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Three-phonon and four-phonon interaction processes in a pair-condensed Fermi gas

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Abstract

We study the interactions among phonons and the phonon lifetime in a pair-condensed Fermi gas in the BEC-BCS crossover in the collisionless regime. To compute the phonon-phonon coupling amplitudes we use a microscopic model based on a generalized BCS Ansatz including moving pairs, which allows for a systematic expansion around the mean field BCS approximation of the ground state. We show that the quantum hydrodynamic expression of the amplitudes obtained by Landau and Khalatnikov apply only on the energy shell, that is for resonant processes that conserve energy. The microscopic model yields the same excitation spectrum as the Random Phase Approximation, with a linear (phononic) start and a concavity at low wave number that changes from upwards to downwards in the BEC-BCS crossover. When the concavity of the dispersion relation is upwards at low wave number, the leading damping mechanism at low temperature is the Beliaev-Landau process $2 \text{ phonons} \leftrightarrow 1 \text{ phonon}$ while, when the concavity is downwards, it is the Landau-Khalatnikov process $2 \text{ phonons} \leftrightarrow 2 \text{ phonons}$. In both cases, by rescaling the wave vectors to absorb the dependence on the interaction strength, we obtain a universal formula for the damping rate. This universal formula corrects and extends the original analytic results of Landau and Khalatnikov [ZhETF **19**, 637 (1949)] for the $2 \leftrightarrow 2$ processes in the downward concavity case. In the upward concavity case, for the Beliaev $1 \leftrightarrow 2$ process for the unitary gas at zero temperature, we calculate the damping rate of an excitation with wave number q including the first correction proportional to q^7 to the q^5 hydrodynamic prediction, which was never done before in a systematic way.

1. Introduction

In several many-body systems the low-lying collective excitations are phonons. At low temperature, interactions among phonons determine their lifetime, correlation time and mean-free path. Therefore, they play a central role in transport phenomena, such as thermal conduction in dielectric solids, in hydrodynamic properties, such as temperature dependent viscosity and attenuation of sound in liquid helium [1, 2, 3, 4, 5, 6], and in macroscopic coherence properties of degenerate gases, as they determine the intrinsic coherence time of the condensate both in bosonic and fermionic gases [7, 8]. The dominant phonon decay channel differs among physical systems, and depends in particular on the curvature of the phonon excitation branch [9, 10, 11]. For a convex dispersion relation at low wave-number, Beliaev-Landau $2 \leftrightarrow 1$ processes involving three quasiparticles dominate [12, 13], while for a concave dispersion relation at low wave-number, Beliaev-Landau interactions are not resonant, and the Landau-Khalatnikov $2 \leftrightarrow 2$ processes involving four quasiparticles dominate the dynamics.

Compared to other many-body systems, cold atomic gases offer the unique possibility to control and tune some microscopic parameters, in particular the interaction strength. For a cold atomic Fermi gas in two spin states \uparrow and \downarrow , interactions occur only in the s -wave between fermions of opposite spins. We restrict ourselves to the universal regime of interactions of negligible range b , fully characterized by a single parameter, the s -wave scattering length a , $|a| \gg b$. Using an external magnetic field, one can adjust this parameter of the interaction strength close to a so-called Feshbach resonance. This feature has allowed cold Fermi gases experiments to study the crossover between the Bardeen-Cooper-Schrieffer (BCS) regime $1/a \rightarrow -\infty$, where the superfluid pairs ($\mathbf{k} \uparrow, -\mathbf{k} \downarrow$) are localized in Fourier space close to the Fermi surface, and the Bose-Einstein Condensate (BEC) regime $1/a \rightarrow +\infty$, of dilute tightly bound dimers that behave as bosonic particles [14, 15, 16, 17, 18, 19, 20, 21, 22]. For the unpolarized gas of spin-1/2

fermions, the excitation spectrum consists of two branches: a gapped fermionic branch describing the excitation of the internal degrees of freedom of the $\uparrow\downarrow$ pairs, and a bosonic branch describing the collective motion of the pair center of mass, which has a phononic behavior at small wave vectors [23, 24, 25, 26, 27, 11]. If the density of excitations is sufficiently small, the elementary excitations are long-lived weakly interacting quasiparticles.

In cold gases of spin-1/2 fermions, the dispersion relation of phonons changes from convex in the BEC regime [25] to concave in the BCS regime [24]; it is generally expected at unitarity to be convex at low wave number [28, 29, 30, 31, 26, 11], although the predicted values of the curvature parameter significantly differ and there is no experimental verdict yet. A complete experimental investigation of the interactions between phonons, in both the concave and convex cases, seems in any case possible within this same physical system. In particular, experiments should observe a sharp increase in the phonon lifetime by switching from the convex to the concave case through a tuning of the scattering length. Experimental studies of collective excitations in cold gases have been performed both in the spectral domain with Bragg scattering [32, 33] and in the time domain [34, 35]. Those studies combined with the possibility to realize homogeneous systems in flat-bottom traps [36] make the measurement of the decay caused by either the $2 \leftrightarrow 1$ or the $2 \leftrightarrow 2$ processes a concrete prospect.

To date, an exhaustive theoretical study of phonon interactions and a general expression of the phonon damping rates at low temperature for any interaction strength is missing. The most natural example of a convex dispersion relation is the Bogoliubov excitation branch of the weakly interacting Bose gas. In this case, Beliaev obtained the coupling amplitudes in a microscopic framework and, from those, the damping rate associated to the $1 \rightarrow 2$ processes [12]. The contribution of the $2 \rightarrow 1$ Landau processes, which exist only at nonzero temperature, was obtained later in [37, 38, 39]. In superfluid spin-1/2 Fermi gases, reference [13] calculated the zero-temperature Beliaev $1 \rightarrow 2$ damping rate by a phenomenological low-energy effective theory going one step beyond quantum hydrodynamics in order to include a curvature in the phonon dispersion relation. However, the authors omitted to include corrections to the phonon coupling amplitude, which makes their treatment inconsistent. In the bosonic case of phase II of ^4He , whose dispersion relation was originally believed to be concave, Landau and Khalatnikov obtained the coupling amplitude for the $2 \leftrightarrow 2$ processes, including the contribution of virtual off-resonant 3-phonon processes and introducing by hand corrections to hydrodynamics in the form of a cubic term in the excitation spectrum. However, the fact that they computed the damping rates only in the low and high wave-number limits makes this study incomplete and motivates us to revisit it. Studies of low-energy $2 \leftrightarrow 2$ processes in ^4He were put on hold after 1970, when it was discovered that the dispersion relation of ^4He is in fact convex at low wave-number [3, 4] at least at not too high pressure [4, 5]. Studies on processes in the wave-number ranges where the dispersion relation becomes concave again, particularly challenging due to the existence of a small convex region, are still an active research topic [10]. To our knowledge, the study of the $2 \leftrightarrow 2$ processes in cold Fermi gases is mainly at the unitary limit and in the collisional hydrodynamic regime where the relevant measure for dissipation is the shear viscosity of the gas [40, 41, 42]. In this article, we restrict to temperatures low enough to be in the collisionless regime. There, we present a complete study of the interaction processes between phonons in cold Fermi gases, for any interaction strength in the BEC-BCS crossover, therefore in both the concave and convex cases.

