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An Anisotropic Mumford-Shah Model

David Vicente

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Abstract

A variational model is introduced for the segmentation problem of thin structures, like tubes or thin plates, in an image. The energy is based on the Mumford-Shah model with a surfacic term perturbed by a Finsler metric. The formulation in the special space of functions with bounded variations is given and, in order to get an energy more adapted for numerics, a result of Γ -convergence is proved.

1 Introduction

This work is motivated by the segmentation problem of sets strongly elongated in some directions as, for instance, tubes or thin plates in an image of dimension $n \in \{2, 3\}$. In Computer Vision, the Mumford-Shah model is one of the most studied [1]. It consists, for a given image $g \in L^\infty(\Omega)$, in finding a couple (u, K) which minimizes the following energy

$$\mathcal{E}_{MS}(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \mathcal{H}^{n-1}(K), \quad (1.1)$$

where $u \in W^{1,2}(\Omega \setminus K)$, K is compact and \mathcal{H}^{n-1} is the $(n-1)$ -dimensional Hausdorff measure. To minimize this energy, K must fit the set of discontinuity of the image and u must represent the regular part of the intensity. In order to adapt this model for the particular case of thin and elongated sets, we have introduced in [2] a Finsler metric φ which must fit the anisotropy of the image. At any point $x \in \Omega$, $\varphi(x, \cdot)$ is a norm whose unit ball coincides with the elongation of the sets we want to detect. So, we set

$$\mathcal{E}(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_K \varphi(x, \nu) d\mathcal{H}^{n-1}, \quad (1.2)$$

where $u \in W^{1,2}(\Omega \setminus K)$, K is a compact $(n-1)$ -dimensional submanifold and ν is a unit vector orthogonal to K . This kind of model which consists in an energy with a volumic and a surfacic parts also arises in Fracture Mechanics Theory ([3, 4, 5] for example). In this setting, the analysis for the case where φ is a constant norm has been treated in [6]. To our knowledge, the inhomogeneous case, where φ also depends on x , has not been done yet.

Since the compact submanifolds of Ω cannot be endowed with a topology which ensures that the direct methods apply, a weak formulation of the problem is needed. To do this, De Giorgi and Ambrosio [7] proposed to set this kind of problem in the space SBV of special functions with bounded variation. Thus, setting $K = J_u$ in (1.2) and defining $E(u) = \mathcal{E}(u, J_u)$, it gives

$$E(u) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}, \quad (1.3)$$

where $u \in \text{SBV}(\Omega)$, ∇u is the derivative of u with respect to the Lebesgue measure, J_u is its jump set and ν_u is a unit vector orthogonal to J_u . The abstract theory in SBV has been developed: Ambrosio established the existence result [8, 9], and regularity for minimizers of this kind of energy has been proved [10, 11, 12, 13]. Those results ensure that any minimizer u of the relaxed problem in SBV provides a couple $(u, \overline{J_u})$ which also minimizes the initial model \mathcal{E} .

The numerical approximation for solutions is hard because of the treatment of the jump set J_u . To overcome this difficulty, the idea is to perform a variational approximation of the functional E in the sense of De Giorgi Γ -convergence [14, 15] with Ambrosio-Tortorelli's approximation.

In order to approximate (1.3), we propose two slightly different families of functionals $(E_\varepsilon)_\varepsilon$ and $(\tilde{E}_\varepsilon)_\varepsilon$ defined by

$$E_\varepsilon(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx, \quad (1.4)$$

$$\tilde{E}_\varepsilon(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 [\eta_\varepsilon + (1 - z^2)^2] dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx. \quad (1.5)$$

For both versions, the function z takes its values in $[0; 1]$ and plays the role of a control on the gradient of u . In the second one, the parameter η_ε is infinitesimal with respect to ε . The first functional is directly inspired by the initial Ambrosio-Tortorelli's approximation [16], while the second one was introduced later in [17] and it was used in various papers, for example [6, 18, 19]. Those functionals are more adapted for numerics since usual finite element methods can be directly applied. They formally differ by the introduction of the term

$$\eta_\varepsilon \int_{\Omega} |\nabla u|^2 dx. \quad (1.6)$$

By this way, \tilde{E}_ε admits as natural domain of definition the classical Sobolev space $(W^{1,2}(\Omega))^2$. However, this term is not strictly necessary in our study because all the results which are proven for \tilde{E}_ε are also true for E_ε . For this simplification, the cost to pay is a slightly longer analysis in order to introduce an adapted domain for E_ε which ensures that its minimization is still a well-posed problem.

The Γ -convergence result when $\varepsilon \rightarrow 0^+$ will be proven for both E_ε and \tilde{E}_ε . However, in the Image Processing context, the parameter ε is devoted to be *small* but *fixed*. Indeed, for stability of the algorithms, ε has to be bounded by below by a positive constant which depends on the size of the grid (see [20]). As in practice we can not take the limit $\varepsilon \rightarrow 0$, we may choose the first version in order to avoid the additional diffusion in the numerics due to the term (1.6).

One may think that such Γ -convergence result is useless for applications because of the lower bound for ε . Nevertheless, this result provides a continuous model (1.2) of the discrete energy which may be used as a mirror to understand the proper geometry of the problem. Then, an automatic determination of the (numerous) parameters of the algorithms is possible if we have an apriori on the image we want to get (see [21, 2]).

In section 2, we introduce the geometric framework and we recall some results on spaces of functions with bounded variation. In section 3, we introduce the family of functionals $(E_\varepsilon)_\varepsilon$ with their domains, we give our main result and its complete proof.

2 Functional framework

In all the paper, Ω is an open and bounded domain with Lipschitz boundaries. We adopt the notations:

- $\langle \cdot, \cdot \rangle$ for the canonical scalar product, $|\cdot|$ for the euclidean norm in \mathbb{R}^n , dist for the associated distance and $\|\cdot\|$ for the associated matricial norm,
- $\mathbb{B}(\Omega)$ for the Borelian functions and \mathcal{L}^n for the Lebesgue measure on Ω ,
- \mathcal{H}^k for the k -dimensional Hausdorff measure,
- \bar{u}^N for the truncation $\min(N, \max(u, -N))$,
- f is superadditive if $f(A \cup B) \geq f(A) + f(B)$ for any disjoint sets A, B ,

- f is non decreasing if $f(A) \leq f(B)$ for any sets A, B such that $A \subset B$.

In all the paper, in order to emphasize on the 1-D specific technic, we denote ∇v by v' for any function v which is defined on \mathbb{R} .

2.1 Local metrics and Minkowski content

We introduce the geometric framework adapted to the anisotropic setting.

Definition 2.1. A function φ defined on $\overline{\Omega} \times \mathbb{R}^n$ and taking its values in $[0; +\infty[$ belongs to $\mathfrak{M}(\Omega)$ if it is continuous and satisfies

- i) for any $x \in \overline{\Omega}$, $\varphi(x, \cdot)$ is a norm on \mathbb{R}^n ,
- ii) there exists $\lambda > 0$, $\Lambda > 0$, such that, for any $(x, \mathbf{v}) \in \Omega \times \mathbb{R}^n$, we have

$$\lambda|\mathbf{v}| \leq \varphi(x, \mathbf{v}) \leq \Lambda|\mathbf{v}|. \quad (2.1)$$

To any metric we may associate its dual as follows.

Definition 2.2. For $\varphi \in \mathfrak{M}(\Omega)$, its dual metric φ^o is defined by

$$\varphi^o(x, \mathbf{v}) = \sup\{\langle \mathbf{v}, \mathbf{v}' \rangle : \varphi(x, \mathbf{v}') \leq 1\}.$$

A well known consequence of the definition is the following assertion: if $\varphi \in \mathfrak{M}(\Omega)$ then $\varphi^o \in \mathfrak{M}(\Omega)$ and $\varphi^{oo} = \varphi$. For $S \subset \Omega$, we may consider the associated geodesic distance:

$$\begin{aligned} d_{\varphi^o}(x, y) &= \inf \left\{ \int_0^1 \varphi^o(\gamma, \gamma') dt : \begin{array}{l} \gamma \in W^{1,1}([0; 1]; \Omega), \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}, \\ d_{\varphi^o}^S(x) &= \inf \{ d_{\varphi^o}(x, y) : y \in S \}. \end{aligned}$$

According to (2.1), $d_{\varphi^o}^S$ is Lipschitz and then it is differentiable almost everywhere. Furthermore, we have the following (see [22])

Proposition 2.1. Let $S \subset \Omega$ be a closed set and $\varphi \in \mathfrak{M}(\Omega)$. Then, we have

$$\varphi(x, \nabla d_{\varphi^o}^S(x)) = 1$$

at each point $x \in \Omega \setminus S$ where $d_{\varphi^o}^S$ is differentiable.

We recall that, for $S \subset \mathbb{R}^n$, the $(n - 1)$ -dimensional upper Minkowski content of S is defined by

$$\mathcal{M}^*(S) = \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{x : \text{dist}(x, S) < \rho\})}{2\rho}.$$

In our anisotropic setting, the associated anisotropic Minkowski $(n - 1)$ -dimensional upper content is defined by the limit

$$\mathcal{M}_{\varphi^o}^*(S) = \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{x \in \Omega : d_{\varphi^o}^S(x) < \rho\})}{2\rho}. \quad (2.2)$$

In [23] (Theorem 3.2.39), under regularity assumptions on S , it is proved that $\mathcal{M}^*(S) \leq \mathcal{H}^{n-1}(S)$. We need the following generalization of this result.

Proposition 2.2. Let $K \subset \Omega$ be a finite union of $(n - 1)$ -dimensional simplexes and $\varphi \in \mathfrak{M}(\Omega)$. Then, we have

$$\mathcal{M}_{\varphi^o}^*(K) \leq \int_K \varphi(x, \nu) d\mathcal{H}^{n-1}.$$

The proof is given in Appendix A.

