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An Anisotropic Mumford-Shah Model

David Vicente

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Abstract

A variational model is introduced for the segmentation problem of thin structures, like tubes or thin plates, in an image. The energy is based on the Mumford-Shah model with a surfacic term perturbed by a Finsler metric. The formulation in the special space of functions with bounded variations is given and, in order to get an energy more adapted for numerics, a result of Γ -convergence is proved.

1 Introduction

This work is motivated by the segmentation problem of sets strongly elongated in some directions as, for instance, tubes or thin plates in an image of dimension $n \in \{2; 3\}$. In Computer Vision, the Mumford-Shah model is one of the most studied [1]. It consists, for a given image $g \in L^\infty(\Omega)$, in finding a couple (u, K) which minimizes the following energy

$$\mathcal{E}_{MS}(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \mathcal{H}^{n-1}(K), \quad (1.1)$$

where $u \in W^{1,2}(\Omega \setminus K)$, K is compact and \mathcal{H}^{n-1} is the $(n - 1)$ -dimensional Hausdorff measure. To minimize this energy, K must fit the set of discontinuity of the image and u must represent the regular part of the intensity. In order to adapt this model for the particular case of thin and elongated sets, we have introduced in [2] a Finsler metric φ which must fit the anisotropy of the image. At any point $x \in \Omega$, $\varphi(x, \cdot)$ is a norm whose unit ball coincides with the elongation of the sets we want to detect. So, we set

$$\mathcal{E}(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_K \varphi(x, \nu) d\mathcal{H}^{n-1}, \quad (1.2)$$

where $u \in W^{1,2}(\Omega \setminus K)$, K is a compact $(n - 1)$ -dimensional submanifold and ν is a unit vector orthogonal to K . This kind of model which consists in an energy with a volumic and a surfacic parts also arises in Fracture Mechanics Theory ([3, 4, 5] for example). In this setting, the analysis for the case where φ is a constant norm has been treated in [6]. To our knowledge, the inhomogeneous case, where φ also depends on x , has not been done yet.

Since the compact submanifolds of Ω cannot be endowed with a topology which ensures that the direct methods apply, a weak formulation of the problem is needed. To do this, De Giorgi and Ambrosio [7] proposed to set this kind of problem in the space SBV of special functions with bounded variation. Thus, setting $K = J_u$ in (1.2) and defining $E(u) = \mathcal{E}(u, J_u)$, it gives

$$E(u) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}, \quad (1.3)$$

where $u \in \text{SBV}(\Omega)$, ∇u is the derivative of u with respect to the Lebesgue measure, J_u is its jump set and ν_u is a unit vector orthogonal to J_u . The abstract theory in SBV has been developed: Ambrosio established the existence result [8, 9], and regularity for minimizers of this kind of energy has been proved [10, 11, 12, 13]. Those results ensure that any minimizer u of the relaxed problem in SBV provides a couple $(u, \overline{J_u})$ which also minimizes the initial model \mathcal{E} .

The numerical approximation for solutions is hard because of the treatment of the jump set J_u . To overcome this difficulty, the idea is to perform a variational approximation of the functional E in the sense of De Giorgi Γ -convergence [14, 15] with Ambrosio-Tortorelli's approximation.

In order to approximate (1.3), we propose two slightly different families of functionals $(E_\varepsilon)_\varepsilon$ and $(\tilde{E}_\varepsilon)_\varepsilon$ defined by

$$E_\varepsilon(u, z) = \int_\Omega (u - g)^2 dx + \int_\Omega |\nabla u|^2 (1 - z^2)^2 dx + \int_\Omega \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx, \quad (1.4)$$

$$\tilde{E}_\varepsilon(u, z) = \int_\Omega (u - g)^2 dx + \int_\Omega |\nabla u|^2 [\eta_\varepsilon + (1 - z^2)^2] dx + \int_\Omega \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx. \quad (1.5)$$

For both versions, the function z takes its values in $[0; 1]$ and plays the role of a control on the gradient of u . In the second one, the parameter η_ε is infinitesimal with respect to ε . The first functional is directly inspired by the initial Ambrosio-Tortorelli's approximation [16], while the second one was introduced later in [17] and it was used in various papers, for example [6, 18, 19]. Those functionals are more adapted for numerics since usual finite element methods can be directly applied. They formally differ by the introduction of the term

$$\eta_\varepsilon \int_\Omega |\nabla u|^2 dx. \quad (1.6)$$

By this way, \tilde{E}_ε admits as natural domain of definition the classical Sobolev space $(W^{1,2}(\Omega))^2$. However, this term is not strictly necessary in our study because all the results which are proven for \tilde{E}_ε are also true for E_ε . For this simplification, the cost to pay is a slightly longer analysis in order to introduce an adapted domain for E_ε which ensures that its minimization is still a well-posed problem.

The Γ -convergence result when $\varepsilon \rightarrow 0^+$ will be proven for both E_ε and \tilde{E}_ε . However, in the Image Processing context, the parameter ε is devoted to be *small* but *fixed*. Indeed, for stability of the algorithms, ε has to be bounded by below by a positive constant which depends on the size of the grid (see [20]). As in practice we can not take the limit $\varepsilon \rightarrow 0$, we may choose the first version in order to avoid the additional diffusion in the numerics due to the term (1.6).

One may think that such Γ -convergence result is useless for applications because of the lower bound for ε . Nevertheless, this result provides a continuous model (1.2) of the discrete energy which may be used as a mirror to understand the proper geometry of the problem. Then, an automatic determination of the (numerous) parameters of the algorithms is possible if we have an a priori on the image we want to get (see [21, 2]).

In section 2, we introduce the geometric framework and we recall some results on spaces of functions with bounded variation. In section 3, we introduce the family of functionals $(E_\varepsilon)_\varepsilon$ with their domains, we give our main result and its complete proof.

2 Functional framework

In all the paper, Ω is an open and bounded domain with Lipschitz boundaries. We adopt the notations:

- $\langle \cdot, \cdot \rangle$ for the canonical scalar product, $|\cdot|$ for the euclidean norm in \mathbb{R}^n , dist for the associated distance and $\|\cdot\|$ for the associated matricial norm,
- $\mathbb{B}(\Omega)$ for the Borelian functions and \mathcal{L}^n for the Lebesgue measure on Ω ,
- \mathcal{H}^k for the k -dimensional Hausdorff measure,
- \bar{u}^N for the truncation $\min(N, \max(u, -N))$,
- f is superadditive if $f(A \cup B) \geq f(A) + f(B)$ for any disjoint sets A, B ,

- f is non decreasing if $f(A) \leq f(B)$ for any sets A, B such that $A \subset B$.

In all the paper, in order to emphasize on the 1-D specific technic, we denote ∇v by v' for any function v which is defined on \mathbb{R} .

2.1 Local metrics and Minkowski content

We introduce the geometric framework adapted to the anisotropic setting.

Definition 2.1. *A function φ defined on $\overline{\Omega} \times \mathbb{R}^n$ and taking its values in $[0; +\infty[$ belongs to $\mathfrak{M}(\Omega)$ if it is continuous and satisfies*

- i) for any $x \in \overline{\Omega}$, $\varphi(x, \cdot)$ is a norm on \mathbb{R}^n ,*
- ii) there exists $\lambda > 0$, $\Lambda > 0$, such that, for any $(x, \mathbf{v}) \in \Omega \times \mathbb{R}^n$, we have*

$$\lambda|\mathbf{v}| \leq \varphi(x, \mathbf{v}) \leq \Lambda|\mathbf{v}|. \quad (2.1)$$

To any metric we may associate its dual as follows.

Definition 2.2. *For $\varphi \in \mathfrak{M}(\Omega)$, its dual metric φ^o is defined by*

$$\varphi^o(x, \mathbf{v}) = \sup\{\langle \mathbf{v}, \mathbf{v}' \rangle : \varphi(x, \mathbf{v}') \leq 1\}.$$

A well known consequence of the definition is the following assertion: if $\varphi \in \mathfrak{M}(\Omega)$ then $\varphi^o \in \mathfrak{M}(\Omega)$ and $\varphi^{oo} = \varphi$. For $S \subset \Omega$, we may consider the associated geodesic distance:

$$\begin{aligned} d_{\varphi^o}(x, y) &= \inf \left\{ \int_0^1 \varphi^o(\gamma, \gamma') dt : \begin{array}{l} \gamma \in W^{1,1}([0; 1]; \Omega), \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}, \\ d_{\varphi^o}^S(x) &= \inf \{d_{\varphi^o}(x, y) : y \in S\}. \end{aligned}$$

According to (2.1), $d_{\varphi^o}^S$ is Lipschitz and then it is differentiable almost everywhere. Furthermore, we have the following (see [22])

Proposition 2.1. *Let $S \subset \Omega$ be a closed set and $\varphi \in \mathfrak{M}(\Omega)$. Then, we have*

$$\varphi(x, \nabla d_{\varphi^o}^S(x)) = 1$$

at each point $x \in \Omega \setminus S$ where $d_{\varphi^o}^S$ is differentiable.

We recall that, for $S \subset \mathbb{R}^n$, the $(n - 1)$ -dimensional upper Minkowski content of S is defined by

$$\mathcal{M}^*(S) = \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{x : \text{dist}(x, S) < \rho\})}{2\rho}.$$

In our anisotropic setting, the associated anisotropic Minkowski $(n - 1)$ -dimensional upper content is defined by the limit

$$\mathcal{M}_{\varphi^o}^*(S) = \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{x \in \Omega : d_{\varphi^o}^S(x) < \rho\})}{2\rho}. \quad (2.2)$$

In [23] (Theorem 3.2.39), under regularity assumptions on S , it is proved that $\mathcal{M}^*(S) \leq \mathcal{H}^{n-1}(S)$. We need the following generalization of this result.

