Persistent homoclinic tangencies and infinitely many sinks for residual sets of automorphisms of low degree of $\mathbb{C}^3$

Abstract

We show that there exists a polynomial automorphism f of $\mathbb{C}^3$ of degree 2 such that for every automorphism g sufficiently close to f, g admits a tangency between the stable and unstable laminations of some hyperbolic set. As a consequence, for each $d \geq 2$, there exists an open set of polynomial automorphisms of degree at most d in which the automorphisms having infinitely many sinks are dense. To prove these results, we give a complex analogous to the notion of blender introduced by Bonatti and Diaz.

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1 Introduction

1.1 Background

Hyperbolic systems such as the horseshoe introduced by Smale were originally conjectured to be dense in the set of parameters in the 1960's. This was quickly discovered to be false in general for diffeomorphisms of manifolds of dimension greater than 2 (see [1]). The discovery in the seventies of the so-called Newhouse phenomenon, i.e. the existence of residual sets of C^2-diffemorphisms of compact surfaces with infinitely many sinks (periodic attractors) in [16] showed it was false in dimension 2 too. The technical core of the proof is the reduction to a line of tangency between the stable and unstable foliations where two Cantor sets must have persistent intersections. This gives persistent homoclinic tangencies between the stable and unstable foliations, ultimately leading to infinitely many sinks. Indeed, it is a well known fact that a sink is created in the unfolding of a generic homoclinic tangency.

Palis and Viana showed in [17] an analogous result for real diffeomorphisms in higher dimensions. We say that a saddle periodic point of multipliers $|\lambda_1| \leq |\lambda_2| < 1 < |\lambda_3|$ is sectionally dissipative if the product of any two of its eigenvalues is less than 1 in modulus, that is, $|\lambda_1\lambda_3| < 1$ and $|\lambda_2\lambda_3| < 1$. More precisely, they proved that near any smooth diffeomorphism of $\mathbb{R}^3$ exhibiting a homoclinic tangency associated to a sectionally dissipative saddle, there is a residual subset of an open set of diffeomorphisms such that each of its elements displays infinitely many coexisting sinks.

In the complex setting, this reduction is not possible anymore and to get persistent homoclinic tangencies, we have to intersect two Cantor sets in the plane. Let us denote by $\mathrm{Aut}_d(\mathbb{C}^k)$ the space of polynomial automorphisms of $\mathbb{C}^k$ of degree d for $d, k \geq 2$. Buzzard proved in [7] that there exists $d > 0$ such that there exists an automorphism $G \in \mathrm{Aut}_d(\mathbb{C}^2)$ and a neighborhood $N \subset \mathrm{Aut}_d(\mathbb{C}^2)$ of G such that N has persistent homoclinic tangencies. Buzzard gives an elegant criterion (see [6]) which generates the intersection of two planar Cantor sets, hence leading to persistent homoclinic tangencies. In his article, Buzzard uses a Runge approximation argument to get a polynomial automorphism, which implies that the degree d remains unknown and is supposedly very high.

In the article [4], Bonatti and Diaz introduce a type of horseshoe they called blender horseshoe. The important property of such hyperbolic sets lies in the fractal configuration of one of their stable/unstable manifold which implies persistent intersections between any well oriented graph and this foliation. In some sense, the foliation behaves just as it had greater Hausdorff dimension than every individual manifold of the foliation. They find how to get robust homoclinic tangencies for some C^r-diffeomorphism of $\mathbb{R}^3$ using blenders in [5]. In the article [8], one can find real polynomial maps of degree 2 with a blender. Other studies of persistent tangencies using blenders include [3] and [2].

1.2 Results and outline

In this article, we generalize Buzzard's Theorem to dimension 3 and show that the degree can be controlled in this case. Here is our main result :

Main Theorem. *There exists a polynomial automorphism f of degree 2 of $\mathbb{C}^3$ such that for every $g \in \mathrm{Aut}(\mathbb{C}^3)$ sufficiently close to f, g admits a tangency between the stable and unstable laminations of some hyperbolic set.*

Notice that in the previous result, g is not assumed to be polynomial.

Corollary 1. *For each $d \geq 2$, there exists an open subset of $\mathrm{Aut}_d(\mathbb{C}^3)$ in which the automorphisms having a homoclinic tangency are dense.*

Corollary 2. *For each $d \geq 2$, there exists an open subset $\mathrm{Aut}_d(\mathbb{C}^3)$ in which the automorphisms having infinitely many sinks are dense.*

Let us present the main ideas of the proof of this result. We consider the following automorphism of $\mathbb{C}^3$:

$$f_0 : (z_1, z_2, z_3) \mapsto (p_c(z_1) + bz_2 + \sigma z_3(z_1 - \alpha), z_1, \lambda z_1 + \mu z_3 + \nu) \tag{1}$$

where p_c is a quadratic polynomial and the coefficients $b, \lambda, \mu, \nu, \alpha$ are complex numbers. We prove that f_0 has a horseshoe $\mathcal{H}_{f_0}$ of index $(2, 1)$: the first direction is strongly expanded, the second one is strongly contracted and the third one is moderately contracted by f_0. Informally speaking, the third projection restricted to $\mathcal{H}_{f_0}$ satisfies a special "open covering property" formalized in the following definition. This is an analogous in the complex setting of the notion of cs-blender in the sense of Bonatti and Diaz.

Definition (Blender Property). *Let f be a polynomial automorphism of $\mathbb{C}^3$, D a tridisk of $\mathbb{C}^3$ and $\mathcal{H}_f = \bigcap_{-\infty}^{+\infty} f^n(D)$ a horseshoe of index $(2, 1)$. We will suppose that there exist $k > 1$ and three cone fields C^u, C^{ss}, C^{cs} such that in D :*

1. *C^u is f-invariant*
2. *C^{ss} is f^{-1}-invariant*
3. *every vector in C^u is expanded by a factor larger than k under f*
4. *every vector in C^{ss} is expanded by a factor larger than k under f^{-1}*
5. *every vector in C^{cs} is expanded by a factor larger than 1 under f^{-1}*

We will say that $\mathcal{H}_f$ is a blender if there exists a non empty open set $D' \Subset D$ such that every curve tangent to C^{ss} which intersects D' intersects the unstable set $W^u(\mathcal{H}_f)$ of $\mathcal{H}_f$.

Besides, we show that f_0 has a periodic point which is sectionally dissipative. Once the blender is constructed, finding persistent tangencies is not trivial. We introduce manifolds with special geometry called folded manifolds. We prove that any folded manifold which is in good position has a tangency with the unstable manifold of a point of $\mathcal{H}_{f_0}$. We choose the parameters c, b and σ in order to create an initial heteroclinic tangency between the unstable manifold of a point of $\mathcal{H}_{f_0}$ and a folded manifold. This folded manifold is in good position and included in the stable manifold of another point of $\mathcal{H}_{f_0}$. This enables us to produce persistent heteroclinic tangencies between stable and unstable manifolds of points of $\mathcal{H}_{f_0}$. This gives rise to homoclinic tangencies associated to the sectionally dissipative point. By a classical argument going back to Newhouse, this provides a subset of the set of automorphisms of degree 2 in which automorphisms with infinitely many sinks are dense.

The plan of the paper is as follows. In section 2, we choose a family of quadratic polynomials and we fix complex coefficients λ, μ, ν. In section 3, we introduce the map f_0 which depends on three parameters c, b, σ and the associated horseshoe and we show show that it has the blender property. Then, in section 4, we introduce the formalism of folded manifolds and we give the mechanism which gives persistent tangencies. In section 5, we prove that it is possible to choose f_0 in order to have a heteroclinic tangency. Finally, we prove the main Theorem in section 6. In Appendix A, we explain how to construct a sink from a sectionally dissipative tangency.

Note: This article is a complete rewriting of a first version released on arXiv in

November 2016. In that version the polynomial automorphism f was of degree 5. To the best of the author's knowledge, the notion of blender was used there for the first time in holomorphic dynamics.

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2 Preliminaries

2.1 Choice of a quadratic polynomial

In the following, we will consider the euclidean norm on $\mathbb{C}^n$ for $n \in \{1,2,3\}$.

Notation 2.1.1. *We denote by $\mathbb{D} \subset \mathbb{C}$ the unit disk, and by $\mathbb{D}(0,r)$ the disk centered at 0 of radius r for any $r > 0$. In particular, $\mathbb{D}(0,1) = \mathbb{D}$.*

Notation 2.1.2. *We will denote by* dist *the distance induced by the euclidean norm on $\mathbb{C}^n$ for $n \in \{1,2,3\}$.*

Notation 2.1.3. *For every $z \in \mathbb{C}^3$ and $i \in \{1,2,3\}$, we denote by $\mathrm{pr}_i(z)$ the i^{th}-coordinate of z.*

In the following proposition, we carefully choose a family of quadratic polynomials with special properties.

Proposition 2.1.4. *For every integer $q > 1$, there exists a disk $\mathcal{C} \subset \mathbb{C}$ of center $c_0 \in \mathbb{C}$, a holomorphic family $(p_c)_{c\in\mathcal{C}}$ of quadratic polynomials, two integers m and r, a disk $\mathbb{D}'$ with $\mathbb{D} \subset \mathbb{D}'$ such that we have :*

1. *for every $c \in \mathcal{C}$, $p_c^{-r}(\mathbb{D})$ (resp. $p_c^{-r}(\mathbb{D}')$) admits two components $\mathbb{D}_1, \mathbb{D}_2$ (resp. $\mathbb{D}'_1, \mathbb{D}'_2$) included in $\mathbb{D}$ (resp. $\mathbb{D}'$) such that p_c^r is univalent on both $\mathbb{D}_1$ and $\mathbb{D}_2$ (resp. $\mathbb{D}'_1$ and $\mathbb{D}'_2$). Denote by $\alpha_c = \bigcap_{n\geq 0}(p_c^r)^{-n}(\mathbb{D}_1)$ and $\gamma_c = \bigcap_{n\geq 0}(p_c^r)^{-n}(\mathbb{D}_2)$ which are two fixed points of p_c^r.*
2. *for every $c \in \mathcal{C}$, α_c is a repulsive fixed point of p_c, $|p'_c(\alpha_c)| > 2$ and we have :*
$$A := r\alpha_{c_0} \neq \gamma_{c_0} + p_{c_0}(\gamma_{c_0}) + \cdots + p_{c_0}^{r-1}(\gamma_{c_0}) := B \text{ and } |A - B| > 1$$
3. *the critical point 0 is preperiodic for $c = c_0$: $p_{c_0}^m(0) = \alpha_{c_0}$ and at $c = c_0$, we have : $\frac{d}{dc}(p_c^m(0)) \neq 0$*
4. *there exists $R > 0$ such that $\mathbb{D}' \subset \mathbb{D}(0,R)$ and such that the Julia set of p_c is included in $\mathbb{D}(0,R)$ for every $c \in \mathcal{C}$*
5. *for every $c \in \mathcal{C}$, p_c has a periodic point δ_c of multiplier of modulus between 1 and $(1 + 10^{-10})^{1/qr}$.*

Proof. We begin by working with the family of quadratic polynomials $p_c(z) = z^2 + c$, we will rescale at the end of the proof. Let us fix an integer $q > 1$. We begin by taking the only real quadratic polynomial $p_a(z) = z^2 + a$ with one parabolic cycle δ_a of period 3. In particular, $a < -1$. Simple calculation show that $z^2 + a$ has the fixed point $\alpha_a = \frac{1}{2}(1 + \sqrt{1-4a}) > 1$. For any $z \in \mathbb{C}$ such that $|z| \geq 10$, we have $|p_a(z)| = |z^2 + a| > 10 \cdot |z| - |a|$ and then $|p_a^n(z)| \to +\infty$. This shows that the Julia set of p_a is included in $\mathbb{D}(0,10)$. We take a ball $\mathbb{B}' \subset \mathbb{D}(0,10)$ around α_a which is disjoint of the orbit of the critical point 0 of $z^2 + a$ (this is possible since the critical orbit tends to the parabolic orbit of $z^2 + a$). We denote by $\mathbb{B}$ the ball of center α_a and radius $\mathrm{rad}(\mathbb{B}) = \frac{1}{1000}\mathrm{rad}(\mathbb{B}')$.

We now choose a repulsive periodic point γ_a of period r_0 inside $\mathbb{B}'$ non equal to α_a with special properties. Let us first point out that since $\mathbb{B}$ intersects the Julia set of p_a, there exists an integer r_0 such that $p_a^{r_0}(\mathbb{B})$ contains the whole Julia set of p_a. In particular, all the periodic orbits of period r_0 have a point in $\mathbb{B}$. It is possible to choose such a periodic point γ_a of period r_0 inside $\mathbb{B}$ non equal to α_a such that $\gamma_a + p_a(\gamma_a) + \cdots + p_a^{r_0-1}(\gamma_a) \neq r_0\alpha_a$. Indeed, every periodic point of period r_0 is a solution of $p_a^{r_0}(z) = z$ whose coefficient of degree $2^{r_0} - 1$ is equal both to 0 and to the sum $z^1 + \cdots + z^{2^{r_0}}$ of all the solutions of $p_a^{r_0}(z) = z$ (with multiplicity). If we had $z^i + \cdots p_a^{r_0}(z^i) = r_0\alpha_a$ for every $z^i \neq \alpha_a$, then we would find $\alpha_a = 0$, which is impossible. Then it is possible to find some periodic γ_a of period r_0 inside $\mathbb{B}$ non equal to α_a satisfying this inequality. We fix this point γ_a. There exists a ball $\tilde{\mathbb{B}}_1 \Subset \mathbb{B}$ (with $\alpha_a \in \tilde{\mathbb{B}}_1$) and an integer r_1 such that $p_a^{r_1}(\tilde{\mathbb{B}}_1)$ contains $\mathbb{B}'$. Since $\mathbb{B}'$ is disjoint of the orbit of the critical point 0 of p_a, it is possible to find a component of $p_a^{-r_1}(\mathbb{B}') \Subset \mathbb{B} \Subset \mathbb{B}'$ where $p_a^{r_1}$ is univalent. Similarly, we can find an integer r_2 and a component of $p_a^{-r_2}(\mathbb{B}') \Subset \mathbb{B} \Subset \mathbb{B}'$ where $p_a^{r_2}$ is univalent. Taking a multiple r of r_0, r_1 and r_2 if necessary, we can suppose that we have two components $\mathbb{B}_1'$ and $\mathbb{B}_2'$ of $p_a^{-r}(\mathbb{B}')$ where p_a^r is univalent, such that $\alpha_a \in \mathbb{B}_1'$, $\gamma_a \in \mathbb{B}_2'$ and both α_a and γ_a are fixed points of p_a^r. There are components $\mathbb{B}_1 \Subset \mathbb{B}_1'$ and $\mathbb{B}_2' \Subset \mathbb{B}_2$ of $p_a^{-r}(\mathbb{B})$ where p_a^r is univalent and with $\alpha_a \in \mathbb{B}_1$ and $\gamma_a \in \mathbb{B}_2$. Since $\gamma_a + p_a(\gamma_a) + \cdots + p_a^{r_0-1}(\gamma_a) \neq r_0\alpha_a$ and r is a multiple of r_0, we have $r\alpha_a \neq \gamma_a + p_a(\gamma_a) + \cdots + p_a^{r-1}(\gamma_a)$. Then, taking a multiple of r if necessary (we will still denote it by r), we have : $|r\alpha_a - (\gamma_a + p_a(\gamma_a) + \cdots + p_a^{r-1}(\gamma_a))| > 10$.

Then it is possible to fix some neighborhood $\mathcal{C}_a$ of a in $\mathbb{C}$ such that for every $c \in \mathcal{C}_a$:

1. the continuation α_c of α_a is a repulsive fixed point in $\mathbb{B}'$ such that $|p_c'(\alpha_c)| > 1$
2. the Julia set of p_c is included in $\mathbb{D}(0, 10)$
3. the continuation δ_c of δ_a is of multiplier of modulus between between $(1 - 10^{-10})^{1/qr}$ and $(1 + 10^{-10})^{1/qr}$
4. $r\alpha_c \neq \gamma_c + p_c(\gamma_c) + \cdots + p_c^{r-1}(\gamma_c)$ and $|r\alpha_c - (\gamma_c + p_c(\gamma_c) + \cdots + p_c^{r-1}(\gamma_c))| > 10$
5. $p_c^{-r}(\mathbb{B})$ (resp. $p_c^{-r}(\mathbb{B}')$) admits two components $\mathbb{B}_1, \mathbb{B}_2$ (resp. $\mathbb{B}_1', \mathbb{B}_2'$) included in $\mathbb{B}$ (resp. $\mathbb{B}'$) such that p_c^r is univalent on both $\mathbb{B}_1$ and $\mathbb{B}_2$ (resp. $\mathbb{B}_1'$ and $\mathbb{B}_2'$)

The parameter a belongs to the Mandelbrot set. Misiurewicz parameters are dense inside the Mandelbrot set so it is possible to find a parameter $\tilde{c}$ inside $\mathcal{C}_a$ such that the critical point 0 is preperiodic for $p_{\tilde{c}}$. The critical point 0 is sent after a finite number of iterations of $p_{\tilde{c}}$ on a periodic orbit. This periodic orbit is accumulated by preimages of $\alpha_{\tilde{c}}$ by iterates of $p_{\tilde{c}}$. Then by the Argument Principle it is possible to take a new Misiurewicz parameter $c_0 \in \mathcal{C}_a$ such that 0 is still preperiodic but with associated orbit the fixed point α_{c_0}. There exists an integer m such that $p_{c_0}^m(0) = \alpha_{c_0}$. The inequality $\frac{d}{dc}(p_c^m(0)) \neq 0$ at $c = c_0$ is a direct consequence of Lemma 1, Chapter 5 of [10]. For the parameter c_0, δ_{c_0} is of multiplier of modulus between between 1 and $(1 + 10^{-10})^{1/qr}$. We pick some ball $\mathcal{C} \subset \mathcal{C}_a$ of center c_0 where this is still true.

For each $c \in \mathcal{C}$, we do a rescaling by an affine map so that after rescaling $\mathbb{B}$ is sent on $\mathbb{D}$. Properties 1, 3 and 5 are still true. Property 4 is still true with a disk $\mathbb{D}(0, R)$ with a fixed $R > 0$ instead of $\mathbb{D}(0, 10)$. Since $r\alpha_c \neq \gamma_c + p_c(\gamma_c) + \cdots + p_c^{r-1}(\gamma_c)$ and $|r\alpha_c - (\gamma_c + p_c(\gamma_c) + \cdots + p_c^{r-1}(\gamma_c))| > 10$ before rescaling, we have $A \neq B$ and $|A - B| > 1$ after and then Property 2 is true. Then Properties 1, 2, 3, 4 and 5 are satisfied for every $c \in \mathcal{C}$. In the following, after rescaling, we will denote $\mathbb{B}, \mathbb{B}', \mathbb{B}'', \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_1', \mathbb{B}_2'$ by $\mathbb{D}, \mathbb{D}', \mathbb{D}'', \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_1', \mathbb{D}_2'$. For simplicity, we will still denote by p_c the polynomial after rescaling. $\square$

2.2 Choice of an IFS

Notation 2.2.1. *For every $c \in \mathcal{C}$, we denote by h_1 and h_2 the two inverse branches of p_c^r on $\mathbb{D}'$ given by Proposition 2.1.4 such that $\alpha_c = \bigcap_{n\geq 0} h_1^n(\mathbb{D})$ and $\gamma_c = \bigcap_{n\geq 0} h_2^n(\mathbb{D})$.*

Notation 2.2.2. *We denote $\mu_0 = (1-10^{-4})^{\frac{1}{qr}} \cdot e^{i\cdot\frac{\pi}{2qr}}$ which depends on the integer q. In particular, we have the following equality : $\mu_0^{qr} = (1-10^{-4}) \cdot e^{i\cdot\frac{\pi}{2}}$.*

In the following result, we iterate q times the maps h_1 and h_2 with a specific choice of the integer q. Remind that A,B and R were defined in Proposition 2.1.4.