We describe the unpolarized pair-condensed Fermi gas using a microscopic semiclassical model, based on a variational state including moving pairs. This model allows for a systematic expansion of the Hamiltonian in terms of two canonically conjugate fields β and β^* that are weak if the density of excitations in the gas is low. The microscopic model and the principle of the method are exposed in section 2.

In section 3 we consider the expansion of the Hamiltonian to the quadratic order in the fields β and β^* . By diagonalizing the quadratic problem, we find the spectrum of the excitations [23, 24, 25, 26, 27, 11]. Beyond the purely spectral results known in the literature, we derive here explicitly the quasiparticle modes and the annihilation/creation amplitudes, which we then quantize. This is rather elegantly done thanks to the Hamiltonian character of the equations in the semiclassical model.

Section 4 is devoted to the calculation of the coupling amplitudes for three-phonon and four-phonon interaction processes. We give the quantum hydrodynamics [1] predictions for the amplitudes of the $2 \leftrightarrow 1$ and the $2 \leftrightarrow 2$ processes, and compare them on the energy shell to a microscopic derivation, based for $2 \leftrightarrow 1$ on the fermionic model of section 3, and for $2 \leftrightarrow 2$ on a model of bosons with finite range interactions designed to have a concave dispersion relation at low wave numbers. Our microscopic test of the hydrodynamic predictions is particularly valuable for the $2 \leftrightarrow 2$ processes, as (i) these processes involve nonresonant $2 \leftrightarrow 1$ and $3 \leftrightarrow 0$ processes, whose amplitudes are not correctly given by hydrodynamics, (ii) genuine hydrodynamics leads to a divergence in the $2 \leftrightarrow 2$ amplitude that

Landau and Khalatnikov had to regularize “by hand” by introducing a curvature in the phonon dispersion relation.

In section 5 we present a direct application of the previous results by calculating the damping rate of phonons in a Fermi gas in the whole BEC-BCS crossover in the collisionless regime. To this aim we use the effective on-shell coupling amplitudes for $2 \leftrightarrow 1$ and $2 \leftrightarrow 2$ processes calculated in section 4 within a Master equation approach. The three main results of section 5 are that (i) both for a convex and concave dispersion, by introducing properly normalized dimensionless quantities, we obtain a universal curve, giving the damping rate as a function of the wave vector, for any interaction strength in the BEC-BCS crossover, (ii) we give all the analytic asymptotic behaviors of the phonon damping rates, for $\hbar c q \ll k_B T$ and $\hbar c q \gg k_B T$ where c is the speed of sound and T is the temperature, correcting in particular the result of Landau and Khalatnikov [1] in the concave case, and (iii) for the unitary gas and at zero temperature only, we calculate the first correction to the hydrodynamic prediction for the phonon damping rate, using the effective field theory of Son and Wingate [43, 44], and improving the result of reference [13].

2. The microscopic approach

2.1. Interaction Hamiltonian in the s -wave

We consider a gas of fermions in two internal states \uparrow and \downarrow on a cubic lattice of step l with periodic boundary conditions in the finite-size volume $[0, L]^3$. Opposite spin fermions have an on-site interaction characterized by a coupling constant g_0

$$V(\mathbf{r}, \mathbf{r}') = g_0 \frac{\delta_{\mathbf{r}, \mathbf{r}'}}{l^3} \quad (1)$$

The discretisation of space automatically provides a momentum cut-off, restricting the single-particle wave vectors to the first Brillouin zone $\text{FBZ} = [-\pi/l, \pi/l]^3$ of the lattice, while allowing one to keep the simplicity of contact interactions. The Hamiltonian of the system in the grand canonical ensemble of chemical potential μ is given by:

$$\hat{H} = l^3 \sum_{\mathbf{r}, \sigma=\uparrow/\downarrow} \hat{\psi}_{\sigma}^{\dagger}(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} - \mu \right) \hat{\psi}_{\sigma}(\mathbf{r}) + g_0 l^3 \sum_{\mathbf{r}} \hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}(\mathbf{r}) \hat{\psi}_{\uparrow}(\mathbf{r}) \quad (2)$$

where the discrete Laplacian operator $\Delta_{\mathbf{r}}$ has the eigenfunctions $e^{i\mathbf{k}\cdot\mathbf{r}}$ with eigenvalues $-k^2$ and the field operator of the fermions has the discrete anticommutation relations $\{\hat{\psi}_{\sigma}(\mathbf{r}), \hat{\psi}_{\sigma'}^{\dagger}(\mathbf{r}')\} = \delta_{\sigma\sigma'} \delta_{\mathbf{r}\mathbf{r}'}/l^3$, with $\sigma, \sigma' = \uparrow$ or \downarrow . The desired regime of zero-range interactions is obtained by taking the continuous space limit $l \rightarrow 0$ for a scattering length a fixed to the experimental value. In practice, one eliminates the bare coupling constant g_0 using the following relation, derived from the open space ($L = \infty$) scattering theory applied to the potential (1) [45, 46]:

$$\frac{1}{g_0} = \frac{m}{4\pi\hbar^2 a} - \int_{\text{FBZ}} \frac{d^3 k}{(2\pi)^3} \frac{m}{\hbar^2 k^2} \quad (3)$$

2.2. Ground state in the BCS approximation

The idea behind the theory of Bardeen-Cooper-Schrieffer (BCS) [47] is to search for an approximation of the ground state of the Hamiltonian (2) within the family of states

$$|\psi^{\text{BCS}}\rangle = \prod_{\mathbf{k} \in \mathcal{D}} \left(U_{\mathbf{k}} - V_{\mathbf{k}} \hat{a}_{\mathbf{k}\uparrow}^{\dagger} \hat{a}_{-\mathbf{k}\downarrow}^{\dagger} \right) |0\rangle \quad (4)$$

where $\mathcal{D} = \frac{2\pi}{L} \mathbb{Z}^3 \cap [-\pi/l, \pi/l]^3$ is the ensemble of wavevectors of the first Brillouin zone satisfying the periodic boundary conditions, the operator $\hat{a}_{\mathbf{k}\sigma}$ is a Fourier component of the field operator $\hat{\psi}_{\sigma}(\mathbf{r})$ and annihilates a fermion of wavevector \mathbf{k} and spin σ , $V_{\mathbf{k}}$ is the probability amplitude of finding a $\mathbf{k}\sigma$ fermion in $|\psi^{\text{BCS}}\rangle$ and $U_{\mathbf{k}} = \sqrt{1 - |V_{\mathbf{k}}|^2}$. The locus of the minimizers of the classical energy functional $\langle \psi^{\text{BCS}} | \hat{H} | \psi^{\text{BCS}} \rangle$ is the circle

$$U_{\mathbf{k}} = U_{\mathbf{k}}^0 \quad V_{\mathbf{k}} = V_{\mathbf{k}}^0 e^{i\phi} \quad (5)$$

where $\phi \in [0, 2\pi[$ and the values of reference $U_{\mathbf{k}}^0$ and $V_{\mathbf{k}}^0$ are chosen real.

