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CONVEX PROJECTIVE GENERALIZED DEHN FILLING

SUHYOUNG CHOI, GYE-SEON LEE, AND LUDOVIC MARQUIS

ABSTRACT. In dimension $d = 4, 5, 6, 7$, we find the first examples of complete finite volume hyperbolic d -dimensional manifolds M with cusps such that an infinite number of orbifolds M_m obtained by generalized Dehn fillings on M admit a properly convex real projective structure. The manifolds M are covering of hyperbolic Coxeter orbifolds and the orbifold fundamental groups Γ_m of M_m are Gromov hyperbolic relative to a collection of subgroups virtually isomorphic to \mathbb{Z}^{d-2} .

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1. INTRODUCTION

1.1. Motivation. Hyperbolic Dehn filling theorem proven by Thurston [Thu02] is a fundamental theorem of hyperbolic 3-manifold theory. It states that if the interior of a compact 3-manifold M with toral boundaries admits a complete hyperbolic structure of finite volume, then except for finitely many Dehn fillings on each boundary component, all Dehn fillings of M admit a hyperbolic structure.

Within the realm of hyperbolic geometry, this phenomenon happens only for 3-manifolds even though there are topological Dehn fillings for every compact d -manifold M with a boundary which is homeomorphic to a $(d-1)$ -dimensional torus \mathbb{T}^{d-1} . Let \mathbb{D}^n be the closed n -ball. Since the compact d -manifold $\mathbb{D}^2 \times \mathbb{T}^{d-2}$ has also a toral boundary, if we glue M and $\mathbb{D}^2 \times \mathbb{T}^{d-2}$ together along their boundaries, then we obtain a closed d -manifold, called a

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Dehn filling of M . In a similar way we can deal with compact d -manifolds with several toral boundaries. Now even if we assume that the interior of M admits a finite volume hyperbolic structure, no Dehn fillings of M admit a hyperbolic structure when $d \geq 4$.

Although Dehn fillings in dimension bigger than 3 admit no hyperbolic structure, they can admit a geometric structure which is “larger” than a hyperbolic structure. A main aim of this article is to show that a real projective structure can be a good candidate for this purpose. In a similar spirit, Anderson [And06] and Bamler [Bam12] proved that many aspects of Dehn filling theory for hyperbolic 3-manifolds can be generalized to Einstein metrics in every dimension. More precisely, if the interior of a compact d -manifold M with toral boundaries admits a finite volume hyperbolic structure, then except for finitely many Dehn fillings on each boundary component, all Dehn fillings of M admit Einstein metrics.

Let \mathbb{S}^d be the d -dimensional projective sphere¹ and note that the group $\mathrm{SL}_{d+1}^{\pm}(\mathbb{R})$ of linear automorphism of determinant ± 1 is the group of projective automorphisms of \mathbb{S}^d . We say that a d -dimensional manifold N admits a properly convex real projective structure if N is homeomorphic to Ω/Γ with Ω a properly convex subset of \mathbb{S}^d and Γ a discrete subgroup of $\mathrm{SL}_{d+1}^{\pm}(\mathbb{R})$ acting properly discontinuously on Ω . Now we can ask the following question:

Question 1.1. *Is there a compact manifold M of dimension $d \geq 4$ with toral boundaries such that the interior of M admits a finite volume hyperbolic structure, and except for finitely many Dehn fillings on each boundary component, all Dehn fillings of M admit a properly convex real projective structure?*

We can generalize Question 1.1 only requiring the interior of M to admit a properly convex real projective structure of finite volume.

1.2. Evidence. In this paper, we give an evidence towards a positive answer to Question 1.1. It is difficult for us to find such a manifold directly, and hence we begin with Coxeter orbifold, also called reflection orbifold or reflectofold (see Chapter 13 of Thurston [Thu02] for more precise definition of orbifold). The definition of Coxeter orbifold is somewhat more complicated than the definition of manifold, however it turns out that convex projective Coxeter orbifolds are easier to build than convex projective manifolds. The first examples towards Question 1.1 are hyperbolic Coxeter d -orbifolds for $d = 4, 5, 6$.

Consider a convex polytope P in the hyperbolic space \mathbb{H}^d with dihedral angles submultiples of π , i.e. each dihedral angle is $\frac{\pi}{m}$ with an integer $m \geq 2$ or $m = \infty$. We call such a polytope a hyperbolic Coxeter polytope. The group Γ generated by the reflections about the codimension one faces of P is a discrete subgroup of the group $\mathrm{Isom}(\mathbb{H}^d)$ of isometries of \mathbb{H}^d , which is isomorphic to $\mathrm{PO}_{d,1}(\mathbb{R})$, and the quotient \mathbb{H}^d/Γ is a hyperbolic Coxeter d -orbifold. Hyperbolic Coxeter orbifolds are useful objects that we can concretely construct. For example, the first example of a closed orientable hyperbolic 3-manifold is an eight-fold cover of a right-angled hyperbolic Coxeter 3-orbifold with 14 faces, which was constructed by Löbell in 1931 [Lob31].

¹i.e. the space of half-lines of \mathbb{R}^{d+1} .

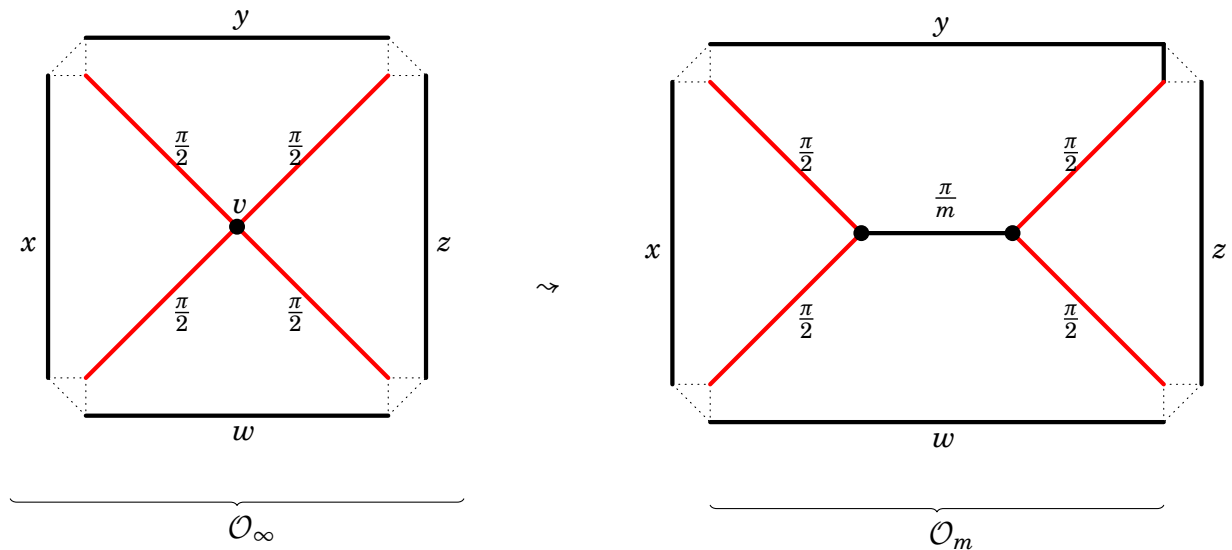


FIGURE 1. Hyperbolic Dehn fillings in dimension 3.

By Andreev's theorems [And70a, And70b], there is a version of the Dehn filling theorem for hyperbolic Coxeter 3-orbifolds (see Chapter 7 of Vinberg–Shvartsman [VS93] and Proposition 2 of Kolpakov [Kol12]): Let \mathcal{O}_∞ be a compact Coxeter 3-orbifold with a boundary $\partial\mathcal{O}_\infty$ that is a closed Coxeter 2-orbifold admitting a Euclidean structure. A m -generalized Dehn filling \mathcal{O}_m of \mathcal{O}_∞ , or simply a m -Dehn filling, is a closed Coxeter 3-orbifold \mathcal{O}_m such that \mathcal{O}_∞ is orbifold diffeomorphic (see Davis [Dav14] or Wiemeler [Wie13]) to the complement of an open neighborhood of an edge r of order m^2 of \mathcal{O}_m (see Figure 1). Note that the existence of a m -Dehn filling implies that the boundary $\partial\mathcal{O}_\infty$ of \mathcal{O}_∞ must be a rectangle with 4 corner reflectors of order 2.

A corollary of Andreev's theorems says that if the interior of \mathcal{O}_∞ admits a hyperbolic structure of finite volume, then there exists a natural number N such that, for each $m > N$, the 3-orbifolds \mathcal{O}_m admit a hyperbolic structure.

Now we can state the main theorem of the paper: let \mathcal{O}_∞ be a compact Coxeter d -orbifold with a boundary $\partial\mathcal{O}_\infty$ that is a closed Coxeter $(d-1)$ -orbifold admitting a Euclidean structure. A m -Dehn filling \mathcal{O}_m of \mathcal{O}_∞ is a Coxeter d -orbifold such that \mathcal{O}_∞ is orbifold diffeomorphic to the complement of an open neighborhood of a face r of codimension 2 of \mathcal{O}_m , and each interior point of r has a neighborhood modeled on $(\mathbb{R}^2/D_m) \times \mathbb{R}^{d-2}$. We abbreviate a connected open set to a *domain*.

Theorem A. *In dimension $d = 4, 5, 6$ (resp. $d = 7$), there exists a complete finite volume hyperbolic Coxeter d -orbifold \mathcal{O}_∞ with orbifold fundamental group W_∞ and holonomy representation $\rho_\infty : W_\infty \rightarrow \mathrm{O}_{d,1}(\mathbb{R}) \subset \mathrm{SL}_{d+1}^\pm(\mathbb{R})$ such that there is a natural number N and a sequence of*

²An edge r of \mathcal{O}_m is said to be of order m if each interior point of r has a neighborhood modelled on $(\mathbb{R}^2/D_m) \times \mathbb{R}$, where D_m is the dihedral group of order $2m$.

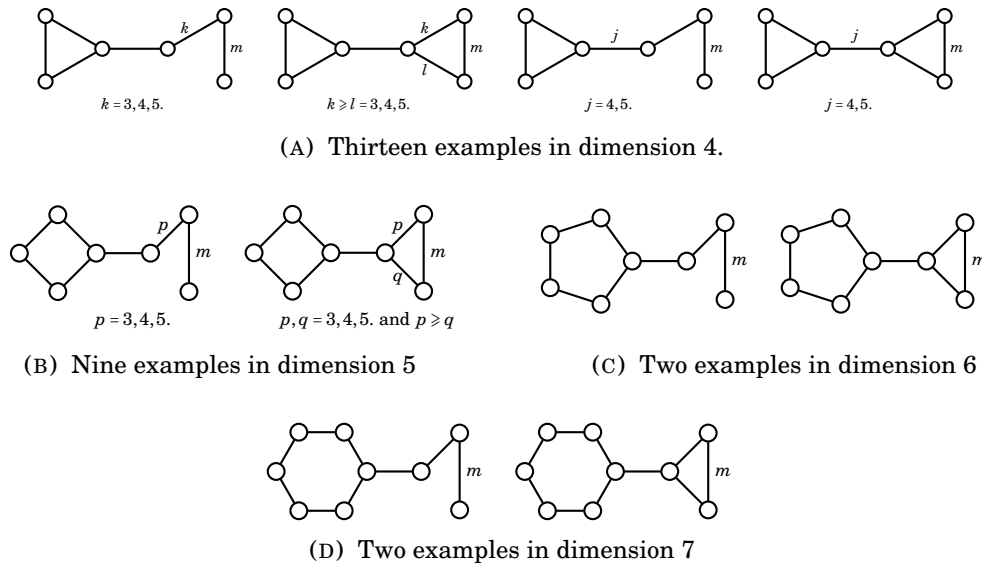


TABLE 1. Examples for Theorem A

representations

$$(\rho_m : W_\infty \rightarrow \mathrm{SL}_{d+1}^\pm(\mathbb{R}))_{m>N}$$

satisfying the following:

- The image $\rho_m(W_\infty)$ is a discrete subgroup of $\mathrm{SL}_{d+1}^\pm(\mathbb{R})$ acting properly discontinuously and cocompactly (resp. with finite covolume) on the unique $\rho_m(W_\infty)$ -invariant properly convex domain $\Omega_m \subset \mathbb{S}^d$.
- The induced representation $W_\infty/\ker(\rho_m) \rightarrow \mathrm{SL}_{d+1}^\pm(\mathbb{R})$ is the holonomy representation of a properly convex real projective structure on a Dehn filling \mathcal{O}_m of \mathcal{O}_∞ .
- The representations $(\rho_m)_{m>N}$ converge algebraically to ρ_∞ .
- The convex domains $(\Omega_m)_{m>N}$ converge to $\Omega_\infty = \mathbb{H}^d \subset \mathbb{S}^d$ in the Hausdorff topology.

Remark 1.2. The authors conjecture that the sequence $(\mathcal{O}_m)_m$ of convex projective orbifolds with Hilbert metrics converge to the convex projective orbifold \mathcal{O}_∞ in the Gromov-Hausdorff topology.

Remark 1.3. The Coxeter orbifolds \mathcal{O}_∞ whose orbifold fundamental groups are the Coxeter groups in Table 1 (assuming $j \neq 5$, $p, q = 3$ and by setting $m = \infty$) are “prime” examples satisfying Theorem A. We describe in Section 9 how to build infinitely many Coxeter orbifolds \mathcal{O}_∞ satisfying Theorem A using a gluing procedure and the Coxeter groups of Table 1 given by setting $j = 5$ or $p = 5$. However, this gluing trick can be applied only in dimension 4 and 5.

Remark 1.4. The reader might wonder what “prime” means in this paper. As an ad-hoc definition, a Coxeter d -orbifold is *prime* if it arises from a d -polytope with $d + 2$ faces of codimension 1, or as a bit more interesting definition, if it cannot be obtained by the gluing procedure described in Section 9.

In Theorem A, changing the target Lie group of representations from $\mathrm{O}_{d,1}(\mathbb{R})$ to $\mathrm{SL}_{d+1}^\pm(\mathbb{R})$ is essential because of the following theorem:

Theorem 1.5 (Garland and Raghunathan [GR70]). *Let d be an integer strictly bigger than 3. Assume that Γ is a lattice in $O_{d,1}(\mathbb{R})$ and $\rho_\infty : \Gamma \rightarrow O_{d,1}(\mathbb{R})$ is the canonical inclusion. Then there is a neighborhood U of ρ_∞ in $\text{Hom}(\Gamma, O_{d,1}(\mathbb{R}))$ such that every $\rho \in U$ is conjugate to ρ_∞ .*

In other words, every sufficiently small deformation of ρ_∞ in $\text{Hom}(\Gamma, O_{d,1}(\mathbb{R}))$ is trivial (see also Bergeron and Gelander [BG04] for an alternative proof).

By Selberg's lemma, Theorem A also allows us to find properly convex projective structures on Dehn fillings of a manifold:

Theorem B. *In dimension $d = 4, 5, 6$, there exists a finite volume hyperbolic d -manifold M_∞ such that an infinite number of closed d -orbifolds M_m obtained by Dehn fillings of M_∞ admit properly convex real projective structures. Moreover, the singular locus of M_m is the union of disjoint $(d - 2)$ -dimensional tori, and hence the orbifold fundamental group of M_m is not Gromov hyperbolic.*

1.3. Another novelty of the examples in Theorem A. A properly convex domain Ω in \mathbb{S}^d is *divisible* if there exists a discrete subgroup Γ of $\text{SL}_{d+1}^\pm(\mathbb{R})$ such that Γ preserves Ω and the quotient Ω/Γ is compact. In that case, we say that Γ *divides* Ω . We call Ω *decomposable* if there exist non-empty closed convex subsets $A, B \subset \mathbb{S}^d$ such that the spans of A and B are disjoint and the convex hull of $A \cup B$ is the closure $\overline{\Omega}$ of Ω . Otherwise, Ω is called *indecomposable*.

Divisible strictly convex domains exist in every dimension (see [Ben04a], even better for a construction of divisible strictly convex domains of dimension $d \geq 4$ not quasi-isometric to the real hyperbolic space, see [Kap07] or [Ben06b] for $d = 4$). On the other hand, the existence of indecomposable divisible properly convex domains of every dimension d which are not strictly convex is still an open question.

At the time of writing this paper, the only known examples of indecomposable divisible properly convex domains which are not strictly convex are the examples introduced by Benoist [Ben06a] in dimension $d = 3, \dots, 7$. The construction of Benoist has been explored by the third author in [Mar10] and by the second author with Ballas and Danciger in [BDL15], both in dimension $d = 3$. Afterwards, the authors [CLM] study further this kind of examples in dimension $d = 4, \dots, 7$. Moreover, the construction in this paper is inspired by the work of Benoist [Ben06b].