Proposition 2.2.3. *Reducing $\mathcal{C}$ if necessary, there exists an integer q such that for every $c \in \mathcal{C}$:*

1. *$\frac{1}{C} < |(h_j^q)'| < \frac{2}{C}$ with $C > 10^{10}$ on a neighborhood $\mathbb{D}''$ of $\mathbb{D}$ with $\mathbb{D} \subset \mathbb{D}'' \subset \mathbb{D}'$*
2. *$\mathrm{diam}((h_j^q(\mathbb{D}')) \leq 10^{-11} \cdot \mathrm{dist}(h_j^q(\mathbb{D}'), \partial\mathbb{D})$*
3. *for every $z \in h_1^q(\mathbb{D}')$ and $0 \leq n \leq qr(1-10^{-10}r^{-1}R^{-1}\min(1,|A-B|))$:*
$$\mu_0^0 p_c^{n+r-1}(z) + \cdots + \mu_0^{r-1} p_c^n(z) \in \mathbb{D}(A, 10^{-10} \cdot |A-B|)$$
4. *for every $z \in h_2^q(\mathbb{D}')$ and $0 \leq n \leq qr(1-10^{-10}r^{-1}R^{-1}\min(1,|A-B|))$:*
$$\mu_0^0 p_c^{n+r-1}(z) + \cdots + \mu_0^{r-1} p_c^n(z) \in \mathbb{D}(B, 10^{-10} \cdot |A-B|)$$

Proof. We first show the result for $c = c_0$. According to the Schwarz Lemma, it is possible to find some ball $\mathbb{D}''$ with $\mathbb{D} \Subset \mathbb{D}'' \Subset \mathbb{D}'$ such that $|h_j'| < e < 1$ on $\mathbb{D}''$. We reduce the radius of $\mathbb{B}''$ so that $\mathrm{rad}(\mathbb{B}'') \leq \frac{1}{100}\mathrm{rad}(\mathbb{B}')$. Then, taking q such that $e^q < 10^{-10}$, we have $|(h_j^q)'| < 10^{-11}$ on $\mathbb{D}''$. Since the radius of $\mathbb{D}$ is smaller than 100 times the radius of $\mathbb{D}'$ and since $\frac{1}{2} < \frac{1-0.01}{(1+0.01)^3} < \frac{1+0.01}{(1-0.01)^3} < 2$, it is a consequence of the Koebe Theorem that there exists C such that $\frac{1}{C} < |(h_j^q)'| < \frac{2}{C}$ with $C > 10^{10}$ on $\mathbb{D}''$. Since h_j is a contraction such that $\bigcap_{n\geq 0} h_1^n(\mathbb{D}') = \{\alpha_c\}$ and $\bigcap_{n\geq 0} h_2^n(\mathbb{D}') = \{\gamma_c\}$, increasing the value of q if necessary, we have that $\mathrm{diam}((h_j^q(\mathbb{D}')) \leq 10^{-11} \cdot \mathrm{dist}(h_j^q(\mathbb{D}'), \partial\mathbb{D})$. When $q \to \infty$, we have that $\mu_0^k \to 1$ and $p_{c_0}^{n+k}(z) \to \alpha_{c_0}$ for $0 \leq k \leq r$, $0 \leq n \leq qr(1-10^{-10}r^{-1}R^{-1}\min(1,|A-B|))$ and $z \in h_1^q(\mathbb{D}')$. Then, increasing the value of q if necessary, we have that $\mu_0^0 p_{c_0}^{n+r-1}(z) + \cdots + \mu_0^{r-1} p_{c_0}^n(z) \in \mathbb{D}(A, 10^{-10}|A-B|)$. The proof of the last item is similar. Since all these conditions are open, reducing the ball $\mathcal{C}$ of center c_0 if necessary, they stay true for every $c \in \mathcal{C}$.

□

Notation 2.2.4. *From now on, we fix such a value of q and the associated value μ_0.*

2.3 Choice of the parameters λ and ν

Notation 2.3.1. *We denote by $A' = (\mu_0^{r-1}\alpha_{c_0} + \cdots + \mu_0^0 p_{c_0}^{r-1}(\alpha_{c_0}))$ and $B' = (\mu_0^{r-1}\gamma_{c_0} + \cdots + \mu_0^0 p_{c_0}^{r-1}(\gamma_{c_0}))$.*

According to Proposition 2.2.3, we have $A' \in \mathbb{D}(A, 10^{-10}|A-B|)$ and $B' \in \mathbb{D}(B, 10^{-10}|A-B|)$. According to item 2 of Proposition 2.1.4, this implies that $|A'-B'| > \frac{1}{2}|A-B| > \frac{1}{2}$.

Proposition 2.3.2. *There exist two constants λ, ν such that $|\lambda| < 1$ and:*

$$\lambda A'(1 + \mu_0^r + \cdots + \mu_0^{qr-r}) + \nu(1 + \mu_0 + \cdots + \mu_0^{qr-1}) = \frac{9}{10} \cdot 10^{-4}$$

$$\lambda B'(1 + \mu_0^r + \cdots + \mu_0^{qr-r}) + \nu(1 + \mu_0 + \cdots + \mu_0^{qr-1}) = -\frac{9}{10} \cdot 10^{-4}$$

Proof. We have : $1+\mu_0^r+\cdots+\mu_0^{qr-r} = (1-\mu_0^{qr})/(1-\mu_0^r) \neq 0$. We also have $1+\mu_0+\cdots+\mu_0^{qr-1} = (1-\mu_0^{qr})/(1-\mu_0) \neq 0$. Since $A' \neq B'$, it is possible to take two coefficients λ and ν so that the images of these two complex numbers by $z \mapsto \lambda(1+\mu_0^r+\cdots+\mu_0^{qr-r})z+\nu(1+\mu_0+\cdots+\mu_0^{qr-1})$ are respectively $\frac{9}{10}\cdot 10^{-4}$ and $-\frac{9}{10}\cdot 10^{-4}$.

Lemma 2.3.3. *We have :*

$$\frac{q}{2} \leq 1+|\mu_0^r|+|\mu_0|^{2r}+\cdots+|\mu_0|^{qr-r} \leq 10\cdot|1+\mu_0^r+\mu_0^{2r}+\cdots+\mu_0^{qr-r}|$$

Proof. We have : $\frac{1}{2}\leq 1, \frac{1}{2}\leq|\mu_0^r|,\ldots,\frac{1}{2}\leq|\mu_0^{qr-r}|$ so the first inequality is trivial. Since every term μ^{nr} $(0\leq n\leq q-1)$ has a positive real part and this real part is larger than $\frac{1}{2}$ for $0\leq n\leq\frac{q-1}{2}$, we have $\mathrm{Re}(1+\mu_0^r+\mu_0^{2r}+\cdots+\mu_0^{qr-r})\geq\frac{q-1}{2}\cdot\frac{1}{2}$ and then $1+|\mu_0^r|+|\mu_0|^{2r}+\cdots+|\mu_0|^{qr-r}\leq q\leq 10\cdot\frac{q-1}{2}\cdot\frac{1}{2}\leq 10\cdot|1+\mu_0^r+\mu_0^{2r}+\cdots+\mu_0^{qr-r}|$. The proof is complete. □

We have : $|\lambda||A'-B'||1+\mu_0^r+\cdots+\mu_0^{qr-r}| = 2\cdot\frac{9}{10}\cdot 10^{-4}$ with $|A'-B'|>\frac{1}{2}$ and $|1+\mu_0^r+\cdots+\mu_0^{qr-r}|\geq 1$ according to the previous lemma. This implies that $|\lambda|<1$. □

Corollary 2.3.4. *Reducing $\mathcal{C}$ if necessary, there exists a neighborhood B_μ of μ_0 such that for every $c\in\mathcal{C}$ and $\mu\in\mathcal{B}_\mu$ we have :*

1. *for every $z\in h_1^q(\mathbb{D}')$, we have :*

$$\nu+\lambda p_c^{qr-1}(z)+\mu(\nu+\cdots+\mu(\nu+\lambda z))\in\mathbb{D}(\frac{9}{10}\cdot 10^{-4},10^{-10})$$

2. *for every $z\in h_2^q(\mathbb{D}')$, we have :*

$$\nu+\lambda p_c^{q-1}(z)+\mu(\nu+\cdots+\mu(\nu+\lambda z))\in\mathbb{D}(-\frac{9}{10}\cdot 10^{-4},10^{-10})$$

Proof. We first prove the result for $c=c_0$ and $\mu=\mu_0$. According to Proposition 2.3.2, we have :

$$\nu+\lambda p_{c_0}^{qr-1}(z)+\mu_0(\nu+\cdots+\mu_0(\nu+\lambda z))-\frac{9}{10}\cdot 10^{-4} =$$
$$\lambda\sum_{n=0}^{l-1}(p_{c_0}^{qr-1-n}(z)-\alpha_{c_0})\mu_0^n+\lambda\sum_{n=l}^{qr-1}(p_{c_0}^{qr-1-n}(z)-\alpha_{c_0})\mu_0^n$$

where l is the largest integer such that $l\leq 10^{-10}\frac{q}{R}\min(1,|A-B|)$ and which is a multiple of r. In particular, $qr-l$ is a multiple of r. Using the third item of Proposition 2.2.3 :

$$|\lambda\sum_{n=l}^{qr-1}(p_{c_0}^{qr-1-n}(z)-\alpha_{c_0})\mu_0^n|\leq|\lambda|\cdot 10^{-10}|A-B|\cdot(1+|\mu_0|^r+|\mu_0|^{2r}+\cdots+|\mu_0|^{qr-r})$$

Since $|\lambda|\cdot|A-B|\cdot|1+\mu_0^r+\mu_0^{2r}+\cdots+\mu_0^{qr-r}|<|\lambda|\cdot 2|A'-B'||1+\mu_0^r+\cdots+\mu_0^{qr-r}| = 2\cdot 2\cdot\frac{9}{10}\cdot 10^{-4}$ and since $1+|\mu_0^r|+|\mu_0|^{2r}+\cdots+|\mu_0|^{qr-r}\leq 10\cdot|1+\mu_0^r+\mu_0^{2r}+\cdots+\mu_0^{qr-r}|$ according to Lemma 2.3.3, we have :

$$|\lambda\cdot\sum_{n=l}^{qr-1}(p_{c_0}^{qr-1-n}(z)-\alpha_{c_0})\mu_0^n|\leq 10^{-10}\cdot 3\cdot 10^{-4}\cdot 10\leq 10^{-11}$$

Then, because $\mathbb{D}'$ and the Julia set of p_{c_0} is included inside $\mathbb{D}(0,R)$ (see item 4 of Proposition 2.1.4), we have :

$$|\lambda \sum_{n=0}^{l-1}(p_{c_0}^{qr-1-n}(z)-\alpha_{c_0})\mu_0^n| \leq |\lambda|\cdot 2\cdot R\cdot(1+|\mu_0|+|\mu_0|^2+\cdots+|\mu_0|^{l-1}) \leq |\lambda|\cdot 2R\cdot l$$

$$\leq |\lambda|\cdot 2R\cdot 10^{-10}\frac{q}{R}\cdot\min(1,|A-B|) \leq 10^{-9}\cdot|\lambda|\cdot\frac{q}{2}\cdot\frac{|A-B|}{2} \leq 10^{-9}\cdot|\lambda|\cdot\frac{q}{2}\cdot|A'-B'|$$

$$\leq 10^{-9}\cdot|\lambda|\cdot|A'-B'|\cdot 10\cdot|1+\mu_0+\mu_0^2+\cdots+\mu_0^{qr-r}| \leq 10^{-8}\cdot 2\cdot\frac{9}{10}\cdot 10^{-4} \leq 3\cdot 10^{-11}$$

since $\frac{q}{2} \leq 10\cdot|1+\mu_0^r+\mu_0^{2r}+\cdots+\mu_0^{qr-r}|$ according to Lemma 2.3.3. Then we just have to sum the two inequalities to get the result for $c=c_0$ and $\mu=\mu_0$. Then by continuity, since the inequality is open, it stays true for every μ in some ball $\mathcal{B}_\mu$ of center μ_0 and $c\in\mathcal{C}$ after reducing $\mathcal{C}$ if necessary. Then item 1 is true and the proof of item 2 is similar. The result is proven.

□

Remark 2.3.5. Since $|\mu_0|=(1-10^{-4})^{\frac{1}{qr}}$, we reduce $\mathcal{B}_\mu$ if necessary so that $|\mu|<(1-10^{-4}+10^{-10})^{\frac{1}{qr}}$ for every $\mu\in\mathcal{B}_\mu$. If necessary, we reduce it another time so that for every $\mu\in\mathcal{B}_\mu$, we have $|\mu|^{2qr}>1-2\cdot 10^{-4}$.

2.4 Choice of the parameter μ

In this subsection, we slightly perturb the coefficient μ_0 into a new value μ in order to satisfy some equality for a product of matrices. Notice that this choice has nothing to do with the next section and the blender property, it will be useful in Section 5.

Notation 2.4.1. *We denote $\beta_0=0,\dots,\beta_m=\alpha_{c_0}$.*

Notation 2.4.2. *For every $\mu\in\mathcal{B}_\mu$, we define : $w_0=0+1\cdot(\beta_0-\beta_m)$, $w_1=p'_{c_0}(\beta_1)w_0+\mu(\beta_1-\beta_m)$ $\cdots$ and $w_{m-1}=p'_{c_0}(\beta_{m-1})w_{m-2}+\mu^m(\beta_{m-1}-\beta_m)$ where $p'_{c_0}(\beta_0)=0$, $p'_{c_0}(\beta_1),\cdots,p'_{c_0}(\beta_{m-1})\neq 0$.*

Definition 2.4.3. *Since $\beta_{m-1}-\beta_m\neq 0$, w_{m-1} is a polynomial of degree m in the variable μ so we fix some $\mu\in\mathcal{B}_\mu$ such that $w_{m-1}=w_{m-1}(\mu)\neq 0$.*

Notation 2.4.4. *We denote for every $\sigma\in\mathbb{C}$, $0\leq n\leq m-1$:*

$$M_n^\sigma=\begin{pmatrix} p'_{c_0}(\beta_n) & 0 & \sigma(\beta_n-\beta_m)\\ 1 & 0 & 0\\ \lambda & 0 & \mu\end{pmatrix}$$

Proposition 2.4.5. *We have : $M_{m-1}^\sigma\cdots M_0^\sigma\cdot(0,0,1)=(\zeta_1(\sigma),\zeta_2(\sigma),\zeta_3(\sigma))$ where ζ_1,ζ_2,ζ_3 are holomorphic such that $\zeta_1(\sigma)=w_{m-1}\cdot\sigma+O(\sigma^2)$ and $\zeta_3(\sigma)=\mu^m+O(\sigma)$.*

Proof. It is a straightforward consequence of Definition 2.4.3. □

Simple calculations yield the following corollary (the important fact is that $p'_{c_0}(\beta_0)=0$). The proof is left to the reader.

Corollary 2.4.6. *Let $\epsilon_1^n,\epsilon_2^n,\epsilon_3^n$ be three holomorphic functions such that $\epsilon_1^n(\sigma)=O(\sigma)$, $\epsilon_2^n(\sigma)=O(\sigma^2)$ and $\epsilon_3^n(\sigma)=O(\sigma)$ for $0\leq n\leq m-1$. Let us denote for every $\sigma\in\mathbb{C}$, $0\leq n\leq m-1$:*

$$N_n^\sigma=\begin{pmatrix} p'_{c_0}(\beta_n)+\epsilon_1^n(\sigma) & \epsilon_2^n(\sigma) & \sigma(\beta_n+\epsilon_3^n(\sigma)-\beta_m)\\ 1 & 0 & 0\\ \lambda & 0 & \mu\end{pmatrix}$$

For every holomorphic maps ξ_1, ξ_2 such that $\xi_1(\sigma) = O(\sigma)$ and $\xi_2(\sigma) = O(\sigma)$, we get :

$$N_{m-1}^{\sigma} \cdots N_0^{\sigma} \cdot (\xi_1(\sigma), \xi_2(\sigma), 1) = (\zeta_1(\sigma), \zeta_2(\sigma), \zeta_3(\sigma))$$

where $\zeta_1, \zeta_2, \zeta_3$ are holomorphic such that $\zeta_1(\sigma) = w_{m-1} \cdot \sigma + O(\sigma^2)$ and $\zeta_3(\sigma) = \mu^m + O(\sigma)$.

3 Construction of a blender

In this section, we construct a polynomial automorphism f_0 of $\mathbb{C}^3$. We show that f_0 has a horseshoe $\mathcal{H}_{f_0}$ and that $\mathcal{H}_{f_0}$ is a blender.

3.1 Three complex dimensions : the map f_0

We consider now the 3 dimensional map f_0 introduced in (1) given by $f_0(z_1, z_2, z_3) = (p_c(z) + bz_2 + \sigma z_3(z_1 - \alpha_{c_0}), z_1, \lambda z_1 + \mu z_3 + \nu)$. It is clear it is a polynomial automorphism for $c \in \mathcal{C}$ and $b \neq 0$. In the following, we will see that the first direction is expanded by f_0 and corresponds to the direction of the unstable manifolds of a hyperbolic set we are going to describe. The second and third directions are contracted by f_0 and correspond to the directions of the stable manifolds of this hyperbolic set.

Notation 3.1.1. $C^u = \{v = (v_1, v_2, v_3) \in \mathbb{C}^3 : \max(|v_2|, |v_3|) \leq 10^{-6} \cdot |v_1|\}$ $C^{ss} = \{v = (v_1, v_2, v_3) \in \mathbb{C}^3 : \max(|v_1|, |v_3|) \leq 10^{-6} \cdot |v_2|\}$ *and* $C^{cs} = \{v = (v_1, v_2, v_3) \in \mathbb{C}^3 : \max(|v_1|, |v_2|) \leq 10^{-6} \cdot |v_3|\}$

We now give a non general definition of a horseshoe which is specific to our context.

Definition 3.1.2. *Given an automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$, a tridisk $D = D_1 \times D_2 \times D_3 \subset \mathbb{D}^3$ and an integer $p \geq 1$, we say that $\mathcal{H}_F = \bigcap_{n \in \mathbb{Z}} F^n(\overline{D})$ is a p-branch horseshoe for F if :*

1. *$F(D) \cap D$ has p components $D^{j,u}$ which do not intersect $D_1 \times \partial(D_2 \times D_3)$*
2. *$F^{-1}(D) \cap D$ has p components $D^{j,s}$ which do not intersect $\partial D_1 \times D_2 \times D_3$*
3. *the cone field C^u is F-invariant on $\bigcup_{1 \leq j \leq p} D^{j,s}$ and the cone field C^{ss} is F^{-1}-invariant on $\bigcup_{1 \leq j \leq p} D^{j,u}$*
4. *there exists $C_F > 2$ such that at every point of $\bigcup_{1 \leq j \leq p} D^{j,s}$, for every $v \in C^u$, we have : $||DF(v)|| > C_F||v||$ and at every point of $\bigcup_{1 \leq j \leq p} D^{j,u}$, for every $v \in C^{ss}$, we have : $||D(F^{-1})(v)|| > C_F||v||$ and for every $v \in C^{cs}$, we have : $2||v|| > ||D(F^{-1})(v)|| > ||v||$.*

Proposition 3.1.3. *If $\mathcal{H}_F = \bigcap_{n \in \mathbb{Z}} F^n(\overline{D})$ is a p-branch horseshoe, then it is a horseshoe in the classical meaning of this term, that is a compact, invariant, transitive, hyperbolic set .*

Proof. The set $\bigcap_{n \in \mathbb{Z}} F^n(\overline{D})$ is compact as an intersection of compact sets and F-invariant by definition. Moreover, one can take the (non necessarily) decomposition $\mathbb{C}^3 \simeq \mathbb{R}^6 = \mathbb{C} \bigoplus \mathbb{C}^2 \simeq \mathbb{R}^2 \bigoplus \mathbb{R}^4$ and the associated constant cone fields $\mathcal{C}_{\mathbb{R}}^u = \{(v_1, v_2, v_3) : ||v_1|| > ||(v_2, v_3)||\}$ and $\mathcal{C}_{\mathbb{R}}^s = \{(v_1, v_2, v_3) : ||(v_2, v_3)|| > ||v_1||\}$. The definition above implies that both $\mathcal{C}_{\mathbb{R}}^u$ is F-invariant and $\mathcal{C}_{\mathbb{R}}^s$ is F^{-1}-invariant. Moreover, they are expanded by a factor larger than 1 respectively under F and F^{-1}. Besides, the sets $D^{j,u}$ do not intersect $D_1 \times \partial(D_2 \times D_3)$ and the sets $D^{j,s}$ do not intersect $\partial D_1 \times D_2 \times D_3$. Then $\bigcap_{n \in \mathbb{Z}} F^n(\overline{D})$ is a horseshoe in the sense of Definition 6.5.2 of [15]. According to the discussion following this definition, $\bigcap_{n \in \mathbb{Z}} F^n(\overline{D})$ is hyperbolic (this is also a straightforward application of the cone field criterion, Corollary 6.4.8 in [15]) and is topologically conjugate to a shift. In particular, it is transitive. This ends the proof of the proposition. $\square$

Remark 3.1.4. For our definition, a p-branch horseshoe has one unstable direction and two stable directions. It is also straightforward that if F has a p-branch horseshoe, then F^2 has a p^2-branch horseshoe.

