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Strategic Voting in a Social Context: Considerate Equilibria

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Abstract. In a voting system, voters may adopt a strategic behaviour in order to manipulate the outcome of the election. This naturally entails a game theoretic conception of voting. The specificity of our work is that we embed the voting game into a social context where agents and their relations are given by a graph, i.e. a social network. We aim at integrating the information provided by the graph in a refinement of the game-theotical analysis of an election. We consider coalitional equilibria immune to deviations performed by realistic coalitions based on the social network, namely the cliques of the graph. Agents are not fully selfish as they have consideration for their relatives. The corresponding notion of equilibrium was introduced by Hoefer *et al.* [12] and called *considerate equilibrium*. We propose to study its existence and the ability of the agents to converge to such an equilibrium in strategic voting games using well-known voting rules: Plurality, Antiplurality, Plurality with runoff, Borda, k-approval, STV, Maximin and Copeland.

1 Introduction

The way of aggregating the preferences of a society in order to make a collective decision is a fundamental issue in every part of the community life. When several alternatives, or candidates, are available, voting systems are classically used to make collective decisions in many different contexts (political elections, decisions within committees, planning of meetings). In this respect, social choice theorists designed various voting rules whose properties have been analyzed in an axiomatic manner (see e.g. [18, 5] for an overview on that topic).

Most of these works implicitly assume that voters truthfully reveal their preferences. However, there is no possibility to ensure that they tell the truth when they vote. Voters can think they will be better off if they do not exactly align their ballot with their preferences (tactical voting), what is called *manipulation*. This situation can lead to an outcome that is more favorable to the manipulators. However, one may loose the good properties of the voting rule (as underlined in [8]). It is highly desirable for a voting rule to be immune to such strategic behaviors. Unfortunately, the Gibbard-Satterthwaite theorem [10, 25] shows that any reasonable non-dictatorial voting system is manipulable.

One way to circumvent this impossibility result is to resort to voting rules which are computationally hard to manipulate. This computational complexity may constitute a barrier to malicious behaviours. However this approach has raised many criticisms in the community of computational social choice [6, 7]. The main reason is that concepts used to define computational complexity are based on worst case analysis, and there may be various voting situations where a manipulation is easy to compute. Another approach consists in perceiving an election as a strategic situation where the voters are the players of a game. In this case, a plausible outcome of the election is a situation of equilibrium: those who can dethrone the current winner prefer the winner to its possible contenders.

An important line of research consists in analyzing the existence of an equilibrium and the ability of the participants to reach it in a natural dynamic process where there is a deviation only if it is profitable. These works have contributed to observe that the guaranteed existence of an equilibrium, together with the convergence of the dynamics to such a stable state, depends on the voting rule, the initial state and the type of deviations that are allowed. A fundamental work about specific solution concepts in voting is due to Myerson and Weber [19] who define a voting equilibrium as a state consistent with a preliminary poll. More classically, the existence of a Nash equilibrium [20] — a stable outcome regarding unilateral deviations — has been well studied, especially around an axiomatic study of voting rules which admit it, see e.g. [15].

Recently, many articles deal with convergence to a Nash equilibrium through an *iterative voting* framework [1], a voting process consisting of several rounds, where at each round one voter is allowed to change her ballot. As an illustration, one can think of online voting procedures like *Doodle* polls (online tool to schedule meetings). Convergence for several voting rules has been established in this way in [14, 16, 24] but convergence is far from being guaranteed, conducting to reconsider the dynamic process with restricted deviations [11, 21, 23]. As far as we know, iterative voting has never been adapted to solution concepts different from the Nash equilibrium.

A drawback of the Nash equilibrium is its weakness against coalitional deviations. Actually, it is interesting to consider coalitions over the agents, modeling for instance some political parties or groups of friends. Well-known solution concepts take into account coalitional deviations, e.g. the *strong equilibrium* [3] and the *super-strong equilibrium* [29]. In these equilibria, every subset of agents is a possible coalition. Unfortunately, the existence of such equilibria cannot be guaranteed for most of the classical voting rules. The conditions of existence of a strong equilibrium in a voting game have been explored for instance in [22]. A characterization is provided by Sertel and Sanver [26] with an axiomatic point of view, as well as in [17] for specific voting rules. In another type of equilibrium called *coalition-proof equilibrium* [4], not every type of deviation is allowed.

Considering that any coalition of voters can form may be questionable or unrealistic, because it supposes that all agents in any subset are able to coordinate their moves in order to manipulate. Indeed, such a coordination requires a high level of communication and trust for the manipulators. In practice, one can reasonably exclude some coalitions from the definition of an equilibrium that is stable against group deviations. For example, this is the case of the *partition equilibrium* introduced by Feldman and Tennenholtz [9] where only the coalitions belonging to a prescribed partition of the voter set have the ability to deviate.

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More generally, one can determine which coalitions can form by exploiting social networks. Exploiting the social relations which bind the members of group is receiving much attention [13]. In a strategic game context, we can make the same observation and consider that the agents are embedded in a social network and then, relations among them are fully characterized by their links in that network. To go further, one can suppose that a voter is tied by social relations which force her to have consideration for other participants. Consequently, we can assume that an agent is not only guided by her own preferences over the candidates (or alternatives) but pursues the goal of optimizing the welfare of the communities where she belongs, as in [2]. A solution concept which naturally arises from this situation is the one where the social network is represented by a graph G whose node set and the agent set coincide. The possible coalitions are prescribed patterns of G and agents have consideration for their neighbors in the graph. When considering cliques of G as coalitions, such an equilibrium is called a *considerate equilibrium*. Introduced by Hoefer *et al.* [12], a considerate equilibrium is a state robust to deviations by the cliques of the social network, but a coalition given by a clique only deviates when it is not harmful for her relatives (neighbors in the graph). As for the partition equilibrium that it extends, the considerate equilibrium has only been studied, as far as we know, for a special case of congestion game.

To our best knowledge, the social context in which the voters are embedded has been surprisingly underestimated in strategic voting games. Some recent papers started to investigate this question, as in [27, 28] where the authors study iterative voting via social networks. They consider that voters are not aware of every voter's ballot but only of the ballot of their neighbors in the social network. Thus, they use social relations from the perspective of a gain of information, but not in terms of defining which coalitions can form. In this article, we propose to fill this gap.

Concretely, we explore the existence of a considerate equilibrium in a strategic voting game and the ability of the game to converge to such an equilibrium for different voting rules. Our main contributions are existence proofs of a considerate equilibrium in a voting game under well-established voting rules, namely Plurality, Antiplurality, Plurality with runoff, STV and Maximin. We also investigate the possibility for the voters to reach a considerate equilibrium in a natural iterative process. In this respect, our results are rather negative because convergence to a partition equilibrium, or convergence to a Nash equilibrium, which are less demanding goals than convergence to a considerate equilibrium, fail.

The article is organized as follows. We first introduce in Section 2 the strategic voting game framework and the notion of considerate equilibrium. Then we study in Section 3 the case of positioning scoring rules such as Plurality and Antiplurality rules. Section 4 is devoted to two voting rules with runoff and in Section 5, two voting rules based on pair-wise comparisons of the alternatives (Copeland and Maximin) are studied. We conclude in Section 6 with a discussion on the global results and the impact of consideration within deviating moves.

2 Strategic voting games

We consider a strategic game defined on the basis of an election $(N, M, \mathcal{F})$ where N is a set of $n \geq 2$ voters (also called agents), M is a set of $m \geq 2$ candidates (also called alternatives) and $\mathcal{F}$ is a voting rule. Each voter is a player. All players have the same strategy space S and every element of S is a possible ballot. A strategy profile σ (also called state) is a member of S^n where σ_i designates the strategy adopted by player $i \in N$. In general, the voting rule is a mapping $\mathcal{F} : S^n \to 2^M$. In this article we restrict ourselves to voting rules that output a singleton, i.e. $\mathcal{F} : S^n \to M$. In the game, the elected candidate is $\mathcal{F}(\sigma)$ for σ being the players' strategy profile.

