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Regularity properties and simulations of Gaussian random fields on the Sphere cross Time

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Abstract We study the regularity properties of Gaussian fields defined over spheres across time. In particular, we consider two alternative spectral decompositions for a Gaussian field on $\mathbb{S}^2 \times \mathbb{R}$. For each decomposition, we establish regularity properties through Sobolev and interpolation spaces. Then, conditions for sample Hölder continuity are also provided. We propose a simulation method and study its level of accuracy in the L^2 sense. The method turns to be both fast and efficient.

Keywords: Gaussian random fields, Global data, Space-time covariance, Karhunen-Lóeve expansion, Spherical harmonic functions, Schoenberg's functions.

MSC2010 subject classifications. Primary 60G60, 60G17, 41A25; secondary 60G15, 33C55, 46E35, 33C45.

1. Introduction

Spatio-temporal variability is of major importance in several applications, in particular for anthropogenic and natural processes such as: earthquakes, geographic evolution of diseases, income distributions, mortality fields, atmospheric pollutant concentrations, hydrological basin characterization and precipitation fields. For many natural phenomena involving, for instance, climate change and atmospheric variables, many branches of applied sciences have been increasingly interested in the analysis of data distributed over the whole spherical shell representing planet Earth and evolving across time. Hence the need for random fields models where the spatial location is continuously indexed through the sphere, and where time can be either continuous or discrete. It is common to consider the observations as a partial realization of a spatio-temporal random field which is usually

considered to be Gaussian (see Christakos (2005); Christakos and Hristopulos (1998); Dimitrakopoulos (1994)). Hence, the dependence structure in space-time is governed by the covariance of the spatio-temporal Gaussian field, and we refer the reader to Gneiting (2002); Stein (2005); Zastavnyi and Porcu (2011) for significant contributions in this direction.

Specifically, let $d \geq 1$ be a positive integer, and let $\mathbb{S}^d = \{\mathbf{x} \in \mathbb{R}^{d+1}, \|\mathbf{x}\| = 1\}$ be the d -dimensional unit sphere in the Euclidean space \mathbb{R}^{d+1} , $\|\cdot\|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^{d+1}$. We denote $Z = \{Z(\mathbf{x}, t), (\mathbf{x}, t) \in \mathbb{S}^d \times \mathbb{R}\}$ a Gaussian field on $\mathbb{S}^d \times \mathbb{R}$.

The *tour de force* in Lang and Schwab (2015) characterizes isotropic Gaussian random fields on the sphere through Karhunen–Loève expansions with respect to the spherical harmonic functions and the angular power spectrum. They show that the smoothness of the covariance is connected to the decay of the angular power spectrum and then discuss the relation to sample Hölder continuity and sample differentiability of the random fields.

The present paper extends the work of Lang and Schwab (2015) to space-time. Such extension is non-trivial and depends on two alternative spectral decompositions of a Gaussian field on spheres across time. In particular, we propose either Hermite or classical Karhunen-Loeve expansions and show how regularity properties as well as sample Hölder continuities can evolve dynamically over time. The crux of our arguments rely on recent advances on the characterization of covariance functions associated to Gaussian fields on spheres across time (see Porcu, Bevilacqua and Genton (2016) and Berg and Porcu (2016)). In particular, the Berg-Porcu representation in terms of Schoenberg functions inspires the proposal of alternative spectral decompositions for the temporal part, which become then crucial to establish the regularity properties of the associated Gaussian field.

The second part of the paper is devoted to simulation methods which should be computationally fast while keeping a reasonable level of accuracy, resulting in a notorious trade off. Efficient simulation methods for random fields defined on the sphere cross time are, currently, almost unexplored. Cholesky decomposition is an attractive alternative since it is an exact method. However, the method has an order of computation of $\mathcal{O}(N^3)$, with N denoting the sample size. This makes computationally challenging its implementation in problems of large scale (the so called Big “ n ” problem). The search for efficient simulation methods is thus mandatory. We propose a simulation method based on a suitable truncation of the proposed double spectral decompositions. We establish its accuracy in the L^2 sense and illustrate how the model keeps a reasonable level of accuracy while being especially fast, even when the number of spatio-temporal locations is very high.

The remainder of the article is as follows. Section 2 provides the basic material for a self-contained exposition. The expansions for the kernel covariances and the random field are presented in Section 3. Section 4 present the regularity results for the kernel covariance functions in terms of weighted Sobolev spaces and weighted bi-sequence spaces. Section 5 illustrates the results on Hölder regularity of the sample paths. In Section 6 our simulation method is developed and its accuracy is studied. Also, we provide some numerical

experiments for illustrative purposes. In Appendix 7.1 we prove a rather general version of the Karhunen-Loève theorem.

2. Preliminaries

This section is largely expository and devoted to the illustration of the framework and notations that will be of major utility through the manuscript. While all the tools presented in this section are valid for $d \in \mathbb{N}$, for the sake of simplicity only the results for the case $d = 2$ are exposed, being the generalization for $d \geq 3$ a major part of the supplementary material.

2.1. Spherical harmonic functions

Spherical harmonic are restrictions to the unit sphere \mathbb{S}^d of the real harmonic polynomials in \mathbb{R}^{d+1} . They are the eigenfunctions of the Laplacian on \mathbb{S}^d (Dai and Xu, 2013).

Let \mathcal{H}_j^d be the space generated by spherical harmonics of degree j over \mathbb{S}^d . Corollary 1.1.4 in Dai and Xu (2013) shows that

$$\dim(\mathcal{H}_j^d) = \frac{(2j + d - 1)(j + d - 2)!}{j!(d - 1)!}, \quad j \geq 1, \quad \dim \mathcal{H}_0^d = 1. \quad (2.1)$$

Denote $\mathcal{Y}_{j,m,d}$, for $m = 1, \dots, \dim(\mathcal{H}_j^d)$, the spherical harmonics of degree j over \mathbb{S}^d . Considering σ_d as the surface area measure and $\|\sigma_d\|$ the surface area of \mathbb{S}^d ,

$$\|\sigma_d\| := \int_{\mathbb{S}^d} d\sigma_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)},$$

then, spherical harmonics satisfy the orthogonality relation

$$\int_{\mathbb{S}^d} \mathcal{Y}_{j,m,d}(\mathbf{x}) \mathcal{Y}_{j',m',d}(\mathbf{x}) d\sigma_d = \delta_{j,j'} \cdot \|\sigma_d\|; \quad j, j' \in \mathbb{N}, \quad m = 0, \dots, \dim(\mathcal{H}_j^d),$$

where $\delta_{j,j'}$ is the Kronecker delta function, being identically equal to one if $j = j'$, and zero otherwise. For $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$, normalized spherical harmonics yield the addition formula,

$$\sum_{m=1}^{\dim(\mathcal{H}_j^d)} \mathcal{Y}_{j,m,d}(\mathbf{x}) \mathcal{Y}_{j,m,d}(\mathbf{y}) = \frac{(2j + d - 1)}{(d - 1)} C_j^{(d-1)/2}(\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}), \quad j \in \mathbb{N}, \quad (2.2)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^{d+1}}$ denotes the inner product in \mathbb{R}^{d+1} . Here C_j^r are the Gegenbauer (or ultraspherical) polynomials of degree j .

In what follows $c_j(d, x)$ denotes the standardized Gegenbauer polynomial, being identically equal to one for $x = 1$ when $r = (d - 1)/2$, that is

$$c_j(d, x) = \frac{C_j^{(d-1)/2}(x)}{C_j^{(d-1)/2}(1)} = \frac{j!}{(d-1)_j} C_j^{(d-1)/2}(x), \quad x \in [-1, 1], \quad (2.3)$$

with $(\cdot)_j$ being the Pochhammer symbol (rising factorial). When $r = 1/2$ (i.e., when $d = 2$ in the previous expression), $C_j^{1/2} = P_j$, where P_j is the Legendre polynomial of degree j .

For a detailed description of the properties of Spherical Harmonics and Gegenbauer polynomials, the reader is referred to [Dai and Xu \(2013\)](#).

During the rest of the manuscript, all the polynomials are considered to be normalized.

2.2. Isotropic stationary Gaussian random fields on the sphere cross time

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider \mathbb{S}^d and $\mathbb{S}^d \times \mathbb{R}$ as manifolds contained in \mathbb{R}^{d+1} and in \mathbb{R}^{d+2} respectively.

Definition 2.1. *The geodesic distance θ on \mathbb{S}^d (“great circle” or “spherical” distance) is defined by,*

$$\theta(\mathbf{x}, \mathbf{y}) = \arccos(\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d. \quad (2.4)$$

Geodesic distance ρ on $\mathbb{S}^d \times \mathbb{R}$ is defined through

$$\rho((\mathbf{x}, s), (\mathbf{y}, t)) = \sqrt{\theta(\mathbf{x}, \mathbf{y})^2 + (t - s)^2}, \quad (\mathbf{x}, s), (\mathbf{y}, t) \in \mathbb{S}^d \times \mathbb{R}. \quad (2.5)$$

Throughout, unless it is explicitly presented in a different way, we write \mathbb{R} instead of (\mathbb{R}, μ) , for μ the Lebesgue measure. Analogously, we write \mathbb{S}^d instead of (\mathbb{S}^d, σ_d) , being σ_d the surface area measure, which is equivalent to others uniformly distributed measures on \mathbb{S}^d , such as the Haar measure or the Lebesgue spherical measure ([Christensen, 1970](#)).

Definition 2.2. *A $\mathcal{F} \times \mathcal{B}(\mathbb{S}^d \times \mathbb{R})$ -measurable mapping, $Z : \Omega \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{R}$, is called a real-valued random field on $\mathbb{S}^d \times \mathbb{R}$.*

A random field $Z = \{Z(\mathbf{x}, t) : \mathbf{x} \in \mathbb{S}^d, t \in \mathbb{R}\}$ is called *Gaussian* if for all $k \in \mathbb{N}$, $(\mathbf{x}_1, t_1), \dots, (\mathbf{x}_k, t_k) \in \mathbb{S}^d \times \mathbb{R}$, the random vector $(Z(\mathbf{x}_1, t_1), \dots, Z(\mathbf{x}_k, t_k))^\top$, is multivariate Gaussian distributed, i.e., $\sum_{i=1}^k a_i Z(\mathbf{x}_i, t_i)$ is a normally distributed random variable for all $a_i \in \mathbb{R}$, $i = 1, \dots, k$. Here, \top denotes the transpose operator.