Definition 3.1.5. *We say that a saddle periodic point of multipliers* $|\lambda_1| \leq |\lambda_2| < 1 < |\lambda_3|$ *is sectionally dissipative if the product of any two of its eigenvalues is less than 1 in modulus, that is,* $|\lambda_1\lambda_3| < 1$ *and* $|\lambda_2\lambda_3| < 1$.

In the next proposition, we prove that if b and σ are sufficiently small, then some iterate of $f_0 = f_{c,b,\sigma}$ has a 2-branch horseshoe. Moreover, we introduce a neighborhood $\mathcal{F}$ of f_0 where this property persists. In section 5, we will make a particular choice of c, b and σ so that the stable manifold of a periodic point of f_0 will have special properties, which will persist in a new neighborhood $\mathcal{F}' \Subset \mathcal{F}$ of f_0 in $\mathrm{Aut}_2(\mathbb{C}^3)$.

Proposition 3.1.6. *Let* $f_0 = f_{c,b,\sigma}$ *be the polynomial automorphism of* $\mathbb{C}^3$ *introduced in* (1) *with* $c \in \mathcal{C}$. *Then, there exists* $10^{-10} > b_0 > 0$ *and* $10^{-10} > \sigma_0 > 0$ *independent of* $c \in \mathcal{C}$ *such that if* $0 < |b| < b_0$ *and* $0 \leq |\sigma| < \sigma_0$, *then* $\mathcal{H}_{f_0} = \bigcap_{n\in\mathbb{Z}} (f_0^{qr})^n(\overline{\mathbb{D}^3})$ *is a* 2*-branch horseshoe. Moreover,* f_0 *has a fixed point* δ_{f_0} *that is sectionally dissipative and belongs to the homoclinic class of the continuation* α_{f_0} *of* α. *These results remain true for any* f *in some neighborhood* $\mathcal{F} = \mathcal{F}(c, b, \sigma)$ *of* f_0 *in* $Aut_2(\mathbb{C}^3)$.

Proof. The limit set of the IFS defined by the two local inverse h_1^q and h_2^q of p_c^{qr} is a basic repeller in $\mathbb{C}$. The map $f_{c,0,0} : (z_1, z_2, z_3) \mapsto (p_c(z_1), z_1, \lambda z_1 + \mu z_3 + \nu)$ is such that $f_{c,0,0}^{qr}(\overline{\mathbb{D}^3})$ has 2 components which are 2 graphs over the coordinate z_1, which are included in $\mathbb{D}^3$ and which do not intersect $\mathbb{D} \times \partial(\mathbb{D} \times \mathbb{D})$. For these two graphs, the coordinate z_2 lies respectively in $\mathbb{D}_1$ and $\mathbb{D}_2$. The third coordinate of $f_{c,0,0}^{qr}$ is equal to $\nu + \lambda p_c^{qr-1}(z_1) + \mu(\nu + \cdots + \mu(\nu + \lambda z_1)) + \mu^{qr} z_3$. According to Corollary 2.3.4, $\nu + \lambda p_c^{qr-1}(z_1) + \mu(\nu + \cdots + \mu(\nu + \lambda z_1)) \in \mathbb{D}(\pm\frac{9}{10} \cdot 10^{-4}, 10^{-10})$ and we have that $|\mu|^{qr} < 1 - 10^{-4} + 10^{-10}$ (see Remark 2.3.5), so the coordinate z_3 lies in $\mathbb{D}$ for these two graphs. Then, by continuity there exists $10^{-10} > \sigma_0 > 0$ such that for every $0 \leq |\sigma| < \sigma_0$, $f_{c,0,\sigma}^{qr}(\overline{\mathbb{D}^3})$ has 2 components which are graphs over the coordinate z_1 which are included in $\mathbb{D}^3$. For these two graphs, the coordinate z_2 lies respectively in $\mathbb{D}_1''$ and $\mathbb{D}_2''$ and the coordinate z_3 in $\mathbb{D}$. The distance to $\mathbb{D} \times \partial(\mathbb{D} \times \mathbb{D})$ of each of these two graphs is strictly positive and bounded independently of $0 \leq |\sigma| < \sigma_0$. Then this implies that there exists $10^{-10} > b_0 > 0$ such that for every $0 < |b| < b_0$ and $0 \leq |\sigma| < \sigma_0$, $f_{c,b,\sigma}^{qr}(\overline{\mathbb{D}^3})$ has two connected components $D^{1,u}$ and $D^{2,u}$ which are two tridisks which do not intersect $\mathbb{D} \times \partial(\mathbb{D} \times \mathbb{D})$. This proves item 1 of Definition 3.1.2. The proof of item 2 is similar and gives two components $D^{1,s}$ and $D^{2,s}$ in $f_{c,b,\sigma}^{-qr}(\overline{\mathbb{D}^3})$. The cone field C^u is f_0^{qr}-invariant on $D^{1,s} \cup D^{2,s}$ and dilated under f_0^{qr} by a factor at least 10^9 because $|h_j'| < 10^{-9}$ on $h_j^q(\mathbb{D}'')$ (see item 1 of Proposition 2.2.3). For the same reason, the cone field C^{ss} is f_0^{-qr}-invariant on $D^{1,u} \cup D^{2,u}$ and dilated under f_0^{-qr} by a factor at least 10^9. This shows both items 3 and 4 of Definition 3.1.2. Then $\mathcal{H}_{f_0} = \bigcap_{n\in\mathbb{Z}} (f_0^{qr})^n(\overline{\mathbb{D}^3})$ is a 2-branch horseshoe.

The periodic point δ_c is of multiplier of modulus between 1 and $(1 + 10^{-10})^{1/qr}$. Then by continuity the periodic point δ_{f_0} continuation of the periodic point δ_c of p_c is a saddle point of expanding eigenvalue between 1 and $(1+2\cdot 10^{-10})^{1/qr}$ after reducing b_0 and σ_0 if necessary. It has two contracting eigenvalues. When $b = \sigma = 0$, one is equal to 0 and the other one is a power of μ with $|\mu| < (1-10^{-4}+10^{10})^{\frac{1}{qr}}$ (see Remark 2.3.5). Then, by continuity, reducing b_0 and σ_0 if necessary, the saddle point δ_{f_0} is sectionally dissipative. We denote by $W^u_{\mathrm{loc}}(\alpha_{f_0})$ the component of $W^u(\alpha_{f_0}) \cap \mathbb{D}^3$ which contains α_{f_0}, and we use the same notation for stable manifolds and other periodic points. For sufficiently values of b and σ, both $W^u_{\mathrm{loc}}(\delta_{f_0})$ and $W^u_{\mathrm{loc}}(\alpha_{f_0})$ are graphs over $z_1 \in \mathbb{D}$ and both $W^s_{\mathrm{loc}}(\delta_{f_0})$ and $W^s_{\mathrm{loc}}(\alpha_{f_0})$ are graphs over $(z_2, z_3) \in \mathbb{D}^2$. Then reducing b_0 and σ_0 if necessary, $W^u_{\mathrm{loc}}(\alpha_{f_0})$ and $W^s_{\mathrm{loc}}(\delta_{f_0})$ intersect and $W^s_{\mathrm{loc}}(\alpha_{f_0})$ and

$W^u_{\mathrm{loc}}(\delta_{f_0})$ intersect.

Finally, since all the conditions of the proposition are open, these results stay true for any f in some neighborhood $\mathcal{F} = \mathcal{F}(c, b, \sigma)$ of f_0 in $\mathrm{Aut}(\mathbb{C}^3)$. This concludes the proof of the proposition.

□

3.2 Blender property

In this subsection, we show that the third coordinate of $f \in \mathcal{F}$ has some kind of open-covering property. As we already said, since $f^{qr} \in \mathcal{F}$ has a 2-branch horseshoe, then f^{2qr} has a 4-branch horseshoe. We are interested in the geometric properties of the third coordinate of f^{2qr}. Here, we use it to show that the 4-branch horseshoe associated to f^{2qr} is a blender. Remind that the maps h_1 and h_2 were defined in Notation 2.2.2.

Definition 3.2.1. *For $f \in \mathcal{F}$, we denote $f^{qr}[j]$ the restriction of f^{qr} on $h^q_j(\mathbb{D}') \times \mathbb{D}' \times \mathbb{D}$. Then, we define $V_1 = f^{qr}[1](h^q_1(\mathbb{D}'') \times \mathbb{D}'' \times \mathbb{D}'') \cap \mathbb{D}^3$ and $V_2 = f^{qr}[2](h^q_2(\mathbb{D}'') \times \mathbb{D}'' \times \mathbb{D}'') \cap \mathbb{D}^3$. We define $U_1 = f^{qr}[1](V_1) \cap V_1$, $U_2 = f^{qr}[2](V_1) \cap V_2$, $U_3 = f^{qr}[1](V_2) \cap V_1$ and $U_4 = f^{qr}[2](V_2) \cap V_2$. We define $g_j = f^{-2qr}_{|U_j}$ for $1 \le j \le 4$.*

Lemma 3.2.2. *Reducing b_0 and σ_0 if necessary, for any $f \in \mathcal{F}$ and $j \in \{1, 2, 3, 4\}$, we have :* $\mathrm{diam}(\mathrm{pr}_2(U_j)) \le 10^{-11} \cdot \mathrm{dist}(U_j, \mathbb{D} \times \partial\mathbb{D} \times \mathbb{D})$.

Proof. Reducing b_0 and σ_0 if necessary, we have $\mathrm{pr}_2(V_1) \subset h^q_1(\mathbb{D}'')$ and $\mathrm{pr}_2(V_2) \subset h^q_2(\mathbb{D}'')$. But we have $\mathrm{diam}((h^q_j(\mathbb{D}'')) \le 10^{-11} \cdot \mathrm{dist}(h^q_j(\mathbb{D}''), \partial\mathbb{D})$ according to Proposition 2.2.3. Then we have : $\mathrm{diam}(\mathrm{pr}_2(U_j)) \le \max(\mathrm{diam}(\mathrm{pr}_2(V_1), \mathrm{diam}(\mathrm{pr}_2(V_2))) \le 10^{-11} \cdot \mathrm{dist}(U_j, \mathbb{D} \times \partial\mathbb{D} \times \mathbb{D})$. □

Lemma 3.2.3. *Reducing b_0, σ_0 and $\mathcal{F}$ if necessary, we have :*

1. $\forall z \in f^{-qr}(V_1), \mathrm{pr}_3(f^{qr}(z)) \in \mathbb{D}(\mu^{qr} z_3 + \frac{9}{10} \cdot 10^{-4}, 10^{-9})$
2. $\forall z \in f^{-qr}(V_2), \mathrm{pr}_3(f^{qr}(z)) \in \mathbb{D}(\mu^{qr} z_3 - \frac{9}{10} \cdot 10^{-4}, 10^{-9})$

Proof. For $\sigma = b = 0$, it is a simple consequence of Corollary 2.3.4. Then, by continuity we just have to take sufficiently small values of $\sigma_0, b_0, \mathcal{F}$ to get the bound 10^{-9}. □

Notation 3.2.4. *We denote $c_1 = \frac{9}{10} \cdot 10^{-4} \cdot (1 + i), c_2 = \frac{9}{10} \cdot 10^{-4} \cdot (-1 + i), c_3 = \frac{9}{10} \cdot 10^{-4} \cdot (1 - i), c_4 = \frac{9}{10} \cdot 10^{-4} \cdot (-1 - i)$*

A double iteration of the previous proposition implies the following result :

Corollary 3.2.5. *For every $z \in U_j$, we have $\mathrm{pr}_3(g_j(z)) \in \mathbb{D}(\frac{1}{\mu^{2qr}} z_3 - c_j, 10^{-6})$.*

We show an open covering property for the affine maps $z \mapsto \frac{1}{\mu^{2qr}} z - c_j$.

Proposition 3.2.6. *For every $z \in \mathbb{D}(0, \frac{1}{10})$, there exists $j \in \{1, 2, 3, 4\}$ with :*

$$\frac{1}{\mu^{2qr}} z - c_j \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$$

Proof. We check that the union of the images of $\mathbb{D}(0, \frac{1}{10} - 10^{-4})$ under the four affine maps $z \mapsto \mu^{2qr} z + c_j$ contains $\mathbb{D}(0, \frac{1}{10})$. It is enough to show that for every point z of the set $\{z : |z| = \frac{1}{10}$ and $0 \le \mathrm{Arg}(z) \le \frac{\pi}{2}\}$, $z - c_1$ belongs to the disk $\mathbb{D}(0, |\mu|^{2qr} \cdot \frac{1}{10} \cdot (1 - 10^{-4}))$. Let us point out that $|\mu|^{2qr} > 1 - 2 \cdot 10^{-4}$ (see Remark 2.3.5). We have $|z - c_1|^2 = (x - \frac{9}{10} \cdot 10^{-4})^2 + (y - \frac{9}{10} \cdot 10^{-4})^2$. Since $x^2 + y^2 = (\frac{1}{10})^2$, at least x or y is larger than $\frac{1}{\sqrt{2}} \cdot \frac{1}{10}$. Then :

$$|z - c_1|^2 < (\frac{1}{10})^2 - 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{10} \cdot \frac{9}{10} \cdot 10^{-4} + 2 \cdot (\frac{9}{10} \cdot 10^{-4})^2$$

$$< (\frac{1}{10})^2(1 - 10 \cdot 10^{-4}) < (\frac{1}{10})^2(1 - 4 \cdot 10^{-4})^2 < (|\mu|^{2qr} \cdot \frac{1}{10} \cdot (1 - 10^{-4}))^2$$

So for every point z of the set $\{z : |z| = \frac{1}{10} \text{ and } 0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{2}\}$, we have that $z - c_1 \in \mathbb{D}(0, |\mu|^{2qr} \cdot \frac{1}{10} \cdot (1 - 10^{-4}))$. We also have $0 - c_1 \in \mathbb{D}(0, |\mu|^{2qr} \cdot \frac{1}{10} \cdot (1 - 10^{-4}))$. Then by convexity, we have that the image of $\mathbb{D}(0, \frac{1}{10} - 10^{-4})$ by the affine maps $z \mapsto \mu^{2qr} z + c_j$ contains the first quadrant of $\mathbb{D}(0, \frac{1}{10})$. Then, this shows the result by symmetry.

□

We are finally in position to prove the main result of this section.

Proposition 3.2.7. *Let Γ be any ss-curve intersecting $\mathbb{D}^2 \times \mathbb{D}(0, \frac{1}{10})$. Then for every $f \in \mathcal{F}$, Γ intersects the unstable manifold of a point of the horseshoe $\mathcal{H}_f$: in other words, $\mathcal{H}_f$ is a blender.*

Proof. Let Γ be any ss-curve intersecting $\mathbb{D}^2 \times \mathbb{D}(0, \frac{1}{10})$ and $f \in \mathcal{F}$. We show that there exists $j \in \{1, 2, 3, 4\}$ such that $g_j(\Gamma)$ contains a ss-curve Γ^1 intersecting $\mathbb{D}^3$. Let $Z = (Z_1, Z_2, Z_3)$ a point of $\Gamma \cap (\mathbb{D}^2 \times \mathbb{D}(0, \frac{1}{10}))$. According to the previous result, there exists $j \in \{1, 2, 3, 4\}$ such that $\frac{1}{\mu^{2qr}} Z_3 - c_j \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$. Since $\mathbb{D}(0, \frac{1}{10}) \subset \mathrm{pr}_3(U_j)$, by continuity, Γ intersects U_j. The cone C^{ss} is g_j-invariant, then $\Gamma^1 = g_j(\Gamma \cap U_j)$ is a ss-curve. Since $\Gamma^1 = g_j(\Gamma \cap U_j)$ is a ss-curve, we have that $\mathrm{diam}(\mathrm{pr}_3(\Gamma^1)) \leq 10^{-6}$. Since $|\mathrm{pr}_3((g_j)(z)) - \frac{1}{\mu^{2qr}} z_3 + c_j| < 10^{-6}$, we have that $\mathrm{pr}_3(\Gamma^1) \subset \mathbb{D}(0, \frac{1}{10})$. Finally, Γ^1 intersects $\mathbb{D}^2 \times \mathbb{D}(0, \frac{1}{10})$. By iteration of this result, we can construct a sequence of ss-curve $\Gamma^n \subset (g_{j_n} \circ ... \circ g_{j_1})(\Gamma)$ intersecting $\mathbb{D}^3$. The set $\bigcap_{n \geq 1} f^{n2qr}(\Gamma^n) \subset \Gamma$ is non empty as the intersection of a decreasing sequence of compact sets. But any point in $\bigcap_{n \geq 1} f^{n2qr}(\Gamma^n)$ is a point of the unstable manifold of the horseshoe $\mathcal{H}_f$. This ends the proof. □

4 Mechanism to get persistent tangencies

In this section, we explain how the blender property obtained in the last section leads to persistent tangencies with certain "folded" surfaces.

4.1 Some definitions

Definition 4.1.1. *A submanifold (or an analytic set) $W \subset \mathbb{D}^n$ is horizontal relatively to a decomposition $\mathbb{D}^n = \mathbb{D}^k \times \mathbb{D}^{n-k}$ if W does not intersect $\mathbb{D}^k \times \partial \mathbb{D}^{n-k}$. We will also say it is horizontal relatively to the projection $\mathbb{D}^n \to \mathbb{D}^k$. If $\dim(W) = k$, then the natural projection on $\mathbb{D}^k$ is a branched covering of degree d. We will say that W is of degree d. We similarly define vertical submanifolds in $\mathbb{D}^k \times \mathbb{D}^{n-k}$ and their degree.*

The two following propositions are classical. For a proof, one can refer to [9]. One can also refer to [11].

Proposition 4.1.2. *Let W be a horizontal curve of degree d and W' be a vertical curve of degree d' in $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$. Then W and W' intersect in dd' points with multiplicity.*

Proposition 4.1.3. *Let W be a horizontal curve (resp. surface) of degree d and W' be a vertical surface (resp.curve) of degree d' in $\mathbb{D}^3 = \mathbb{D}^1 \times \mathbb{D}^2$ (resp. $\mathbb{D}^3 = \mathbb{D}^2 \times \mathbb{D}^1$). Then W and W' intersect in dd' points with multiplicity.*

Remark 4.1.4. In particular, in the two previous propositions, if all the intersections are transverse, there are exactly dd' distinct points of intersection. If it is not the case, there is at least one point of tangency.

We now introduce several definitions specific to our context.