The preference relation that a player i has over M is expressed with a weak linear order $\succsim_i$. For x and $y \in M$, $x \succsim_i y$ means that player i values x at least as much as y. We write $x \sim_i y$ to say that player i is indifferent between x and y. The strict part of $\succsim_i$ is denoted by $\succ_i$. We denote by $\mathcal{L}(M)$ (resp., $L(M)$) the set of all possible weak linear orders (resp., linear orders) over M.

We have $\succsim_i \in \mathcal{L}(M)$ for each player $i \in N$ and $\succsim = (\succsim_1, \ldots, \succsim_n) \in \mathcal{L}(M)^N$ is called the *truthful profile*, i.e. the profile of the true preferences of the players. We say that the preferences are strict if the truthful profile belongs to $L(M)^N$.

The way the strategy set S is defined depends on the voting rule. A ballot can be a full ranking of the candidates or an unordered subset of M. In the voting game, a player can be insincere and she may strategically report a ballot that does not reflect her true preferences.

2.1 Solution concepts

The focus is on pure strategies and we consider that the players indirectly evaluate a strategy profile, in a sense that they have preferences over $\mathcal{F}(\sigma)$ instead of σ. In this article we analyze plausible outcomes of the game, namely the states (strategy profiles) which are at equilibrium. An equilibrium is a state that is immune to a predefined set of possible moves (also called deviations). A single player or a group of players (also called coalition) may deviate. Individual preferences extend to collective preferences for each group C of players. For x and $y \in M$, we have

- $x >_C y \Leftrightarrow [\forall i \in C,\ x \succ_i y]$,
- $x \geq_C y \Leftrightarrow [\forall i \in C,\ x \succsim_i y]$ and $[\exists j \in C,\ x \succ_j y]$,
- $x \geqq_C y \Leftrightarrow [\forall i \in C,\ x \succsim_i y]$,
- $x \sim_C y \Leftrightarrow [\forall i \in C,\ x \sim_i y]$.

Let us give two notions related to the Pareto dominance. An alternative $x \in M$ is said to be *undominated* (resp., *strictly undominated*) over a coalition $C \subseteq N$ if there does not exist any alternative y such that $y \geq_C x$ (resp., $y \geqq_C x$).

Example 1 *Take four alternatives* $\{a, b, c, d\}$ *and three players* $\{1, 2, 3\}$ *with the following preferences.*

1	$(a \sim b) \succ c \succ d$
2	$(a \sim b) \succ d \succ c$
3	$d \succ (a \sim b) \succ c$

For $C = \{1, 2, 3\}$*, alternative d is strictly undominated, a and b are undominated but not strictly undominated, and c is dominated.*

An undominated alternative always exists for any non-empty coalition $C \subseteq N$ but strictly undominated alternatives may be absent.

Let $\mathcal{C}$ be a non-empty family of coalitions of N, i.e. $\mathcal{C} \subseteq 2^N \setminus \emptyset$; it is the collection of coalitions that can possibly deviate. In the following, σ_C and σ_{-C} stand for the restriction of σ to the strategy adopted by the players of C and $N \setminus C$, respectively. Thus, (σ'_C, σ_{-C}) denotes σ in which σ_i is replaced by σ'_i iff $i \in C$.

Definition 1 (Improving move (IM)) *For a coalition* $C \in \mathcal{C}$*, an improving move from a state* σ *is a joint strategy* $\sigma'_C \in S^{|C|}$ *such that* $\mathcal{F}(\sigma'_C, \sigma_{-C}) >_C \mathcal{F}(\sigma)$.

In an improving move, a coalition deviates only if each member strictly prefers the new state. This type of deviation can be relaxed by considering *weak improving moves*.

Definition 2 (Weak improving move (WIM)) *For a coalition $C \in \mathcal{C}$, a weak improving move from a state σ is a joint strategy $\sigma'_C \in S^{|C|}$ such that $\mathcal{F}(\sigma'_C, \sigma_{-C}) \geq_C \mathcal{F}(\sigma)$.*

Weak improving moves are appealing because they allow the participation of some players who do not benefit, but the deviation cannot harm its instigators.

Interestingly, we can use a pair $(\mathcal{C}, \mu)$ for $\mu \in \{\mathbf{IM},\mathbf{WIM}\}$ to define a notion of equilibrium: a state σ is a $(\mathcal{C}, \mu)$-equilibrium if no coalition $C \in \mathcal{C}$ can deviate from σ with a move of type μ. Thus, a *Nash equilibrium* (NE) [20] is a $(\mathcal{C},\mathbf{IM})$-equilibrium where $\mathcal{C} = \{\{i\} : i \in N\}$. Similarly a *strong equilibrium* (SE) [3] is a $(\mathcal{C},\mathbf{IM})$-equilibrium where $\mathcal{C} = 2^N \setminus \emptyset$. A *super strong equilibrium* (SSE) [29, 9] is a $(\mathcal{C},\mathbf{WIM})$-equilibrium where $\mathcal{C} = 2^N \setminus \emptyset$. These three solutions concepts are linked as follows: NE $\supseteq$ SE $\supseteq$ SSE.

On one hand, the strong equilibrium and the super strong equilibrium are more sustainable than the Nash equilibrium because they preclude a larger set of possible deviations. On the other hand, (super) strong equilibria are less likely to exist than Nash equilibria.

An argument against the (super) strong equilibrium is that in many situations, not all coalitions are conceivable. In this article we consider a special collection of coalitions $\mathcal{C}$ that takes into account the social context in which the players are embedded. Given a social network of voters represented by a graph G with node set N and edge set E, an edge $(u, v) \in E$ indicates that players u and v are related, e.g. they have the possibility to communicate, so these two players can participate in a deviating coalition. As done in [9, 12], it makes sense to define the cliques of graph G as the possible coalitions. In that case $\mathcal{C} = \{C \in 2^N \mid \forall i, j \in C,\ (i, j) \in E\}$ and each coalition $C \in \mathcal{C}$ has a set of neighbors $\mathcal{N}(C) = \{i \in N \setminus C \mid \exists j \in C \text{ such that } (i, j) \in E\}$.

Interestingly, the fact that the players are related implies that each individual is not only guided by her own preferences, but she can also care about how deviating can negatively impact her relatives. In order to take into account the social context and the fact that a player can have consideration for other players (her neighbors in the graph), an appropriate notion of deviation, called *considerate improving move*, was introduced in [12].

Definition 3 (Considerate improving move (CIM)) *For coalition $C \in \mathcal{C}$, a considerate improving move σ'_C from a state σ is a weak improving move where in addition, $\mathcal{F}(\sigma'_C, \sigma_{-C}) \geqq_{\mathcal{N}(C)} \mathcal{F}(\sigma)$.*

In a considerate improving move by coalition C, at least one player in C is better off and no player of $C \cup \mathcal{N}(C)$ can be worse off.

Consequently, the pair $(\mathcal{C},\mathbf{CIM})$, where $\mathcal{C}$ contains all the cliques of the social network $G = (N, E)$, leads to a new type of equilibrium called *considerate equilibrium* [12]. In what follows, we will say that a game always admits a considerate equilibrium if for any instance of the game, and any social network $G = (N, E)$, there exists at least one state σ for which no coalition of players forming a clique in G has a considerate improving move.

Example 2 *Consider an instance where $N = \{1, 2, 3\}$ and $M = \{a, b\}$. The social network is a path, so the possible coalitions are $\mathcal{C} = \{\{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}\}$. The profile of preferences is:*

1	$b \sim a$
2	$b \succ a$
3	$a \succ b$

*Consider a state σ where a is elected. Coalitions $\{1, 2\}$ and $\{2\}$ are the only coalitions having incentive to move, they want to make b win since $b \geq_{\{1,2\}} a$ and $b \geq_{\{2\}} a$. Because $\{3\} \in \mathcal{N}(\{1, 2\}) \cap \mathcal{N}(\{2\})$ and $a \succ_3 b$, coalitions $\{1, 2\}$ and $\{2\}$ cannot perform a **CIM** (without even taking into account the ability of the coalitions to change the outcome). Thus, σ is a considerate equilibrium.*

Following the same idea, if the social network $G = (N, E)$ is composed of a set of disjoint cliques and only maximal cliques of G are considered in $\mathcal{C}$, then an equilibrium associated with $(\mathcal{C},\mathbf{WIM})$ is called a *partition equilibrium* [9] (in this case, a **CIM** move corresponds to a **WIM** move). Clearly, if a considerate equilibrium exists, then a partition equilibrium exists. Furthermore, if $E = \emptyset$, then a considerate equilibrium corresponds to a Nash equilibrium. Thus, a Nash equilibrium is a special case of a considerate equilibrium and if a considerate equilibrium exists then a Nash equilibrium exists.