The difference between the previously known examples and the examples in Theorem A can be visually seen from the notion of properly embedded simplex of dimension e (shorten to e -simplex). A simplex Δ is *properly embedded* in a properly convex domain Ω if $\Delta \subset \Omega$ and $\partial\Delta \subset \partial\Omega$. A properly embedded e -simplex Δ of a convex domain Ω is *tight* if Δ is not included in a e' -properly embedded simplex Δ' of Ω with $e' > e$.

All the pairs (Γ, Ω) built in [Ben06a, Mar10, BDL15, CLM], assuming that Γ divides Ω , satisfy the following properties:

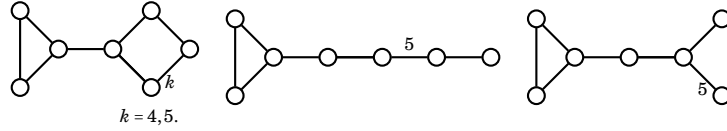
- The d -dimensional convex domain Ω contains a properly embedded $(d - 1)$ -simplex.

- The group Γ is Gromov hyperbolic relative to a finite collection of subgroups of Γ virtually isomorphic to \mathbb{Z}^{d-1} .
- The quotient orbifold Ω/Γ is homeomorphic to the union along the boundaries of finitely many d -orbifolds each of which admits a finite volume hyperbolic structure on its interior.

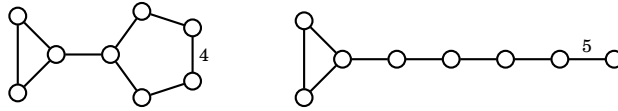
Theorem C. *Assume that $d = 4, 5, 6$, $\Gamma = \rho_m(W_\infty)$ and $\Omega = \Omega_m$ in Theorem A. Then (Γ, Ω) satisfies the following:*

- *The d -dimensional convex domain Ω contains a properly embedded tight $(d-2)$ -simplex.*
- *The group Γ is Gromov hyperbolic relative to a finite collection of subgroups virtually isomorphic to \mathbb{Z}^{d-2} .*
- *The quotient orbifold Ω/Γ is not homeomorphic to the union along the boundaries of finitely many d -orbifolds each of which admits a finite volume hyperbolic structure on its interior.*

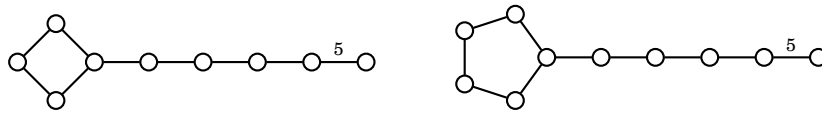
The computations involved in the proof of Theorem A in fact easily extend to give a two (small) new families of pairs (Γ, Ω) (see the Coxeter graphs in Table 2 for Theorem D and Table 3 for Theorem E).



(A) Four examples in dimension 5



(B) Two examples in dimension 6



(C) One example in dimension 7 and one example in dimension 8

TABLE 2. Coxeter groups of Theorem D

Theorem D. *For each $(d, e) \in \{(5, 2), (6, 2), (7, 3), (8, 4)\}$, there exists a discrete group $\Gamma \subset \mathrm{SL}_{d+1}^\pm(\mathbb{R})$ which divides $\Omega \subset \mathbb{S}^d$ such that:*

- *The d -dimensional convex domain Ω contains a properly embedded tight e -simplex.*
- *The group Γ is Gromov hyperbolic relative to a finite collection of subgroups virtually isomorphic to \mathbb{Z}^e .*
- *The quotient orbifold Ω/Γ is not homeomorphic to the union along the boundaries of finitely many d -orbifolds each of which admits a finite volume hyperbolic structure on its interior.*

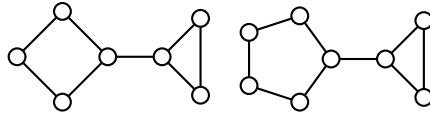


TABLE 3. Coxeter groups of Theorem E

Theorem E. *For each $d = 5, 6$, there exists a discrete group $\Gamma \subset \mathrm{SL}_{d+1}^{\pm}(\mathbb{R})$ which divides $\Omega \subset \mathbb{S}^d$ such that:*

- *The d -dimensional convex domain Ω contains a properly embedded tight $(d - 2)$ -simplex and a properly embedded tight triangle.*
- *The group Γ is Gromov hyperbolic relative to a finite collection of subgroups virtually isomorphic to \mathbb{Z}^2 and \mathbb{Z}^{d-2} .*
- *The quotient orbifold Ω/Γ is not homeomorphic to the union along the boundaries of finitely many d -orbifolds each of which admits a finite volume hyperbolic structure on its interior.*

Finally, we would like to propose the following question:

Question 1.6. *For which integers (d, m, e_1, \dots, e_m) with $m \geq 1$ and $2 \leq e_1 < e_2 < \dots < e_m \leq d - 1$, is there a d -dimensional divisible convex domain $\Omega \subset \mathbb{S}^d$ which contains a properly embedded tight e_i -simplex for all $i = 1, \dots, m$?*

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2. PRELIMINARY

2.1. Coxeter groups. A *Coxeter system* is a pair (S, M) of a finite set S and a symmetric matrix $M = (M_{st})_{s,t \in S}$ whose diagonal entries $M_{ss} = 1$ and whose off-diagonal entries $M_{st} \in \{2, 3, \dots, m, \dots, \infty\}$. To a Coxeter system (S, M) we can associate a *Coxeter group*, denoted by W_S (or simply W): the group generated by S with the relations $(st)^{M_{st}} = 1$ for every $(s, t) \in S \times S$ such that $M_{st} \neq \infty$. The *rank* of W is the cardinality $|S|$ of S .

The *Coxeter diagram* (or *Coxeter graph*) of W is a labeled graph, also denoted by W , such that the set of vertices of W is S , an edge \overline{st} connects two vertices $s, t \in S$ if and only if $M_{st} \neq 2$,

and the label of the edge \overline{st} is $M_{st} \in \{3, \dots, m, \dots, \infty\}$. A Coxeter group W is *irreducible* if the Coxeter graph W is connected. The *Gram matrix* of W , denoted by $\text{Cos}(W)$, is an $|S| \times |S|$ symmetric matrix whose entries are given by

$$(\text{Cos}(W))_{st} = -2 \cos\left(\frac{\pi}{M_{st}}\right) \quad \text{for every } s, t \in S.$$

An irreducible Coxeter group W (or a connected Coxeter graph W) is *spherical* (resp. *affine* or *Lannér*) if for every $s \in S$, the (s, s) -minor of $\text{Cos}(W)$ is positive definite and the determinant of $\text{Cos}(W)$ is positive (resp. zero or negative). If an irreducible Coxeter group W is not spherical nor affine, then W is *large*, i.e. there is a homomorphism of a finite index subgroup of W onto a non-abelian free group (see Margulis–Vinberg [MV00]). Remark that an irreducible affine Coxeter group is virtually $\mathbb{Z}^{|S|-1}$. A Coxeter graph W (or a Coxeter group W) is called *spherical* (resp. *affine*) if each connected component of W is spherical (resp. affine). Note that a Coxeter group W is spherical if and only if W is a finite group.

We refer to Appendix A for the list of all the irreducible spherical, irreducible affine and Lannér Coxeter groups, which we will use frequently.

2.2. Coxeter polytopes. Let V be a $(d+1)$ -dimensional vector space over \mathbb{R} , and let $\mathbb{S}(V)$ be the projective sphere, i.e. the space of half-lines in V . In order to indicate the dimension of $\mathbb{S}(V)$, we use the notation \mathbb{S}^d instead of $\mathbb{S}(V)$. Denote by $\text{SL}^\pm(V)$ (resp. $\text{SL}(V)$) the group of matrices with determinant ± 1 (resp. 1). The projective sphere $\mathbb{S}(V)$ and $\text{SL}^\pm(V)$ are two-fold covers of the real projective space $\mathbb{P}(V)$ and the group $\text{PGL}(V)$ of projective transformations of V , respectively.

The natural projection of $V \setminus \{0\}$ onto $\mathbb{S}(V)$ is denoted by \mathbb{S} , and for every subspace W of V , $\mathbb{S}(W)$ denotes $\mathbb{S}(W \setminus \{0\})$ for the simplicity of the notation. A subset P of $\mathbb{S}(V)$ is *convex* if for every 2-dimensional subspace Π of V , $P \cap \mathbb{S}(\Pi)$ is connected, and moreover a convex subset P is *properly convex* if the closure \overline{P} of P does not contain a pair of antipodal points. In other words, P is properly convex if and only if there is an affine chart \mathbb{A} of $\mathbb{S}(V)$ such that $\overline{P} \subset \mathbb{A}$ and P is convex in \mathbb{A} in the usual sense. A properly convex set P is *strictly convex* if there is no nontrivial segment in the boundary ∂P of P .

A *projective polytope* is a properly convex subset P of $\mathbb{S}(V)$ such that P has a non-empty interior and

$$P = \bigcap_{i=1}^r \mathbb{S}(\{v \in V \mid \alpha_i(v) \leq 0\})$$

where α_i , $i = 1, \dots, r$, are linear forms on V . We always assume that the set of linear forms is *minimal*, i.e. none of the half spaces $\mathbb{S}(\{v \in V \mid \alpha_i(v) \leq 0\})$ contains the intersection of all the others. A *facet* (resp. *ridge*) of a polytope is a face of codimension 1 (resp. 2). Two facets s, t of a polytope P are *adjacent* if $s \cap t$ is a ridge of P .

A *projective reflection* is an element of $\text{SL}^\pm(V)$ of order 2 which is the identity on a hyperplane. Each projective reflection σ can be written as:

$$\sigma = \text{Id} - a \otimes b$$

where α is a linear form on V and b is a vector in V such that $\alpha(b) = 2$. The projective hyperplane $\mathbb{S}(\ker(\alpha))$ is the *support* of σ and the half-line $\mathbb{S}(b)$ the *polar* of σ . A *projective rotation* is an element of $\mathrm{SL}(V)$ which is the identity on a subspace H of codimension 2 and which is conjugate to a matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ on V/H with $0 < \theta < 2\pi$. The real number θ is called the *angle* of rotation.

A *mirror polytope* is a pair of a projective polytope P of $\mathbb{S}(V)$ and a finite number of projective reflections $(\sigma_s = \mathrm{Id} - \alpha_s \otimes b_s)_{s \in S}$ with $\alpha_s(b_s) = 2$ such that:

- The index set S consists of all the facets of P .
- For each facet $s \in S$, the support of σ_s is the supporting hyperplane of s .
- For every two distinct facets s, t of P , $\alpha_s(b_t)$ and $\alpha_t(b_s)$ are either both negative or both zero, and $\alpha_s(b_t)\alpha_t(b_s) \geq 4$ or $= 4\cos^2\theta$ with $0 < \theta \leq \frac{\pi}{2}$.

In fact, $\alpha_s(b_t)\alpha_t(b_s) = 4\cos^2\theta$ if and only if the product $\sigma_s\sigma_t$ is a rotation of angle 2θ . Remark that the definition of mirror polytope does not come out of the blue: It is motivated by Tits–Vinberg’s Theorem 2.1 (see Proposition 6 of Vinberg [Vin71] or Lemma 1.2 of Benoist [Ben04b] for more details).

The *dihedral angle* of a ridge $s \cap t$ of a mirror polytope P is θ if $\sigma_s\sigma_t$ is a rotation of angle 2θ , and 0 otherwise. A *Coxeter polytope* is a mirror polytope whose dihedral angles are 0 or submultiples of π . If P is a Coxeter polytope, then the *Coxeter system* of P is the Coxeter system (S, M) such that S is the set of facets of P and for every two distinct facets s, t of S , $M_{st} = m_{st}$ if $\sigma_s\sigma_t$ is a rotation of angle $\frac{2\pi}{m_{st}}$, and $M_{st} = \infty$ otherwise. We denote by W_P (or simply W) the *Coxeter group* of P , and we call P *irreducible* if W_P is irreducible.

2.3. Tits’s simplex. To a Coxeter group W , we can associate a Coxeter polytope Δ_W . The polytope Δ_W is a simplex of dimension the rank of W minus 1 and the Coxeter group of Δ_W is W .

Consider the vector space $V = (\mathbb{R}^S)^*$ and denote by $(e_s)_{s \in S}$ the canonical basis of \mathbb{R}^S . The simplex we want to obtain is the projectivization $P = \mathbb{S}(\mathcal{C})$ of the simplicial cone $\mathcal{C} = \{\varphi \in (\mathbb{R}^S)^* \mid \varphi(e_s) \leq 0, \forall s \in S\}$. For each element $s \in S$, the reflection about the facet $P \cap \mathbb{S}(\{\varphi \in (\mathbb{R}^S)^* \mid \varphi(e_s) = 0\})$ is defined by the formula $\sigma_s(\varphi) = \varphi - 2\varphi(e_s)B_W(e_s, \cdot)$, where B_W is the symmetric bilinear form given by $B_W(e_s, e_t) = -2\cos\left(\frac{\pi}{M_{st}}\right)$. The matrix $(B_W(e_s, e_t))_{s, t \in S}$ is the matrix of *Tits bilinear form* of W , and the resulting Coxeter polytope is the *Tits simplex* Δ_W of W . Remark that the group generated by the reflections σ_s preserves the symmetric bilinear form B_W .

2.4. Tits–Vinberg’s Theorem. Let (S, M) be a Coxeter system. For each subset S' of S , we consider the Coxeter group $W_{S'}$ of the Coxeter subsystem (S', M') , where M' is the restriction of M to S' . Theorem 2.1 shows that the natural homomorphism $W_{S'} \rightarrow W_S$ is injective, and therefore $W_{S'}$ can be identified with the subgroup of W_S generated by S' . These subgroups $W_{S'}$ are called *standard subgroups* of W_S . If P is a Coxeter polytope and f is a proper face of P (i.e. $f \neq \emptyset, P$), then we write $S_f = \{s \in S \mid f \subset s\}$ and $W_f := W_{S_f}$.

Theorem 2.1 (Chapter V of Tits [Bou68] for Tits simplex, and Vinberg [Vin71]). *Let P be a Coxeter polytope of $\mathbb{S}(V)$ with Coxeter group W_P and let Γ_P be the subgroup of $\mathrm{SL}^\pm(V)$ generated by the projective reflections $(\sigma_s)_{s \in S}$. Then:*

- *The homomorphism $\sigma : W_P \rightarrow \Gamma_P$ defined by $\sigma(s) = \sigma_s$ is an isomorphism.*
- *The family of polytopes $(\gamma(P))_{\gamma \in \Gamma_P}$ is a tiling of a convex subset \mathcal{C}_P of $\mathbb{S}(V)$.*
- ∴ *The group Γ_P is a discrete subgroup of $\mathrm{SL}^\pm(V)$ acting properly discontinuously on the interior Ω_P of \mathcal{C}_P .*
- ∴ *An open proper face f of P lies in Ω_P if and only if the Coxeter group W_f is finite.*

We call Γ_P the *projective Coxeter group* of P . Theorem 2.1 tells us that Ω_P is a convex domain of $\mathbb{S}(V)$ and Ω_P/Γ_P is a convex real projective Coxeter orbifold.

2.5. The link of a Coxeter polytope at a vertex. Given a mirror polytope P of dimension d in \mathbb{S}^d , we can associate to a vertex v of P a new mirror polytope P_v of dimension $d - 1$ which is “ P seen from v ”. We call P_v the *link of P at v* . In order to build P_v , first, look at the projective sphere $\mathbb{S}_v = \mathbb{S}(\mathbb{R}^{d+1}/\langle v \rangle)$, where $\langle v \rangle$ is the subspace spanned by v , and second, observe that the reflections $(\sigma_s)_{s \in S_v}$ induce reflections on \mathbb{S}_v . Finally, we can construct a projective polytope in \mathbb{S}_v :

$$\bigcap_{s \in S_v} \mathbb{S}(\{x \in \mathbb{R}^{d+1} \mid \alpha_s(x) \leq 0\} / \langle v \rangle).$$

This projective polytope together with the reflections induced by $(\sigma_s)_{s \in S_v}$ on \mathbb{S}_v give us the link P_v of P at v , which is a mirror polytope of dimension $d - 1$. It is obvious that if P is a Coxeter polytope, then P_v is also a Coxeter polytope. Note that if W_v denotes W_{P_v} , then the Coxeter graph W_v is obtained from the Coxeter graph W_P by keeping only the vertices in S_v and the edges connecting two vertices of S_v .

2.6. The Cartan matrix of a Coxeter polytope. An $m \times m$ matrix $A = (a_{ij})_{i,j=1,\dots,m}$ is a *Cartan matrix* if every diagonal entry of A is 2, every off-diagonal entry is negative or zero, and for all $i, j = 1, \dots, m$, $a_{ij} = 0$ if and only if $a_{ji} = 0$.

