to prove a structure theorem (2.5) on doubly saturated (see 2.4) subgroup schemes \mathfrak{G} of reductive groups G . The proof makes crucial use of a result of Zhiwei Yun on root systems. The appendix contains the result.

In Section 5, we consider a reductive group acting on an affine variety X and a point x of X whose orbit $G.x$ is closed in X . We prove a schematic analogue of [BR, Proposition 7.4,7.6] under some conditions on the characteristic of k . More precisely, if X embeds in a G -module V of low height, then we obtain, as a consequence of 2.5(2), an analogue of Luna's étale slice theorem (5.1, 5.6). In [BR] the language of schemes was not used and as a consequence the orbit $G.x$ had to be assumed "separable". An orbit $G.x$ is separable if and only if the stabilizer G_x is reduced.

2. SATURATION AND INFINITESIMAL SATURATION

2.1. Let G be a reductive algebraic group over k . Our terminology is that of [SGA3]: *reductive* implies *smooth, and connected* and an *algebraic group* will mean an affine group scheme of finite type over k . Fix a maximal torus T_G and a Borel subgroup containing T_G which determines a root system R and a set of positive roots R^+ . Let $\langle \cdot, \cdot \rangle$ be the natural pairing between the characters and co-characters and for each root α , and let α^\vee be the corresponding coroot.

If the root system R associated to G is irreducible, the Coxeter number h_G of R and of G , admits the following equivalent descriptions:

- (1) It is the order of the Coxeter elements of the Weyl group W . This shows that R and the dual root system R^\vee have the same Coxeter number.
- (2) Let α_0 be the highest root and $\sum n_i \alpha_i$ its expression as a linear combination of the simple roots. One has:

$$h_G = 1 + \sum n_i \tag{2.1.1}$$

- (3) Applying this to the dual root system, one gets $h_G = \langle \rho, \beta^\vee \rangle + 1$ where ρ is half the sum of positive roots and β^\vee is the highest coroot. Indeed, ρ is also the sum of the fundamental weights ω_i , and $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

For a general reductive group G , define the Coxeter number h_G to be the largest among the ones for the irreducible components of R .

It follows from (2) above that if G is a reductive group, U the unipotent radical of a Borel subgroup, and $\mathfrak{u} := \text{Lie}(U)$, the descending central

series of u , defined by $Z^1 u = u$ and $Z^i u = [u, Z^{i-1} u]$, satisfies

$$Z^{\mathfrak{h}_G}(u) = 0 \quad (2.1.2)$$

For the group U , similarly $Z^{\mathfrak{h}_G}(U) = (1)$.

The Lie algebra \mathfrak{g} of G is a p -Lie algebra. If \mathfrak{n} in \mathfrak{g} is nilpotent and if $p \geq \mathfrak{h}_G$, then $\mathfrak{n}^p = 0$. To check this, we may assume that \mathfrak{n} is in the Lie algebra \mathfrak{u} of the unipotent radical U of a Borel subgroup. For each positive root α , let X_α be a basis for the root subspace \mathfrak{g}_α . Express \mathfrak{n} as $\sum_{\alpha \in R^+} a_\alpha \cdot X_\alpha$. Each X_α is the infinitesimal generator of an additive group. It follows that $X_\alpha^p = 0$ for each α . Observe that

$$\mathfrak{n}^p = \sum_{\alpha \in R^+} a_\alpha^p \cdot X_\alpha^p \pmod{Z^p \mathfrak{u}} \quad (2.1.3)$$

and $Z^p \mathfrak{u} = 0$, as $p \geq \mathfrak{h}_G$. (see [Mc 1, page 10]).

2.2. Let \mathfrak{g}_{nilp} (resp. G^u) be the reduced subscheme of $\text{Lie}(G)$ (resp. G) with points the nilpotent (resp. unipotent) elements. Let U be the unipotent radical of a Borel subgroup. For $p \geq \mathfrak{h}_G$, the Campbell-Hausdorff group law \circ makes sense in characteristic p and turns $\mathfrak{u} := \text{Lie}(U)$ into an algebraic group over k . This is so since $Z^p \mathfrak{u} = 0$. Further, there is a unique isomorphism

$$\exp : (\text{Lie}(U), \circ) \xrightarrow{\sim} U \quad (2.2.1)$$

equivariant for the action of B and whose differential at the origin is the identity. If in addition the simply connected covering of the derived group of G is an étale covering (which is the case for $p > \mathfrak{h}_G$, and could fail when $p = \mathfrak{h}_G$ due to the presence of $\text{SL}(p)$ as factors in the covering), then there is a unique G -equivariant isomorphism ([S2, Theorem 3, page 21]):

$$\exp : \mathfrak{g}_{nilp} \rightarrow G^u \quad (2.2.2)$$

which induces (2.2.1) on each unipotent radical of a Borel subgroup. Let $\log : G^u \rightarrow \mathfrak{g}_{nilp}$ denote its inverse.

For u a unipotent element of $G(k)$, one defines the “ t -power map” $t \mapsto u^t$, from \mathbb{G}_a to G , by

$$t \mapsto \exp(t \log u). \quad (2.2.3)$$

For $G = \text{GL}(V)$ such a map $t \mapsto u^t$ is more generally defined for any u in G such that $u^p = 1$. It is given by the truncated binomial expression ([S1, 4.1.1, page 524]):

$$t \mapsto u^t := \sum_{i < p} \binom{t}{i} (u - 1)^i. \quad (2.2.4)$$

Similarly, X in $\text{End}(V)$ such that $X^p = 0$ defines a morphism $t \mapsto \exp(tX)$ from \mathbb{G}_a to $\text{GL}(V)$ given by the truncated exponential series:

$$t \mapsto I + tX + \frac{(tX)^2}{2!} + \dots + \frac{(tX)^{p-1}}{(p-1)!}. \quad (2.2.5)$$

Until the end of §3, we make the following assumptions on the reductive group G .

ASSUMPTION 2.3. Let \tilde{G} be the simply connected covering of the derived group G' . We assume that $p \geq h_G$ and that the map $\tilde{G} \rightarrow G'$ is étale.

In particular, by 2.1 and 2.2, the exponential map (2.2.2) is defined, every unipotent element in $G(k)$ is of order p , and every nilpotent in $\text{Lie}(G)$ is a p -nilpotent. One can then define the notions of saturation and infinitesimal saturation of subgroup schemes $\mathfrak{G} \subset G$ as follows (see Remark 2.19 for the case when $G = \text{PGL}(p)$).

DEFINITION 2.4. ([S1, §4], [D, Definition 1.5])

- (1) A subgroup scheme $\mathfrak{G} \subset G$ is called saturated if for every u in $\mathfrak{G}(k)$ which is unipotent, the homomorphism $t \mapsto u^t$ (2.2.3) from \mathbb{G}_a to G factors through \mathfrak{G} .
- (2) A subgroup scheme $\mathfrak{G} \subset G$ is called infinitesimally saturated if for every nilpotent X in $\text{Lie}(\mathfrak{G})$, the morphism $t \mapsto \exp(tX)$ (2.2.2) from \mathbb{G}_a to G factors through \mathfrak{G} .
- (3) \mathfrak{G} is doubly saturated if it is saturated and infinitesimally saturated.

An element of $\text{Lie}(\mathfrak{G})$ is nilpotent if and only if it is nilpotent as an element of $\text{Lie}(G)$. The reference to the exponential map (2.2.2) in (2) therefore makes sense. One way to see it is to observe that the inclusion of $\text{Lie}(\mathfrak{G})$ in $\text{Lie}(G)$ is a morphism of p -Lie algebras and that X is nilpotent if and only if it is killed by an iterated p -power map, i.e. $X^{p^n} = 0$.

Let \mathfrak{G}^0 be the identity component of \mathfrak{G} and \mathfrak{G}_{red}^0 the reduced subscheme of \mathfrak{G}^0 .

THEOREM 2.5. Let $\mathfrak{G} \subset G$ be a k -subgroup scheme which is infinitesimally saturated. Assume that if $p = h_G$, \mathfrak{G}_{red}^0 is reductive. Then

- (1) The group \mathfrak{G}_{red}^0 and its unipotent radical $R_u(\mathfrak{G}_{red}^0)$ are normal subgroup schemes of \mathfrak{G} and the quotient group scheme $\mathfrak{G}^0 / \mathfrak{G}_{red}^0$ is of multiplicative type.
- (2) If \mathfrak{G}_{red}^0 is reductive, there exists a central, connected subgroup scheme of multiplicative type $M \subset \mathfrak{G}^0$ such that the morphism $M \times \mathfrak{G}_{red}^0 \rightarrow \mathfrak{G}^0$ realizes \mathfrak{G}^0 as a quotient of $M \times \mathfrak{G}_{red}^0$.

The proof of part (2) of 2.5 will occupy most section 2, until 2.16. Part (1) will be proven in section 3.

By [D, Lemma 2.3], the conclusions of 2.5 hold for \mathfrak{G} if and only if they hold for the identity component \mathfrak{G}^o . Until the end of section 3, we will assume that \mathfrak{G} is connected.

LEMMA 2.6. *If \mathfrak{G} is an infinitesimally saturated subgroup scheme of G , every nilpotent element \mathfrak{n} of $\text{Lie}(\mathfrak{G})$ is in $\text{Lie}(\mathfrak{G}_{red})$.*

Proof. As \mathbb{G}_a is reduced, the morphism $t \mapsto \exp(t\mathfrak{n})$ maps \mathbb{G}_a to $\mathfrak{G}_{red} \subset G$. The image of 1 in $\text{Lie}(\mathbb{G}_a)$ is \mathfrak{n} . \square

Part (2) of 2.5 is a corollary of 2.6 and of the following theorem, which does not refer to G anymore.

THEOREM 2.7. *Let \mathfrak{G} be a connected algebraic group such that*

- (a) \mathfrak{G}_{red} is reductive
- (b) any nilpotent element of $\text{Lie}(\mathfrak{G})$ is in $\text{Lie}(\mathfrak{G}_{red})$.

Then, the conclusion of 2.5(2) hold. As a consequence, \mathfrak{G}_{red} is a normal subgroup scheme of \mathfrak{G} and $\mathfrak{G}/\mathfrak{G}_{red}$ is of multiplicative type.