Proposition 2.2. *Let $K \subset \Omega$ be a finite union of $(n - 1)$ -dimensional simplexes and $\varphi \in \mathfrak{M}(\Omega)$. Then, we have*

$$\mathcal{M}_{\varphi^o}^*(K) \leq \int_K \varphi(x, \nu) d\mathcal{H}^{n-1}.$$

The proof is given in Appendix A.

2.2 BV spaces and slicing results

For the classical definitions and results on BV and SBV we refer to [24]. We recall that, if $u \in \text{BV}(\Omega)$, then its derivative Du belongs to the space of vectorial Radon measures and, if $u \in \text{SBV}(\Omega)$, then the Cantor part of Du is null and we obtain

$$Du = \nabla u \cdot \mathcal{L}^n + (u^+ - u^-)\nu_u \cdot \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the density of Du with respect to the Lebesgue measure \mathcal{L}^n , u^+ (resp. u^-) is the approximate upper (resp. lower) limit and $\mathcal{H}^{n-1} \llcorner J_u$ is the restriction of \mathcal{H}^{n-1} to its jump set J_u . The *slicing* results are crucial in our study.

Definition 2.3. Let $\nu \in \mathbb{S}^{n-1}$ be fixed. We denote by Π_ν the hyperplane

$$\{x \in \mathbb{R}^n : \langle x, \nu \rangle = 0\}.$$

If $x \in \Pi_\nu$, we set

$$\begin{aligned} \Omega_x &= \{t \in \mathbb{R} : x + t\nu \in \Omega\}, \\ \Omega_\nu &= \{x \in \Pi_\nu : \Omega_x \neq \emptyset\}. \end{aligned}$$

For any function u defined on Ω and any $x \in \Omega_\nu$, we set

$$\begin{aligned} (u)_x : \Omega_x &\rightarrow \mathbb{R} \\ t &\mapsto u(x + t\nu). \end{aligned}$$

The following Theorem is proved in [8].

Theorem 2.1. Let $u \in L^\infty(\Omega)$ be a function such that, for all $\nu \in \mathbb{S}^{n-1}$,

i) $(u)_x \in \text{SBV}(\Omega_x)$ for \mathcal{H}^{n-1} a.e. $x \in \Omega_\nu$,

ii) $\int_{\Omega_\nu} \left[\int_{\Omega_x} |(u)'_x| dt + \mathcal{H}^0(J_{(u)_x}) \right] d\mathcal{H}^{n-1}(x) < +\infty$;

then, $u \in \text{SBV}(\Omega)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. Conversely, let $u \in \text{SBV}(\Omega) \cap L^\infty(\Omega)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$. Then i) and ii) are satisfied. Moreover, we have

iii) $\langle \nabla u(x + t\nu), \nu \rangle = (u)'_x(t)$, for a.e. $t \in \Omega_x$ and \mathcal{H}^{n-1} -a.e. $x \in \Omega_\nu$,

iv) $\int_{J_u} |\langle \nu_u, \nu \rangle| d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(J_{(u)_x}) d\mathcal{H}^{n-1}(x)$.

We state a result (Proposition 1.16 in [25]) which will simplify the presentation of the slicing technics.

Lemma 2.1. Let $\mathcal{A}(\Omega)$ be the family of open sets of Ω , $\mu : \mathcal{A}(\Omega) \rightarrow [0; +\infty[$ be a superadditive function on disjoint open sets, λ be a positive measure on Ω and $\psi_k : \Omega \rightarrow [0; +\infty]$ be a countable family of Borel functions such that $\mu(A) \geq \int_A \psi_k d\lambda$ for every $A \in \mathcal{A}(\Omega)$. Set $\psi = \sup_k \psi_k$, then

$$\mu(A) \geq \int_A \psi d\lambda$$

for every $A \in \mathcal{A}(\Omega)$.

Let us now introduce a sub-class of SBV functions.

Definition 2.4. Let $\mathcal{W}(\Omega)$ be the space of all $u \in \text{SBV}(\Omega)$ such that

i) $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$;

ii) $\overline{J_u}$ is the intersection of Ω with a finite number of $(n-1)$ -dimensional simplexes;

iii) $u \in W^{k,\infty}(\Omega \setminus \overline{J_u})$ for every $k \in \mathbb{N}$.

The following Theorem (see [26]) provides a density result which will be useful for the approximation proof.

Theorem 2.2. *Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that*

$$\mathcal{H}^{n-1}(J_u) < +\infty, \quad \nabla u \in L^2(\Omega),$$

then there exists a sequence $(u_k)_k \subset \mathcal{W}(\Omega)$ such that

i) $u_k \rightarrow u$ strongly in $L^1(\Omega)$;

ii) $\nabla u_k \rightarrow \nabla u$ strongly in $L^2(\Omega)$;

iii) $\limsup_k \|u_k\|_\infty \leq \|u\|_\infty$;

iv) *for every $A \subset\subset \Omega$ and for every uppersemicontinuous function $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbf{S}^{n-1} \rightarrow [0; +\infty]$ such that $\varphi(x, a, b, \nu) = \varphi(x, a, b, -\nu)$ for every $x \in \Omega$, $a, b \in \mathbb{R}$ and $\nu \in \mathbf{S}^{n-1}$ there holds*

$$\limsup_{k \rightarrow \infty} \int_{A \cap J_{u_k}} \varphi(x, u_k^-, u_k^+, \nu_{u_k}) d\mathcal{H}^{n-1} \leq \int_{A \cap J_u} \varphi(x, u^-, u^+, \nu_u) d\mathcal{H}^{n-1}.$$

2.3 Approximate Differentiability and GSBV space

Due to the non coercivity of $\int_\Omega |\nabla u|^2 (1 - z^2)^2 dx$ in $W^{1,2}(\Omega)$, we need to introduce a weaker definition for ∇u and a larger domain for (1.3).

Definition 2.5. *Let $x \in \Omega$ be a Lebesgue point of $u \in L^1(\Omega)$; we say that u is approximately differentiable at x if there exists $L \in \mathbb{R}^n$ such that*

$$\lim_{r \rightarrow 0^+} \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} \frac{|u(y) - u(x) - \langle L, y - x \rangle|}{r} dy = 0.$$

If u is approximately differentiable at x then L , uniquely determined by this equality, is called the approximate differential of u at x and denoted by $\nabla u(x)$.

Remark 2.1. *The concept of approximate differentiability of u at x can be rephrased as the convergence in $L^1_{loc}(\mathbb{R}^n)$ of the functions $v_r(y) = [u(x + ry) - u(x)]/r$ to the linear function Ly . As $(v_r)_r$ converges to Du in the sense of distributions, if $u \in L^2(\Omega)$ and its approximate differential exists and belongs to $L^2(\Omega; \mathbb{R}^n)$, then u belongs to $W^{1,2}(\Omega)$ and its weak derivative coincides with its approximate differential almost everywhere.*

This definition of the differentiability satisfies the following result [24] (Theorem 3.83).

Theorem 2.3. *If $v \in BV(\Omega)$, then v is approximately differentiable almost everywhere in Ω . Moreover, the approximate differential ∇v is the density of the absolutely continuous part of Dv with respect to the Lebesgue measure.*

We need the three following technical Lemmas.

Lemma 2.2. *If $u, v \in \mathbb{B}(\Omega)$, u is approximately differentiable at x and*

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(\{y \in B_r(x) : u(y) \neq v(y)\})}{r^n} = 0,$$

then v is approximately differentiable at x and $\nabla u(x) = \nabla v(x)$.

Lemma 2.3. *If $u, v \in \mathbb{B}(\Omega)$ are approximately differentiable almost everywhere in Ω and $u \in L^\infty(\Omega)$, then uv is approximately differentiable almost everywhere in Ω and we get the Leibniz rule $\nabla(uv) = u\nabla v + v\nabla u$ almost everywhere in Ω .*

Lemma 2.4. *If v is approximately differentiable almost everywhere, then v^{-1} is also approximately differentiable almost everywhere in $\{x: v(x) \neq 0\}$.*

While the proof of Lemma 2.3 is given in Appendix B, the proofs of Lemmas 2.2 and 2.4 may be found in [24] (Proposition 3.71 and 3.73). Eventually, as no L^∞ bound is imposed by our functional (1.3), then it is useful to consider the following wider class

$$\text{GSBV}(\Omega) = \{u \in \mathbb{B}(\Omega) : \bar{u}^N \in \text{SBV}(\Omega) \text{ for any } N \in \mathbb{N}\}.$$

The structure of the generalized derivative of a GSBV function is similar to that of a SBV function [24] (Proposition 3.73 and Theorem 4.34). In this case, we have

$$J_u = \bigcup_{M=1}^{\infty} J_{\bar{u}^M} \quad (2.3)$$

and J_u is also countably \mathcal{H}^{n-1} -rectifiable; for any $N, M \geq 1$, it satisfies

$$\nu_{\bar{u}^N}(x) = \nu_{\bar{u}^M}(x), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_{\bar{u}^N} \cap J_{\bar{u}^M}. \quad (2.4)$$

Moreover, u is also approximately differentiable \mathcal{L}^n -a.e. in Ω and

$$\nabla u(x) = \nabla \bar{u}^N(x), \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|u| \leq N\}, \quad (2.5)$$

$$\nabla \bar{u}^N(x) = 0, \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|u| > N\}, \quad (2.6)$$

Thus, relations (2.3), (2.4) and (2.5) provide a slightly more general meaning for ∇u , J_u and ν_u and then it ensures that our functional (1.3) is well defined in $\text{GSBV}(\Omega)$.