Notation 4.1.5. *Let i, j be two distinct integers in $\{1, 2, 3\}$. We will denote by π_i the projection over the i^{th} coordinate and by (π_i, π_j) the projection over the i^{th} and the j^{th} coordinates.*

Definition 4.1.6. *Let i, j be two distinct integers in $\{1, 2, 3\}$. A (i, j)-surface $\mathcal{S}$ is a complex surface horizontal relatively to the projection (π_i, π_j). A $(2, 3)$-surface will be called a s-surface.*

Remark 4.1.7. A (i, j)-surface $\mathcal{S}$ is a ramified covering of degree d over $\mathbb{D}^2$. In the rest of this article, we will only consider ramified coverings of degree 1 or 2.

In the next definition, it is convenient to fix once forall a numerical constant 10^{-4}.

Definition 4.1.8. *Let i, j, k three integers such that $\{i, j, k\} = \{1, 2, 3\}$. A (i, j)-quasi plane $\mathcal{V}$ is a holomorphic graph $\{z_k = v(z_i, z_j)\}$ over $\mathbb{D}^2$ such that there exists v^0 with*

$$\forall (z_i, z_j) \in \mathbb{D}^2, |v^0 - v(z_i, z_j)| < 10^{-4}$$

Definition 4.1.9. *Let $k \in \{2, 3\}$. A k-folded curve is a holomorphic curve horizontal relatively to π_k which is a ramified 2-covering over $z_k \in \mathbb{D}$ with exactly one point of ramification $z_{k,ram}$. We denote* $\text{fold}(\Gamma) = z_{k,ram}$ *the* fold *of* Γ.

Definition 4.1.10. *Let $k \in \{2, 3\}$. A k-folded $(2, 3)$-surface $\mathcal{W}$ is a complex surface of degree 2 that is horizontal relatively to (π_2, π_3) and such that for every $(1, k)$-quasi plane $\mathcal{V}$, $\Gamma = \mathcal{V} \cap \mathcal{W}$ is a k-folded curve. We denote :*

$$\text{Fold}(\mathcal{W}) = \{\text{fold}(\Gamma) : \Gamma \textit{ is a k-folded curve included in } \mathcal{W}\}$$

which is a subset of $\mathbb{D}$. $\mathcal{W}$ is said to be concentrated if $\text{diam}(\text{Fold}(\mathcal{W})) \leq 10^{-5}$.

Definition 4.1.11. *A u-curve (resp. ss-curve, cs-curve) Γ is a holomorphic graph over z_1 (resp. z_2, z_3) which has all its tangent vectors in C^u (resp. C^{ss}, C^{cs}) (recall that these cones were defined in Notation 3.1.1).*

4.2 Preparatory lemmas

In this subsection we gather some simple technical results that will be useful in the following subsections.

Lemma 4.2.1. *Let $\Gamma = \gamma(\mathbb{D})$ be a k-folded curve (with $k \in \{2, 3\}$) included inside $(1, k)$-quasi plane. Then there exists a disk $\mathbb{D}_\Gamma \Subset \mathbb{D}$ such that : for every $Z \in \mathbb{D}_\Gamma$ such that $\text{pr}_k(\gamma(Z)) \in \mathbb{D}_\Gamma$ we have $|\gamma_1'(Z)| < 10|\gamma_k'(Z)|$. In particular, Γ is the union of two graphs upon the coordinate $z_k \in \mathbb{D}_\Gamma$. Moreover, if $\text{diam}(pr_1(\Gamma)) \leq 10^{-8}$, for every $Z \in \mathbb{D}$ such that $|\text{pr}_k(\gamma(Z)) - \text{fold}(\Gamma)| \geq 10^{-6}$ we have : $|\gamma_1'(Z)| < |\gamma_k'(Z)|$.*

Proof. Let $\mathbb{D}_\Gamma \Subset \mathbb{D}$ be a ball of radius $\frac{1}{10}$ such that $\text{fold}(\Gamma) \notin \mathbb{D}_\Gamma$. We have that Γ is the union of two graphs over $z_k \in \mathbb{D}_\Gamma$. Let us denote $z_1 = \xi(z_k)$ one of them. On $\mathbb{D}_\Gamma$, by the Cauchy inequality, we have $|\xi'(z_k)| < 10 \cdot 1 = 10$. This implies that $|\gamma_1'| < 10|\gamma_k'|$ at Z and the result follows. The proof of the second inequality is the same, with this time $|\zeta'(z_k)| < 2 \cdot 10^6 \cdot 10^{-8} < 1$.

□

Let us point out for convenience the following obvious consequences of Propositions 4.1.2 and 4.1.3.

Lemma 4.2.2. *Let Γ be a graph over the k^{th}-coordinate (with $k \in \{2, 3\}$) included in some $(1, k)$-quasi-plane, such that $|\gamma_1'| < 10|\gamma_k'|$. Let us denote $k' = \{2, 3\} - k$. Let $\mathcal{V}$ be a $(1, k')$-quasi plane. Then $\Gamma \cap \mathcal{V}$ is a singleton.*

Lemma 4.2.3. *Let $\mathcal{V}$ be a $(1,k)$-quasi plane. Let Γ be a u-curve and $\tilde{\Gamma}$ be a graph over z_k with $|\tilde{\gamma}_1'| < 10|\tilde{\gamma}_k'|$, both included in $\mathcal{V}$. Then $\Gamma \cap \tilde{\Gamma}$ is a singleton.*

Lemma 4.2.4. *Let $\mathcal{V}$ be a $(1,k)$-quasi plane foliated by u-curves $\mathcal{V}$. Let Γ be a k-folded curve included in $\mathcal{V}$. Then the map $\mathcal{T} : \mathbb{D} \to \mathbb{D}$ defined by $\gamma(z) \in \mathcal{V}_{\mathcal{T}(z)}$ is a 2-covering with exactly one point of ramification.*

Proof. The map $\mathcal{T}$ is holomorphic by the Implicit Function Theorem. Then it is a 2-covering with exactly one point of ramification by Proposition 4.1.2.

□

Proposition 4.2.5. *Let $\mathcal{W}$ be a k-folded surface with $k \in \{2,3\}$. Let Γ be a u-curve. Then $\mathcal{W} \cap \Gamma$ has one or two points.*

4.3 Main result

Here is the main result of this section : we show that any concentrated 3-folded $(2,3)$-surface with its fold in good position has a point of tangency with the unstable manifold of a point of the horseshoe $\mathcal{H}_f$.

Proposition 4.3.1. *Let $\mathcal{W}$ be a concentrated 3-folded $(2,3)$-surface such that $\mathrm{Fold}(\mathcal{W}) \subset \mathbb{D}(0, \frac{1}{10})$ and $\mathrm{diam}(pr_1(\mathcal{W})) \leq 10^{-8}$. Then there exists a point κ_f of the horseshoe $\mathcal{H}_f$ such that $\mathcal{W}$ has a point of tangency with the unstable manifold of κ_f.*

We begin by a lemma showing that the image of a folded curve contains a folded curve. Remind that the sets U_j were defined in Definition 3.2.1.

Lemma 4.3.2. *Let Γ be a 3-folded curve included in a $(1,3)$-quasi plane $\mathcal{V}$ and included in U_j for $j \in \{1,2,3,4\}$. We suppose $\mathrm{diam}(\mathrm{pr}_1(\Gamma)) \leq 10^{-8}$ and that $\mathcal{V}' = g_j(\mathcal{V}) \cap \mathbb{D}^3$ is a $(1,3)$-quasi plane. Then $\Gamma' = g_j(\Gamma) \cap \mathbb{D}^3$ is a 3-folded curve and :*

$$\mathrm{fold}(\Gamma') \in \mathbb{D}(\frac{1}{\mu^{2qr}}\mathrm{fold}(\Gamma) - c_j, 2 \cdot 10^{-6})$$

Proof. According to Lemma 4.2.1, for every $Z \in \mathbb{D}$ such that $|\mathrm{pr}_3(\gamma(Z)) - \mathrm{fold}(\Gamma)| \geq 10^{-6}$ we have that $|\gamma_1'(Z)| < |\gamma_3'(Z)|$ and Γ is locally the union of two graphs. We can foliate $\mathcal{V}'$ by u-curves $(\mathcal{V}_t')_{t\in\mathbb{D}}$ where $\mathcal{V}_t' = \mathcal{V}' \cap \{z_3 = t\}$. For any $t \in \mathbb{D}$, $\mathcal{V}_t = f^{2qr}(\mathcal{V}_t') \cap \mathbb{D}^3$ is a u-curve. For every $t \in \mathbb{D}$ such that $\mathcal{V}_t = f^{2qr}(\mathcal{V}_t') \cap \mathbb{D}^3$ does not intersect $(\mathbb{D}^2 \times \mathbb{D}(\mathrm{fold}(\Gamma), 10^{-6})$, $\mathcal{V}_t$ intersects Γ in exactly two points by Lemma 4.2.3. It is clear there exists infinitely many $t \in \mathbb{D}$ satisfying this property. This implies that $\mathrm{pr}_3(\Gamma')$ is a 2-covering. By the Riemann-Hürwitz formula, $\mathrm{pr}_3(\Gamma')$ is a 2-covering with only one point of ramification. Moreover this shows that $\mathrm{fold}(\Gamma') \in \mathrm{pr}_3(g_j(\mathbb{D}^2 \times \mathbb{D}(\mathrm{fold}(\Gamma), 10^{-6}))$. According to Corollary 3.2.5, we have that $\mathrm{fold}(\Gamma') \in \mathbb{D}(\frac{1}{\mu^{2qr}}\mathrm{fold}(\Gamma) - c_j, 2 \cdot 10^{-6})$.

□

From now on, we prove lemmas that will show that the image under g_j of a 3-folded $(2,3)$-surface with its fold in $\mathbb{D}(0, \frac{1}{10})$ is concentrated and that it is possible to choose $j \in \{1,2,3,4\}$ so that the new fold is still in $\mathbb{D}(0, \frac{1}{10})$.

Lemma 4.3.3. *Let $\mathcal{V}^0$ and $\mathcal{V}^1$ be two $(1,3)$-quasi planes included in U_j for some $j \in \{1,2,3,4\}$. Then there exists some holomorphic family $(\mathcal{V}_t)_{t\in\mathbb{D}(0,10^6)}$ where $\mathcal{V}^0 = \mathcal{V}_0$ and $\mathcal{V}^1 = \mathcal{V}_1$, and for every $t \in \mathbb{D}(0, 10^6)$, $\mathcal{V}_t$ is a $(1,3)$-quasi plane.*

Proof. $\mathcal{V}^0$ is the graph of $(z_1, z_3) \mapsto v_0(z_1, z_3)$ and $\mathcal{V}^1$ is the graph of $(z_1, z_3) \mapsto v_1(z_1, z_3)$. Then let us denote for each $t \in \mathbb{D}(0, 10^6)$: $v_t(z_1, z_3) = (1-t) \cdot v_0(z_1, z_3) + t \cdot v_1(z_1, z_3)$. Let us call $\mathcal{V}_t$ the graph of v_t. Then it is clear that $\mathcal{V}^0 = \mathcal{V}_0$ and $\mathcal{V}^1 = \mathcal{V}_1$. According to Lemma 3.2.2, we have $\mathrm{diam}(\mathrm{pr}_2(U_j)) \leq 10^{-11} \cdot \mathrm{dist}(U_j, \mathbb{D} \times \partial\mathbb{D} \times \mathbb{D})$

and then $\mathcal{V}_t \subset \mathbb{D}^3$ for every $t \in \mathbb{D}(0, 10^6)$. Besides, for each $t \in \mathbb{D}(0, 10^6)$, for each $(z_1, z_3) \in \mathbb{D}^2$, for each $(z_1', z_3') \in \mathbb{D}^2$, we have :

$$|v_t(z_1, z_3) - v_t(z_1', z_3')| \leq |1 - t| \cdot |v_0(z_1, z_3) - v_0(z_1', z_3')| + |t| \cdot |v_1(z_1, z_3) - v_1(z_1', z_3')|$$

Then $|v_t(z_1, z_3) - v_t(z_1', z_3')| \leq 2 \cdot 10^6 \cdot 10^{-11} < 10^{-4}$ and $\mathcal{V}_t$ is a $(1, 3)$-quasi plane. $\square$

Lemma 4.3.4. *Let $\mathcal{V}^0$ and $\mathcal{V}^1$ be two $(1,3)$-quasi planes and a holomorphic family $(\mathcal{V}_t)_{t\in\mathbb{D}(0,10^6)}$ such that $\mathcal{V}^0 = \mathcal{V}_0$, $\mathcal{V}^1 = \mathcal{V}_1$ and $\mathcal{V}_t$ is a $(1,3)$-quasi plane for $t \in \mathbb{D}(0, 10^6)$. Let $\mathcal{W}$ be a 3-folded $(2,3)$-surface. Then $|\mathrm{fold}(\mathcal{V}^1 \cap \mathcal{W}) - \mathrm{fold}(\mathcal{V}^0 \cap \mathcal{W})| < 10^{-6}$.*

Proof. This is a consequence of the Cauchy inequality applied to the function $t \mapsto \mathrm{fold}(\mathcal{V}^t \cap \mathcal{W})$ defined on $\mathbb{D}(0, 10^6)$. It is holomorphic by the Implicit Function Theorem and its image has its diameter bounded by 1 because $\mathrm{Fold}(\mathcal{W}) \subset \mathbb{D}$. Then by the Cauchy inequality its derivative is smaller than 10^{-6}. Then we have that $|\mathrm{fold}(\mathcal{V}^1 \cap \mathcal{W}) - \mathrm{fold}(\mathcal{V}^0 \cap \mathcal{W})| < 1 \cdot 10^{-6} = 10^{-6}$ $\square$

The following proposition is important because it says that the image under g_j of a 3-folded $(2,3)$-surface is a concentrated folded surface. We use it to prove Proposition 4.3.6 and we will also use it in Section 5.

Proposition 4.3.5. *Let $1 \leq j \leq 4$. Let $\mathcal{W}$ be a 3-folded $(2,3)$ surface such that $\mathrm{diam}(\mathrm{pr}_1(\mathcal{W})) \leq 10^{-8}$. Then $g_j(\mathcal{W} \cap U_j)$ is a concentrated 3-folded $(2,3)$ surface for any $f \in \mathcal{F}$.*

Proof. Let $\mathcal{V}$ and $\mathcal{V}'$ be two $(1,3)$-quasi planes. Then the sets $f^{2qr}(\mathcal{V} \cap g_j(\mathbb{D}^3))$ and $f^{2qr}(\mathcal{V}' \cap g_j(\mathbb{D}^3))$ are $(1,3)$-quasi planes included in U_j. The sets $\Gamma = f^{2qr}(\mathcal{V} \cap g_j(\mathbb{D}^3)) \cap \mathcal{W}$ and $\Gamma' = f^{2qr}(\mathcal{V}' \cap g_j(\mathbb{D}^3)) \cap \mathcal{W}$ are 3-folded curves. According to Lemmas 4.3.3 and 4.3.4, $|\mathrm{fold}(\Gamma') - \mathrm{fold}(\Gamma)| < 10^{-6}$. The result of Lemma 4.3.2 implies that we have $\mathrm{fold}(\mathcal{V} \cap g_j(\mathcal{W})) \in \mathbb{D}(\frac{1}{\mu^{2qr}}\mathrm{fold}(\Gamma) - c_j, 2 \cdot 10^{-6})$ and the analogous for $\mathcal{V}'$. This implies that $\mathrm{diam}(\mathrm{Fold}(g_j(\mathcal{W}))) \leq 2 \cdot 2 \cdot 10^{-6} + 2 \cdot 10^{-6} < 10^{-5}$. Then $g_j(\mathcal{W})$ is a concentrated 3-folded $(2,3)$-surface. $\square$

Corollary 4.3.6. *Let $\mathcal{W}$ be a concentrated 3-folded $(2,3)$-surface such that such that $\mathrm{diam}(\mathrm{pr}_1(\mathcal{W})) \leq 10^{-8}$ and $\mathrm{Fold}(\mathcal{W}) \subset \mathbb{D}(0, \frac{1}{10})$. Then there exists $j \in \{1, 2, 3, 4\}$ such that $g_j(\mathcal{W} \cap U_j)$ is a concentrated 3-folded $(2,3)$-surface (such that $\mathrm{diam}(\mathrm{pr}_1(g_j(\mathcal{W} \cap U_j)) \leq 10^{-8}$) and $\mathrm{Fold}(g_j(\mathcal{W} \cap U_j)) \subset \mathbb{D}(0, \frac{1}{10})$.*

Proof. Let Γ be a 3-folded curve included in $\mathcal{W}$. According to Proposition 3.2.6, there exists $j \in \{1, 2, 3, 4\}$ such that $\frac{1}{\mu^{2qr}}\mathrm{fold}(\Gamma) - c_j \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$. According to Lemma 4.3.2 we have $\mathrm{fold}(g_j(\Gamma)) \in \mathbb{D}(\frac{1}{\mu^{2qr}}\mathrm{fold}(\Gamma) - c_j, 2 \cdot 10^{-6})$. According to Lemma 4.3.5, $g_j(\mathcal{W})$ is concentrated. Then $\mathrm{Fold}(g_j(\mathcal{W} \cap U_j))$ is included in $\mathbb{D}(\frac{1}{\mu^{2qr}}\mathrm{fold}(\Gamma) - c_j, 2 \cdot 10^{-6} + 10^{-5}) \subset \mathbb{D}(0, \frac{1}{10})$ for $f \in \mathcal{F}$. The proof is done. $\square$

Proof of Proposition 4.3.1. By iteration of Corollary 4.3.6 there exists a sequence $(j_n)_{n\geq 1}$ of digits in $\{1, 2, 3, 4\}$ such that the sequence $(\mathcal{W}_n)_{n\geq 0}$ defined by $\mathcal{W}_0 = \mathcal{W}$ and $\mathcal{W}_{n+1} = g_{j_n}(\mathcal{W}_n \cap U_{j_n})$ is a sequence of concentrated 3-folded $(2,3)$-surfaces with $\mathrm{Fold}(\mathcal{W}^n) \subset \mathbb{D}(0, \frac{1}{10}) \subset \mathbb{D}$ and $\mathrm{diam}(\mathrm{pr}_1(\mathcal{W}^n)) \leq 10^{-8}$ for every $n \geq 1$. We define for every $n \geq 1$ $\tilde{\mathcal{W}}_n = f^n(\mathcal{W}_n) \subset \mathcal{W}_0$. We have for every $n \geq 1$ the inclusions $\tilde{\mathcal{W}}_{n+1} \subset \tilde{\mathcal{W}}_n \subset \tilde{\mathcal{W}}_0$. The sequence $(\tilde{\mathcal{W}}_n)_{n\geq 1}$ is a decreasing sequence of compact sets. $\tilde{\mathcal{W}}_\infty = \bigcap_{n\geq 1} \tilde{\mathcal{W}}_n$ is a connected compact set. Since $\mathcal{W}_n$ is a 3-folded $(2,3)$-surface for every $n \geq 0$, there exists $z_n \in \mathcal{W}_n$ and a non zero vector $v_n \in T_{z_n}\mathcal{W}_n$ such that $v_n \in C^u$. We denote for every $n \geq 1$, $\tilde{z}_n = f^n(z_n) \in \tilde{\mathcal{W}}_n \subset \tilde{\mathcal{W}}^0$ and $\tilde{v}_n$

an unitary vector parallel to $D_{z_n} f^n(v_n)$. We have $\tilde{v}_n \in T_{\tilde{z}_n}\tilde{\mathcal{W}}_n$. Taking a subsequence if necessary we can suppose $\tilde{z}_n \to \tilde{z}_\infty \in \tilde{\mathcal{W}}_\infty$ and $\tilde{v}_n \to \tilde{v}_\infty$ for some point $\tilde{z}_\infty$ and some vector $\tilde{v}_\infty$. Clearly $\tilde{z}_\infty \in \mathcal{W}$. By construction the whole forward orbit of $\tilde{\mathcal{W}}_\infty$ is in $\mathbb{D}^3$. Then, necessarily, $\tilde{z}_\infty$ is included in some unstable manifold $W^u(\kappa_f)$ for some point κ_f of the horseshoe $\mathcal{H}_f$. By construction $\tilde{v}_\infty \in T_{\tilde{z}_\infty}\mathcal{W}$ and $\tilde{v}_\infty \in \bigcap_{n\geq 0} Df^n(C^u(z_n)) = T_{\tilde{z}_\infty} W^u_{\text{loc}}(\tilde{z}_\infty)$. This proves that $\mathcal{W}$ has a point of tangency with the unstable manifold of κ_f.