Let T be a maximal torus of \mathfrak{G}_{red} , and let H be the centralizer of T in \mathfrak{G} . One has $H \cap \mathfrak{G}_{red} = T$. It follows from (b) that any nilpotent element \mathfrak{n} of $\text{Lie}(H)$ is in $\text{Lie}(T)$, hence vanishes. By the following lemma, H is of multiplicative type, and in particular commutative.

LEMMA 2.8. *Let H be a connected algebraic group over k . If all the elements of $\text{Lie}(H)$ are semi-simple, then H is of multiplicative type.*

Proof. (see also ([DG, IV, §3, Lemma 3.7])) $\text{Lie}(H)$ is commutative: Fix x in $\text{Lie}(H)$, and let us show that it is central in $\text{Lie}(H)$. As adx is semi-simple, it suffices to show that if y is in an eigenspace of adx , i.e. $[x, y] = \lambda y$, then x and y commute, i.e. $\lambda = 0$. Let W be the vector subspace of $\text{Lie}(H)$ generated by the $y^{[p^\ell]}$ ($\ell \geq 0$). The $y^{[p^\ell]}$ commute. The map $z \mapsto z^p$ therefore induces a p -linear map from W to itself, injective by assumption. It follows that W has a basis $e_i, (1 \leq i \leq N)$ consisting of elements such that $e_i^p = e_i$, and $(\sum a^i e_i)^p = \sum (a^i)^p e_i$.

LEMMA 2.9. *For $b = (b_i)$ in k^N , define $b^p := (b_i^p)$. Then, any b in k^N is a linear combination of the $b^{[p^\ell]}$ for $\ell \geq 1$.*

Proof. The $b^{[p^a]}$ ($a \geq 0$) are linearly dependent. A linear dependence relation can be written

$$\sum_{j \geq m} c_j b^{p^j} = 0 \tag{2.9.1}$$

with $c_m \neq 0$. Extracting p^m -roots, we get

$$\left(\sum_{j \geq 0} d_j b^{p^j}\right) = 0 \quad (2.9.2)$$

where $d_j = c_{m+j}^{1/p^m}$. In particular, $d_0 \neq 0$, proving the lemma. \square

End of proof of commutativity From the lemma above, y is a linear combination of the $y^{|p^\ell|}$ ($\ell > 0$). The bracket $[y^{|p^\ell|}, x]$ vanishes for ($\ell > 0$). Indeed, it is $(ady)^{|p^\ell|}(x)$ which vanishes because $[y, [y, x]] = [y, -\lambda y] = 0$. It follows that $[y, x]$ vanishes too.

The p -Lie algebra $\text{Lie}(H)$ determines the kernel of the Frobenius morphism $F : H \rightarrow H^{(p)}$, where $H^{(p)}$ is obtained from H by extension of scalars $\lambda \mapsto \lambda^p$, $k \rightarrow k$. As $\text{Lie}(H)$ is commutative semi-simple, this kernel of the Frobenius morphism is of multiplicative type. The same holds for $H^{(p)}$, which is obtained from H using an automorphism of k . The same holds for each $H^{(p^\ell)}$.

For any n , the kernel K_n of the iterated Frobenius map $F^n : H \rightarrow H^{(p^\ell)}$ is an iterated extension of subgroups of the kernel of the Frobenius of the $H^{(p^i)}$, $i < n$. It is hence of multiplicative type, being an iterated extension of connected groups of multiplicative type ([SGA3, XVII, 7.1.1]).

The K_n form an increasing sequence. By the proof of [D, Proposition 1.1], there exist subgroup of multiplicative type M of H containing all K_n . As H is connected and as M contains all infinitesimal neighbourhoods of the identity element, one has $M = H$. \square

Whenever a group M of multiplicative type acts on a group K , its action on $\text{Lie}(K)$ defines a weight decomposition:

$$\text{Lie}(K) = \bigoplus_{\beta \in X(M)} \text{Lie}(K)^\beta \quad (2.9.3)$$

If $v \in \text{Lie}(K)^\beta$, then v^p is in $\text{Lie}(K)^{p\beta}$. Indeed, after any extension of scalars R/k , if m is in $M(R)$,

$$m(v^p) = (m(v))^p = (\beta(m)v)^p = \beta(m)^p v^p \quad (2.9.4)$$

LEMMA 2.10. *With the above notations, if β is not torsion, $\text{Lie}(K)^\beta$ consists of nilpotent elements.*

Proof. Indeed, if β is not torsion, the $p^\ell \beta$ are all distinct and $\text{Lie}(K)^{p^\ell \beta}$ must vanish for $\ell \gg 0$. It follows that the elements of $\text{Lie}(K)^\beta$ are nilpotent. \square

Let us apply this to the action of T on \mathfrak{G} by inner automorphisms.

LEMMA 2.11. *If β in $X(T)$ is not zero, the weight spaces $\text{Lie}(\mathfrak{G})^\beta$ equals $\text{Lie}(\mathfrak{G}_{red})^\beta$.*

Proof. Indeed, $\text{Lie}(\mathfrak{G})^\beta$ consists of nilpotent elements. By our assumption 2.7(b), it is contained in $\text{Lie}(\mathfrak{G}_{red})$. \square

Let B a Borel subgroup of \mathfrak{G}_{red} containing T , and let U be its unipotent radical.

LEMMA 2.12. *Under the assumption of 2.7, H normalizes U .*

Proof. Let C in $X(T) \otimes \mathbb{R}$ be the cone generated by the positive roots, relative to B and define

$$C^* := C \setminus \{0\} \tag{2.12.1}$$

As in 2.11, we let T act on \mathfrak{G} by conjugation (t acts by $g \mapsto t g t^{-1}$). This action induces actions on $\mathfrak{g} := \text{Lie}(\mathfrak{G})$, the affine algebra A of \mathfrak{G} , its augmentation ideal \mathfrak{m} (defining the unit element), and the dual $\mathfrak{g}^\vee = \mathfrak{m}/\mathfrak{m}^2$ of \mathfrak{g} . Similarly T acts on U , its Lie algebra \mathfrak{u} , its affine algebra A_U and its augmentation ideal \mathfrak{m}_U . For the action on affine algebras, t in T transforms $f(g)$ into $f(t^{-1} g t)$. From these actions, we get $X(T)$ -gradings. By 2.7(b)

$$\mathfrak{u} = \bigoplus_{\beta \in C^*} \mathfrak{g}^\beta. \tag{2.12.2}$$

It follows that $\mathfrak{u}^\vee = \mathfrak{m}_U/\mathfrak{m}_U^2$ is the sum of the $(\mathfrak{g}^\vee)^\beta$, for β in the negative $-C^*$ of C^* . For $n > 0$, the weights by which T acts on $\mathfrak{m}_U^n/\mathfrak{m}_U^{n+1}$ are in $-C^*$. As U is connected, the intersection of the \mathfrak{m}_U^n is reduced to 0, and the weights by which T acts on \mathfrak{m}_U are also in $-C^*$.

Let I be the ideal of A generated by the graded components \mathfrak{m}^β of \mathfrak{m} for β not in $-C^*$, and put $A_1 := A/I$. The image in \mathfrak{m}_U of a \mathfrak{m}^β as above is contained in \mathfrak{m}_U^β , hence vanishes. It follows that U is contained in the closed subscheme $U_1 = \text{Spec } A_1$ of \mathfrak{G} defined by I . As the graded component defined \mathfrak{m}^0 of \mathfrak{m} is contained in I , the graded component A_1^0 of A_1 (the T -invariants) is reduced to the constants. As T is connected it follows that U_1 is connected. As the image of \mathfrak{m}_U^β in $\mathfrak{m}/\mathfrak{m}^2$ is $(\mathfrak{m}/\mathfrak{m}^2)^\beta$, the image of I in $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}^\vee$ is the orthogonal of $\mathfrak{u} \subset \mathfrak{g}$ and the tangent space at the origin of U_1 is \mathfrak{u} .

Claim: The subscheme U_1 of \mathfrak{G} is a subgroup scheme, i.e. the coproduct $\Delta : A \rightarrow A \otimes A$, $f(g) \mapsto f(gh)$ maps I to $I \otimes A \oplus A \otimes I$. Indeed, Δ respects

the gradings; for $k \subset A$ the constants, one has $A = k \oplus \mathfrak{m}$, and Δ maps \mathfrak{m} to $(k \otimes \mathfrak{m}) \oplus (\mathfrak{m} \otimes k) \oplus (\mathfrak{m} \otimes \mathfrak{m})$, and hence \mathfrak{m}^β maps to

$$(k \otimes \mathfrak{m}^\beta) \oplus (\mathfrak{m}^\beta \otimes k) \oplus \sum_{\beta=\beta'+\beta''} \mathfrak{m}^{\beta'} \otimes \mathfrak{m}^{\beta''}. \quad (2.12.3)$$

As $-C^*$ is stable by addition, if $\beta = \beta' + \beta''$ and that β is not in $-C^*$, one of β' or β'' is not in $-C^*$, and the corresponding $\mathfrak{m}^{\beta'}$ or $\mathfrak{m}^{\beta''}$ is contained in I . The claim follows.

To summarize, U_1 is connected, and the inclusion $U \subset U_1$ induces an isomorphism $\mathrm{Lie}(U) \xrightarrow{\sim} \mathrm{Lie}(U_1)$. As U is smooth, this implies that $U = U_1$.

Since H centralizes T , the ideal I is stable by H , meaning that H normalizes U . □

COROLLARY 2.13. *Under the assumption of 2.7, H normalizes \mathfrak{G}_{red} .*

Proof. Let B^- be the Borel subgroup of \mathfrak{G}_{red} containing T and opposite to B , and let U^- be its unipotent radical. As U^- , T and U are normalized by H , the big cell $U^-TU \subset \mathfrak{G}_{red}$ is stable by the conjugation action of H , and so is its schematic closure \mathfrak{G}_{red} . □

In what follows, we identify schemes with the corresponding fppf sheaves. A quotient such as \mathfrak{G}/H represents the quotient of the sheaf of groups \mathfrak{G} by the subsheaf H : \mathfrak{G} is a H -torsor over \mathfrak{G}/H .

LEMMA 2.14. *The morphism of schemes*

$$\mathfrak{G}_{red}/T \rightarrow \mathfrak{G}/H \quad (2.14.1)$$

is an isomorphism.