To study the low-energy behavior of the system, we select on this circle the state of phase $\phi = 0$

$$|\psi_0^{\text{BCS}}\rangle = \prod_{\mathbf{k}} \left(U_{\mathbf{k}}^0 - V_{\mathbf{k}}^0 \hat{a}_{\mathbf{k}\uparrow}^\dagger \hat{a}_{-\mathbf{k}\downarrow}^\dagger \right) |0\rangle \quad (6)$$

as the origin of our expansion. In this symmetry-breaking state, we define the *order parameter* of the BCS theory

$$\Delta \equiv g_0 \langle \psi_0^{\text{BCS}} | \hat{\psi}_\downarrow \hat{\psi}_\uparrow | \psi_0^{\text{BCS}} \rangle = -\frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} U_{\mathbf{k}}^0 V_{\mathbf{k}}^0, \quad (7)$$

This quantity replaces g_0 or a as the most natural parameter characterizing the interaction strength in the BCS theory. It can be used to express the coefficients $U_{\mathbf{k}}^0, V_{\mathbf{k}}^0$ in the form

$$V_{\mathbf{k}}^0 = \sqrt{\frac{1}{2} \left(1 - \frac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \right)} \quad \text{and} \quad U_{\mathbf{k}}^0 = \sqrt{\frac{1}{2} \left(1 + \frac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \right)} \quad (8)$$

with the energies

$$\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - \mu + \frac{g_0 \rho}{2} \quad (9)$$

$$\epsilon_{\mathbf{k}} = \sqrt{\Delta^2 + \xi_{\mathbf{k}}^2} \quad (10)$$

With the useful relation $U_{\mathbf{k}}^0 V_{\mathbf{k}}^0 = \Delta/2\epsilon_{\mathbf{k}}$, the gap equation (7) takes its more usual form:

$$\frac{1}{g_0} = -\frac{1}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} \frac{1}{2\epsilon_{\mathbf{k}}} \quad (11)$$

In the state (6), the average total density ρ , or average total number of particles $\langle \hat{N} \rangle$ per unit volume, is given by

$$\rho \equiv \frac{\langle \hat{N} \rangle}{L^3} \equiv \frac{k_F^3}{3\pi^2} = \frac{2}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} (V_{\mathbf{k}}^0)^2 \quad (12)$$

where k_F is the Fermi wave number of the ideal gas of density ρ . Combined with equation (8) this leads to the BCS equation of state which relates ρ to μ and Δ .

Performing a Bogoliubov rotation on the particle creation and annihilation operators, one finally defines the creation and annihilation operators of fermionic excitations:

$$\hat{\gamma}_{\mathbf{k}\uparrow} = U_{\mathbf{k}}^0 \hat{a}_{\mathbf{k}\uparrow} + V_{\mathbf{k}}^0 \hat{a}_{-\mathbf{k}\downarrow}^\dagger \quad (13)$$

$$\hat{\gamma}_{-\mathbf{k}\downarrow} = -V_{\mathbf{k}}^0 \hat{a}_{\mathbf{k}\uparrow}^\dagger + U_{\mathbf{k}}^0 \hat{a}_{-\mathbf{k}\downarrow} \quad (14)$$

The BCS ground state (6) is the vacuum of these operators, that annihilate $\mathbf{k} \uparrow$ and $-\mathbf{k} \downarrow$ quasiparticles of energy $\epsilon_{\mathbf{k}}$. For completeness we give the corresponding modal expansion of the fermionic fields:

$$\begin{pmatrix} \psi_\uparrow(\mathbf{r}) \\ \psi_\downarrow^\dagger(\mathbf{r}) \end{pmatrix} = \frac{1}{L^{3/2}} \sum_{\mathbf{k} \in \mathcal{D}} \hat{\gamma}_{\mathbf{k}\uparrow} \begin{pmatrix} U_{\mathbf{k}}^0 \\ V_{\mathbf{k}}^0 \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{\gamma}_{\mathbf{k}\downarrow}^\dagger \begin{pmatrix} -V_{\mathbf{k}}^0 \\ U_{\mathbf{k}}^0 \end{pmatrix} e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (15)$$

2.3. Ansatz of moving pairs

To study the fluctuations of the system, we view the symmetry-breaking BCS ground state as the quasiparticle vacuum and we construct a coherent state of pairs of quasiparticles, or squeezed vacuum, from it:

$$|\psi(t)\rangle = \mathcal{N}(t) \exp \left(\sum_{\mathbf{k}, \mathbf{q}} z_{\mathbf{k}+\mathbf{q}, \mathbf{k}}(t) \hat{\gamma}_{\mathbf{k}+\mathbf{q}\uparrow}^\dagger \hat{\gamma}_{-\mathbf{k}\downarrow}^\dagger \right) |\psi_{\text{BCS}}^0\rangle \quad (16)$$

where the independent variational parameters $z_{\mathbf{k}+\mathbf{q},\mathbf{k}}$ are the coherent state complex amplitudes and $\mathcal{N}(t)$ is the normalisation factor. This Ansatz is inspired from section 9.9b of reference [48]. Contrarily to the BCS ground state (6) in which all the pairs are at rest it includes pairs of quasiparticles with nonzero center-of-mass wavevector \mathbf{q} . This will allow us to study the gapless, phononlike, collective modes of the system and their interactions. We do not include in the Ansatz elementary fermionic excitations obtained by the direct action of $\hat{\gamma}_{\mathbf{k}\sigma}^\dagger$ on the BCS ground state $|\psi_{\text{BCS}}^0\rangle$ since they are gapped, and we are interested here in excitation energies much smaller than the gap. Note that the quasiparticle annihilation and creation operators $\hat{\gamma}_{\mathbf{k}\sigma}$ and $\hat{\gamma}_{\mathbf{k}\sigma}^\dagger$ are linear combinations of the particle annihilation and creation operators $\hat{a}_{\mathbf{k}\sigma}$ and $\hat{a}_{\mathbf{k}\sigma}^\dagger$, see equations (13,14), so that considering a coherent state of quasiparticle pairs (16) is equivalent to considering a coherent state of pairs of particles, that is *in fine* a general time-dependent BCS Ansatz. In the end, all these Ansatz are Gaussian in the fermionic field operators, but the form (16) leads to much simpler calculations in the weakly excited regime as it immediately identifies the $z_{\mathbf{k}+\mathbf{q},\mathbf{k}}$ as small parameters of the expansion.

References [48] and [49] explain how to apply the variational principle to the state (16). One cleverly introduces the variables

$$\beta_{\mathbf{k},\mathbf{k}'} = -\left(z(1+z^\dagger z)^{-1/2}\right)_{\mathbf{k},\mathbf{k}'} \quad (17)$$

with z the matrix $(z_{\mathbf{k},\mathbf{k}'}),_{\mathbf{k},\mathbf{k}' \in \mathcal{D}}$, so that the equations of motion take a Hamiltonian form

$$i\hbar \frac{d\beta_{\mathbf{k}',\mathbf{k}}}{dt} = \frac{\partial E}{\partial \beta_{\mathbf{k}',\mathbf{k}}^*} \quad (18)$$

$$-i\hbar \frac{d\beta_{\mathbf{k}',\mathbf{k}}^*}{dt} = \frac{\partial E}{\partial \beta_{\mathbf{k}',\mathbf{k}}} \quad (19)$$