3 Γ -convergence result

This section is entirely devoted to the approximation process. In subsection 3.1 we define the domains for both E_ε and \tilde{E}_ε . Then, we give the main Theorem of this paper. In subsection 3.2 we prove, for $\varepsilon > 0$ fixed, that the minimization problems admit a solution. Eventually, in subsection 3.3, we give the complete proof of Γ -convergence for both E_ε and \tilde{E}_ε .

3.1 The functionals, their domains and the main Theorem

Formally, we define

$$E_\varepsilon(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx,$$

$$\tilde{E}_\varepsilon(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 [\eta_\varepsilon + (1 - z^2)^2] dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx.$$

Both functionals E_ε and \tilde{E}_ε are well defined in $\tilde{\mathcal{D}}_n(\Omega) = W^{1,2}(\Omega) \times W^{1,2}(\Omega; [0; 1])$. Contrary to \tilde{E}_ε , the functional E_ε is not coercive for the Sobolev norm because the coefficient $(1 - z^2)^2$ removes the coercivity with respect to u . For that, we need to introduce a specific domain for E_ε that ensures the existence of a minimizer. If, in addition, u is bounded, we have

$$\begin{aligned} |\nabla(u(1 - z^2))|^2 &= |\nabla u(1 - z^2) - 2uz\nabla z|^2, \\ &\leq 2|\nabla u|^2(1 - z^2)^2 + 4\|u\|_\infty |\nabla z|^2. \end{aligned}$$

It gives

$$\int_{\Omega} |\nabla(u(1-z^2))|^2 dx \leq \left(2 + \frac{4\|u\|_{\infty}}{\lambda\varepsilon}\right) E_{\varepsilon}(u, z).$$

So, it is natural to set

$$\mathcal{D}_n(\Omega) = \left\{ (u, z) : u \in \mathbb{B}(\Omega), z \in W^{1,2}(\Omega; [0; 1]), \bar{u}^N(1-z^2) \in W^{1,2}(\Omega) \text{ for any } N \geq 1 \right\}.$$

Assuming $(u, z) \in \mathcal{D}_n(\Omega)$ does not ensure that $u \in W^{1,2}(\Omega)$ and ∇u can not be defined as the gradient of u in the Sobolev space. For that, the gradient of u is interpreted as in Definition 2.5 and then, the following Proposition ensures that $E_{\varepsilon}(u, z)$ is well defined in $\mathcal{D}_n(\Omega)$.

Proposition 3.1. *If $(u, z) \in \mathcal{D}_n(\Omega)$, then u is approximately differentiable in $\{x \in \Omega : z(x) \neq 1\}$ and z is approximately differentiable in Ω . Moreover, almost everywhere in $\{x \in \Omega : z(x) \neq 1\}$, we have*

$$(1-z^2)\nabla u = \nabla[u(1-z^2)] + 2uz\nabla z, \quad (3.1)$$

where ∇ is the approximate derivation on the left hand side and the weak derivation in $W^{1,2}(\Omega)$ on the right hand side.

Proof. As Ω is bounded, then the inclusion $W^{1,2}(\Omega) \subset \text{BV}(\Omega)$ is ensured. So, if $(u, z) \in \mathcal{D}_n(\Omega)$, according to Theorem 2.3, z and $\bar{u}^N(1-z^2)$ are approximately differentiable almost everywhere in Ω . According to Lemmas 2.3 and 2.4, we deduce that \bar{u}^N is approximately differentiable almost everywhere in the set $\{x : z(x) \neq 1\}$ and, according to the Leibniz rule, we have

$$(1-z^2)\nabla \bar{u}^N = \nabla[\bar{u}^N(1-z^2)] + 2\bar{u}^N z \nabla z,$$

where ∇ is the approximate derivation. As $z, \bar{u}^N(1-z^2) \in W^{1,2}(\Omega)$, according to Theorem 2.3, their approximate derivatives correspond to their weak derivatives almost everywhere. According to Lemma 2.2, u and $u(1-z^2)$ are a.e. approximately differentiable in $\{|u| < N\}$ and we have

$$\nabla \bar{u}^N(x) = \nabla u(x), \quad \nabla[\bar{u}^N(1-z^2)](x) = \nabla[u(1-z^2)](x), \quad \bar{u}^N(x) = u(x),$$

for a.e. every $x \in \{|u| < N\}$, which concludes the proof. \square

We consider the following functionals:

- $E : \mathbb{B}(\Omega) \rightarrow [0; +\infty]$ defined by

$$E(u) = \int_{\Omega} (u-g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}$$

if $u \in \text{GSBV}(\Omega)$, and $E(u) = +\infty$ otherwise;

- $E_{\varepsilon} : \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \rightarrow [0; +\infty]$ defined by

$$E_{\varepsilon}(u, z) = \int_{\Omega} (u-g)^2 dx + \int_{\Omega} |\nabla u|^2 (1-z^2)^2 dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx$$

if $(u, z) \in \mathcal{D}_n(\Omega)$, where $(1-z^2)\nabla u$ is defined by the right hand side of (3.1). Moreover, we set $E_{\varepsilon}(u, z) = +\infty$ otherwise;

- $\tilde{E}_{\varepsilon} : \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \rightarrow [0; +\infty]$ defined by

$$\tilde{E}_{\varepsilon}(u, z) = \int_{\Omega} (u-g)^2 dx + \int_{\Omega} |\nabla u|^2 [\eta_{\varepsilon} + (1-z^2)^2] dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx$$

if $(u, z) \in \tilde{\mathcal{D}}_n(\Omega)$, $\tilde{E}_{\varepsilon}(u, z) = +\infty$ otherwise.

The main result of the paper is the following

Theorem 3.1. *If $g \in L^\infty(\Omega)$ and $\varphi \in \mathfrak{M}(\Omega)$, then we have the following assertions.*

- i) For any $\varepsilon > 0$, E_ε admits a minimizer, denoted by $(u_\varepsilon, z_\varepsilon)$. Moreover, we can assume that $u_\varepsilon(x) = g(x)$ on $\{x \in \Omega: z_\varepsilon(x) = 1\}$.*
- ii) For any $(\varepsilon_k)_k$ converging to 0^+ , there exists a subsequence, still denoted by $(\varepsilon_k)_k$, and $u \in \text{GSBV}(\Omega)$ such that $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$ converges to $(u, 0)$ almost everywhere and u is a minimizer of E .*

If, by addition, $\eta_\varepsilon = o(\varepsilon)$, the same result holds true for \tilde{E}_ε .

We set

$$\begin{aligned} (\mathcal{P}) : & \quad \text{Min}\{E(u) : u \in \mathbb{B}(\Omega)\}, \\ (\mathcal{P}_\varepsilon) : & \quad \text{Min}\{E_\varepsilon(u, z) : (u, z) \in \mathbb{B}(\Omega) \times \mathbb{B}(\Omega)\}, \\ (\tilde{\mathcal{P}}_\varepsilon) : & \quad \text{Min}\{\tilde{E}_\varepsilon(u, z) : (u, z) \in \mathbb{B}(\Omega) \times \mathbb{B}(\Omega)\}. \end{aligned}$$

Theorem 3.1 i) implies that, for $\varepsilon > 0$ fixed, $(\mathcal{P}_\varepsilon)$ and $(\tilde{\mathcal{P}}_\varepsilon)$ are well posed problems, its proof is given in Section 3.2. Theorem ii) implies that, up to the extraction of a subsequence, the sequence of solutions of $(\mathcal{P}_{\varepsilon_k})$ and $(\tilde{\mathcal{P}}_{\varepsilon_k})$ converge to a solution of (\mathcal{P}) , its proof is given in Section 3.4.

3.2 Existence result for $(\mathcal{P}_\varepsilon)$ and $(\tilde{\mathcal{P}}_\varepsilon)$

We prove Theorem 3.1 i) for E_ε , which is a straightforward consequence of Propositions 3.2 and 3.3. The arguments for \tilde{E}_ε are the same, so its proof is omitted.

Proposition 3.2. *Let $\varphi \in \mathfrak{M}(\Omega)$ and $\varepsilon > 0$ be fixed. There exists $(u_k, z_k)_k$ a minimizing sequence of E_ε such that $(u_k)_k$ is a bounded sequence in $L^\infty(\Omega)$, $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, $(u_k, z_k)_k$ converges almost everywhere to $(u, z) \in \mathcal{D}_n(\Omega)$ and $u(x) = g(x)$ on $\{x: z(x) = 1\}$.*

To prove it, the following Lemma is needed.

Lemma 3.1. *For $(u, z) \in \mathcal{D}_n(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$ fixed, we have $(u_x, z_x) \in \mathcal{D}_1(\Omega_x)$ for \mathcal{H}^{n-1} -almost every $x \in \Omega_\nu$ (see the notations of definition 2.3), and*

$$\begin{aligned} u'_x(t) &= \langle \nabla u(x + t\nu), \nu \rangle, \\ z'_x(t) &= \langle \nabla z(x + t\nu), \nu \rangle, \end{aligned}$$

for almost every $t \in \Omega_x \setminus \{s: z(x + s\nu) = 1\}$.

The proof is given in Appendix C. Now, we prove Proposition 3.2.