2.2 Dynamics

Beyond the existence of an equilibrium, we also take into account the *dynamics* of the voting game. Starting from an inital state, the players (or coalitions of players) can manipulate by successively changing their strategy. Each such move is supposed to be an improvement for the deviator(s) and we will exclusively study the moves defined above: **IM**, **WIM** and **CIM**. We will resort to indices to stress the step at which a state occurs. Namely σ^0 is the initial state whereas σ^t denotes the state at step t. A dynamics is a sequence $\sigma^0\sigma^1 \ldots \sigma^r$ such that each pair of consecutive steps $i, i + 1$ is associated with an improving move for some coalition[2] that turns σ^i into σ^{i+1}. If the possible deviating coalitions belong to $\mathcal{C}$ and moves are of type μ then the dynamics is said to be associated with the $(\mathcal{C}, \mu)$-equilibrium. A dynamics ends when no further move is possible and therefore a $(\mathcal{C}, \mu)$-equilibrium is reached. We have *convergence* when the dynamics is finite for every initial state. A natural restriction (see e.g. [16]) is to suppose that the initial state is truthful. Indeed, the players start by giving their true opinion and if they are not satisfied with the outcome then they reconsider their vote. A dynamics fails to converge when a state appears more than once.

In a non-equilibrium state σ, a coalition C (which can be a singleton) may have more than one possible improving moves. For improving moves of type μ, the *better replies* are the set of all joint strategies σ'_C such that $\mathcal{F}(\sigma_{-C}, \sigma'_C) \geq_C \mathcal{F}(\sigma)$, provided that a move of type μ turns σ into (σ_{-C}, σ'_C). A better reply σ'_C is a *best reply* if no other joint strategy σ''_C satisfies $\mathcal{F}(\sigma_{-C}, \sigma''_C) \geq_C \mathcal{F}(\sigma_{-C}, \sigma'_C)$, i.e. it leads to an outcome that is undominated for C. We say that an improving move is *unanimous* regarding a coalition C if all members of C adopt the same strategy.

2.3 Voting rules

Numerous voting systems use *scores* and the rule is to elect the candidates who reach the highest score. Let Sc be a function which associates score $Sc(\sigma, x) \in \mathbb{R}$ with every pair $(\sigma, x) \in S^n \times M$. Since we focus on voting functions that output a single alternative, we use in this work a deterministic tie-breaking rule which follows an absolute priority ranking over the alternatives. Namely, a fixed ranking $\rhd$ of the candidates is employed and amongst the candidates that attain the best score, the one coming first in $\rhd$ wins the election.

Let us assume for the moment that $S = L(M)$, i.e. every player is asked to report a strict ordering of M. In *positioning scoring rules*

[2] No two coalitions can move simultaneously.

(PSR), the score $Sc(\sigma, x)$ of alternative x under profile σ depends on the absolute position of x in each ballot. Concretely, we are given a vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that $\alpha_1 \geq \cdots \geq \alpha_m$ and $\alpha_1 > \alpha_m$. If x is placed at position k in a ballot, then x receives α_k points. The score of x is defined as the sum of these points over all ballots.

Thus, each PSR is characterized by its vector α. We can mention in particular the *Borda* rule in which $\alpha = (m-1, m-2, \ldots, 0)$ and the *k-approval* rule for $k \in [m-1] = \{1, \ldots, m-1\}$ with $\alpha = (1, \ldots, 1, 0, \ldots, 0)$ in which k consecutive ones are followed by $m-k$ consecutive zeros. If k is equal to 1, then the associated voting rule is called *Plurality*. If k is equal to $m-1$, then the associated voting rule is called *Antiplurality* (also known as *Veto*).

Note that for Plurality and Antiplurality, only a single alternative in each ballot is useful. This alternative is respectively the first one (the only candidate approved by the voter) and the last one (the only candidate vetoed by the voter). Thus, in these two cases, we assume that the strategy space S is equal to M instead of $L(M)$, and therefore a stategy profile σ belongs to M^n.

In *runoff voting rules*, the election runs in several rounds. We can mention in particular the well-known *Plurality with runoff* which is actually used for political elections in many countries. In such a rule, given a strategy profile in $L(M)^n$ as an input, the first round selects the two best ranked alternatives regarding Plurality (use $\rhd$ to break ties). Then, all eliminated candidates are removed from the profile, providing a profile where only the two selected alternatives are present. The second round elects the winner of this resulting profile under Plurality. Note that only the first round is necessary if an alternative gets a majority of the votes. The *Single Transferable Vote (STV)* rule is an iterated process taking as an input a strategy profile in $L(M)^n$. At each step, the loser of Plurality gets eliminated (use $\rhd$ to break ties), and the profile is updated by removing this alternative from the ballots of the agents. The process continues until an alternative obtains an absolute majority of votes under Plurality and thus gets elected.

Given a strategy profile σ with the associated linear order $\succ^\sigma \in L(M)^n$ and two alternatives $a, b \in M$, let $W(a, b)$ and $\omega(a, b)$ be the set of voters who prefer a to b and the number of voters who prefer a to b, respectively. That is, $W(a, b) = \{i \in N \mid a \succ_i^\sigma b\}$ and $\omega(a, b) = |W(a, b)|$.

Some voting rules, instead of taking into account the position of an alternative in a ballot like PSRs, are based on pair-wise comparisons. The *Copeland* rule assigns to each alternative x the number of alternatives that x beats in a pair-wise election in a given state $\sigma \in L(M)^n$, that is $Sc(\sigma, x) = |\{y \in M \setminus x \mid \omega(x, y) > \frac{n}{2}\}|$. The *Maximin* rule assigns to x the mimimum number of voters in favour of x in any pair-wise comparison in σ, i.e. $Sc(\sigma, x) = \min_{y \in M \setminus x} \omega(x, y)$.

3 Positionning Scoring Rules (PSRs)

3.1 Plurality

A profile σ is said to be *unanimous* if there exists a strategy $s \in S$ such that $\sigma_i = s$ for all $i \in N$. Since $S = M$ in the voting game with Plurality, every alternative x induces a unanimous profile $(x, \ldots, x)$.

The voting game under Plurality always admits a Nash equilibrium, e.g. the unanimous profile $(x, \ldots, x)$ where x is the best alternative in the tie breaking rule $\rhd$. However, it is known that a strong equilibrium is not guaranteed to exist (see e.g. [26]), as we can see in the following example showing a well-known Condorcet paradox.

Example 3 *$N = \{1, 2, 3\}$, $M = \{a, b, c\}$, $a \rhd b \rhd c$ and the profile of preferences is:*

1	$a \succ b \succ c$
2	$c \succ a \succ b$
3	$b \succ c \succ a$

If a is elected then players 2 and 3 have incentive to deviate to c. If c is elected then players 1 and 3 have incentive to deviate to b. If b is elected then players 1 and 2 have incentive to deviate to a.

This example also rules out the existence of a super strong equilibrium but we shall see that a considerate equilibrium must exist.