A function $h : \mathbb{S}^d \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is *positive definite* if

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j h(\mathbf{x}_i, t_i, \mathbf{x}_j, t_j) \geq 0, \quad (2.6)$$

for all finite systems of pairwise distinct points $\{(\mathbf{x}_k, t_k)\}_{k=1}^n \subset \mathbb{S}^d \times \mathbb{R}$ and constants $c_1, \dots, c_n \in \mathbb{R}$. A positive definite function h is *strictly positive definite* if the inequality (2.6) is strict, unless $c_1 = \dots = c_n = 0$. We call a function $h : \mathbb{S}^d \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{R}$ *spatially isotropic and temporally stationary* if there exists a function $\psi : [-1, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h(\mathbf{x}, t, \mathbf{y}, s) = \psi(\cos \theta(\mathbf{x}, \mathbf{y}), t - s), \quad (\mathbf{x}, t), (\mathbf{y}, s) \in \mathbb{S}^d \times \mathbb{R}. \quad (2.7)$$

Definition 2.3. We call a random field $Z = \{Z(\mathbf{x}, t) : \mathbf{x} \in \mathbb{S}^d, t \in \mathbb{R}\}$ *isotropic stationary* if $\mathbb{E}[Z(\mathbf{x}, t)]$ is constant for all $(\mathbf{x}, t) \in \mathbb{S}^d \times \mathbb{R}$, and if the associated covariance h is spatially isotropic and temporally stationary.

We call the function ψ *covariance kernel* or simply the *kernel* associated to the covariance h .

Through the manuscript we work with zero-mean random fields, with no loss of generality.

Remark 2.1. A Gaussian isotropic stationary random field on $\mathbb{S}^d \times \mathbb{R}$ has an invariant distribution under rotations on the spatial variable, and under translations on the temporal variable. Thus, its covariance function depends on its arguments via the great circle distance $\theta(\mathbf{x}, \mathbf{y})$ and the time lag, or equivalently, via the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}$ and the time lag.

2.3. Kernel covariance functions on the sphere cross time

In his celebrated paper, [Schoenberg \(1942\)](#) characterized the class of continuous functions $f : [-1, 1] \rightarrow \mathbb{R}$ such that $f(\cos \theta)$ is positive definite over the product space $\mathbb{S}^d \times \mathbb{S}^d$, with θ defined in 2.4. Recently, [Berg and Porcu \(2016\)](#) extended Schoenberg's characterization by considering the product space $\mathbb{S}^d \times G$, with G being a locally compact group. They defined the class $\Psi(\mathbb{S}^d, G)$ of continuous functions $\psi : [-1, 1] \times G \rightarrow \mathbb{C}$ such that $\psi(\cos \theta, u^{-1} \cdot v)$ is positive definite on $(\mathbb{S}^d \times G)^2$. In particular, the case $G = \mathbb{R}$ offers a characterization of spatio-temporal covariance functions of centred *isotropic stationary* Gaussian random fields over the sphere cross time.

Here we use the class $\Psi(\mathbb{S}^d, \mathbb{R})$ of continuous functions $\psi : [-1, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ such that the associated isotropic function h in (2.7) is positive definite.

The following result is rephrased from [Berg and Porcu \(2016\)](#).

Theorem 2.1. *Let $d \in \mathbb{N}$ and let $\psi : [-1, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ a mapping. Then, $\psi \in \Psi(\mathbb{S}^d, \mathbb{R})$ if and only if there exists a sequence $(\varphi_{j,d})_{j \geq 0}$ of continuous and positive definite functions on \mathbb{R} such that*

$$\psi(\cos \theta, t) = \sum_{j=0}^{\infty} \varphi_{j,d}(t) c_j(d, \cos \theta), \quad \theta \in [0, \pi], \quad t \in \mathbb{R}, \quad (2.8)$$

where

$$\varphi_{j,d}(t) = \frac{\dim(\mathcal{H}_j^d) \cdot \|\sigma_{d-1}\|}{\|\sigma_d\|} \int_{-1}^1 \psi(x, t) c_j(d, x) (1-x^2)^{d/2-1} dx. \quad (2.9)$$

The series in (2.8) is uniformly convergent for $(\theta, t) \in [0, \pi] \times \mathbb{R}$, and $\sum_{j=0}^{\infty} \varphi_{j,d}(0) < +\infty$.

Remark 2.2. Some comments are in order:

- For $\varepsilon_1 = (1, 0, \dots, 0)$ a unit vector in \mathbb{R}^{d+1} , we may consider the mapping $\psi_{\varepsilon_1} : \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\psi_{\varepsilon_1}(\mathbf{y}, t) = \psi(\langle \varepsilon_1, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}, t), \quad (\mathbf{y}, t) \in \mathbb{S}^d \times \mathbb{R}.$$

Then, arguments in page 13 of [Berg and Porcu \(2016\)](#), show that

$$\begin{aligned} \varphi_{j,d}(t) &= \dim(\mathcal{H}_j^d) \int_{\mathbb{S}^d} \psi(\langle \varepsilon_1, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}, t) c_j(d, \langle \varepsilon_1, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}) d\sigma_d(\mathbf{y}) \\ &= \dim(\mathcal{H}_j^d) \langle \psi_{\varepsilon_1}(\cdot, t), c_j(d, \langle \varepsilon_1, \cdot \rangle_{\mathbb{R}^{d+1}}) \rangle_{L^2(\mathbb{S}^d, \sigma_d)}. \end{aligned}$$

Hence, for each $t \in \mathbb{R}$, $\varphi_{j,d}(t)$ may be understood as the orthogonal projection of $\psi_{\varepsilon_1}(\cdot, t)$ onto \mathcal{H}_j^d .

- We also observe that, in comparison with the representation of covariance functions for isotropic random fields on the sphere, representation (3.1) does not consider Schoenberg coefficients $\varphi_{j,d}$ but Schoenberg functions $\varphi_{j,d}(\cdot)$, which will play a fundamental role subsequently.
- Let $\mathcal{P}(\mathbb{R})$ be the set of bounded, continuous and positive definite functions on \mathbb{R} . It is well known that any $\varphi \in \mathcal{P}(\mathbb{R})$ satisfies $|\varphi(t)| \leq \varphi(0)$, ([Sasvari, 1994](#)). Therefore, $\varphi \in \mathcal{P}(\mathbb{R})$, ensures that $\varphi \in L^2(\mathbb{R}, \nu)$ for any finite measure ν .

3. Expansions for isotropic stationary Gaussian random fields and kernel covariance functions

Apparently from Theorem 2.1, the kernel ψ of a isotropic stationary random field Z on $\mathbb{S}^d \times \mathbb{R}$ admits the following representation:

$$\psi(\theta(\mathbf{x}, \mathbf{y}), t - s) = \sum_{j=0}^{\infty} \varphi_{j,d}(t - s) c_j(d, \cos \theta), \quad (\mathbf{x}, t), (\mathbf{y}, s) \in \mathbb{S}^d \times \mathbb{R}, \quad (3.1)$$

where $\{\varphi_{j,d}\}_{j \geq 0}$ is a sequence of continuous positive definite functions on \mathbb{R} , such that the series is uniformly convergent.

Expression (3.1) allows to consider different expansions for the kernel. Before introducing these representations, we present the expansion for the random field which motivates the simulation methodology.

3.1. Karhunen-Loève expansions for isotropic stationary Gaussian random fields on the sphere cross time

For an isotropic stationary Gaussian random field Z over $\mathbb{S}^2 \times \mathbb{R}$, Jones (1963) proposed the following Karhunen-Loève representation

$$Z(\mathbf{x}, t) = \sum_{j=0}^{\infty} \sum_{m=-j}^j X_{j,m}(t) \mathcal{Y}_{j,m}(\mathbf{x}), \quad (\mathbf{x}, t) \in \mathbb{S}^2 \times \mathbb{R}, \quad (3.2)$$

where $\{X_{j,m}(t)\}_{j,m}$ is a sequence of one-dimensional mutually independent stochastic processes. The set of all $X_{j,m}(t)$ forms a denumerable infinite dimensional stochastic process which completely defines the process on the sphere, and $\mathcal{Y}_{j,m}(\mathbf{x})$ are the spherical harmonics.

Representation (3.1) for the spatio-temporal covariance and (3.2) for the random field, suggests a spectral simulation method. Let us introduce the following family of Gaussian random fields on $\mathbb{S}^d \times \mathbb{R}$.

Definition 3.1. Let Z be a random field on $\mathbb{S}^d \times \mathbb{R}$ defined as

$$Z(\mathbf{x}, t) = \sum_{j=0}^{\infty} c_{j,d} \sum_{m=0}^{\dim(\mathcal{H}_j^d)} X_{j,m,d}(t) \mathcal{Y}_{j,m,d}(\mathbf{x}), \quad (3.3)$$

where for each j, m and d , $\{X_{j,m,d}(t), t \in \mathbb{R}\}$ is a zero-mean stationary Gaussian process such that

$$\text{cov}\{X_{j,m,d}(t), X_{j',m',d}(s)\} := \mathbb{E}\{X_{j,m,d}(t) X_{j',m',d}(s)\} = \varphi_{j,d}(t - s) \delta_{jj'} \delta_{mm'}. \quad (3.4)$$

Here, δ_{ij} denotes the Kronecker delta function, $\{\varphi_{j,d}\}_{j=0}^{\infty}$ represents the Schoenberg's functions associated to the mapping ψ in Equation (3.1), and $c_{j,d} = \left(\frac{(d-1)}{(2j+d-1)C_j^{(d-1)/2}(1)} \right)^{1/2}$.