Proof. As T is the intersection of \mathfrak{G}_{red} and H , (2.14.1), as a morphism of fppf sheaves, is injective. Testing on $\mathrm{Spec}(k)$ and $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$, one sees that it is bijective on points and injective on tangent space at each point. It is hence radicial and unramified, hence on some open set of \mathfrak{G}_{red}/T an immersion. The \mathfrak{G}_{red} homogeneity then shows that it is a closed embedding.

The tangent space at the origin of \mathfrak{G}_{red}/T is $\mathrm{Lie}(\mathfrak{G}_{red})/\mathrm{Lie}(T)$. For \mathfrak{G}/H if J is the ideal defining H , as \mathfrak{G}_{red} is a H -torsor on \mathfrak{G}_{red}/H , if η is the ideal defining the origin in \mathfrak{G}_{red}/H , the pull-back of η/η^2 to \mathfrak{G}_{red}/H is J/J^2 , and η/η^2 is the fiber of J/J^2 at the origin e . By [D, 1.15] applied to T acting on \mathfrak{G} by conjugation, $(J/J^2)_e \xrightarrow{\sim} \bigoplus_{\beta \neq 0} (\mathfrak{m}/\mathfrak{m}^2)^\beta = (\mathrm{Lie}(\mathfrak{G}_{red})/\mathrm{Lie}(T))^\vee$. Thus, the closed embedding (2.14.1) is an isomorphism near the origin, hence everywhere by homogeneity, proving 2.14.

□

COROLLARY 2.15. *Under the assumptions of 2.7, \mathfrak{G}_{red} is normal in \mathfrak{G} , and there exists in \mathfrak{G} a central connected subgroup scheme of multiplicative type M such that the morphism $M \times \mathfrak{G}_{red} \rightarrow \mathfrak{G}$ realizes \mathfrak{G} as a quotient of $M \times \mathfrak{G}_{red}$.*

Proof. By 2.14, the product map $\mathfrak{G}_{red} \times H \rightarrow \mathfrak{G}$ is onto, as a morphism of sheaves. As both \mathfrak{G}_{red} and H normalize $\mathfrak{G}_{red} \subset \mathfrak{G}$, so does \mathfrak{G} .

To complete the proof of 2.15 (and thereby of 2.7), we follow [D, §2.25]. Let M be the subgroup of H which centralizes \mathfrak{G}_{red} . Since \mathfrak{G}_{red} is reductive, the group scheme $\text{Aut}_T(\mathfrak{G}_{red})$ of automorphisms which preserve T is precisely T^{ad} , the image of T in the adjoint group. Hence the conjugation action of H on \mathfrak{G}_{red} gives the exact sequence:

$$1 \rightarrow M \rightarrow H \rightarrow T^{ad} \rightarrow 1 \quad (2.15.1)$$

and T surjects onto T^{ad} implying that M and T generate H . Since M is generated by $M_{red} \subset \mathfrak{G}_{red}$ and M^0 , and since H and \mathfrak{G}_{red} generate \mathfrak{G} we see that M^0 and \mathfrak{G}_{red} generate \mathfrak{G} . Moreover, M is central. Thus

$$M^0 \times \mathfrak{G}_{red} \rightarrow \mathfrak{G} \quad (2.15.2)$$

is an epimorphism. This concludes the proof of 2.7 and in particular, 2.5(2). □

LEMMA 2.16. *Suppose that H is a maximal connected subgroup scheme of multiplicative type of an algebraic group \mathfrak{G} . Let $Z_{\mathfrak{G}}^0(H)$ be the identity component of the centralizer of H in \mathfrak{G} and define $U := Z_{\mathfrak{G}}^0(H)/H$. Then the sequence:*

$$0 \rightarrow \text{Lie}(H) \rightarrow \text{Lie}(Z_{\mathfrak{G}}^0(H)) \rightarrow \text{Lie}(U) \rightarrow 0 \quad (2.16.1)$$

associated to the central extension

$$1 \rightarrow H \rightarrow Z_{\mathfrak{G}}^0(H) \rightarrow U \rightarrow 1. \quad (2.16.2)$$

is exact.

Proof. Left exactness of (2.16.1) is clear. The maximality of H implies that U is unipotent (see [D, §2.5, page 590]).

Embed H in \mathbb{G}_m^r as a subgroup scheme. The quotient $H'' = \mathbb{G}_m^r/H$ is a torus, being a quotient of one. The central extension (2.16.2), by a push forward, gives a central extension ([SGA3, Exposé XVII, Lemma 6.2.4])

$$1 \rightarrow \mathbb{G}_m^r \rightarrow E \rightarrow U \rightarrow 1 \quad (2.16.3)$$

and a diagram of groups:

$$\begin{array}{ccccccc}
& & 1 & & 1 & & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H & \longrightarrow & Z_{\mathfrak{G}}^0(H) & \longrightarrow & U \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow = \\
1 & \longrightarrow & \mathbb{G}_m^r & \longrightarrow & E & \longrightarrow & U \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H'' & \xrightarrow{=} & H'' & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \\
& & 1 & & 1 & &
\end{array} \tag{2.16.4}$$

Since H'' is multiplicative every nilpotent in $\text{Lie}(E)$ maps to 0 in $\text{Lie}(H'')$ and hence comes from a nilpotent in $\text{Lie}(Z_{\mathfrak{G}}^0(H))$.

Since \mathbb{G}_m^r is smooth, by [SGA3, Exposé VII, Proposition 8.2] the sequence (2.16.3) gives an exact sequence

$$0 \rightarrow \text{Lie}(\mathbb{G}_m^r) \rightarrow \text{Lie}(E) \rightarrow \text{Lie}(U) \rightarrow 0 \tag{2.16.5}$$

Since U is unipotent, any element z in $\text{Lie}(U)$ is nilpotent. Let z' in $\text{Lie}(E)$ be a lift of z . The Jordan decomposition makes sense for any p -Lie algebra over a perfect field k and uses only the p -power map (see for example [W2, Corollary 4.5.9, page 135]). Thus, by using the Jordan decomposition of the lift z' in $\text{Lie}(E)$ and noting that the semi-simple part gets mapped to zero in $\text{Lie}(U)$, we can assume that z' can also be chosen to be nilpotent.

Since every nilpotent in $\text{Lie}(E)$ comes from a nilpotent in $\text{Lie}(Z_{\mathfrak{G}}^0(H))$, we conclude that z gets lifted to a nilpotent in $\text{Lie}(Z_{\mathfrak{G}}^0(H))$. This implies that (2.16.1) is also right exact. \square

LEMMA 2.17. *Suppose that \mathfrak{G} is a subgroup scheme of G which is infinitesimally saturated in G . Then the subgroup scheme $Z_{\mathfrak{G}}^0(H)$ is infinitesimally saturated in G ; in particular, every non-zero nilpotent in $\text{Lie}(Z_{\mathfrak{G}}^0(H))$ lies in $\text{Lie}(Z_{\mathfrak{G}}^0(H)_{red})$.*

Proof. If \mathfrak{n} in $\text{Lie}(Z_{\mathfrak{G}}^0(H))$ is nilpotent, the map $\rho : t \mapsto \exp(t\mathfrak{n}) : \mathbb{G}_a \rightarrow G$ factors through \mathfrak{G} . Since H is central in $Z_{\mathfrak{G}}^0(H)$, the action of H by inner automorphisms fixes \mathfrak{n} . Since the map “exp” is compatible with conjugation the entire curve ρ is fixed by H . Therefore, ρ factors through $Z_{\mathfrak{G}}^0(H)$ and hence through $Z_{\mathfrak{G}}^0(H)_{red}$, since \mathbb{G}_a is reduced and

connected, and hence $\mathfrak{n} \in \text{Lie}(Z_{\mathfrak{G}}^0(H)_{red})$. This completes the proof of the lemma. \square

We have the following extension of ([D, Lemma 2.7, Corollary 2.11]).

LEMMA 2.18. *Let \mathfrak{G} be as in 2.17. Let $H \subset \mathfrak{G}$ be a maximal connected subgroup scheme of multiplicative type. Then the central extension (2.16.2) splits and U is a smooth unipotent group.*

Proof. We claim that U is smooth. By Lemma 2.16, every element z in $\text{Lie}(U)$ comes from a nilpotent in \mathfrak{n} in $\text{Lie}(Z_{\mathfrak{G}}^0(H))$. By 2.17, \mathfrak{n} is in $\text{Lie}(Z_{\mathfrak{G}}^0(H)_{red})$. Its image z is hence in $\text{Lie}(U_{red})$. Thus, $\text{Lie}(U) = \text{Lie}(U_{red})$ proving the claim.

Now we have a central extension (2.16.2) with the added feature that U is smooth. By [SGA3, Exposé XVII, Theorem 6.1.1] it follows that (2.16.2) splits (uniquely) and

$$Z_{\mathfrak{G}}^0(H) = H \times U. \quad (2.18.1)$$

\square

Remark 2.19. Recall that in 2.2, for the existence of an exponential map (2.2.2) we assumed that the covering morphism $\tilde{G} \rightarrow G'$ is étale, which therefore became a part of the standing assumption (2.3). The case which gets excluded is when the simply connected cover of the derived group has factors of $\text{SL}(p)$. For instance, when G' is simple with $p = \mathfrak{h}_G$, the only case excluded is $G' = \text{PGL}(p)$.

Let G be a reductive group for which $p = \mathfrak{h}_G$. The tables of Coxeter numbers of simple groups show that except for type A , where $\mathfrak{h}_{\text{SL}(n)} = n$, Coxeter numbers are even greater than 2. It follows that \tilde{G} is a product of $\text{SL}(p)$'s and other simple factors with p larger than their Coxeter numbers. Even when the morphism $\tilde{G} \rightarrow G'$ is *not étale*, we can still define the notions of *saturation* (resp *infinitesimal saturation*) of subgroup schemes $\mathfrak{G} \subset G$ as follows.

Say $\mathfrak{G} \subset G$ is *saturated* (resp *infinitesimally saturated*) if the inverse image of \mathfrak{G} in \tilde{G} is saturated (resp infinitesimally saturated). With this definition, Theorem 2.5 remains true for $p \geq \mathfrak{h}_G$.