Theorem 1 *Every instance of the voting game with players' preferences in $\mathcal{L}(M)$ and $\mathcal{F}$=Plurality admits a considerate equilibrium.*

Proof: The social network is a graph $G = (N, E)$. Let $a \in M$ be the best alternative w.r.t. $\rhd$. Under Plurality, the only coalitions which are able to change the outcome by deviating from a unanimous state are cliques of size at least $n/2$. For the unanimous state $(a, \ldots, a)$, these *powerful coalitions* are only those of size strictly larger than $n/2$, that we denote $\mathcal{C}_p$. Let $\mathcal{C}_{n/2}$ be the set of cliques of size exactly $n/2$. Let us denote by $\mathcal{Q}$ the set of all agents belonging to a coalition which can change the outcome of a unanimous state, that is $\mathcal{Q} = \bigcup_{Q \in \mathcal{C}_p \cup \mathcal{C}_{n/2}} Q$. If $\mathcal{C}_p = \emptyset$ then the unanimous profile $(a, \ldots, a)$ is a considerate equilibrium. Suppose from now on that $\mathcal{C}_p \neq \emptyset$.

For any coalitions $C \in \mathcal{C}_p$ and $C' \in (\mathcal{C}_p \cup \mathcal{C}_{n/2})$, C and C' have at least one member in common, therefore $C \subseteq (\mathcal{N}(C') \cup C')$ and $\mathcal{Q} \subseteq (C \cup \mathcal{N}(C))$, for all $C \in \mathcal{C}_p$. Hence, C' cannot deviate so that a worse candidate, from the viewpoint of at least one member of C, is elected. If an alternative x strictly undominated over $C \in \mathcal{C}_p$ exists, then the unanimous profile $(x, \ldots, x)$ is a considerate equilibrium.

Suppose from now on that for every coalition $C \in \mathcal{C}_p$, there does not exist any strictly undominated alternative over C. However, an undominated alternative over C must exist. For a coalition C and an alternative x, let I_x^C be the *indifference set* of x within C, i.e. $I_x^C = \{y \in M \setminus \{x\} : y \sim_C x\}$. Let ND be the subset of alternatives which are both undominated over at least one coalition of $\mathcal{C}_p$ and undominated over $\mathcal{Q}$. The set ND is never empty because $\mathcal{C}_p \neq \emptyset$. We denote by x the best alternative of ND w.r.t $\rhd$ and $C \in \mathcal{C}_p$ is the coalition for which x is undominated. We will analyze deviations from the unanimous profile $(x, \ldots, x)$ and every time a deviation is performed, then we will consider for the next step the unanimous profile of the corresponding winner. Since we start from unanimous profiles, the deviations are only performed by coalitions in $(\mathcal{C}_p \cup \mathcal{C}_{n/2})$. The winner of every deviation belongs to $I_x^C \cup \{x\}$ because the deviating coalitions have consideration for C. A coalition $C' \in \mathcal{C}_p$ cannot deviate from $(x, \ldots, x)$, because $\mathcal{Q} \subseteq (C' \cup \mathcal{N}(C'))$, x is undominated over $\mathcal{Q}$ by definition of ND and $C' \subseteq \mathcal{Q}$. However, a coalition $C' \in \mathcal{C}_{n/2}$ can deviate if there exists $y \in I_x^C$ such that $y \geq_{C'} x$ and $y \rhd x$. If $y \in$ ND then $y \rhd x$ contradicts the fact that x is the best alternative of ND w.r.t. $\rhd$, so $y \notin$ ND. Since $y \geq_{C'} x$ and x is undominated over $\mathcal{Q}$, there exists $j \in \mathcal{Q} \setminus (C \cup C')$ such that $x \succ_j y$. If there exists $C'' \in (\mathcal{C}_p \cup \mathcal{C}_{n/2})$ for which $j \in C''$ with $C'' \cap C' \neq \emptyset$, then C' cannot deviate to y by consideration for j because $j \in \mathcal{N}(C')$, therefore $\overline{C'} = N \setminus C'$ is a coalition belonging to $\mathcal{C}_{n/2}$ and $j \in \overline{C'}$. It follows that $\mathcal{Q} = N$. Obviously, $C' \cap C \neq \emptyset$ and $\overline{C'} \cap C \neq \emptyset$. These voters cannot be the beneficiaries of a deviation since they are indifferent among alternatives of I_x^C. Let $I = C' \setminus C$ and $J = \overline{C'} \setminus C$ denote respectively the voters of C' and $\overline{C'}$ who do not belong to C. Then, C' deviates to y and we consider for the next step the profile $(y, \ldots, y)$. Observe that for a given succession of such deviations performed by coalitions in $\mathcal{C}_{n/2}$, the winners are not in ND and the rank of the winner in $\rhd$ strictly improves. Thus, there is only a finite number of such deviations. Now, every time a coalition $C'' \in \mathcal{C}_p$ can deviate to $z \in I_x^C$ from a unanimous profile $(y^t, \dots, y^t)$ where y^t is the winner at step t (we can verify that C'' cannot deviate to x), it follows that $z \geq_N y^t$ because $\mathcal{Q} \subseteq (C'' \cup \mathcal{N}(C''))$. There exists a voter i such that $z \succ_i y^t$, thus $i \in I \cup J$, say $i \in J$, implying that $j \in \mathcal{N}(C')$. So, no coalition can make j less satisfied. Hence, the improvements within the sequence of the defined deviations follow the preferences of every such j and during a succession of deviations performed by coalitions of $\mathcal{C}_{n/2}$ the rank of the winner in $\rhd$ is improved. Since we have a finite number of alternatives, we finally reach an alternative for which the unanimous associated profile is a considerate equilibrium. ■

Note that when $n > 2$, the considerate equilibria that we constructed in the previous proof are also Nash equilibria because they are unanimous profiles, showing that we can combine two requirements which may be conflicting. Indeed, there exist instances for $n = 2$ where the set of considerate equilibria and the set of Nash equilibria do not intersect.

Theorem 1 gives the existence of a considerate equilibrium and therefore, existence of a partition equilibrium, in any instance of the voting game under Plurality. However, if we let the players deviate, do we reach this equilibrium? Unfortunately, the answer may be negative even for the partition equilibrium and additional natural restrictions. We impose for instance that every deviation is a unanimous direct best reply[3], the initial profile is truthful and the players' preferences are strict.

Proposition 1 *The dynamics associated with the partition equilibrium may not converge in the voting game with $\mathcal{F}$=Plurality, even if the initial profile is truthful, each move is a unanimous direct best reply and the preferences are strict, single-peaked and single-crossing.*

Proof: Consider an instance where $N = \{1, 2, \dots, 12\}$, $M = \{a, b, c, d\}$ and $c \rhd d \rhd b \rhd a$. The profile of preferences is:

1, 2, 3	$a \succ b \succ d \succ c$
4	$b \succ a \succ d \succ c$
5, 6	$b \succ a \succ d \succ c$
7, 8, 9	$c \succ d \succ b \succ a$
10, 11	$d \succ b \succ a \succ c$
12	$d \succ b \succ a \succ c$

The preferences are single-peaked using the axis over the candidates (c, d, b, a) and single-crossing using the axis over the voters $(1, 2, 3, 4, 5, 6, 10, 11, 12, 7, 8, 9)$. The partition over the voters is $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7, 8, 9\}, \{10, 11\}, \{12\}\}$. The next table gives a sequence of states where the first and last ones coincide. Deviations are marked with bold letters.

Steps \ Players	1	2	3	4	5	6	7	8	9	10	11	12	Winner
0	a	a	a	b	b	b	c	c	c	d	d	d	c
1	a	a	a	**a**	b	b	c	c	c	d	d	d	a
2	a	a	a	a	b	b	c	c	c	**b**	**b**	d	b
3	a	a	a	a	b	b	**d**	**d**	**d**	b	b	d	d
4	a	a	a	**b**	b	b	d	d	d	b	b	d	b
5	a	a	a	b	b	b	d	d	d	**d**	**d**	d	d
6	a	a	a	b	b	b	**c**	**c**	**c**	d	d	d	c

■

Interestingly, Proposition 1 can be mitigated if we consider a special partition of N where all coalitions have the same size. The result follows from a simple extension of a proof given in [16].