For each j, m, d , the process $\{X_{j,m,d}(t), t \in \mathbb{R}\}$ has the following Karhunen-Loève expansion (see Appendix 7.1),

$$X_{j,m,d}(t) = \sum_{k=0}^{\infty} \lambda_{k,j,m,d} \zeta_k(t), \quad t \in \mathbb{R}, \quad (3.5)$$

where for each $j, m, d \in \mathbb{N}$, $\{\lambda_{k,j,m,d}\}_k$ is a sequence of independent random variables defined by

$$\lambda_{k,j,m,d} := \int_{\mathbb{R}} X_{j,m,d}(t) \zeta_k(t) d\mu(t).$$

Also, $\lambda_{j,m,d} \sim \mathcal{N}(0, a_{k,j,d})$, where $\{a_{k,j,d}\}_k$ and $\{\zeta_k\}_k$ are the eigenvalues and eigenvectors (respectively) of the integral operator associated to $\varphi_{j,d}$.

Proposition 3.1. *Let Z be a random field as in Definition 3.1. Then, Z is a Gaussian, isotropic stationary random field, with zero-mean and covariance function given by ψ as in (3.1).*

Proof. The proof follows straightforward by using the properties of the process $X_{j,m,d}$ and the addition formula in Equation (2.2). Details are left to the reader. \square

Remark 3.1. An alternative way to write (3.3) is:

$$Z(\mathbf{x}, t) = \sum_{j=0}^{\infty} c_{j,d} \sum_{m=1}^{\dim(\mathcal{H}_j^d)} \sum_{k=0}^{\infty} \lambda_{k,j,m,d} \zeta_k(t) \mathcal{Y}_{j,m,d}(\mathbf{x}), \quad (\mathbf{x}, t) \in \mathbb{S}^d \times \mathbb{R}. \quad (3.6)$$

Expressions (3.3) or (3.6) represent a way to construct isotropic stationary Gaussian random fields on the sphere cross time. However, it is not yet proved that any random field with this properties can be written in this way.

3.2. Hermite expansion of kernel covariance functions

The Schoenberg's functions $\{\varphi_{j,d}\}_{j=0}^{\infty}$ associated to ψ in Equation (3.1) belongs to the class $\mathcal{P}(\mathbb{R})$. Thus, $\{\varphi_{j,d}\}_{j=0}^{\infty} \subset L^2(\mathbb{R}, \nu)$, with ν the standard Gaussian measure on \mathbb{R} . Hence, they can be expanded in terms of normalized Hermite polynomials. For each $j \in \mathbb{N}$ we write

$$\varphi_{j,d}(u) = \sum_{k=0}^{\infty} \alpha_{j,k,d} H_k(u), \quad u \in \mathbb{R}, \quad (3.7)$$

where H_k is the normalized Hermite polynomial of degree k given by

$$H_k(u) = \frac{(-1)^n}{\sqrt{k!}\sqrt{2\pi}} e^{\frac{u^2}{2}} \frac{d^n}{du^n} e^{-\frac{u^2}{2}}. \quad (3.8)$$

Consequently, the kernel ψ in (3.1) can be reformulated as

$$\psi(\theta, t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{j,k,d} H_k(t) c_{j,d}(\cos \theta), \quad (\theta, t) \in [0, \pi] \times \mathbb{R}. \quad (3.9)$$

We call the series $\{\alpha_{k,j,d}\}_{k,j \in \mathbb{N}}$ the *Spatio-temporal Hermite power spectrum*.

3.3. Double Karhunen-Loève expansion of kernel covariance functions

Representation (3.5) allows to re-write the Schoenberg's functions as

$$\varphi_{j,d}(t-s) = \sum_{k=0}^{\infty} a_{k,j,d} \zeta_k(t) \zeta_k(s), \quad t, s \in \mathbb{R}. \quad (3.10)$$

By recalling the stationarity condition, the reader may verify that the last expression can be reformulated to

$$\varphi_{j,d}(t-s) = \sum_{k=0}^{\infty} a_{k,j,d} \varepsilon_k(t-s), \quad (3.11)$$

where $\{\varepsilon_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Hence, the kernel covariance function ψ in (3.1) also admits the expansion

$$\psi(\theta, t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{k,j,d} \varepsilon_k(t) c_{j,d}(\cos \theta), \quad (\theta, t) \in [0, \pi] \times \mathbb{R}. \quad (3.12)$$

Following Lang and Schwab (2015), we call the series $\{a_{k,j,d}\}_{k,l \in \mathbb{N}}$ the *Spatio-temporal angular power spectrum*.

In this context, the properties satisfied by the Schoenberg's functions (Theorem 2.1) are translated to the fact that the family of functions $\{\sum_{k=0}^{\infty} a_{k,j,d} \varepsilon_k(\cdot)\}_{j \geq 0}$ is a sequence of continuous and positive definite functions on \mathbb{R} , and

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{k,j,d} \varepsilon_k(0) < +\infty. \quad (3.13)$$

4. Regularity properties

This section is devoted to study the behaviour of the kernel covariance function associated to an isotropic stationary GRF Z . It will be shown that the regularity of such kernels is closely related to the decay of the Hermite power spectrum (or the angular power spectrum). Moreover, the latter characterizes also the $(K - J)$ -term truncation of a GRF Z as in Equation (3.6). We recall that we have introduced two expansions for the kernel covariance function of an isotropic stationary GRF Z :

- i.- Hermite expansion according to Formula (3.9), valid for any isotropic stationary GRF Z over $(\mathbb{S}^d, \sigma_d) \times (\mathbb{R}, \nu)$ with ν the standard Gaussian measure.
- ii.- Double Kaurhunen-Loève expansion according to Formula (3.12), valid for isotropic stationary GRF as in (3.6) over $(\mathbb{S}^d, \sigma_d) \times (\mathbb{R}, \mu)$ with μ the Lebesgue measure.

We follow Lang and Schwab (2015) when using an alternative parametrization in terms of the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}$ and the time lag. Define, $\psi_I : [-1, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\psi_I(u, t) := \psi(\arccos u, t), \quad \text{for } \psi \in \Psi(\mathbb{S}^d, \mathbb{R}). \quad (4.1)$$

Hence, for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$ and $t, s \in \mathbb{R}$,

$$\text{cov}(Z(\mathbf{x}, t), Z(\mathbf{y}, s)) = \psi(\theta(\mathbf{x}, \mathbf{y}), t - s) = \psi_I(\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^{d+1}}, t - s). \quad (4.2)$$

Remark 4.1. Considering the relations between Gegenbauer and Legendre polynomials, in the case $\mathbb{S}^2 \times \mathbb{R}$, the kernel (3.1) turns out to be

$$\sum_{j=0}^{\infty} \varphi_{j,2}(t - s) P_j(\cos \theta(\mathbf{x}, \mathbf{y})) = \psi_I(\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^3}, t - s), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^2, t, s \in \mathbb{R}. \quad (4.3)$$

Remark 4.2. In what follows we present the regularity analysis of the kernel in relation to the behaviour of the two proposed expansions for the kernels. We consider the case $d = 2$. The general case $d \geq 2$ being similar but the computations consider Gegenbauer polynomials instead of Legendre polynomials, is included in a supplementary material.

4.1. Regularity analysis for the Hermite expansion

In this part of the manuscript we consider the measure space (\mathbb{R}, ν) with ν the standard Gaussian measure.

For $n, m \in \mathbb{N}$ let the spaces $H^n(-1, 1) \subset L^2(-1, 1)$ and $H^m(\mathbb{R}) \subset L^2(\mathbb{R})$ be the standard Sobolev spaces. We extend the proposal in Lang and Schwab (2015) and consider the

function spaces $W^{n,m} := W^{n,m}((-1,1) \times \mathbb{R})$ as the closure of $H^n(-1,1) \times H^m(\mathbb{R})$ with respect to the weighted norm $\|\cdot\|_{W^{n,m}((-1,1) \times \mathbb{R})}$ given by

$$\|f\|_{W^{n,m}}^2 := \sum_{j,k=0}^{n,m} |f|_{W^{j,k}}^2, \quad (4.4)$$

where for $j, k \in \mathbb{N}$

$$|f|_{W^{j,k}}^2 := \int_{\mathbb{R}} \int_{-1}^1 \left(\frac{\partial^k}{\partial t^k} \frac{\partial^j}{\partial x^j} f(x,t) \right)^2 (1-x^2)^j dx d\nu. \quad (4.5)$$

Analogously, we abuse of notation by writing L^2 instead of $L^2((-1,1) \times \mathbb{R})$.

We consider the canonical partial order relation in \mathbb{N}^2 : $(n,m) \leq (n',m')$ iff $[n \leq n'$ and $m \leq m']$. Note that $(W^{n,n}, n \in \mathbb{N})$ is a decreasing sequence of separable Hilbert spaces, i.e.

$$L^2(-1,1) \times L^2(\mathbb{R}) = W^{0,0} \supset W^{1,1} \supset \dots \supset W^{n,n} \supset \dots$$

By Theorem 5.2 in [Adams and Fournier \(2003\)](#), the norm of $W^{n,m}$ is equivalent to the first and the last element of the sum, i.e.

$$\|f\|_{W^{n,m}}^2 \simeq \|f\|_{L^2}^2 + |f|_{W^{n,m}}^2, \quad f \in W^{n,m}. \quad (4.6)$$

Let $f \in L^2$. As the normalized Hermite polynomials $\{H_k\}_{k \in \mathbb{N}}$ constitute a orthonormal basis of $L^2(\mathbb{R})$, and the normalized Legendre polynomials $\{P_j\}_{j \in \mathbb{N}}$ a orthonormal basis for $L^2(-1,1)$, it is clear that $\{H_k \cdot P_j\}_{k,j \in \mathbb{N}}$ is an orthonormal basis for $L^2((-1,1) \times \mathbb{R}) \cong L^2(-1,1) \times L^2(\mathbb{R})$. Therefore, f can be expanded in the series

$$f(x,t) = \sum_{k,j=0}^{\infty} b_{k,j} H_k(t) P_j(x), \quad (x,t) \in [-1,1] \times \mathbb{R}, \quad (4.7)$$

with

$$b_{k,l} := \int_{\mathbb{R}} \int_{-1}^1 f(x,t) H_k(t) P_j(x) dx d\nu, \quad k,l \in \mathbb{N}. \quad (4.8)$$

Hence, f is a valid covariance kernel of the type (4.2), provided $b_{k,j}$ in (4.7) are non-negative.