2.2 BV spaces and slicing results

For the classical definitions and results on BV and SBV we refer to [24]. We recall that, if $u \in \text{BV}(\Omega)$, then its derivative Du belongs to the space of vectorial Radon measures and, if $u \in \text{SBV}(\Omega)$, then the Cantor part of Du is null and we obtain

$$Du = \nabla u \cdot \mathcal{L}^n + (u^+ - u^-)\nu_u \cdot \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the density of Du with respect to the Lebesgue measure \mathcal{L}^n , u^+ (resp. u^-) is the approximate upper (resp. lower) limit and $\mathcal{H}^{n-1} \llcorner J_u$ is the restriction of \mathcal{H}^{n-1} to its jump set J_u . The *slicing* results are crucial in our study.

Definition 2.3. Let $\nu \in \mathbb{S}^{n-1}$ be fixed. We denote by Π_ν the hyperplane

$$\{x \in \mathbb{R}^n : \langle x, \nu \rangle = 0\}.$$

If $x \in \Pi_\nu$, we set

$$\begin{aligned} \Omega_x &= \{t \in \mathbb{R} : x + t\nu \in \Omega\}, \\ \Omega_\nu &= \{x \in \Pi_\nu : \Omega_x \neq \emptyset\}. \end{aligned}$$

For any function u defined on Ω and any $x \in \Omega_\nu$, we set

$$\begin{aligned} (u)_x : \Omega_x &\rightarrow \mathbb{R} \\ t &\mapsto u(x + t\nu). \end{aligned}$$

The following Theorem is proved in [8].

Theorem 2.1. Let $u \in L^\infty(\Omega)$ be a function such that, for all $\nu \in \mathbb{S}^{n-1}$,

i) $(u)_x \in \text{SBV}(\Omega_x)$ for \mathcal{H}^{n-1} a.e. $x \in \Omega_\nu$,

ii) $\int_{\Omega_\nu} \left[\int_{\Omega_x} |(u)'_x| dt + \mathcal{H}^0(J_{(u)_x}) \right] d\mathcal{H}^{n-1}(x) < +\infty$;

then, $u \in \text{SBV}(\Omega)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. Conversely, let $u \in \text{SBV}(\Omega) \cap L^\infty(\Omega)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$. Then i) and ii) are satisfied. Moreover, we have

iii) $\langle \nabla u(x + t\nu), \nu \rangle = (u)'_x(t)$, for a.e. $t \in \Omega_x$ and \mathcal{H}^{n-1} -a.e. $x \in \Omega_\nu$,

iv) $\int_{J_u} |\langle \nu_u, \nu \rangle| d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(J_{(u)_x}) d\mathcal{H}^{n-1}(x)$.

We state a result (Proposition 1.16 in [25]) which will simplify the presentation of the slicing technics.

Lemma 2.1. Let $\mathcal{A}(\Omega)$ be the family of open sets of Ω , $\mu : \mathcal{A}(\Omega) \rightarrow [0; +\infty[$ be a superadditive function on disjoint open sets, λ be a positive measure on Ω and $\psi_k : \Omega \rightarrow [0; +\infty]$ be a countable family of Borel functions such that $\mu(A) \geq \int_A \psi_k d\lambda$ for every $A \in \mathcal{A}(\Omega)$. Set $\psi = \sup_k \psi_k$, then

$$\mu(A) \geq \int_A \psi d\lambda$$

for every $A \in \mathcal{A}(\Omega)$.

Let us now introduce a sub-class of SBV functions.

Definition 2.4. Let $\mathcal{W}(\Omega)$ be the space of all $u \in \text{SBV}(\Omega)$ such that

i) $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$;

ii) $\overline{J_u}$ is the intersection of Ω with a finite number of $(n-1)$ -dimensional simplexes;

iii) $u \in W^{k,\infty}(\Omega \setminus \overline{J_u})$ for every $k \in \mathbb{N}$.

The following Theorem (see [26]) provides a density result which will be useful for the approximation proof.

Theorem 2.2. *Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that*

$$\mathcal{H}^{n-1}(J_u) < +\infty, \quad \nabla u \in L^2(\Omega),$$

then there exists a sequence $(u_k)_k \subset \mathcal{W}(\Omega)$ such that

i) $u_k \rightarrow u$ strongly in $L^1(\Omega)$;

ii) $\nabla u_k \rightarrow \nabla u$ strongly in $L^2(\Omega)$;

iii) $\limsup_k \|u_k\|_\infty \leq \|u\|_\infty$;

iv) *for every $A \subset\subset \Omega$ and for every uppersemicontinuous function $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbf{S}^{n-1} \rightarrow [0; +\infty]$ such that $\varphi(x, a, b, \nu) = \varphi(x, a, b, -\nu)$ for every $x \in \Omega$, $a, b \in \mathbb{R}$ and $\nu \in \mathbf{S}^{n-1}$ there holds*

$$\limsup_{k \rightarrow \infty} \int_{A \cap J_{u_k}} \varphi(x, u_k^-, u_k^+, \nu_{u_k}) d\mathcal{H}^{n-1} \leq \int_{A \cap J_u} \varphi(x, u^-, u^+, \nu_u) d\mathcal{H}^{n-1}.$$

2.3 Approximate Differentiability and GSBV space

Due to the non coercivity of $\int_\Omega |\nabla u|^2 (1 - z^2)^2 dx$ in $W^{1,2}(\Omega)$, we need to introduce a weaker definition for ∇u and a larger domain for (1.3).

Definition 2.5. *Let $x \in \Omega$ be a Lebesgue point of $u \in L^1(\Omega)$; we say that u is approximately differentiable at x if there exists $L \in \mathbb{R}^n$ such that*

$$\lim_{r \rightarrow 0^+} \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} \frac{|u(y) - u(x) - \langle L, y - x \rangle|}{r} dy = 0.$$

If u is approximately differentiable at x then L , uniquely determined by this equality, is called the approximate differential of u at x and denoted by $\nabla u(x)$.

Remark 2.1. *The concept of approximate differentiability of u at x can be rephrased as the convergence in $L^1_{loc}(\mathbb{R}^n)$ of the functions $v_r(y) = [u(x + ry) - u(x)]/r$ to the linear function Ly . As $(v_r)_r$ converges to Du in the sense of distributions, if $u \in L^2(\Omega)$ and its approximate differential exists and belongs to $L^2(\Omega; \mathbb{R}^n)$, then u belongs to $W^{1,2}(\Omega)$ and its weak derivative coincides with its approximate differential almost everywhere.*

This definition of the differentiability satisfies the following result [24] (Theorem 3.83).

Theorem 2.3. *If $v \in BV(\Omega)$, then v is approximately differentiable almost everywhere in Ω . Moreover, the approximate differential ∇v is the density of the absolutely continuous part of Dv with respect to the Lebesgue measure.*

We need the three following technical Lemmas.

Lemma 2.2. *If $u, v \in \mathbb{B}(\Omega)$, u is approximately differentiable at x and*

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(\{y \in B_r(x) : u(y) \neq v(y)\})}{r^n} = 0,$$

then v is approximately differentiable at x and $\nabla u(x) = \nabla v(x)$.

Lemma 2.3. *If $u, v \in \mathbb{B}(\Omega)$ are approximately differentiable almost everywhere in Ω and $u \in L^\infty(\Omega)$, then uv is approximately differentiable almost everywhere in Ω and we get the Leibniz rule $\nabla(uv) = u\nabla v + v\nabla u$ almost everywhere in Ω .*

Lemma 2.4. *If v is approximately differentiable almost everywhere, then v^{-1} is also approximately differentiable almost everywhere in $\{x: v(x) \neq 0\}$.*

While the proof of Lemma 2.3 is given in Appendix B, the proofs of Lemmas 2.2 and 2.4 may be found in [24] (Proposition 3.71 and 3.73). Eventually, as no L^∞ bound is imposed by our functional (1.3), then it is useful to consider the following wider class

$$\text{GSBV}(\Omega) = \{u \in \mathbb{B}(\Omega) : \bar{u}^N \in \text{SBV}(\Omega) \text{ for any } N \in \mathbb{N}\}.$$

The structure of the generalized derivative of a GSBV function is similar to that of a SBV function [24] (Proposition 3.73 and Theorem 4.34). In this case, we have

$$J_u = \bigcup_{M=1}^{\infty} J_{\bar{u}^M} \quad (2.3)$$

and J_u is also countably \mathcal{H}^{n-1} -rectifiable; for any $N, M \geq 1$, it satisfies

$$\nu_{\bar{u}^N}(x) = \nu_{\bar{u}^M}(x), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_{\bar{u}^N} \cap J_{\bar{u}^M}. \quad (2.4)$$

Moreover, u is also approximately differentiable \mathcal{L}^n -a.e. in Ω and

$$\nabla u(x) = \nabla \bar{u}^N(x), \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|u| \leq N\}, \quad (2.5)$$

$$\nabla \bar{u}^N(x) = 0, \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|u| > N\}, \quad (2.6)$$

Thus, relations (2.3), (2.4) and (2.5) provide a slightly more general meaning for ∇u , J_u and ν_u and then it ensures that our functional (1.3) is well defined in $\text{GSBV}(\Omega)$.

3 Γ -convergence result

This section is entirely devoted to the approximation process. In subsection 3.1 we define the domains for both E_ε and \tilde{E}_ε . Then, we give the main Theorem of this paper. In subsection 3.2 we prove, for $\varepsilon > 0$ fixed, that the minimization problems admit a solution. Eventually, in subsection 3.3, we give the complete proof of Γ -convergence for both E_ε and \tilde{E}_ε .