Proof. Let $(u_k, z_k)_k$ be a minimizing sequence for E_ε . We fix $N \geq \|g\|_\infty$ and we consider the truncated functions \bar{u}_k^N . As for the proof of Proposition 3.1, we get that $\bar{u}_k^N(1 - z_k^2)$ and $1 - z_k^2$ are approximately differentiable almost everywhere and \bar{u}_k^N is approximately differentiable almost everywhere in $\{x: z_k(x) \neq 1\}$. According to relations (2.5) and (2.6), $\nabla \bar{u}_k^N(x) = 0$ almost everywhere in $\{x: |\bar{u}_k^N(x)| = N\}$ and $\nabla \bar{u}_k^N(x) = \nabla u_k(x)$ almost everywhere in $\{x: |\bar{u}_k^N(x)| < N\}$. So, we get

$$\int_{\Omega} |\nabla \bar{u}_k^N|^2 (1 - z_k^2)^2 dx \leq \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx. \quad (3.2)$$

We deduce that $E_\varepsilon(\bar{u}_k^N, z_k) \leq E_\varepsilon(u_k, z_k)$ and then $(\bar{u}_k^N, z_k)_k$ is also a minimizing sequence. According to (2.1), we have

$$\int_{\Omega} |\nabla z_k|^2 dx + \int_{\Omega} z_k^2 dx \leq \left(\frac{1}{\lambda\varepsilon} + 4\varepsilon \right) E_\varepsilon(u_k, z_k),$$

and then $(z_k)_k$ is a bounded sequence of $W^{1,2}(\Omega)$. So, there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to $z \in W^{1,2}(\Omega)$ and z takes also its values in $[0; 1]$. For $w_k = \bar{u}_k^N(1 - z_k^2)$, we have

$$\int_{\Omega} |\nabla w_k|^2 dx + \int_{\Omega} w_k^2 dx \leq 2 \int_{\Omega} |\nabla \bar{u}_k^N|^2 (1 - z_k^2)^2 dx + 2N^2 \int_{\Omega} |\nabla z_k|^2 dx + N^2 \int_{\Omega} (1 - z_k^2)^2 dx$$

and then $(w_k)_k$ is a bounded sequence of $W^{1,2}(\Omega)$. Then, we may extract a subsequence, still denoted by $(w_k)_k$, which converges almost everywhere to $w \in W^{1,2}(\Omega)$. In particular, $(\bar{u}_k^N(x))_k$ converges for almost every $x \in \{y: z(y) \neq 1\}$ to $u(x)$. If $x \in \{y: z(y) = 1\}$, we set $u(x) = g(x)$. This construction ensures that $(u, z) \in \mathcal{D}_n(\Omega)$. \square

Proposition 3.3. *Let $\varphi \in \mathfrak{M}(\Omega)$ and $\varepsilon > 0$ be fixed. If $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converges almost everywhere to $(u, z) \in \mathcal{D}_n(\Omega)$, $(u_k)_k$ is a bounded sequence in $L^\infty(\Omega)$ and $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, then*

$$\liminf_{k \rightarrow \infty} E_\varepsilon(u_k, z_k) \geq E_\varepsilon(u, z).$$

Proof. Fatou Lemma gives

$$\liminf_{k \rightarrow \infty} \int_{\Omega} (u_k - g)^2 dx \geq \int_{\Omega} (u - g)^2 dx, \quad \liminf_{k \rightarrow \infty} \int_{\Omega} \frac{z_k^2}{4\varepsilon} dx \geq \int_{\Omega} \frac{z^2}{4\varepsilon} dx. \quad (3.3)$$

As $\varphi^{oo} = \varphi$, we have

$$\varphi(x, \mathbf{v}) = \sup_{\nu \in \mathbb{S}^{n-1}} \frac{1}{\varphi^o(x, \nu)} \langle \mathbf{v}, \nu \rangle.$$

Let $\nu \in \mathbb{S}^{n-1}$ be fixed and $A \subset \Omega$ an open set. It gives

$$\int_A \varphi(x, \nabla z_k)^2 dx \geq \int_A \frac{1}{\varphi^o(x, \nu)^2} \langle \nabla z_k, \nu \rangle^2 dx. \quad (3.4)$$

As $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, then $(\langle \nabla z_k, \nu \rangle)_k$ weakly converges to $\langle \nabla z, \nu \rangle$ in $L^2(A)$. Moreover, $x \rightarrow 1/\varphi^o(x, \nu)$ belongs to $L^\infty(A)$, so we deduce that $(\langle \nabla z_k, \nu \rangle / \varphi^o(x, \nu))_k$ weakly converges to $\langle \nabla z, \nu \rangle / \varphi^o(x, \nu)$ in $L^2(A)$. As the L^2 -norm is lower semi-continuous with respect to the weak convergence, we deduce that

$$\liminf_{k \rightarrow \infty} \int_A \frac{1}{\varphi^o(x, \nu)^2} \langle \nabla z_k, \nu \rangle^2 dx \geq \int_A \frac{1}{\varphi^o(x, \nu)^2} \langle \nabla z, \nu \rangle^2 dx \quad (3.5)$$

Let $(\nu_i)_i$ be a countable dense family of \mathbb{S}^{n-1} . We set

$$\mu(A) = \liminf_{k \rightarrow \infty} \int_A \varphi(x, \nabla z_k)^2 dx, \quad \psi_i(x) = \frac{1}{\varphi^o(x, \nu_i)} \langle \nabla z(x), \nu_i \rangle.$$

As $\sup_i \psi_i(x) = \varphi(x, \nabla z(x))^2$, according to Lemmas 2.1 and inequalities (3.4), (3.5), we deduce that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \varphi(x, \nabla z_k)^2 dx \geq \int_{\Omega} \varphi(x, \nabla z)^2 dx. \quad (3.6)$$

Thus, according to (3.3) and (3.6), to show Proposition 3.3, it suffices to prove that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx \geq \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx. \quad (3.7)$$

We first consider the one-dimensional case $n = 1$ and then by a slicing argument we get the lower semi-continuity for the general case $n \geq 1$.

For any $\delta \in]0; 1[$, we set $A_\delta = \{x \in \Omega : z(x) < 1 - \delta\}$. As $n = 1$, then z is a continuous function, A_δ is an open set and $(z_k)_k$ uniformly converges to z . In particular, we have

$$A \subset \{x \in \Omega : z_k(x) \leq 1 - \delta/2\}$$

for k large enough, and so it yields

$$\int_{A_\delta} |u'_k|^2 dx \leq \frac{1}{1 - (1 - \delta/2)^2} E_\varepsilon(u_k, z_k).$$

With the same argument as for Remark 2.1, we deduce that $(u_k)_k$ is a bounded sequence of $W^{1,2}(A_\delta)$. We may assume that this sequence is weakly convergent to u in $W^{1,2}(A_\delta)$ and as $(z_k)_k$ is uniformly convergent to z , we get

$$\int_{A_\delta} |u'|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{A_\delta} |u'_k|^2 (1 - z_k^2)^2 dx.$$

Passing to the limit $\delta \rightarrow 0^+$ gives

$$\int_\Omega |u'|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_\Omega |u'_k|^2 (1 - z_k^2)^2 dx.$$

Now, we generalize this result to the dimension $n \geq 1$. With the notation u_x introduced in Definition 2.3, using the previous result obtained in dimension 1, Lemma 3.1 and Fatou Lemma, give

$$\begin{aligned} \int_A |\langle \nabla u, \nu \rangle|^2 (1 - z^2)^2 dx &= \int_{A_\nu} \int_{A_x} |u'_x(t)|^2 (1 - z_x(t))^2 dt dx, \\ &\leq \int_{A_\nu} \liminf_{k \rightarrow \infty} \int_{A_x} |(u_k)'_x(t)|^2 (1 - (z_k)_x(t))^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_\nu} \int_{A_x} |(u_k)'_x(t)|^2 (1 - (z_k)_x(t))^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_\nu} \int_{A_x} |\langle \nabla u_k(x + t\nu), \nu \rangle|^2 (1 - z_k(x + t\nu))^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_A |\langle \nabla u_k, \nu \rangle|^2 (1 - z_k^2)^2 dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_A |\nabla u_k|^2 (1 - z_k^2)^2 dx, \end{aligned}$$

for any open set $A \subset \Omega$ and every $\nu \in \mathbb{S}^{n-1}$. Let $(\nu_i)_i$ be a dense and countable family of \mathbb{S}^{n-1} . We set

$$\psi_i = |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2, \quad \psi = |\nabla u|^2 (1 - z^2)^2, \quad \mu(A) = \liminf_{k \rightarrow \infty} \int_A |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

As μ is superadditive on $\mathcal{A}(\Omega)$, according to Lemma 2.1, we may conclude

$$\int_\Omega |\nabla u|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

□

3.3 Γ -convergence result for $\varepsilon \rightarrow 0^+$

This section is dedicated to the proof of Theorem 3.1 *ii*). For that, we will prove the following Γ -convergence result.

Theorem 3.2. *Let $\varphi \in \mathfrak{M}(\Omega)$ be fixed and $(\varepsilon_k)_k$ be a sequence which converges to 0^+ . Then, we have*

i) if $u \in \mathbb{B}(\Omega)$, $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ and $(u_k, z_k)_k$ converges to $(u, 0)$ in $L^1(\Omega)$, then

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(u_k, z_k) \geq E(u); \quad (3.8)$$

ii) for any $u \in \mathbb{B}(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ in $L^1(\Omega)$ and

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(u_k, z_k) \leq E(u). \quad (3.9)$$

If $\eta_\varepsilon = o(\varepsilon)$, the same result holds true for $\tilde{E}_{\varepsilon_k}$. Moreover, the same sequence $(u_k, z_k)_k$ than for E_{ε_k} may be used for (3.9).