One evaluates the associated classical Hamiltonian

$$E \equiv \langle \psi | \hat{H} | \psi \rangle \quad (20)$$

through Wick's theorem, that relates E to averages of operators that are bilinear in $\hat{\gamma}_{\mathbf{k}\sigma}$. Finally one obtains an elegant expression for these averages by performing a Schmidt decomposition of the matrix z and expressing the result in terms of the field β , in matrix notation:

$$\langle \hat{\gamma}_{-\mathbf{k}\downarrow} \hat{\gamma}_{\mathbf{k}'\uparrow} \rangle = -\left(\beta(1-\beta^\dagger \beta)^{1/2}\right)_{\mathbf{k}',\mathbf{k}} \quad (21)$$

$$\langle \hat{\gamma}_{\mathbf{k}'\uparrow}^\dagger \hat{\gamma}_{-\mathbf{k}\downarrow}^\dagger \rangle = -\left(\beta(1-\beta^\dagger \beta)^{1/2}\right)_{\mathbf{k},\mathbf{k}'}^* \quad (22)$$

$$\langle \hat{\gamma}_{\mathbf{k}'\uparrow}^\dagger \hat{\gamma}_{\mathbf{k}'\uparrow} \rangle = \left(\beta \beta^\dagger\right)_{\mathbf{k},\mathbf{k}'} \quad (23)$$

$$\langle \hat{\gamma}_{-\mathbf{k}'\downarrow}^\dagger \hat{\gamma}_{-\mathbf{k}\downarrow} \rangle = \left(\beta^\dagger \beta\right)_{\mathbf{k}',\mathbf{k}}. \quad (24)$$

3. Quadratic order of the classical Hamiltonian: normal variables of the bosonic branch and phonon operators

3.1. Linearized equations of motion

To linearize the equations of motion (18) and (19), we assume that the state (16) differs only slightly from the BCS ground state, so that

$$\forall \mathbf{k}, \mathbf{k}', |z_{\mathbf{k},\mathbf{k}'}| \ll 1 \quad \text{and} \quad |\beta_{\mathbf{k},\mathbf{k}'}| \ll 1, \quad (25)$$

We expand the energy functional in powers of the field β

$$E = E_0 + E_2 + O(\beta^3) \quad (26)$$

where $E_0 = \langle \psi_0^{\text{BCS}} | \hat{H} | \psi_0^{\text{BCS}} \rangle$ is a constant and E_2 is bilinear in β . There are no linear terms in this expansion since the BCS ground state (corresponding to $\beta = 0$) is a minimizer of E . As in the Random Phase Approximation (RPA) [23], the linearized equations of motion, which follow from (18) and (19) after replacement of E by E_2 , are decoupled

according to the total wavevector \mathbf{q} . For this reason, we rewrite the coordinates of the field $\beta_{\mathbf{k}_1, \mathbf{k}_2}$ in the Anderson fashion, with the relative wavevector $(\mathbf{k}_1 + \mathbf{k}_2)/2$ in the index and the center-of-mass one $\mathbf{k}_1 - \mathbf{k}_2$ in the exponent, keeping in mind that $\beta_{\mathbf{k}_1, \mathbf{k}_2}$ actually involves the physical wave vectors \mathbf{k}_1 and $-\mathbf{k}_2$, see equation (21), so that:

$$\beta_{\mathbf{k}}^{\mathbf{q}} \equiv \beta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} \quad (27)$$

In terms of the vectors $\beta^{\mathbf{q}} = (\beta_{\mathbf{k}}^{\mathbf{q}})_{\mathbf{k} \in \mathcal{D}}$ and $\bar{\beta}^{\mathbf{q}} = ((\beta_{\mathbf{k}}^{\mathbf{q}})^*)_{\mathbf{k} \in \mathcal{D}}$, the equations of motion take the matrix form

$$i\hbar \frac{d}{dt} \begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix} = \mathcal{L}^{\mathbf{q}} \begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix}, \quad (28)$$

with an evolution operator $\mathcal{L}^{\mathbf{q}}$ that is symplectic

$$\sigma_z \mathcal{L}^{\mathbf{q}} \sigma_z = (\mathcal{L}^{\mathbf{q}})^\dagger \quad (29)$$

and particle-hole symmetric

$$\sigma_x \mathcal{L}^{\mathbf{q}} \sigma_x = -(\mathcal{L}^{\mathbf{q}})^* \quad (30)$$

We have introduced $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in block notations, and we termed (30) a particle-hole symmetry because σ_x exchanges the field β with the complex-conjugate field $\bar{\beta}$, which is reminiscent of the fact that, in the quantum world, the hole field operator is the hermitian conjugate of the particle field operator.

We give an explicit expression of the evolution operator $\mathcal{L}^{\mathbf{q}}$ in the sum-and-difference basis

$$y_{\mathbf{k}}^{\mathbf{q}} = \beta_{\mathbf{k}}^{\mathbf{q}} - \bar{\beta}_{\mathbf{k}}^{\mathbf{q}} \quad (31)$$

$$s_{\mathbf{k}}^{\mathbf{q}} = \beta_{\mathbf{k}}^{\mathbf{q}} + \bar{\beta}_{\mathbf{k}}^{\mathbf{q}} \quad (32)$$

We have

$$i\hbar \frac{dy_{\mathbf{k}}^{\mathbf{q}}}{dt} = \epsilon_{\mathbf{kq}} s_{\mathbf{k}}^{\mathbf{q}} + \frac{g_0}{L^3} \sum_{\mathbf{k}' \in \mathcal{D}} (W_{\mathbf{kq}}^- W_{\mathbf{k}'\mathbf{q}}^- + w_{\mathbf{kq}}^+ w_{\mathbf{k}'\mathbf{q}}^+) s_{\mathbf{k}'}^{\mathbf{q}} \quad (33)$$

$$i\hbar \frac{ds_{\mathbf{k}}^{\mathbf{q}}}{dt} = \epsilon_{\mathbf{kq}} y_{\mathbf{k}}^{\mathbf{q}} + \frac{g_0}{L^3} \sum_{\mathbf{k}' \in \mathcal{D}} (W_{\mathbf{kq}}^+ W_{\mathbf{k}'\mathbf{q}}^+ - w_{\mathbf{kq}}^- w_{\mathbf{k}'\mathbf{q}}^-) y_{\mathbf{k}'}^{\mathbf{q}} \quad (34)$$

The coefficients $W_{\mathbf{kq}}^\pm$ and $w_{\mathbf{kq}}^\pm$ are cleverly chosen combinations of the coefficients $U_{\mathbf{k}}^0$ and $V_{\mathbf{k}}^0$

$$W_{\mathbf{kq}}^\pm = U_{\mathbf{k}+\mathbf{q}/2}^0 U_{\mathbf{k}-\mathbf{q}/2}^0 \pm V_{\mathbf{k}+\mathbf{q}/2}^0 V_{\mathbf{k}-\mathbf{q}/2}^0 \quad (35)$$

$$w_{\mathbf{kq}}^\pm = U_{\mathbf{k}+\mathbf{q}/2}^0 V_{\mathbf{k}-\mathbf{q}/2}^0 \pm U_{\mathbf{k}-\mathbf{q}/2}^0 V_{\mathbf{k}+\mathbf{q}/2}^0 \quad (36)$$

and the energies $\epsilon_{\mathbf{kq}}$ are those of the continuum of two fermionic quasiparticles,

$$\epsilon_{\mathbf{kq}} = \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2} \quad (37)$$

The equations of motion (33,34) contain two terms: first, an individual part that couples the amplitudes of same relative and center-of-mass wavevectors \mathbf{k} and \mathbf{q} and contains the trivial evolution of the operators $\hat{\gamma}_{\mathbf{k}\sigma}$ under the BCS Hamiltonian $\hat{H}_{\text{BCS}} = E_0 + \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} \hat{\gamma}_{\mathbf{k}\sigma}^\dagger \hat{\gamma}_{\mathbf{k}\sigma}$, and second, a collective part that couples the normal amplitudes $y_{\mathbf{k}}^{\mathbf{q}}$ and $s_{\mathbf{k}}^{\mathbf{q}}$ to collective amplitudes of same total wavevector \mathbf{q} . Our semi-classical equations of motion coincide with the quantum average in state (16) of the RPA equations [23]. This can be seen by using the definitions (13) and (14) to express equations (78a-d) of reference [23] in terms of the quasiparticle operators and by neglecting the expectation values of the $\hat{\gamma}_{\mathbf{k}\sigma}^\dagger \hat{\gamma}_{\mathbf{k}'\sigma}$ operators, which, by virtue of equations (23) and (24), is justified at the linear order of the variational theory.