3.1 The functionals, their domains and the main Theorem

Formally, we define

$$E_\varepsilon(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx,$$

$$\tilde{E}_\varepsilon(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 [\eta_\varepsilon + (1 - z^2)^2] dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx.$$

Both functionals E_ε and \tilde{E}_ε are well defined in $\tilde{\mathcal{D}}_n(\Omega) = W^{1,2}(\Omega) \times W^{1,2}(\Omega; [0; 1])$. Contrary to \tilde{E}_ε , the functional E_ε is not coercive for the Sobolev norm because the coefficient $(1 - z^2)^2$ removes the coercivity with respect to u . For that, we need to introduce a specific domain for E_ε that ensures the existence of a minimizer. If, in addition, u is bounded, we have

$$\begin{aligned} |\nabla(u(1 - z^2))|^2 &= |\nabla u(1 - z^2) - 2uz\nabla z|^2, \\ &\leq 2|\nabla u|^2(1 - z^2)^2 + 4\|u\|_\infty |\nabla z|^2. \end{aligned}$$

It gives

$$\int_{\Omega} |\nabla(u(1 - z^2))|^2 dx \leq \left(2 + \frac{4\|u\|_{\infty}}{\lambda\varepsilon}\right) E_{\varepsilon}(u, z).$$

So, it is natural to set

$$\mathcal{D}_n(\Omega) = \left\{ (u, z) : u \in \mathbb{B}(\Omega), z \in W^{1,2}(\Omega; [0; 1]), \bar{u}^N(1 - z^2) \in W^{1,2}(\Omega) \text{ for any } N \geq 1 \right\}.$$

Assuming $(u, z) \in \mathcal{D}_n(\Omega)$ does not ensure that $u \in W^{1,2}(\Omega)$ and ∇u can not be defined as the gradient of u in the Sobolev space. For that, the gradient of u is interpreted as in Definition 2.5 and then, the following Proposition ensures that $E_{\varepsilon}(u, z)$ is well defined in $\mathcal{D}_n(\Omega)$.

Proposition 3.1. *If $(u, z) \in \mathcal{D}_n(\Omega)$, then u is approximately differentiable in $\{x \in \Omega : z(x) \neq 1\}$ and z is approximately differentiable in Ω . Moreover, almost everywhere in $\{x \in \Omega : z(x) \neq 1\}$, we have*

$$(1 - z^2)\nabla u = \nabla[u(1 - z^2)] + 2uz\nabla z, \quad (3.1)$$

where ∇ is the approximate derivation on the left hand side and the weak derivation in $W^{1,2}(\Omega)$ on the right hand side.

Proof. As Ω is bounded, then the inclusion $W^{1,2}(\Omega) \subset \text{BV}(\Omega)$ is ensured. So, if $(u, z) \in \mathcal{D}_n(\Omega)$, according to Theorem 2.3, z and $\bar{u}^N(1 - z^2)$ are approximately differentiable almost everywhere in Ω . According to Lemmas 2.3 and 2.4, we deduce that \bar{u}^N is approximately differentiable almost everywhere in the set $\{x : z(x) \neq 1\}$ and, according to the Leibniz rule, we have

$$(1 - z^2)\nabla \bar{u}^N = \nabla[\bar{u}^N(1 - z^2)] + 2\bar{u}^N z \nabla z,$$

where ∇ is the approximate derivation. As $z, \bar{u}^N(1 - z^2) \in W^{1,2}(\Omega)$, according to Theorem 2.3, their approximate derivatives correspond to their weak derivatives almost everywhere. According to Lemma 2.2, u and $u(1 - z^2)$ are a.e. approximately differentiable in $\{|u| < N\}$ and we have

$$\nabla \bar{u}^N(x) = \nabla u(x), \quad \nabla[\bar{u}^N(1 - z^2)](x) = \nabla[u(1 - z^2)](x), \quad \bar{u}^N(x) = u(x),$$

for a.e. every $x \in \{|u| < N\}$, which concludes the proof. \square

We consider the following functionals:

- $E : \mathbb{B}(\Omega) \rightarrow [0; +\infty]$ defined by

$$E(u) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}$$

if $u \in \text{GSBV}(\Omega)$, and $E(u) = +\infty$ otherwise;

- $E_{\varepsilon} : \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \rightarrow [0; +\infty]$ defined by

$$E_{\varepsilon}(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx$$

if $(u, z) \in \mathcal{D}_n(\Omega)$, where $(1 - z^2)\nabla u$ is defined by the right hand side of (3.1). Moreover, we set $E_{\varepsilon}(u, z) = +\infty$ otherwise;

- $\tilde{E}_{\varepsilon} : \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \rightarrow [0; +\infty]$ defined by

$$\tilde{E}_{\varepsilon}(u, z) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 [\eta_{\varepsilon} + (1 - z^2)^2] dx + \int_{\Omega} \left(\varepsilon \varphi(x, \nabla z)^2 + \frac{z^2}{4\varepsilon} \right) dx$$

if $(u, z) \in \tilde{\mathcal{D}}_n(\Omega)$, $\tilde{E}_{\varepsilon}(u, z) = +\infty$ otherwise.

The main result of the paper is the following

Theorem 3.1. *If $g \in L^\infty(\Omega)$ and $\varphi \in \mathfrak{M}(\Omega)$, then we have the following assertions.*

- i) *For any $\varepsilon > 0$, E_ε admits a minimizer, denoted by $(u_\varepsilon, z_\varepsilon)$. Moreover, we can assume that $u_\varepsilon(x) = g(x)$ on $\{x \in \Omega: z_\varepsilon(x) = 1\}$.*
- ii) *For any $(\varepsilon_k)_k$ converging to 0^+ , there exists a subsequence, still denoted by $(\varepsilon_k)_k$, and $u \in \text{GSBV}(\Omega)$ such that $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$ converges to $(u, 0)$ almost everywhere and u is a minimizer of E .*

If, by addition, $\eta_\varepsilon = o(\varepsilon)$, the same result holds true for \tilde{E}_ε .

We set

$$\begin{aligned} (\mathcal{P}) : & \quad \text{Min}\{E(u) : u \in \mathbb{B}(\Omega)\}, \\ (\mathcal{P}_\varepsilon) : & \quad \text{Min}\{E_\varepsilon(u, z) : (u, z) \in \mathbb{B}(\Omega) \times \mathbb{B}(\Omega)\}, \\ (\tilde{\mathcal{P}}_\varepsilon) : & \quad \text{Min}\{\tilde{E}_\varepsilon(u, z) : (u, z) \in \mathbb{B}(\Omega) \times \mathbb{B}(\Omega)\}. \end{aligned}$$

Theorem 3.1 i) implies that, for $\varepsilon > 0$ fixed, $(\mathcal{P}_\varepsilon)$ and $(\tilde{\mathcal{P}}_\varepsilon)$ are well posed problems, its proof is given in Section 3.2. Theorem ii) implies that, up to the extraction of a subsequence, the sequence of solutions of $(\mathcal{P}_{\varepsilon_k})$ and $(\tilde{\mathcal{P}}_{\varepsilon_k})$ converge to a solution of (\mathcal{P}) , its proof is given in Section 3.4.

3.2 Existence result for $(\mathcal{P}_\varepsilon)$ and $(\tilde{\mathcal{P}}_\varepsilon)$

We prove Theorem 3.1 i) for E_ε , which is a straightforward consequence of Propositions 3.2 and 3.3. The arguments for \tilde{E}_ε are the same, so its proof is omitted.

Proposition 3.2. *Let $\varphi \in \mathfrak{M}(\Omega)$ and $\varepsilon > 0$ be fixed. There exists $(u_k, z_k)_k$ a minimizing sequence of E_ε such that $(u_k)_k$ is a bounded sequence in $L^\infty(\Omega)$, $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, $(u_k, z_k)_k$ converges almost everywhere to $(u, z) \in \mathcal{D}_n(\Omega)$ and $u(x) = g(x)$ on $\{x: z(x) = 1\}$.*

To prove it, the following Lemma is needed.

Lemma 3.1. *For $(u, z) \in \mathcal{D}_n(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$ fixed, we have $(u_x, z_x) \in \mathcal{D}_1(\Omega_x)$ for \mathcal{H}^{n-1} -almost every $x \in \Omega_\nu$ (see the notations of definition 2.3), and*

$$\begin{aligned} u'_x(t) &= \langle \nabla u(x + t\nu), \nu \rangle, \\ z'_x(t) &= \langle \nabla z(x + t\nu), \nu \rangle, \end{aligned}$$

for almost every $t \in \Omega_x \setminus \{s: z(x + s\nu) = 1\}$.

The proof is given in Appendix C. Now, we prove Proposition 3.2.