Proposition 2 *If $\mathcal{P}$ is a partition of N such that all coalitions of $\mathcal{P}$ have the same size, then the dynamics associated with a ($\mathcal{P}$,***WIM***)-equilibrium converges for any profile of preferences, any initial profile, and if unanimous direct replies are performed.*

A *Condorcet winner* is an alternative x which wins with an absolute majority against any other alternative, i.e. $\omega(x, y) > \frac{n}{2}$ for all $y \in M \setminus \{x\}$. Plurality is not *Condorcet consistent* which means that it does not always elect the Condorcet winner when it exists. A unanimous profile with the Condorcet winner is always a considerate equilibrium (as stated in [26] for strong equilibria) since no absolute majority of voters prefer another alternative. However, when a Condorcet winner exists, the game may not converge to a state electing it, even with a truthful initial profile. This observation is due to the non-Condorcet consistency of the Plurality voting rule.

Example 4 *Let $N = \{1, 2, 3\}$, $M = \{a, b, c, d\}$, $a \rhd b \rhd c \rhd d$, the coalitions are $\{\{1\}, \{2\}, \{3\}\}$ and the profile of preferences is:*

1	$a \succ b \succ c \succ d$
2	$c \succ b \succ d \succ a$
3	$d \succ b \succ a \succ c$

From the truthful initial profile $\{a, c, d\}$ electing a, the only possible deviation is that agent 2 deviates to alternative d, making d winner. We reach the state $\{a, d, d\}$ electing d, which is a considerate equilibrium whereas b is Condorcet winner.

3.2 Antiplurality

Before going into detailled analysis of antiplurality, we provide the definition of a *feasible elimination procedure* (f.e.p.), a tool introduced in social choice theory (see e.g. [22]).

Definition 4 *For a mapping $\beta : M \to \mathbb{N}$ such that $\sum_{x \in M} \beta(x) = n+1$, an f.e.p. is a sequence $(x_1, C_1; x_2, C_2; \dots, C_{m-1}; x_m)$ where $\forall i \in [m-1], C_i \subseteq N$ is such that: (i) $|C_i| = \beta(x_i)$ and $\forall j \in [m-1] \setminus \{i\}, C_i \cap C_j = \emptyset$, (ii) $M = \bigcup_{k \in [m]} x_k$, and (iii) $\forall l \in C_i$ and $\forall k \in \{i+1, \dots, m\}, x_k \succ_l x_i$.*

It has been shown that an f.e.p. exists for any preference profile in $L(M)^n$ and for any mapping β such that $\sum_{x \in M} \beta(x) = n + 1$ (see e.g. Remark 9.2.1 in [22]). We will use this fact[4] to show the following proposition:

Proposition 3 *A strong equilibrium exists in any instance of the voting game with players' preferences in $L(M)$ and $\mathcal{F}$=Antiplurality.*

Proof: In order to prove the proposition, we show that it is possible to construct, from a given f.e.p., a state σ which is a strong equilibrium for antiplurality. First of all, let us define the mapping β used to define the f.e.p. The value $\beta(x)$ corresponds to the minimum amount of vetos required to ensure that x cannot be chosen by Antiplurality (whatever the other ballots). Let q and r be the quotient and the rest of the euclidean division of n by m, respectively. It is easy to check that $\beta(x) = q + 1$ if x is ranked among the $r + 1$ first alternatives

[3] Convergence to a Nash equilibrium is not guaranteed with better or best replies but it converges with direct replies [16]. For Plurality, replies are direct when the voters deviate by voting for the new winner.

[4] Note that the condition $n + 1 \geq m$ appears in existence result of [22] but it is easy to fulfill this condition by only keeping the $n + 1$ first candidates in the tie-breaking rule. Indeed, only these alternatives can be elected.

in the tie-breaking rule $\rhd$, and $\beta(x) = q$ otherwise. Note that by the definition of q and r, $\sum_{x \in M} \beta(x) = n + 1$ holds.

Let $(x_1, C_1, x_2, C_2, \ldots, C_{m-1}, x_m)$ be an f.e.p. for the mapping β described above, and let σ be a state such that $\forall i \in [m-1]$ and $\forall j \in C_i$, $\sigma_j = x_i$. Note that by the definition of β, $\mathcal{F}(\sigma) = x_m$. We conclude the proof by showing that σ is a strong equilibrium. By contradiction, if σ is not a strong equilibrium then there exists a coalition $C \subseteq N$ and a joint strategy σ'_C such that $\mathcal{F}(\sigma'_C, \sigma_{-C}) >_C x_m$. Let y denote the candidate $\mathcal{F}(\sigma'_C, \sigma_{-C})$. By the definition of β, there must be $i \in [m-1]$ and $l \in C_i$ such that $y = x_i$ and $l \in C$, because otherwise y is vetoed by at least $\beta(y)$ voters and cannot be chosen by $\mathcal{F}$. But by (iii) of Definition 4, this implies that $x_m \succ_l x_i = y$, a contradiction with $y >_C x_m$. ■

Note that the strong equilibrium exhibited in the above proof can be constructed in polynomial time.

Observe that when a Condorcet winner exists in the preference profile, a strong equilibrium electing it may not exist, as we can see in the next example. Actually, the notion of winner under Antiplurality is far from being related to the concept of Condorcet winner, and a minority of voters can have a significant power within this rule.

Example 5 *Consider an instance where $N = \{1, 2, 3\}$ and $M = \{a, b, c, d\}$. The profile of preferences is:*

1	$b \succ a \succ c \succ d$
2	$c \succ d \succ a \succ b$
3	$b \succ d \succ c \succ a$

Alternative b is the Condorcet winner. However, from a state where b is elected, voter 2 has always incentive to deviate by vetoing b, and this veto is sufficient to avoid the election of b. Hence, there is no strong equilibrium electing b.

Even if we relax the assumption of strict preferences, Proposition 3 continues to hold. However, this proposition does not ensure the existence of a super strong equilibrium when the preferences are not strict, as illustrated by the following counterexample.

Example 6 *Let $N = \{1, 2, 3\}$, $M = \{a, b\}$ and the profile of preferences is:*

1	$a \succ b$
2	$b \succ a$
3	$a \sim b$

If a is elected then coalition $\{2, 3\}$ can deviate and veto a. If b is elected then coalition $\{1, 3\}$ can deviate and veto b.

Nevertheless, the following theorem shows that a considerate equilibrium is guaranteed to exist even with non-strict preferences.

Theorem 2 *A considerate equilibrium exists in any instance of the voting game with players' preferences in $\mathcal{L}(M)$ and $\mathcal{F}$=Antiplurality.*

Proof: The proof relies on a refinement of the f.e.p. to the concept of considerate equilibrium. The *considerate f.e.p.* (c.f.e.p.) is defined in a similar fashion as f.e.p., except that condition (iii) is replaced by (iii') $\forall l \in C_i$ and $\forall k \in \{i+1, \ldots, m\}$, $x_k \succ_l x_i$, or $[x_k \sim_l x_i$ and $x_i \not\gtrsim_{\delta(l)} x_k]$, where $\delta(l)$ is the set of neighbors of l in G. First of all, let us show that a c.f.e.p. exists for any preference profile $\succsim \in \mathcal{L}(M)^n$. To this end, for any voter $i \in N$, we construct a strict preference $\succ'_i$ which is consistent with the strict part of $\succsim_i$, and where ties are broken by $\geq_{\delta(i)}$ (or arbitrarily in case of incomparability for $\geq_{\delta(i)}$). This construction leads to a profile of strict preferences $\succ'$. By Remark 9.2.1 of [22], we know that an f.e.p. exists for $\succ'$. We state that such an f.e.p. is also a c.f.e.p. for $\succsim$. Indeed, conditions (i) and (ii) trivially hold because they are similar for both f.e.p. and c.f.e.p., and they do not depend on the considered profile of preferences. Furthermore, the construction of $\succ'$ ensures that for any $l \in N$, $x \succ'_l y$ implies $x \succ_l y$, or $[x \sim_l y$ and $y \not\gtrsim_{\delta(l)} x]$. Therefore, (iii') holds and the f.e.p. for $\succ'$ is also a c.f.e.p. for $\succsim$.