A Cartan matrix A is *irreducible* if there are no simultaneous permutations of the rows and the columns of A such that A is a non-trivial block diagonal matrix, i.e. it is not a direct sum of two matrices. Every Cartan matrix A decomposes into a direct sum of irreducible matrices, which are called the *components* of A . By Perron–Frobenius theorem, an irreducible Cartan matrix A has a real eigenvalue. An irreducible Cartan matrix A is of *positive type*, *zero type* or *negative type* when the smallest real eigenvalue of A is positive, zero or negative, respectively. If A^+ (resp. A^0 , A^-) denotes the direct sum of the components of positive (resp. zero, negative) type of a Cartan matrix A , then $A = A^+ \oplus A^0 \oplus A^-$, i.e. the direct sum of A^+ , A^0 and A^- .

For each Coxeter polytope P , we can define the *Cartan matrix A_P of P* by $(A_P)_{st} = \alpha_s(b_t)$. Note that A_P is irreducible if and only if P is irreducible. The pairs $(\alpha_s, b_s)_{s \in S}$ that determine the Coxeter polytope P are unique up to the action of $|S|$ positive numbers $(\lambda_s)_{s \in S}$ given by:

$$(1) \quad (\alpha_s, b_s)_{s \in S} \mapsto (\lambda_s \alpha_s, \lambda_s^{-1} b_s)_{s \in S}$$

This leads to define an equivalence relation on Cartan matrices: Two Cartan matrices A and A' are equivalent if there exists a positive diagonal matrix D such that $A' = DAD^{-1}$. We denote by $[A]$ the equivalence class of the Cartan matrix A .

For every sequence (s_1, \dots, s_k) of elements of S , the number $A_{s_1 s_2} A_{s_2 s_3} \cdots A_{s_k s_1}$ is independent of the choice of a Cartan matrix in the class $[A]$. These invariants are called the *cyclic products* of A , which determine $[A]$ (see Proposition 16 of [Vin71]). The importance of Cartan matrix can be seen from:

Theorem 2.2. [Corollary 1 of Vinberg [Vin71]] *Let A be an irreducible Cartan matrix of negative type and of rank $d + 1$. Then there exists a mirror polytope P of dimension d such that $A_P = A$. Moreover, P is unique up to conjugations by $\mathrm{SL}^\pm(V)$.*

2.7. Elliptic and Parabolic Coxeter polytopes. A Coxeter polytope P is *elliptic* (resp. *parabolic*) if the projective Coxeter group Γ_P of P is derived from a discrete cocompact group generated by reflections in the sphere (resp. the Euclidean space). These polytopes were classified by Coxeter:

Theorem 2.3 (Coxeter [Cox49] and Propositions 21 and 22 of Vinberg [Vin71]). *Let P be a Coxeter polytope with Coxeter group W . Then P is elliptic if and only if $A_P = A_P^+$. If this is the case, then P is isomorphic to the Tits simplex Δ_W .*

In order to describe parabolic Coxeter polytopes, we need to introduce a notation: If W is an irreducible affine Coxeter group, then Ω_{Δ_W} is an affine chart of dimension $\mathrm{rank}(W) - 1$ on which W acts properly and cocompactly, and W preserves an Euclidean norm on Ω_{Δ_W} induced by the Tits bilinear form B_W . Now let W be an affine Coxeter group. If we write $W = W_1 \times \cdots \times W_r$ the decomposition of W into irreducible components, then

$$\hat{\Delta}_W := \Delta_{W_1} \times \cdots \times \Delta_{W_r} \subset \Omega_{\Delta_{W_1}} \times \cdots \times \Omega_{\Delta_{W_r}}$$

is a Coxeter polytope in the Euclidean space \mathbb{E}^e of dimension $e = \mathrm{rank}(W) - r$, where the symbol \times stands for the usual Cartesian product. The Euclidean norm on the affine space $\Omega_{\Delta_{W_1}} \times \cdots \times \Omega_{\Delta_{W_r}}$ is the product of the Euclidean norms of the factors, and the action of W on \mathbb{E}^e is again proper and cocompact.

Theorem 2.4 (Coxeter [Cox49] and Propositions 21 and 23 of Vinberg [Vin71]). *Let P be a Coxeter polytope of dimension d with Coxeter group W . Then P is parabolic if and only if $A_P = A_P^0$ and A_P is of rank d . If this is the case, then P is isomorphic to $\hat{\Delta}_W$.*

2.8. Hilbert geometry. A Coxeter polytope P of \mathbb{S}^d is said to be *loxodromic* if $A_P = A_P^-$ and of rank $d + 1$. Before going any further, we remark that if P is an irreducible loxodromic Coxeter polytope of dimension d , then the representation $\sigma : W_P \rightarrow \mathrm{SL}_{d+1}^\pm(\mathbb{R})$ in Theorem 2.1 is irreducible and the convex domain Ω_P is properly convex (see Lemma 15 of Vinberg [Vin71]). This situation leads us to study Hilbert geometry.

Given a properly convex domain Ω , we can use the cross-ratio to define a metric d_Ω on Ω : For any two distinct points x, y in Ω , the line l passing through x and y meets the boundary $\partial\Omega$ of Ω in two other points p and q . Assume that p and q separate x and y on the line l (see

Figure 2). The *Hilbert metric* $d_\Omega : \Omega \times \Omega \rightarrow [0, +\infty)$ is defined by:

$$d_\Omega(x, y) = \frac{1}{2} \log([p : x : y : q]) \quad \text{for every } x, y \in \Omega,$$

where $[p : x : y : q]$ is the cross-ratio of the points p, x, y, q . The metric topology of Ω is the same as the one inherited from $\mathbb{S}(V)$. The metric space (Ω, d_Ω) is complete, the closed balls are compact, the group $\text{Aut}(\Omega)$ of projective transformations that preserve Ω acts by isometries on Ω , and therefore acts properly.

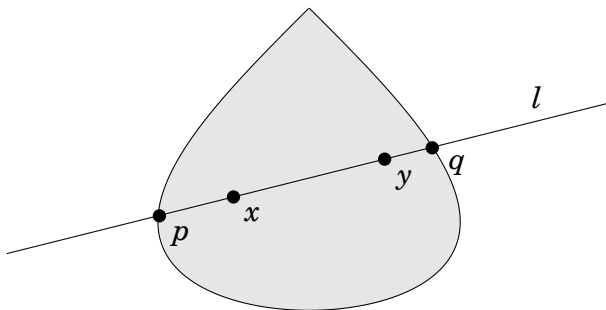


FIGURE 2. The Hilbert metric

The Hilbert metric comes from a Finsler norm on Ω , and this Finsler metric gives rise to an absolutely continuous measure μ_Ω with respect to Lebesgue measure, called the *Busemann measure*. We refer to de la Harpe [dlH93] or Vernicos [Ver05] for more details on Hilbert geometry.

2.9. Perfect, quasi-perfect and 2-perfect Coxeter polytope. Vinberg [Vin71] also shows that if P is a Coxeter polytope, then the following are equivalent:

- For every vertex v of P , the standard subgroup W_v of W is finite.
- The convex subset \mathcal{C}_P of $\mathbb{S}(V)$ is open, i.e. $\mathcal{C}_P = \Omega_P$.
- The action of Γ_P on Ω_P is cocompact.

Following him, we call such a Coxeter polytope P *perfect* (see Definition 8 of [Vin71]). It is known that a perfect Coxeter polytope is either elliptic, parabolic, or irreducible loxodromic (see Proposition 26 of [Vin71]).

A Coxeter polytope P is *2-perfect* if every vertex link of P is perfect, and it is *quasi-perfect* if every vertex link of P is either elliptic or parabolic. It is obvious that a quasi-perfect Coxeter polytope is 2-perfect. In order to describe the geometric property of 2-perfect projective Coxeter groups, we introduce the following terminology: Let $C(\Lambda_P)$ be the convex hull of the limit set³ Λ_P of Γ_P . The action of Γ_P on Ω_P is *geometrically finite* if $\mu_{\Omega_P}(C(\Lambda_P) \cap P) < \infty$, *finite covolume* if $\mu_{\Omega_P}(P) < \infty$, and *convex-cocompact* if $C(\Lambda_P) \cap P \subset \Omega_P$.

Theorem 2.5 (Theorem A, Marquis [Mar14]). *Let P be an irreducible, loxodromic, 2-perfect Coxeter polytope, let Γ_P be the projective Coxeter group of P , and let Ω_P be the interior of the Γ -orbit of P . Then the action of Γ_P on Ω_P is*

³The limit set of Γ_P is the closure of the set of attractive points of loxodromic elements of Γ_P . It is also the smallest closed Γ_P -invariant subset of the projective space. See Lemma 2.9 and 3.3 of Benoist [Ben00].

- *geometrically finite,*
- *finite covolume if and only if P is quasi-perfect,*
- *convex-cocompact if and only if every vertex link of P is elliptic or loxodromic.*

2.10. Deformation spaces of marked Coxeter polytopes. Two projective polytopes \mathcal{G} and \mathcal{G}' are *combinatorially equivalent* if there exists a bijection δ between the set \mathcal{F} of all faces of \mathcal{G} and the set \mathcal{F}' of all faces of \mathcal{G}' such that δ preserves the inclusion relation, i.e. for every $F_1, F_2 \in \mathcal{F}$, $F_1 \subset F_2$ if and only if $\delta(F_1) \subset \delta(F_2)$. We call δ a *lattice isomorphism* between \mathcal{G} and \mathcal{G}' . A *combinatorial polytope* is a combinatorial equivalence class of polytopes. A *labeled polytope* is a pair of a combinatorial polytope \mathcal{G} and a ridge labeling on \mathcal{G} , which is a function of the set of the ridges of \mathcal{G} to $\{\frac{\pi}{m} \mid m = 2, 3, \dots, \infty\}$.

Let \mathcal{G} be a labeled d -polytope. A *marked Coxeter polytope realizing \mathcal{G}* is a pair of a Coxeter d -polytope P of \mathbb{S}^d and a lattice isomorphism ϕ between \mathcal{G} and P such that the label of each ridge r of \mathcal{G} is the dihedral angle of the ridge $\phi(r)$ of P . Two marked Coxeter polytopes $(P, \phi: \mathcal{G} \rightarrow P)$ and $(P', \phi': \mathcal{G} \rightarrow P')$ realizing \mathcal{G} are *isomorphic* if there is a projective automorphism ψ of \mathbb{S}^d such that $\psi(P) = P'$ and $\hat{\psi} \circ \phi = \phi'$, where $\hat{\psi}$ is the lattice isomorphism between P and P' induced by ψ .

Definition 2.6. The *deformation space* $\mathfrak{C}(\mathcal{G})$ of a labeled d -polytope \mathcal{G} is the space of isomorphism classes of marked Coxeter d -polytopes realizing \mathcal{G} .

A labeled polytope \mathcal{G} is *rigid* if the deformation space $\mathfrak{C}(\mathcal{G})$ is the singleton. Otherwise, \mathcal{G} is *flexible*.

Remark 2.7. A labeled polytope \mathcal{G} and the deformation space $\mathfrak{C}(\mathcal{G})$ can be thought of as a Coxeter orbifold \mathcal{O} and the deformation space of marked convex real projective structures on \mathcal{O} , respectively (see Thurston [Thu02]).

In a way similar to Coxeter polytopes, we can associate to a labeled polytope \mathcal{G} a Coxeter system and therefore the Coxeter group $W_{\mathcal{G}}$: the generators and the relations of $W_{\mathcal{G}}$ corresponds to the facets and the ridge labeling of \mathcal{G} , respectively. A labeled polytope \mathcal{G} is *irreducible* (resp. *spherical, affine, large, Lannér*) if $W_{\mathcal{G}}$ is irreducible (resp. spherical, affine, large, Lannér). We can also define the *link \mathcal{G}_v of a labeled polytope \mathcal{G} at a vertex v* using the usual notion of the link of a combinatorial polytope and labeling \mathcal{G}_v accordingly. A labeled polytope \mathcal{G} is *perfect* (resp. *2-perfect*) if every vertex link of \mathcal{G} is spherical (resp. perfect). An easy observation is:

Remark 2.8. Let \mathcal{G} be a labeled polytope of dimension d . Then:

- If $W_{\mathcal{G}}$ is spherical or Lannér, then \mathcal{G} is perfect.
- If $W_{\mathcal{G}}$ is affine and virtually isomorphic to \mathbb{Z}^d , then \mathcal{G} is perfect.
- If \mathcal{G} is a perfect d -simplex, then $W_{\mathcal{G}}$ is spherical, irreducible affine or Lannér.

Note that we *cannot* define the notion of quasi-perfect labeled polytope because, for example, the labeled triangle $(\frac{\pi}{\infty}, \frac{\pi}{\infty}, \frac{\pi}{\infty})$ admits a realization for which the link of every vertex is parabolic (e.g. an ideal triangle of \mathbb{H}^2) or loxodromic (e.g. a hyperideal triangle of \mathbb{H}^2).

To be short, we say that a vertex v of a Coxeter (or labeled) polytope is “*something*” if the corresponding link is “*something*”. Here, the word “*something*” can be: irreducible, spherical, affine, large, Lannér, rigid, flexible, perfect, 2-perfect and so on.

2.11. The space of Cartan matrices. Let (S, M) be a Coxeter system with Coxeter group W . A Cartan matrix $A = (a_{st})_{s,t \in S}$ realizes (S, M) (or simply W) if $a_{st}a_{ts} = 4 \cos^2(\frac{\pi}{M_{st}})$ when $M_{st} \neq \infty$ and $a_{st}a_{ts} \geq 4$ when $M_{st} = \infty$. The space of Cartan matrices of W of rank $d + 1$, denoted by $\mathfrak{C}(W, d)$, is the space of equivalence classes of Cartan matrices of rank $d + 1$ which realize W (see (1) in Section 2.6 for the equivalence relation).

If \mathcal{G} is a labeled d -polytope with Coxeter group $W_{\mathcal{G}}$, then there is a map

$$\Lambda: \mathfrak{C}(\mathcal{G}) \rightarrow \bigcup_{e \leq d} \mathfrak{C}(W_{\mathcal{G}}, e)$$

given by $\Lambda(P) = [A_P]$. We will see that in the cases we are interested in, the image of $\mathfrak{C}(\mathcal{G})$ under Λ is exactly $\mathfrak{C}(W_{\mathcal{G}}, d)$ and moreover $\mathfrak{C}(\mathcal{G})$ is homeomorphic to $\mathfrak{C}(W_{\mathcal{G}}, d)$.

2.12. Dehn filling of Coxeter polytope. Let P_{∞} be a Coxeter d -polytope and let $\mathcal{V} = \{v_1, \dots, v_k\}$ be a non-empty subset of the set $\mathcal{V}_{\mathcal{P}}$ of all parabolic vertices of P_{∞} .

Definition 2.9. A projective generalized Dehn filling (or simply Dehn filling) of P_{∞} at \mathcal{V} is a Coxeter d -polytope P_{m_1, \dots, m_k} with $m_i \in \mathbb{N}$, $i = 1, \dots, k$, such that the underlying labeled polytope of P_{∞} is obtained from the underlying labeled polytope of P_{m_1, \dots, m_k} by collapsing a ridge r_i of label $\frac{\pi}{m_i}$ to the vertex v_i for each $i = 1, \dots, k$ (see Figure 1 for Dehn filling in dimension 3).

Remark 2.10. Since every vertex $v \in \mathcal{V}$ is parabolic, we know from Marquis [Mar14] that v is *cusped*, i.e. there exists a neighbourhood \mathcal{U} of v such that

$$\mu_{\Omega_{P_{\infty}}}(P_{\infty} \cap \mathcal{U}) < \infty.$$

In Section 8, we show that the existence of a Dehn filling *implies some condition on the cusp*: Let P_{∞} be an irreducible Coxeter d -polytope with Coxeter group W_{∞} . If there is a Dehn filling of P_{∞} at $\mathcal{V} \subset \mathcal{V}_{\mathcal{P}}$, then for each $v \in \mathcal{V}$, the Coxeter group $W_{\infty, v}$ of the link of P_{∞} at v is equal to $\tilde{A}_1 \times \tilde{A}_{d-2}$ (see Appendix A for the notation of the spherical and affine Coxeter groups).

2.13. Combinatorial 4-polytopes with 6 facets. By taking a cone with basis a polytope Q of dimension $d - 1$, we obtain a polytope $\text{Pyr}(Q)$ of dimension d . Every non-trivial choice of apex of the cone gives the same combinatorial structure for $\text{Pyr}(Q)$. Such a polytope $\text{Pyr}(Q)$ is called the *pyramid over Q* . We use the notation Δ_d to denote the simplex of dimension d . A polytope P is *simple* if every vertex link of P is the simplex.