Proof. Let $(u_k, z_k)_k$ be a minimizing sequence for E_ε . We fix $N \geq \|g\|_\infty$ and we consider the truncated functions \bar{u}_k^N . As for the proof of Proposition 3.1, we get that $\bar{u}_k^N(1 - z_k^2)$ and $1 - z_k^2$ are approximately differentiable almost everywhere and \bar{u}_k^N is approximately differentiable almost everywhere in $\{x: z_k(x) \neq 1\}$. According to relations (2.5) and (2.6), $\nabla \bar{u}_k^N(x) = 0$ almost everywhere in $\{x: |\bar{u}_k^N(x)| = N\}$ and $\nabla \bar{u}_k^N(x) = \nabla u_k(x)$ almost everywhere in $\{x: |\bar{u}_k^N(x)| < N\}$. So, we get

$$\int_\Omega |\nabla \bar{u}_k^N|^2 (1 - z_k^2)^2 dx \leq \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 dx. \quad (3.2)$$

We deduce that $E_\varepsilon(\bar{u}_k^N, z_k) \leq E_\varepsilon(u_k, z_k)$ and then $(\bar{u}_k^N, z_k)_k$ is also a minimizing sequence. According to (2.1), we have

$$\int_\Omega |\nabla z_k|^2 dx + \int_\Omega z_k^2 dx \leq \left(\frac{1}{\lambda\varepsilon} + 4\varepsilon \right) E_\varepsilon(u_k, z_k),$$

and then $(z_k)_k$ is a bounded sequence of $W^{1,2}(\Omega)$. So, there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to $z \in W^{1,2}(\Omega)$ and z takes also its values in $[0; 1]$. For $w_k = \bar{u}_k^N(1 - z_k^2)$, we have

$$\begin{aligned} \int_{\Omega} |\nabla w_k|^2 dx + \int_{\Omega} w_k^2 dx &\leq 2 \int_{\Omega} |\nabla \bar{u}_k^N|^2 (1 - z_k^2)^2 dx + 2N^2 \int_{\Omega} |\nabla z_k|^2 dx \\ &\quad + N^2 \int_{\Omega} (1 - z_k^2)^2 dx \end{aligned}$$

and then $(w_k)_k$ is a bounded sequence of $W^{1,2}(\Omega)$. Then, we may extract a subsequence, still denoted by $(w_k)_k$, which converges almost everywhere to $w \in W^{1,2}(\Omega)$. In particular, $(\bar{u}_k^N(x))_k$ converges for almost every $x \in \{y: z(y) \neq 1\}$ to $u(x)$. If $x \in \{y: z(y) = 1\}$, we set $u(x) = g(x)$. This construction ensures that $(u, z) \in \mathcal{D}_n(\Omega)$. \square

Proposition 3.3. *Let $\varphi \in \mathfrak{M}(\Omega)$ and $\varepsilon > 0$ be fixed. If $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converges almost everywhere to $(u, z) \in \mathcal{D}_n(\Omega)$, $(u_k)_k$ is a bounded sequence in $L^\infty(\Omega)$ and $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, then*

$$\liminf_{k \rightarrow \infty} E_\varepsilon(u_k, z_k) \geq E_\varepsilon(u, z).$$

Proof. Fatou Lemma gives

$$\liminf_{k \rightarrow \infty} \int_{\Omega} (u_k - g)^2 dx \geq \int_{\Omega} (u - g)^2 dx, \quad \liminf_{k \rightarrow \infty} \int_{\Omega} \frac{z_k^2}{4\varepsilon} dx \geq \int_{\Omega} \frac{z^2}{4\varepsilon} dx. \quad (3.3)$$

As $\varphi^{oo} = \varphi$, we have

$$\varphi(x, \mathbf{v}) = \sup_{\nu \in \mathbb{S}^{n-1}} \frac{1}{\varphi^o(x, \mathbf{v})} \langle \mathbf{v}, \nu \rangle.$$

Let $\nu \in \mathbb{S}^{n-1}$ be fixed and $A \subset \Omega$ an open set. It gives

$$\int_A \varphi(x, \nabla z_k)^2 dx \geq \int_A \frac{1}{\varphi^o(x, \nu)^2} \langle \nabla z_k, \nu \rangle^2 dx. \quad (3.4)$$

As $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, then $(\langle \nabla z_k, \nu \rangle)_k$ weakly converges to $\langle \nabla z, \nu \rangle$ in $L^2(A)$. Moreover, $x \rightarrow 1/\varphi^o(x, \nu)$ belongs to $L^\infty(A)$, so we deduce that $(\langle \nabla z_k, \nu \rangle / \varphi^o(x, \nu))_k$ weakly converges to $\langle \nabla z, \nu \rangle / \varphi^o(x, \nu)$ in $L^2(A)$. As the L^2 -norm is lower semi-continuous with respect to the weak convergence, we deduce that

$$\liminf_{k \rightarrow \infty} \int_A \frac{1}{\varphi^o(x, \nu)^2} \langle \nabla z_k, \nu \rangle^2 dx \geq \int_A \frac{1}{\varphi^o(x, \nu)^2} \langle \nabla z, \nu \rangle^2 dx \quad (3.5)$$

Let $(\nu_i)_i$ be a countable dense family of \mathbb{S}^{n-1} . We set

$$\mu(A) = \liminf_{k \rightarrow \infty} \int_A \varphi(x, \nabla z_k)^2 dx, \quad \psi_i(x) = \frac{1}{\varphi^o(x, \nu_i)} \langle \nabla z(x), \nu_i \rangle.$$

As $\sup_i \psi_i(x) = \varphi(x, \nabla z(x))^2$, according to Lemmas 2.1 and inequalities (3.4), (3.5), we deduce that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \varphi(x, \nabla z_k)^2 dx \geq \int_{\Omega} \varphi(x, \nabla z)^2 dx. \quad (3.6)$$

Thus, according to (3.3) and (3.6), to show Proposition 3.3, it suffices to prove that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx \geq \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx. \quad (3.7)$$

We first consider the one-dimensional case $n = 1$ and then by a slicing argument we get the lower semi-continuity for the general case $n \geq 1$.

For any $\delta \in]0; 1[$, we set $A_\delta = \{x \in \Omega : z(x) < 1 - \delta\}$. As $n = 1$, then z is a continuous function, A_δ is an open set and $(z_k)_k$ uniformly converges to z . In particular, we have

$$A \subset \{x \in \Omega : z_k(x) \leq 1 - \delta/2\}$$

for k large enough, and so it yields

$$\int_{A_\delta} |u'_k|^2 dx \leq \frac{1}{1 - (1 - \delta/2)^2} E_\varepsilon(u_k, z_k).$$

With the same argument as for Remark 2.1, we deduce that $(u_k)_k$ is a bounded sequence of $W^{1,2}(A_\delta)$. We may assume that this sequence is weakly convergent to u in $W^{1,2}(A_\delta)$ and as $(z_k)_k$ is uniformly convergent to z , we get

$$\int_{A_\delta} |u'|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{A_\delta} |u'_k|^2 (1 - z_k^2)^2 dx.$$

Passing to the limit $\delta \rightarrow 0^+$ gives

$$\int_{\Omega} |u'|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |u'_k|^2 (1 - z_k^2)^2 dx.$$

Now, we generalize this result to the dimension $n \geq 1$. With the notation u_x introduced in Definition 2.3, using the previous result obtained in dimension 1, Lemma 3.1 and Fatou Lemma, give

$$\begin{aligned} \int_A |\langle \nabla u, \nu \rangle|^2 (1 - z^2)^2 dx &= \int_{A_\nu} \int_{A_x} |u'_x(t)|^2 (1 - z_x(t)^2)^2 dt dx, \\ &\leq \int_{A_\nu} \liminf_{k \rightarrow \infty} \int_{A_x} |(u_k)'_x(t)|^2 (1 - (z_k)_x(t)^2)^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_\nu} \int_{A_x} |(u_k)'_x(t)|^2 (1 - (z_k)_x(t)^2)^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_\nu} \int_{A_x} |\langle \nabla u_k(x + t\nu), \nu \rangle|^2 (1 - z_k(x + t\nu)^2)^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_A |\langle \nabla u_k, \nu \rangle|^2 (1 - z_k^2)^2 dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_A |\nabla u_k|^2 (1 - z_k^2)^2 dx, \end{aligned}$$

for any open set $A \subset \Omega$ and every $\nu \in \mathbb{S}^{n-1}$. Let $(\nu_i)_i$ be a dense and countable family of \mathbb{S}^{n-1} . We set

$$\psi_i = |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2, \quad \psi = |\nabla u|^2 (1 - z^2)^2, \quad \mu(A) = \liminf_{k \rightarrow \infty} \int_A |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

As μ is superadditive on $\mathcal{A}(\Omega)$, according to Lemma 2.1, we may conclude

$$\int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

□

3.3 Γ -convergence result for $\varepsilon \rightarrow 0^+$

This section is dedicated to the proof of Theorem 3.1 ii). For that, we will prove the following Γ -convergence result.

Theorem 3.2. *Let $\varphi \in \mathfrak{M}(\Omega)$ be fixed and $(\varepsilon_k)_k$ be a sequence which converges to 0^+ . Then, we have*

i) if $u \in \mathbb{B}(\Omega)$, $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ and $(u_k, z_k)_k$ converges to $(u, 0)$ in $L^1(\Omega)$, then

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(u_k, z_k) \geq E(u); \quad (3.8)$$

ii) for any $u \in \mathbb{B}(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ in $L^1(\Omega)$ and

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(u_k, z_k) \leq E(u). \quad (3.9)$$

If $\eta_\varepsilon = o(\varepsilon)$, the same result holds true for $\tilde{E}_{\varepsilon_k}$. Moreover, the same sequence $(u_k, z_k)_k$ than for E_{ε_k} may be used for (3.9).