This allows to tackle the problem in a different way. Instead of using spectral techniques, regularity of kernels might be shown through an isomorphism between the spaces $W^{n,m}((-1,1) \times \mathbb{R})$ and the weighted bi-sequence spaces

$$(kj)_{m,n} := \ell^2 \left(k^{m/2} \cdot j^n; \quad k,j \in \mathbb{N} \right), \quad (4.9)$$

where $(k^{m/2} \cdot j^n; \quad k,j \in \mathbb{N})$ denotes the sequence of weights.

From now on, for the sake of simplicity, we only consider weighted Sobolev spaces $W^n := W^{n,n}$, and the weighted bi-sequence spaces $(kj)_n = (kj)_{n,n}$, obtained as a special case of Equation (4.9) when $n = m$. The careful reader will observe that the following results may be extended to the case $n \neq m$.

In order to extend the isomorphism to spaces W^η with η being not an integer, following Triebel (1999) we now introduce the interpolation spaces $W^\eta := W^\eta((-1, 1) \times \mathbb{R})$ for $n < \eta < n + 1$, defined through:

$$W^\eta := (W^n, W^{n+1})_{\eta-n, 2}, \quad (4.10)$$

equipped with the norm $\|f\|_{W^\eta}$ given by

$$\|f\|_{W^\eta} = \int_0^\infty r^{-2(\eta-n)} |J(r, f)|^2 \frac{dr}{r}, \quad (4.11)$$

where the functional J is defined by

$$J(r, f) = \inf_{f=v+w} (\|v\|_{W^n} + r \cdot \|w\|_{W^{n+1}}), \quad r > 0, v \in W^n, w \in W^{n+1}. \quad (4.12)$$

The definition of the interpolation spaces $(kj)_\eta$ for η non-integer is carried out in analogous way.

The *interpolation property* (see section 2.4.1 in Triebel (1983)), implies that, if the spaces W^n and $(kl)_n$ are isomorphic for all $n \in \mathbb{N}$, then they are isomorphic for all $\eta \in \mathbb{R}_+$.

Theorem 4.1. *Let $f \in L^2$ and $\eta \in \mathbb{R}_+$ be given. Then, $f \in W^\eta$ if and only if*

$$\sum_{k,j=0}^{\infty} b_{k,j}^2 k^\eta \cdot j^{2\eta} < \infty,$$

i.e.

$$\|f\|_{W^\eta}^2 \simeq \sum_{k,j=0}^{\infty} b_{k,j}^2 k^\eta \cdot j^{2\eta}$$

is an equivalent norm in W^η .

For $\psi_I \in W^n$, $n \in \mathbb{N}$, the equivalence is reduced to: $(\alpha_{k,j} k^{n/2} j^n, k, j \in \mathbb{N})$ is in $\ell^2(\mathbb{N}^2)$ if and only if $\frac{\partial^n}{\partial t^n} \frac{\partial^n}{\partial x^n} \psi_I(x, t)(1-x^2)^{n/2}$ is in $L^2[(-1, 1) \times \mathbb{R}]$, i.e.

$$\sum_{k,j=n}^{\infty} \alpha_{k,j}^2 k^n \cdot j^{2n} < +\infty$$

if and only if

$$\int_{\mathbb{R}} \int_{-1}^1 \left(\frac{\partial^n}{\partial t^n} \frac{\partial^n}{\partial x^n} \psi_I(x, t) \right)^2 (1-x^2)^n dx d\nu < +\infty.$$

Proof. Assume first that the claim is already proved for $\eta \in \mathbb{N}$, i.e., W^η is isomorphic to the weighted bi-sequence space $(kj)_n$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$, let $n < \eta < n + 1$ and set $\kappa := \eta - n$. By the interpolation theorem of Stein-Weiss (see Theorem 5.4.1 in [Bergh and Löfström \(1976\)](#)), the weights of $(kj)_\eta$ are given by

$$(k^n j^{2n})^{1-\kappa} \cdot (k^{n+1} j^{2(n+1)})^\kappa = k^\eta j^{2\eta}.$$

Now, we prove the isomorphism between W^n and $(kj)_n$ for $n \in \mathbb{N}$, which is equivalent to prove the second formulation of the theorem. We have that,

$$\begin{aligned} & \int_{\mathbb{R}} \int_{-1}^1 \left(\frac{\partial^n}{\partial t^n} \frac{\partial^n}{\partial x^n} \psi_I(x, t) \right)^2 (1-x^2)^n dx d\nu \\ &= \int_{\mathbb{R}} \int_{-1}^1 \left(\sum_{k,j=0}^{\infty} \alpha_{k,j} \frac{\partial^n}{\partial t^n} H_k(t) \frac{\partial^n}{\partial x^n} P_j(x) \right)^2 (1-x^2)^n dx d\nu \\ &= \sum_{k,k',j,j'=0}^{\infty} \alpha_{k,j} \alpha_{k',j'} \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} H_k(t) \frac{\partial^n}{\partial t^n} H_{k'}(t) d\nu \\ & \quad \cdot \int_{-1}^1 \frac{\partial^n}{\partial x^n} P_j(x) \frac{\partial^n}{\partial x^n} P_{j'}(x) (1-x^2)^n dx \\ &= \sum_{k,k',j,j'=0}^{\infty} \alpha_{k,j} \alpha_{k',j'} \cdot I_{k,k'}(t) \cdot I_{j,j'}(x), \end{aligned} \tag{4.13}$$

where

$$I_{k,k'}(t) = \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} H_k(t) \frac{\partial^n}{\partial t^n} H_{k'}(t) d\nu$$

and

$$I_{j,j'}(x) = \int_{-1}^1 \frac{\partial^n}{\partial x^n} P_j(x) \frac{\partial^n}{\partial x^n} P_{j'}(x) (1-x^2)^n dx.$$

Standard properties of normalized Hermite polynomials show that $I_{k,k'}(t) = 0$ if $k < n$, and

$$I_{k,k'}(t) = \frac{k!}{(k-n)!} \delta_{k,k'} ; \quad k \geq n.$$

Now, we observe that

$$\frac{k^n}{(k-n)^{k-n}} = \frac{k^n}{(1-n/k)^{k(1-n/k)}}.$$

Then, by Stirling inequality

$$\sqrt{2\pi} k^{k+1/2} e^{-k} \leq k! \leq e k^{k+1/2} e^{-k},$$

shows that, for $k \geq n$, there exists constants $c_1(n)$ and $c_2(n)$ such that

$$c_1(n)k^n \leq \frac{k!}{(k-n)!} \leq c_2(n)k^n.$$

Hence,

$$I_{k,k'}(t) \simeq k^n \delta_{k,k'} \quad , \quad k \geq n. \quad (4.14)$$

Similar arguments as the previous calculations for $I_{k,k'}(t)$, and the second step in the proof of Theorem 3.1 in [Lang and Schwab \(2015\)](#), allow to conclude that $I_{j,j'}(x) = 0$ if $j < n$, and

$$I_{j,j'}(x) \simeq j^{2n} \delta_{j,j'} \quad , \quad j \geq n. \quad (4.15)$$

Therefore, from (4.13), (4.14) and (4.15) we deduce that

$$\int_{\mathbb{R}} \int_{-1}^1 \left(\frac{\partial^n}{\partial t^n} \frac{\partial^n}{\partial x^n} \psi_I(x, t) \right)^2 (1-x^2)^n dx d\nu \simeq \sum_{k,j=n}^{\infty} \alpha_{k,j}^2 k^n j^{2n}, \quad (4.16)$$

which concludes the proof. \square

4.2. Regularity analysis for the double Karhunen-Loève expansion

With the exception of minor details on the definitions of the spaces, this part of the manuscript follows quite similarly to Section 4.1.

For $n \in \mathbb{N}$ consider the spaces $H^n(-1, 1) \times L^2(\mathbb{R}) \subset L^2(-1, 1) \times L^2(\mathbb{R}) \cong L^2((-1, 1) \times \mathbb{R})$, and define the function spaces $V_T^n := V_T^n((-1, 1) \times \mathbb{R})$ as the closures of $H^n(-1, 1) \times L^2(\mathbb{R})$ with respect to the weighted norms $\|\cdot\|_{V_T^n}$ given by

$$\|f\|_{V_T^n((-1,1) \times \mathbb{R})}^2 := \sum_{j=0}^n |f|_{V_T^j((-1,1) \times \mathbb{R})}^2, \quad (4.17)$$

where for $j \in \mathbb{N}$

$$|f|_{V_T^j}^2 := \int_{\mathbb{R}} \int_{-1}^1 \left(\frac{\partial^j}{\partial x^j} f(x, t) \right)^2 (1-x^2)^j dx d\mu. \quad (4.18)$$

$(V_T^n, n \in \mathbb{N})$ is a decreasing sequence of separable Hilbert spaces, and

$$\|f\|_{V_T^n}^2 \simeq \|f\|_{L^2}^2 + |f|_{V_T^n}^2, \quad f \in V_T^n. \quad (4.19)$$

Now we look at the weighted bi-sequence spaces $(kj)_{T,n} := \ell^2((j^n; k, j \in \mathbb{N}))$.

Analogously to Section 4.1, we consider the interpolation spaces V_T^η for $n < \eta < n+1$. Next result follows exactly the same lines as the Theorem 4.1. The proof is omitted.

Theorem 4.2. *Let $f \in L^2$ and $\eta \in \mathbb{R}_+$ be given. Then, $f \in V_T^\eta$ if and only if*

$$\sum_{k,j=0}^{\infty} b_{k,j}^2 j^{2\eta} < \infty,$$

i.e.

$$\|f\|_{V_T^\eta}^2 \simeq \sum_{k,j=0}^{\infty} b_{k,j}^2 j^{2\eta}$$

is an equivalent norm in V_T^η .