This notion of saturation (resp infinitesimal saturation) can also be seen in terms of suitably defined t -power maps and exponential maps. We restrict ourselves to the case when $G' = \text{PGL}(p)$. At the level of Lie algebras, the induced morphism:

$$\mathfrak{sl}(p) \rightarrow \mathfrak{pgl}(p) \quad (2.19.1)$$

is a radicial map on the locus of nilpotent elements. If $A \in \mathfrak{gl}(p)$ is a matrix representing an element of $\mathfrak{pgl}(p)$, it is nilpotent if all but

the constant coefficients of the characteristic polynomial vanishes, i.e. $Tr(\wedge^i(A)) = 0, \forall 1 \leq i < p$ and the characteristic polynomial reduces to $T^p - \det(A)$. This condition is stable under $A \mapsto A + \lambda.I$ as $(T - \lambda)^p - \det(A) = T^p - (\det(A) + \lambda^p) = T^p - \det(A + \lambda.I)$.

We get the unique lift $\tilde{A} \in \mathfrak{sl}(p)$ by taking $\tilde{A} := A - \det(A)^{1/p}.I$. Now one has $Tr(\wedge^i(\tilde{A})) = 0$ for $1 \leq i \leq p$ and hence we can define the exponential morphism $\mathbb{G}_a \rightarrow \mathrm{PGL}(p)$ as:

$$t \mapsto \exp(tA) := \exp(t\tilde{A}). \quad (2.19.2)$$

Likewise, in the case of unipotents in the group $\mathrm{PGL}(p)$, the restriction of this map to the locus of unipotent elements:

$$\mathrm{SL}(p)^u \rightarrow \mathrm{PGL}(p)^u \quad (2.19.3)$$

is radicial. Thus, any unipotent $u \in \mathrm{PGL}(p)$ has a unique unipotent lift $\tilde{u} \in \mathrm{SL}(p)$ and one can define the t -power map $\mathbb{G}_a \rightarrow \mathrm{PGL}(p)$ as:

$$t \mapsto u^t := \mathrm{image}(\tilde{u}^t) \quad (2.19.4)$$

With the notions of t -power map and exponential morphisms in place, we can define the notions of *saturation* (resp *infinitesimal saturation*) of subgroup schemes $\mathfrak{G} \subset G'$ exactly as in Definition 2.4 using (2.19.4) (resp (2.19.2)) and these coincide with the definitions made above.

We note however that these ‘‘punctual’’ maps, i.e. defined for each nilpotent A (resp each unipotent u), are not induced by a morphism from $\mathfrak{pgl}(p)_{\mathrm{nilp}}$ to $\mathrm{PGL}(p)^u$ (resp. $\mathbb{A}^1 \times \mathrm{PGL}(p)^u \rightarrow \mathrm{PGL}(p)^u : (t, u) \mapsto u^t$ is not a morphism).

3. COMPLETION OF PROOF OF THE STRUCTURE THEOREM 2.5

We begin by stating a general result on root systems whose proof is given in the appendix.

PROPOSITION 3.1. *Let R be an irreducible root system with Coxeter number h and let X be the lattice spanned by R . Let $\phi : X \rightarrow \mathbb{R}/\mathbb{Z}$ be a homomorphism. Then there exists a basis B for R such that if $\alpha \in R$ satisfies $\phi(\alpha) \in (0, 1/h) \bmod \mathbb{Z}$, then α is positive with respect to B .*

3.2. Assumption 2.3 on the reductive group G continues to be in force. Let \mathfrak{G} be a subgroup scheme of G , and H a maximal connected subgroup scheme of multiplicative type of \mathfrak{G} . For the existence of H see [D, Proposition 2.1].

Let $X(H)$ be the group of characters of H . The action of H on $\mathrm{Lie}(\mathfrak{G})$ by conjugation gives an $X(H)$ -gradation $\mathrm{Lie}(\mathfrak{G}) = \mathrm{Lie}(Z_{\mathfrak{G}}^0(H)) \oplus \bigoplus_{\alpha \neq 0} \mathrm{Lie}(\mathfrak{G})^\alpha$.

COROLLARY 3.3. (cf. [D, Lemma 2.12]) *Let M be a connected subgroup scheme of multiplicative type of G and suppose that $p > h_G$. The action of M on $\mathrm{Lie}(G)$ by conjugation gives an $X(M)$ -gradation $\mathrm{Lie}(G) = \bigoplus_{\alpha \in X(M)} \mathrm{Lie}(G)^\alpha$. If $\alpha \neq 0$, then for $\gamma \in \mathrm{Lie}(G)^\alpha$ one has $\gamma^p = 0$. In particular, if $M = H$ and $\gamma \in \mathrm{Lie}(\mathfrak{G})^\alpha$ we have $\gamma^p = 0$.*

Proof. In a smooth algebraic group, the maximal connected subgroups of multiplicative type are (maximal) tori. Indeed, if T is such a maximal subgroup, its centralizer $Z_G(T)$ is smooth, being the fixed locus of a linearly reductive group acting on a smooth variety (see for example [DG, Theorem 2.8, Chapter II, §5]). Define $U := Z_G(T)^\circ / T$. The group scheme U is smooth, as a quotient of $Z_G(T)^\circ$ which is smooth, and is unipotent by maximality of T . By [SGA3, Exposé XVII, Theorem 6.1.1], we have a splitting $Z_G(T)^\circ \simeq T \times U$, and hence T is smooth.

In our case M is hence contained in a maximal torus T of G . By the assumption 2.3 on G , it suffices to prove that γ is nilpotent. Suppose first that M is isomorphic to μ_p . In that case, the non-trivial character α of M induces an isomorphism from M to μ_p . Let $\phi: X(T) \rightarrow X(M) \stackrel{\alpha}{\cong} X(\mu_p) = \mathbb{Z}/p$ be induced by the inclusion of M into T . Then, the M -weight space $\mathrm{Lie}(G)^\alpha$, is the sum of T -weight spaces $\mathrm{Lie}(G)^\beta$ for β a root such that $\phi(\beta) = 1$. By 3.1, applied to $\frac{1}{p}\phi$, with values in $\frac{1}{p}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$, this sum is contained in the Lie algebra of the unipotent radical of a Borel subgroup of G . In particular, it consists of nilpotent elements.

For the general case, if M is trivial, the claim is empty. If M is not trivial, pick $A \subset M$ isomorphic to μ_p . If the restriction of α to A is non-trivial, one applies 3.3 to the already proven case of A and the restriction of α to A . If the restriction of α to A is trivial, one considers $M/A \subset Z_G(\mu_p)^\circ / \mu_p$ and repeat the argument. \square

3.4. The proof of 2.5 now follows [D, page 594-599] verbatim with a sole alteration; recall that in [D] the group G was the linear group $\mathrm{GL}(V)$ and the condition on the characteristic was $p > \dim(V)$. For an arbitrary connected reductive G , this condition now gets replaced by $p > h_G$, which makes 3.3 applicable.

Remark 3.5. If $G = \mathrm{GL}(V)$ one has $h_G = \dim(V)$. In the case $G = \prod \mathrm{GL}(V_i)$ with for each i , $p > \dim(V_i)$, the case $p > h_G$ of Theorem 2.5 gives us [D, Theorem 1.7].

Example 3.6. (Brian Conrad) Here is an example in any characteristic $p > 0$, of a connected group of multiplicative type M acting on a reductive group G , and of a non-trivial character α of M , such that the weight space $\mathrm{Lie}(G)^\alpha$ contains elements which are not nilpotent. We

take $G = \mathrm{SL}_p$ (so that $\mathfrak{h}_G = \mathfrak{p}$) and $M = \mu_{p^2}$, and for α the character $\zeta \mapsto \zeta^p$.

We embed M in the maximal torus of diagonal matrices of SL_p by

$$\zeta \mapsto \mathrm{diag}(\zeta^0, \zeta^{-p}, \zeta^{-2p}, \dots, \zeta^{-(p-1)p})$$

The restriction to M of each simple root and of the lowest root is the character $\alpha : \zeta \mapsto \zeta^p$, and $\mathrm{Lie}(G)^\alpha$ is the sum of the corresponding root spaces. In the standard visualization of SL_p this weight space inside $\mathrm{Lie}(G) = \mathfrak{sl}_p$ is the span of the super-diagonal entries and the lower-left entry.

A sum of nonzero elements in those root lines contributing to $\mathrm{Lie}(G)^\alpha$ is a $p \times p$ -matrix $X \in \mathfrak{sl}_p$ which satisfies

$$X(e_1) = t_p e_p, X(e_2) = t_1 e_1, \dots, X(e_p) = t_{p-1} e_{p-1} \quad (3.6.1)$$

Iterating p times gives $X^p = \mathrm{diag}(t, \dots, t)$, with $t := \prod t_j \neq 0$. Hence, $X \in \mathrm{Lie}(G)^\alpha$ is not nilpotent.

Example 3.7. A variant of the above example leads to an example of an infinitesimally saturated group scheme $\mathfrak{G} \subset \mathrm{SL}(V)$ with $\dim(V) = p$ and such that V is an irreducible representation and such that \mathfrak{G}_{red} is a unipotent group. This in particular implies that \mathfrak{G}_{red} is not normal in \mathfrak{G} .

Let V be the affine algebra of μ_p , that is

$$V = \mathcal{O}(\mu_p) := \frac{k[u]}{(u^p - 1)} \quad (3.7.1)$$

The vector space V admits the basis $\{u^i \mid i \in \mathbb{Z}/p\}$.

The multiplicative group $\mathcal{O}(\mu_p)^*$ acts by multiplication on V . For $f \in \mathcal{O}(\mu_p)^*$, f^p is constant. Define Nf to be the constant value of f^p . It is in fact the norm of f . The action of $\mathcal{O}(\mu_p)^*$ on V induces an action of the subgroup $N \subset \mathcal{O}(\mu_p)^*$ for which $Nf = 1$.

On V we have also the action of μ_p by translations and this action normalizes the group $\mathcal{O}(\mu_p)^*$ and its subgroup N . Consider the group scheme:

$$\mathfrak{G} := \mu_p \times N \quad (3.7.2)$$

We make a few observations on \mathfrak{G} :

- (1) \mathfrak{G} is infinitesimally saturated;
- (2) \mathfrak{G} contains the group generated by the first factor μ_p and $\{u^i \mid i \in \mathbb{Z}/p\} \simeq \mathbb{Z}/p$. This group is a Heisenberg type central extension of $\mu_p \times \mathbb{Z}/p$ by μ_p .