3.3.1 The inequality for the lower Γ -limit

We prove the first inequality of Γ -convergence (3.8). As the domain of $\tilde{E}_{\varepsilon_k}$ is included in the domain of E_{ε_k} , it is sufficient to prove the result only for E_{ε_k} . Moreover, it is also sufficient to prove this result with $(u_k)_k$ and u uniformly bounded in $L^\infty(\Omega)$. Indeed, if

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(\bar{u}_k^N, z_k) \geq E(\bar{u}^N)$$

holds true for any $N \geq 1$, as $E_{\varepsilon_k}(\bar{u}_k^N, z_k) \leq E_{\varepsilon_k}(u_k, z_k)$ for any $N \geq \|g\|_\infty$ and $(E(\bar{u}^N))_N$ converges to $E(u)$ when N converges to $+\infty$, then we get (3.8). Let $u \in L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and $(u_k, z_k)_k$ converges to $(u, 0)$ in $L^1(\Omega)$. In the sequel, we emphasize on the domain of the function: for U an open subset of Ω , we adopt the following notation

$$\begin{aligned} F(u; U) &= \int_U |\nabla u|^2 dx + \int_{J_u \cap U} \varphi(x, \nu_u) d\mathcal{H}^{n-1}, \\ F_{\varepsilon_k}(u_k, z_k; U) &= \int_U |\nabla u_k|^2 (1 - z_k^2)^2 dx + \int_U \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx. \end{aligned}$$

It is implicit that $U = \Omega$ in the previous notation if it is not mentioned. Fatou Lemma yields

$$\liminf_{k \rightarrow \infty} \int_\Omega (u_k - g)^2 dx \geq \int_\Omega (u - g)^2 dx$$

and then it suffices to prove that $\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \geq F(u)$. We perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension $n \geq 2$.

The one-dimensional case. We first give a lower bound for the surface term in dimension $n = 1$. As we argue like in [16], [17], [25], we only mention the result. In this paragraph, we assume that $\Omega = I$ is an open interval and that φ is a constant $m > 0$. Moreover, we assume in this section that m is fixed. We denote the 1-D approximating functional by

$$G_\varepsilon(u, z; I) = \int_I |u'(t)|^2 (1 - z(t)^2)^2 dt + \int_I \left(m\varepsilon |z'(t)|^2 + \frac{z(t)^2}{4\varepsilon} \right) dt, \quad (3.10)$$

where the domain is

$$\mathcal{D}_1(I) = \{(u, z) : u \in L^\infty(I), z \in W^{1,2}(I; [0; 1]), u(1 - z^2) \in W^{1,2}(I)\}.$$

We denote the lower Γ -limit, by

$$G_-(u; I) = \inf \left\{ \liminf_{k \rightarrow \infty} G_{\varepsilon_k}(u_k, z_k; I) \right\},$$

where the inf is taken over all sequence $(u_k, z_k)_k \subset \mathcal{D}_1(I)$ such that (u_k, z_k) converges almost everywhere to $(u, 0)$ in I . We have the following

Proposition 3.4. *Let $I \subset \mathbb{R}$ be an open interval and $u \in \mathbb{B}(I)$. If $G_-(u; I) < \infty$, then $u \in SBV(I)$ and*

$$\int_I |u'(t)|^2 dt + m^{1/2} \mathcal{H}^0(J_u \cap I) \leq G_-(u; I).$$

Generalization to dimension $n \geq 2$. We give the proof of (3.8). Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converging in $L^1(\Omega)$ to $(u, 0)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$. Up to the extraction of a subsequence, we may assume that its convergence is pointwise almost everywhere. We have to prove that

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \geq F(u). \quad (3.11)$$

We assume that $\liminf F_{\varepsilon_k}(u_k, z_k)$ is finite, otherwise the result is ensured.

Claim: We denote $\mu(A) = \liminf F_{\varepsilon_k}(u_k, z_k; A)$. For any $A \in \mathcal{A}(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$, we have

$$\mu(A) \geq \int_A \langle \nabla u, \nu \rangle^2 dx + \int_{J_u \cap A} \frac{1}{\varphi^o(x, \nu)^2} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}. \quad (3.12)$$

Let $\delta > 0$ be fixed. As φ is uniformly continuous, there exists a finite family $(A_i)_i \subset A$ such that $\text{diam}(A_i) \leq \delta$, $\cup_i \overline{A_i} = A$ and, for any $i \in I$, it satisfies

$$|\varphi(x, \mathbf{v}) - \varphi(y, \mathbf{v})| \leq \delta |\mathbf{v}|, \quad (3.13)$$

for any $(x, y) \in A_i^2$ and $\mathbf{v} \in \mathbb{R}^n$. We fix $a_i \in A_i$ for all $i \in I$. Let $\nu \in \mathbb{S}^{n-1}$ be fixed. According to (3.13), (2.1), for any $(x, \mathbf{v}) \in A_i \times \mathbb{R}^n$, we have

$$\begin{aligned} |\varphi(x, \mathbf{v}) - \varphi(a_i, \mathbf{v})| &\leq \delta |\mathbf{v}|, \\ &\leq \delta \lambda^{-1} \varphi(a_i, \mathbf{v}). \end{aligned}$$

We set $C(\delta) = (1 - \delta \lambda^{-1})^2$, then, for any $x \in A_i$, we get

$$\varphi(x, \nabla z_k(x))^2 \geq C(\delta) \varphi(a_i, \nabla z_k(x))^2. \quad (3.14)$$

As $(\varphi^o)^o = \varphi$, we get

$$\varphi(x, \mathbf{v}) = \sup_{\nu \in \mathbb{S}^{n-1}} \frac{1}{\varphi^o(x, \nu)} \langle \mathbf{v}, \nu \rangle$$

and then

$$\varphi(a_i, \nabla z_k(x))^2 = \sup_{\nu \in \mathbb{S}^{n-1}} \frac{1}{\varphi^o(a_i, \nu)^2} \langle \nabla z_k(x), \nu \rangle^2 \quad (3.15)$$

According to (3.14) and (3.15), $F_{\varepsilon_k}(u_k, z_k; A_i)$ is greater than

$$\int_{A_i} \left(|\nabla u_k|^2 (1 - z_k^2)^2 + C(\delta) \frac{1}{\varphi^o(a_i, \nu)^2} \varepsilon_k \langle \nabla z_k, \nu \rangle^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx.$$

With the notation introduced in Definition 2.3, $(v)_y$ is the function defined on $(A_i)_\nu^y$ as $(v)_y(t) = v(y + t\nu)$. According to Lemma 3.1, we have $(u_k)_y'(t) = \langle \nabla u(y + t\nu), \nu \rangle$ and $(z_k)_y'(t) = \langle \nabla z(y + t\nu), \nu \rangle$, so Fubini Theorem implies that $F_{\varepsilon_k}(u_k, z_k; A_i)$ is greater than

$$\int_{(A_i)_\nu} \int_{(A_i)_\nu^y} \left(|(u_k)_y'|^2 (1 - ((z_k)_y')^2)^2 + C(\delta) \frac{1}{\varphi^o(a_i, \nu)^2} \varepsilon_k |(z_k)_y'|^2 + \frac{((z_k)_y')^2}{4\varepsilon_k} \right),$$

where the integration is done over the product measure $dt d\mathcal{H}^{n-1}(y)$. With the one-dimensional notations (3.10), it gives

$$F_{\varepsilon_k}(u_k, z_k; A_i) \geq \int_{(A_i)_\nu} G_{\varepsilon_k}((u_k)_y, (z_k)_y; (A_i)_\nu^y) d\mathcal{H}^{n-1}(y),$$

where $m = C(\delta) \frac{1}{\varphi^o(a_i, \nu)^2}$. Fatou lemma yields

$$\mu(A_i) \geq \int_{A_\nu} \liminf_{k \rightarrow \infty} G_{\varepsilon_k}((u_k)_y, (z_k)_y; (A_i)_\nu^y) d\mathcal{H}^{n-1}(y)$$

and then

$$\mu(A_i) \geq \int_{(A_i)_\nu} G_-((u)_y; (A_i)_\nu^y) d\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; A_i)$ is finite, we deduce that $G_-((u)_y; (A_i)_\nu^y)$ is finite for \mathcal{H}^{n-1} almost every $y \in (A_i)_\nu$. We may apply Lemma 3.4, with $I = (A_i)_\nu^y$ and $u = (u)_y$, it gives that $(u)_y \in \text{SBV}((A_i)_\nu^y)$, for \mathcal{H}^{n-1} almost every $y \in (A_i)_\nu$ and we have

$$\mu(A_i) \geq \int_{(A_i)_\nu} \left[\int_{(A_i)_\nu^y} |(u)_y'|^2 dt + m^{1/2} \mathcal{H}^0(J_{(u)_y} \cap (A_i)_\nu^y) \right] d\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; A_i)$ is finite, Theorem 2.1 implies that

$$\int_{(A_i)_\nu} \left[\int_{(A_i)_\nu^y} |(u)_y'|^2 dt + m^{1/2} \mathcal{H}^0(J_{(u)_y} \cap (A_i)_\nu^y) \right] d\mathcal{H}^{n-1}(y)$$

is equal to

$$\int_\Omega \langle \nabla u, \nu \rangle^2 dx + \int_{J_u \cap A_i} m^{1/2} |\langle \nu_u, \nu \rangle| d\mathcal{H}^{n-1}.$$

We deduce

$$\mu(A_i) \geq \int_{A_i} \langle \nabla u, \nu \rangle^2 dx + \int_{J_u \cap A_i} m^{1/2} |\langle \nu_u, \nu \rangle| d\mathcal{H}^{n-1}.$$

If we replace m by its value, it gives

$$\mu(A_i) \geq \int_{A_i} \langle \nabla u, \nu \rangle^2 dx + C(\delta) \int_{J_u \cap A_i} \frac{1}{\varphi^o(a_i, \nu)} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}.$$

As the function φ is uniformly continuous, then φ^o is also uniformly continuous and there exists a function, still denoted by $C(\delta)$, such that $C(\delta) \rightarrow 1$ for $\delta \rightarrow 0^+$ and which satisfies

$$\mu(A_i) \geq \int_{A_i} \langle \nabla u, \nu \rangle^2 dx + C(\delta) \int_{J_u \cap A_i} \frac{1}{\varphi^o(x, \nu)} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}.$$

As μ is superadditive, we get

$$\mu(A) \geq \int_A \langle \nabla u, \nu \rangle^2 dx + C(\delta) \int_{J_u \cap A} \frac{1}{\varphi^o(x, \nu)} |\langle \nu, \nu_u \rangle| d\mathcal{H}^{n-1}.$$

We take $\delta \rightarrow 0^+$, it concludes the proof of the *Claim*. Hereafter, we divide the remaining part of the proof in two steps corresponding on the estimate on the regular part and on the jump part of u .