3.3.1 The inequality for the lower Γ -limit

We prove the first inequality of Γ -convergence (3.8). As the domain of $\tilde{E}_{\varepsilon_k}$ is included in the domain of E_{ε_k} , it is sufficient to prove the result only for E_{ε_k} . Moreover, it is also sufficient to prove this result with $(u_k)_k$ and u uniformly bounded in $L^\infty(\Omega)$. Indeed, if

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(\bar{u}_k^N, z_k) \geq E(\bar{u}^N)$$

holds true for any $N \geq 1$, as $E_{\varepsilon_k}(\bar{u}_k^N, z_k) \leq E_{\varepsilon_k}(u_k, z_k)$ for any $N \geq \|g\|_\infty$ and $(E(\bar{u}^N))_N$ converges to $E(u)$ when N converges to $+\infty$, then we get (3.8). Let $u \in L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and $(u_k, z_k)_k$ converges to $(u, 0)$ in $L^1(\Omega)$. In the sequel, we emphasize on the domain of the function: for U an open subset of Ω , we adopt the following notation

$$\begin{aligned} F(u; U) &= \int_U |\nabla u|^2 dx + \int_{J_u \cap U} \varphi(x, \nu_u) d\mathcal{H}^{n-1}, \\ F_{\varepsilon_k}(u_k, z_k; U) &= \int_U |\nabla u_k|^2 (1 - z_k^2)^2 dx + \int_U \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx. \end{aligned}$$

It is implicit that $U = \Omega$ in the previous notation if it is not mentioned. Fatou Lemma yields

$$\liminf_{k \rightarrow \infty} \int_\Omega (u_k - g)^2 dx \geq \int_\Omega (u - g)^2 dx$$

and then it suffices to prove that $\liminf F_{\varepsilon_k}(u_k, z_k) \geq F(u)$. We perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension $n \geq 2$.

The one-dimensional case. We first give a lower bound for the surface term in dimension $n = 1$. As we argue like in [16], [17], [25], we only mention the result. In this paragraph, we assume that $\Omega = I$ is an open interval and that φ is a constant $m > 0$. Moreover, we assume in this section that m is fixed. We denote the 1-D approximating functional by

$$G_\varepsilon(u, z; I) = \int_I |u'(t)|^2 (1 - z(t)^2)^2 dt + \int_I \left(m\varepsilon |z'(t)|^2 + \frac{z(t)^2}{4\varepsilon} \right) dt, \quad (3.10)$$

where the domain is

$$\mathcal{D}_1(I) = \{(u, z) : u \in L^\infty(I), z \in W^{1,2}(I; [0; 1]), u(1 - z^2) \in W^{1,2}(I)\}.$$

We denote the lower Γ -limit, by

$$G_-(u; I) = \inf \left\{ \liminf_{k \rightarrow \infty} G_{\varepsilon_k}(u_k, z_k; I) \right\},$$

where the inf is taken over all sequence $(u_k, z_k)_k \subset \mathcal{D}_1(I)$ such that (u_k, z_k) converges almost everywhere to $(u, 0)$ in I . We have the following

Proposition 3.4. *Let $I \subset \mathbb{R}$ be an open interval and $u \in \mathbb{B}(I)$. If $G_-(u; I) < \infty$, then $u \in SBV(I)$ and*

$$\int_I |u'(t)|^2 dt + m^{1/2} \mathcal{H}^0(J_u \cap I) \leq G_-(u; I).$$

Generalization to dimension $n \geq 2$. We give the proof of (3.8). Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converging in $L^1(\Omega)$ to $(u, 0)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$. Up to the extraction of a subsequence, we may assume that its convergence is pointwise almost everywhere. We have to prove that

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \geq F(u). \quad (3.11)$$

We assume that $\liminf F_{\varepsilon_k}(u_k, z_k)$ is finite, otherwise the result is ensured.

Claim: We denote $\mu(A) = \liminf F_{\varepsilon_k}(u_k, z_k; A)$. For any $A \in \mathcal{A}(\Omega)$ and $\nu \in \mathbf{S}^{n-1}$, we have

$$\mu(A) \geq \int_A \langle \nabla u, \nu \rangle^2 dx + \int_{J_u \cap A} \frac{1}{\varphi^o(x, \nu)^2} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}. \quad (3.12)$$

Let $\delta > 0$ be fixed. As φ is uniformly continuous, there exists a finite family $(A_i)_i \subset A$ such that $\text{diam}(A_i) \leq \delta$, $\cup_i \overline{A_i} = A$ and, for any $i \in I$, it satisfies

$$|\varphi(x, \mathbf{v}) - \varphi(y, \mathbf{v})| \leq \delta |\mathbf{v}|, \quad (3.13)$$

for any $(x, y) \in A_i^2$ and $\mathbf{v} \in \mathbb{R}^n$. We fix $a_i \in A_i$ for all $i \in I$. Let $\nu \in \mathbf{S}^{n-1}$ be fixed. According to (3.13), (2.1), for any $(x, \mathbf{v}) \in A_i \times \mathbb{R}^n$, we have

$$\begin{aligned} |\varphi(x, \mathbf{v}) - \varphi(a_i, \mathbf{v})| &\leq \delta |\mathbf{v}|, \\ &\leq \delta \lambda^{-1} \varphi(a_i, \mathbf{v}). \end{aligned}$$

We set $C(\delta) = (1 - \delta \lambda^{-1})^2$, then, for any $x \in A_i$, we get

$$\varphi(x, \nabla z_k(x))^2 \geq C(\delta) \varphi(a_i, \nabla z_k(x))^2. \quad (3.14)$$

As $(\varphi^o)^o = \varphi$, we get

$$\varphi(x, \mathbf{v}) = \sup_{\nu \in \mathbf{S}^{n-1}} \frac{1}{\varphi^o(x, \nu)} \langle \mathbf{v}, \nu \rangle$$

and then

$$\varphi(a_i, \nabla z_k(x))^2 = \sup_{\nu \in \mathbf{S}^{n-1}} \frac{1}{\varphi^o(a_i, \nu)^2} \langle \nabla z_k(x), \nu \rangle^2 \quad (3.15)$$

According to (3.14) and (3.15), $F_{\varepsilon_k}(u_k, z_k; A_i)$ is greater than

$$\int_{A_i} \left(|\nabla u_k|^2 (1 - z_k^2)^2 + C(\delta) \frac{1}{\varphi^o(a_i, \nu)^2} \varepsilon_k \langle \nabla z_k, \nu \rangle^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx.$$

With the notation introduced in Definition 2.3, $(v)_y$ is the function defined on $(A_i)_\nu^y$ as $(v)_y(t) = v(y + t\nu)$. According to Lemma 3.1, we have $(u_k)'_y(t) = \langle \nabla u(y + t\nu), \nu \rangle$ and $(z_k)'_y(t) = \langle \nabla z(y + t\nu), \nu \rangle$, so Fubini Theorem implies that $F_{\varepsilon_k}(u_k, z_k; A_i)$ is greater than

$$\int_{(A_i)_\nu} \int_{(A_i)_\nu^y} \left(|(u_k)'_y|^2 (1 - ((z_k)_y)^2)^2 + C(\delta) \frac{1}{\varphi^o(a_i, \nu)^2} \varepsilon_k |(z_k)'_y|^2 + \frac{((z_k)_y)^2}{4\varepsilon_k} \right),$$

where the integration is done over the product measure $dt d\mathcal{H}^{n-1}(y)$. With the one-dimensional notations (3.10), it gives

$$F_{\varepsilon_k}(u_k, z_k; A_i) \geq \int_{(A_i)_\nu} G_{\varepsilon_k}((u_k)_y, (z_k)_y; (A_i)_\nu^y) d\mathcal{H}^{n-1}(y),$$

where $m = C(\delta) \frac{1}{\varphi^o(a_i, \nu)^2}$. Fatou lemma yields

$$\mu(A_i) \geq \int_{A_\nu} \liminf_{k \rightarrow \infty} G_{\varepsilon_k}((u_k)_y, (z_k)_y; (A_i)_\nu^y) d\mathcal{H}^{n-1}(y)$$

and then

$$\mu(A_i) \geq \int_{(A_i)_\nu} G_-((u)_y; (A_i)_\nu^y) d\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; A_i)$ is finite, we deduce that $G_-((u)_y; (A_i)_\nu^y)$ is finite for \mathcal{H}^{n-1} almost every $y \in (A_i)_\nu$. We may apply Lemma 3.4, with $I = (A_i)_\nu^y$ and $u = (u)_y$, it gives that $(u)_y \in \text{SBV}((A_i)_\nu^y)$, for \mathcal{H}^{n-1} almost every $y \in (A_i)_\nu$ and we have

$$\mu(A_i) \geq \int_{(A_i)_\nu} \left[\int_{(A_i)_\nu^y} |(u)_y'|^2 dt + m^{1/2} \mathcal{H}^0(J_{(u)_y} \cap (A_i)_\nu^y) \right] d\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; A_i)$ is finite, Theorem 2.1 implies that

$$\int_{(A_i)_\nu} \left[\int_{(A_i)_\nu^y} |(u)_y'|^2 dt + m^{1/2} \mathcal{H}^0(J_{(u)_y} \cap (A_i)_\nu^y) \right] d\mathcal{H}^{n-1}(y)$$

is equal to

$$\int_{\Omega} \langle \nabla u, \nu \rangle^2 dx + \int_{J_u \cap A_i} m^{1/2} |\langle \nu_u, \nu \rangle| d\mathcal{H}^{n-1}.$$

We deduce

$$\mu(A_i) \geq \int_{A_i} \langle \nabla u, \nu \rangle^2 dx + \int_{J_u \cap A_i} m^{1/2} |\langle \nu_u, \nu \rangle| d\mathcal{H}^{n-1}.$$

If we replace m by its value, it gives

$$\mu(A_i) \geq \int_{A_i} \langle \nabla u, \nu \rangle^2 dx + C(\delta) \int_{J_u \cap A_i} \frac{1}{\varphi^o(a_i, \nu)} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}.$$

As the function φ is uniformly continuous, then φ^o is also uniformly continuous and there exists a function, still denoted by $C(\delta)$, such that $C(\delta) \rightarrow 1$ for $\delta \rightarrow 0^+$ and which satisfies

$$\mu(A_i) \geq \int_{A_i} \langle \nabla u, \nu \rangle^2 dx + C(\delta) \int_{J_u \cap A_i} \frac{1}{\varphi^o(x, \nu)} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}.$$

As μ is superadditive, we get

$$\mu(A) \geq \int_A \langle \nabla u, \nu \rangle^2 dx + C(\delta) \int_{J_u \cap A} \frac{1}{\varphi^o(x, \nu)} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}.$$

We take $\delta \rightarrow 0^+$, it concludes the proof of the *Claim*. Hereafter, we divide the remaining part of the proof in two steps corresponding on the estimate on the regular part and on the jump part of u .