3.2. Branch of collective excitations

We now look for the eigenmodes of the linear system (33,34) with a positive energy $\hbar\omega_{\mathbf{q}}$ below the continuum $k \mapsto \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}$ of two fermionic excitations, which is the spectrum we would obtain by restricting ourselves to the individual part of the system (33,34)

$$0 < \hbar\omega_{\mathbf{q}} < \inf_{\mathbf{k}} (\epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}) \quad (38)$$

The eigenvalue problem associated to (33,34) is solved in full generality in reference [49]. We give the main steps of the solution in the simplifying continuous space limit $l \rightarrow 0$ in Appendix A. The implicit equation on the mode eigenfrequency reads

$$I_{++}(\omega_{\mathbf{q}}, q) I_{--}(\omega_{\mathbf{q}}, q) = \hbar^2 \omega_{\mathbf{q}}^2 \left[I_{+-}(\omega_{\mathbf{q}}, q) \right]^2 \quad (39)$$

where we introduced the integrals

$$I_{++}(\omega, q) = \int_{\mathbb{R}^3} d^3k \left[\frac{\epsilon_{\mathbf{k}\mathbf{q}} (W_{\mathbf{k}\mathbf{q}}^+)^2}{(\hbar\omega)^2 - (\epsilon_{\mathbf{k}\mathbf{q}})^2} + \frac{1}{2\epsilon_{\mathbf{k}}} \right] \quad (40)$$

$$I_{--}(\omega, q) = \int_{\mathbb{R}^3} d^3k \left[\frac{\epsilon_{\mathbf{k}\mathbf{q}} (W_{\mathbf{k}\mathbf{q}}^-)^2}{(\hbar\omega)^2 - (\epsilon_{\mathbf{k}\mathbf{q}})^2} + \frac{1}{2\epsilon_{\mathbf{k}}} \right] \quad (41)$$

$$I_{+-}(\omega, q) = \int_{\mathbb{R}^3} d^3k \frac{W_{\mathbf{k}\mathbf{q}}^+ W_{\mathbf{k}\mathbf{q}}^-}{(\hbar\omega)^2 - (\epsilon_{\mathbf{k}\mathbf{q}})^2} \quad (42)$$

We note that the same equation (39) can be obtained by the RPA; this was done by Anderson [23] in the weak coupling limit and by reference [49] in the general case. Although the linear system of the RPA contains extra terms proportional to the operators $\hat{\gamma}_{\mathbf{k}+\mathbf{q}\uparrow}^\dagger \hat{\gamma}_{\mathbf{k}\uparrow}$ and $\hat{\gamma}_{-\mathbf{k}\downarrow}^\dagger \hat{\gamma}_{-\mathbf{k}-\mathbf{q}\downarrow}$, which are absent in (33,34) because their expectation values in the Ansatz (16) are quadratic in the field β , these extra terms can be treated as source terms and do not affect the spectrum. Equation (39) can also be obtained by a Gaussian approximation of the action in a path integral framework [24, 27] and a Greens functions approach associated with a diagrammatic approximation [25]. The conditions on q for the existence of a solution $\omega_{\mathbf{q}}$ are discussed in reference [25], while the concavity of the spectrum is studied in reference [11]. Beyond those previous works, we construct here in the next section the quantum operators associated to the collective modes.

3.3. Construction of the normal variables of the collective branch

3.3.1. General case

Using the symmetries (29) and (30) of the evolution operator, we obtain the normal amplitudes exactly as in the bosonic case with the Bogoliubov theory [50]. The first step is to derive the eigenvector \vec{e}_+ of energy $\hbar\omega_{\mathbf{q}} > 0$

$$\vec{e}_+(\mathbf{q}) = \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (43)$$

To obtain the analytic expression of the coefficients M and N in the continuous limit $l \rightarrow 0$, we solve the system (33,34) setting $d/dt \rightarrow -i\omega_{\mathbf{q}}$ and we use equations (A.6) and (A.7) to eliminate the collective amplitudes. We have

$$M_{\mathbf{k}}^{\mathbf{q}} - N_{\mathbf{k}}^{\mathbf{q}} = \frac{2\Delta \left(\epsilon_{\mathbf{k}\mathbf{q}} W_{\mathbf{k}\mathbf{q}}^+ - W_{\mathbf{k}\mathbf{q}}^- \frac{I_{++}(\omega_{\mathbf{q}}, q)}{I_{+-}(\omega_{\mathbf{q}}, q)} \right)}{\sqrt{\mathcal{N}_{\mathbf{q}}} (\epsilon_{\mathbf{k}\mathbf{q}}^2 - \hbar^2 \omega_{\mathbf{q}}^2)} \quad (44)$$

$$M_{\mathbf{k}}^{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}} = \frac{2\Delta \left(\hbar^2 \omega_{\mathbf{q}}^2 W_{\mathbf{k}\mathbf{q}}^+ - \epsilon_{\mathbf{k}\mathbf{q}} W_{\mathbf{k}\mathbf{q}}^- \frac{I_{++}(\omega_{\mathbf{q}}, q)}{I_{+-}(\omega_{\mathbf{q}}, q)} \right)}{\hbar\omega_{\mathbf{q}} \sqrt{\mathcal{N}_{\mathbf{q}}} (\epsilon_{\mathbf{k}\mathbf{q}}^2 - \hbar^2 \omega_{\mathbf{q}}^2)} \quad (45)$$

where the dimensionless normalization constant $\mathcal{N}_{\mathbf{q}}$ will be determined by equation (51). Important properties of M and N are their invariance by internal ($\mathbf{k} \rightarrow -\mathbf{k}$) and external ($\mathbf{q} \rightarrow -\mathbf{q}$) parity, where internal and external refers to the structure of the pairs:

$$M_{\mathbf{k}}^{\mathbf{q}} = M_{-\mathbf{k}}^{\mathbf{q}} = M_{\mathbf{k}}^{-\mathbf{q}} \quad (46)$$

$$N_{\mathbf{k}}^{\mathbf{q}} = N_{-\mathbf{k}}^{\mathbf{q}} = N_{\mathbf{k}}^{-\mathbf{q}} \quad (47)$$

This is a consequence of the parity invariance of the problem and of the s -wave pairing. Due to the particle-hole symmetry (30), one can associate to the eigenvector \vec{e}_+ another eigenvector \vec{e}_- of energy $-\hbar\omega_{\mathbf{q}}$ by a multiplication by σ_x :

$$\vec{e}_-(\mathbf{q}) = \sigma_x \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} = \begin{pmatrix} N_{\mathbf{k}}^{\mathbf{q}} \\ M_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (48)$$