For $\psi_I \in V_T^n$, $n \in \mathbb{N}$, the equivalence is reduced to: $(a_{k,j} j^n, k, j \in \mathbb{N})$ is in $j^2(\mathbb{N}^2)$ if and only if $\frac{\partial^n}{\partial x^n} \psi_I(x, t)(1-x^2)^{n/2}$ is in $L^2[(-1, 1) \times \mathbb{R}]$, i.e.

$$\sum_{k,j=n}^{\infty} a_{k,j}^2 j^{2n} < +\infty$$

if and only if

$$\int_{\mathbb{R}} \int_{-1}^1 \left(\frac{\partial^n}{\partial x^n} \psi_I(x, t) \right)^2 (1-x^2)^n dx d\nu < +\infty.$$

Remark 4.3. By taking into account the normalizing constants, all the previous results encompasses the results in Section 3 of [Lang and Schwab \(2015\)](#).

Remark 4.4. Previous result are also valid for the general case, $d \geq 2$. The proofs follows similar arguments but this time considering Gegenbauer polynomials instead of Legendre polynomials. Details are left to the reader.

5. Sample Hölder continuity for the case $d = 2$

We recall the *geodesic* distance ρ on $\mathbb{S}^2 \times \mathbb{R}$ introduced in Section 2,

$$\rho((\mathbf{x}, s), (\mathbf{y}, t)) = \sqrt{\theta(\mathbf{x}, \mathbf{y})^2 + (t-s)^2}, \quad (\mathbf{x}, s), (\mathbf{y}, t) \in \mathbb{S}^2 \times \mathbb{R}.$$

The geodesic distance naturally induces a metric on $[0, \pi] \times \mathbb{R}$, by

$$\rho((0, 0), (\theta, u)) = \rho(\theta, u) = \sqrt{\theta^2 + u^2}, \quad (\theta, u) \in [0, \pi] \times \mathbb{R}.$$

Hypothesis 5.1. *There exists a $\delta > 0$ such that*

$$\sum_{k,j}^{\infty} \alpha_{k,j} \cdot k^{\delta/8} \cdot j^{(d-1+\delta)/2} < +\infty.$$

Proposition 5.1. *Let $\{\alpha_{k,j}, k, j \in \mathbb{N}^2\}$ be the Hermite power spectrum of an isotropic stationary GRF on $\mathbb{S}^2 \times \mathbb{R}$ which satisfies Hypothesis 4.1 with $d = 2$ for some $\delta \in]0, 2]$. Then, there exists a positive constant C_δ such that for all $(\theta, u) \in [0, \pi] \times \mathbb{R}$ the corresponding kernel function ψ satisfies*

$$|\psi(0, 0) - \psi(\theta, u)| \leq C_\delta \rho(\theta, u)^\delta. \quad (5.1)$$

Proof. Without loss of generality, we may consider $|u| < 1$.

$$\begin{aligned} |\psi(0, 0) - \psi(\theta, u)| &= \left| \sum_{k,j}^{\infty} \alpha_{k,j} H_k(0) P_j(1) - \sum_{k,j}^{\infty} \alpha_{k,j} H_k(u) P_j(\cos \theta) \right| \\ &\leq \sum_{k,j}^{\infty} \alpha_{k,j} |H_k(0) \cdot P_j(1) - H_k(u) P_j(\cos \theta)| \\ &\leq \sum_{k,j}^{\infty} \alpha_{k,j} \left(|H_k(0) P_j(1) - H_k(u) P_j(1)| \right. \\ &\quad \left. + |H_k(u) P_j(1) - H_k(u) P_j(\cos \theta)| \right). \end{aligned} \quad (5.2)$$

From Theorem 1 in [Bonan and Clark \(1990\)](#), there exists a positive constant c such that

$$|H_n(u)| \leq \frac{c \cdot e^{u^2}}{(2n+1-u^2)^{1/4}}, \quad \text{for } |u| \leq \sqrt{2n+1}. \quad (5.3)$$

Therefore,

$$\begin{aligned} |H_k(0) - H_k(u)| &= \left| \int_0^u H'_k(x) dx \right| \\ &= \left| \int_0^u \sqrt{k} H_{k-1}(x) dx \right| \\ &\leq |u| \sqrt{k} \cdot \frac{c}{(2(k-1))^{1/4}} \\ &\leq c \cdot |u| \cdot k^{1/4}. \end{aligned} \quad (5.4)$$

Moreover, as $|u| \leq 1$, from (5.3) we deduce that there exists a constant $c' > 0$ such that

$$|H_k(0) - H_k(u)| \leq c'.$$

Hence, by interpolation we have that for all $\gamma \in [0, 1]$,

$$|H_k(0) - H_k(u)| \leq \left(c \cdot |u| \cdot k^{1/4} \right)^\gamma (c')^{1-\gamma} \leq C \cdot |u|^\gamma \cdot k^{\gamma/4}.$$

Therefore, by setting $\tilde{P}_j = P_j \cdot \|P_j\|$ as the non-normalized Legendre polynomial, recalling that $\tilde{P}_j(1) = 1$ for all $j \in \mathbb{N}$, the bounds developed in the proof of Lemma 4.2 in [Lang and Schwab \(2015\)](#), and equivalence of norms in \mathbb{R} , we have that for all $\gamma \in [0, 1]$, (the constant c need not to be the same from line to line),

$$\begin{aligned}
& |H_k(0)P_j(1) - H_k(u)P_j(1)| + |H_k(u)P_j(1) - H_k(u)P_j(\cos \theta)| \\
& \leq c \cdot \sqrt{2j+1} \cdot |u|^\gamma \cdot k^{\gamma/4} + C \cdot \sqrt{2j+1} \cdot k^{\gamma/4} \cdot |1 - \tilde{P}_j(\cos \theta)| \\
& \leq c \cdot \sqrt{2j+1} \cdot |u|^\gamma \cdot k^{\gamma/4} + C \cdot \sqrt{2j+1} \cdot k^{\gamma/4} \cdot |1 - \cos \theta|^\gamma \cdot (j(j+1))^\gamma \\
& \leq c \cdot k^{\gamma/4} \cdot \sqrt{2j+1} \cdot (j(j+1))^\gamma \cdot (|u|^\gamma + |1 - \cos \theta|^\gamma) \\
& \leq c \cdot k^{\gamma/4} \cdot \sqrt{2j+1} \cdot (j(j+1))^\gamma \cdot (|u|^{2\gamma} + \theta^{2\gamma}) \\
& \leq c \cdot k^{\gamma/4} \cdot \sqrt{2j+1} \cdot (j(j+1))^\gamma \cdot (|u|^2 + \theta^2)^\gamma \\
& = c \cdot k^{\gamma/4} \cdot \sqrt{2j+1} \cdot (j(j+1))^\gamma \cdot \rho(\theta, u)^{2\gamma}.
\end{aligned} \tag{5.5}$$

In consequence, from (5.2) and (5.5) we have that

$$|\psi(0, 0) - \psi(\theta, u)| \leq c \cdot \sum_{k,j}^{\infty} \alpha_{k,j} \sqrt{2j+1} \cdot k^{\gamma/4} \cdot (j(j+1))^\gamma \cdot \rho(\theta, u)^{2\gamma}.$$

We observe that, by hypothesis (5.1), the series in the last inequality converges. Therefore, by choosing $\delta = 2\gamma$ and

$$C_\delta := c \cdot \sum_{k,j}^{\infty} \alpha_{k,j} \sqrt{2j+1} \cdot k^{\delta/8} \cdot (j(j+1))^{\delta/2},$$

we have

$$|\psi(0, 0) - \psi(\theta, u)| \leq C_\delta \cdot \rho(\theta, u)^\delta.$$

□

We now have the components to prove the sample Hölder regularity of a isotropic stationary GRF.

Proposition 5.2. *Let Z be an isotropic stationary GRF on $\mathbb{S}^2 \times \mathbb{R}$ with Hermite power spectrum $\{\alpha_{k,j}, k, j \in \mathbb{N}^2\}$. If Hypothesis (5.1) is satisfied with $d = 2$ for some $\delta \in [0, 2]$, then for all $p \in \mathbb{N}$ there exists a constant $C_{\delta,p}$ such that for all $(\mathbf{x}, t), (\mathbf{y}, s) \in \mathbb{S}^2 \times \mathbb{R}$,*

$$\mathbb{E} (|Z(\mathbf{x}, t) - Z(\mathbf{y}, s)|^{2p}) \leq C_{\delta,p} \rho((\mathbf{x}, s), (\mathbf{y}, t)). \tag{5.6}$$

Proof 5.1. *Apparently, $Z(\mathbf{x}, t) - Z(\mathbf{y}, s)$ is a centered Gaussian random variable. On the other hand, if X is a $\mathcal{N}(0, \sigma^2)$ distributed random variable, then*

$$\mathbb{E} (|X|^{2p}) = \mathbb{E} (|\sigma Y|^{2p}) = (\sigma^2)^p \mathbb{E} (|Y|^{2p}) = \mathbb{E} (|X|^{2p})^p c_{2p},$$

for $p \in \mathbb{N}$, where Y is a standard normal random variable and $c_{2,p}$ denotes its second moment. We may also note that

$$\begin{aligned} \mathbb{E}(|Z(\mathbf{x}, t) - Z(\mathbf{y}, s)|^2) &= \mathbb{E}(Z(\mathbf{x}, t)^2) - 2\mathbb{E}(Z(\mathbf{x}, t)Z(\mathbf{y}, s)) + \mathbb{E}(Z(\mathbf{y}, s)^2) \\ &= \psi(\mathbf{x}, t, \mathbf{x}, t) - 2\psi(\mathbf{x}, t, \mathbf{y}, s) + \psi(\mathbf{y}, s, \mathbf{y}, s) \\ &= 2(\psi(0, 0) - \psi(\theta, t - s)). \end{aligned}$$

Combining the two previous observations, we conclude that

$$\begin{aligned} \mathbb{E}(|Z(\mathbf{x}, t) - Z(\mathbf{y}, s)|^{2p}) &= c_{2,p}\mathbb{E}(|Z(\mathbf{x}, t) - Z(\mathbf{y}, s)|^2)^p \\ &= 2c_{2,p}(\psi(0, 0) - \psi(\theta, t - s))^p \\ &\leq 2c_{2,p}C_\delta^p\rho((\mathbf{x}, s), (\mathbf{y}, t)), \end{aligned}$$

where the last inequality is due to Proposition 5.1.