The remainder of this proof follows the same line as the proof of Proposition 3. Let $(x_1, C_1; x_2, C_2; \ldots, C_{m-1}; x_m)$ be a c.f.e.p. for the mapping β described in the proof of Proposition 3, and let σ be a state such that $\sigma_j = x_i$, $\forall i \in [m-1]$ and $\forall j \in C_i$. Let us show that σ is a considerate equilibrium. By contradiction, if σ is not a considerate equilibrium then there exists a coalition $C \in \mathcal{C}$ and a joint strategy σ'_C such that $\mathcal{F}(\sigma'_C, \sigma_{-C}) \geq_C x_m$ and $\mathcal{F}(\sigma'_C, \sigma_{-C}) \geqq_{\mathcal{N}(C)} x_m$. Let y denote the candidate $\mathcal{F}(\sigma'_C, \sigma_{-C})$. There must be $i \in [m-1]$ and $l \in C_i$ such that $y = x_i$ and $l \in C$. Furthermore, any neighbor of l in G belongs to C or $\mathcal{N}(C)$, and no other voter belongs to C. Therefore, $C \cup \mathcal{N}(C) = \delta(i) \cup \{i\}$. But by condition (iii') of the definition of a c.f.e.p., this implies that $x_m \succ_l x_i = y$ or $x_m \sim_i x_i$ and $y = x_i \not\gtrsim_{C \cup \mathcal{N}(C)} x_m$, a contradiction with $y \geq_C x$ and $y \geqq_{\mathcal{N}(C)} x_m$. ■

Unfortunately, the dynamics associated with the considerate equilibrium is not guaranteed to converge, even if we restrict ourselves to the dynamics associated with the partition equilibrium and we consider unanimous direct best replies[5].

Proposition 4 *The dynamics associated with the partition equilibrium may not converge in the voting game with $\mathcal{F}$=Antiplurality, even if the initial profile is truthful, each move is a unanimous direct best reply and the preferences are strict, single-peaked and single-crossing.*

Proof: Consider an instance where $N = \{1, 2, \ldots, 6\}$, $M = \{a, b, c, d\}$ and $a \rhd b \rhd c \rhd d$. The preferences are single-peaked using the axis over the candidates (c, d, a, b) and single-crossing using the axis over the voters $(2, 1, 6, 3, 4, 5)$. The profile of preferences is:

1	$d \succ c \succ a \succ b$
2	$c \succ d \succ a \succ b$
3	$a \succ d \succ b \succ c$
4, 5	$b \succ a \succ d \succ c$
6	$a \succ d \succ c \succ b$

The partition over the voters is $\{\{1\}, \{2\}, \{3\}, \{4, 5\}, \{6\}\}$. The next table gives a sequence of states where the steps 1 to 7 form a cycle. Deviations are marked with bold letters. Step 0 corresponds to the truthful profile.

Steps \ Players	1	2	3	4	5	6	Winner
0	$-b$	$-b$	$-c$	$-c$	$-c$	$-b$	a
1	$-b$	$-\mathbf{a}$	$-c$	$-c$	$-c$	$-b$	d
2	$-b$	$-a$	$-c$	$-c$	$-c$	$-\mathbf{d}$	a
3	$-b$	$-a$	$-c$	$-\mathbf{a}$	$-\mathbf{a}$	$-d$	b
4	$-b$	$-a$	$-c$	$-a$	$-a$	$-\mathbf{b}$	d
5	$-b$	$-\mathbf{d}$	$-c$	$-a$	$-a$	$-b$	c
6	$-b$	$-d$	$-c$	$-\mathbf{c}$	$-\mathbf{c}$	$-b$	a
7	$-b$	$-\mathbf{a}$	$-c$	$-c$	$-c$	$-b$	d

■

[5] Replies are direct under Antiplurality when the voters veto the current winner. Convergence to a Nash equilibrium is guaranteed when direct replies are performed [14, 24].

3.3 Other PSRs

In this section, the strategy space S is $L(M)$. Each PSR is characterized by its score vector α. After normalization, the score vector of every PSR can be written as $\alpha = (1, \alpha_2, \dots, \alpha_{m-1}, 0)$ where $\alpha_i \in [0, 1]$ and the α_is remain non increasing. The PSR is neither Plurality nor Antiplurality iff $\alpha_2 > 0$ and $\alpha_{m-1} < 1$. If $\alpha_i \in \{0, 1\}$ for all $i \in \{2, \dots, m-1\}$, then the associated PSR is k-approval.

We know that for Borda and k-approval (k is a constant), a voting game may not converge to a Nash equilibrium [14], even if the initial state is truthful and only best replies are used. Therefore, this result holds too for a considerate equilibrium. Furthermore, we can prove that a Nash equilibrium (and thus, a considerate equilibrium) is not guaranteed to exist for Borda and k-approval where k is fixed according to either the number of consecutive ones, or to the number of consecutive zeros.

Proposition 5 *Let l be an integer such that $l > 1$ and consider the k-approval rule where $k = l$ or $k = m - l$. A Nash equilibrium is not guaranteed to exist in the voting game, even if the players' preferences are strict and single-peaked.*

Proof: Consider an instance where $N = \{1, 2\}$, $M = \{x_1, x_2, \dots, x_{2k}\}$, and $x_1 \rhd x_2 \rhd \cdots \rhd x_{2k}$. Thus, $m = 2k$. The profile of preferences is:

$$\begin{array}{ll} 1 & x_1 \succ x_2 \succ \cdots \succ x_m \\ 2 & x_m \succ x_{m-1} \succ \cdots \succ x_1 \end{array}$$

With such a k and m, either the winner is present in the k first ranked candidates of both voters, or the winner is x_1. Moreover, there is always one agent who prefers at least k alternatives to the current winner. Suppose alternative x_i is elected. If $i \leq k$, then a better reply for agent 2 is to rank candidates $x_1, \dots, x_i$ last in her new ballot and put at the first position her preferred candidate within the top k of agent 1. If $i > k$, a better reply for agent 1 is to place candidates $x_i, \dots, x_m$ among the k last ranked of her new ballot. ■

Proposition 6 *A Nash equilibrium is not guaranteed to exist in the voting game where $\mathcal{F}$ is a PSR with $0 < \alpha_2 \leq \frac{1}{2}$, even if the players' preferences are strict and single-peaked.*

Proof: Consider a 2-player instance where $M = \{a, b, c\}$, $a \rhd b \rhd c$ and the profile of preferences is:

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & c \succ b \succ a \end{array}$$

If a is elected then agent 2 puts a last in her new ballot and puts first the best ranked alternative within agent 1's ballot between b and c. If c is elected then agent 1 puts c last in her new ballot and puts first the best ranked alternative within agent 2's ballot between a and b. Finally, if b is elected then agent 1 can make a the winner: if c is ranked first by agent 2 then agent 1 plays $a \succ b \succ c$, otherwise she plays $a \succ c \succ b$. ■

Note that Borda is covered by this proposition so a Nash equilibrium (and thus, a considerate equilibrium) is not guaranteed to exist.

4 Runoff voting rules

The voting game under Plurality with runoff and STV always admits a Nash equilibrium, e.g. the strategy profile where the best alternative coming first in the tie breaking rule $\rhd$ is placed on the top of each ballot. More generally, we prove that a considerate equilibrium is guaranteed to exist in any instance of the game under Plurality with run-off and STV. We cannot generalize again to the existence of a strong equilibrium, as we can see for the profile given in Example 3.

Theorem 3 *Every instance of the voting game with players' preferences in $\mathcal{L}(M)$ and $\mathcal{F} \in$ {STV, Plurality with runoff} admits a considerate equilibrium.*

Proof: The proof follows the same idea as the proof for Plurality (Theorem 1). It suffices to show that under STV and Plurality with runoff the set of powerful coalitions $\mathcal{C}_p$ also corresponds to the set of coalitions with size strictly larger than $n/2$. Indeed, if a is the best alternative w.r.t. $\rhd$, then in the unanimous profile where a is placed on the top of each ballot, a is elected without any further round because it gets the absolute majority of votes. Thus, only a coalition of size strictly larger than $n/2$ can make the outcome change by placing another alternative unanimously on the top of the new ballot. ■

We have existence of a considerate equilibrium. Unfortunately, the game may not converge to it since it may not even converge to a Nash equilibrium, as the next proposition states. Moreover, this is still the case for restricted best replies.