We now derive the dual vectors with which we will project the field β on the eigenmodes \vec{e}_+ and \vec{e}_- . The symplectic symmetry (29) ensures that they are obtained by a mere multiplication by σ_z

$$\vec{d}_+(\mathbf{q}) = \sigma_z \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} = \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ -N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (49)$$

$$\vec{d}_-(\mathbf{q}) = -\sigma_z \begin{pmatrix} N_{\mathbf{k}}^{\mathbf{q}} \\ M_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} = \begin{pmatrix} -N_{\mathbf{k}}^{\mathbf{q}} \\ M_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (50)$$

Finally the eigenvectors and their dual vectors are normalized by choosing $\mathcal{N}_{\mathbf{q}}$ so that

$$(\vec{d}_{\pm})^* \cdot \vec{e}_{\pm} = \sum_{\mathbf{k} \in \mathcal{D}} [(M_{\mathbf{k}}^{\mathbf{q}})^2 - (N_{\mathbf{k}}^{\mathbf{q}})^2] = 1 \quad (51)$$

To obtain the amplitudes $b_{\mathbf{q}}$ of the collective modes, we project the field β :

$$b_{\mathbf{q}} = \vec{d}_+(\mathbf{q}) \cdot \begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix} = \sum_{\mathbf{k} \in \mathcal{D}} [M_{\mathbf{k}}^{\mathbf{q}} \beta_{\mathbf{k}}^{\mathbf{q}} - N_{\mathbf{k}}^{\mathbf{q}} (\beta_{\mathbf{k}}^{-\mathbf{q}})^*] \quad (52)$$

$$b_{-\mathbf{q}}^* = \vec{d}_-(\mathbf{q}) \cdot \begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix} = \sum_{\mathbf{k} \in \mathcal{D}} [-N_{\mathbf{k}}^{\mathbf{q}} \beta_{\mathbf{k}}^{\mathbf{q}} + M_{\mathbf{k}}^{\mathbf{q}} (\beta_{\mathbf{k}}^{-\mathbf{q}})^*] \quad (53)$$

This is the first central result of this paper. The equality $b_{\mathbf{q}}^* = (b_{-\mathbf{q}})^*$ suggested by our notation is a consequence of the invariance by external parity (46–47).

Conversely, to express the classical field in terms of the phonon amplitudes, we expand it on the eigenvectors:

$$\begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix} = b_{\mathbf{q}} \vec{e}_+(\mathbf{q}) + b_{-\mathbf{q}}^* \vec{e}_-(\mathbf{q}) + \dots \quad (54)$$

where in the ellipsis \dots we omitted the component of the field on the other eigenmodes of total wavevector \mathbf{q} . One can show [49] that in the continuous limit those omitted modes are nothing else than the continuum $\mathbf{k} \mapsto \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}$ of fermionic-quasiparticle biexcitations with center-of-mass wavevector \mathbf{q} . Projecting the vectorial equation (54), we obtain:

$$\beta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} = M_{\mathbf{k}}^{\mathbf{q}} b_{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}} b_{-\mathbf{q}}^* + \dots \quad (55)$$

$$\beta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}+\mathbf{q}/2}^* = N_{\mathbf{k}}^{\mathbf{q}} b_{\mathbf{q}} + M_{\mathbf{k}}^{\mathbf{q}} b_{-\mathbf{q}}^* + \dots \quad (56)$$

After quantization (see Sec.3.4 below), M and N appear in these expressions as the coefficients of a new Bogoliubov rotation rearranging the fermionic bilinear operators appearing in equations (21,22) into bosonic quasiparticle operators. It comes on top of the rotation induced by the $U_{\mathbf{k}}^0$ and $V_{\mathbf{k}}^0$ coefficients rearranging the particle operators into fermionic quasiparticle operators. This new rotation acts on the pairs of fermions, hence the two indices of M and N .

3.3.2. Long-wave limit

In the long-wave limit $q \rightarrow 0$, we recall the expansion of the energy $\hbar\omega_{\mathbf{q}}$ obtained by reference [11]:

$$\hbar\omega_{\mathbf{q}} \underset{q \rightarrow 0}{=} \hbar c q \left[1 + \frac{\gamma}{8} \left(\frac{\hbar q}{mc} \right)^2 + O \left(\frac{\hbar q}{mc} \right)^4 \right]. \quad (57)$$

In this expression, c is the speed of sound, derived, as in any superfluid, from the equation of state via the hydrodynamic formula

$$mc^2 = \rho \frac{d\mu}{d\rho}, \quad (58)$$

where the derivative with respect to ρ is taken for a fixed scattering length a . The coefficient γ of the cubic order of the expansion (57) is a rational fraction [11] of the variables

$$x = \frac{\Delta}{\mu} \quad \text{and} \quad y = \frac{d\Delta}{d\mu} = \frac{\int_{\mathbb{R}^3} d^3 k \frac{\epsilon_{\mathbf{k}}}{\epsilon_{\mathbf{k}}^3}}{\int_{\mathbb{R}^3} d^3 k \frac{\Delta}{\epsilon_{\mathbf{k}}^3}} \quad (59)$$

Note that y can be related to x by the use of the BCS equation of state (12) in the last equality of (59).

The $q \rightarrow 0$ expansion of the sum (44) and difference (45) of $M_{\mathbf{k}}^{\mathbf{q}}$ and $N_{\mathbf{k}}^{\mathbf{q}}$ then read:

$$\sqrt{\mathcal{N}_{\mathbf{q}}(M_{\mathbf{k}}^{\mathbf{q}} - N_{\mathbf{k}}^{\mathbf{q}})} = \frac{\Delta}{\epsilon_{\mathbf{k}}} + O(q^2) \quad (60)$$

$$\sqrt{\mathcal{N}_{\mathbf{q}}(M_{\mathbf{k}}^{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}})} = -\frac{\hbar c q \epsilon_{\mathbf{k}}}{2\Delta} \frac{dW_{\mathbf{k}\mathbf{0}}^-}{d\mu} + O(q^3) \quad (61)$$

$$\mathcal{N}_{\mathbf{q}}/L^3 = \frac{\hbar c q}{2} \frac{d\rho}{d\mu} + O(q^3) \quad (62)$$

Note that to lowest order these expressions coincide with the coefficients of the zero-energy mode $\vec{\epsilon}_n$ and of the anomalous mode $\vec{\epsilon}_a$ of the zero-momentum subspace evolution operator \mathcal{L}^0 obtained in reference [51].