- (3) the representation V is irreducible as a \mathfrak{G} -module since it is so for the Heisenberg type subgroup, being simply its standard representation.
- (4) The reduced group \mathfrak{G}_{red} can be identified with the unipotent group $\{f \in \mathcal{O}(\mu_p)^* \mid f(1) = 1\}$.
- (5) The subgroup \mathfrak{G}_{red} is not normal as the point 1 of μ_p is not invariant by translations.

4. SEMI-SIMPLICITY STATEMENTS

Let G be a reductive group. Let C be an algebraic group and $\rho : C \rightarrow G$ be a morphism.

DEFINITION 4.1. ([S2, Page 20]) *One says that ρ is cr if, whenever ρ factors through a parabolic P of G , it factors through a Levi subgroup of P .*

When $G = \mathrm{GL}(V)$, ρ is cr if and only if the representation V of C is completely reducible (or equivalently, semi-simple) and hence the terminology.

The property of ρ being cr depends only on the subgroup scheme of G which is the (schematic) image of C . It in fact only depends on the image of C in the adjoint group G^{ad} . Indeed, the parabolic subgroups of G are the inverse images of the parabolic subgroups of G^{ad} , and similarly for the Levi subgroups. A subgroup scheme \mathfrak{G} of G will be called cr if its inclusion in G is so.

For an irreducible root system R , let α_0 be the highest root and $\sum n_i \alpha_i$ its expression as a linear combination of the simple roots. The characteristic p of k is called *good* for R if p is larger than each n_i . For a general root system R , p is good if it is so for each irreducible component of R .

PROPOSITION 4.2. *Suppose C is an extension*

$$1 \rightarrow B \rightarrow C \rightarrow A \rightarrow 1 \tag{4.2.1}$$

with A° of multiplicative type and A/A° a finite group of order prime to p , and suppose that p is good. Let $\rho : C \rightarrow G$ be a morphism. If the restriction of ρ to B is cr, then ρ is cr.

We don't know whether the proposition holds without the assumption that p is good.

Proof. Let P be a parabolic subgroup, U its unipotent radical, and \mathfrak{u} the Lie algebra of U . The parabolic P is said to be *restricted* if the nilpotence class of U is less than p . If P is a maximal parabolic corresponding to a simple root α , the nilpotence class of U is the coefficient of α in the

highest root. It follows that p is *good* if and only if all the maximal parabolic subgroups are restricted. By [Sei, Proposition 5.3] (credited by the author to Serre), if P is restricted, one obtains by specialization from characteristic 0, a P -equivariant isomorphism:

$$\exp : (\mathfrak{u}, \circ) \xrightarrow{\sim} U \tag{4.2.2}$$

from \mathfrak{u} endowed with the Campbell-Hausdorff group law, to U .

We will first show that whenever ρ factors through a restricted parabolic subgroup P as above, if the restriction of ρ to B factors through some Levi subgroup of P , the ρ itself factors through some Levi subgroup of P .

The group $U(k)$ acts on the right on the set $\mathcal{L}(k)$ of Levi subgroups of P by

$$u \text{ in } U(k) \text{ acts by } L \mapsto u^{-1}Lu$$

This action turns $\mathcal{L}(k)$ into a $U(k)$ -torsor. This expresses the fact that two Levi subgroups are conjugate by a unique element of $U(k)$. The group $P(k)$ acts on $\mathcal{L}(k)$ and on $U(k)$ by conjugation. This turns $\mathcal{L}(k)$ into an equivariant $U(k)$ -torsor.

We will need a scheme-theoretic version of the above. Fix a Levi subgroup L_o . Let \mathcal{L} be the trivial U -torsor (i.e. U with the right action of U by right translations). We have the family of Levi subgroups $L_u := u^{-1}L_o u$ parametrized by $\mathcal{L} = U$. We let P act on U by conjugation, and on \mathcal{L} as follows: $p = v\ell$ in $P = UL_o$ acts on $\mathcal{L} = U$ by $u \mapsto v^{-1}pup^{-1}$. This turns \mathcal{L} into an equivariant U -torsor. When we pass to k -points, and attach to u in \mathcal{L} the Levi subgroup $uL_o u^{-1}$, we recover the previously described situation.

The morphism $\rho : C \rightarrow P$ turns \mathcal{L} into an equivariant U -torsor. A point x of \mathcal{L} corresponding to a Levi subgroup L_x is fixed by C (scheme-theoretically) if and only if ρ factors through L_x . This expresses the fact that a Levi subgroup is its own normalizer in P .

We want to prove that if B has a fixed point in \mathcal{L} , so does C . Let U^B be the subgroup of U fixed by B , for the conjugation action. If B has a fixed point in \mathcal{L} , the fixed locus \mathcal{L}^B is a U^B -torsor. As B is a normal subgroup of C , C acts on \mathcal{L}^B and U^B , and the action factors through A .

The isomorphism $\exp : (\mathfrak{u}, \circ) \rightarrow U$ is compatible with the action of B by conjugation. Hence it induces an isomorphism from $(\mathfrak{u}^B, \circ) \rightarrow U^B$. Let $Z^i(\mathfrak{u}^B)$ be the central series of \mathfrak{u}^B , and define $Z^i(U^B) := \exp(Z^i(\mathfrak{u}^B))$. The isomorphism \exp induces an isomorphism between the vector group

$Gr_Z^i(u^B)$ and $Z^i(U^B)/Z^{i+1}(U^B)$, compatible with the action of A . On $Gr_Z^i(u^B)$, this action is linear.

The assumption on A amounts to saying that A is linearly reductive, that is, all its representations are semi-simple. Equivalently, if k is the trivial representation, any extension

$$0 \rightarrow V \xrightarrow{a} E \xrightarrow{b} k \rightarrow 0 \quad (4.2.3)$$

splits. Passing from E to $b^{-1}(1)$, such extensions correspond to A -equivariant V -torsors, and the extension splits if and only if A has a scheme-theoretic fixed point on the corresponding torsor.

Define U_i^B to be $U^B/Z^i(U^B)$, and \mathcal{L}_i^B to be the U_i^B -torsor obtained from \mathcal{L} by pushing by $U^B \rightarrow U_i^B$. We prove by induction on i that A has a fixed point on \mathcal{L}_i^B .

As U_1^B is trivial, the case $i = 1$ is trivial. If x is a fixed point of A in \mathcal{L}_i^B , the inverse image of x in \mathcal{L}_{i+1}^B is an equivariant A -torsor on $Gr_Z^i(u^B) \sim Z^i(U^B)/Z^{i+1}(U^B)$. By linear reductivity, A has a fixed point on the inverse image. As the central descending series of u , and hence of u^B terminates, this proves 4.2 for restricted parabolic subgroups.

We now prove 4.2 by induction on (the dimension) of G . Suppose that ρ factors through a proper parabolic subgroup P . As p is good, there exists a restricted proper parabolic Q containing P , and P is the inverse image by the projection $Q \rightarrow Q/R_u(Q)$ of a parabolic P' of $Q/R_u(Q)$. Let L be a Levi subgroup of Q through which ρ factors and let P'_L be the parabolic subgroup of L obtained as the inverse image of P' by the isomorphism $L \xrightarrow{\sim} Q/R_u(Q)$. Levi subgroups of P'_L are Levi subgroups of P , and it remains to apply the induction hypothesis to L , for which p is good too.

□

Remark 4.3. For several results related to 4.2 but in the setting of reduced subgroups, see [BMR1, Theorem 3.10] and [BMR2, Theorem 1.1 and Corollary 3.7].

Fix in the reductive group G , a maximal torus T , and a system of simple roots corresponding to a Borel subgroup B containing T . Let U be the unipotent radical of B .

DEFINITION 4.4. (cf. [Dy], [S1],[IMP],) *The Dynkin height $ht_G(V)$ of a representation V of G is the largest among $\{\sum_{\alpha>0} \langle \lambda, \alpha^\vee \rangle\}$, for λ a weight for the action of T on V .*

This notion and this terminology go back to Dynkin [Dy, Page 331-332] where it is called “height” as in [IMP], while in Serre ([S1] and [S2]), it is simply “ $n(V)$ ”. If V is an irreducible representation, with dominant weight λ^+ and smallest weight λ^- , it is the sum of the coefficients of $\lambda^+ - \lambda^-$, expressed as linear combination of the simple roots.

It follows that the product in $\text{End}(V)$ of the action of $ht_G(V) + 1$ elements of $\text{Lie}(U)$ vanishes and for \mathfrak{n} nilpotent in $\text{Lie}(G)$, one has $\mathfrak{n}^{ht(V)+1} = 0$ in $\text{End}(V)$.

4.5. The representation $\rho : G \rightarrow \text{GL}(V)$ is said to be of *low height* if $p > ht_G(V)$. By [S2, Theorem 6, page 25], representations of low height are semi-simple. One can show that if G admits a representation V of low height which is *almost faithful*, meaning that its kernel is of multiplicative type, then G satisfies the assumption 2.3. That $p \geq h_G$ results from the more precise statement that $ht_G(V) \geq h_G - 1$ ([S3, (5.2.4), Page 213]). For the property that \tilde{G}/G' is étale, one uses the fact that the non-trivial irreducible representation of $\text{PGL}(p)$ of the smallest height is the adjoint representation which is of height $2p - 2$.

We now assume that V is of low height, and the assumption 2.3 on G . It follows that any nilpotent \mathfrak{n} in $\text{Lie}(G)$ satisfies $\mathfrak{n}^p = 0$, and further, the exponential map (2.2.2) is defined. The image $d\rho(\mathfrak{n})$ of \mathfrak{n} in $\text{Lie}(\text{GL}(V)) = \text{End}(V)$ also has a vanishing p^{th} -power, hence $\exp(d\rho(\mathfrak{n}).t)$ is defined. By [S2, Theorem 5, page 24], one has the following compatibility statement.

4.6. Compatibility If \mathfrak{n} in $\text{Lie}(G)$ is nilpotent

$$\rho(\exp(t\mathfrak{n})) = \exp(td\rho(\mathfrak{n})). \quad (4.6.1)$$

As a consequence, if $u^p = 1$ in G , one has

$$\rho(u^t) = \rho(u)^t \quad (4.6.2)$$

The following theorem is a schematic analogue of [BT], for p large enough.