6. Spatio-temporal spectral simulation

Our simulation method consists in a double truncation for the Karhunen-Loève expansion defined through Equation (3.6). Given two positive integers J and K , we can simulate a spatio-temporal Gaussian random field over $\mathbb{S}^2 \times [0, T]$, where T is the time horizon, using the following approximation

$$\widehat{Z}(\mathbf{x}, t) = \sum_{j=0}^J c_j \sum_{m=-j}^j \sum_{k=0}^K \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2, t \in [0, T], \quad (6.1)$$

where $c_j = \sqrt{\frac{4\pi}{2j+1}}$. For a neater exposition, along this section we omit the subscript associated to the spatial dimension $d = 2$. In the subsequent sections, we derive an explicit expression of Equation (6.1) for a real-valued random field. Also, we study the accuracy of the truncation in terms of J and K using the $L^2(\mathbb{S}^2 \times [0, T])$ -norm. Finally, we give some examples.

6.1. An explicit expression for real-valued fields

Expressions for spherical harmonics in arbitrary dimensions are complicated. When $d = 2$, algebraically closed forms are area feasible. We call $\mathcal{Y}_{j,m}$ the *surface spherical harmonics*. In order to introduce such functions, we need to define the associated Legendre polynomials. Let j be a non-negative integer and $m \in \{0, 1, \dots, j\}$. For $\mu \in [-1, 1]$, the associated Legendre polynomials $P_{j,m}$ are defined through

$$\begin{aligned} P_{j,m}(\mu) &= (-1)^m (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} (P_j(\mu)), \\ &= (2m - 1)!! (-1)^m (1 - \mu^2)^{m/2} C_{j-m}^{m+1/2}(\mu), \end{aligned} \quad (6.2)$$

where !! denotes the double factorial.

Every point on \mathbb{S}^2 can be characterized by its spherical coordinates, that is the pair $(\beta_1, \beta_2) \in [0, \pi] \times [0, 2\pi)$. Thus, the mapping $\mathcal{Y}_{j,m} : [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{C}$ is defined by

$$\begin{aligned}\mathcal{Y}_{j,m}(\beta_1, \beta_2) &= \sqrt{\frac{2j+1}{4\pi} \frac{(j-m)!}{(j+m)!}} P_{j,m}(\cos \beta_1) \exp(im\beta_2), \quad j \in \mathbb{N}_0, m \in \{0, \dots, j\}, \\ \mathcal{Y}_{j,m}(\beta_1, \beta_2) &= (-1)^m \overline{\mathcal{Y}_{j,-m}(\beta_1, \beta_2)}, \quad j \in \mathbb{N}, m \in \{-j, \dots, -1\},\end{aligned}\tag{6.3}$$

where \bar{c} denotes the conjugate of $c \in \mathbb{C}$. In the real-valued case, the spherical harmonics functions are reduced to

$$\mathcal{Y}_{j,m}(\beta_1, \beta_2) = \begin{cases} \sqrt{2} \sqrt{\frac{2j+1}{4\pi} \frac{(j-m)!}{(j+m)!}} P_{j,m}(\cos \beta_1) \cos(m\beta_2), & m > 0 \\ \sqrt{\frac{2j+1}{4\pi}} P_{j,0}(\cos \beta_1), & m = 0 \\ \sqrt{2} \sqrt{\frac{2j+1}{4\pi} \frac{(j-|m|)!}{(j+|m|)!}} P_{j,m}(\cos \beta_1) \sin(|m|\beta_2), & m < 0. \end{cases}\tag{6.4}$$

On the other hand, we consider the temporal mappings, $\zeta_k(\cdot)$, as a trigonometric Fourier basis of $L^2([0, T])$. Let $(\beta_1, \beta_2) \in [0, \pi] \times [0, 2\pi)$ be the angles associated to the spherical coordinates of the location $\mathbf{x} = \mathbf{x}(\beta_1, \beta_2) \in \mathbb{S}^2$. We have the following simulation method for a real-valued random field defined on $\mathbb{S}^2 \times [0, T]$

$$\begin{aligned}\widehat{Z}(\mathbf{x}, t) &= \sum_{j=0}^J c_j \sqrt{a_{0,j}} \left\{ A_{0,j,0}^{(1)} \tilde{P}_{j,0}(\cos \beta_1) + \sqrt{2} \sum_{m=1}^j \tilde{P}_{j,m}(\cos \beta_1) \cdot \left[A_{0,j,m}^{(1)} \cos(m\beta_2) \right. \right. \\ &\quad \left. \left. + B_{0,j,m}^{(1)} \sin(m\beta_2) \right] \right\} \\ &+ \sum_{j=0}^J c_j \sum_{k=1}^K \sqrt{a_{k,j}} \left\{ \tilde{P}_{j,0}(\cos \beta_1) \left[A_{k,j,0}^{(1)} \cos\left(\frac{\pi kt}{2T}\right) + A_{k,j,0}^{(2)} \sin\left(\frac{\pi kt}{2T}\right) \right] \right. \\ &\quad \left. + \sqrt{2} \sum_{m=1}^j \tilde{P}_{j,m}(\cos \beta_1) \left(\cos(m\beta_2) \left[A_{k,j,m}^{(1)} \cos\left(\frac{\pi kt}{2T}\right) + A_{k,j,m}^{(2)} \sin\left(\frac{\pi kt}{2T}\right) \right] \right. \right. \\ &\quad \left. \left. + \sin(m\beta_2) \left[B_{k,j,m}^{(1)} \cos\left(\frac{\pi kt}{2T}\right) + B_{k,j,m}^{(2)} \sin\left(\frac{\pi kt}{2T}\right) \right] \right) \right\},\end{aligned}$$

where $\{A_{k,j,m}^{(r)}\}$ and $\{B_{k,j,m}^{(r)}\}$ are sequences of independent standard Gaussian random variables, for $r = 1, 2$, $\{a_{k,j}\}_{k,j}$ is the spectrum, and $\tilde{P}_{j,m}$ represents the normalized Legendre function

$$\tilde{P}_{j,m}(\cdot) = \sqrt{\frac{4\pi}{2j+1} \frac{(j-m)!}{(j+m)!}} P_{j,m}(\cdot).$$

6.2. Accuracy of the method

We assess the $L^2(\mathbb{S}^2 \times [0, T])$ -error associated to the truncated expansion. We consider the Karhunen-Loève representation of Z and its truncation \widehat{Z} given in Equation (6.1), for two positive integers J and K . We follow the scheme used by [Kozachenko and Kozachenko \(2001\)](#) in the spatial context and extend the result to the spatio-temporal case. First, we need the following lemma.

Lemma 6.1. *Let ξ_1, ξ_2, \dots be a sequence of independent Gaussian random variables such that $\mathbb{E}(\xi_i) = 0$ and $\mathbb{E}(\xi_i^2) = \sigma_i^2$, with $\sum_{i=1}^{\infty} \sigma_i^2 < +\infty$, then for any $0 \leq u \leq 1$, we have*

$$\mathbb{E} \left(\exp \left\{ \frac{u}{2(\sum_{i=1}^{\infty} \sigma_i^4)} \left(\sum_{i=1}^{\infty} \xi_i^2 - \sum_{i=1}^{\infty} \sigma_i^2 \right) \right\} \right) \leq \exp\{-u/2\}(1-u)^{-1/2}. \quad (6.5)$$

The proof of Lemma 6.1 is given in Corollary 1 in [Kozachenko and Kozachenko \(2001\)](#). Next, we state the main result of this section.

Theorem 6.1. *Suppose that $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{k,j} < +\infty$. Then, for any $\epsilon > 0$, the following inequality holds*

$$\mathbb{P} \left\{ \|Z(\mathbf{x}, t) - \widehat{Z}(\mathbf{x}, t)\|_{L^2(\mathbb{S}^2 \times [0, T])} > \sqrt{P_{J,K} + \epsilon Q_{J,K}} \right\} \leq \exp(-\epsilon/2)\sqrt{\epsilon + 1}, \quad (6.6)$$

where

$$P_{J,K} = 4\pi \left(\sum_{j=J+1}^{\infty} \sum_{k=0}^{\infty} a_{k,j} + \sum_{j=0}^J \sum_{k=K+1}^{\infty} a_{k,j} \right),$$

and

$$Q_{J,K} = 4\pi \left(\sum_{j=J+1}^{\infty} c_j^2 \sum_{k=0}^{\infty} a_{k,j}^2 + \sum_{j=0}^J c_j^2 \sum_{k=K+1}^{\infty} a_{k,j}^2 \right)^{1/2}.$$

Proof. First, note that the difference between the full expansion and the truncated

representation is given by

$$\begin{aligned}
Z(\mathbf{x}, t) - \widehat{Z}(\mathbf{x}, t) &= \sum_{j=0}^{\infty} c_j \sum_{k=0}^{\infty} \sum_{m=-j}^j \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}) \\
&\quad - \sum_{j=0}^J c_j \sum_{k=0}^K \sum_{m=-j}^j \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}) \\
&= \sum_{j=J+1}^{\infty} c_j \sum_{k=0}^{\infty} \sum_{m=-j}^j \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}) \\
&\quad + \sum_{j=0}^J c_j \sum_{k=K+1}^{\infty} \sum_{m=-j}^j \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}).
\end{aligned}$$