As an application of the modal expansion (55,56) and of the above low- q limit (60,61) of the corresponding amplitudes, we derive a modal expansion for the density

$$\rho(\mathbf{r}, t) = \sum_{\sigma} \langle \hat{\psi}_{\sigma}^{\dagger}(\mathbf{r}) \hat{\psi}_{\sigma}(\mathbf{r}) \rangle \quad (63)$$

and for the phase $\theta(\mathbf{r}, t)$ of the order parameter, such that

$$g_0 \langle \hat{\psi}_{\downarrow}(\mathbf{r}) \hat{\psi}_{\uparrow}(\mathbf{r}) \rangle = |g_0 \langle \hat{\psi}_{\downarrow}(\mathbf{r}) \hat{\psi}_{\uparrow}(\mathbf{r}) \rangle| e^{i\theta(\mathbf{r}, t)}, \quad (64)$$

where the expectation value is taken in the time-dependent variational Ansatz (16). Expanding the Ansatz to first order in the amplitudes $z_{\mathbf{k}+\mathbf{q}, \mathbf{k}}(t) \simeq -\beta_{\mathbf{k}+\mathbf{q}, \mathbf{k}}(t)$, using Wick's theorem to calculate the expectation value in the BCS ground state and replacing the fermionic fields $\hat{\psi}_{\sigma}(\mathbf{r})$ by their modal expansion from equation (15), we obtain to leading order the deviations from the BCS ground state values:

$$\delta\rho(\mathbf{r}, t) = \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{q}} w_{\mathbf{k}\mathbf{q}}^+ [\beta_{\mathbf{k}+\frac{\mathbf{q}}{2}, \mathbf{k}-\frac{\mathbf{q}}{2}}(t) e^{i\mathbf{q}\cdot\mathbf{r}} + \text{c.c.}] + O(\beta^2) \quad (65)$$

$$\delta\theta(\mathbf{r}, t) = \frac{-g_0}{2i\Delta L^3} \sum_{\mathbf{k}, \mathbf{q}} W_{\mathbf{k}\mathbf{q}}^+ [\beta_{\mathbf{k}+\frac{\mathbf{q}}{2}, \mathbf{k}-\frac{\mathbf{q}}{2}}(t) e^{i\mathbf{q}\cdot\mathbf{r}} - \text{c.c.}] + O(\beta^2) \quad (66)$$

where the coefficients $w_{\mathbf{k}\mathbf{q}}^+$ and $W_{\mathbf{k}\mathbf{q}}^+$ are defined by equations (35,36). Next, we replace the field β by its expansion (55,56) on the collective modes to obtain, using the symmetry property (47),

$$\delta\rho(\mathbf{r}, t) = \frac{1}{L^{3/2}} \sum_{\mathbf{q}} \mathcal{B}_{\mathbf{q}} (b_{\mathbf{q}} + b_{-\mathbf{q}}^*) e^{i\mathbf{q}\cdot\mathbf{r}} + O(\beta^2) \quad (67)$$

$$\delta\theta(\mathbf{r}, t) = \frac{-i}{L^{3/2}} \sum_{\mathbf{q}} \mathcal{C}_{\mathbf{q}} (b_{\mathbf{q}} - b_{-\mathbf{q}}^*) e^{i\mathbf{q}\cdot\mathbf{r}} + O(\beta^2) \quad (68)$$

with the coefficients

$$\mathcal{B}_{\mathbf{q}} = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} w_{\mathbf{kq}}^+ (M_{\mathbf{k}}^{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}}) \quad (69)$$

$$\mathcal{C}_{\mathbf{q}} = \frac{g_0}{2\Delta L^{3/2}} \sum_{\mathbf{k}} W_{\mathbf{kq}}^+ (N_{\mathbf{k}}^{\mathbf{q}} - M_{\mathbf{k}}^{\mathbf{q}}) \quad (70)$$

In the low- q limit, also using the density (12), the gap equation (11) and the thermodynamic expression (58) of the sound velocity, we obtain the simple scaling laws

$$\mathcal{B}_{\mathbf{q}} = \left(\frac{\hbar q \rho}{2mc} \right)^{1/2} [1 + O(q^2)] \quad (71)$$

$$\mathcal{C}_{\mathbf{q}} = \left(\frac{2mc}{\hbar q \rho} \right)^{1/2} [1 + O(q^2)] \quad (72)$$

This result will be compared to the quantum hydrodynamic theory in section 4.1.2.

3.4. Quantization of the normal variables

To quantize the amplitudes of the bosonic modes obtained in our semi-classical approach, we remember that, at the linear order of the small amplitude approximation (25), the field $\beta_{\mathbf{k}}^{\mathbf{q}}$ reduces to $-\langle \hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow} \rangle$, that is to the expectation value of a quasiparticle pair operator, see equation (21). We therefore perform the substitution

$$\beta_{\mathbf{k}}^{\mathbf{q}} = \beta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} \rightarrow -\hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow} \quad (73)$$

which transforms the amplitude $b_{\mathbf{q}}$ in a quantum operator:

$$\hat{b}_{\mathbf{q}} = - \sum_{\mathbf{k} \in \mathcal{D}} \left(M_{\mathbf{k}}^{\mathbf{q}} \hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow} - N_{\mathbf{k}}^{\mathbf{q}} \hat{\gamma}_{\mathbf{k}-\mathbf{q}/2\uparrow}^{\dagger} \hat{\gamma}_{-\mathbf{k}-\mathbf{q}/2\downarrow}^{\dagger} \right) \quad (74)$$

This intuitive action can in fact be justified by several arguments. First, the quantum operator (74) can also be obtained in the framework of the RPA by diagonalizing the homogeneous system for the quasiparticle pair creation and annihilation operators $\hat{\gamma}_{\mathbf{k}\uparrow} \hat{\gamma}_{\mathbf{k}'\downarrow}$ and $\hat{\gamma}_{\mathbf{k}'\downarrow}^{\dagger} \hat{\gamma}_{\mathbf{k}\uparrow}^{\dagger}$, and treating the operators $\hat{\gamma}_{\mathbf{k}\sigma}^{\dagger} \hat{\gamma}_{\mathbf{k}'\sigma}$, whose dynamics is trivial in the RPA and whose average value is sub-leading in the variational theory, as source terms. Second, one can apply the quantization procedure based on bosonic images as described in chapter 11 of reference [48] where the quantum version of the field β is taken to be a bosonic field operator \mathbb{B} (not to be confused with the bosonic excitation operators $\hat{b}_{\mathbf{q}}$). The expression of the *bosonic image* of a two-body fermionic operator such as $\hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow}$ in terms of the field \mathbb{B} , which would provide an exact version of the substitution (73), is not simple in the general case since it involves an infinite series expansion in powers of \mathbb{B} . Fortunately, in the limit of a weakly excited gas, one can neglect the operator population of the bosonic images $\mathbb{B} \mathbb{B}^{\dagger}$, and thus justify the substitution (73), once we assimilate the two-body fermionic operators and their bosonic images. Last, we highlight the bosonic nature of the operator $\hat{b}_{\mathbf{q}}$ (74) when the gas is weakly excited. The commutator of $\hat{b}_{\mathbf{q}}$ and $\hat{b}_{\mathbf{q}}^{\dagger}$ reads:

$$[\hat{b}_{\mathbf{q}}, \hat{b}_{\mathbf{q}}^{\dagger}] - 1 = \sum_{\mathbf{k} \in \mathcal{D}} \left[\left(N_{\mathbf{k}}^{\mathbf{q}} \right)^2 \left(\hat{\gamma}_{\mathbf{k}-\mathbf{q}/2\uparrow}^{\dagger} \hat{\gamma}_{\mathbf{k}-\mathbf{q}/2\uparrow} + \hat{\gamma}_{-\mathbf{k}-\mathbf{q}/2\downarrow}^{\dagger} \hat{\gamma}_{-\mathbf{k}-\mathbf{q}/2\downarrow} \right) - \left(M_{\mathbf{k}}^{\mathbf{q}} \right)^2 \left(\hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow}^{\dagger} \hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow} + \hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow}^{\dagger} \hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \right) \right] \quad (75)$$

It differs from unity by fermionic quasiparticle population operators, that are exactly zero in the BCS ground state and of second order in the field $z = O(\beta)$ in a quasiparticle coherent state such as (16).