There are 4 different combinatorial polytopes of dimension 4 with 6 facets (see Paragraph 6.5 of Ziegler [Zie95]). Two of them are simple: $\Delta_1 \times \Delta_3$ and $\Delta_2 \times \Delta_2$, and two others are pyramids: $\text{Pyr}(\Delta_1 \times \Delta_2)$ and $\text{Pyr}(\text{Pyr}(\Delta_1 \times \Delta_1))$.

We emphasize the following distinction between these four polytopes.

- In $\Delta_1 \times \Delta_3$: *there exists a pair of facets that do not intersect.*

- In $\Delta_2 \times \Delta_2$: *any two facets are adjacent.*
- In $\text{Pyr}(\Delta_1 \times \Delta_2)$ and $\text{Pyr}(\text{Pyr}(\Delta_1 \times \Delta_1))$: *any two facets intersect.*
- In $\text{Pyr}(\Delta_1 \times \Delta_2)$: *there is exactly one pair of non-adjacent facets.*
- In $\text{Pyr}(\text{Pyr}(\Delta_1 \times \Delta_1))$: *there are exactly two pairs of non-adjacent facets.*

A *triangular prism* (or simply *prism*) is the combinatorial polytope $\Delta_1 \times \Delta_2$, and hence $\text{Pyr}(\Delta_1 \times \Delta_2)$ is the 4-pyramid over the prism. More generally, $\Delta_1 \times \Delta_{d-1}$ is the *d-prism* (or a *simplicial prism*).

3. LABELED POLYTOPES \mathcal{G}_m^i

In Sections 3, 4, 5 and 6, for the sake of demonstrating the construction of the examples in Theorem A without too many technical details, we concentrate our attention on three families of labeled 4-polytopes \mathcal{G}_m^i with $i = 1, 2, 3$ such that $7 \leq m \leq \infty$ if $i = 1, 2$, and $3 \leq m \leq \infty$ if $i = 3$. The arguments can easily be extended to cover the other families in Table 1.

If m is finite, then the underlying polytope of \mathcal{G}_m^i is the product of two triangles, and if $m = \infty$, then the underlying polytope of \mathcal{G}_∞^i is the pyramid over the prism (see Figure 3 for the Schlegel diagrams of these polytopes, and refer to Chapter 5 of Ziegler [Zie95] for more details about Schlegel diagram). We explain below how to give labels on the ridges of \mathcal{G}_m^i using the Coxeter groups W_m^i in Table 4.

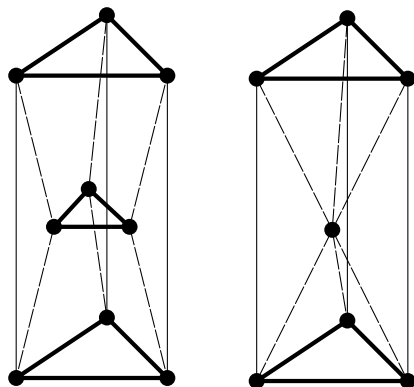


FIGURE 3. The Schlegel diagram of the product of two triangles and of a pyramid over a prism.

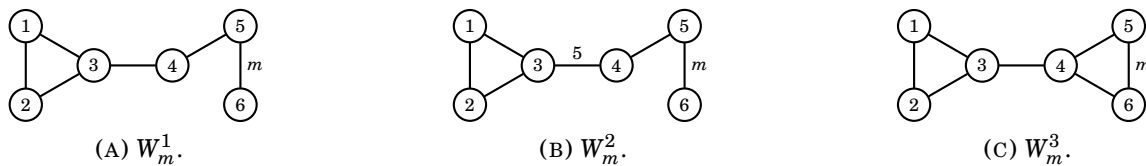


TABLE 4. The Coxeter graph of W_m^1, W_m^2, W_m^3

3.1. The labeled polytopes \mathcal{G}_m^i for m finite. The underlying polytope of \mathcal{G}_m^i is the Cartesian product \mathcal{G} of two triangles. It is a 4-dimensional polytope with 6 facets, 15 ridges and 9 vertices.

We describe more precisely the combinatorial structure of \mathcal{G} : Let T_1, T_2 be two triangles and let $\mathcal{G} = T_1 \times T_2$. Denote the edges of T_1 by e_1, e_2, e_3 and the edges of T_2 by e_4, e_5, e_6 . The 6 facets of \mathcal{G} are $F_k = e_k \times T_2$, $k = 1, 2, 3$, and $F_l = T_1 \times e_l$, $l = 4, 5, 6$, and the 15 ridges of \mathcal{G} can be divided into 2 families:

- 9 ridges (quadrilaterals): $e_k \times e_l$ for every $k = 1, 2, 3$ and $l = 4, 5, 6$.
- 6 ridges (triangles): $T_1 \times w$ and $v \times T_2$ for every vertex w (resp. v) of T_2 (resp. T_1).

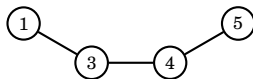
The ridge $F_k \cap F_l$ is $(e_k \cap e_l) \times T_2$ if $(k, l) \in \{(1, 2), (2, 3), (3, 1)\}$, $T_1 \times (e_k \cap e_l)$ if $(k, l) \in \{(4, 5), (5, 6), (6, 4)\}$, and $e_k \times e_l$ if $k = 1, 2, 3$ and $l = 4, 5, 6$. Finally, the 9 vertices of \mathcal{G} are $v_{klst} = F_k \cap F_l \cap F_s \cap F_t$ for $(k, l) \in \{(1, 2), (2, 3), (3, 1)\}$ and $(s, t) \in \{(4, 5), (5, 6), (6, 4)\}$.

Note that every two distinct facets of \mathcal{G} are adjacent, and hence we can label the ridges of \mathcal{G} according to the Coxeter graph W_m^i in Table 4 to obtain the labeled polytope \mathcal{G}_m^i .

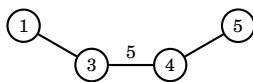
Proposition 3.1. *Let \mathcal{G}_m^i be the labeled polytope with m finite. Then:*

- The labeled polytopes \mathcal{G}_m^1 and \mathcal{G}_m^3 are perfect.
- The labeled polytope \mathcal{G}_m^2 is 2-perfect with exactly two Lannér vertices v_{1345} and v_{2345} .

Proof. First, remark that \mathcal{G}_m^i is simple, and so every vertex link is a 3-simplex. Hence, by Remark 2.8, the proof immediately follows from comparing the Coxeter diagrams of the links of \mathcal{G}_m^i and the Coxeter diagrams in Appendix A. For example, on the one hand, the Coxeter group of the link of \mathcal{G}_m^1 (or \mathcal{G}_m^3) at the vertex v_{1345} is:



which is the spherical Coxeter diagram A_4 in Table 6, and on the other hand, the Coxeter group of the link of \mathcal{G}_m^2 at the vertex v_{1345} is:

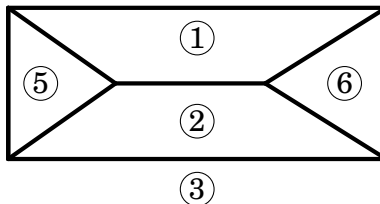


which is a Lannér Coxeter diagram in Table 9. □

3.2. The labeled polytopes \mathcal{G}_m^i for m infinite. The underlying polytope of the labeled polytope \mathcal{G}_∞^i , $i = 1, 2, 3$, is the pyramid \mathcal{G}_∞ over the prism. It is a 4-dimensional polytope with 6 facets, 14 ridges and 7 vertices.

More precise description of the combinatorial structure of \mathcal{G}_∞ is as follows: Let I be a segment with two endpoints a_5 and a_6 , and Δ be a triangle with three edges b_1, b_2 and b_3 . The prism $I \times \Delta$ has 5 faces labeled in the obvious way (see Figure 4). Let $\mathcal{G}_\infty = \text{Pyr}(I \times \Delta)$. Among the 6 facets of \mathcal{G}_∞ , the five of them, F_i , $i = 1, 2, 3, 5, 6$, correspond to the facets of $I \times \Delta$, and F_4 denotes the remaining one. The 14 ridges of \mathcal{G}_∞ can be divided into 2 families:

- 5 ridges correspond to the faces of $I \times \Delta$ and these ridges are in the facet F_4 .


 FIGURE 4. The basis of \mathcal{G}_∞

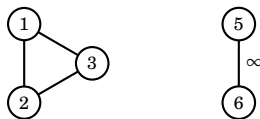
- 9 ridges correspond to the edges of $I \times \Delta$ and these ridges are $F_1 \cap F_2, F_2 \cap F_3, F_3 \cap F_1$ and $F_k \cap F_l$ with $k = 1, 2, 3$ and $l = 5, 6$.

We remark that every two facets F_k and F_l are adjacent except when $\{k, l\} = \{5, 6\}$ and $F_5 \cap F_6$ is the apex s of the pyramid \mathcal{G}_∞ , and therefore we can label the ridges of \mathcal{G}_∞ according to the Coxeter graph W_∞^i in Table 4 to obtain \mathcal{G}_∞^i .

Proposition 3.2. *Let \mathcal{G}_∞^i be the labeled polytope \mathcal{G}_m^i with m infinite. Then:*

- For every $i = 1, 2, 3$, the labeled polytope \mathcal{G}_∞^i is 2-perfect.
- For each $i = 1, 3$, the only non-spherical vertex of \mathcal{G}_∞^i is the apex s , which is affine of type $\tilde{A}_1 \times \tilde{A}_2$.
- The labeled polytope \mathcal{G}_∞^2 has 3 non-spherical vertices v_{1345}, v_{2345} and the apex s . Two of them, v_{1345} and v_{2345} , are Lannér, and s is affine of type $\tilde{A}_1 \times \tilde{A}_2$.

Proof. Remark that except the apex s of \mathcal{G}_∞^i , every vertex link is a 3-simplex, and the link of s is a prism. Once again, by Remark 2.8, the proof follows from comparing the Coxeter diagrams of the links of \mathcal{G}_∞^i and the Coxeter diagrams in Appendix A. For the apex s , the Coxeter group of the link of \mathcal{G}_∞^i at $s = F_1 \cap F_2 \cap F_3 \cap F_5 \cap F_6$ is:



which is the affine Coxeter group $\tilde{A}_2 \times \tilde{A}_1$ (see Table 7) virtually isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}^1 = \mathbb{Z}^3$, and hence s is perfect. \square

4. DEFORMATION SPACES $\mathcal{C}(\mathcal{G}_m^i)$

This section is devoted to the proof of the following proposition:

Proposition 4.1. *Let $\mathcal{C}(\mathcal{G}_m^i)$ be the deformation space of the labeled polytope \mathcal{G}_m^i with m finite. Then:*

- The deformation space $\mathcal{C}(\mathcal{G}_m^i)$, $i = 1, 2$, consists of two points for every $6 < m < \infty$.
- The deformation space $\mathcal{C}(\mathcal{G}_m^3)$ consists of two disjoint lines for every $3 < m < \infty$.

In order to prove Proposition 4.1 and others, we need Theorem 4.2, which tells us how the combinatorial structure of a Coxeter polytope P is influenced by the Cartan matrix of P . Recall that S is the set of facets of P , $A = A_P$ the Cartan matrix of P and for every face f of

triangles is the only 4-polytope with 6 facets such that every two distinct facets are adjacent, which completes the proof. \square

4.2. The computation. For $\alpha, \beta, \gamma \in \mathbb{R}$, let:

$$\Psi_{\alpha, \beta, \gamma} = 1 - \cos(\alpha)^2 - \cos(\beta)^2 - \cos(\gamma)^2 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

The following equality is remarkable:

$$\Psi_{\alpha, \beta, \gamma} = -4\cos\left(\frac{\alpha + \beta + \gamma}{2}\right)\cos\left(\frac{-\alpha + \beta + \gamma}{2}\right)\cos\left(\frac{\alpha - \beta + \gamma}{2}\right)\cos\left(\frac{\alpha + \beta - \gamma}{2}\right)$$

If $\alpha, \beta, \gamma \in (0, \frac{\pi}{2}]$, then $\Psi_{\alpha, \beta, \gamma} < 0$ (resp. $= 0, > 0$) if and only if $\alpha + \beta + \gamma < \pi$ (resp. $= \pi, > \pi$). We introduce the following two matrices:

$$A_\lambda^1 = \begin{pmatrix} 1 & -c_{12} & -\lambda^{-1}c_{31} \\ -c_{12} & 1 & -c_{23} \\ -\lambda c_{31} & -c_{23} & 1 \end{pmatrix} \quad A_\mu^2 = \begin{pmatrix} 1 & -c_{45} & -\mu^{-1}c_{64} \\ -c_{45} & 1 & -c_{56} \\ -\mu c_{64} & -c_{56} & 1 \end{pmatrix}$$

Implicitly, we assume that $m_{12} = m_{23} = m_{13} = m_{45} = 3$, but we keep the notation of double index in order to use the remarkable equality. By a direct computation, we have:

$$\frac{1}{64} \det(A_{\lambda, \mu}) = \det(A_\lambda^1) \det(A_\mu^2) - s_{12}^2 c_{34}^2 s_{56}^2$$

Another direct computation gives us:

$$\begin{aligned} \det(A_\lambda^1) &= 1 - (c_{12}^2 + c_{23}^2 + c_{31}^2) - c_{12}c_{23}c_{31}(\lambda + \lambda^{-1}) = \Psi_{123} - c_{12}c_{23}c_{31}(\lambda + \lambda^{-1} - 2) \\ \det(A_\mu^2) &= 1 - (c_{45}^2 + c_{56}^2 + c_{64}^2) - c_{45}c_{56}c_{64}(\mu + \mu^{-1}) = \Psi_{456} - c_{45}c_{56}c_{64}(\mu + \mu^{-1} - 2) \end{aligned}$$

where Ψ_{123} (resp. Ψ_{456}) is $\Psi_{\alpha, \beta, \gamma}$ with $(\alpha, \beta, \gamma) = (\frac{\pi}{m_{12}}, \frac{\pi}{m_{23}}, \frac{\pi}{m_{31}})$ (resp. $(\frac{\pi}{m_{45}}, \frac{\pi}{m_{56}}, \frac{\pi}{m_{64}})$).

Remark 4.4. The quantity Ψ_{123} is negative (resp. zero, positive) if and only if the triangle $\{1, 2, 3\}$ is hyperbolic (resp. affine, spherical). Of course, the analogous statement is true for the quantity Ψ_{456} and the triangle $\{4, 5, 6\}$.

4.3. Proof of Proposition 4.1.

Proof of Proposition 4.1. Assume that $i = 1, 2$. By Proposition 4.3, we know that $\mathcal{C}(\mathcal{G}_m^i)$ is homeomorphic to:

$$\{\lambda \in \mathbb{R} \mid \lambda > 0 \text{ and } \det(A_\lambda) = 0\}$$

Moreover, since $c_{64} = 0$ and the triangle $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ is affine, i.e. $\Psi_{123} = 0$, we have:

$$\frac{1}{64} \det(A_\lambda) = \frac{1}{8} (2 - (\lambda + \lambda^{-1})) \Psi_{456} - s_{12}^2 c_{34}^2 s_{56}^2$$

Observe that $m > 6$ if and only if the triangle $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m})$ is hyperbolic, i.e. $\Psi_{456} < 0$. Let $x = \lambda + \lambda^{-1}$. The function $[2, +\infty) \ni x \mapsto \det(A_\lambda) \in \mathbb{R}$ is an unbounded strictly increasing function. If $x = 2$, i.e. $\lambda = 1$, then $\det(A_\lambda) = -64s_{12}^2 c_{34}^2 s_{56}^2 < 0$. Hence, there is a unique $x > 2$ and exactly two $\lambda \in \mathbb{R}_+^* \setminus \{1\}$ such that $\det(A_\lambda) = 0$.