Step 1: Regular Part. For any open set $A \subset \Omega$, we prove the following inequality

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_A |\nabla u|^2 dx. \quad (3.16)$$

Let $(\nu_i)_i \subset \mathbb{S}^{n-1}$ be a countable set of point which is dense in \mathbb{S}^{n-1} . We set

$$\mu(A) = \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; A), \quad \psi_i(x) = \langle \nabla u(x), \nu_i \rangle^2.$$

According to *Claim*, $\mu(A) \geq \int_A \psi_i dx$. As $\sup_i \psi_i(x) = |\nabla u(x)|^2$, Lemma 2.1 yields (3.16).

Step 2: Jump Part. For any $A \in \mathcal{A}(\Omega)$, we prove that

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{J_u \cap A} \varphi(x, \nu_u) d\mathcal{H}^{n-1}. \quad (3.17)$$

We consider $(\nu_i)_i \subset \mathbb{S}^{n-1}$ and μ as in *Step 1* and we set

$$\widetilde{\psi}_i(x) = \frac{1}{\varphi^o(x, \nu_i)} |\langle \nu_u(x), \nu_i \rangle|.$$

According to *Claim*, we have $\mu(A) \geq \int_A \widetilde{\psi}_i d\mathcal{H}^{n-1}$. As $\sup_i \widetilde{\psi}_i(x) = \varphi(x, \nu_u(x))$, Lemma 2.1 yields (3.17).

Conclusion: Let μ be the Borel measure on Ω defined for any Borelian $A \subset \Omega$ by

$$\mu(A) = \int_{J_u \cap A} \varphi(x, \nu_u) d\mathcal{H}^{n-1}.$$

Let $\delta > 0$ be fixed, according to the inner regularity of Borel measures (Proposition 1.43 in [24]), there exists a compact $K \subset J_u$ such that $\mu(J_u \setminus K) \leq \delta$, that is

$$\int_{J_u \setminus K} \varphi(x, \nu_u) d\mathcal{H}^{n-1} \leq \delta. \quad (3.18)$$

We set $K_\tau = \{x \in \Omega : \text{dist}(x, K) < \tau\}$. *First Step* implies that $\int_\Omega |\nabla u|^2 dx$ is finite. As $K \subset J_u$, then we get $\mathcal{H}^{n-1}(K) < +\infty$ and, in particular, $\int_K |\nabla u|^2 dx = 0$. Thus, there exists $\tau > 0$ such that

$$\int_{K_\tau} |\nabla u|^2 dx \leq \delta. \quad (3.19)$$

According to (3.18), (3.19), *First Step* and *Second Step*, we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; \Omega) &\geq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; K_\tau) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k; \Omega \setminus \overline{K_\tau}), \\ &\geq \int_{J_u \cap K_\tau} \varphi(x, \nu_u) d\mathcal{H}^{n-1} + \int_{\Omega \setminus \overline{K_\tau}} |\nabla u|^2 dx, \\ &\geq \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1} + \int_\Omega |\nabla u|^2 dx - 2\delta. \end{aligned}$$

Taking the limit $\delta \rightarrow 0^+$ concludes the proof.

3.3.2 The inequality for the upper Γ -limit

In this section we prove the *upper* inequality of Γ -convergence, that is ii) of Theorem 3.2, for both E_{ε_k} and $\widetilde{E}_{\varepsilon_k}$. As for the *lower* inequality case, it is sufficient to prove it with $u \in L^\infty(\Omega)$. Indeed, if for any $N \geq 1$ there exists a sequence $(u_{k,N}, z_{k,N})_k$ which converges to $(\bar{u}^N, 0)$ in $L^1(\Omega)$ and which satisfies

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(u_{k,N}, z_{k,N}) \leq E(\bar{u}^N),$$

then by a diagonal extraction, there exists a sequence $(u_k, z_k)_k$ which converges to $(u, 0)$ in $L^1(\Omega)$ and which satisfies (3.9). So, let $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ be fixed. We prove that there exists a sequence $(u_k, z_k)_k \subset \widetilde{\mathcal{D}}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere,

$$\limsup_{k \rightarrow \infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 dx + \int_\Omega \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx$$

is lower than

$$\int_\Omega |\nabla u|^2 dx + \int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}$$

and

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \eta_{\varepsilon_k} dx = 0.$$

According to Theorem 2.2 and Proposition 2.2, with a diagonal extraction, it suffices to prove a weaker result, where $\int_{J_u} \varphi(x, \nu_u) d\mathcal{H}^{n-1}$ is replaced by its approximation with a Minkowski content. More precisely, it suffices to prove the following

Proposition 3.5. *Let $\varphi \in \mathfrak{M}(\Omega)$ be fixed, $(\varepsilon_k)_k$ be a sequence converging to 0^+ and $\eta_k = o(\varepsilon_k)$. For $u \in SBV(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \tilde{\mathcal{D}}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere, $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and*

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \leq \int_{\Omega} |\nabla u|^2 dx + \mathcal{M}_{\varphi^o}^*(J_u), \quad \limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \eta_k dx = 0. \quad (3.20)$$

Proof. Without loss of generality, we may assume that $|\nabla u| \in L^2(\Omega)$ and $\mathcal{M}_{\varphi^o}^*(J_u) < +\infty$, otherwise the result is obvious. If $u \in W^{1,2}(\Omega)$, then $J_u = \emptyset$ and the stationary sequence $u_k = u$, $z_k = 0$ is a solution. If $u \notin W^{1,2}(\Omega)$, then $J_u \neq \emptyset$ and $(1 - z_k^2)^2$ has to be infinitesimal near of J_u . For $\rho > 0$, we set

$$(J_u)_\rho = \{x \in \Omega : d_{\varphi^o}^{J_u}(x) < \rho\}. \quad (3.21)$$

We separate Ω in three parts:

$$(J_u)_{b_k}, \quad (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}, \quad \Omega \setminus (J_u)_{a_k+b_k} \quad (3.22)$$

with

$$a_k = -4\varepsilon_k \ln(\varepsilon_k), \quad b_k = \sqrt{\varepsilon_k \eta_k}. \quad (3.23)$$

Let Ψ_k defined by

$$\Psi_k(x) = \begin{cases} 1 & \forall x \in (J_u)_{b_k/2}, \\ 2 - 2d_{\varphi^o}^{J_u}(x)/b_k & \forall x \in (J_u)_{b_k} \setminus (J_u)_{b_k/2}, \\ 0 & \forall x \in \Omega \setminus (J_u)_{b_k}. \end{cases} \quad (3.24)$$

This definition ensures that Ψ_k is a continuous function with support in $(J_u)_{b_k}$, which is equal to 1 in a neighborhood $(J_u)_{b_k/2}$ of J_u and which *shrinks* to J_u when $k \rightarrow \infty$. Ellipticity inequality (2.1) yields

$$\begin{aligned} |d_{\varphi^o}^{J_u}(x) - d_{\varphi^o}^{J_u}(y)| &\leq d_{\varphi^o}(x, y), \\ &\leq \lambda^{-1}|x - y|. \end{aligned}$$

So, $d_{\varphi^o}^{J_u}$ is Lipschitzian and Rademacher Theorem ensures that $\nabla d_{\varphi^o}^{J_u}$ exists for almost every $x \in \Omega$. It gives

$$|\nabla \Psi_k| \leq \frac{2}{b_k \lambda}.$$

We set $u_k = (1 - \Psi_k)u$ and then $u_k \equiv u$ in $\Omega \setminus (J_u)_{b_k}$. As $(b_k)_k$ converges to 0 then u_k converges to u almost everywhere. As we have $\nabla u_k = -\nabla \Psi_k u + (1 - \Psi_k)\nabla u$, we get

$$\int_{\Omega} |\nabla u_k|^2 \eta_k dx \leq \eta_k \left(\|u\|_\infty^2 \frac{8}{b_k^2 \lambda^2} \mathcal{L}^n((J_u)_{b_k}) + 2 \int_{\Omega} |\nabla u|^2 dx \right).$$

As $\mathcal{M}_{\varphi^o}^*(J_u) < +\infty$, there exists $(\omega_k)_k$ a sequence which converges to 0^+ such that

$$\mathcal{L}^n((J_u)_{b_k}) \leq 2b_k(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k), \quad (3.25)$$

it gives

$$\int_{\Omega} |\nabla u_k|^2 \eta_k dx \leq \eta_k \left(\|u\|_\infty^2 \frac{16}{b_k \lambda^2} (\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k) + 2 \int_{\Omega} |\nabla u|^2 dx \right).$$

As $\eta_k/b_k \rightarrow 0$ and $\eta_k \rightarrow 0$, we deduce that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \eta_k dx = 0$$

and then the second inequality of (3.20) is proven. It remains to prove the first one. For that, we set $z_k = 1$ in $(J_u)_{b_k}$ and $z_k = \varepsilon_k^2$ in $\Omega \setminus (J_u)_{a_k+b_k}$. In $(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$ we adopt the following construction: we introduce

$$\theta_k(t) = \varepsilon_k^2 \exp\left(\frac{t}{2\varepsilon_k}\right)$$

and we set

$$\tilde{z}_k(t) = \begin{cases} 1 & \forall t \in [0; b_k], \\ \theta_k(a_k + b_k - t) & \forall t \in]b_k; a_k + b_k], \\ \varepsilon_k^2 & \forall t \in]a_k + b_k; +\infty[. \end{cases} \quad (3.26)$$