Then, we measure the error in terms of the $L^2(\mathbb{S}^2 \times [0, T])$ -norm:

$$\begin{aligned}
\|Z(\mathbf{x}, t) - \widehat{Z}(\mathbf{x}, t)\|_{L^2(\mathbb{S}^2 \times [0, T])}^2 &= \int_{[0, T]} \int_{\mathbb{S}^2} \left\{ \sum_{j=J+1}^{\infty} c_j \sum_{k=0}^{\infty} \sum_{m=-j}^j \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}) \right. \\
&\quad \left. + \sum_{j=0}^J c_j \sum_{k=K+1}^{\infty} \sum_{m=-j}^j \lambda_{k,j,m} \zeta_k(t) \mathcal{Y}_{j,m}(\mathbf{x}) \right\}^2 dx dt.
\end{aligned} \tag{6.7}$$

Next, we use the orthonormality of the spatial and temporal basis to obtain the following equality:

$$\begin{aligned}
\|Z(\mathbf{x}, t) - \widehat{Z}(\mathbf{x}, t)\|_{L^2(\mathbb{S}^2 \times [0, T])}^2 &= \int_{[0, T]} \left(\sum_{j=J+1}^{\infty} c_j^2 \sum_{m=-j}^j \left[\sum_{k=0}^{\infty} \lambda_{k,j,m} \zeta_k(t) \right]^2 \right. \\
&\quad \left. + \sum_{j=0}^J c_j^2 \sum_{m=-j}^j \left[\sum_{k=K+1}^{\infty} \lambda_{k,j,m} \zeta_k(t) \right]^2 \right) dt \\
&= \sum_{j=J+1}^{\infty} c_j^2 \sum_{k=0}^{\infty} \sum_{m=-j}^j (\lambda_{k,j,m})^2 + \sum_{j=0}^J c_j^2 \sum_{k=K+1}^{\infty} \sum_{m=-j}^j (\lambda_{k,j,m})^2.
\end{aligned}$$

Therefore, $\|Z(\mathbf{x}, t) - \widehat{Z}(\mathbf{x}, t)\|_{J^2(\mathbb{S}^2 \times [0, T])}^2$ is the sum of the square of independent centered Gaussian random variables, so that, using Lemma 6.1 we obtain the bound

$$\mathbb{E} \left(\exp \left\{ \frac{u \|Z(\mathbf{x}, t) - \widehat{Z}(\mathbf{x}, t)\|_{L^2(\mathbb{S}^2 \times [0, T])}^2 - P_{J,K}}{2Q_{J,K}} \right\} \right) \leq \exp\{-u/2\} (1-u)^{-1/2}. \tag{6.8}$$

Finally, the proof follows from Chebyshev inequality and the analogous arguments used in [Kozachenko and Kozachenko \(2001\)](#).

□

6.3. Examples

We illustrate some examples generated from our proposal. All numerical experiments were carried out on a 2.7 GHz processor with 8 GB of memory and implemented coupling R functions and C routines. Simple examples can be generated by choosing the coefficients with a determined structure. For instance, consider the following options

$$a_{k,j} = \frac{\xi}{(1+j)^{\nu_1}(1+k)^{\nu_2}}, \quad (6.9)$$

or

$$a_{k,j} = \frac{\xi}{(1+j^{\nu_1}+k^{\nu_2})^\tau}, \quad (6.10)$$

with $\xi > 0$, $\nu_i > 1$, for $i = 1, 2$, and $\tau > 1$, in order to ensure the summability of this coefficients. In [Lang and Schwab \(2015\)](#) some realizations are illustrated using a similar spectrum, but in a merely spatial context. In [Figure 1](#), we show a spatio-temporal realization on $\mathbb{S}^2 \times \{1, 2\}$, over 16000 spatial locations, with the coefficients in [Equation \(6.9\)](#) in two cases: (a) $\nu_1 = 3$ and $\nu_2 = 2$, and (b) $\nu_1 = 2$ and $\nu_2 = 3$. In [Figure 2](#), a similar realization is obtained from the spectrum [\(6.10\)](#), with $\tau = 3/2$. For each case, we truncate the series using $K = J = 50$ and choose ξ such that the realizations have unit variance. Note that the parameter ν_1 is the responsible of the spatial scale and smoothness of the realization. [Table 1](#) shows the execution times in terms of the number of observations.

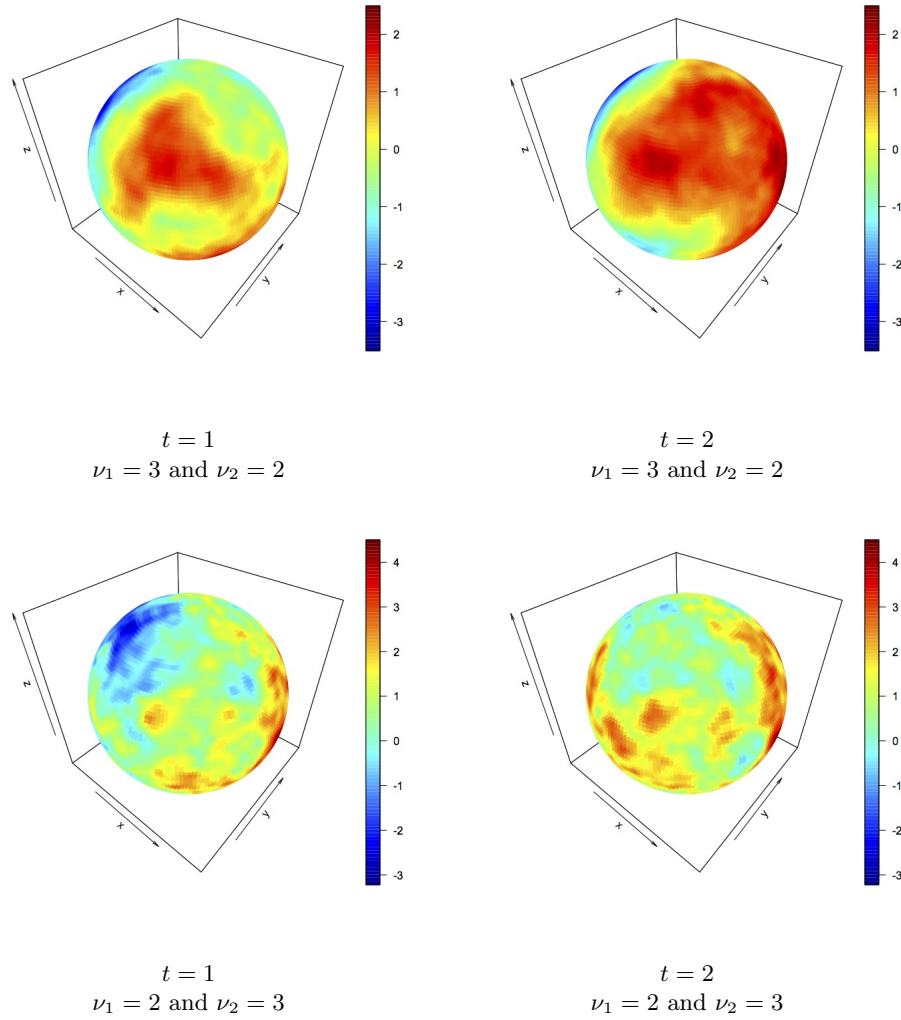


Figure 1. Spatio-temporal realization on $\mathbb{S}^2 \times \{1, 2\}$ with spectrum as in Equation (6.9).

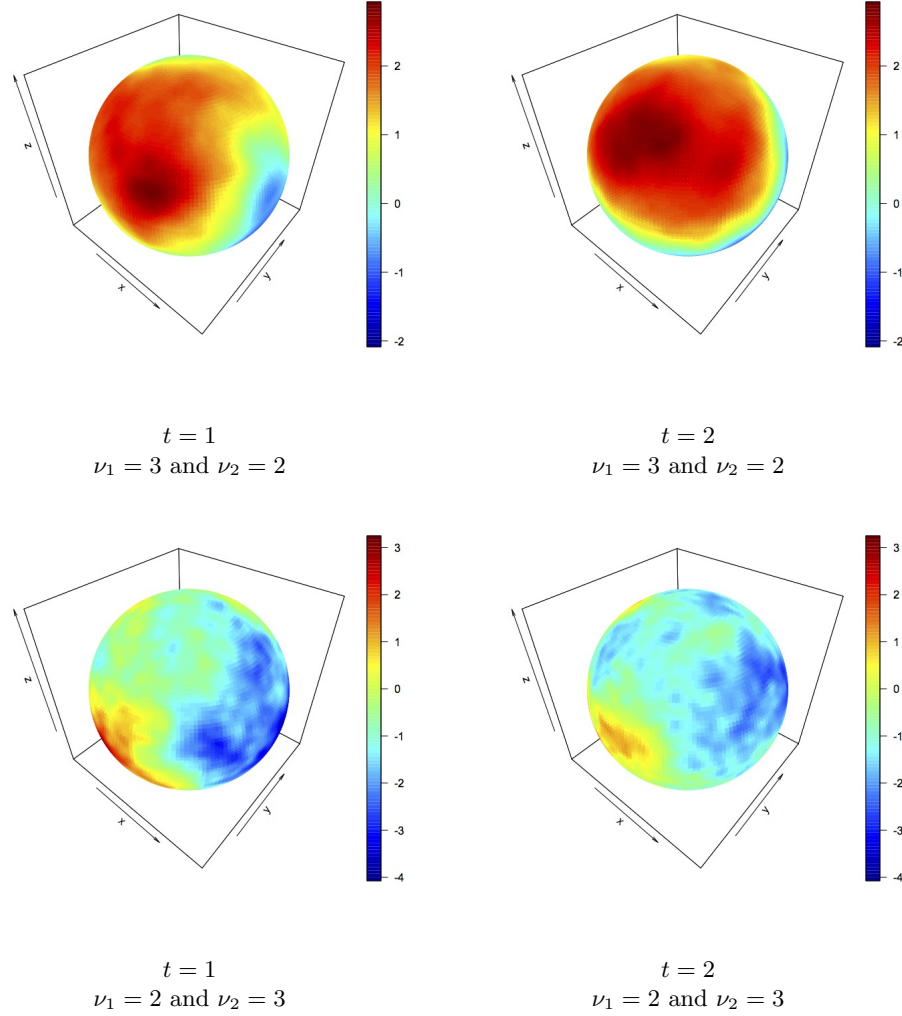


Figure 2. Spatio-temporal realization on $\mathbb{S}^2 \times \{1, 2\}$ with spectrum as in Equation (6.10).