Now assume that $i = 3$. Once again, by Proposition 4.3, we need to solve the equation:

$$\frac{1}{64} \det(A_{\lambda, \mu}) = \frac{1}{8} (\lambda + \lambda^{-1} - 2)(c_{45}c_{56}c_{64}(\mu + \mu^{-1} - 2) - \Psi_{456}) - s_{12}^2 c_{34}^2 s_{56}^2 = 0$$

on the set $\{(\lambda, \mu) \in \mathbb{R}^2 \mid \lambda, \mu > 0\}$. Let $x = \lambda + \lambda^{-1}$ and $y = \mu + \mu^{-1}$. The equation is of the form:

$$(x - 2)(y - A) = B$$

with $B > 0$ and $A < 2$, and thus the solution space of the equation on $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 2\}$ is homeomorphic to a closed half-line whose extremity E is a point $(x_0, 2)$ with $x_0 > 2$ (see the blue curve in Figure 5 (A)). It follows that the set of solutions in (λ, μ) is homeomorphic to two disjoint lines (see the two blue curves in Figure 5 (B)), which completes the proof. \square

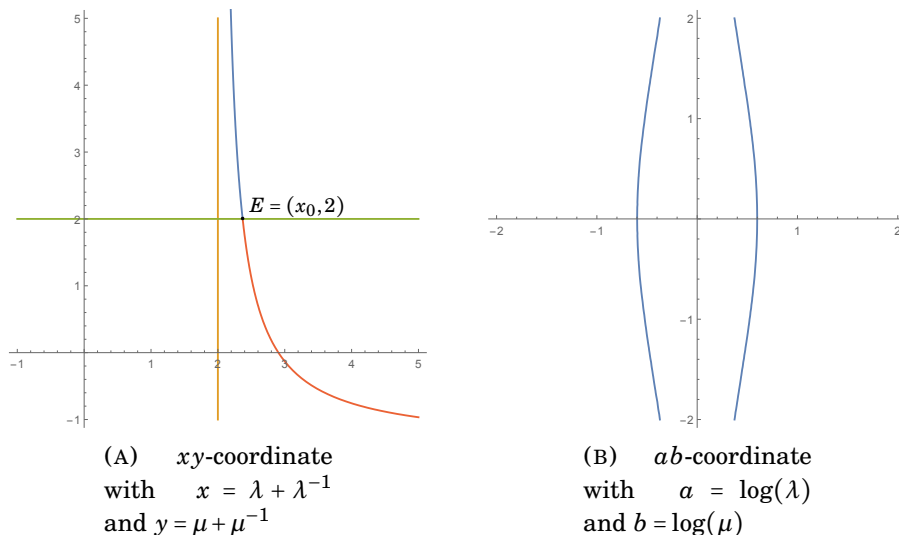


FIGURE 5. The deformation space $\mathcal{C}(\mathcal{G}_m^3)$ with $m = 7$

Remark 4.5. If $i = 1, 2$ and $3 \leq m \leq 6$, then $\mathcal{C}(\mathcal{G}_m^i) = \emptyset$.

Remark 4.6. If $i = 3$ and $m = 3$, then the equation to solve is of the form:

$$(x-2)(y-2) = B$$

with $B > 0$, and thus the set of solutions in (λ, μ) is homeomorphic to four disjoint lines.

Definition 4.7. For each Coxeter polytope $P \in \mathcal{C}(\mathcal{G}_m^3)$, the Cartan matrix A_P is equivalent to the unique standard form $A_{\lambda, \mu}$, and thus the real number μ is an invariant of P . It is called *the μ -invariant of P* and denoted by $\mu(P)$.

By the parametrization of $\mathcal{C}(\mathcal{G}_m^3)$, the map $\mathcal{C}(\mathcal{G}_m^3) \rightarrow \mathbb{R}_+^*$ given by $P \mapsto \mu(P)$ is a two-to-one covering map. More precisely, for any $\mu > 0$, there exist exactly two Coxeter polytopes P_m and P'_m in $\mathcal{C}(\mathcal{G}_m^3)$ such that $A_{P_m} = A_{\lambda, \mu}$ and $A_{P'_m} = A_{\lambda^{-1}, \mu}$ with some $\lambda > 0$.

Remark 4.8 (Duality). If $i = 1, 2$, then $\mathcal{C}(\mathcal{G}_m^i) = \{P, P^\sharp\}$ and A_{P^\sharp} is the transpose of A_P , and so P and P^\sharp are *dual* in a sense that there is an outer automorphism of $\mathrm{SL}_5^\pm(\mathbb{R})$ which exchange them. If $i = 3$, then the analogous statement is true for each pair $\{P, P^\sharp\}$ of Coxeter polytopes in $\mathcal{C}(\mathcal{G}_m^i)$ such that $A_P = A_{\lambda, \mu}$ and $A_{P^\sharp} = A_{\lambda^{-1}, \mu^{-1}}$ (up to equivalence).

5. LIMITS OF P_m

In this section, we study the limit of the sequence $(P_m)_m$.

Proposition 5.1.

- Let $i = 1, 2$. Assume P_m is one of the two points of $\mathcal{C}(\mathcal{G}_m^i)$. If A_{P_m} has the standard form, then as m goes to infinity, A_{P_m} converges to the symmetric matrix:

$$A_\infty^i = \begin{pmatrix} 2 & -1 & -1 & & & & \\ -1 & 2 & -1 & & & & \\ -1 & -1 & 2 & -2c_{34} & & & \\ & & -2c_{34} & 2 & -1 & & \\ & & & -1 & 2 & -2 & \\ & & & & -2 & 2 & \end{pmatrix}$$

- Let $i = 3$ and $\mu > 0$. Assume P_m is one of the two points of $\mathcal{C}(\mathcal{G}_m^3)$ such that $\mu(P_m) = \mu$. If A_{P_m} has the standard form, then as m goes to infinity, A_{P_m} converges to the matrix:

$$A_{\infty, \mu}^3 = \begin{pmatrix} 2 & -1 & -1 & & & & \\ -1 & 2 & -1 & & & & \\ -1 & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & -\mu^{-1} & \\ & & & -1 & 2 & -2 & \\ & & & -\mu & -2 & 2 & \end{pmatrix}$$

Proof. This is a straightforward computation. □

For each $m \in \mathbb{N} \cup \{\infty\}$, let P_m be a Coxeter polytope with Coxeter group W_m and Coxeter system (S_m, M_m) , and let $\rho_m : W_m \rightarrow \mathrm{SL}_{d+1}^\pm(\mathbb{R})$ be the corresponding representation. A sequence of Coxeter polytopes $(P_m)_m$ converges to P_∞ as $m \rightarrow \infty$ if:

- There is a bijection $\hat{\phi} : S_\infty \rightarrow S_m$ which induces the homomorphism $\phi : W_\infty \rightarrow W_m$.
- The sequence of representations $(\rho_m \circ \phi)_m$ converges algebraically to $\rho_\infty \in \mathrm{Hom}(W_\infty, \mathrm{SL}^\pm(V))$.

Note that $\mathrm{Hom}(W_\infty, \mathrm{SL}^\pm(V))$ is the space of all homomorphisms of W_∞ into $\mathrm{SL}^\pm(V)$ with the topology of pointwise convergence.

Corollary 5.2. *Let $i = 1, 2$ or 3 and $\mu > 0$. Let P_m be one of the two points of $\mathcal{C}(\mathcal{G}_m^i)$ (such that $\mu(P_m) = \mu$ if $i = 3$). As m goes to infinity, the sequence $(P_m)_m$ converges to a Coxeter polytope P_∞ whose Cartan matrix is A_∞^i if $i = 1, 2$ and $A_{\infty, \mu}^3$ if $i = 3$. Moreover, the underlying polytope of P_∞ is the pyramid over the prism.*

Proof. We ignore the subscript i and the index μ because the proof does not depend on i and μ .

The Cartan matrix A_∞ is irreducible, of negative type and of rank $d + 1$. By Theorem 2.2, there exists a unique Coxeter polytope P_∞ realizing A_∞ . The obvious bijection between S_∞ and S_m and Proposition 5.1 immediately tell us that the sequence $(P_m)_m$ converges to a Coxeter polytope P_∞ . The only remaining part is to prove that the underlying polytope of P_∞ is the pyramid over the prism. Let $T = \{1, 2, 3, 5, 6\}$. The Cartan matrix $A_T = A_T^0$ and $W_T = \tilde{A}_1 \times \tilde{A}_2$ virtually isomorphic to \mathbb{Z}^3 , and thus the third part of Theorem 4.2 shows that $T \in \sigma(P)$ and the intersection of the five facets corresponding to T is a vertex v of P_∞ . By Theorem 2.4, the link of P_∞ at v_∞ is a parabolic Coxeter 3-polytope whose underlying polytope is a prism, and so P_∞ is indeed the pyramid over the prism. □

6. GEOMETRIC DESCRIPTION

In this section, we explain the geometric properties of the Coxeter polytopes P_m and of the action of the projective reflection groups Γ_{P_m} of P_m on Ω_{P_m} . To do so, we need the following theorem:

Theorem 6.1 (Theorem E of Marquis [Mar14] using Benoist [Ben04a] and Cooper–Long–Tillmann [CLT15]). *Let P be a quasi-perfect loxodromic Coxeter polytope. Then the following are equivalent:*

- *The group W_P is Gromov hyperbolic relative to the standard subgroups of W_P corresponding to the parabolic vertices of P .*
- *The convex domain Ω_P is strictly convex.*
- *The boundary $\partial\Omega_P$ of Ω_P is of class \mathcal{C}^1 .*

6.1. Description of the geometry of the first example. Assume $i = 1$, and drop the index i . The Coxeter polytope P_∞ is a quasi-perfect hyperbolic polytope of dimension 4, listed in Tumarkin [Tum04]. In particular, P_∞ is a hyperbolic Coxeter polytope of finite volume, i.e. the projective Coxeter group of P_∞ is a non-uniform lattice of $\text{PO}_{d,1}(\mathbb{R})$.

The Coxeter 4-polytope P_m is perfect. The group Γ_m is not Gromov hyperbolic since the group generated by the reflections corresponding to $\{1, 2, 3\}$ is an affine Coxeter group virtually isomorphic to \mathbb{Z}^2 , and so by Theorem 6.1, the convex domain Ω_m is not strictly convex nor with \mathcal{C}^1 boundary.

6.2. Description of the geometry of the second example. Assume $i = 2$, and drop the index i . The Coxeter polytope P_∞ is a 2-perfect hyperbolic polytope of dimension 4 with four spherical vertices, one parabolic vertex and two loxodromic vertices, and the Coxeter polytope P_m is a 2-perfect Coxeter polytope of dimension 4 with seven spherical vertices and two loxodromic vertices (see the right subfigure 6). We will come back on these examples in Section 9.

6.3. Description of the geometry of the third example. Assume $i = 3$, and drop the index i . The Coxeter polytope P_∞ is a quasi-perfect polytope of dimension 4 with six spherical vertices, one parabolic vertex (see the left subfigure 6). Moreover, Γ_∞ is a non-uniform lattice of $\text{PO}_{d,1}(\mathbb{R})$ if and only if $\mu(P_\infty) = 1$. In other words, if $\mu(P_\infty) \neq 1$, then P_∞ is not a hyperbolic Coxeter polytope and the convex domain Ω_{P_∞} is not an ellipsoid, however by Theorem 6.1, it is strictly convex with \mathcal{C}^1 boundary.

The Coxeter polytope P_m is a perfect Coxeter polytope of dimension 4. Once again, by Theorem 6.1, the convex domain Ω_m is not strictly convex nor with \mathcal{C}^1 boundary. Note that compare to Section 6.1, in the case when $i = 3$, there is a one-parameter family of deformations of the convex domains.

7. MORE PRIME EXAMPLES IN DIMENSION 4

In Table 1A, we give a list of Coxeter groups W_m , which is not redundant but contains the previous three examples. For each of them, if m is an integer greater than or equal to 7, then we can label the product of two triangles in order to obtain a labeled polytope \mathcal{G}_m . The technique we previously describe easily extends to show:

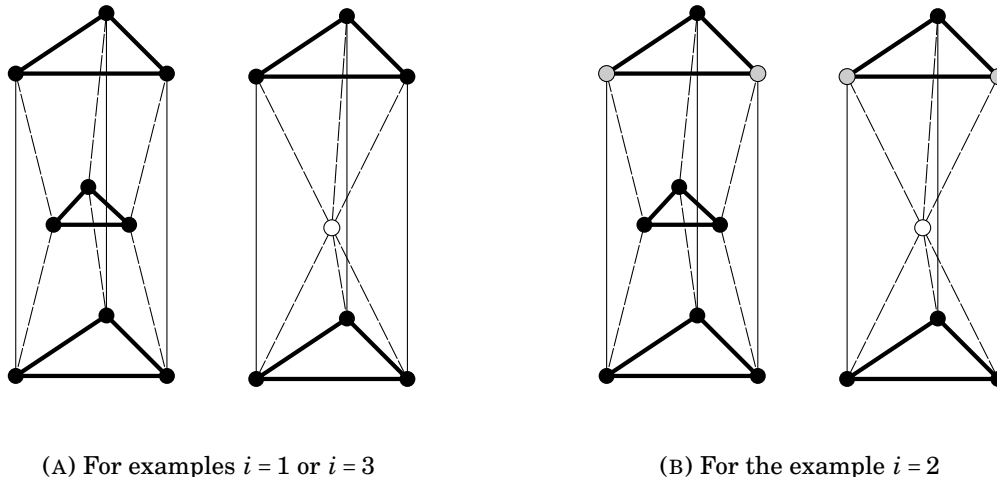


FIGURE 6. The spherical vertices are in black, the parabolic vertices are in white, the loxodromic vertices are in grey.

Theorem 7.1. *Choose one of the 13 families $(W_m)_{m \geq 7}$ in Table 1A. If the Coxeter graphs of the family have one loop, then $\mathfrak{C}(\mathcal{G}_m)$ consists of two points. If the Coxeter graphs of the family have two loops, then $\mathfrak{C}(\mathcal{G}_m)$ consists of two disjoint lines.*

If $m = \infty$, then we can label a pyramid over a prism in order to obtain a labeled polytope \mathcal{G}_∞ . Eventually, we can show:

Theorem 7.2. *Choose one of the 13 families $(W_m)_{m \geq 7}$ in Table 1A.*

- *Assume that the Coxeter graphs of the family have one loop. If P_m is one of the two points of $\mathfrak{C}(\mathcal{G}_m)$, then the sequence $(P_m)_m$ converges to the unique hyperbolic polytope P_∞ whose Coxeter graph is W_∞ .*
- *Assume that the Coxeter graphs of the family have two loops. If $\mu > 0$ and P_m is one of the two points of $\mathfrak{C}(\mathcal{G}_m)$ with $\mu(P_m) = \mu$, then $(P_m)_m$ converges to P_∞ which is the unique Coxeter 4-polytope such that $\mu(P_\infty) = \mu$ and $W_{P_\infty} = W_\infty$.*

8. NECESSARY CONDITIONS FOR DEHN FILLINGS

Let P_∞ be a Coxeter polytope with Coxeter group W_∞ and for some $m_i \in \mathbb{N}$, let P_{m_1, \dots, m_k} be a Dehn filling of P_∞ at a non-empty set $\mathcal{V} = \{v_1, \dots, v_k\}$ of some parabolic vertices of P_∞ , i.e. the underlying labeled polytope of P_∞ is obtained from P_{m_1, \dots, m_k} by collapsing a ridge r_i of label $\frac{\pi}{m_i}$ to the vertex v_i for every $v_i \in \mathcal{V}$.

Proposition 8.1. *Let $d \geq 3$, and let P_∞ be an irreducible Coxeter polytope of dimension d . If there exists a Dehn filling of P_∞ at \mathcal{V} , then for every $v \in \mathcal{V}$, $W_{\infty, v} = \tilde{A}_1 \times \tilde{A}_{d-2}$. Moreover, the link $P_{\infty, v}$ of P_∞ at v is a $(d-1)$ -prism.*

Before beginning the proof, we recall the following notations and a lemma of Vinberg: Let P be a Coxeter polytope with Coxeter group W , and S the set of facets of P . For each face f of P , let $S_f = \{s \in S \mid f \subset s\}$, and $\sigma(P) = \{S_f \mid f \text{ is a face of } P\} \subset 2^S$. If $T \subset S$, then $Z(T) = \{s \in S \mid s \perp t \forall t \in T\}$. Finally, W_T and A_T denote the Coxeter group and the Cartan matrix restricted to T , respectively, and S^0 denote the subset T of S such that $A_T = A_S^0$.

Lemma 8.2 (Lemma 13 and Proposition 21 of Vinberg [Vin71]). *Let W be an irreducible affine Coxeter group. If A_0 is the Tits matrix of W and A is a Cartan matrix realizing W , then the following is true:*

- $\text{rank}(A) \leq \text{rank}(A_0) + 1$.
- $\text{rank}(A) \geq \text{rank}(A_0)$, and the equality holds if and only if A and A_0 are equivalent.
- If A is not of zero type, then $W = \tilde{A}_k$ for some $k \geq 1$.

Proof of Proposition 8.1. For the simplicity of the notation, we assume that $|\mathcal{V}| = 1$ and denote by P_m a Dehn filling of P_∞ at \mathcal{V} . The proof of the general case is the same.

Let S_v be the set of facets of P_∞ containing $v \in \mathcal{V}$. First, look at the v -link Coxeter group $W_{\infty,v}$ of W_∞ , i.e. the Coxeter group of the link $P_{\infty,v}$ of P_∞ at v . Since v is a parabolic vertex, each component of the decomposition of the Coxeter group $W_{\infty,v}$:

$$W_{\infty,v} = W_{\infty,v}^1 \times \cdots \times W_{\infty,v}^l \quad \text{for some } l \in \mathbb{N}$$

is an irreducible affine Coxeter group (see Table 7) and $W_{\infty,v}$ is virtually isomorphic to \mathbb{Z}^{d-1} .