□

5 Initial heteroclinic tangency

In this section, we show that :

1. for every sufficiently small values of b and σ, we can find $c_1 = c_1(b, \sigma)$ such that $f_1 = f_{c_1,b,\sigma}$ has a point of heteroclinic tangency τ between $W^s(\alpha_{f_1})$ and $W^u(\phi_{f_1})$ (where ϕ_{f_1} is a periodic point in $\mathcal{H}_{f_1}$)
2. we can take iterates of a neighborhood of τ inside $W^s(\alpha_{f_1})$ under f_1^{-1} in order to create a concentrated 3-folded $(2, 3)$-surface inside $W^s(\alpha_{f_1})$

5.1 Initial tangency

Remind we defined the parameter disk $\mathcal{C}$ and the integer m in Proposition 2.1.4.

Proposition 5.1.1. *Reducing $\mathcal{C}$ if necessary, there exist $0 < b_1 < b_0$, $0 < \sigma_1 < \sigma_0$ and an integer s such that for every $0 < |b| < b_1$ and $0 \leq |\sigma| < \sigma_1$:*

1. *for every u-curve $\mathcal{U}$, $f_0^{s+m}(\mathcal{U})$ is a degree 2 curve over the first coordinate*
2. *for every holomorphic family of u-curves $(\mathcal{U}_c)_{c\in\mathcal{C}}$, there exists $c_1 = c_1(b, \sigma)$ such that there exists a quadratic tangency τ between $W^s(\alpha_{f_1})$ and $\mathcal{U}_{c_1}$ where $f_1 = f_{c_1,b,\sigma}$ and $\tau \in \mathcal{U}_{c_1}$. Moreover every iterate of $f_1^{s+m}(\tau)$ under f_1 is in $\mathbb{D}(\alpha_{c_0}, 10^{-10}|w_{m-1}|) \times \mathbb{C}^2$ (where the constant w_{m-1} was introduced in Definition 2.4.3) and the mapping $(b, \sigma) \mapsto \tau$ is holomorphic.*

Proof. In the following, we are going to reduce several times the bounds σ_0 and b_0 into bounds σ_1 and b_1 to satisfy the two items. We begin by taking $\sigma_1 = \frac{\sigma_0}{2}$ and $b_1 = \frac{b_0}{2}$. Let us take any u-curve $\mathcal{U}$ or any holomorphic family of u-curves $(\mathcal{U}_c)_{c\in\mathcal{C}}$. Since $\mathbb{D}'$ intersects the Julia set of p_c for every $c \in \mathcal{C}$, it is possible to find an integer s and a holomorphic map β_{-s} defined on $\mathcal{C}$ such that for every $c \in \mathcal{C}$, we have $p_c^s(\beta_{-s}(c)) = 0$. In particular, for the parameter c_0, we have $p_{c_0}^{s+m}(\beta_{-s}(c_0)) = \alpha_{c_0}$ and the map defined on $\mathcal{C}$ which sends c to $p_c^{s+m}(\beta_{-s}(c)) - \alpha_c$ is an open set which contains 0 in its interior. Then reducing sufficiently the bounds $0 \leq |\sigma| < \sigma_1$ and $0 < |b| < b_1$ for σ and b, we have that for any $c \in \mathcal{C}$, there exists a neighborhood of the point of $\mathcal{U}$ of first coordinate $z_1 = \beta_{-s}(c_0)$ inside $\mathcal{U}$ whose image under f_0^s is of the form $\{(z_1, u^2(z_1), u^3(z_1)), z_1 \in \mathbb{D}(0, \rho)\}$. Indeed, $\beta_{-s}(c), p_c(\beta_{-s}(c)), \cdots, p_c^{s-1}(\beta_{-s}(c))$ are not critical points of p_c. The image of the curve $\{(z_1, u^2(z_1), u^3(z_1)), z_1 \in \mathbb{D}(0, \rho)\}$ under f_0 is the curve $\{(p_c(z_1)+bu^2(z_1)+\sigma u^3(z_1)(u^1(z_1)-\alpha_{c_0}), z_1, \lambda z_1+\mu u^3(z_1)+\nu), z_1 \in \mathbb{D}(0, \rho)\}$. Then it has a point of quadratic tangency with the foliation $z_1 = C^{st}$ if b and σ are sufficiently small. Indeed, reducing b_1 and σ_1 if necessary, by continuity the derivative of the first coordinate $p_c(z_1) + bu^2(z_1) + \sigma u^3(z_1)(u^1(z_1) - \alpha_{c_0})$ vanishes for some value $\overline{z_1} \in \mathbb{D}(0, \rho)$. We can iterate this curve $(m - 1)$ times under f_0. Since p_c has no other critical point, this will still be a degree 2 curve upon the first coordinate, reducing b_1 and σ_1 if necessary.

In the case of a holomorphic family of u-curves $(\mathcal{U}_c)_{c\in\mathcal{C}}$, there exists a neighborhood of the point of $\mathcal{U}_c$ of first coordinate $z_1 = \beta_{-s}(c)$ inside $\mathcal{U}_c$ which is sent under f_0^s

on a u-curve $\{(z_1, u_c^2(z_1), u_c^3(z_1)), z_1 \in \mathbb{D}(0,\rho)\}$ but by the Cauchy inequality $(u_c^2)'$ and $(u_c^3)'$ are uniformly (relatively to c) bounded and the conclusion is the same. In particular, this proves the first item of the result.

We call Tan_{f_0} the first coordinate of the point of vertical tangency. The map defined on $\mathcal{C}$ which sends c to $p_c^{s+m}(\beta_{-s}(c)) - \alpha_c$ is an open set which contains 0 in its interior. Let us denote by $2l$ the distance of 0 to the image of $\partial\mathcal{C}$. Reducing b_1 and σ_1 another time if necessary, by continuity this implies that $\{\mathrm{Tan}_{f_0} - \alpha_c, c \in \partial\mathcal{C}\}$ is a curve in the plane. We have that 0 is in a bounded connected component of its complement and at distance at least l from $\{\mathrm{Tan}_{f_0} - \alpha_c, c \in \partial\mathcal{C}\}$.

Notation 5.1.2. *Let $\kappa_f \in \mathcal{H}_f$ for $f \in \mathcal{F}$. We denote by $W^s_{loc}(\kappa_f)$ the component of $W^s(\kappa_f) \cap \mathbb{D}^3$ which contains κ_f and by $W^u_{loc}(\kappa_f)$ the component of $W^u(\kappa_f) \cap \mathbb{D}^3$ which contains κ_f.*

Lemma 5.1.3. *Reducing b_1 and σ_1 if necessary, $W^s_{loc}(\alpha_{f_0})$ is a graph over $(z_2, z_3) \in \mathbb{D}^2$ included in $\mathbb{D}(\alpha_c, \min(10^{-11} \cdot |w_{m-1}|, 10^{-1} \cdot l)) \times \mathbb{D}^2$.*

Proof. For $\sigma = 0$, $W^s(\alpha_{f_0})$ is the product of $W^s(\alpha_H)$ by the z_3 axis where H is the Hénon map $H : (z_1, z_2) \mapsto (p_c(z_1) + bz_2, z_1)$. Moreover, for every $\epsilon > 0$, we can reduce b_1 such that if $|b| < b_1$, then the cone $C^{ss,\epsilon}$ centered at e_2 of opening $\frac{\epsilon}{2}$ is H^{-1}-invariant in $\mathbb{D}^2$. This implies that $W^s_{\mathrm{loc}}(\alpha_H)$ is a ss-curve included in $\mathbb{D}(\alpha_c, \epsilon) \times \mathbb{D}$ since all its tangent vectors lie in $C^{ss,\epsilon}$. Then for $\sigma = 0$, the skew-product structure implies that $W^s_{\mathrm{loc}}(\alpha_{f_0})$ is a $(2,3)$-surface included in $\mathbb{D}(\alpha_c, \epsilon) \times \mathbb{D} \times \mathbb{D}$ which is the product of $W^s_{\mathrm{loc}}(\alpha_H)$ by the z_3 axis. Then, by continuity, it is possible to reduce σ_1 such that for every f_0 with $0 < |b| < b_1$ and $0 \leq |\sigma| < \sigma_1$, $W^s_{loc}(\alpha_{f_0})$ is a graph over $(z_2, z_3) \in \mathbb{D}^2$ included in $\mathbb{D}(\alpha_c, \epsilon) \times \mathbb{D} \times \mathbb{C}$. We take $\epsilon = \min(10^{-11} \cdot |w_{m-1}|, 10^{-1} \cdot l)$. ☐

The set $\{\mathrm{Tan}_{f_0}, c \in \partial\mathcal{C}\}$ is such that α_c is in a bounded connected component of its complement. In particular, there is a parameter $\overline{c} \in \mathcal{C}$ such that Tan_{f_0} belongs to $\mathbb{D}(\alpha_{\overline{c}}, \frac{l}{2})$ (for this parameter $\overline{c}$, $f_0^{s+m}(\mathcal{U}_{\overline{c}}) \cap (\mathbb{D}(\alpha_{\overline{c}}, \frac{l}{2}) \times \mathbb{D}^2)$ is not the union of two graphs upon $z_1 \in \mathbb{D}(\alpha_{\overline{c}}, \frac{l}{2})$ and for $c \in \partial\mathcal{C}$, $f_0^{s+m}(\mathcal{U}_c)$ is the union of two graphs over $z_1 \in \mathbb{D}(\alpha_c, \frac{l}{2})$. According to Lemma 5.1.3, $W^s_{loc}(\alpha_{f_0})$ is a graph over $(z_2, z_3) \in \mathbb{D}^2$ included in $\mathbb{D}(\alpha_c, \frac{l}{2}) \times \mathbb{D}^2$. The following lemma is the analogous in dimension 3 of Proposition 8.1 of [12]. The proof is essentially the same and relies on the continuity of the intersection index of properly intersecting analytic sets of complementary dimensions.

Lemma 5.1.4. *Let $(\Gamma_c)_{c \in \mathcal{C}}$ be a holomorphic family of curves of degree 2 over the first coordinate. We assume that:*
(i) there exists a compact subset $\mathcal{C}' \subset \mathcal{C}$ such that if $c \in \mathcal{C} - \mathcal{C}'$, Γ_c is the union of 2 graphs over $z_1 \in \mathbb{D}(\alpha_c, \frac{l}{2})$
(ii) there exists $\overline{c} \in \mathcal{C}$ such that $\Gamma_{\overline{c}}$ is not the union of 2 graphs over $z_1 \in \mathbb{D}(\alpha_c, \frac{l}{2})$
Then, if $(\mathcal{V}_c)_{c \in \mathcal{C}}$ is any holomorphic family of graphs over $(z_2, z_3) \in \mathbb{D}^2$ contained in $\mathbb{D}(\alpha_c, \frac{l}{2}) \times \mathbb{D}^2$, there exists $c_1 \in \mathcal{C}$ such that Γ_{c_1} and $\mathcal{V}_{c_1}$ admit a point of tangency.

We apply the previous lemma, taking the family $(\Gamma_c)_{c \in \mathcal{C}} = (f_0^{s+m}(\mathcal{U}_c) \cap \mathbb{D}^3)_{c \in C}$ as curves and the family of stable manifolds $W^s_{loc}(\alpha_{f_0})$ as graphs over $(z_2, z_3) \in \mathbb{D}^2$. We can conclude that there is a parameter c_1 such that there is a quadratic tangency $f_1^{s+m}(\tau)$ between $W^s_{\mathrm{loc}}(\alpha_{f_1})$ and $f_1^{s+m}(\mathcal{U}_{c_1})$ where $\tau \in \mathcal{U}_{c_1}$ for $f_1 = f_{c_1,b,\sigma}$. Then τ is a point of tangency between $W^s(\alpha_{f_1})$ and $\mathcal{U}_{c_1}$. Since $W^s_{\mathrm{loc}}(\alpha_{f_1})$ is included in $\mathbb{D}(\alpha_{c_1}, \min(10^{-10}|w_{m-1}|)) \times \mathbb{D} \times \mathbb{C}$, we have that all the iterates of $f_1^{s+m}(\tau)$ under f_1 are in $\mathbb{D}(\alpha_{c_1}, 10^{-11} \cdot |w_{m-1}|) \times \mathbb{C}^2$. Reducing $\mathcal{C}$ if necessary, we have that $\alpha_{c_1} \in \mathbb{D}(\alpha_{c_0}, 10^{-11}|w_{m-1}|)$ for every $c \in \mathcal{C}$ and then all the iterates of $f_1^{s+m}(\tau)$ under f_1 are in $\mathbb{D}(\alpha_{c_0}, 10^{-10}|w_{m-1}|) \times \mathbb{C}^2$. The map $(b, \sigma) \mapsto c_1$ is holomorphic according to the

Implicit Function Theorem and then $(b, \sigma) \mapsto \tau$ is holomorphic too. This shows item 2 and ends the proof of the proposition. $\square$

In the next proposition, we show that the tangencies we created in the previous result are generically unfolded. Beware that in the next result, the map f_1 is associated to a family of u-curves $(\mathcal{U}_c)_{c\in\mathcal{C}}$ which can be distinct from the other family $(\mathcal{U}'_f)_{f\in\mathcal{F}'}$.

Definition 5.1.5. *Let $(U_f)_{f\in\mathcal{F}}$ be a holomorphic family of u-curves and $(S_f)_{f\in\mathcal{F}}$ be a holomorphic family of s-surfaces. We suppose that for $f \in \mathcal{F}$, there is a point of quadratic tangency between U_f and S_f. We say that this tangency is generically unfolded if there exists a one-dimensional holomorphic family $(f^t)_{t\in\mathbb{D}}$ of polynomial automorphisms and a holomorphic family of local biholomorphisms $(\Psi_t)_{t\in\mathbb{D}}$ with :*

1. *$f^0 = f$ and $f^t \in \mathcal{F}$ for every $t \in \mathbb{D}$*
2. *$\Psi_t(S_{f^t})$ is a vertical plane $\{z_1 = C^{st}\}$ where C^{st} does not depend on $t \in \mathbb{D}$*
3. *if we denote by $\tan_t$ the first coordinate of the point of tangency of $\Psi_t(U_{f^t})$ with $\{z_1 = C^{st}\}$ (there exists exactly one such point because the tangency is quadratic), then $|\frac{d\tan_t}{dt}|$ is uniformly bounded from below by a strictly positive constant for $t \in \mathbb{D}$*

Proposition 5.1.6. *Reducing $\mathcal{C}$, b_1 and σ_1 if necessary, for every $0 < |b| < b_1$ and $0 \leq |\sigma| < \sigma_1$, for every holomorphic family of u-curves $(\mathcal{U}_c)_{c\in\mathcal{C}}$, there exists a neighborhood $\mathcal{F}' \Subset \mathcal{F}$ of the map $f_1 = f_{c_1(b,\sigma),b,\sigma}$ defined in item 2 of Proposition 5.1.1 associated to $(\mathcal{U}_c)_{c\in\mathcal{C}}$ such that : for every holomorphic family of u-curves $(\mathcal{U}'_f)_{f\in\mathcal{F}'}$, if $f \in \mathcal{F}'$ has a point of tangency τ' between $W^s(\alpha_f)$ and $\mathcal{U}'_f$ such that $f^{s+m}(\tau') \in W^s_{loc}(\alpha_f)$, then the tangency τ' is generically unfolded. In particular, the tangency τ obtained in item 2 of Proposition 5.1.1 is generically unfolded.*

Proof. For every $f \in \mathcal{F}$, after a holomorphic change of coordinates Ψ_f, we have that $W^s_{\mathrm{loc}}(\alpha_f)$ is the plane $\{z_1 = \alpha_{c_0}\}$. We first show the result for the map f_1. We work in the one-dimensional family $(f_0)_{c\in\mathcal{C}} = (f_{c,b,\sigma})_{c\in\mathcal{C}}$ where b and σ are fixed and we take a holomorphic family of u-curves $(\mathcal{U}'_f)_{f\in\mathcal{F}'}$. It is a consequence of Lemma 5.1.3 that Ψ_{f_0} tends to Id when b and σ tend to 0. When b and σ tend to 0, the curve $\Psi_{f_0} \circ f_0^{s+m}(\mathcal{U}'_{f_0})$ tends to a curve of degree 2 over z_1 of the form $\{(p_c^{s+m}(z_1), z_1, v(z_1)) : z_1 \in \mathbb{D}(0, \rho)\}$ where v is holomorphic. We call $\tan_c$ the first coordinate of the point of vertical tangency of $\Psi_{f_0} \circ f_0^{s+m}(\mathcal{U}'_{f_0})$. We have $p^m_{c_0}(0) = \alpha_{c_0}$, $(p^m_{c_0})'(0) = 0$, $(p^m_{c_0})''(0) \neq 0$ and at $c = c_0$, we have : $\frac{d}{dc}(p^m_c(0)) \neq 0$ (see Proposition 2.1.4). Moreover, c_1 tends to c_0 when b and σ tend to 0. This implies that for sufficiently small fixed values of b and σ, the map $c \mapsto \tan_c$ is such that $\frac{d\tan_c}{dc} \neq 0$ for every $c \in \mathcal{C}$ (reducing $\mathcal{C}$ if necessary). Since the estimates on the derivatives of the u-curves $(\mathcal{U}'_f)_{f\in\mathcal{F}}$ are uniform, we can reduce uniformly b_1 and σ_1 so that this inequality is true no matter the choice of $(\mathcal{U}'_f)_{f\in\mathcal{F}}$. Then, by continuity, there exists some new neighborhood $\mathcal{F}' \Subset \mathcal{F}$ of f_1 such that every $f \in \mathcal{F}'$ belongs to a one-dimensional family $(f^t)_{t\in\mathbb{D}}$ such that : $f^0 = f$, $f^t \in \mathcal{F}'$ and $\frac{d\tan_c}{dt} \neq 0$ ($t \in \mathbb{D}$). This implies in particular that if $f \in \mathcal{F}'$ has a point of tangency τ' between $W^s(\alpha_f)$ and $\mathcal{U}'_f$ such that $f^{s+m}(\tau') \in W^s_{\mathrm{loc}}(\alpha_f)$, then the tangency $f^{s+m}(\tau')$ is generically unfolded and then also τ'. The proof is over. $\square$

5.2 A transversality result

From now on, we are going to construct from this initial heteroclinic tangency a 3-folded $(2, 3)$-surface inside a stable manifold with its fold inside $\mathbb{D}(0, \frac{1}{10})$. In this subsection, we begin by proving that $W^s(\alpha_{f_1})$ has some special geometry. We start by choosing a periodic point $\phi_{f_1} \in \mathcal{H}_{f_1}$. The main point is that its third coordinate is in $\mathbb{D}(0, \frac{1}{10} - 10^{-4})$, which will be used later in the proof of Proposition 5.3.10.