Proposition 7 *The dynamics associated with the Nash equilibrium is not guaranteed to converge in the voting game with $\mathcal{F} \in$ {STV, Plurality with runoff}, even if the initial profile is truthful, the players' preferences are strict, single-peaked and single-crossing, and each move is a best reply minimizing the distance to the truthful ballot in terms of number of differences in pair-wise comparisons.*

Proof: Consider an instance where $N = \{1, 2, 3, 4\}$, $M = \{a, b, c, d\}$ and $a \rhd b \rhd c \rhd d$. The preferences are single-peaked using the axis over the candidates (a, c, d, b) and single-crossing using the axis over the voters $(2, 3, 4, 1)$. The next table gives a sequence of states where the first and last ones coincide. Deviations are marked with bold letters. Step 0 represents the truthful profile. Each linear order is the ballot of a voter and the last line of the table specifies the winner at each step.

Step 0	Step 1	Step 2	Step 3	Step 4
$c \succ a \succ d \succ b$	$\mathbf{a} \succ \mathbf{c} \succ \mathbf{d} \succ \mathbf{b}$	$a \succ c \succ d \succ b$	$\mathbf{c} \succ \mathbf{a} \succ \mathbf{d} \succ \mathbf{b}$	$c \succ a \succ d \succ b$
$b \succ d \succ c \succ a$	$b \succ d \succ c \succ a$	$\mathbf{d} \succ \mathbf{b} \succ \mathbf{c} \succ \mathbf{a}$	$d \succ b \succ c \succ a$	$\mathbf{b} \succ \mathbf{d} \succ \mathbf{c} \succ \mathbf{a}$
$d \succ b \succ c \succ a$	$d \succ b \succ c \succ a$	$d \succ b \succ c \succ a$	$d \succ b \succ c \succ a$	$d \succ b \succ c \succ a$
$c \succ d \succ a \succ b$	$c \succ d \succ a \succ b$	$c \succ d \succ a \succ b$	$c \succ d \succ a \succ b$	$c \succ d \succ a \succ b$
b	a	d	c	b

This counterexample works for STV and Plurality with runoff. ■

5 Voting rules based on pair-wise comparisons

5.1 Copeland

We show by a simple counterexample that even a Nash equilibrium is not guaranteed to exist in the voting game when using the Copeland rule. Therefore, a considerate equilibrium may not exist.

Proposition 8 *A Nash equilibrium is not guaranteed to exist in the strategic voting game where $\mathcal{F}$=Copeland, even if the preferences are strict and single-peaked.*

Proof: Consider an instance where $N = \{1, 2\}$, $M = \{a, b, c\}$ and $a \rhd b \rhd c$. The profile of preferences is:

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & c \succ b \succ a \end{array}$$

An alternative x wins against y in a pair-wise election if x is ranked before y in the two ballots. If a wins, then agent 2 deviates by placing a at the last position of her new ballot and placing on top of her new ballot the first ranked candidate in the ballot of agent 1 between b and c. If c wins, then it suffices for agent 1 to place c at the last position of her new ballot because c is ranked last in the tie-breaking. Finally, if b wins, then if c is not ranked first in agent 2's ballot, then agent 1 plays $a \succ c \succ b$; otherwise, agent 1 plays $a \succ b \succ c$. In conclusion, there is no Nash equilibrium in this instance. ■

5.2 Maximin

The voting game under Maximin always admits a Nash equilibrium, e.g. the profile where the best alternative w.r.t. $\rhd$ is placed on top of each ballot. However, a strong equilibrium is not guaranteed to exist as we can see with the profile given in Example 3. Nevertheless, a considerate equilibrium is guaranteed to exist in any instance.

Theorem 4 *Every instance of the voting game where the players' preferences are in $\mathcal{L}(M)$ and $\mathcal{F}$=Maximin admits a considerate equilibrium.*

Proof: The proof follows the same idea as the proof for Plurality (Theorem 1). It suffices to show that under Maximin the set of powerful coalitions $\mathcal{C}_p$ also corresponds to the set of coalitions with size strictly larger than $n/2$. Indeed, if a is the best alternative w.r.t $\rhd$ then a gets elected in the unanimous profile where a is placed on top of each ballot. From such a profile, only a coalition of size strictly larger than $n/2$ can change the outcome by placing another alternative y unanimously on top of the new ballot. If the coalition has only $n/2$ agents, it is not sufficient because a still has a maximin score of $n/2$ and no other candidate can have a strictly better score. ■

It is known that the strategic voting game under the Maximin rule is not guaranteed to converge to a Nash equilibrium with an arbitrary deterministic tie-breaking [14]. However, even with a deterministic tie-breaking which is a linear order, the game under Maximin is not guaranteed to converge, as we can see in the next proposition. Moreover, we do not use a general best reply but a restricted one.

Proposition 9 *The dynamics associated with the Nash equilibrium is not guaranteed to converge in the voting game with $\mathcal{F}$=Maximin, even if the initial profile is truthful, the players' preferences are strict, single-peaked and single-crossing, and each move is a best reply minimizing the distance to the truthful ballot in terms of number of differences in pair-wise comparisons.*

Proof: Consider an instance where $N = \{1, 2, 3, 4, 5\}$, $M = \{a, b, c, d\}$ and $a \rhd b \rhd c \rhd d$. The preferences are single-peaked using the axis over the candidates (d, b, c, a) and single-crossing using the axis over the voters $(1, 4, 3, 2, 5)$. The next table gives a sequence of states where the steps 1 to 5 form a cycle. Deviations are marked with bold letters. Step 0 corresponds to the truthful profile. Each linear order is the ballot of a voter and the last line of the table specifies the winner at each step.

Step 0	Step 1	Step 2	Step 3	Step 4	Step 5
$d \succ b \succ c \succ a$	$d \succ b \succ c \succ a$	$\mathbf{c} \succ \mathbf{d} \succ \mathbf{b} \succ \mathbf{a}$	$c \succ d \succ b \succ a$	$\mathbf{d} \succ \mathbf{b} \succ \mathbf{c} \succ \mathbf{a}$	$d \succ b \succ c \succ a$
$a \succ c \succ b \succ d$	$\mathbf{a} \succ \mathbf{d} \succ \mathbf{c} \succ \mathbf{b}$	$a \succ d \succ c \succ b$	$a \succ d \succ c \succ b$	$a \succ d \succ c \succ b$	$a \succ d \succ c \succ b$
$c \succ b \succ a \succ d$	$c \succ b \succ a \succ d$	$c \succ b \succ a \succ d$	$c \succ b \succ a \succ d$	$c \succ b \succ a \succ d$	$c \succ b \succ a \succ d$
$b \succ d \succ c \succ a$	$b \succ d \succ c \succ a$	$b \succ d \succ c \succ a$	$b \succ d \succ c \succ a$	$b \succ d \succ c \succ a$	$b \succ d \succ c \succ a$
$a \succ c \succ b \succ d$	$a \succ c \succ b \succ d$	$a \succ c \succ b \succ d$	$\mathbf{a} \succ \mathbf{b} \succ \mathbf{d} \succ \mathbf{c}$	$a \succ b \succ d \succ c$	$\mathbf{a} \succ \mathbf{c} \succ \mathbf{b} \succ \mathbf{d}$
c	a	c	a	b	a

■

6 Discussion and perspectives

We proposed to explore voting games from a strategical and social point of view. The considerate equilibrium captures a stable outcome regarding every set of coalitions arising from a social network.