The facets of P_∞ are in correspondence with the facets of P_m , and so we can consider the Coxeter subgroup $W_{m,v}$ of W_m corresponding to the set S_v . Moreover, two adjacent facets of S_v in P_∞ are also adjacent in P_m with the same dihedral angle. Remark that the converse statement is false. Also, the decomposition of $W_{\infty,v}$ gives a partition of the set $S_v = S_v^1 \sqcup \cdots \sqcup S_v^l$, where $W_{\infty,v}^i$ corresponds to S_v^i for each $i = 1, \dots, l$, with the geometric interpretation that if $s \in S_v^i$ and $t \in S_v^j$ with $i \neq j$, then s and t are adjacent in P_∞ and the dihedral angle is $\frac{\pi}{2}$. As a consequence, the decomposition of $W_{\infty,v}$ induces a decomposition of the Coxeter group $W_{m,v}$:

$$W_{m,v} = W_{m,v}^1 \times \cdots \times W_{m,v}^l$$

At this moment, it is not obvious that each factor of this decomposition is irreducible, however we will see in the next paragraph that this is the case.

Let r be the ridge of P_m that collapses to the vertex v to obtain P_∞ , and $r = s \cap t$ with two facets $s, t \in S_v$. Then $W_{\{s,t\}} = I_2(m)$ in W_m and $W_{\{s,t\}} = \tilde{A}_1$ in W_∞ , and so we can assume that $W_{m,v}^1 = I_2(m)$ and $W_{\infty,v}^1 = \tilde{A}_1$. Finally, collapsing the ridge r is the only change of the adjacency rule, and hence for all $i = 2, \dots, l$, we have $W_{m,v}^i = W_{\infty,v}^i$.

Let $A_{\infty,v}$ (resp. $A_{m,v}$) be the Cartan submatrix of $A_\infty := A_{P_\infty}$ (resp. $A_m := A_{P_m}$) corresponding to S_v . According to the decomposition of $W_{\infty,v}$, we can decompose $A_{\infty,v}$ and $A_{m,v}$ as the direct sum:

$$A_{\infty,v} = A_{\infty,v}^1 \oplus \cdots \oplus A_{\infty,v}^l \quad \text{and} \quad A_{m,v} = A_{m,v}^1 \oplus \cdots \oplus A_{m,v}^l$$

where each $A_{\infty,v}^i$ and $A_{m,v}^i$, $i = 1, \dots, l$, corresponds to S_v^i , respectively, i.e. each $A_{\infty,v}^i$ (resp. $A_{m,v}^i$) is a component of $A_{\infty,v}$ (resp. $A_{m,v}$). Observe that $l \geq 2$: Otherwise,

$$d - 1 = \text{rank}(A_{\infty,v}) = \text{rank}(A_{\infty,v}^1) = 1,$$

which contradicts to the fact that $d \geq 3$.

Now, we have the following equalities or inequalities for rank:

- $\text{rank}(A_{m,v}) \leq \text{rank}(A_m) = d + 1$,
- $\text{rank}(A_{\infty,v}) = d - 1$,
- $\text{rank}(A_{m,v}^1) = \text{rank}(A_{\infty,v}^1) + 1$,
- $\text{rank}(A_{m,v}^i) \geq \text{rank}(A_{\infty,v}^i)$ for every $i \in \{2, \dots, l\}$,
- $\text{rank}(A_{m,v}^i) = \text{rank}(A_{\infty,v}^i) + 1$ for at least one $i \in \{2, \dots, l\}$.

The first item is trivial. The second item follows from the fact that v is a parabolic vertex of P_∞ . The third item is also trivial. The fourth item is a consequence of the second item of Lemma 8.2 and $W_{m,v}^i = W_{\infty,v}^i$.

In order to prove the fifth item, we assume on the contrary that for every $i \in \{2, \dots, l\}$, $\text{rank}(A_{m,v}^i) = \text{rank}(A_{\infty,v}^i)$. Let $T = S_v^2 \sqcup \dots \sqcup S_v^l$. First, by the second item of Lemma 8.2, for each $i \in \{2, \dots, l\}$, the Cartan matrix $A_{m,v}^i$ is of zero type, and hence $A_{m,T} = A_{m,T}^0$. If $U = T \cup Z(T)^0$, then $A_{m,U} = A_{m,U}^0$ and $Z(U)^0 = \emptyset$, so by the second item of Theorem 4.2, the intersection of the facets of P_m corresponding to U is non-empty. It follows that the intersection of the facets of P_m corresponding to T is also non-empty, contradicting the combinatorial structure of the Dehn filling P_m .

From the fifth item, we can assume without loss of generality that:

$$\text{rank}(A_{m,v}^2) = \text{rank}(A_{\infty,v}^2) + 1.$$

Now we have to show that $l = 2$. Suppose for contradiction that $l \geq 3$. Since

$$\text{rank}(A_{\infty,v}) = \sum_{i=1}^l \text{rank}(A_{\infty,v}^i) \quad \text{and} \quad \text{rank}(A_{m,v}) = \sum_{i=1}^l \text{rank}(A_{m,v}^i),$$

the previous five equalities and inequalities implies that:

$$\text{rank}(A_m) = \text{rank}(A_{m,v}) = d + 1 \quad \text{and} \quad \text{rank}(A_{m,v}^i) = \text{rank}(A_{\infty,v}^i) \quad \text{for every } i \in \{3, \dots, l\}$$

and hence by the second item of Lemma 8.2, we conclude that for every $i = 3, \dots, l$, $A_{m,v}^i$ is equivalent to $A_{\infty,v}^i$, which is of zero type.

Now look at the Cartan matrix A_m :

$$A_m = \begin{bmatrix} A_m^1 & 0 & 0 & \dots & 0 & C_1 \\ 0 & A_m^2 & 0 & \dots & 0 & C_2 \\ 0 & 0 & A_m^3 & \dots & 0 & C_3 \\ \dots & \dots & \dots & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & A_m^l & C_l \\ D_1 & D_2 & D_3 & \dots & D_l & F \end{bmatrix}$$

where $C_i, D_i, i = 1, \dots, l$, and F are some matrices. First, all entries of C_i and D_i are negative or null. Second, since A_m^i is of zero type, for every $i = 3, \dots, l$, there is a vector $X_i > 0$ such that $A_m^i X_i = 0$. Recall that $X > 0$ if every entry of X is strictly positive. Moreover, since $\text{rank}(A_m) = \text{rank}(A_{m,v})$, we also have that

$$\text{rank} \begin{bmatrix} A_m^l \\ D_l \end{bmatrix} = \text{rank}(A_m^l)$$

and so the vector $D_l X$ should be zero. As a consequence, D_l is a zero matrix, and so is C_l , which contradicts the fact that A_m and A_∞ are irreducible. Finally, we prove that $l = 2$.

In the previous paragraph, we show that $A_{m,v}^2$ cannot be of zero type, and therefore the Coxeter group $W_{\infty,v}^2 = W_{m,v}^2$ must be an irreducible affine Coxeter group which can be realized by a Cartan matrix not of zero type. By the third item of Lemma 8.2, only one family of irreducible affine Coxeter group satisfies this property: \tilde{A}_k (see Table 7). After all, $W_{\infty,v} = \tilde{A}_1 \times \tilde{A}_k$ virtually isomorphic to \mathbb{Z}^{k+1} , and therefore $k = d - 2$.

Finally, the Coxeter polytope $P_{\infty,v}$ satisfies the hypothesis of Theorem 2.4, and so $P_{\infty,v}$ is isomorphic to $\hat{\Delta}_{\tilde{A}_1 \times \tilde{A}_{d-2}}$, which is a prism. \square

9. HOW TO MAKE MORE EXAMPLES

9.1. Some definitions.

Truncation. Let \mathcal{G} be a combinatorial polytope, and let v be a vertex of \mathcal{G} . A (combinatorial) truncation of \mathcal{G} at v is an operation that cuts the vertex v (without touching the facets not containing v), creating a new facet in place of v (see Figure 7). The truncated polytope of \mathcal{G} at v is denoted by $\mathcal{G}^{\dagger v}$. If \mathcal{G} is a labeled polytope, then we attach the labels $\frac{\pi}{2}$ to all new ridges of $\mathcal{G}^{\dagger v}$, and we denote this new labeled polytope again by $\mathcal{G}^{\dagger v}$. Given a set \mathcal{V} of vertices of \mathcal{G} , in a similar way, we define the truncated polytope $\mathcal{G}^{\dagger \mathcal{V}}$ of \mathcal{G} at \mathcal{V} .

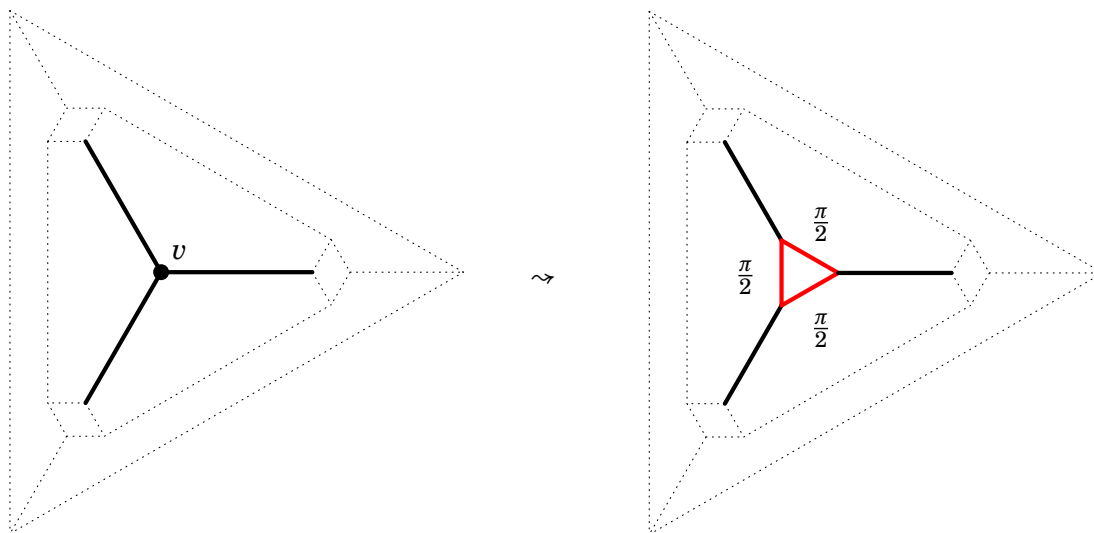


FIGURE 7. An example of truncation: $\mathcal{G} \rightsquigarrow \mathcal{G}^{\dagger v}$

If P is a Coxeter polytope of \mathbb{S}^d and v is a vertex of P , then we can also define a (geometric) truncation applied to P in the case when:

Definition 9.1. The vertex v of P is *truncable* if the projective subspace Π_v spanned by the polars $[b_s]$ for all facets s of P containing v is a hyperplane such that for every (closed) edge $e \ni v$, the set $\Pi_v \cap e$ consists of exactly one point in the interior of e .

Assume that a vertex v of P is truncable. Then a new Coxeter polytope $P^{\dagger v}$ can be obtained from P by the following operation of *truncation*: Let Π_v^- (resp. Π_v^+) be the connected component of $\mathbb{S}^d \setminus \Pi_v$ which does not contain v (resp. which contains v), and let $\overline{\Pi_v^-}$ be the closure of Π_v^- . The underlying polytope of $P^{\dagger v}$ is $\overline{\Pi_v^-} \cap P$, which has one *new facet* given by the hyperplane Π_v and the *old facets* given by P . The reflections about the old facets of $P^{\dagger v}$ are unchanged, and the reflection about the new facet is determined by the support Π_v and the polar v . It is easy to see that the dihedral angles of the ridges in the new facet of $P^{\dagger v}$ are $\frac{\pi}{2}$.

Remark 9.2. The hyperplane Π_v is invariant by the reflections about the facets of P containing v , and thus the intersection $\Pi_v \cap P$ is a Coxeter polytope of Π_v isomorphic to P_v . Also, note that the polytope $\overline{\Pi_v^+} \cap P$ is the pyramid over $\Pi_v \cap P$.

In a similar way, if \mathcal{V} is a subset of the set of truncable vertices of P , then we define the truncated Coxeter polytope $P^{\dagger \mathcal{V}}$ of P at \mathcal{V} . The following lemma is an easy corollary of Proposition 4.14 and Lemma 4.17 in Marquis [Mar14].

Lemma 9.3. Let \mathcal{G} be a labeled polytope with Coxeter group W , and let \mathcal{V} be a subset of the set of simple vertices of \mathcal{G} . Assume that W is irreducible and for every $v \in \mathcal{V}$, the v -link Coxeter group W_v is Lannér. Then the map $\chi : \mathcal{C}(\mathcal{G}) \rightarrow \mathcal{C}(\mathcal{G}^{\dagger \mathcal{V}})$ given by $\chi(P) = P^{\dagger \mathcal{V}}$ is well-defined and χ is a homeomorphism.

Remark 9.4. If \mathcal{G} is a labeled polytope with only spherical and Lannér vertices and \mathcal{V} is the set of Lannér vertices, then $\mathcal{G}^{\dagger \mathcal{V}}$ is perfect.

Gluing. Let \mathcal{G}_1 and \mathcal{G}_2 be two labeled d -polytopes, and let v_i , $i = 1, 2$, be a vertex of \mathcal{G}_i . If there is an isomorphism $f : (\mathcal{G}_1)_{v_1} \rightarrow (\mathcal{G}_2)_{v_2}$ as labeled polytope, then we can construct a new labeled d -polytope, denoted by $\mathcal{G}_1 \#_{v_1 \# v_2} \mathcal{G}_2$ (or simply $\mathcal{G}_1 \# \mathcal{G}_2$), by gluing together two truncated polytopes $\mathcal{G}_1^{\dagger v_1}$ and $\mathcal{G}_2^{\dagger v_2}$ via f (see Figure 8).

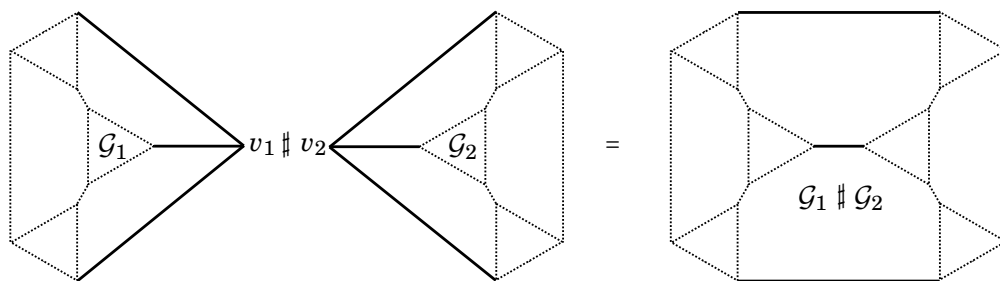


FIGURE 8. Gluing two 3-polytopes \mathcal{G}_1 and \mathcal{G}_2 along v_1 and v_2

Similarly, we can also define a (geometric) gluing applied to Coxeter polytopes: let P_1 and P_2 be Coxeter polytopes of \mathbb{S}^d and let v_i , $i = 1, 2$, be a truncable vertex of P_i . If there is an isomorphism $g : (P_1)_{v_1} \rightarrow (P_2)_{v_2}$ as Coxeter polytopes, then the Coxeter polytope $P_1 \# P_2$ is obtained by gluing two truncated polytopes $P_1^{\dagger v_1}$ and $P_2^{\dagger v_2}$ via g . For more details about truncation, we refer to [CLM].

9.2. Examples of truncable polytopes. In order to check whether the Coxeter group is relatively Gromov hyperbolic or not, we need the following theorem. Recall that if (S, M) is a Coxeter system, then a subset T of S gives us a Coxeter subsystem (T, M_T) , where M_T is the restriction of M to T . Here, we abbreviate (T, M_T) to T . If $s, t \in S$ are commuting elements of W_S , then we express this fact by writing $s \perp t$. In a natural way, we can understand the following notations: $S_1 \perp S_2$ for $S_1, S_2 \subset S$, and U^\perp for $U \subset S$.

Theorem 9.5 (Moussong [Mou88] for Gromov hyperbolic case and Caprace [Cap09, Cap15]). *Let (S, M) be a Coxeter system, and let \mathcal{T} be a collection of subsets of S . Then the group W_S is Gromov hyperbolic relative to $\{W_T \mid T \in \mathcal{T}\}$ if and only if the following hold:*

- If U is an affine subsystem of S of rank ≥ 3 , then $U \subset T$ for some $T \in \mathcal{T}$.
- If S_1, S_2 are irreducible non-spherical subsystems such that $S_1 \perp S_2$, then $S_1 \cup S_2 \subset T$ for some $T \in \mathcal{T}$.
- If T, T' are two distinct elements of \mathcal{T} , then $T \cap T'$ is a spherical subsystem of S .
- If $T \in \mathcal{T}$ and U is an irreducible non-spherical subsystem of T , then $U^\perp \subset T$.