Lemma 5.2.1. *For every $c \in \mathcal{C}$, $0 < |b| < b_1$, $0 \leq |\sigma| < \sigma_1$, there exists a periodic point ϕ_{f_0} inside $\mathcal{H}_{f_0}$ such that $\mathrm{pr}_3(\phi_{f_0}) \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$.*

Proof. According to Proposition 3.2.7, the ss-curve $\{0\}\times\mathbb{D}\times\{0\}$ intersects the unstable set of the horseshoe $\mathcal{H}_{f_0}$. Then, there exists a u-curve $\mathcal{L}$ which is a component of an unstable manifold of a point of $\mathcal{H}_{f_0}$ which has a point of third coordinate 0. Since $\mathcal{L}$ is a u-curve, it is included in $\mathbb{D}^2 \times \mathbb{D}(0, \frac{1}{10} - 10^{-4})$. It intersects $W^s_{\mathrm{loc}}(\alpha_{f_0})$ in a point of $\mathcal{H}_{f_0}$ of third coordinate in $\mathbb{D}(0, \frac{1}{10} - 10^{-4})$. Since $\mathcal{H}_{f_0}$ is a horseshoe, its periodic points are dense and so there is a periodic point $\phi_{f_0} \in \mathcal{H}_{f_0}$ with $\mathrm{pr}_3(\phi_{f_0}) \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$. $\square$

Notation 5.2.2. *We fix a periodic point of the horseshoe ϕ_{f_0} with $\mathrm{pr}_3(\phi_{f_0}) \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$. Reducing $\mathcal{F}$ if necessary, we have $\mathrm{pr}_3(\phi_f) \in \mathbb{D}(0, \frac{1}{10} - 10^{-4})$ for $f \in \mathcal{F}$.*

We now choose the parameter b in function of the parameter σ :

Notation 5.2.3. *In the following, we will take $b = b(\sigma) = \sigma^2$. For technical reasons, we reduce σ_1 such that $\sigma_1 < \sqrt{b_1}$.*

In particular, we can use Proposition 5.1.1 : there exists a map $\sigma \mapsto c_1(\sigma)$ such that there is a heteroclinic tangency τ between the local unstable manifold $\mathcal{U}_{f_1}$ of ϕ_{f_1} and the stable manifold $W^s(\alpha_{f_1})$ of α_{f_1}. Moreover, $\sigma \mapsto \tau$ is holomorphic. We have $D(f_1^s)_\tau \cdot (0,0,1) = \mu^s(\xi_1(\sigma), \xi_2(\sigma), 1)$ with $\xi_1(\sigma) = O(\sigma)$ and $\xi_2(\sigma) = O(\sigma)$. The maps $\sigma \mapsto \mathrm{pr}_1(f_1^s(\tau)) - \beta_0, \cdots, \sigma \mapsto \mathrm{pr}_1(f_1^{s+m-1}(\tau)) - \beta_{m-1}$ are holomorphic (remind that the constants β_i were defined in 2.4.1). Since they vanish when $\sigma = 0$, they are also $O(\sigma)$. The map $\sigma \mapsto b(\sigma)$ is $O(\sigma^2)$ because $b(\sigma) = \sigma^2$. Then the differentials at $f_1^s(\tau), \cdots, f_1^{s+m-1}(\tau)$ verify the conditions of Corollary 2.4.6 so from Corollary 2.4.6 we immediately get the following (remind that w_{m-1} was defined in Definition 2.4.3) :

Lemma 5.2.4. *We have : $D(f_1^{s+m})_\tau \cdot (0,0,1) = (\zeta_1(\sigma), \zeta_2(\sigma), \zeta_3(\sigma))$ where ζ_1, ζ_2, ζ_3 are holomorphic, $\zeta_1(\sigma) = w_{m-1} \cdot \sigma + O(\sigma^2)$, $\zeta_2(0) = 0 + O(\sigma)$ and $\zeta_3(\sigma) = \mu^m + O(\sigma)$.*

Notation 5.2.5. *We reduce σ_1 so that $|\zeta_2(\sigma)| < 1$, $|\zeta_3(\sigma) - \mu^m| < \frac{1}{10}|\mu^m|$ and $\sigma_1 < \frac{1}{10^{10}} \cdot |w_{m-1}|$. From now on, we fix $\sigma = \frac{\sigma_1}{2}$, $b = b(\sigma) = \sigma^2$, $c = c_1(\sigma)$, the associated map $f_1 = f_{c_1(\sigma), b(\sigma), \sigma}$.*

Lemma 5.2.6. *For every vector v^0 such that $|v_1^0 - w_{m-1}\sigma| < \frac{1}{10}|w_{m-1}||\sigma|$, $|v_2^0| < 1$ and $|v_3^0 - \mu^m| < \frac{1}{10}|\mu|^m$, at any point of $W^s_{loc}(\alpha_{f_1}) \cap \mathbb{D}^3$, v^0 is transverse to $W^s_{loc}(\alpha_{f_1})$.*

Proof. Let ψ^0 be a point of $W^s_{\mathrm{loc}}(\alpha_{f_1})$ and let us consider the sequence of points defined by $\psi^n = f_1^n(\psi^0)$. According to Lemma 5.1.3, for every $n \geq 0$, ψ^n is in $\mathbb{D}(\alpha_{c_0}, 10^{-10}|w_{m-1}|) \times \mathbb{C}^2$. Then for every $n \geq 1$, the differential at ψ^n of f_1 is of the form :

$$I_n = \begin{pmatrix} m_n & b(\sigma) & \sigma(z_1 - \alpha_{c_0}) \\ 1 & 0 & 0 \\ \lambda & 0 & \mu \end{pmatrix}$$

where $b(\sigma) = \sigma^2$ and $|m_n| > 2$ for every $n \geq 1$. Since $\sigma_1 < \frac{1}{10^{10}}|w_{m-1}|$ we have $|b| < 10^{-10}|w_{m-1}||\sigma|$. We denote $v^n = (I_n \cdot ... \cdot I_1)(v^0)$ for every $n \geq 1$. Let us show that there exists an integer i such that $|v_1^i| \geq |v_3^i|$. Let us suppose this is false and we show by induction the following properties for $n \geq 0$:

1. $1 \geq |v_1^{n+1}| > \frac{3}{2}|v_1^n| \geq \frac{1}{100}|w_{m-1}||\sigma|$
2. $|v_2^n| \leq 1$
3. $|v_3^{n+1}| \leq |v_3^n| \leq 1$

For $n = 0$, item 2 is satisfied, $|v_3^n| \leq 1$ and $|v_1^0| \geq \frac{1}{100}|w_{m-1}||\sigma|$. Since $v_1^1 = m_1 v_1^0 + bv_2^0 + \sigma(\xi_1^0 - \alpha_{c_0})$, $|bv_2^0| < |b| < 10^{-10}|w_{m-1}||\sigma|$ and $|\sigma(\psi_1^0 - \alpha_{c_0})| < 10^{-10}|w_{m-1}||\sigma|$, we have $|v_1^1| > \frac{3}{2}|v_1^0|$. Moreover $|v_1^n| < |v_3^n| < 1$ by hypothesis. Them item 1 is true. We have : $|v_3^1| < |\lambda||v_1^0| + |\mu||v_3^0| < |\lambda||v_3^0| + |\mu||v_3^0| < |v_3^0|$. Then item 3 is true and the induction procedure is true for $n = 1$.

Let us suppose it is true for some n. Then $|v_2^{n+1}| = |v_1^n| \leq 1$. Moreover $|v_3^{n+1}| < |\lambda||v_1^n| + |\mu||v_3^n| < |\lambda||v_3^n| + |\mu||v_3^n| < |v_3^n|$. We have $v_1^{n+1} = m_n v_1^n + b v_2^n + \sigma(\psi_1^n - \alpha_{c_0})$. Since $|m_n| > 2$, $|v_2^n| < |b| < 10^{-10}|w_{m-1}||\sigma|$ and $|\sigma(\psi_1^n - \alpha_{c_0})| < 10^{-10}|w_{m-1}||\sigma|$, we have $|v_1^{n+1}| > \frac{3}{2}|v_1^n|$. We still have $|v_1^{n+1}| < |v_3^{n+1}| < 1$ by hypothesis. Then for every n, we have $\frac{|v_1^{n+1}|}{|v_3^{n+1}|} \geq \frac{3}{2}\frac{|v_1^n|}{|v_3^n|}$. This implies a contradiction : there must exist some integer i such that $|v_1^i| \geq |v_3^i|$. Then v^i is transverse to $W^s_{\mathrm{loc}}(\alpha_{f_1})$ since $W^s_{\mathrm{loc}}(\alpha_{f_1})$ is a $(2,3)$-surface and this implies that v^0 itself is transverse to $W^s_{\mathrm{loc}}(\alpha_{f_1})$. □

We can now state the main result of this subsection :

Proposition 5.2.7. *The vector* $(0,0,1)$ *is transverse to* $W^s(\alpha_{f_1})$ *at* τ.

Proof. From Lemmas 5.2.4 and 5.2.6, we know that the image under f_1^{s+m} of a neighborhood of τ in $W^s(\alpha_{f_1})$ is transverse to $D(f_1^{s+m})_\tau \cdot (0,0,1)$. Applying f_1^{-s-m}, this immediately implies the result.

□

5.3 Orientation of the fold of $W^s(\alpha_f)$

In this subsection, we take iterates of the initial tangency τ under f_1^{-1} in order to create a concentrated 3-folded $(2,3)$-surface inside a stable manifold.

We have that $W^u_{\mathrm{loc}}(\phi_{f_1})$ is a graph $\{(z_1, u^2(z_1), u^3(z_1) : z_1 \in \mathbb{D}\}$ over $z_1 \in \mathbb{D}$. We consider the biholomorphism Ψ defined by $\Psi(z_1,z_2,z_3) = (z_1, z_2 - u^2(z_1), z_3 - u^3(z_1))$ which sends $W^u_{\mathrm{loc}}(\phi_{f_1})$ into the z_1 axis. We denote $\mathcal{W}_f$ the component of $W^s(\alpha_f)$ which contains τ for $f \in \mathcal{F}'$. We denote by Tan' the subset of $\Psi(\mathcal{W}_f)$ where $\Psi(\mathcal{W}_f)$ is tangent to some line $\{z_2 = C^{st}, z_3 = C^{st}\}$ and by $\mathrm{Tan} = \Psi^{-1}(\mathrm{Tan}')$. When $f = f_1$, we denote them by Tan^0 and $(\mathrm{Tan}')^0$. We denote by $e^u(\tau)$ a tangent vector of $W^u_{\mathrm{loc}}(\phi_{f_1})$ at τ.

Lemma 5.3.1. *Tan^0 is a complex curve regular at τ of tangent vector v_{tan} at τ such that $v_{tan} \notin \mathbb{C} \cdot (0,0,1)$ and $v_{tan} \notin \mathbb{C} \cdot e^u(\tau)$.*

Proof. We are working in the projectivized tangent bundle $\mathbb{P}T\mathbb{C}^3 \simeq \mathbb{C}^3 \times \mathbb{P}^2(\mathbb{C})$ of $\mathbb{C}^3$ which is of dimension 5. The lift $\hat{\mathcal{W}_{f_1}}$ of $\mathcal{W}_{f_1}$ to $\mathbb{P}T\mathbb{C}^3$ is a complex submanifold of dimension 3. The lift of every complex curve $C_{x,y} = \Psi^{-1}(\mathbb{D} \times \{(x,y)\})$ to $\mathbb{P}T\mathbb{C}^3$ is a complex curve $\hat{C}_{x,y}$. Then $\bigcup_{(x,y)\in\mathbb{D}^2} \hat{C}_{x,y}$ is a complex submanifold of dimension 3. Moreover, according to Proposition 5.1.1, $\mathcal{W}_{f_1}$ has a point of quadratic tangency with $C_{0,0}$ and then $\hat{\mathcal{W}_{f_1}}$ is transverse to $\bigcup_{(x,y)\in\mathbb{D}^2} \hat{C}_{x,y}$. Then $\hat{\mathrm{Tan}^0} = \hat{\mathcal{W}_{f_1}} \cap \big(\bigcup_{(x,y)\in\mathbb{D}^2} \hat{C}_{x,y}\big)$ is a regular complex curve. Then its projection Tan^0 is a regular complex curve of tangent vector v_{tan} at τ. According to Lemma 5.2.7, $v_{\mathrm{tan}} \notin \mathbb{C} \cdot (0,0,1)$. By construction, $e^u(\tau)$ is a tangent vector of $W^u_{\mathrm{loc}}(\phi_{f_1})$ at τ so $v_{\mathrm{tan}} \notin \mathbb{C} \cdot e^u(\tau)$. The proof is done.

□

Notation 5.3.2. *For every $\epsilon > 0$, we denote $C^{v_{tan},\epsilon} = \{u \in \mathbb{C}^3 : |\langle u, v_{tan}\rangle| < \epsilon||u||\}$.*

Corollary 5.3.3. *For every $\epsilon > 0$, reducing the neighborhood $\mathcal{F}'$ of f_1 if necessary, there exists a real number $\eta > 0$ such that : for every $f \in \mathcal{F}'$, for every holomorphic foliation $(\mathcal{V}_{x,y})_{(x,y)}$ of a neighborhood of τ such that every $\mathcal{V}_{x,y}$ is η-close to $W^u_{loc}(\phi_f)$, the set Tan of points of $\mathcal{W}_f$ where $\mathcal{W}_f$ is tangent to some $\mathcal{V}_{x,y}$ is a complex curve regular at τ which has its tangent vectors in $C^{v_{tan},\epsilon}$.*

Proof. The intersection $\hat{\mathrm{Tan}^0} = \hat{\mathcal{W}_{f_1}} \cap \big(\bigcup_{(x,y)\in\mathbb{D}^2} \hat{C}_{x,y}\big)$ persists as a regular complex curve since the intersection is transverse. Then it is still true for its projection Tan. □

The following result is a geometric technical lemma.

Lemma 5.3.4. *There exists constants $\epsilon, \rho, \rho_1, \rho_2, \rho_3 > 0$ such that for every $t > 0$, we have the following property : for every regular complex curve Γ going through $\tau + (\mathbb{D}(0, t\rho))^3$, if Γ has all its tangent vectors in $C^{v_{tan}, \epsilon}$, then Γ is horizontal relatively to π_2 in $\tau + \mathbb{D}(0, t\rho_1)e^u(\tau) + \mathbb{D}(0, t\rho_2)(0, 1, 0) + \mathbb{D}(0, t\rho_3)(0, 0, 1)$.*

Proof. According to Lemma 5.3.1, $v_{\tan} \notin \mathbb{C} \cdot (0, 0, 1)$ and $v_{\tan} \notin \mathbb{C} \cdot e^u(\tau)$. □

Notation 5.3.5. *We fix the value of ϵ give by Lemma 5.3.4 and the associated value of η given by Lemma 5.3.3.*

Lemma 5.3.6. *There exists $t_0 > 0$ such that for every $t < t_0$, there exists an integer $j = j(t)$ such that :*

1. $f_1^{-j}\Big(\tau + \mathbb{D}(0, t\rho_1)e^u(\tau) + \partial\mathbb{D}(0, t\rho_2)(0, 1, 0) + \mathbb{D}(0, t\rho_3)(0, 0, 1)\Big) \cap \mathbb{D}^3 = \emptyset$
2. $f_1^{-j}\Big(\tau + \mathbb{D}(0, t\rho_1)e^u(\tau) + \mathbb{D}(0, t\rho_2)(0, 1, 0) + \mathbb{D}(0, t\rho_3)(0, 0, 1)\Big) \cap \partial\mathbb{D}^3 \Subset \mathbb{D} \times \partial\mathbb{D} \times \mathbb{D}$

Moreover, $j(t)$ tends to ∞ when t tends to 0. Reducing $\mathcal{F}'$, by continuity this remains true for any $f \in \mathcal{F}'$ for a given $t < t_0$.

In other words, f^{-j} realizes a crossed mapping between $\tau + \mathbb{D}(0, t\rho_1)e^u(\tau) + \partial\mathbb{D}(0, t\rho_2)(0, 1, 0) + \mathbb{D}(0, t\rho_3)(0, 0, 1)$ and $\mathbb{D}^3$ (see [14] for this notion).

Proof. We foliate $\mathbb{D}_{t,\rho_1,\rho_2,\rho_3} = \tau + \mathbb{D}(0, t\rho_1)e^u(\tau) + \mathbb{D}(0, t\rho_2)(0, 1, 0) + \mathbb{D}(0, t\rho_3)(0, 0, 1)$ by disks $\mathcal{L}_{z_1,z_3}$ parallel to the z_2 axis. Since C^{ss} is f_1^{-1}-invariant in $\mathbb{D}^3$, for every $n \geq 0$, the image of this tridisk under f_1^{-n} is foliated by the ss-curves $f_1^{-n}(\mathcal{L}_{z_1,z_3}) \cap \mathbb{D}^3$. We call the length of a ss-curve the radius of the maximal (for the inclusion) disk included in this ss-curve. For every $n \geq 0$, we denote by l_n the minimum of the lengths of all the s-curves $f_1^{-n}(\mathcal{L}_{z_1,z_3})$. For every $n \geq 0$, we denote by d_n the maximum of the diameters of the sets $\mathcal{V} \cap f_1^{-n}(\mathbb{D}_{t,\rho_1,\rho_2,\rho_3})$ where $\mathcal{V}$ varies in the set of $(2, 3)$-quasi planes. Since every vector in C^{ss} is dilated by a factor larger than $C/(2|\sigma|^2)$ (see Proposition 2.2.3 for the definition of the constant C) under f_1^{-1} with $C > 10^{10}$, we have that $l_{n+1} \geq C/(2|\sigma|^2) \cdot l_n$. Since C^{ss} is f_1^{-1}-invariant, for every $(2, 3)$-quasi plane $\mathcal{V}$, $f_1(\mathcal{V}) \cap \mathbb{D}^3$ is a $(1, 3)$-quasi plane. Every tangent vector u to the $(2, 3)$-surface $f_1(\mathcal{V}) \cap \mathbb{D}^3$ is of the form $u = u^1 \cdot (1, 0, 0) + u^2 \cdot (0, 1, 0) + u^3 \cdot (0, 0, 1)$ with $|u^2| \leq 10^{-4} \max(|u^1|, |u^3|)$. Then the vector u is dilated under f_1^{-1} by no more than $\max(1, 10^{-4}C/(|\sigma|^2), 2) \leq 10^{-4}C/(|\sigma|^2)$. Then $d_{n+1} \leq 10^{-4}C/(|\sigma|^2) \cdot d_n$. This implies that if t is smaller than some value t_0, there exists an integer $j = j(t)$ such that $\frac{l_j}{d_j} \cdot \frac{\rho_3}{\rho_2} \geq (\frac{10^4}{2})^j \cdot \frac{\rho_3}{\rho_2} \geq 10$, $l_j \geq 10$ and $d_j \leq \frac{1}{10}$. Let us consider any $(2, 3)$-quasi plane $\mathcal{V}$ going through $f_1^{-j}(\tau)$. Then $\mathcal{V}' = f_1^j(\mathcal{V}) \cap \mathbb{D}^3$ is a $(2, 3)$-quasi plane going through τ which then intersects every disk $\mathcal{L}_{z_1,z_3}$. Then $\mathcal{V}$ intersects all the ss-curves $f_1^{-j}(\mathcal{L}_{z_1,z_3}) \cap \mathbb{D}^3$. Since $l_j \geq 10$, this implies that for every (z_1, z_3), we have $\partial(f_1^{-j}(\mathcal{L}_{z_1,z_3})) \subset \mathbb{C}^3 - \mathbb{D}^3$. Since $d_j \leq \frac{1}{10}$, this implies that $f_1^{-j}\Big(\mathbb{D}_{t,\rho_1,\rho_2,\rho_3}\Big) \cap \partial\mathbb{D}^3 \Subset \mathbb{D} \times \partial\mathbb{D} \times \mathbb{D}$. When t tends to 0, the lengths of the disks $\mathcal{L}_{z_1,z_3}$ tends to 0 and then $j = j(t)$ tends to ∞. Reducing $\mathcal{F}'$, by continuity this stays true for $f \in \mathcal{F}'$ for a given t. The proof is done. □

Lemma 5.3.7. *Let $\mathcal{V}$ be a $(1, 3)$-quasi plane foliated by u-curves $(\mathcal{V}^x)_{x\in\mathbb{D}}$. Let ϑ be a point such that $\vartheta \notin \mathcal{V}$ and let us take a vector $u \in C^u$. Then there exists a holomorphic foliation of a neighborhood of ϑ by graphs $\mathcal{V}_y$ over $(z_1, z_3) \in \mathbb{D}^2$ such that each $\mathcal{V}_y$ is foliated by graphs $(\mathcal{V}_{x,y})_{(x,y)\in\mathbb{D}^2}$ over $z_1 \in \mathbb{D}$ of slope bounded by 10^{-1} with :*

1. *for every $x \in \mathbb{D}$, $\mathcal{V}^x = \mathcal{V}_{x,0}$*
2. *the leaf going through ϑ has u as tangent vector*

Proof. We consider two consecutive biholomorphisms. The first one φ_1 sends the $(1,3)$-quasi plane $\mathcal{V}$ to the plane $\{z_2 = 0\}$ and each curve $\mathcal{V}^x$ to the line $\{z_2 = 0, z_3 = x\}$. For a given $z \in \mathbb{D}^3$, we denote by $\mathrm{pr}(z)$ the projection (parallel to $(0,1,0)$) of z on $\mathcal{V}$. We denote by $(\varphi_1)_2(z)$ the complex number such that $z - \mathrm{pr}(z) = (\varphi_1)_2(z) \cdot (0,1,0)$. We denote by by $(\varphi_1)_3(z)$ the complex number x such that $\mathrm{pr}(z) \in \mathcal{V}^x$. We denote $\varphi_1(z) = (z_1, (\varphi_1)_2(z), (\varphi_1)_3(z))$. The restriction of $(\varphi_1)_2$ to a line $\{z_2 = C^{st}, z_3 = C^{st}\}$ is a holomorphic map defined on $\mathbb{D}$ of image in a disk of radius 10^{-4} since $\mathcal{V}$ is a $(1,3)$-quasi plane. By the Cauchy inequality, this implies that $||\frac{\partial(\varphi_1)_2}{\partial z_1}|| < 10^{-4}$. The restriction of $(\varphi_1)_3$ to a line $\{z_2 = C^{st}, z_3 = C^{st}\}$ is a holomorphic map defined on $\mathbb{D}$ of image in a disk of radius 10^{-4} since every $\mathcal{V}^x$ is a u-curve. Then : $||\frac{\partial(\varphi_1)_3}{\partial z_1}|| < 10^{-4}$. Finally, φ_1 is a holomorphic map from $\mathbb{D}^3$ to $\mathbb{D}^3$. Then : $||D\varphi_1|| < 1$.