THEOREM 4.7. *Suppose that the reductive group G admits a low height almost faithful representation $\rho : G \rightarrow \text{GL}(V)$, and that $p > h_G$. Then, for any non-trivial unipotent subgroup U of G , there exists a proper parabolic subgroup P of G containing the normalizer $N_G(U)$ of U , and whose unipotent radical contains U .*

The condition $p > h_G$ implies that G satisfies the assumption 2.3. If G is simple simply connected, it implies the existence of an almost faithful low height representation except for the G of type F_4 , E_6 , E_7 or E_8 ,

in which case the lowest height of a non-trivial representation and the Coxeter number are respectively $16 > 12$, $16 > 12$, $27 > 18$ and $58 > 30$. For these groups, we do not know whether the conclusion of the theorem is valid assuming only $p > h_G$.

Proof. Let V^U be the invariants of U acting on V . It is not zero, because U is unipotent. It is not V , because the representation V is almost faithful, hence faithful on U . It does not have a U -stable supplement V' in V , because U would have invariants in V' .

Let H be the subgroup scheme of G which stabilizes V^U . It contains the normalizer $N_G(U)$ of U . It is a doubly saturated subgroup scheme of G . Indeed, if h in $H(k)$ is of order p , by (4.6.2), $\rho(h^t) = \rho(h)^t = \sum_{i < p} \binom{t}{i} (\rho(h) - 1)^i$, which stabilizes V^U , and similarly if \mathfrak{n} in $\text{Lie}(H)$ is nilpotent, the $\exp(t\mathfrak{n})$ are in H .

As $p > h_G$, theorem 2.5 ensures that H_{red}^0 is a normal subgroup scheme of H and that the quotient H/H_{red}^0 is an extension of a finite group of order prime to p by a group of multiplicative type. It follows that $U \subset H_{red}^0$.

LEMMA 4.8. H_{red}^0 is not reductive.

Proof. If it were, V would be a representation of low height of H_{red}^0 ([S2, Corollary 1, page 25]), hence a semi-simple representation of H_{red}^0 , and V^U would have in V a H_{red}^0 -stable supplement. As V^U does not admit a supplement stable under $U \subset H_{red}^0$, this is absurd. \square

Proof of 4.7 continued: If S is a doubly saturated subgroup scheme of G , we will call $R_u(S_{red}^0)$ the *unipotent radical* of S and denote it simply by $R_u(S)$. By 2.5, it is a normal subgroup of S and $S/R_u(S_{red}^0)$ does not contain any normal unipotent subgroup. This justifies the terminology.

Define $U_1 := R_u(H)$. By 4.8, it is a non-trivial unipotent subgroup of G , and we can iterate the construction. We define for $i \geq 1$

$$\begin{aligned} H_i &:= \text{stabilizer of } V^{U_i} \subset V \\ U_{i+1} &:= R_u(H_i) \end{aligned}$$

One has $U \subset H_{red}^0$ and

$$U \subset N_G(U) \subset H \subset N_G(U_1) \subset H_1 \subset N_G(U_2) \subset H_2 \dots \quad (4.8.1)$$

The $H_{i,red}^0$ form an increasing sequence of smooth connected subgroups of G . It stabilizes, hence so do the sequences of the U_i and of the H_i . If $H_i = H_{i+1}$, one has $H_i = N_G(U_{i+1}) = H_{i+1}$ and

$$U_{i+1} = R_u(H_i) = R_u(H_{i+1}) = R_u(N(U_{i+1})_{red}^o)$$

By [BT, Proposition 2.3, page 99] (or for example [H, Section 30.3, Proposition on page 186]), this implies that $N(U_{i+1})_{red} = (H_{i+1})_{red}$ is a proper parabolic subgroup of G . Call it Q . A parabolic subgroup of G is its own normalizer scheme (cf. [SGA3, XII, 7.9], [CGP, page 469]). As $(H_{i+1})_{red} = Q$ is normal in H_{i+1} , it follows that $H_{i+1} = Q$ and that $N_G(U) \subset Q$.

A Levi subgroup L of Q is a reductive subgroup of G , hence satisfies the assumptions of 4.7. It is isomorphic to $Q/R_u(Q)$. If U is not contained in $R_u(Q)$, we can repeat the argument for the image \bar{U} of U in $Q/R_u(Q)$, which is isomorphic to L . One obtains a proper parabolic subgroup of $Q/R_u(Q)$ which contains the normalizer of \bar{U} . Its inverse image in Q is a parabolic subgroup, properly contained in Q and containing the normalizer of U . Iterating, one eventually finds a parabolic P containing $N_G(U)$ and such that $U \subset R_u(P)$. \square

Remark 4.9. Let \tilde{G} be the simply connected central extension of the derived group G' , and let \tilde{G}_i be its simple factors: $\tilde{G} = \prod \tilde{G}_i$. A representation of low height V of G is almost faithful if and only if its restriction to each \tilde{G}_i is not trivial. It suffices to check this for each \tilde{G}_i separately. Thus we may assume G is simply connected. The existence of a non-trivial V of low height implies that $\underline{p} > 2$ for G of type $B_n, C_n (n \geq 2)$ or F_4 and $\underline{p} > 3$ for G of type G_2 . Let \bar{G} be the image of G in $\text{GL}(V)$. If V is non-trivial, $u : G \rightarrow \bar{G}$ is an isogeny. We want to show that it is a central isogeny. If it is not, the structure of isogenies ([SGA3, XXII, 4.2.13]) show that $\ker(u)$ contains the kernel of the Frobenius. The weights of V are the p^{th} -powers and $ht_G(V)$ is a multiple of p , contradicting the low height assumption.

4.10. If G is a reductive group for which assumption 2.3 holds, and \mathfrak{G} is a subgroup scheme of G , the *double saturation* \mathfrak{G}^* of \mathfrak{G} is the smallest doubly saturated subgroup scheme of G containing \mathfrak{G} . It is the intersection of the doubly saturated subgroup schemes containing \mathfrak{G} , and is obtained from \mathfrak{G} by iterating the construction of taking the group generated by \mathfrak{G} , the additive groups $\exp(t\mathfrak{n})$ for \mathfrak{n} nilpotent in $\text{Lie}(\mathfrak{G})$ and u^t for u of order p in $\mathfrak{G}(k)$.

COROLLARY 4.11. *Let V be a low height almost faithful representation of a reductive group G . Assume that $\underline{p} > h_G$. Let \mathfrak{G} be a subgroup scheme of G , and let \mathfrak{G}^* be its double saturation. Then the following conditions are equivalent.*

- (i) V is a semi-simple representation of \mathfrak{G}
- (ii) V is a semi-simple representation of \mathfrak{G}^*

- (iii) \mathfrak{G} is cr in G
- (iv) \mathfrak{G}^* is cr in G
- (v) the unipotent radical of $(\mathfrak{G}^*)_{red}^0$ is trivial

Proof. (i) \iff (ii): If W is a subspace of V , the stabilizer in G of W is doubly saturated, as we saw in the beginning of the proof of 4.7. If \mathfrak{G} stabilizes W , it follows that \mathfrak{G}^* also stabilizes W : the lattice of subrepresentations of V is the same for \mathfrak{G} and \mathfrak{G}^* , hence the claim.

(iii) \iff (iv): Similarly, the parabolic subgroups of G and their Levi subgroups are doubly saturated, hence contain \mathfrak{G} if and only if they contain \mathfrak{G}^* .

not (v) \implies not (ii): Let U be the unipotent radical of $(\mathfrak{G}^*)_{red}^0$. If it is non-trivial, $V^U \neq V$, because V is faithful on U and does not have a U -stable supplement. As U is normal in \mathfrak{G}^* (2.5), V^U is a subrepresentation for the action of \mathfrak{G}^* on V . This contradicts the semi-simplicity of V .

(v) \implies (ii): The representation V of the reductive group $(\mathfrak{G}^*)_{red}^0$ is of low height, hence semi-simple. By 2.5, $(\mathfrak{G}^*)_{red}^0$ is a normal subgroup of \mathfrak{G}^* and the quotient A is linearly reductive. If W is a sub- \mathfrak{G}^* -representation of V , A acts on the affine space of $(\mathfrak{G}^*)_{red}^0$ -invariant retracts $V \rightarrow W$. It has a fixed point, whose kernel is a supplement to W .

not (v) \implies not (iv): Let U be the unipotent radical of $(\mathfrak{G}^*)_{red}^0$. If it is not trivial, there exists a parabolic P containing its normalizer, hence \mathfrak{G}^* , and the unipotent radical of P contains U (4.7). Thus, no Levi subgroup of P can contain \mathfrak{G}^* .

(v) \implies (iv): By [S2, Theorem 7, page 26], $(\mathfrak{G}^*)_{red}^0$ is cr in G , and one applies 4.2. □

COROLLARY 4.12. *Let $\rho : G \rightarrow GL(V)$ be an almost faithful low height representation, and let v in V be an element such that the G -orbit of v in V is closed. Then there exists a connected multiplicative central subgroup scheme $M \subset G_v^0$ and a surjective homomorphism $M \times G_{v,red}^0 \rightarrow G_v^0$.*

Proof. The orbit being closed in V and hence affine, the reduced stabilizer $G_{v,red}$ is reductive ([Bo]). Since $p \geq h_G$ and since stabilizers are doubly saturated and 2.5(2) holds for $\mathfrak{G} = G_v$, we get the required result. □

We now observe that the results of [D, Section 6] can be obtained as a consequence of 4.11. Note that by the remarks in [D, Page 607] it suffices

to prove the semisimplicity results in the case when k is algebraically closed.

THEOREM 4.13. *\mathfrak{G} be an algebraic group. Let $(V_i)_{i \in I}$ be a finite family of semi-simple \mathfrak{G} -modules and let m_i be integers ≥ 0 . If*

$$\sum m_i(\dim V_i - m_i) < p \quad (4.13.1)$$

the \mathfrak{G} -module $\bigotimes_j \bigwedge^{m_j} V_j$ is semi-simple.