The next table summarizes the existence and convergence results for the different voting rules that we explored. They are classified by solution concepts, which allows to spot the gap between existence/non-existence and convergence/non-convergence since the different equilibria are related: $\exists$ *considerate equilibrium* $\Rightarrow$ $\exists$ *partition equilibrium* $\Rightarrow$ $\exists$ *Nash equilibrium.*

		Plurality	Veto	k-approval	Borda
strong	Existence	×	✔(Prop.3)	×	×
	Convergence	×	×	×	×
considerate	Existence	✔(Th.1)	✔(Th.2)	×	×
	Convergence	×	×	×	×
partition	Existence	✔	✔	×	×
	Convergence	×(Prop.1)	×(Prop.4)	×	×
Nash	Existence	✔	✔	×(Prop.5)	×(Prop.6)
	Convergence	✔[16]	✔[14, 24]	×[14, 24]	×[14, 24]
		STV	PwRO	Copeland	Maximin
strong	Existence	×	×	×	×
	Convergence	×	×	×	×
considerate	Existence	✔(Th.3)	✔(Th.3)	×	✔(Th.4)
	Convergence	×	×	×	×
partition	Existence	✔	✔	×	✔
	Convergence	×	×	×	×
Nash	Existence	✔	✔	×(Prop.8)	✔
	Convergence	×(Prop.7)	×(Prop.7)	×	×(Prop.9)

We can remark that convergence under general best replies (even if they are refined as in Prop. 7 and 9) is difficult to achieve. We restricted ourselves to almost general best replies. A possible extension would be to analyze convergence to a considerate equilibrium for some restricted manipulation moves, as studied in [23, 11, 21] for Nash equilibria. Another possible way to achieve convergence is to focus on specific classes of graphs or coalition families, provided that it matches with an actual social structure.

On the positive side, we were able to prove the existence of a considerate equilibrium for a significant number of voting rules, namely Plurality, Antiplurality, Plurality with runoff, STV and Maximin. These results are encouraging because the notion of considerate equilibrium covers a large spectrum of families of coalitions.

As a balance, the assumption of consideration within this equilibrium — the fact no coalition harms its neighbors — is rather strong. Actually, without the consideration assumption, it is not possible to generalize the existence of such an equilibrium to every class of graph. As an example, take the complete graph which corresponds to a super strong equilibrium and see Example 6. This counterexample holds for every voting rule for which we proved the existence of a considerate equilibrium, showing the importance of the consideration assumption.

Nevertheless, the consideration assumption is relevant if we assume that agents are not fully selfish and that they care about their relatives. Moreover, it allows to integrate a social dimension into the game. If agents are connected in the social network, then they are reluctent to act in a way that harm their partners. If one wants to escape from the consideration assumption, then an option is to restrict to specific classes of graphs, or to specific families of coalitions. For example, with the partition equilibrium, our existence results hold without the consideration assumption. One can also think of *laminar structures* where there is either inclusion or empty intersection between every pair of coalitions. Another possibility is to relax the consideration assumption, e.g. a coalition C has consideration for an agent $i \notin C$ if i has at least a given number of neighbors in C.

REFERENCES

[1] S. Airiau and U. Endriss, 'Iterated majority voting', in *Proceedings of the First International Conference on Algorithm Decision Theory (ADT-09)*, volume 5783 of *LNCS*, pp. 38–49, (October 2009).

[2] I. Ashlagi, P. Krysta, and M. Tennenholtz, 'Social context games', in *Internet and Network Economics*, 675–683, Springer, (2008).

[3] R. J. Aumann, 'Acceptable points in general cooperative n-person games', in *Contribution to the Theory of Games*, eds., A. W. Tucker and R. D. Luce, volume IV of *Annals of Mathematics Studies, 40*, pp. 287–324. Princeton Univ. Press, (1959).

[4] B. D. Bernheim, B. Peleg, and M. D. Whinston, 'Coalition-proof nash equilibria i. concepts', *Journal of Economic Theory*, **42**(1), 1–12, (1987).

[5] F. Brandt, V. Conitzer, U. Endriss, J. Lang, A.D. Procaccia, and H. Moulin, *Handbook of Computational Social Choice*, Cambridge University Press, 2016.

[6] P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra, 'Using complexity to protect elections', *Communications of the ACM*, **53**(11), 74–82, (2010).

[7] P. Faliszewski and A. D. Procaccia, 'AIs war on manipulation: Are we winning', *AI Magazine*, **31**(4), 53–64, (2010).

[8] A. M. Feldman and R. Serrano, *Welfare economics and social choice theory*, Springer Science & Business Media, 2006.

[9] M. Feldman and M. Tennenholtz, 'Partition equilibrium', in *Algorithmic Game Theory, Second International Symposium, SAGT 2009, Paphos, Cyprus, October 18-20, 2009. Proceedings*, eds., M. Mavronicolas and V. Papadopoulou, volume 5814 of *LNCS*, pp. 48–59. Springer, (2009).

[10] A. Gibbard, 'Manipulation of voting schemes: a general result', *Econometrica: journal of the Econometric Society*, 587–601, (1973).

[11] U. Grandi, A. Loreggia, F. Rossi, K. B. Venable, and T. Walsh, 'Restricted manipulation in iterative voting: Condorcet efficiency and borda score', in *Algorithmic Decision Theory*, 181–192, Springer, (2013).

[12] M. Hoefer, M. Penn, M. Polukarov, A. Skopalik, and B. Vöcking, 'Considerate equilibrium', in *22nd International Joint Conference on Artificial Intelligence (IJCAI)*, (July 2011).

[13] M. O. Jackson et al., *Social and economic networks*, volume 3, Princeton university press Princeton, 2008.

[14] O. Lev and J. S. Rosenschein, 'Convergence of iterative voting', in *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pp. 611–618. International Foundation for Autonomous Agents and Multiagent Systems, (2012).

[15] E. Maskin, 'Nash equilibrium and welfare optimality*', *The Review of Economic Studies*, **66**(1), 23–38, (1999).

[16] R. Meir, M. Polukarov, J. Rosenschein, and N. Jennings, 'Convergence to equilibria in plurality voting', in *Proc. of 24th Conference on Artificial Intelligence (AAAI-10)*, pp. 823–828, (2010).

[17] M. Messner and M. K. Polborn, 'Strong and coalition-proof political equilibria under plurality and runoff rule', *International Journal of Game Theory*, **35**(2), 287–314, (2007).

[18] H. Moulin, *Axioms of cooperative decision making*, Cambridge University Press, 1991.

[19] R. B. Myerson and R. J. Weber, 'A theory of voting equilibria', *American Political Science Review*, **87**(01), 102–114, (1993).

[20] J. Nash, 'Non-cooperative Games', *The Annals of Mathematics*, **54**(2), 286–295, (1951).

[21] S. Obraztsova, E. Markakis, M. Polukarov, Z. Rabinovich, and N. R. Jennings, 'On the convergence of iterative voting: how restrictive should restricted dynamics be?', in *AAAI 2015: Twenty-Ninth AAAI Conference on Artificial Intelligence*, pp. 993–999, (2015).

[22] B. Peleg and H. Peters, *Strategic Social Choice: Stable Representations of Constitutions*, Springer-Verlag Berlin Heidelberg, 2010.

[23] A. Reijngoud and U. Endriss, 'Voter response to iterated poll information', in *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pp. 635–644. International Foundation for Autonomous Agents and Multiagent Systems, (2012).

[24] R. Reyhani and M. Wilson, 'Best-reply dynamics for scoring rules', in *20th European Conference on Artificial Intelligence*. IOS Press, (2012).

[25] M. A. Satterthwaite, 'Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions', *Journal of economic theory*, **10**(2), 187–217, (1975).

[26] M. Sertel and R. Sanver, 'Strong equilibrium outcomes of voting games are the generalized condorcet winners', *Social Choice and Welfare*, **22**(2), 331–347, (2004).

[27] S. Sina, N. Hazon, A. Hassidim, and S. Kraus, 'Adapting the social network to affect elections', in *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pp. 705–713. International Foundation for Autonomous Agents and Multiagent Systems, (2015).

[28] A. Tsang and K. Larson, 'The echo chamber: Strategic voting and homophily in social networks', *to appear in AAMAS 2016*, (2016).

[29] M. Voorneveld, *Potential Games and Interactive Decisions with Multiple Criteria*, Ph.D. dissertation, Tilburg University, 1999.