Step 1: Regular Part. For any open set $A \subset \Omega$, we prove the following inequality

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_A |\nabla u|^2 dx. \quad (3.16)$$

Let $(\nu_i)_i \subset \mathbb{S}^{n-1}$ be a countable set of point which is dense in \mathbb{S}^{n-1} . We set

$$\mu(A) = \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; A), \quad \psi_i(x) = \langle \nabla u(x), \nu_i \rangle^2.$$

According to *Claim*, $\mu(A) \geq \int_A \psi_i dx$. As $\sup_i \psi_i(x) = |\nabla u(x)|^2$, Lemma 2.1 yields (3.16).

Step 2: Jump Part. For any $A \in \mathcal{A}(\Omega)$, we prove that

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{J_u \cap A} \varphi(x, \nu_u) d\mathcal{H}^{n-1}. \quad (3.17)$$

We consider $(\nu_i)_i \subset \mathbb{S}^{n-1}$ and μ as in *Step 1* and we set

$$\widetilde{\psi}_i(x) = \frac{1}{\varphi^o(x, \nu_i)} |\langle \nu_u(x), \nu_i \rangle|.$$

According to *Claim*, we have $\mu(A) \geq \int_A \widetilde{\psi}_i d\mathcal{H}^{n-1}$. As $\sup_i \widetilde{\psi}_i(x) = \varphi(x, \nu_u(x))$, Lemma 2.1 yields (3.17).

Conclusion: Let μ be the Borel measure on Ω defined for any Borelian $A \subset \Omega$ by

$$\mu(A) = \int_{J_u \cap A} \varphi(x, \nu_u) d\mathcal{H}^{n-1}.$$

Let $\delta > 0$ be fixed, according to the inner regularity of Borel measures (Proposition 1.43 in [24]), there exists a compact $K \subset J_u$ such that $\mu(J_u \setminus K) \leq \delta$, that is

$$\int_{J_u \setminus K} \varphi(x, \nu_u) d\mathcal{H}^{n-1} \leq \delta. \quad (3.18)$$

We set $K_\tau = \{x \in \Omega : \text{dist}(x, K) < \tau\}$. *First Step* implies that $\int_\Omega |\nabla u|^2 dx$ is finite. As $K \subset J_u$, then we get $\mathcal{H}^{n-1}(K) < +\infty$ and, in particular, $\int_K |\nabla u|^2 dx = 0$. Thus, there exists $\tau > 0$ such that

$$\int_{K_\tau} |\nabla u|^2 dx \leq \delta. \quad (3.19)$$

According to (3.18), (3.19), *First Step* and *Second Step*, we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; \Omega) &\geq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; K_\tau) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; \Omega \setminus \overline{K_\tau}), \\ &\geq \int_{J_u \cap K_\tau} \varphi(x, \nu_u) d\mathcal{H}^{n-1} + \int_{\Omega \setminus \overline{K_\tau}} |\nabla u|^2 dx, \\ &\geq \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1} + \int_\Omega |\nabla u|^2 dx - 2\delta. \end{aligned}$$

Taking the limit $\delta \rightarrow 0^+$ concludes the proof.

3.3.2 The inequality for the upper Γ -limit

In this section we prove the *upper* inequality of Γ -convergence, that is ii) of Theorem 3.2, for both E_{ε_k} and $\widetilde{E}_{\varepsilon_k}$. As for the *lower* inequality case, it is sufficient to prove it with $u \in L^\infty(\Omega)$. Indeed, if for any $N \geq 1$ there exists a sequence $(u_{k,N}, z_{k,N})_k$ which converges to $(\bar{u}^N, 0)$ in $L^1(\Omega)$ and which satisfies

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(u_{k,N}, z_k) \leq E(\bar{u}^N),$$

then by a diagonal extraction, there exists a sequence $(u_k, z_k)_k$ which converges to $(u, 0)$ in $L^1(\Omega)$ and which satisfies (3.9). So, let $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ be fixed. We prove that there exists a sequence $(u_k, z_k)_k \subset \widetilde{\mathcal{D}}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere,

$$\limsup_{k \rightarrow \infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 dx + \int_\Omega \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx$$

is lower than

$$\int_\Omega |\nabla u|^2 dx + \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}$$

and

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \eta_{\varepsilon_k} dx = 0.$$

According to Theorem 2.2 and Proposition 2.2, with a diagonal extraction, it suffices to prove a weaker result, where $\int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}$ is replaced by its approximation with a Minkowski content. More precisely, it suffices to prove the following

Proposition 3.5. *Let $\varphi \in \mathfrak{M}(\Omega)$ be fixed, $(\varepsilon_k)_k$ be a sequence converging to 0^+ and $\eta_k = o(\varepsilon_k)$. For $u \in SBV(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \tilde{\mathcal{D}}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere, $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and*

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \leq \int_{\Omega} |\nabla u|^2 dx + \mathcal{M}_{\varphi^o}^*(J_u), \quad \limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \eta_k dx = 0. \quad (3.20)$$

Proof. Without loss of generality, we may assume that $|\nabla u| \in L^2(\Omega)$ and $\mathcal{M}_{\varphi^o}^*(J_u) < +\infty$, otherwise the result is obvious. If $u \in W^{1,2}(\Omega)$, then $J_u = \emptyset$ and the stationary sequence $u_k = u$, $z_k = 0$ is a solution. If $u \notin W^{1,2}(\Omega)$, then $J_u \neq \emptyset$ and $(1 - z_k^2)^2$ has to be infinitesimal near of J_u . For $\rho > 0$, we set

$$(J_u)_\rho = \{x \in \Omega: d_{\varphi^o}^{J_u}(x) < \rho\}. \quad (3.21)$$

We separate Ω in three parts:

$$(J_u)_{b_k}, \quad (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}, \quad \Omega \setminus (J_u)_{a_k+b_k} \quad (3.22)$$

with

$$a_k = -4\varepsilon_k \ln(\varepsilon_k), \quad b_k = \sqrt{\varepsilon_k \eta_k}. \quad (3.23)$$

Let Ψ_k defined by

$$\Psi_k(x) = \begin{cases} 1 & \forall x \in (J_u)_{b_k/2}, \\ 2 - 2d_{\varphi^o}^{J_u}(x)/b_k & \forall x \in (J_u)_{b_k} \setminus (J_u)_{b_k/2}, \\ 0 & \forall x \in \Omega \setminus (J_u)_{b_k}. \end{cases} \quad (3.24)$$

This definition ensures that Ψ_k is a continuous function with support in $(J_u)_{b_k}$, which is equal to 1 in a neighborhood $(J_u)_{b_k/2}$ of J_u and which *shrinks* to J_u when $k \rightarrow \infty$. Ellipticity inequality (2.1) yields

$$\begin{aligned} |d_{\varphi^o}^{J_u}(x) - d_{\varphi^o}^{J_u}(y)| &\leq d_{\varphi^o}(x, y), \\ &\leq \lambda^{-1}|x - y|. \end{aligned}$$

So, $d_{\varphi^o}^{J_u}$ is Lipschitzian and Rademacher Theorem ensures that $\nabla d_{\varphi^o}^{J_u}$ exists for almost every $x \in \Omega$. It gives

$$|\nabla \Psi_k| \leq \frac{2}{b_k \lambda}.$$

We set $u_k = (1 - \Psi_k)u$ and then $u_k \equiv u$ in $\Omega \setminus (J_u)_{b_k}$. As $(b_k)_k$ converges to 0 then u_k converges to u almost everywhere. As we have $\nabla u_k = -\nabla \Psi_k u + (1 - \Psi_k)\nabla u$, we get

$$\int_{\Omega} |\nabla u_k|^2 \eta_k dx \leq \eta_k \left(\|u\|_\infty^2 \frac{8}{b_k^2 \lambda^2} \mathcal{L}^n((J_u)_{b_k}) + 2 \int_{\Omega} |\nabla u|^2 dx \right).$$

As $\mathcal{M}_{\varphi^o}^*(J_u) < +\infty$, there exists $(\omega_k)_k$ a sequence which converges to 0^+ such that

$$\mathcal{L}^n((J_u)_{b_k}) \leq 2b_k(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k), \quad (3.25)$$

it gives

$$\int_{\Omega} |\nabla u_k|^2 \eta_k dx \leq \eta_k \left(\|u\|_\infty^2 \frac{16}{b_k \lambda^2} (\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k) + 2 \int_{\Omega} |\nabla u|^2 dx \right).$$

As $\eta_k/b_k \rightarrow 0$ and $\eta_k \rightarrow 0$, we deduce that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \eta_k dx = 0$$

and then the second inequality of (3.20) is proven. It remains to prove the first one. For that, we set $z_k = 1$ in $(J_u)_{b_k}$ and $z_k = \varepsilon_k^2$ in $\Omega \setminus (J_u)_{a_k+b_k}$. In $(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$ we adopt the following construction: we introduce

$$\theta_k(t) = \varepsilon_k^2 \exp\left(\frac{t}{2\varepsilon_k}\right)$$

and we set

$$\tilde{z}_k(t) = \begin{cases} 1 & \forall t \in [0; b_k], \\ \theta_k(a_k + b_k - t) & \forall t \in]b_k; a_k + b_k], \\ \varepsilon_k^2 & \forall t \in]a_k + b_k; +\infty[. \end{cases} \quad (3.26)$$