Proof. Let $G = \prod_j \mathrm{GL}(V_j)$ and $V = \bigotimes_j \bigwedge^{m_j} V_j$. Then, (4.13.1) is simply the inequality $p > \mathrm{ht}_G(V)$. Replacing \mathfrak{G} by its image in G we may and shall assume that \mathfrak{G} is a subgroup scheme of G . Since V_i are semi-simple \mathfrak{G} -modules, it follows that \mathfrak{G} is G -cr, for $G = \prod_j \mathrm{GL}(V_j)$. By [D, §6.2], we may also assume that $p > \dim(V_j) = \mathrm{h}_{\mathrm{GL}(V_j)} \forall j$.

Hence by working with the image of G (and \mathfrak{G}) in $\mathrm{GL}(V)$ and applying 4.11, we conclude that V is semi-simple as a \mathfrak{G} -module. \square

Remark 4.14. If (V_i, q_i) is a non-degenerate quadratic space with $\dim V_i = 2d_i$ on which \mathfrak{G} acts by similitudes, then by passing to a subgroup of index at most 2 and mapping to the group of similitudes rather than $\mathrm{GL}(V_i)$, one can replace the term $m_i(\dim V_i - m_i)$ by $m_i(\dim V_i - m_i - 1)$, when $m_i \leq 2d_i$.

Complete reducibility in the classical case. By 4.1, a subgroup scheme $\mathfrak{G} \subset G$ is called cr if for every parabolic subgroup $P \subset G$ containing \mathfrak{G} , there exists an opposite parabolic subgroup P' such that $\mathfrak{G} \subset P \cap P'$. Suppose that $\mathrm{char}(k) \neq 2$ and let G be $\mathrm{SO}(V)$ (or $\mathrm{Sp}(V)$ in any characteristic), relative to a non-degenerate symmetric or alternating bilinear form B on V . In this situation, the notion of cr can be interpreted as follows: \mathfrak{G} is cr in G if and only if for every \mathfrak{G} -submodule $W \subset V$ which is totally isotropic, there exists a totally isotropic \mathfrak{G} -submodule W' of the same dimension, such that the restriction of B to $W + W'$ is non-degenerate (cf. [S3, Example 3.3.3, page 206]).

LEMMA 4.15. *Let the subgroup scheme \mathfrak{G} of G be cr. Then the \mathfrak{G} -module V is semi-simple and conversely.*

Proof. Let $W \subset V$ be a \mathfrak{G} -submodule. Then we need to produce a \mathfrak{G} -complement.

Consider $W_1 := W \cap W^\perp$. If $W_1 = (0)$, then $W \oplus W^\perp = V$ and we are done. So let $W_1 \neq (0)$. Then W_1 is a \mathfrak{G} -submodule which is totally isotropic and hence by the cr property, we have a totally isotropic \mathfrak{G} -submodule W'_1 of the same dimension as W_1 , such that the form B is

non-degenerate on $W_1 + W'_1$. In particular, $W_1 \cap W'_1 = (0)$. Since $W_1 \subset W^\perp$, we see that $W \subset W_1^\perp$.

Let $w \in W \cap W'_1 \subset W_1^\perp \cap W'_1$ and suppose $w \neq 0$. Since $w \in W'_1$, there exists $w' \in W_1$ such that $B(w, w') \neq 0$. On the other hand, since $w \in W_1^\perp$, $B(w, v) = 0$ for all $v \in W_1$ and in particular $B(w, w') = 0$ which contradicts the assumption that $w \neq 0$. Hence it follows that $W \cap W'_1 = (0)$. Thus, $W \subsetneq X = W \oplus W'_1 \subset V$ is a \mathfrak{G} -submodule. We proceed similarly and get $X_1 = X \cap X^\perp$ such that $X \oplus X_1 \subset V$. If $X_1 = (0)$, then $V = X \oplus X^\perp$, so get a \mathfrak{G} -decomposition of V as $W \oplus W'_1 \oplus \dots$, that is a \mathfrak{G} -complement of W in V .

Conversely, let V be semi-simple as a \mathfrak{G} -module. Let $W \subset V$ be a totally isotropic \mathfrak{G} -submodule of $\dim(W) = d$. Note that $d < \dim(V)/2$. We therefore have a \mathfrak{G} -submodule $Z \subset V$ such $V = W \oplus Z$. The non-degenerate form B gives an \mathfrak{G} -equivariant isomorphism $\phi : V \rightarrow V^* = W^* \oplus Z^*$ and since W is totally isotropic $\phi(W) \cap W^* = (0)$. Hence $\phi(W) \subset Z$.

Again, since W is totally isotropic, the restriction of B to Z is non-degenerate and hence we get an isomorphism $\psi : Z^* \rightarrow Z$. Define $W' := \psi \circ \phi(W)$.

Then it is easily seen that W' is of dimension d and also totally isotropic \mathfrak{G} -invariant submodule of V . Finally, B is non-degenerate on $W \oplus W'$. Hence \mathfrak{G} is cr in G . □

5. ÉTALE SLICES IN POSITIVE CHARACTERISTICS

We begin this section with the following (linear) analogue of the Luna étale slice theorem in positive characteristics.

THEOREM 5.1. *Let V be a G -module with low height i.e. such that $p > ht_G(V)$. Let v in V be an element such that the orbit $G.v$ is a closed orbit in V . Then there exists a G_v -invariant linear subspace S of V giving rise to a commutative diagram:*

$$\begin{array}{ccc} G \times^{G_v} S & \xrightarrow{\phi} & V \\ f \downarrow & & \downarrow q \\ (G \times^{G_v} S) \parallel G & \xrightarrow{\ell} & V \parallel G \end{array} \quad (5.1.1)$$

and G -equivariant open subsets $U \subset (G \times^{G_v} S)$ containing the closed orbit $G.v$ and an open subset U' of V containing v , for which (5.1.1) induces a

cartesian diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\phi} & U' \\
 f \downarrow & & \downarrow q \\
 U \parallel G & \xrightarrow{\ell|_U} & U' \parallel G
 \end{array} \tag{5.1.2}$$

such that the morphism $\ell|_U$ is étale.

Remark 5.2. The above theorem was stated in the note [MP] whose proof contained serious gaps (as was pointed to the authors by Serre in a private correspondence).

We note that G/G_v is constructed in ([DG, III, Proposition 3.5.2]). It represents the quotient in the category of fppf sheaves. Furthermore if $\pi_v : G \rightarrow V$, $g \mapsto g.v$, the image $\text{im}(\pi_v)$, as a locally closed sub-scheme of V with its *reduced scheme* structure, can be identified with the scheme G/G_v . We call this locally closed sub-scheme, the *orbit* $G.v$ and have the identification $G/G_v \simeq G.v$ (see also [DG, Proposition and Definition 1.6, III, §3, page 325]).

PROPOSITION 5.3. *Let V be an arbitrary G -module. Let $v \in V$ and suppose that there exists a G_v -submodule S of the tangent space $T_v(V)$, such that $T_v(V)$ splits as $T_v(G.v) \oplus S$. Let G act on $G \times S$ by $h.(g, s) := (h.g, s)$. Then the G -morphism $\Phi : G \times S \rightarrow V$ given by $(g, s) \mapsto g.v + g.s$ descends to a G -morphism*

$$\phi : G \times^{G_v} S \rightarrow V. \tag{5.3.1}$$

which is étale at $(e, 0)$, e being the identity of G .

Proof. To check the morphic properties, we need to check for all A -valued points of G_v where A is a k -algebra and we will suppress the A .

We will check that it is constant on the G_v -orbits. Let α in G_v act on $G \times S$ by $\alpha.(g, s) = (g.\alpha, \alpha^{-1}.s)$. Observe that $\Phi(\alpha.(g, s)) = \Phi(g.\alpha, \alpha^{-1}.s) = g.\alpha.v + g.\alpha.\alpha^{-1}.s = g.v + g.s$ (since α fixes v). Therefore it is constant on the G_v -orbits. Since the action of G_v on $G \times S$ is scheme-theoretically free, Φ descends to a morphism $\phi : G \times^{G_v} S \rightarrow V$. Clearly the actions of G and G_v on $G \times S$ commute and hence the descended morphism is also a G -morphism.

Observe that the quotient morphism $G \rightarrow G/G_v$ is a torsor for the group scheme G_v , locally trivial under the fppf topology. Since the action of G_v on S is linear, we see that the associated fibre space $\psi : G \times^{G_v} S \rightarrow G/G_v$ is a locally free sheaf of rank = $\dim(S)$. In particular, $G \times^{G_v} S$ is a smooth k -scheme of finite type. Observe further that under the

morphism ϕ , the zero section of the vector bundle $\psi : G \times^{G_v} S \rightarrow G/G_v$ canonically maps onto the orbit $G.v \subset V$, while the fibre $\psi^{-1}(e.G_v)$ of the identity coset $e.G_v \in G/G_v$ maps isomorphically to the affine subspace $S + v \subset V$. Since $T_v(V) = T_v(G.v) \oplus S$ by assumption, it follows that the differential $d\phi_z$, at $z = (e, 0)$, is an isomorphism.

We now apply [D, Lemme 2.9], to conclude that the morphism $\phi : G \times^{G_v} S \rightarrow V$ is étale at $z = (e, 0)$. \square

PROPOSITION 5.4. *Let V be a G -module such that $p > ht_G(V)$, and let v in V be an element such that the G -orbit of v in V is closed. Then there exists a G_v -submodule $S \subset V$ such that $V = T_v(G.v) \oplus S$ as a G_v -module. In particular, the consequences of 5.3 hold good.*

Proof. One has the exact sequence:

$$1 \rightarrow G_v^0 \rightarrow G_v \rightarrow \pi_0(G_v) \rightarrow 1 \quad (5.4.1)$$

the quotient $\pi_0(G_v)$ being the group of connected components. Further, we note that $|G_v/G_v^0| = |G_{v,red}/G_{v,red}^0|$. Note also that since $G_{v,red}$ is a saturated subgroup of G , by [S2, Property 3, Page 23] the index $|G_{v,red}/G_{v,red}^0|$ is prime to p . Thus, $\pi_0(G_v)$ is linearly reductive.

Since $G.v$ is a closed orbit, by 4.12 we see that $G_{v,red}^0$ is reductive and by 2.5(1), we have the exact sequence:

$$1 \rightarrow G_{v,red}^0 \rightarrow G_v^0 \rightarrow \tau \rightarrow 1 \quad (5.4.2)$$

where τ is a multiplicative group scheme.