We denote $\varphi_1(\vartheta) = (\tilde{\vartheta_1}, \tilde{\vartheta_2}, \tilde{\vartheta_3})$ (with $\tilde{\vartheta_2} \neq 0$) and $D_{\varphi_1} u = (1, \epsilon_2, \epsilon_3)$. The previous estimates imply that $|\epsilon_2| < 10^{-4}$ and $|\epsilon_3| < 10^{-4}$. We take a second biholomorphism :

$$\varphi_2 : (z_1, z_2, z_3) \mapsto (z_1, z_2 - \epsilon_2(z_1 - \tilde{\vartheta_1})\frac{z_2}{\tilde{\vartheta_2}}, z_3 - \epsilon_3(z_1 - \tilde{\vartheta_1})\frac{z_2}{\tilde{\vartheta_2}})$$

It is easy to show it is a biholomorphism on $\mathbb{D} \times \mathbb{D}(0, 10\tilde{\vartheta_2}) \times \mathbb{D}$ since $|\epsilon_2| < 10^{-4}$ and $|\epsilon_3| < 10^{-4}$. We have : $D\varphi_2(D\varphi_1(u)) = (1,0,0)$. We denote $\mathcal{V}_{x,y} = (\varphi_2 \circ \varphi_1)^{-1}(\{z_3 = x, z_2 = y\})$. Then $(\mathcal{V}_{x,y})_{x,y}$ is a holomorphic foliation of a neighborhood of ϑ such that for every $x \in \mathbb{D}$, $\mathcal{V}^x = \mathcal{V}_{x,0}$ and the leaf going through ϑ has u as tangent vector. We have $||\frac{\partial(\varphi_2)_2}{\partial z_1}|| < 10\epsilon_2$ and $||\frac{\partial(\varphi_2)_3}{\partial z_1}|| < 10\epsilon_3$ on $z_1 \in \mathbb{D}, z_2 \in \mathbb{D}(0, 10\vartheta_2), z_3 \in \mathbb{D}$, then we have that $||D\varphi_2 \cdot (1,0,0) - (1,0,0)|| < 2 \cdot 10^{-3}$. We have $||\frac{\partial(\varphi_1)_2}{\partial z_1}|| < 10^{-4}$, $||\frac{\partial(\varphi_1)_3}{\partial z_1}|| < 10^{-4}$ and $||D\varphi_1|| < 10$. Then $||D(\varphi_2 \circ \varphi_1) \cdot (1,0,0) - (1,0,0)|| < 10^{-1}$ on $\mathbb{D} \times \mathbb{D}(0, 10\tilde{\vartheta_2}) \times \mathbb{D}$. This implies that every $(\mathcal{V}_{x,y})_{(x,y)}$ where $x \in \mathbb{D}, y \in \mathbb{D}(0, 10\vartheta_2)$ is a graph over $z_1 \in \mathbb{D}$ of slope bounded by 10^{-1}. Every $\mathcal{V}_y = \bigcup_{x \in \mathbb{D}} \mathcal{V}_{x,y}$ is a graph over $(z_1, z_3) \in \mathbb{D}^2$. The proof is done. $\square$

The following is a simple consequence of the inclination Lemma.

Lemma 5.3.8. *There exists an integer k such that for every graph $\mathcal{L}$ of slope bounded by 10^{-1} over $z_1 \in \mathbb{D}$, for every $f \in \mathcal{F}'$, $f^k(\mathcal{L}) \cap \mathbb{D}^3$ has a component which is a graph over $z_1 \in \mathbb{D}$ which is η-close to $W^u_{loc}(\phi_f)$.*

Notation 5.3.9. *We fix this integer k and we fix some $t > 0$ such that $j(t) = k$ (see Lemma 5.3.6). We reduce $\mathcal{F}'$ so that for every $f \in \mathcal{F}'$: there is a point $\vartheta \in W^s(\alpha_f)$ such that $f^k(\vartheta) \in (\tau + (\mathbb{D}(0, t\rho))^3)$ and $W^s(\alpha_f)$ has a tangent vector $u \in C^u$ at ϑ (it is possible by continuity since it is the case for f_1).*

Proposition 5.3.10. *For every $f \in \mathcal{F}'$, $W^s(\alpha_f)$ contains a concentrated 3-folded $(2,3)$-surface $\mathcal{W}''_f$ such that* $\mathrm{Fold}(\mathcal{W}''_f) \subset \mathbb{D}(0, \frac{1}{10})$ *and* $\mathrm{diam}(\mathrm{pr}_1(\mathcal{W}''_f)) < 10^{-8}$.

Proof. Let $f \in \mathcal{F}'$. Let $\mathcal{V}$ be a $(1,3)$-quasi plane foliated by the u-curves $\mathcal{V}^x = \mathcal{V} \cap \{z_3 = x\}$. We take $\vartheta \in W^s(\alpha_f)$ such that $f^k(\vartheta) \in (\tau + (\mathbb{D}(0, t\rho))^3)$ such that $W^s(\alpha_f)$ has a tangent vector $u \in C^u$ at ϑ. To find such a point, we just have to remark that $W^s(\alpha_f)$ is not included in $\mathcal{V}$. In particular, if $\tau \notin \mathcal{V}$, we choose $\vartheta = \tau$.

According to Lemma 5.3.7 it is possible to find some holomorphic foliation of a neighborhood of ϑ by graphs $\mathcal{V}_y$ over (z_1, z_3) foliated by graphs $\mathcal{V}_{x,y}$ over z_1 of slope bounded by 10^{-1} such that $\mathcal{V}^x = \mathcal{V}_{x,0}$ and the leaf going through ϑ is tangent to u and then to $W^s(\alpha_f)$. Since $\vartheta = \tau$ if $\tau \notin \mathcal{V}$, it is also a holomorphic foliation of a neighborhood of τ in both cases. We apply f^k to all these curves $\mathcal{V}_{x,y}$. According to Lemma 5.3.8, the sets $f^k(\mathcal{V}_{x,y})$ have components which are graphs over $z_1 \in \mathbb{D}$ η-close to $W^u_{\mathrm{loc}}(\phi_f)$. Then according to Lemma 5.3.3, the set Tan of points of $f^k(\mathcal{V}_{x,y})$ where $f^k(\mathcal{V}_{x,y})$ is tangent to $W^s(\alpha_f)$ is a regular complex curve which has all its tangent vectors in $C^{v_{\tan},\epsilon}$ and goes through the point $f^k(\vartheta) \in \tau + (\mathbb{D}(0, t\rho))^3$. According to Lemma 5.3.4, Tan can intersect $\tau+\mathbb{D}(0,t\rho_1)e^u(\tau)+\mathbb{D}(0,t\rho_2)(0,1,0)+\mathbb{D}(0,t\rho_3)(0,0,1)$ only in $\tau + \mathbb{D}(0, t\rho_1)e^u(\tau) + \partial\mathbb{D}(0, t\rho_2)(0, 1, 0) + \mathbb{D}(0, t\rho_3)(0, 0, 1)$. According to Lemma 5.3.6, we have both $f^{-k}\Big(\tau+\mathbb{D}(0,t\rho_1)e^u(\tau)+\partial\mathbb{D}(0,t\rho_2)(0,1,0)+\mathbb{D}(0,t\rho_3)(0,0,1)\Big) \subset \mathbb{C}^3-\mathbb{D}^3$ and $f^{-k}\Big(\tau+\mathbb{D}(0,t\rho_1)e^u(\tau)+\mathbb{D}(0,t\rho_2)(0,1,0)+\mathbb{D}(0,t\rho_3)(0,0,1)\Big)\cap\partial\mathbb{D}^3 \Subset \mathbb{D}\times\partial\mathbb{D}\times\mathbb{D}$. Then $f^{-k}(\text{Tan})$ intersects every surface $\mathcal{V}_y$ but in exactly one point which is not in $\mathcal{V}_y \cap ((\partial\mathbb{D}\times\mathbb{D}\times\mathbb{D})\cup(\mathbb{D}\times\mathbb{D}\times\partial\mathbb{D}))$. We denote by $\mathcal{W}'_f$ the continuation of the component of $W^s(\alpha_f)$ containing $f_1^{-k}(\tau)$ for $f \in \mathcal{F}'$. Then $\mathcal{W}'_f$ is tangent to at least one but not infinitely many u-curves $\mathcal{V}^x$. By the Riemann-Hürwitz formula, $\mathcal{W}'_f$ is tangent to exactly one $\mathcal{V}^x = \mathcal{V} \cap \{z_3 = x\}$. This implies that pr_3 restricted on $\mathcal{W}'_f \cap \mathcal{V}$ is a two-covering with exactly one point of ramification. Since this is true for any $(1, 3)$-quasi plane $\mathcal{V}$, by definition, $\mathcal{W}'_f$ is a 3-folded $(2, 3)$-surface for every $f \in \mathcal{F}'$. Then according to Proposition 4.3.5, we can iterate under f^{-1} to get a component $\mathcal{W}''_f$ of $W^s(\alpha_f)$ which is a concentrated 3-folded $(2, 3)$-surface. Since $\text{pr}_3(\phi_f) \in \mathbb{D}(0, \frac{1}{10}-10^{-4})$ we have that $\text{Fold}(\mathcal{W}''_f) \subset \mathbb{D}(0, \frac{1}{10})$. Since the first direction is contracted under f, iterating, we can have the additional property that $\text{diam}(\text{pr}_1(\mathcal{W}''_f)) < 10^{-8}$.

□

6 Proof of the main results

Proof of the main Theorem and Corollaries 1 and 2. According to Proposition 5.3.10, for every $f \in \mathcal{F}'$, $W^s(\alpha_f)$ contains some concentrated 3-folded $(2, 3)$-surface $\mathcal{W}^0$ with $\text{Fold}(\mathcal{W}^0) \subset \mathbb{D}(0, \frac{1}{10})$ and moreover $\text{diam}(\text{pr}_1(\mathcal{W}^0)) < 10^{-8}$. This implies persistent heteroclinic tangencies. Indeed, by Proposition 4.3.1 there exists a point κ_f of the horseshoe $\mathcal{H}_f$ such that there is a point of quadratic tangency τ' between $W^u(\kappa_f)$ and $\mathcal{W}^0 \subset W^s(\alpha_f)$. The proof of the main Theorem is complete.

We now prove Corollary 1. We call $\mathcal{U}'_f$ the component of $W^u(\kappa_f) \cap \mathbb{D}^3$ containing the point of tangency τ' and we consider the family $(\mathcal{U}'_{f'})_{f'\in\mathcal{F}'}$ of holomorphic u-curves given by the continuation of $\mathcal{U}'_f$. According to Proposition 5.1.6, the tangency τ' is generically unfolded. We apply a standard argument to obtain homoclinic tangencies. Let us take any neighborhood $\mathcal{F}''$ of f. Since the tangency τ' is generically unfolded, we can rely on the following well-know lemma :

Lemma 6.0.11. *There exists $\epsilon'' > 0$ such that for every holomorphic family $(\mathcal{U}''_{f'})_{f'\in\mathcal{F}'}$ of u-curves ϵ''-close to $(\mathcal{U}'_{f'})_{f'\in\mathcal{F}'}$ and every holomorphic family $(\mathcal{W}''_{f'})_{f'\in\mathcal{F}'}$ of complex surfaces ϵ''-close to $(\mathcal{W}^0)_{f'\in\mathcal{F}'}$, there exists a point of quadratic tangency between $\mathcal{U}''_{f''}$ and $\mathcal{W}''_{f''}$ where $f'' \in \mathcal{F}''$.*

We claim that there exists a holomorphic family of u-curves $(\mathcal{U}''_{f'})_{f'\in\mathcal{F}'}$ ϵ''-close to $(\mathcal{U}'_{f'})_{f'\in\mathcal{F}'}$ such that for every $f' \in \mathcal{F}'$, $\mathcal{U}''_{f'}$ is a component of $W^u(\delta_{f'}) \cap \mathbb{D}^3$ and a holomorphic family of complex surfaces $(\mathcal{W}''_{f'})_{f'\in\mathcal{F}'}$ ϵ''-close to $((\mathcal{W}^0)_{f'})_{f'\in\mathcal{F}'}$ such that for every $f' \in \mathcal{F}'$, $\mathcal{W}''_{f'}$ is a component of $W^s(\delta_{f'})$. We prove this fact for the case of u-curves, the proof is the same for complex surfaces. According to Proposition 3.1.6, $\alpha_{f'}$, and then $\kappa_{f'}$, is in the homoclinic class of $\delta_{f'}$. This implies that there exists a transversal intersection between $W^s(\kappa_{f'})$ and $W^u(\delta_{f'})$ for every $f' \in \mathcal{F}'$. For a given $f' \in \mathcal{F}'$, by the inclination lemma, there exists an integer $n_{f'}$ such that for every $n \geq n_{f'}$, $(f')^{n_{f'}}(W^u(\delta_{f'}))$ contains a u-curve ϵ''-close to $\mathcal{U}'_{f'}$. By continuity, $n_{f'}$ can be taken locally constant. By compactness, it is possible to take the maximal value n of $n_{f'}$ on a finite open covering of $\mathcal{F}' \Subset \mathcal{F}$. Then we have that for every $f' \in \mathcal{F}'$, there is a u-curve $\mathcal{U}''_{f'}$ ϵ''-close to $\mathcal{U}'_{f'}$ which is a component of $W^u(\delta_{f'})$ (inside $(f')^n(W^u(\delta_{f'}))$). The proof of Corollary 1 is complete.

We now prove Corollary 2. This shows that there exists $f'' \in \mathcal{F}''$ such that f'' has a point of homoclinic tangency between $W^s(\delta_{f''})$ and $W^u(\delta_{f''})$. In particular maps with homoclinic tangencies are dense in $\mathcal{F}'$. The point δ_f has the property of being sectionally dissipative. Then, the Proposition in the Appendix which gives the creation of sinks from homoclinic tangencies (the proof is given in Appendix for the convenience of the reader) with a classical Baire category argument already used in [7] allows us to conclude to the existence of a residual set of $\mathrm{Aut}_2(\mathbb{C}^3)$ of automorphisms displaying infinitely many sinks. The proof is complete. $\square$

A From homoclinic tangencies to sinks

To show the main Theorem, we need the following result. It is known since the work of Newhouse how to get a sink from a homoclinic tangency. The adaptation to the case of $\mathbb{C}^2$ was obtained by Gavosto in [13]. Here we adapt her proof to the case of $\mathbb{C}^3$. Remind that a generically unfolded tangency is a tangency which is unfolded with a positive speed.

Proposition A.1. *Let $(F_t)_{t\in\mathbb{D}}$ be a family of polynomial automorphisms of $\mathbb{C}^3$. We suppose that for the parameter $t = 0$, there is a sectionally dissipative periodic point P_0 and a generically unfolded homoclinic quadratic tangency $Q \in W^s(P_0) \cap W^u(P_0)$ between $W^s(P_0)$ and $W^u(P_0)$. We also suppose that the three eigenvalues λ_0^{ss}, λ_0^{cs} and λ_0^u at P_0 satisfy $|\lambda_0^{ss}| < |\lambda_0^{cs}| < 1 < |\lambda_0^u|$ (and $|\lambda_0^{cs}| \cdot |\lambda_0^u| < 1$ by sectional dissipativity). Then for every neighborhood $\mathcal{Q}$ of Q and every neighborhood $\mathcal{T}$ of 0, there exists $t \in \mathcal{T}$ such that F_t admits an attracting periodic point in $\mathcal{Q}$.*

Proof. Step 1 : Construction of cone fields

In the following, iterating if necessary, we will suppose that P_0 is a fixed point of F_0. We fix a constant $\eta > 0$ such that we have $(|\lambda_0^{cs}| + \eta)(|\lambda_0^u| + \eta) < 1$ (the periodic point P_0 is sectionally dissipative) and $|\lambda_0^{ss}| + \eta < |\lambda_0^{cs}| - \eta < |\lambda_0^{cs}| + \eta < 1$. We can fix a neighborhood $\mathcal{P}$ of P_0, a neighborhood $\mathcal{T}_0$ of 0, cones C^u, C^{ss} and C^{cs} centered at the three eigenvectors of $DF_0(P_0)$ and an integer N_0 with the following properties : for every $n \geq N_0$, $t \in \mathcal{T}_0$, for every matrix $M = M_n \cdot \ldots \cdot M_1$ where M_i is the differential of F_t at a point in $\mathcal{P}$, we have :

1. C^u is invariant under M and there is exactly one eigenvector (up to multiplication) of M in C^u of eigenvalue $(|\lambda_0^u| - \eta)^n < |\lambda_1| < (|\lambda_0^u| + \eta)^n$
2. C^{ss} is invariant under M^{-1} and there is exactly one eigenvector (up to multiplication) of M in C^{ss} of eigenvalue $(|\lambda_0^{ss}| - \eta)^n < |\lambda_2| < (|\lambda_0^{ss}| + \eta)^n$
3. for every vector v which is not in $C^u \cup C^{ss} \cup C^{cs}$, we have : $M \cdot v \in C^u$ or $M^{-1} \cdot v \in C^{ss}$
4. $(|\lambda_0^u| - \eta)^n (|\lambda_0^{cs}| - \eta)^n (|\lambda_0^{ss}| - \eta)^n < |\det(M)| < (|\lambda_0^u| + \eta)^n (|\lambda_0^{cs}| + \eta)^n (|\lambda_0^{ss}| + \eta)^n$

Reducing η and increasing N_0 if necessary, we have for $n \geq N_0$:

$$(|\lambda_0^u| + \eta)^n (|\lambda_0^{ss}| + \eta)^{2n} < (|\lambda_0^u| - \eta)^n (|\lambda_0^{cs}| - \eta)^n (|\lambda_0^{ss}| - \eta)^n$$

$$(|\lambda_0^u| + \eta)^n (|\lambda_0^{cs}| + \eta)^n (|\lambda_0^{ss}| + \eta)^n < (|\lambda_0^u| - \eta)^{2n} (|\lambda_0^{ss}| - \eta)^n$$

This implies that M is diagonalizable. Indeed, if it was not the case, M would be triangularizable with a double eigenvalue and we would have :

$$(|\lambda_0^u| + \eta)^n (|\lambda_0^{ss}| + \eta)^{2n} > |\det(M)| \text{ or } |\det(M)| > (|\lambda_0^u| - \eta)^{2n} (|\lambda_0^{ss}| - \eta)^n$$

But we have :

$$(|\lambda_0^u| - \eta)^n (|\lambda_0^{cs}| - \eta)^n (|\lambda_0^{ss}| - \eta)^n < |\det(M)| < (|\lambda_0^u| + \eta)^n (|\lambda_0^{cs}| + \eta)^n (|\lambda_0^{ss}| + \eta)^n$$

Then, there would be a contradiction with the previous inequalities. Moreover the third eigenvector has to be C^{cs} according to item 3 above.