Number of observations	500	1000	2000	4000	8000	16000	32000
Time (in seconds)	2.58	5.56	10.61	22.39	43.29	89.37	176.33

Table 1. Evaluating time for the spectral simulation in terms of the number of observations.

Finally, we analyze the accuracy of the simulation method. We consider Theorem 6.1 with $\epsilon = 8.2$, in order to obtain approximately a significance level of 0.05 in the probability

	$K = 50$			$K = 100$			$K = 150$		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
$J = 50$	0.26455	0.08743	0.10742	0.08967	0.08638	0.10612	0.03723	0.08621	0.10591
$J = 100$	0.26347	0.02946	0.03620	0.08858	0.02793	0.03431	0.03614	0.02774	0.03408
$J = 150$	0.26327	0.01178	0.01448	0.08838	0.00967	0.01189	0.03593	0.00946	0.01162

Table 2. Error of the truncation in terms of J and K for the spectrum (6.9) in three scenarios: (a) $(\nu_1, \nu_2) = (3, 2)$, (b) $(\nu_1, \nu_2) = (2, 2)$ and (c) $(\nu_1, \nu_2) = (2, 3)$. For each case, we consider a significance level of 0.05.

	$K = 50$			$K = 100$			$K = 150$		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
$J = 50$	0.02257	0.09927	0.01821	0.00599	0.06953	0.01813	0.00205	0.06359	0.01812
$J = 100$	0.02249	0.07277	0.00515	0.00585	0.03233	0.00501	0.00183	0.02234	0.00500
$J = 150$	0.02249	0.06704	0.00177	0.00583	0.02288	0.00155	0.00182	0.01078	0.00153

Table 3. Error of the truncation in terms of J and K for the spectrum (6.10) in three scenarios: (a) $(\nu_1, \nu_2) = (3, 2)$, (b) $(\nu_1, \nu_2) = (2, 2)$ and (c) $(\nu_1, \nu_2) = (2, 3)$. For each case, we consider a significance level of 0.05.

bound. Then, we calculate the maximum possible squared error (with that significance level) in terms of J and K . Tables 2 and 3 show the error of the truncations for the spectrums (6.9) and (6.10), respectively, under three scenarios: (a) $(\nu_1, \nu_2) = (3, 2)$, (b) $(\nu_1, \nu_2) = (2, 2)$ and (c) $(\nu_1, \nu_2) = (2, 3)$. We set $J, K = 50, 100, 150$. Naturally, the error decreases on the diagonal of the table, and the results are satisfactory, noting that we are working with random fields with unit variance.

7. Appendix

7.1. Karhunen-Loève theorem

Recent results on Functional Analysis (see Ferreira and Menegatto (2009) and Ferreira and Menegatto (2013)) allows to construct Mercer’s Kernels in more general contexts. A proper interpretation of these results allows to generalize the classic Karhunen-Loève theorem in a very neat way. We first introduce the basics notations from Ferreira and Menegatto (2009).

Let S be a non-empty set and K a *positive definite* kernel on S , *i. e.*, a function $K : S \times S \rightarrow \mathbb{C}$ satisfying the inequality

$$\sum_{i,j}^n \bar{c}_i c_j K(x_i, x_j) \geq 0,$$

whenever $n \geq 1$, $\{x_1, x_2, \dots, x_n\}$ is a subset of S and $\{c_1, c_2, \dots, c_n\}$ is a subset of \mathbb{C} . The set of all positive definite kernels over S is denoted by $\mathcal{PD}(S)$.

If S is endowed with a measure ν , denote by $L^2\mathcal{PD}(S, \nu)$ the class of kernels such that the associated integral operator

$$\mathcal{K}(f)(x) := \int_S K(x, y)f(y)d\nu(y), \quad f \in L^2(S, \nu), \quad x \in S,$$

is positive, that is, when the following conditions holds

$$\int_S \left(\int_S K(x, y)f(y)d\nu(y) \right) \overline{f(x)}d\nu(x) \geq 0, \quad f \in L^2(S, \nu),$$

i.e.,

$$\langle \mathcal{K}(f), f \rangle_{L^2(S, \nu)} \geq 0, \quad f \in L^2(S, \nu).$$

Finally we define what is a *Mercer's kernel* according to [Ferreira and Menegatto \(2013\)](#): A continuous kernel K on S is a *Mercer's kernel* when it possesses a series representation of the form

$$K(x, y) = \sum_{j=1}^{\infty} a_n(\mathcal{K})\zeta_j(x, y)\overline{\zeta_j(y)}, \quad x, y \in S, \quad (7.1)$$

in which $\{\zeta_j\}$ is an $L^2(S, \nu)$ -orthonormal basis of continuous functions on S , $\{a_n(\mathcal{K})\}$ decreases to 0 and the series converges uniformly and absolutely on compact subsets of $S \times S$.

For the rest of this manuscript we will consider S to be a topological space endowed with a strictly positive measure ν , that is, a complete Borel measure on S for which two properties hold: every open non-empty subset of S has positive measure and every $x \in S$ belongs to an open subset of S having finite measure.

Theorem 7.1. *Let $(X_s)_{s \in S}$ be a centered stochastic process with continuous covariance function and such that*

$$\int_S \mathbb{E}(X_s^2)d\nu(s) < +\infty. \quad (7.2)$$

Then, the kernel K associated with the covariance function of X is a Mercer's Kernel, therefore, X admits a Karhunen-Loève expansion

$$X = \sum_{j=1}^{+\infty} \lambda_j \zeta_j \quad (7.3)$$

where $\{\zeta_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2(S, \nu)$,

$$\lambda_j = \int_S X_s \zeta_j(s) d\nu(s), \quad (7.4)$$

and satisfy $\mathbb{E}[\lambda_j] = 0$, $\mathbb{E}[\lambda_j \lambda_k] = \delta_{jk} a_j$ and $\text{Var}[\lambda_j] = a_j$.

The series expansion (7.3) converges in $L^2(\Omega \times S; \mathbb{C})$, i.e.,

$$\lim_{J \rightarrow +\infty} \mathbb{E} \left(\int_S \left(X_s - \sum_{j=1}^J \lambda_j \zeta_j(s) \right)^2 dv(s) \right) = 0.$$

The series expansion (7.3) converges in $L^2(\Omega; \mathbb{C})$ for all $s \in S$, i.e., for all $s \in S$

$$\lim_{J \rightarrow +\infty} \mathbb{E} \left(\left(X_s - \sum_{j=1}^J \lambda_j \zeta_j(s) \right)^2 \right) = 0.$$

The convergence of the series expansion (7.3) is absolute and uniform on compact subsets of S .

Proof. Let K be the kernel associated to the covariance of the stochastic process X , i.e.

$$\begin{aligned} K : S \times S &\longrightarrow \mathbb{C} \\ (t, s) &\longrightarrow K(t, s) = \mathbb{E}(X_t X_s), \end{aligned}$$

and let \mathcal{K} be it's associated integral operator. From hypothesis (7.2) it is direct to see that the mapping κ , such that $s \in S \rightarrow \mathbb{C} \ni \kappa(s) := K(s, s)$, belongs to $L^1(S, v)$. Since K is positive definite in the usual sense, then

$$\begin{aligned} \int_{S \times S} |K(s, t)|^2 d(v \times v)(s, t) &\leq \int_{S \times S} K(s, s) K(t, t) d(v \times v)(s, t) \\ &= \left(\int_S \kappa(s) dv(s) \right)^2 < \infty, \end{aligned}$$

that is, $K \in L^2(S \times S, v \times v)$.

Now, consider $f, g \in L^2(S, v)$ and the classic tensor product of functions, i.e.,

$$(f \otimes g)(s, t) = f(s)g(t), \quad s, t \in S,$$

with inner product given by

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{L^2(S \times S, v \times v)} = \langle f_1, g_1 \rangle_{L^2(S, v)} \langle f_2, g_2 \rangle_{L^2(S, v)}.$$

It is immediate to see that $f \otimes g \in L^2(S \times S, v \times v)$, and thus, by Cauchy-Schwarz inequality we obtain $K \cdot f \otimes g \in L^1(S \times S, v \times v)$. This last condition allows to use

Fubini's theorem and observe that for $f \in L^2(S, \nu)$

$$\begin{aligned} \langle \mathcal{K}(f), f \rangle_2 &= \int_S \int_S K(s, t) f(t) f(s) d\nu(t) d\nu(s) \\ &= \mathbb{E} \int_S \int_S X_s f(s) X_t f(t) d\nu(t) d\nu(s) \\ &= \mathbb{E} \left(\int_S X_s f(s) d\nu(s) \right)^2 \geq 0, \end{aligned}$$

hence, $K \in L^2\mathcal{PD}(S, \nu)$.

In conclusion, the kernel K is continuous and $L^2(S, \nu)$ -positive definite on S , and the mapping κ belongs to $L^1(S, \nu)$. Then, by theorem 3.1 in [Ferreira and Menegatto \(2013\)](#) K is a Mercer's kernel.

The rest of the proof concerns the Karhunen-Loève expansion of the process X and it follows well-known arguments. So, it is left to the reader for the sake of conciseness. \square

Remark 7.1. Karhunen-Loève expansion (or Karhunen-Loève theorem), usually require extra hypothesis, like compactness of the associated space S or some kind of invariance of the field. In that line the *Stochastic Peter-Weyl theorem* (theorem 5.5 introduced in [Marinucci and Peccati \(2011\)](#)) may be understood as a Karhunen-Loève theorem for 2-weakly isotropic fields over G a topological compact group with associated Haar measure of unit mass. Theorem 7.1 only require condition (7.2) and the continuity of the covariance function.

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