First example. Let $P \in \mathcal{C}(\mathcal{G}_m^2)$ and let \mathcal{V} be the set of the two Lannér vertices of P . By Lemma 9.3, every vertex $v \in \mathcal{V}$ is truncable, and hence we obtain the perfect polytope $P^{\dagger\mathcal{V}}$.

Second example. Take the Coxeter group U in the left diagram of Table 5. As in Section 3, we can label the product of two triangles according to the Coxeter diagram U to obtain a labeled 4-polytope \mathcal{G}_U , which has 5 spherical vertices and 4 Lannér vertices. By Moussong's criterion (see Theorem 9.5), the Coxeter group U is Gromov hyperbolic, however we will see that U is not hyperbolizable, i.e. U cannot be the Coxeter group of a hyperbolic Coxeter 4-polytope, by computing the deformation space of \mathcal{G}_U .

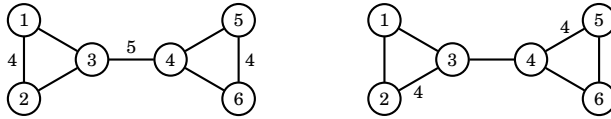


TABLE 5. Coxeter diagram of U and V

We will compute the deformation space $\mathcal{C}(\mathcal{G}_U)$ of \mathcal{G}_U and see that $\mathcal{C}(\mathcal{G}_U)$ satisfies the surprising property of being homeomorphic to a circle. In a way similar to Section 4, the deformation space $\mathcal{C}(\mathcal{G}_U)$ is homeomorphic to $\{(\lambda, \mu) \in \mathbb{R}^2 \mid \lambda, \mu > 0 \text{ and } \det(A_{\lambda, \mu}) = 0\}$, where:

$$A_{\lambda, \mu} = \begin{pmatrix} 2 & -\sqrt{2} & -\lambda^{-1} & & & & \\ -\sqrt{2} & 2 & -1 & & & & \\ -\lambda & -1 & 2 & -2c_5 & & & \\ & & -2c_5 & 2 & -1 & -\mu^{-1} & \\ & & & -1 & 2 & -\sqrt{2} & \\ & & & -\mu & -\sqrt{2} & 2 & \end{pmatrix}$$

A straightforward computation gives us:

$$\begin{aligned} \frac{1}{64} \det(A_{\lambda, \mu}) &= \left(1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} - \frac{11}{22} \frac{\sqrt{2}}{2} (\lambda + \lambda^{-1})\right) \left(-\frac{\sqrt{2}}{8} (\mu + \mu^{-1})\right) - c_5^2 s_4^2 s_4^2 \\ \frac{1}{2} \det(A_{\lambda, \mu}) &= (\lambda + \lambda^{-1})(\mu + \mu^{-1}) - 8c_5^2 \end{aligned}$$

In other words, the deformation space $\mathcal{C}(\mathcal{G}_U)$ is homeomorphic to:

$$\{(\lambda, \mu) \in \mathbb{R}^2 \mid \lambda, \mu > 0 \text{ and } (\lambda + \lambda^{-1})(\mu + \mu^{-1}) - 8c_5^2 = 0\}$$

which is homeomorphic to a circle (see Figure 9).

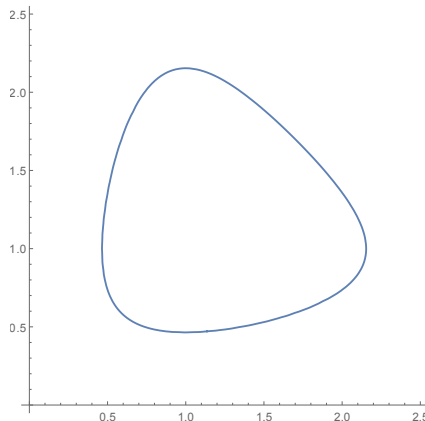


FIGURE 9. The deformation space \mathcal{G}_U

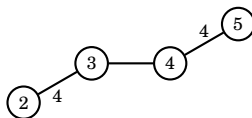
As a consequence, first \mathcal{G}_U is not hyperbolizable since $\lambda = \mu = 1$ is not in $\mathcal{C}(\mathcal{G}_U)$, and second we can answer the following question asked by Benoist if we weaken slightly the assumption:

Question 9.6. *Is there a perfect labeled polytope \mathcal{G} such that \mathcal{G} can be realized as a hyperbolic Coxeter polytope and the deformation space $\mathcal{C}(\mathcal{G})$ is compact and is of dimension ≥ 1 ?*

Theorem 9.7. *There exists a perfect labeled 4-polytope \mathcal{G} such that $W_{\mathcal{G}}$ is Gromov hyperbolic and $\mathcal{C}(\mathcal{G})$ is homeomorphic to the circle.*

Proof. Let \mathcal{V} be the set of all Lannér vertices of \mathcal{G}_U . Then the truncated polytope $\mathcal{G}_U^{\dagger\mathcal{V}}$ is perfect, and by Lemma 9.3, $\mathcal{C}(\mathcal{G}_U^{\dagger\mathcal{V}}) = \mathcal{C}(\mathcal{G}_U)$, which is homeomorphic to the circle. \square

Remark that if we take the Coxeter group V in the right diagram of Table 5, then $\mathcal{C}(\mathcal{G}_V)$ is also homeomorphic to a circle and every Coxeter polytope $P \in \mathcal{C}(\mathcal{G}_V)$ is quasi-perfect and of finite volume. Note that the link of \mathcal{G}_V at the vertex v_{2345} has the Coxeter group:



which is the affine Coxeter group \tilde{C}_4 virtually isomorphic to \mathbb{Z}^3 (see Table 7).

9.3. Playing Lego.

First game. Let N be an integer ≥ 2 , and let T be a line tree with N vertices, i.e. each vertex of T is of valence 1 or 2, and enumerate the vertices t_1, \dots, t_N of T . Choose N integers

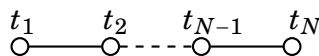


FIGURE 10. First game

$m_1, \dots, m_N > 6$, and for each $i = 1, \dots, N$, select a Coxeter polytope P_i from two polytopes in

$\mathcal{C}(\mathcal{G}_{m_i}^2)$ (see Section 4). Each P_i has two truncable vertices whose links are isomorphic as Coxeter polytope because every Lannér vertex of $\mathcal{G}_{m_i}^2$ is rigid. Let \mathcal{V}_i be the set of truncable vertices of P_i . If each vertex t_i of T corresponds to P_i , then the tree T naturally gives us a pattern to glue the Coxeter polytopes $P_i^{\dagger\mathcal{V}_i}$, $i = 1, \dots, N$, along the new facets of $P_i^{\dagger\mathcal{V}_i}$ in order to obtain a perfect Coxeter 4-polytope P_{m_1, \dots, m_N} . Remark that for each t_i , there are two possible positions of P_i .

If we make every m_i go to infinity, then the limit $P_{\infty, \dots, \infty}$ is a finite volume hyperbolic Coxeter 4-polytope with N cusps. In other words, the Coxeter polytopes P_{m_1, \dots, m_N} are Dehn fillings of $P_{\infty, \dots, \infty}$. An important remark is that since the choice of the integers $m_1, \dots, m_N > 6$ are independent each other, we can apply the Dehn filling operation to each cusp of $P_{\infty, \dots, \infty}$ *independently*.

Second game. Choose two integers $N, N' \geq 1$ and a tree T with $N + N'$ vertices such that N vertices, denoted by $\{\tau_1, \dots, \tau_N\} =: \tau$, are of valence 1 or 2 and N' vertices, denoted by $\{\tau'_1, \dots, \tau'_{N'}\} =: \tau'$, are of valence 1, 2, 3 or 4. Select N integers $m_1, \dots, m_N > 6$. For each vertex

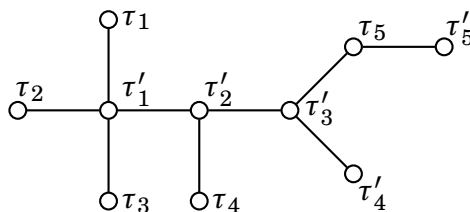


FIGURE 11. Second game with $N = N' = 5$

$t \in \tau$ (resp. τ'), we put a Coxeter polytope P_i (resp. Q_j) among the polytopes in $\mathcal{C}(\mathcal{G}_{m_i}^2)$ (resp. $\mathcal{C}(\mathcal{G}_U)$). Each Coxeter polytope P_i (resp. Q_j) has two (resp. four) truncable rigid vertices, and all the links of P_i and Q_j at the truncable vertices are isomorphic to each other. Let \mathcal{V}_i (resp. \mathcal{V}'_j) be the set of truncable vertices of P_i (resp. Q_j). Then the tree T gives us a pattern to glue the Coxeter polytopes $P_i^{\dagger\mathcal{V}_i}$, $i = 1, \dots, N$, and $Q_j^{\dagger\mathcal{V}'_j}$, $j = 1, \dots, N'$, along their new facets in order to obtain a perfect Coxeter 4-polytope P'_{m_1, \dots, m_N} .

Once again, if every m_i goes to infinity, then by Theorem 2.5, the limit polytope $P'_{\infty, \dots, \infty}$ is of finite volume in $\Omega_{P'_{\infty, \dots, \infty}}$. Remark that by Theorem 9.5 of Caprace, the Coxeter group of $P'_{\infty, \dots, \infty}$ is Gromov hyperbolic relative to the standard subgroups corresponding to the parabolic vertices of $P'_{\infty, \dots, \infty}$, however $P'_{\infty, \dots, \infty}$ is *not* hyperbolizable since \mathcal{G}_U is not.

10. HIGHER DIMENSIONAL EXAMPLES

10.1. Tumarkin's classification. The simplest polytope is the simplex, however we know from Proposition 8.1 that a hyperbolic Coxeter d -polytope which allows us to get a Dehn filling must have a vertex whose link is a $(d - 1)$ -prism, but the simplex has only simple vertices.

The next simplest polytopes have $d + 2$ facets, and Tumarkin [Tum04] classified all finite volume hyperbolic Coxeter d -polytopes with $d + 2$ facets: They exist only in dimension $d \leq 17$, and among them, hyperbolic Coxeter d -polytopes P_∞ with a $\tilde{A}_1 \times \tilde{A}_{d-2}$ cusp exist only in dimension $d \leq 7$. In this paper, for prime examples, we restrict ourself to d -polytopes with $d + 2$ facets, and thus Proposition 8.1 (or the work of Tumarkin) implies that the underlying

polytope of P_∞ is $\text{Pyr}(\Delta_1 \times \Delta_{d-2})$. We will see that the underlying polytopes of the Dehn fillings P_m of P_∞ are $\Delta_2 \times \Delta_{d-2}$.

10.2. The labelling of the \mathcal{G}_m . In a way similar to Section 3, we can label the product of a $(d-2)$ -simplex and a triangle according to each of the 12 families of Coxeter groups $(W_m)_{m \geq 7}$ in Tables 1 B,C,D in order to get a labeled polytope \mathcal{G}_m :

First, let a_1, \dots, a_{d-1} be the vertices in the loop \tilde{A}_{d-2} of W_m , and b_1, b_2, b_3 the remaining vertices of W_m . Next, denote the facets of the $(d-2)$ -simplex by the same symbols a_1, \dots, a_{d-1} and the edges of the triangle by the symbols b_1, b_2, b_3 . Then the symbols $a_1, \dots, a_{d-1}, b_1, b_2, b_3$ are naturally assigned to the facets of $\Delta_{d-2} \times \Delta_2$. Finally, we obtain the labeled polytopes \mathcal{G}_m with a ridge labelling on $\Delta_{d-2} \times \Delta_2$ according to W_m . The techniques we describe before can be easily extended to show:

Theorem 10.1. *Choose one of the 12 families $(W_m)_{m \geq 7}$ in Tables 1 B,C,D. If the Coxeter graphs of the family have one loop, then $\mathfrak{C}(\mathcal{G}_m)$ consists of two points. If the Coxeter graphs of the family have two loops, then $\mathfrak{C}(\mathcal{G}_m)$ consists of two disjoint lines.*

If $m = \infty$, then we can label a pyramid over a prism according to W_∞ in order to obtain a labeled polytope \mathcal{G}_∞ . Eventually, we can show:

Theorem 10.2. *Choose one of the 12 families $(W_m)_{m \geq 7}$ in Tables 1 B,C,D.*

- *Assume that the Coxeter graphs of the family have one loop. If P_m is one of the two points of $\mathfrak{C}(\mathcal{G}_m)$, then the sequence $(P_m)_m$ converges to the unique hyperbolic polytope P_∞ whose Coxeter graph is W_∞ .*
- *Assume that the Coxeter graphs of the family have two loops. If $\mu > 0$ and P_m is one of the two points of $\mathfrak{C}(\mathcal{G}_m)$ such that $\mu(P_m) = \mu$, then $(P_m)_m$ converges to P_∞ which is the unique Coxeter polytope such that $\mu(P_\infty) = \mu$ and whose Coxeter graph is W_∞ .*

Proof of Theorems 10.1 and 10.2. We first need to compute $\mathfrak{C}(\mathcal{G}_m)$, the proof is identical to the proof of Proposition 4.1, so we leave it to the reader. To help the reader do the actual computation, we remark that if c_1, \dots, c_n are real numbers and D_λ is the determinant of the following matrix M_λ

$$M_\lambda = \begin{pmatrix} 2 & -c_1 & & & -c_n \lambda^{-1} \\ -c_1 & 2 & -c_2 & & & \\ & -c_2 & 2 & -c_3 & & \\ & & -c_3 & 2 & \ddots & \\ -c_n \lambda & & & \ddots & 2 & -c_{n-1} \\ & & & & -c_{n-1} & 2 \end{pmatrix}$$

then we have the following equality:

$$D_\lambda = D_1 - c_1 c_2 \cdots c_n ((\lambda + \lambda^{-1}) - 2)$$

Now we need to check that \mathcal{G}_m is the underlying labeled polytope of any Coxeter polytopes realizing W_m . In the case when $m = \infty$, there is a standard subgroup of W_∞ that is an affine Coxeter group $\tilde{A}_1 \times \tilde{A}_{d-2}$, and hence as in the proof of Corollary 5.2, P_∞ is a pyramid over a prism (see also Proposition 8.1).

In the case when $m < \infty$, we need to be more careful. The shortest path is probably to do a bit of book-keeping and to separate dimension 5 from dimensions 6, 7. If we denote by $\text{Pyr}^g(Q)$ the pyramid over ... the pyramid over Q (g times), then every d -polytope with $d+2$ facets is of the form $\text{Pyr}^g(\Delta_e \times \Delta_f)$ with $e+f+g=d$ (see Section 6.5 of Ziegler [Zie95]). It is easy to show that $\text{Pyr}^g(\Delta_e \times \Delta_f)$ is a polytope with $(e+1)(f+1)+g$ vertices and among them exactly $(e+1)(f+1)$ vertices are simple. Moreover, the number of edges of $\text{Pyr}^g(\Delta_e \times \Delta_f)$ is $(e+1)(f+1)\left(g + \frac{e+f}{2}\right) + \frac{g(g-1)}{2}$.

First in dimension 6 or 7, remark that the $3(d-1)$ standard subgroups of W_m given by erasing one node of the \tilde{A}_{d-2} loop and one node of the remaining part are spherical Coxeter groups of rank d . It follows that the Coxeter group W_m contains at least $3(d-1)$ standard spherical subgroups of rank d , and by Theorem 4.2, P_m has at least $3(d-1)$ simple vertices.

However, a careful inspection shows that: *if $d = 6$ or 7 , then the only d -polytope with at least $3(d-1)$ vertices are: $\Delta_2 \times \Delta_4$, $\Delta_3 \times \Delta_3$, $\Delta_2 \times \Delta_5$ and $\Delta_3 \times \Delta_4$.* In particular, P must be simple, and we just have to eliminate the possibility of $\Delta_3 \times \Delta_3$ and $\Delta_3 \times \Delta_4$.

In order to so, we need to introduce the following: Let S be the set of facets of a polytope P . A subset A of $(k+1)$ elements of S is k -adjacent if for every subset B of k elements of A , the intersection of the facets in B is a $(d-k)$ -dimensional face of P . Notice that for the polytope $\Delta_e \times \Delta_f$ with $e \leq f$, there is a f -adjacent subset of $(f+1)$ facets but no $(f+1)$ -adjacent subset of $(f+2)$ facets. However, the Coxeter group W_m contains \tilde{A}_{d-2} and so by Theorem 4.2, there exists a $(d-2)$ -adjacent subset of $(d-1)$ facets of P_m . As a consequence, P must be $\Delta_2 \times \Delta_{d-2}$.