This is a continuous and decreasing function defined on $[0; +\infty[$, moreover, for any $t \in]b_k; a_k + b_k[$, it satisfies

$$\varepsilon_k (\tilde{z}'_k(t))^2 = \frac{(\tilde{z}_k(t))^2}{4\varepsilon_k}. \quad (3.27)$$

We set $z_k = \tilde{z}_k \circ d_{\varphi^o}^{J_u}$. As z_k is constant in $(J_u)_{b_k} \cup (\Omega \setminus (J_u)_{a_k+b_k})$, we have

$$\begin{aligned} F_{\varepsilon_k}(u_k, z_k) &= \int_{\Omega \setminus (J_u)_{a_k+b_k}} |\nabla u|^2 (1 - \varepsilon_k^4)^2 dx \\ &+ \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} |\nabla u|^2 (1 - z_k^2)^2 dx \\ &+ \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx \\ &+ \frac{\varepsilon_k^3}{4} \mathcal{L}^n(\Omega \setminus (J_u)_{a_k+b_k}) + \frac{1}{4\varepsilon_k} \mathcal{L}^n((J_u)_{b_k}). \end{aligned} \quad (3.28)$$

As $|\nabla u| \in L^2(\Omega)$ and $(a_k + b_k)_k$ converges to 0, the first term of (3.28) converges to $\int_{\Omega} |\nabla u|^2 dx$. As $\|z_k\|_{\infty} \leq 1$, the second term converges to 0. As Ω is a bounded domain, the fourth term converges to 0. According to (3.25), the fifth term is lower than $\frac{1}{2}\varepsilon_k(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)$. So, the fifth term converges to 0. To compute the limit of $(F_{\varepsilon_k}(u_k, z_k))_k$, it remains to study the convergence of

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k \varphi(x, \nabla z_k)^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx.$$

For almost every $x \in (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$, we have

$$\nabla z_k = \tilde{z}'_k \circ d_{\varphi^o}^{J_u} \nabla d_{\varphi^o}^{J_u}.$$

It gives

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k (\tilde{z}'_k \circ d_{\varphi^o}^{J_u})^2 \varphi(x, \nabla d_{\varphi^o}^{J_u})^2 + \frac{(\tilde{z}_k \circ d_{\varphi^o}^{J_u})^2}{4\varepsilon_k} \right) dx.$$

According to Proposition 2.1, $\varphi(x, \nabla d_{\varphi^o}^{J_u}(x)) = 1$ for almost every x . So, we may write

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left(\varepsilon_k (\tilde{z}'_k \circ d_{\varphi^o}^{J_u})^2 + \frac{(\tilde{z}_k \circ d_{\varphi^o}^{J_u})^2}{4\varepsilon_k} \right) \varphi(x, \nabla d_{\varphi^o}^{J_u}) dx.$$

We may apply an anisotropic version of the coarea formula (see [27]), it gives

$$A_k(z_k) = \int_{b_k}^{a_k+b_k} \left(\varepsilon_k \tilde{z}'_k(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \left[\int_{\Omega} \varphi(x, D\mathbf{1}_{(J_u)_t}) \right] dt. \quad (3.29)$$

We set

$$\mathcal{H}_{\varphi}(t) = \int_{\Omega} \varphi(x, D\mathbf{1}_{(J_u)_t}), \quad \mathcal{A}_{\varphi}(s) = \int_0^s \mathcal{H}_{\varphi}(t) dt.$$

Applying another time coarea formula gives

$$\begin{aligned} \mathcal{A}_{\varphi}(s_2) - \mathcal{A}_{\varphi}(s_1) &= \int_{s_1}^{s_2} \left[\int_{\Omega} \varphi(x, D\mathbf{1}_{(J_u)_t}) \right] dt, \\ &= \int_{(J_u)_{s_2} \setminus (J_u)_{s_1}} \varphi(x, \nabla d_{\varphi^o}^{J_u}) dx, \\ &= \mathcal{L}^n((J_u)_{s_2} \setminus (J_u)_{s_1}). \end{aligned}$$

So, $\mathcal{A}_{\varphi} \in W_{\text{loc}}^{1,1}([0; +\infty[)$ and $\nabla \mathcal{A}_{\varphi} = \mathcal{H}_{\varphi}$ almost everywhere. Using equality (3.27) and then integrating by parts (3.29) gives

$$\begin{aligned} A_k(z_k) &= \int_{b_k}^{a_k+b_k} \left(\varepsilon_k \tilde{z}'_k(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \mathcal{H}_{\varphi}(t) dt, \\ &= \int_{b_k}^{a_k+b_k} \frac{\tilde{z}_k(t)^2}{2\varepsilon_k} \mathcal{H}_{\varphi}(t) dt, \\ &= \frac{\tilde{z}_k^2(a_k + b_k)}{2\varepsilon_k} \mathcal{A}_{\varphi}(a_k + b_k) - \frac{\tilde{z}_k^2(b_k)}{2\varepsilon_k} \mathcal{A}_{\varphi}(b_k) - \frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}'_k(t) \tilde{z}_k(t) \mathcal{A}_{\varphi}(t) dt. \end{aligned}$$

The first term obviously converges to 0. As for (3.25), we have

$$\mathcal{A}_{\varphi}(b_k) \leq 2b_k(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)$$

and then the second term converges to 0 too. As $s \rightarrow \mathcal{A}_{\varphi}(s)$ is non decreasing, then

$$\mathcal{A}_{\varphi}(t) \leq 2t(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)$$

for any $t \in [b_k; a_k + b_k]$. For the last term, we apply another time this inequality, it gives

$$-\frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}'_k(t) \tilde{z}_k(t) \mathcal{A}_{\varphi}(t) dt \leq -\frac{(\mathcal{M}_{\varphi^o}^*(J_u) + \omega_k)}{\varepsilon_k} \int_{b_k}^{a_k+b_k} 2t \tilde{z}'_k(t) \tilde{z}_k(t) dt. \quad (3.30)$$

Integrating by parts yields

$$\int_{b_k}^{a_k+b_k} 2t \tilde{z}'_k(t) \tilde{z}_k(t) dt = (a_k + b_k) \tilde{z}_k(a_k + b_k)^2 - b_k \tilde{z}_k(b_k)^2 - \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2 dt. \quad (3.31)$$

According to the definitions of (a_k, b_k, z_k) , (3.23) and (3.26), we have

$$(a_k + b_k) \tilde{z}_k(a_k + b_k)^2 - b_k \tilde{z}_k(b_k)^2 = o(\varepsilon_k) \quad (3.32)$$

and equation (3.27) gives

$$\begin{aligned} \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2 dt &= 2\varepsilon_k \int_{b_k}^{a_k+b_k} |\tilde{z}'_k(t)| \tilde{z}_k(t) dt, \\ &= \varepsilon_k (1 - \varepsilon_k^2). \end{aligned} \quad (3.33)$$

From (3.30), (3.31), (3.32) and (3.33) we deduce that $\limsup_k A_k(z_k) \leq \mathcal{M}_{\varphi^o}^*(J_u)$ and, according to the decomposition (3.28), we have

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k) \leq \int_{\Omega} |\nabla u|^2 + \mathcal{M}_{\varphi^o}^*(J_u).$$

To conclude the proof, it suffices to notice that $(u_k, z_k)_k \subset \tilde{\mathcal{D}}_n(\Omega)$.

□

3.4 Compactness result for $(\mathcal{P}_\varepsilon)_\varepsilon$ and $(\tilde{\mathcal{P}}_\varepsilon)_\varepsilon$

In this Section we prove Theorem 3.1, ii). As the same arguments hold for both cases, we only give the proof for $(\mathcal{P}_\varepsilon)_\varepsilon$. According to Theorem 3.1 i), for any $\varepsilon > 0$, there exists $(u_\varepsilon, z_\varepsilon)$ a minimizer of E_ε . According to (3.2), with $N \geq \|g\|_\infty$, we have

$$\mathcal{L}^n(\{x \in \Omega : |u_\varepsilon(x)| > N\}) > 0 \quad \Rightarrow \quad E_\varepsilon(\bar{u}_\varepsilon^N, z_\varepsilon) < E_\varepsilon(u_\varepsilon, z_\varepsilon).$$

We deduce that $\|u_\varepsilon\|_\infty \leq N$ for any $\varepsilon > 0$. For $\omega_\varepsilon = u_\varepsilon(1 - z_\varepsilon^2)$, we get

$$\nabla \omega_\varepsilon = \nabla u_\varepsilon(1 - z_\varepsilon^2) - 2u_\varepsilon z_\varepsilon \nabla z_\varepsilon.$$

It yields

$$\int_\Omega |\nabla \omega_\varepsilon| dx \leq \mathcal{L}^n(\Omega)^{1/2} \left(\int_\Omega |\nabla u_\varepsilon|^2 (1 - z_\varepsilon^2)^2 dx \right)^{1/2} + 2N \int_\Omega |\nabla z_\varepsilon| z_\varepsilon dx. \quad (3.34)$$

Applying the inequality $2ab \leq a^2 + b^2$ with $a = \frac{z_\varepsilon}{2\varepsilon^{1/2}}$ and $b = \varepsilon^{1/2} |\nabla z_\varepsilon|$ gives

$$\int_\Omega |\nabla z_\varepsilon| z_\varepsilon dx \leq \int_\Omega \varepsilon |\nabla z_\varepsilon|^2 dx + \int_\Omega \frac{z_\varepsilon^2}{4\varepsilon} dx. \quad (3.35)$$

According to ellipticity inequality (2.1), we get

$$\int_\Omega \varepsilon |\nabla z_\varepsilon|^2 dx \leq \frac{1}{\lambda} E_\varepsilon(u_\varepsilon, z_\varepsilon). \quad (3.36)$$

By (3.34), (3.35) and (3.36), we deduce

$$\int_\Omega |\nabla \omega_\varepsilon| dx \leq \mathcal{L}^n(\Omega)^{1/2} (E_\varepsilon(u_\varepsilon, z_\varepsilon))^{1/2} + \left(1 + \frac{1}{\lambda}\right) E_\varepsilon(u_\varepsilon, z_\varepsilon).$$

Let $(\varepsilon_k)_k$ be a sequence converging to 0^+ . As $E \not\equiv +\infty$, according to Theorem 3.2 ii), we deduce that $(E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}))_k$ is bounded. So, $(\omega_{\varepsilon_k})_k$ is bounded in $BV(\Omega)$ and there exists a subsequence, still denoted by $(\omega_{\varepsilon_k})_k$ which converges almost everywhere to $\omega \in BV(\Omega)$. As $\int_\Omega z_k^2 dx \leq \varepsilon_k E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k})$, then $(z_k)_k$ converges to 0 in $L^2(\Omega)$ and there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to 0. As $\omega_{\varepsilon_k} = u_{\varepsilon_k}(1 - z_{\varepsilon_k}^2)$, then $(u_{\varepsilon_k})_k$ converges almost everywhere to $u \in L^\infty(\Omega)$. According to Theorem 3.2, u is obviously a minimizer of E .