Since $G_{v,red}^0$ is a reductive saturated subgroup of G , by [S2, Corollary 1, page 25] we get the inequality $p > ht_G(V) \geq ht_{G_{v,red}^0}(V)$. Being of low

height, V is semi-simple as a $G_{v,red}^0$ -module, .

Further, since τ is multiplicative (and hence linearly reductive), by [D, Lemma 4.2] and 5.4.2, we deduce that V is semi-simple as a G_v^0 -module. Using 5.4.1, we conclude that V is semi-simple as a G_v -module as well.

In particular, we have a G_v -supplement S for the G_v -invariant subspace $T_v(G.v) \subset T_v(V) = V$, i.e. we have a G_v -decomposition $S \oplus T_v(G.v)$ for V . \square

We recall the ‘‘Fundamental lemma of Luna’’ which holds in positive characteristics as well and which is essential to complete the proof of Theorem 5.1.

LEMMA 5.5. ([GIT, Page 152]) *Let $\Delta : X \rightarrow Y$ be a G -morphism of affine G -schemes. Let $F \subset X$ be a closed orbit such that:*

- (1) δ is étale at some point of F
- (2) $\delta(F)$ is closed in Y
- (3) δ is injective on F
- (4) X is normal along F .

then there are affine G -invariant open subsets $U \subset X$ and $U' \subset Y$ with $F \subset U$ such that $\delta \parallel G : U \parallel G \rightarrow U' \parallel G$ is étale and $(\delta, p_U) : U \rightarrow U' \times_{U' \parallel G} U \parallel G$ is an isomorphism (where $p_U : U \rightarrow U \parallel G$ is the quotient morphism).

Proof. (of 5.1) As a first step, we need to show that the categorical quotient $G \times^{G_v} S \parallel G$ exists as a scheme. As we have seen in 5.3, action of G on $G \times S$ (via $g.(a, s) = (g.a, s)$) and the (twisted) action of G_v on $G \times S$ commute and hence

$$k[G \times^{G_v} S]^G = (k[G \times S]^{G_v})^G = (k[G \times S]^G)^{G_v} = k[S]^{G_v}. \quad (5.5.1)$$

Thus it is enough to show that $k[S]^{G_v}$ is finitely generated and we would have

$$G \times^{G_v} S \parallel G \simeq S \parallel G_v. \quad (5.5.2)$$

As we have observed in the proof of 5.4, $G_v^0 \subset G_v$ is a normal subgroup of finite index. Thus, the finite generation of $k[S]^{G_v}$ is reduced to checking finite generation of $k[S]^{G_v^0}$. We may therefore replace G_v by G_v^0 in the proof. Further, since $G.v$ is a closed orbit, $G_{v,red}$ is reductive. Therefore, by 4.12, we have a surjection $M \times G_{v,red} \rightarrow G_v^0$. Since M in G_v is central $k[S]^{G_v} = \left(k[S]^{G_{v,red}} \right)^M$.

Again, since $G_{v,red}$ is reductive, the ring of invariants $k[S]^{G_{v,red}}$ is a finitely generated k -algebra. Since M is a group of multiplicative type over an algebraically closed field, it is a product of \mathbb{G}_m 's and a finite group scheme. Therefore, by [GIT, Theorem 1.1] and [AV, Page 113], the ring of invariants $\left(k[S]^{G_{v,red}} \right)^M$ is also finitely generated. This proves the finite generation of $k[S]^{G_v}$.

The commutativity of the diagram (5.1.1) now follows by the property of categorical quotients by the group G .

Now we check the conditions of Lemma 5.5. We consider the given closed G -orbit $F = G \times^{G_v} \{v\} \subset G \times^{G_v} S$. Then we need to check that $\phi(F)$ is a closed orbit in V . In fact, $\phi(F)$ is precisely the closed orbit $G.\phi(v) = G.v \subset V$. Thus we have verified conditions (2) and (3) of Lemma 5.5.

Condition (1) is precisely the content of 5.4 and (4) holds since $G \times^{G_v} S$ has been seen to be smooth.

The isomorphism (δ, p_U) shows that the diagram (5.1.2) is cartesian. This completes the proof of 5.1. \square

COROLLARY 5.6. *Let X be an affine G -scheme embeddable as a closed G -subscheme in low height G -module. Let $x \in X$ be such that the orbit $G.x \subset X$ is closed. Then there exists a locally closed G_x -invariant subscheme (a "slice") X_1 , of X with $x \in X_1 \subset X$ such that the conclusions of 5.1 hold for the G -morphism $G \times^{G_x} X_1 \rightarrow X$.*

Proof. This follows from 5.1 (which corresponds to [BR, Proposition 7.4]) exactly as in [BR, Proposition 7.6]. \square

The following consequence of the slice theorem has many applications so we state it here without proof.

COROLLARY 5.7. (see [BR, Proposition 8.5, p.312]) *Let F be an affine G -subvariety of $\mathbb{P}(V)$, with V as a G -module with low height, and suppose that F contains a unique closed orbit F^{cl} . Then there exists a G -retract $F \rightarrow F^{cl}$.*

Let V be a finite dimensional G -module and let

$$ht_G(\wedge(V)) := \max_i \{ht_G(\wedge^i(V))\} \quad (5.7.1)$$

We then have the following application to the theory of semistable principal bundles in positive characteristics.

THEOREM 5.8. *Let E be a stable G -bundle with G semi-simple and $\rho_V : G \rightarrow SL(V)$ be a representation such that $p > ht_G(\wedge(V))$. Then the associated bundle $E(V)$ is polystable of degree 0.*

Proof. The proof follows [BP, Theorem 9.11] verbatim where the notion of "separable index" is replaced by the Dynkin height, the key ingredient being Corollary 5.7. \square

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APPENDIX A. COXETER NUMBER AND ROOT SYSTEMS, ZHIWEI YUN

PROPOSITION A.1. *Let R be an irreducible root system associated to a simple group G with Coxeter number h and let X be the lattice spanned by R . Let $\phi : X \rightarrow \mathbb{R}/\mathbb{Z}$ be a homomorphism. Then there exists a basis B for R*

such that if $\alpha \in R$ satisfies $\phi(\alpha) \in (0, 1/\mathfrak{h}) \bmod \mathbb{Z}$, then α is positive with respect to B .

Proof. Pick any basis $B_0 = \{\alpha_1, \dots, \alpha_r\}$ of R . We lift ϕ to a linear map $y: X \rightarrow \mathbb{R}$ which is viewed as a point in $V = \text{Hom}(X, \mathbb{R})$.

Let W be the Weyl group of R . Then there is an action of the extended Weyl group $\text{Hom}(X, \mathbb{Z}) \rtimes W$ on V and the validity of the statement for the linear map y is invariant for this action. Therefore we may assume that y lies in the fundamental alcove attached to B_0 , i.e.

$$\alpha_j(y) \geq 0, \quad \forall 1 \leq j \leq r \quad (\text{A.1.1})$$

$$\theta(y) \leq 1 \quad (\text{A.1.2})$$

where θ is the highest root with respect to B_0 . Then for all $\alpha \in R$, we have

$$|\alpha(y)| \leq 1. \quad (\text{A.1.3})$$

We *claim* that there exists a basis B of R such that for any $\beta \in R$ with

$$\beta(y) \in (-1, -1 + 1/\mathfrak{h}) \cup (0, 1/\mathfrak{h}), \quad (\text{A.1.4})$$

β positive with respect to B .

Write $\theta = \sum_i n_i \alpha_i$ and let $\alpha_0 = 1 - \theta$ and $n_0 = 1$ as usual. Then we have the equality $\sum_{i=0}^r n_i \alpha_i = 1$, viewed as affine functions on V and we have

$$\mathfrak{h} = \sum_{i=0}^r n_i. \quad (\text{A.1.5})$$

Therefore, for some $0 \leq i \leq r$, we have $\alpha_i(y) \geq 1/\mathfrak{h}$.

If $i = 0$, then $\theta(y) \leq 1 - 1/\mathfrak{h}$ and in this case, we can take $B = B_0$. In fact, for any negative root $-\beta$ with respect to B_0 , we have $0 \geq -\beta(y) \geq 1 - 1/\mathfrak{h}$, so that $-\beta$ cannot satisfy the condition (A.1.4).

If $i > 0$, let v_i be the vertex of the fundamental alcove opposite to the root hyperplane α_i . So $\alpha_j(v_i) = 0$ if $j \neq i$ and $\alpha_i(v_i) = 1/n_i$. We take a basis B of R such that the vector $y - v_i$ is dominant, i.e. for any $\beta \in R$ positive for B , we have the relation: $\beta(y) \geq \beta(v_i)$.

Such a basis B of R satisfies the requirements of the claim. In fact, let $\beta \in R$ be negative with respect to B .

Write $\beta = \sum_{j=1}^r m_j \alpha_j$ for some $-n_j \leq m_j \leq n_j$. Then, $\beta(v_i) = m_i/n_i$. On the other hand, $\beta(y) = \sum_{j=1}^r m_j \alpha_j(y)$. Since β is negative for B , we have

$$\beta(y) \leq \beta(v_i) = m_i/n_i. \quad (\text{A.1.6})$$

and we have two cases:

- If $\beta > 0$, with respect to B_0 , then $\beta(y) \geq 0$ and $0 \leq m_j \leq n_j$ for all j . If $m_i = 0$, then (A.1.6) implies that $\beta(y) = 0$ and so β fails to satisfy (A.1.4). If $m_i \geq 1$, then $\beta(y) \geq \alpha_i(y) \geq 1/h$, again not satisfying (A.1.4).
- If $\beta < 0$, with respect to B_0 , then $-1 \leq \beta(y) \leq 0$ and $-n_j \leq m_j \leq 0$ for all j . If $m_i = -n_i$, then $\beta(v_i) = -1$ which forces $\beta(y) = -1$ by (A.1.6). If $m_i \geq -n_i + 1$, then comparing $|\beta(y)|$ with $|\theta(y)|$, it misses at least one copy of $\alpha_i(y)$, therefore $|\beta(y)| \leq \theta(y) - \alpha_i(y) \leq 1 - 1/h$ and yet again, $\beta(y)$ does not satisfy (A.1.4).

This proves the claim and the proposition. □

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