This is a continuous and decreasing function defined on $[0; +\infty[$, moreover, for any $t \in]b_k; a_k + b_k[$, it satisfies

$$\varepsilon_k (\tilde{z}'_k(t))^2 = \frac{(\tilde{z}_k(t))^2}{4\varepsilon_k}. \quad (3.27)$$

We set $z_k = \tilde{z}_k \circ d_{\varphi^0}^{J_u}$. As z_k is constant in $(J_u)_{b_k} \cup (\Omega \setminus (J_u)_{a_k+b_k})$, we have

$$\begin{aligned} F_{\varepsilon_k}(u_k, z_k) &= \int_{\Omega \setminus (J_u)_{a_k+b_k}} |\nabla u|^2 (1 - \varepsilon_k^4)^2 dx \\ &+ \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} |\nabla u|^2 (1 - z_k^2)^2 dx \\ &+ \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx \\ &+ \frac{\varepsilon_k^3}{4} \mathcal{L}^n(\Omega \setminus (J_u)_{a_k+b_k}) + \frac{1}{4\varepsilon_k} \mathcal{L}^n((J_u)_{b_k}). \end{aligned} \quad (3.28)$$

As $|\nabla u| \in L^2(\Omega)$ and $(a_k + b_k)_k$ converges to 0, the first term of (3.28) converges to $\int_{\Omega} |\nabla u|^2 dx$. As $\|z_k\|_{\infty} \leq 1$, the second term converges to 0. As Ω is a bounded domain, the fourth term converges to 0. According to (3.25), the fifth term is lower than $\frac{1}{2}\varepsilon_k(\mathcal{M}_{\varphi^0}^*(J_u) + \omega_k)$. So, the fifth term converges to 0. To compute the limit of $(F_{\varepsilon_k}(u_k, z_k))_k$, it remains to study the convergence of

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx.$$

For almost every $x \in (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$, we have

$$\nabla z_k = \tilde{z}'_k \circ d_{\varphi^0}^{J_u} \nabla d_{\varphi^0}^{J_u}.$$

It gives

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k (\tilde{z}'_k \circ d_{\varphi^0}^{J_u})^2 \varphi(x, \nabla d_{\varphi^0}^{J_u})^2 + \frac{(\tilde{z}_k \circ d_{\varphi^0}^{J_u})^2}{4\varepsilon_k} \right) dx.$$

According to Proposition 2.1, $\varphi(x, \nabla d_{\varphi^0}^{J_u}(x)) = 1$ for almost every x . So, we may write

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k (\tilde{z}'_k \circ d_{\varphi^0}^{J_u})^2 + \frac{(\tilde{z}_k \circ d_{\varphi^0}^{J_u})^2}{4\varepsilon_k} \right) \varphi(x, \nabla d_{\varphi^0}^{J_u}) dx.$$

We may apply an anisotropic version of the coarea formula (see [27]), it gives

$$A_k(z_k) = \int_{b_k}^{a_k+b_k} \left(\varepsilon_k \tilde{z}'_k(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \left[\int_{\Omega} \varphi(x, D\mathbf{1}_{(J_u)_t}) \right] dt. \quad (3.29)$$

We set

$$\mathcal{H}_{\varphi}(t) = \int_{\Omega} \varphi(x, D\mathbf{1}_{(J_u)_t}), \quad \mathcal{A}_{\varphi}(s) = \int_0^s \mathcal{H}_{\varphi}(t) dt.$$

Applying another time coarea formula gives

$$\begin{aligned} \mathcal{A}_{\varphi}(s_2) - \mathcal{A}_{\varphi}(s_1) &= \int_{s_1}^{s_2} \left[\int_{\Omega} \varphi(x, D\mathbf{1}_{(J_u)_t}) \right] dt, \\ &= \int_{(J_u)_{s_2} \setminus (J_u)_{s_1}} \varphi(x, \nabla d_{\varphi^o}^{J_u}) dx, \\ &= \mathcal{L}^n((J_u)_{s_2} \setminus (J_u)_{s_1}). \end{aligned}$$

So, $\mathcal{A}_{\varphi} \in W_{\text{loc}}^{1,1}([0; +\infty[)$ and $\nabla \mathcal{A}_{\varphi} = \mathcal{H}_{\varphi}$ almost everywhere. Using equality (3.27) and then integrating by parts (3.29) gives

$$\begin{aligned} A_k(z_k) &= \int_{b_k}^{a_k+b_k} \left(\varepsilon_k \tilde{z}'_k(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \mathcal{H}_{\varphi}(t) dt, \\ &= \int_{b_k}^{a_k+b_k} \frac{\tilde{z}_k(t)^2}{2\varepsilon_k} \mathcal{H}_{\varphi}(t) dt, \\ &= \frac{\tilde{z}_k^2(a_k + b_k)}{2\varepsilon_k} \mathcal{A}_{\varphi}(a_k + b_k) - \frac{\tilde{z}_k^2(b_k)}{2\varepsilon_k} \mathcal{A}_{\varphi}(b_k) - \frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}'_k(t) \tilde{z}_k(t) \mathcal{A}_{\varphi}(t) dt. \end{aligned}$$

The first term obviously converges to 0. As for (3.25), we have

$$\mathcal{A}_{\varphi}(b_k) \leq 2b_k(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)$$

and then the second term converges to 0 too. As $s \rightarrow \mathcal{A}_{\varphi}(s)$ is non decreasing, then

$$\mathcal{A}_{\varphi}(t) \leq 2t(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)$$

for any $t \in [b_k; a_k + b_k]$. For the last term, we apply another time this inequality, it gives

$$-\frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}'_k(t) \tilde{z}_k(t) \mathcal{A}_{\varphi}(t) dt \leq -\frac{(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)}{\varepsilon_k} \int_{b_k}^{a_k+b_k} 2t \tilde{z}'_k(t) \tilde{z}_k(t) dt. \quad (3.30)$$

Integrating by parts yields

$$\int_{b_k}^{a_k+b_k} 2t \tilde{z}'_k(t) \tilde{z}_k(t) dt = (a_k + b_k) \tilde{z}_k(a_k + b_k)^2 - b_k \tilde{z}_k(b_k)^2 - \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2 dt. \quad (3.31)$$

According to the definitions of (a_k, b_k, z_k) , (3.23) and (3.26), we have

$$(a_k + b_k) \tilde{z}_k(a_k + b_k)^2 - b_k \tilde{z}_k(b_k)^2 = o(\varepsilon_k) \quad (3.32)$$

and equation (3.27) gives

$$\begin{aligned} \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2 dt &= 2\varepsilon_k \int_{b_k}^{a_k+b_k} |\tilde{z}'_k(t)| \tilde{z}_k(t) dt, \\ &= \varepsilon_k (1 - \varepsilon_k^2). \end{aligned} \quad (3.33)$$

From (3.30), (3.31), (3.32) and (3.33) we deduce that $\limsup_k A_k(z_k) \leq \mathcal{M}_{\varphi^o}^*(J_u)$ and, according to the decomposition (3.28), we have

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \leq \int_{\Omega} |\nabla u|^2 + \mathcal{M}_{\varphi^o}^*(J_u).$$

To conclude the proof, it suffices to notice that $(u_k, z_k)_k \subset \tilde{\mathcal{D}}_n(\Omega)$. □

3.4 Compactness result for $(\mathcal{P}_\varepsilon)_\varepsilon$ and $(\tilde{\mathcal{P}}_\varepsilon)_\varepsilon$

In this Section we prove Theorem 3.1, ii). As the same arguments hold for both cases, we only give the proof for $(\mathcal{P}_\varepsilon)_\varepsilon$. According to Theorem 3.1 i), for any $\varepsilon > 0$, there exists $(u_\varepsilon, z_\varepsilon)$ a minimizer of E_ε . According to (3.2), with $N \geq \|g\|_\infty$, we have

$$\mathcal{L}^n(\{x \in \Omega: |u_\varepsilon(x)| > N\}) > 0 \quad \Rightarrow \quad E_\varepsilon(\bar{u}_\varepsilon^N, z_\varepsilon) < E_\varepsilon(u_\varepsilon, z_\varepsilon).$$

We deduce that $\|u_\varepsilon\|_\infty \leq N$ for any $\varepsilon > 0$. For $\omega_\varepsilon = u_\varepsilon(1 - z_\varepsilon^2)$, we get

$$\nabla \omega_\varepsilon = \nabla u_\varepsilon(1 - z_\varepsilon^2) - 2u_\varepsilon z_\varepsilon \nabla z_\varepsilon.$$

It yields

$$\int_\Omega |\nabla \omega_\varepsilon| dx \leq \mathcal{L}^n(\Omega)^{1/2} \left(\int_\Omega |\nabla u_\varepsilon|^2 (1 - z_\varepsilon^2)^2 dx \right)^{1/2} + 2N \int_\Omega |\nabla z_\varepsilon| z_\varepsilon dx. \quad (3.34)$$

Applying the inequality $2ab \leq a^2 + b^2$ with $a = \frac{z_\varepsilon}{2\varepsilon^{1/2}}$ and $b = \varepsilon^{1/2} |\nabla z_\varepsilon|$ gives

$$\int_\Omega |\nabla z_\varepsilon| z_\varepsilon dx \leq \int_\Omega \varepsilon |\nabla z_\varepsilon|^2 dx + \int_\Omega \frac{z_\varepsilon^2}{4\varepsilon} dx. \quad (3.35)$$

According to ellipticity inequality (2.1), we get

$$\int_\Omega \varepsilon |\nabla z_\varepsilon|^2 dx \leq \frac{1}{\lambda} E_\varepsilon(u_\varepsilon, z_\varepsilon). \quad (3.36)$$

By (3.34), (3.35) and (3.36), we deduce

$$\int_\Omega |\nabla \omega_\varepsilon| dx \leq \mathcal{L}^n(\Omega)^{1/2} (E_\varepsilon(u_\varepsilon, z_\varepsilon))^{1/2} + \left(1 + \frac{1}{\lambda}\right) E_\varepsilon(u_\varepsilon, z_\varepsilon).$$

Let $(\varepsilon_k)_k$ be a sequence converging to 0^+ . As $E \not\equiv +\infty$, according to Theorem 3.2 ii), we deduce that $(E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}))_k$ is bounded. So, $(\omega_{\varepsilon_k})_k$ is bounded in $\text{BV}(\Omega)$ and there exists a subsequence, still denoted by $(\omega_{\varepsilon_k})_k$ which converges almost everywhere to $\omega \in \text{BV}(\Omega)$. As $\int_\Omega z_k^2 dx \leq \varepsilon_k E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k})$, then $(z_k)_k$ converges to 0 in $L^2(\Omega)$ and there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to 0. As $\omega_{\varepsilon_k} = u_{\varepsilon_k}(1 - z_{\varepsilon_k}^2)$, then $(u_{\varepsilon_k})_k$ converges almost everywhere to $u \in L^\infty(\Omega)$. According to Theorem 3.2, u is obviously a minimizer of E .