Step 2 : Local coordinates

Iterating if necessary, we can suppose that $Q \in \mathcal{P}$. We are going to make several local changes of coordinates so that the stable and unstable manifolds of P_t have a simple form near the tangency. We will use this local change of coordinates in Steps 2 and 3. In Step 4, we will mainly use the coordinates in the canonical basis but we will need the local change of coordinates a last time at some point so we will denote it by Ψ_τ in Step 4 to make the distinction between the two systems of coordinates. In the following, to construct the local coordinates, we will keep the notation z_1, z_2, z_3 by simplification.

We first pick local coordinates such that $Q = 0$ and in the neighborhood of Q, the stable manifold is $\{z_1 = 0\}$. Up to a linear invertible second change of coordinates, we can assure that the tangent vector of $W^u(P_0)$ at Q is $(0, 0, 1)$. Then we have that locally near 0 the unstable manifold is given by a graph over z_3 of the form $\{(w_1(z_3), w_2(z_3), z_3) : z_3\}$. Since the tangency is quadratic, for each t in a neighborhood of 0, there exists exactly one $z_{3,t}$ such that $\frac{\partial w_1}{\partial z_3}(z_{3,t}, t) = 0$. We pick new coordinates a third time by changing the coordinate z_3 into $z_3 - z_{3,t}$. The unstable manifold is :

$$w_1(z_3) = w_1(z_3, t) = w_1(0, t) + \frac{\partial^2 w_1}{\partial z_3^2}(0, t)z_3^2 + h(z_3, t) \text{ with } h(z_3, t) = o(z_3^2)$$

The stable manifold is still locally equal to $\{z_1 = 0\}$. Since there is a quadratic tangency for $t = 0$ which is generically unfolded, we have $w_1(0, 0) = 0$ and $\frac{\partial w_1}{\partial t}(0, 0) \neq 0$. Then we change coordinates a fourth time : z_1 becomes $\frac{t}{w_1(0,t)} z_1$ and z_2 and z_3 are unchanged. In these new coordinates, the unstable manifold is given by :

$$w_1(z_3) = t + z_3^2 \tilde{h}(z_3, t)$$

where $\tilde{h}(z_3, t) \neq 0$ in a neighborhood of 0. The stable manifold is still locally equal to $\{z_1 = 0\}$. The fifth change of coordinates is given by z_3 becoming $z_3(\tilde{h}(z_3, t))^{1/2}$, where $(\tilde{h}(z_3, t))^{1/2}$ is a complex square root of $\tilde{h}(z_3, t))$ (which is well defined since $\tilde{h}(z_3, t) \neq 0$ in a neighborhood of 0). We finally get that the unstable manifold is given by $z_1 = z_3^2 + t$ and $z_2 = w_2(z_3, t)$. The stable manifold is still locally equal to $\{z_1 = 0\}$ and the tangent vector of $W^u(P_0)$ at Q is still $(0, 0, 1)$. The last change of coordinates is given by $z_2 - w_2(z_3, t)$. The unstable manifold is given by $z_1 = z_3^2 + t$ and $z_2 = 0$ and the stable manifold by $z_1 = 0$. The tangent vector of $W^u(P_0)$ at Q is still $(0, 0, 1)$.

Step 3 : Construction of a periodic point

We take a tridisk B around Q in the coordinates that we just defined : $B = \{(z_1, z_2, z_3) : |z_1| < \delta, |z_2| < \delta, |z_3| < \delta\}$ where $0 < \delta < 1$. Since the tangent vector of $W^u(P_0)$ at Q is $(0, 0, 1)$, reducing δ if necessary, there exists a neighborhood $\mathcal{T}_1 \subset \mathcal{T}_0$ such that the component of $W^u(P_t) \cap B$ containing Q (for $t = 0$) or its continuation (for $t \neq 0$) is horizontal in B relatively to the third projection and is included in $\{(z_1, z_2, z_3) \in B : |z_2| < \frac{1}{10}\delta\}$. Using the Inclination Lemma, there exists $N_1 \geq N_0$ such that for $t \in \mathcal{T}_1$ and $n \geq N_1$, $F_t^n(B)$ will intersect B and $F_t(B) \cap B$ is horizontal relatively to the third projection.

In the following, we show that for every sufficiently high n, a periodic point for F_t (where $t \in \mathcal{T}_1$) is created. Let us now denote $\Delta_{z_2,z_3} = \{(z_1, z_2, z_3) : |z_1| < \delta\}$ which is a disk, for $|z_2| < \delta, |z_3| < \delta$. It is possible to increase $N_2 \geq N_1$ such that for any $|z_2| < \delta, |z_3| < \delta$, for every $n \geq N_2$, $F_t^n(\Delta_{z_2,z_3}) \cap B$ is horizontal relatively to the third projection (of degree 1) and included in $\{(z_1, z_2, z_3) \in B : |z_2| < \frac{1}{2}\delta\}$. Since $F_t^n(\Delta_{z_2,z_3}) \cap B$ is horizontal relatively to the third projection and of degree 1, it intersects exactly one disk Δ_{z_2',z_3} where $z_2' = z_2'(z_2)$ with $z_2' \in \mathbb{D}(0, \delta/2)$. This defines for each fixed $t \in \mathcal{T}_1$ a holomorphic map $z_2 \mapsto z_2'$. Then the map $z_2 \mapsto z_2 - z_2'$ defined on $\mathbb{D}(0, \delta)$ is holomorphic and the image of $\partial\mathbb{D}(0, \delta)$ contains a loop around $\mathbb{D}(0, \delta/2)$. Then by the Argument principle there exists $z_2(z_3)$ such that $\Delta_{z_2(z_3),z_3}$ intersects $F_t^n(\Delta_{z_2(z_3),z_3})$.

We are going to choose z_3 in order to create a periodic point in $\Delta_{z_2(z_3),z_3}$. There is a point $R_{z_3} = (f_t(z_3), z_2(z_3), z_3) \in \Delta_{z_2(z_3),z_3}$ which is sent on $S_{z_3} \in \Delta_{z_2(z_3),z_3}$ where $S_{z_3} = (g_t(z_3), z_2(z_3), z_3)$. We are going to choose z_3 (in function of t) such that $R_{z_3} = S_{z_3}$. When N goes to infinity, $f_t(z_3)$ tends to 0 and $g_t(z_3)$ tends to $z_3^2 + t$. Then $g_t(z_3) - f_t(z_3)$ tends to $z_3^2 + t$. In particular, if N is sufficiently high, for every $t \in \mathbb{D}$, the graph of $z_3 \mapsto g_t(z_3) - f_t(z_3)$ is a curve of degree 2 over z_3 which has exactly one point of tangency with the horizontal foliation. We take a new bound N_3 on n such that this is the case, we increase N_3 in the rest of Step 3 in order to satisfy more assumptions. The second coordinate of this point of tangency is a holomorphic function of t which tends to t when increasing N_3. In particular, this implies that there exists exactly one value t_0 of t for which the equation $g_{t_0}(z_3) - f_{t_0}(z_3) = 0$ has one double solution, and for every other value of $t \in \mathbb{D}$, there are two distinct solutions. Then, for F_t, we have two periodic points which are equal when $t = t_0$. We do a reparametrization of the family of maps $(F_t)_{t\in\mathbb{D}}$ by taking $t = \tau^2$. From now on, we are working with the family of maps $\tilde{F}_\tau = F_{\tau^2}$. By simplicity, we will simply denote it by F_τ. For F_τ, we have two periodic points which are equal when $\tau = \tau_0$ where $\tau_0^2 = t_0$. We can increase N_3 if necessary so that for each $\tau \in \mathbb{D} - \mathbb{D}(0, \frac{1}{2})$, the two solutions of $g_{\tau^2}(z_3) - f_{\tau^2}(z_3) = 0$ are respectively $\frac{1}{100}$-close to $\pm i\tau$. For $\tau \in \mathbb{D} - \mathbb{D}(0, \frac{1}{2})$, we denote by R^τ the periodic point corresponding to the solution $\frac{1}{100}$-close to $i\tau$. It is clear that the map $\tau \mapsto R^\tau$ restricted on $\mathbb{D} - \mathbb{D}(0, \frac{1}{2})$ is continuous. For $\tau = \tau_0$, we denote by R^{τ_0} the unique periodic point corresponding to the double solution of $g_{\tau_0^2}(z_3) - f_{\tau_0^2}(z_3) = 0$. Finally, for $\tau \in \mathbb{D}(0, \frac{1}{2})$ not equal to τ_0, there are two distinct periodic points in $\Delta_{z_2(z_3),z_3}$. We pick any path in $\mathbb{C}$ from τ to a point τ_1 in $\mathbb{D} - \mathbb{D}(0, \frac{1}{2})$ which does not contain τ_0. For τ_1, the periodic point R^{τ_1} is defined. Since the path does not contain τ_0, there is exactly one of the two distinct periodic points in $\Delta_{z_2(z_3),z_3}$ for τ which is the continuation of R^{τ_1}. We denote it by R^τ. Since the map $\tau \mapsto R^\tau$ restricted on $\mathbb{D} - \mathbb{D}(0, \frac{1}{2})$ is continuous, this choice is independent of a particular choice of τ_1 in $\mathbb{D} - \mathbb{D}(0, \frac{1}{2})$. Then we have defined a map $\tau \mapsto R^\tau$ on $\mathbb{D}$. It is clear that near any point of $\mathbb{D} - \{\tau_0\}$, R^τ is locally the continuation of the same periodic point of F_τ, then it is holomorphic. The map $\tau \mapsto R^\tau$ is holomorphic on $\mathbb{D} - \{\tau_0\}$. It is trivial that it is continuous at τ_0. Then it is holomorphic on $\mathbb{D}$. As we already said, from now on, we go back to the coordinates in the canonical basis and we call Ψ_τ the local coordinates we just used near the tangency point Q. We will use Ψ_τ a last time at the end of Step 4.

Step 4 : R^τ is a sink

We now show it is possible to pick τ such that R^τ is a sink. From now on, we fix a neighborhood $\mathcal{T} \subset \mathcal{T}_1$ of 0. Recall that Q belongs to the small neighborhood $\mathcal{P}$ of P_0 defined in Step 1. We denote by $n = n_1 + n_2$ such that for $k = 1, ..., n_1$, $F_\tau^k(R^\tau)$ is in $\mathcal{P}$ and $F_\tau^{n_1+1}(R^\tau) \notin \mathcal{P}$. We express all matrices in the τ-dependent basis given by the 3 eigenvectors e_1',e_2', e_3' of $D(F_\tau^{n_1})(R^\tau)$ (the matrix $DF_\tau^{n_1}(R^\tau)$ is diagonalizable according to Step 1). We have that $e_1' \in C^u$, $e_2' \in C^{ss}$ and $e_3' \in C^{cs}$. Then, in this basis, the matrix $DF_\tau^n(R^\tau)$ is of the form $DF_\tau^{n_1+n_2}(R^\tau) = DF_\tau^{n_2}(F_\tau^{n_1}(R^\tau)) \cdot DF_\tau^{n_1}(R^\tau)$. The matrix $DF_\tau^{n_1}(R^\tau)$ is of the form :

$$\begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix}$$

We have that $(|\lambda_0^u|-\eta)^{n_1} < |\Lambda_1| < (|\lambda_0^u|+\eta)^{n_1}$, $(|\lambda_0^{ss}|-\eta)^{n_1} < |\Lambda_2| < (|\lambda_0^{ss}|+\eta)^{n_1}$ and $(|\lambda_0^{cs}|-\eta)^{n_1} < |\Lambda_3| < (|\lambda_0^{cs}|+\eta)^{n_1}$. The matrix $DF_\tau^{n_2}(F_\tau^{n_1}(R^\tau))$ is of the form :

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

Since $e_1' \in C^u$, $e_2' \in C^{ss}$ and $e_3' \in C^{cs}$ and these cones are disjoint, the coefficients A,B,C,D,E,F,G,H,I are bounded in modulus by some constant K which is independant of τ, n and n_1. Then :

$$DF_\tau^{n_1+n_2}(R^\tau) = \begin{pmatrix} A\Lambda_1 & B\Lambda_2 & C\Lambda_3 \\ D\Lambda_1 & E\Lambda_2 & F\Lambda_3 \\ G\Lambda_1 & H\Lambda_2 & I\Lambda_3 \end{pmatrix}$$

Let us suppose that $A\Lambda_1 = 0$. There exists $\epsilon > 0$ such that if the characteristic polynomial of $DF_\tau^{n_1+n_2}(R^\tau)$ is $X^3 + a_2X^2 + a_1X + a_0$ with $|a_0| < \epsilon, |a_1| < \epsilon, |a_2| < \epsilon$, then the 3 eigenvalues of $DF_\tau^{n_1+n_2}(R^\tau)$ are of modulus lower than 1 and then R^τ is a sink. We are going to show that it is possible to get a lower bound on n_1 so that it is always the case. The coefficients of the characteristic polynomial of $DF_\tau^{n_1+n_2}(R^\tau)$ are :

$$a_2 = -(A\Lambda_1 + E\Lambda_2 + I\Lambda_3)$$
$$a_1 = (EI\Lambda_2\Lambda_3 - FH\Lambda_2\Lambda_3 + AI\Lambda_1\Lambda_3 - CG\Lambda_1\Lambda_3 + AE\Lambda_1\Lambda_2 - BD\Lambda_1\Lambda_2)$$
$$a_0 = -\det(DF_\tau^{n_2}(F_\tau^{n_1}(R^\tau))$$

We have : $|a_0| = |\det(DF_\tau^{n_2}(F_\tau^{n_1}(R^\tau)))| < K^3|\Lambda_1\Lambda_2\Lambda_3| < K^3\big((|\lambda_0^{cs}|+\eta)(|\lambda_0^u|+\eta)\big)^{n_1}$ which tends to 0 when $n_1 \to +\infty$ because $(|\lambda_0^{cs}|+\eta)(|\lambda_0^u|+\eta) < 1$ (remind that P_0 is a sectionally dissipative periodic point). We increase n_1 such that $|a_0| < \epsilon/2$. We have that :

$$|a_1| < 6K^2 \max(|\Lambda_1\Lambda_2|, |\Lambda_1\Lambda_3|, |\Lambda_2\Lambda_3|) < 6K^2\big((|\lambda_0^{cs}|+\eta)(|\lambda_0^u|+\eta)\big)^{n_1}$$

which tends to 0 when $n_1 \to +\infty$ because $(|\lambda_0^{cs}|+\eta)(|\lambda_0^u|+\eta) < 1$. We increase n_1 further such that $|a_1| < \epsilon/2$. Finally, in the term a_2, both $E\Lambda_2$ and $I\Lambda_3$ tend to 0 when $n_1 \to +\infty$. The term $A\Lambda_1$ is equal to 0 by hypothesis. We increase n_1 a last time such that $|a_2| < \epsilon/2$. Finally, we pick $N_4 \geq N_3$ such that n_1 is sufficiently high in order to satisfy the previous inequalities. Finally, the 3 eigenvalues of $DF_\tau^{n_1+n_2}(R^\tau)$ are of modulus lower than 1 and R^τ is a sink.

It remains us to show that for a given neighborhood $t \in \mathcal{T}$ of 0 and the corresponding neighborhood of 0 for τ (remind that $t = \tau^2$), it is possible to pick a new bound N_6 on n, such that for $n \geq N_6$, there is a parameter $\tau = \tau(n)$ such that the differential $DF_\tau^n(R^\tau)$ satisfies $A\Lambda_1 = 0$. To show this, we work again in the coordinates Ψ_τ for the map F_τ. We denote by Π the projection of the plane $\Psi_\tau(W^s(P_{\tau^2})) = \{z_1 = 0\}$ into $\mathbb{P}^2(\mathbb{C})$. The projection Π is a holomorphic curve. We denote by Γ the holomorphic curve in $\mathbb{P}^2(\mathbb{C})$ given by the tangent directions to the curve $\Psi_\tau(W^u(P_0)) = \{z_1 = z_3^2, z_2 = 0\}$. Since there is a generically unfolded quadratic tangency at $\tau = 0$ between $W^s(P_0)$ and $W^u(P_0)$, there is a transverse intersection between Π and Γ. Moreover, $\mathbb{P}^2(\mathbb{C})$ is of dimension 2. We take a small disk Δ' going through $\Psi_\tau(R^\tau)$ of direction $D\Psi_\tau(R^\tau) \cdot e_1'$. For a given $n \geq N_4$, we call Γ_n' the complex curve in $\mathbb{P}^2(\mathbb{C})$ given by the tangent directions to the curve $\Psi_\tau(F_\tau^n(\Delta'))$ at the periodic point $\Psi_\tau(R^\tau)$ when τ varies. By the Inclination Lemma, we can increase the bound $N_5 \geq N_4$ on n such that if $n \geq N_5$, $\mathrm{pr}_3(\Psi_\tau(R^\tau))$ can be made as close to $i\tau$ as wanted and $\Psi_\tau(F_\tau^n(\Delta'))$ can be taken as close to $\{z_1 = z_3^2 + t, z_2 = 0\}$ in the C^1 topology. In particular, this shows that the graph Γ_n' can be taken as close to the graph Γ as wanted (in the C^1-topology) by increasing the bound N_5. In particular, we can pick N_5 such that there is a transverse intersection between Γ_n' and Π

if $n \geq N_5$. There is a last step to get $A\Lambda_1 = 0$. We denote by Π' the projection of the plane $\mathrm{Vect}(D\Psi_\tau(R^\tau) \cdot e'_2, D\Psi_\tau(R^\tau) \cdot e'_3)$ in $\mathbb{P}^2(\mathbb{C})$, Π' is a complex curve. Since e'_2 and e'_3 are the two stable eigenvectors of $DF_\tau^{n_1}(R^\tau)$, it is an easy consequence of the Inclination Lemma that if n and then n_1 tend to infinity, then R^τ tends to $W^s(P_{\tau^2})$ and Π' tends to Π (locally as graphs in the C^1-topology). In particular, it is possible to pick a last bound $N_6 \geq N_5$ on n such that if $n \geq N_6$, then Γ'_n and Π' have a transverse intersection. This implies that there exists τ with $t = \tau^2 \in \mathcal{T}$ such that $DF_\tau^n(R^\tau) \cdot e'_1 \in \mathrm{Vect}(e'_2, e'_3)$ which is equivalent to $A\Lambda_1 = 0$. For this parameter τ, we saw that R^τ is a sink.

This is true for every neighborhood $\mathcal{T}$ of 0 and it is clear that R^τ tends to Q when reducing $\mathcal{T}$. The result is proven : for every neighborhood $\mathcal{Q}$ of Q and every neighborhood $\mathcal{T}$ of 0, we can create a sink in $\mathcal{Q}$ for F_t with $t \in \mathcal{T}$. $\square$

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