Second, in dimension 5, we should remark that the 30 standard subgroups of W_m given by erasing one node (resp. two nodes) of the \tilde{A}_3 loop and two nodes (resp. one node) of the remaining part are spherical of rank $d-1$. So the Coxeter group W_m contains at least 30 standard spherical subgroups of rank $d-1$. By Theorem 4.2, this implies that P_m has at least 30 edges. But, a careful inspection show that the only 5-polytope with 7-facets with at least 30 edges is $\Delta_2 \times \Delta_3$. \square

Remark 10.3. The following is the reason why we separate the arguments for dimension 5 and dimensions 6,7: The right Coxeter group in Table 1B with $k=l=5$ has only 6 spherical subsystem of rank 5; and the polytope $\text{Pyr}(\Delta_3 \times \Delta_3)$ has 64 edges, however $\Delta_2 \times \Delta_5$ has only 63 edges.

10.3. Nine examples in dimension 5. See the left diagram in Table 1B. If $p=3$, then the Coxeter polytopes P_m with m finite are perfect and Ω_{P_m} is not strictly convex nor with \mathcal{C}^1 -boundary. The polytope P_∞ is a quasi-perfect hyperbolic Coxeter polytope with one parabolic vertex.

If $p=4$, then the polytopes P_m with m finite are quasi-perfect with one parabolic vertex and Ω_{P_m} is not strictly convex nor with \mathcal{C}^1 -boundary. The polytope P_∞ is a quasi-perfect hyperbolic Coxeter polytope with two parabolic vertices.

If $p=5$, then the polytopes P_m with m finite are 2-perfect with three rigid loxodromic vertices and Ω_{P_m} is not strictly convex. The polytope P_∞ is a 2-perfect hyperbolic Coxeter polytope with one parabolic vertex and three rigid loxodromic vertices.

Now look at the right diagram in Table 1B. Here we will only mention the properties of the Coxeter polytope P_m when m is finite. When $(p,q) = (3,3)$, the Coxeter polytopes

P_m are perfect. When $(p, q) = (4, 3)$, the Coxeter polytopes P_m are quasi-perfect with one parabolic vertex. When $(p, q) = (4, 4)$, the Coxeter polytopes P_m are quasi-perfect with two parabolic vertices. When $(p, q) = (5, 3)$, the Coxeter polytopes P_m are 2-perfect with three rigid loxodromic vertices. When $(p, q) = (5, 4)$, the Coxeter polytopes P_m are 2-perfect with one parabolic vertex and three rigid loxodromic vertices. Finally, when $(p, q) = (5, 5)$, the Coxeter polytopes P_m are 2-perfect with six rigid loxodromic vertices.

An important remark is that when $p = 5$, we can apply the operation of truncation to P_m in order to build infinitely many non-prime examples.

10.4. Two examples in dimension 6. The two Coxeter polytopes we find in dimension 6 are perfect when m is finite (see Table 1C). The following remark explains why we *cannot* construct infinitely many non-prime examples in dimension ≥ 6 . In other words, it is hard to find truncable vertices of polytope of dimension ≥ 6 .

Remark 10.4. By the work of Lannér, we know that large perfect Coxeter simplex, which corresponds to Lannér Coxeter group, exists only up to dimension 4. However, Lemma 9.3 which is the starting point for truncation requires a truncable vertex to be simple, and hence the operation of truncation/gluing works only up to dimension 5. Remark that there exists a Coxeter polytope with a *non-truncable non-simple* loxodromic perfect vertex.

10.5. Two examples in dimension 7. The two Coxeter polytopes we find in dimension 7 are quasi-perfect when m is finite (see Table 1D). Note P_m is *not* perfect.

11. PROOF OF THE MAIN THEOREMS

11.1. Proof of Theorem A.

Proof of Theorem A. In Sections 5 and 10, we provide finite volume hyperbolic Coxeter polytopes P_∞ of dimension $d = 4, 5, 6, 7$ that admit Dehn fillings $(P_m)_{m \geq 7}$.

Here, we remark that since the Coxeter polytopes $(P_m)_{m \geq 7}$ and P_∞ are quasi-perfect, by Theorem 8.2 of [Mar14], the convex domains $(\Omega_m)_{m \geq 7}$ (resp. Ω_∞) that are different from \mathbb{S}^d and that are invariant by $\rho_m(W_\infty)$ (resp. $\rho_\infty(W_\infty)$) are unique.

Let (S_∞, M_∞) be the Coxeter system of P_∞ . We know that the representations $(\rho_m)_{m \geq 7}$ converge algebraically to ρ_∞ , and therefore for each $s \in S_\infty$, the supports and the polars of reflections $\rho_m(\sigma_s)$ converges to the support and the polar of reflections $\rho_\infty(\sigma_s)$. It follows that the underlying projective polytopes of $(P_m)_{m \geq 7}$ converge to the underlying projective polytope of P_∞ in the Hausdorff topology for the closed subsets of \mathbb{S}^d . Moreover, the properly convex closed subsets $(\overline{\Omega}_m)_{m \geq 7}$ of \mathbb{S}^d converges to a convex closed set K in some affine chart of \mathbb{S}^d (up to subsequence), and so the uniqueness of $\rho_\infty(W_\infty)$ -invariant convex domain Ω_∞ implies that $K = \overline{\Omega}_\infty$ and $(\Omega_m)_{m \geq 7}$ converges to Ω_∞ . \square

11.2. Proof of Theorem B.

Proof of Theorem B. Let W_∞ (resp. W_m) be the orbifold fundamental group of \mathcal{O}_∞ (resp. \mathcal{O}_m). By Selberg's Lemma, there is a torsion-free finite-index subgroup Γ_∞ of W_∞ such that the quotient $M_\infty = \Omega_\infty/\Gamma_\infty$ is a manifold. Let Γ_m be the projection of Γ_∞ via the surjective homomorphism $W_\infty \rightarrow W_m$. The subgroup Γ_m of W_m is of finite index bounded above by the index of Γ_∞ in W_∞ , and the closed d -orbifolds $M_m = \Omega_m/\Gamma_m$ are Dehn fillings of M_∞ . \square

Remark 11.1. In the proof of Theorem B, the orbifold M_m is not a manifold.

Proof of Remark 11.1. We can assume that Γ_∞ is normal in W_∞ , i.e. Γ_m is a finite-index normal subgroup of W_m . Assume for contradiction that Γ_m is torsion-free. Since Γ_m is torsion-free, the restriction of the homomorphism $W_m \rightarrow W_m/\Gamma_m$ to D_m is injective. This implies that the index of Γ_m inside W_m is unbounded, which is a contradiction. \square

11.3. Proof of Theorems C, D and E. The main tools to prove Theorem C are Theorem 9.5 of Moussong–Caprace and a Theorem of Druţu–Sapir. We state a simplified version of the main theorem in Druţu–Sapir [DS05]. Recall the definition of quasi-isometry: Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \rightarrow Y$ is called a (c, D) -quasi-isometric embedding if there exist constants $c \geq 1$ and $D \geq 0$ such that for all $x, x' \in X$

$$\frac{1}{c}d_X(x, x') - D \leq d_Y(f(x), f(x')) \leq cd_X(x, x') + D.$$

Moreover, a quasi-isometric embedding $f : X \rightarrow Y$ is a *quasi-isometry* if there is $E > 0$ such that the E -neighborhood of $f(X)$ is Y .

Theorem 11.2 (Druţu–Sapir [DS05]). *Let Γ be a Gromov hyperbolic group relative to a family A_1, \dots, A_r of infinite abelian subgroup of finite type. Let Λ be a group of finite type and B an abelian group of finite type.*

- *For every quasi-isometric embedding of B into Γ , the image of B is at bounded distance from a conjugate of A_i for some i .*
- *If Λ is quasi-isometric to Γ , then Λ is Gromov hyperbolic relative to a family B_1, \dots, B_s of subgroups, each of which is quasi-isometric to A_i for some i .*

Proof of Theorem C. The fact that the group Γ is Gromov hyperbolic relative to a family of subgroups virtually isomorphic to \mathbb{Z}^{d-2} is an easy consequence of Theorem 9.5 of Caprace.

By Lemma 8.19 of Marquis [Mar14], the convex domain Ω contain a properly embedded $(d-2)$ -simplex. Now, we show that the convex domain Ω does not contain a properly embedded $(d-1)$ -simplex.

First, note that a $(d-1)$ -simplex equipped with Hilbert metric is bilipshitz equivalent to the Euclidean space \mathbb{R}^{d-1} (see de la Harpe [dlH93]). Second, since Γ divides Ω , by Švarc–Milnor lemma, the group Γ is quasi-isometric to the metric space (Ω, d_Ω) . Hence, any properly embedded $(d-1)$ -simplex induces a quasi-isometric embedding f of \mathbb{R}^{d-1} into Γ , which is Gromov hyperbolic to a family of subgroups virtually isomorphic to \mathbb{Z}^{d-2} . By the first item of Theorem 11.2, the embedding f induces a quasi-isometric embedding of \mathbb{R}^{d-1} into a peripheral \mathbb{Z}^{d-2} of Γ , but there is no quasi-isometric embedding of \mathbb{R}^{d-1} into \mathbb{Z}^{d-2} .

The quotient orbifold is not homeomorphic to a union along the boundaries of finitely many d -orbifolds each of which admits a finite volume hyperbolic structure on its interior because otherwise Γ would be Gromov hyperbolic relative to a finite collection of subgroups virtually isomorphic to \mathbb{Z}^{d-1} , and Γ is Gromov hyperbolic relative to a finite collection of subgroups virtually isomorphic to \mathbb{Z}^{d-2} , which contradicts the item 2 of Theorem 11.2. \square

Proof of Theorem D. See the Coxeter graphs of Table 2. Each of those Coxeter graph W is obtained by linking a Coxeter subgraph W_{left} which is an \hat{A}_2, \hat{A}_3 or \hat{A}_4 Coxeter graph to a disjoint Coxeter subgraph W_{right} which is Lannér, by an edge with no label.

One can label the product \mathcal{G} of two simplex $\Delta_e \times \Delta_f$ with e (resp. f) being the rank of W_{left} (resp. W_{right}) minus one using W as in section 3, after that the same computation as the one in section 4 shows that $\mathcal{C}(\mathcal{G})$ is two points if W has one loop and two disjoint lines if W has two loops.

The fact that \mathcal{G} is perfect is a consequence of the fact that for every node s of W_{left} , and every node t of W_{right} , the Coxeter group obtained by erasing the node s and t of W is spherical.

Finally, the same arguments than for Theorem C show that the group W is relatively hyperbolic and that Ω contains only tight properly embedded simplex of the expected dimension. \square

Proof of Theorem E. See the Coxeter graphs of Table 3. The only difference with Theorem D is that the computation that gives the existence of such Coxeter polytope is this time an extension of the computation of Remark 4.6. \square

APPENDIX A. USEFUL COXETER DIAGRAMS

A.1. Spherical and affine Coxeter diagram. The irreducible spherical and affine Coxeter group were classified by Coxeter [Cox34]. We reproduce the list of those Coxeter diagrams in Figures 6 and 7. We use the usual convention that an edge that should be labeled by 3 has in fact no label.

A.2. Lannér Coxeter groups. A Coxeter group associated to a large irreducible perfect simplex is called a Lannér Coxeter group, after Lannér [Lan50] first enumerated them in 1950. These groups are very limited, for example they exist only in dimension 2, 3, 4, and there exist infinitely many in dimension 2, 9 in dimension 3 and 5 in dimension 4.

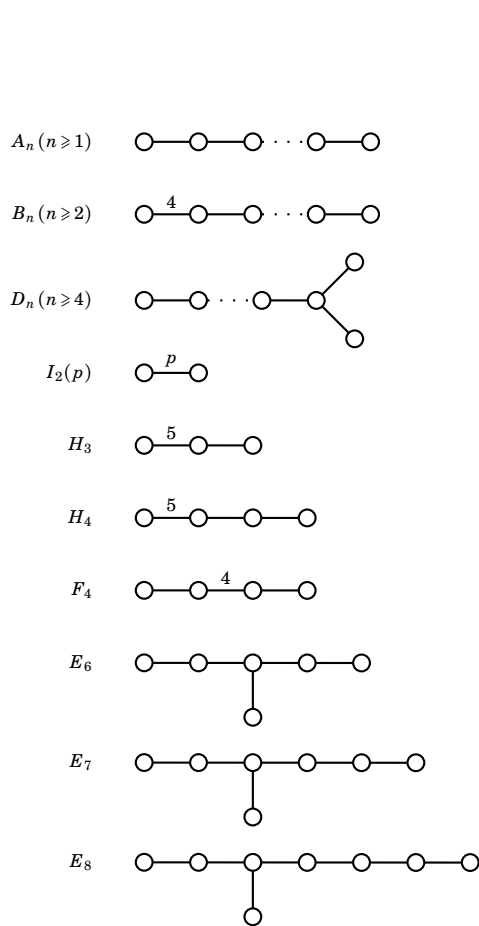


TABLE 6. Irreducible spherical Coxeter diagram

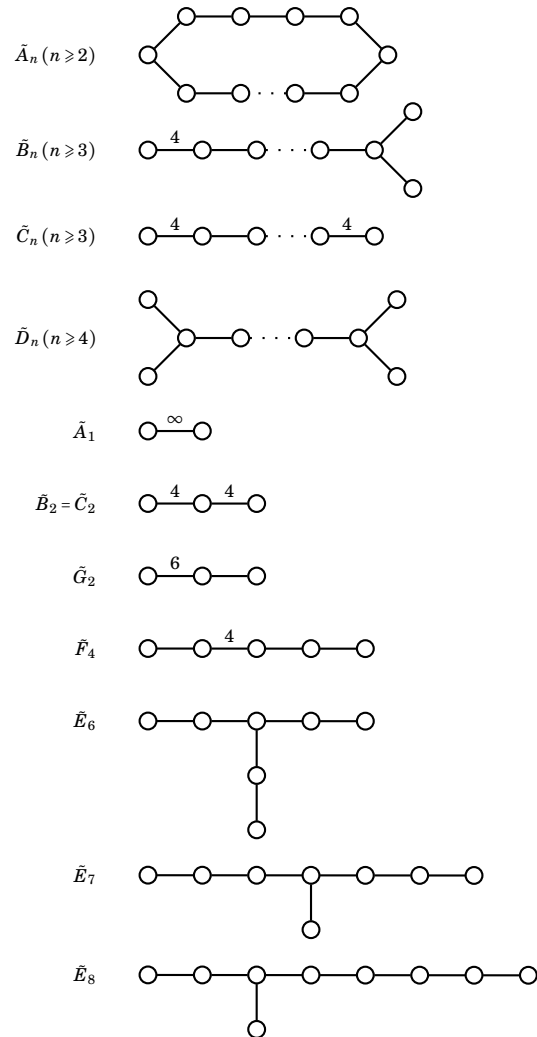


TABLE 7. Irreducible affine Coxeter diagram

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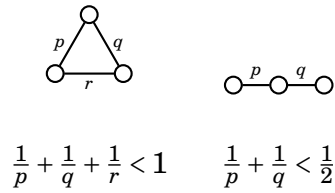


TABLE 8. The Lannér Coxeter groups of the Lannér polygons

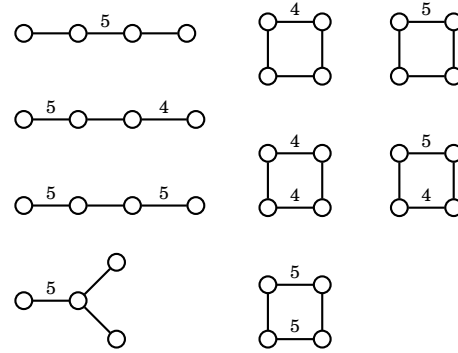


TABLE 9. The Lannér Coxeter groups of the Lannér polyhedra

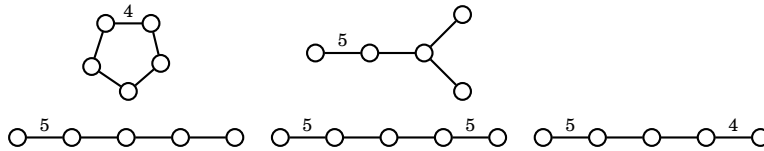


TABLE 10. The Lannér Coxeter groups of the Lannér 4-polytopes

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DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, REPUBLIC OF KOREA

E-mail address: schoi@math.kaist.ac.kr

MATHEMATISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, GERMANY

E-mail address: lee@mathi.uni-heidelberg.de

IRMAR, UNIVERSITÉ DE RENNES 1, FRANCE

E-mail address: ludovic.marquis@univ-rennes1.fr