Conclusion

This paper is the theoretical background needed to introduce our new model. A forthcoming paper will present the numerical point of view. The main assumption of this work is the existence of an anisotropic metric φ which must represent the anisotropy of the image. For that, we may consider the case where φ is a Riemannian metric. That is, φ is given by a field of symmetric definite positive matrices \mathbf{M} such that

$$\varphi(x, \mathbf{v}) = \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v} \rangle^{1/2}.$$

In order to favorize elongated sets, \mathbf{M} is constrained to take its values in a compact sub manifold of symmetric definite positive matrices with fixed spectrum. In order to combine the segmentation task with the computation of this metric, the couple (u, \mathbf{M}) must minimize the functional

$$\int_\Omega (u - g)^2 dx + \int_\Omega |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} dx,$$

where $\alpha > 0$ is fixed in order to satisfy the regularity condition $\varphi \in \mathfrak{M}(\Omega)$. Indded, according to the Sobolev embedding Theorem, the finiteness of the last term ensures that the metric is uniformly continuous. Existence of a minimizer of this energy and approximation with Γ -convergence are straightforward consequences of what we have presented in this article.

A Proof of Proposition 2.2

We prove this result in two steps: we first assume that φ is homogeneous and after we generalize the result in the inhomogeneous setting.

First Step: We assume that $K \subset \Omega$ is a $(n-1)$ -simplex and φ is homogeneous, i.e. it does not depend on $x \in \Omega$.

With the same notations as in (3.21), we set $K_r = \{x \in \Omega: d_{\varphi^o}^K(x) < r\}$. We separate K_r in two parts:

$$K_r^b = \{x \in K_r: \exists y \in \partial K, d_{\varphi^o}(x, y) = d_{\varphi^o}^{K_r}(x)\}, \quad K_r^i = K_r \setminus K_r^b.$$

As φ is an elliptic metric (φ is comparable to the euclidean distance) and ∂K is a finite union of $(n-2)$ -dimensional simplexes, we have

$$\mathcal{L}^n(K_r^b) \sim r^2. \quad (\text{A.1})$$

As we assume that φ is homogeneous, K_r^i is a n -dimensional simplex whose basis is K and height is

$$\sup\{2\langle \nu, \mathbf{v} \rangle: \varphi^o(\mathbf{v}) \leq r\},$$

where ν is an unitary and normal vector to K . As $\varphi^{oo} = \varphi$, we deduce that

$$\mathcal{L}^n(K_r^i) = 2r\varphi(\nu)\mathcal{H}^{n-1}(K). \quad (\text{A.2})$$

According to (A.1) and (A.2), we conclude that $\mathcal{M}_{\varphi^o}^*(K) \leq \varphi(\nu)\mathcal{H}^{n-1}(K)$.

Second Step: We assume that $K = \cup_i K_i$ is a finite union of $(n-1)$ -simplexes and that φ depends on $x \in \Omega$.

We may assume that $\int_K \varphi(x, \nu) d\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. As $\varphi \in \mathfrak{M}(\Omega)$, there exists $\lambda > 0$ such that

$$\lambda^{1/2}\mathcal{H}^{n-1}(K) \leq \int_K \varphi(x, \nu) d\mathcal{H}^{n-1}$$

and then $\mathcal{H}^{n-1}(K)$ is also finite. For $t \in \mathbb{R}$ and $i \in \{1, \dots, n\}$, we set $\Pi_t^i = \{x \in \Omega: \langle x, e_i \rangle = t\}$. Thus, for $k \in \mathbb{N}$ fixed, $\{t \in \mathbb{R}: \mathcal{H}^{n-1}(K \cap \Pi_t^i) > 1/k\}$ is finite and then, for any $i \in \{1, \dots, n\}$, the set $\{t \in \mathbb{R}: \mathcal{H}^{n-1}(K \cap \Pi_t^i) > 0\}$ is at most countable. So, if we consider the cubes whose faces are orthogonal to the vectors of the orthogonal basis, there exists a partition \mathcal{C} of Ω by cubes with diameter less than η , such that for any $C \in \mathcal{C}$, we have

$$\mathcal{H}^{n-1}(K \cap \partial C) = 0. \quad (\text{A.3})$$

For any $C \in \mathcal{C}$, we fix $a_C \in C$ and we set

$$\varphi_C = \varphi(a_C, \cdot). \quad (\text{A.4})$$

As $\varphi \in \mathfrak{M}(\Omega)$, there exists $\varepsilon(\eta)$ such that $\varepsilon(\eta) \rightarrow 0$ for $\eta \rightarrow 0$ and

$$|\varphi(x, \mathbf{v}) - \varphi(y, \mathbf{v})| \leq \varepsilon(\eta)|\mathbf{v}| \quad (\text{A.5})$$

for any $x, y \in C$ and $\mathbf{v} \in \mathbb{R}^n$. It gives

$$\begin{aligned}
\mathcal{M}_{\varphi^\circ}^*(K_i) &\leq \sum_{C \in \mathcal{C}} \mathcal{M}_{\varphi^\circ}^*(K_i \cap C), \\
&\leq (1 + \varepsilon(\eta)) \sum_{C \in \mathcal{C}} \mathcal{M}_{\varphi_C}^*(K_i \cap C), \\
&\leq (1 + \varepsilon(\eta)) \sum_{C \in \mathcal{C}} \int_{K_i \cap C} \varphi_C(\nu) d\mathcal{H}^{n-1}, \\
&\leq (1 + \varepsilon(\eta))^2 \sum_{C \in \mathcal{C}} \int_{K_i \cap C} \varphi(x, \nu) d\mathcal{H}^{n-1}, \\
&\leq (1 + \varepsilon(\eta))^2 \int_{K_i} \varphi(x, \nu) d\mathcal{H}^{n-1}.
\end{aligned}$$

Taking the limit $\eta \rightarrow 0^+$ gives the result of *Second Step* for each compact K_i . To conclude the proof of Proposition 2.2, it suffices to notice that

$$\mathcal{M}_{\varphi^\circ}^*(K) \leq \sum_i \mathcal{M}_{\varphi^\circ}^*(K_i).$$

B Proof of Lemma 2.3

Let $x \in \Omega$ be a point such that v_1, v_2 are approximatively differentiable at x . First, we remark that x is a Lebesgue point for v_1 , that is

$$\lim_{r \rightarrow 0^+} \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} |v_1(y) - v_1(x)| dy = 0. \quad (\text{B.1})$$

Then, we decompose

$$\begin{aligned}
&\mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} \frac{|v_1(y)v_2(y) - v_1(x)v_2(x) - \langle v_1(x)\nabla v_2(x) + v_2(x)\nabla v_1(x), y - x \rangle|}{r} dy \\
&\leq \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} |v_2(x)| \frac{|v_1(y) - v_1(x) - \langle \nabla v_1(x), y - x \rangle|}{r} dy \\
&\quad + \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} |v_1(y)| \frac{|v_2(y) - v_2(x) - \langle \nabla v_2(x), y - x \rangle|}{r} dy \\
&\quad + \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} \frac{|(v_1(y) - v_1(x))\langle \nabla v_2(x), y - x \rangle|}{r} dy.
\end{aligned}$$

As v_1, v_2 are approximatively differentiable at x and $v_1 \in L^\infty(\Omega)$ the two first terms converge to 0. As we have $|\langle \nabla v_2(x), y - x \rangle|/r \leq |\nabla v_2(x)|$ for any $y \in B(x, r)$, according to (B.1), the last term also converges to 0.

C Proof of Lemma 3.1

As Ω is bounded, then $W^{1,2}(\Omega) \subset \text{SBV}(\Omega)$ and then, according to Calderón-Zygmund Theorem (3.83 in [24]), the derivative in the Sobolev sense is equal to the approximate differential for almost every point in Ω . Moreover, according to Theorem 3.107 of [24], for $v \in \text{SBV}(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$, we have

$$\langle \nabla v(x + t\nu), \nu \rangle = \nabla v_x(t) \text{ a.e. } t \in \Omega_x$$

for almost every $x \in \Omega_\nu$. According to Fubini Theorem, if $v \in W^{1,2}(\Omega)$, then $v_x \in W^{1,2}(\Omega_x)$ for almost every $x \in \Omega$. Applying this property with $v = z$ and $v = u(1 - z^2)$ concludes the proof of Lemma 3.1.

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