Conclusion

This paper is the theoretical background needed to introduce our new model. A forthcoming paper will present the numerical point of view. The main assumption of this work is the existence of an anisotropic metric φ which must represent the anisotropy of the image. For that, we may consider the case where φ is a Riemannian metric. That is, φ is given by a field of symmetric definite positive matrices \mathbf{M} such that

$$\varphi(x, \mathbf{v}) = \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v} \rangle^{1/2}.$$

In order to favorize elongated sets, \mathbf{M} is constrained to take its values in a compact sub manifold of symmetric definite positive matrices with fixed spectrum. In order to combine the segmentation task with the computation of this metric, the couple (u, \mathbf{M}) must minimize the functional

$$\int_\Omega (u - g)^2 dx + \int_\Omega |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} dx,$$

where $\alpha > 0$ is fixed in order to satisfy the regularity condition $\varphi \in \mathfrak{M}(\Omega)$. Indded, according to the Sobolev embedding Theorem, the finiteness of the last term ensures that the metric is uniformly continuous. Existence of a minimizer of this energy and approximation with Γ -convergence are straightforward consequences of what we have presented in this article.

A Proof of Proposition 2.2

We prove this result in two steps: we first assume that φ is homogeneous and after we generalize the result in the inhomogeneous setting.

First Step: We assume that $K \subset \Omega$ is a $(n-1)$ -simplex and φ is homogeneous, i.e. it does not depend on $x \in \Omega$.

With the same notations as in (3.21), we set $K_r = \{x \in \Omega: d_{\varphi^o}^K(x) < r\}$. We separate K_r in two parts:

$$K_r^b = \{x \in K_r: \exists y \in \partial K, d_{\varphi^o}(x, y) = d_{\varphi^o}^K(x)\}, \quad K_r^i = K_r \setminus K_r^b.$$

As φ is an elliptic metric (φ is comparable to the euclidean distance) and ∂K is a finite union of $(n-2)$ -dimensional simplexes, we have

$$\mathcal{L}^n(K_r^b) \sim r^2. \quad (\text{A.1})$$

As we assume that φ is homogeneous, K_r^i is a n -dimensional simplex whose basis is K and height is

$$\sup\{2\langle \nu, \mathbf{v} \rangle: \varphi^o(\mathbf{v}) \leq r\},$$

where ν is an unitary and normal vector to K . As $\varphi^{oo} = \varphi$, we deduce that

$$\mathcal{L}^n(K_r^i) = 2r\varphi(\nu)\mathcal{H}^{n-1}(K). \quad (\text{A.2})$$

According to (A.1) and (A.2), we conclude that $\mathcal{M}_{\varphi^o}^*(K) \leq \varphi(\nu)\mathcal{H}^{n-1}(K)$.

Second Step: We assume that $K = \cup_i K_i$ is a finite union of $(n-1)$ -simplexes and that φ depends on $x \in \Omega$.

We may assume that $\int_K \varphi(x, \nu) d\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. As $\varphi \in \mathfrak{M}(\Omega)$, there exists $\lambda > 0$ such that

$$\lambda^{1/2}\mathcal{H}^{n-1}(K) \leq \int_K \varphi(x, \nu) d\mathcal{H}^{n-1}$$

and then $\mathcal{H}^{n-1}(K)$ is also finite. For $t \in \mathbb{R}$ and $i \in \{1, \dots, n\}$, we set $\Pi_t^i = \{x \in \Omega: \langle x, e_i \rangle = t\}$. Thus, for $k \in \mathbb{N}$ fixed, $\{t \in \mathbb{R}: \mathcal{H}^{n-1}(K \cap \Pi_t^i) > 1/k\}$ is finite and then, for any $i \in \{1, \dots, n\}$, the set $\{t \in \mathbb{R}: \mathcal{H}^{n-1}(K \cap \Pi_t^i) > 0\}$ is at most countable. So, if we consider the cubes whose faces are orthogonal to the vectors of the orthogonal basis, there exists a partition \mathcal{C} of Ω by cubes with diameter less than η , such that for any $C \in \mathcal{C}$, we have

$$\mathcal{H}^{n-1}(K \cap \partial C) = 0. \quad (\text{A.3})$$

For any $C \in \mathcal{C}$, we fix $a_C \in C$ and we set

$$\varphi_C = \varphi(a_C, \cdot). \quad (\text{A.4})$$

As $\varphi \in \mathfrak{M}(\Omega)$, there exists $\varepsilon(\eta)$ such that $\varepsilon(\eta) \rightarrow 0$ for $\eta \rightarrow 0$ and

$$|\varphi(x, \mathbf{v}) - \varphi(y, \mathbf{v})| \leq \varepsilon(\eta)|\mathbf{v}| \quad (\text{A.5})$$

for any $x, y \in C$ and $\mathbf{v} \in \mathbb{R}^n$. It gives

$$\begin{aligned}
\mathcal{M}_{\varphi^o}^*(K_i) &\leq \sum_{C \in \mathcal{C}} \mathcal{M}_{\varphi^o}^*(K_i \cap C), \\
&\leq (1 + \varepsilon(\eta)) \sum_{C \in \mathcal{C}} \mathcal{M}_{\varphi_C^o}^*(K_i \cap C), \\
&\leq (1 + \varepsilon(\eta)) \sum_{C \in \mathcal{C}} \int_{K_i \cap C} \varphi_C(\nu) d\mathcal{H}^{n-1}, \\
&\leq (1 + \varepsilon(\eta))^2 \sum_{C \in \mathcal{C}} \int_{K_i \cap C} \varphi(x, \nu) d\mathcal{H}^{n-1}, \\
&\leq (1 + \varepsilon(\eta))^2 \int_{K_i} \varphi(x, \nu) d\mathcal{H}^{n-1}.
\end{aligned}$$

Taking the limit $\eta \rightarrow 0^+$ gives the result of *Second Step* for each compact K_i . To conclude the proof of Proposition 2.2, it suffices to notice that

$$\mathcal{M}_{\varphi^o}^*(K) \leq \sum_i \mathcal{M}_{\varphi^o}^*(K_i).$$

B Proof of Lemma 2.3

Let $x \in \Omega$ be a point such that v_1, v_2 are approximatively differentiable at x . First, we remark that x is a Lebesgue point for v_1 , that is

$$\lim_{r \rightarrow 0^+} \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} |v_1(y) - v_1(x)| dy = 0. \quad (\text{B.1})$$

Then, we decompose

$$\begin{aligned}
&\mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} \frac{|v_1(y)v_2(y) - v_1(x)v_2(x) - \langle v_1(x)\nabla v_2(x) + v_2(x)\nabla v_1(x), y - x \rangle|}{r} dy \\
&\leq \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} |v_2(x)| \frac{|v_1(y) - v_1(x) - \langle \nabla v_1(x), y - x \rangle|}{r} dy \\
&\quad + \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} |v_1(y)| \frac{|v_2(y) - v_2(x) - \langle \nabla v_2(x), y - x \rangle|}{r} dy \\
&\quad + \mathcal{L}^n(B(x, r))^{-1} \int_{B(x, r)} \frac{|(v_1(y) - v_1(x))\langle \nabla v_2(x), y - x \rangle|}{r} dy.
\end{aligned}$$

As v_1, v_2 are approximatively differentiable at x and $v_1 \in L^\infty(\Omega)$ the two first terms converge to 0. As we have $|\langle \nabla v_2(x), y - x \rangle|/r \leq |\nabla v_2(x)|$ for any $y \in B(x, r)$, according to (B.1), the last term also converges to 0.

C Proof of Lemma 3.1

As Ω is bounded, then $W^{1,2}(\Omega) \subset \text{SBV}(\Omega)$ and then, according to Calderón-Zygmund Theorem (3.83 in [24]), the derivative in the Sobolev sense is equal to the approximate differential for almost every point in Ω . Moreover, according to Theorem 3.107 of [24], for $v \in \text{SBV}(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$, we have

$$\langle \nabla v(x + t\nu), \nu \rangle = \nabla v_x(t) \text{ a.e. } t \in \Omega_x$$

for almost every $x \in \Omega_\nu$. According to Fubini Theorem, if $v \in W^{1,2}(\Omega)$, then $v_x \in W^{1,2}(\Omega_x)$ for almost every $x \in \Omega$. Applying this property with $v = z$ and $v = u(1 - z^2)$ concludes the proof of Lemma 3.1.

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