4. Beyond the quadratic order: Interactions between phonons and comparison to hydrodynamics

The central actor in this work is the phonon-phonon interaction, that eventually causes phonon damping. It is derived in this section, in the form of three-phonon or four-phonon coupling amplitudes in phonon-interaction Hamiltonians that are cubic or quartic in the phonon field. Although the results involve sometimes lengthy formulas, the

calculations are conceptually very simple: starting on one side from the microscopic energy functional (20) or the easier-to-handle bosonic model (E.1) and on the other side from the quantum hydrodynamic Hamiltonian (87), one performs an expansion up to order three or four in the fields, and one inserts in the cubic or quartic terms the expansion of the fields on the phonon normal modes involving as linear coefficients the phonon creation and annihilation operators. That is all for the three-phonon processes in subsection 4.1. In the four-phonon processes of subsection 4.2, there is the additional subtlety that the three-phonon processes treated to second order have to be added to the direct four-phonon ones to get the correct effective four-phonon coupling as already understood in reference [1]. An important conclusion of this section will be that the three-phonon and four-phonon coupling amplitudes of the microscopic model and of quantum hydrodynamics coincide in the low wave number limit on the energy shell (that is for processes conserving the unperturbed phonon energy), although they strongly differ outside the energy shell.

4.1. Three-phonon processes

In this subsection, we study the processes involving three bosonic quasiparticles: both the $2 \leftrightarrow 1$ Beliaev-Landau processes and the off-resonant $3 \leftrightarrow 0$ processes. Our goal is to identify these processes in the expansion of the Hamiltonian in powers of the excitation field, and to extract the associated matrix elements. We shall use two independent theories and compare them. First, we will use our microscopic variational theory and expand the energy functional E (20) up to order three in β ,

$$E = E_0 + E_2 + E_3 + O(\beta^4) \quad (76)$$

and we will insert the expansion (54) of the field on the collective eigenmodes in the trilinear term E_3 . We shall perform this microscopic calculation in the case $2 \leftrightarrow 1$ only. Second, we will use the quantum hydrodynamics of Landau and Khalatnikov [1]. This mesoscopic theory treats the pairs of fermions at large spatial scale as bosonic fields. It has the advantage of relying on the exact equation of state. It will be applied to the $2 \leftrightarrow 1$ and $3 \leftrightarrow 0$ processes. The comparison of the two theories will allow us to discuss the validity of quantum hydrodynamics.

4.1.1. Microscopic approach

The idea behind the microscopic calculation of the three-phonon coupling amplitudes is simple: we express the classical Hamiltonian (20) in terms of the amplitudes $b_{\mathbf{q}}$ of the collective modes (see Sec.3.3), isolate the terms containing the creation and annihilation amplitudes $b_{\mathbf{q}}^*$ and $b_{\mathbf{q}}$ corresponding to the processes we study, and extract their coefficient. For the three-phonon processes, we focus on the cubic terms gathered in E_3 .

General case. The cubic part of E can be written as

$$E_3 = \frac{g_0}{L^3} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q} \in \mathcal{D}} \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}} \left[(\beta_{\mathbf{k}'+\mathbf{q}/2, \mathbf{k}'-\mathbf{q}/2} \langle \hat{\gamma}_{\mathbf{k}+\mathbf{q}/2\uparrow}^\dagger \hat{\gamma}_{\mathbf{k}-\mathbf{q}/2\uparrow} \rangle + \beta_{\mathbf{k}'-\mathbf{q}/2, \mathbf{k}'+\mathbf{q}/2} \langle \hat{\gamma}_{-\mathbf{k}-\mathbf{q}/2\downarrow}^\dagger \hat{\gamma}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \rangle) + \text{c.c.} \right] \quad (77)$$

where we introduced the tensor

$$\mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}} = \frac{W_{\mathbf{k}\mathbf{q}}^- w_{\mathbf{k}'\mathbf{q}}^+ - w_{\mathbf{k}\mathbf{q}}^+ W_{\mathbf{k}'\mathbf{q}}^- - w_{\mathbf{k}\mathbf{q}}^- W_{\mathbf{k}'\mathbf{q}}^+ - W_{\mathbf{k}\mathbf{q}}^+ w_{\mathbf{k}'\mathbf{q}}^-}{2}. \quad (78)$$

We use the relations (23,24) and we insert the expansion (54) in the expression (77) to obtain the following result ¹:

$$E_3 = \frac{mc^2}{(\rho L^3)^{1/2}} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathcal{D}} \delta_{\mathbf{q}_1+\mathbf{q}_2, \mathbf{q}_3} \left(\mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) b_{\mathbf{q}_1}^* b_{\mathbf{q}_2}^* b_{\mathbf{q}_3} + \text{c.c.} \right) + \dots \quad (79)$$

The vectors \mathbf{q}_1 , \mathbf{q}_2 and \mathbf{q}_3 are the three wavevectors involved in the Beliaev-Landau process, with \mathbf{q}_3 the wavevector of the phonon which decays into phonons of wavevectors \mathbf{q}_1 and \mathbf{q}_2 or which originates from the merging of two such phonons. In the ellipsis \dots , we omit the three-body processes involving non-bosonic excitations and the terms

1. Everywhere in this article the coupling amplitudes $\mathcal{A}^{m \leftrightarrow n}$ are symmetric functions of their m first arguments, the wavevectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$ of the incoming phonons, as well as of their n last arguments, the wavevectors $\mathbf{q}_{m+1}, \mathbf{q}_{m+2}, \dots, \mathbf{q}_{m+n}$ of the outgoing phonons. With this property the amplitudes are uniquely determined.

proportional to $b_{\mathbf{q}_1} b_{\mathbf{q}_2} b_{\mathbf{q}_3}$ or $b_{\mathbf{q}_1}^* b_{\mathbf{q}_2}^* b_{\mathbf{q}_3}^*$ describing the off-resonant $3 \leftrightarrow 0$ processes which we do not study with the microscopic theory. We have factorized the quantity $mc^2/(\rho L^3)^{1/2}$ and defined the dimensionless coupling amplitude $\mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3)$ of the process $b_{\mathbf{q}_1}^* b_{\mathbf{q}_2}^* b_{\mathbf{q}_3}$, such that it is finite and nonzero in the thermodynamic limit:

$$\begin{aligned} \frac{mc^2}{(\rho L^3)^{1/2}} \mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) &= \frac{g_0}{L^3} \sum_{\mathbf{k}, \mathbf{k}' \in \mathcal{D}} \left[\mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}_1} M_{\mathbf{k}'}^{\mathbf{q}_1} \left(M_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_3} M_{\mathbf{k}-\mathbf{q}_3/2}^{\mathbf{q}_2} + N_{\mathbf{k}+\mathbf{q}_3/2}^{\mathbf{q}_2} N_{\mathbf{k}+\mathbf{q}_2/2}^{\mathbf{q}_3} \right) \right. \\ &\left. + \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}_1} N_{\mathbf{k}'}^{\mathbf{q}_1} \left(M_{\mathbf{k}+\mathbf{q}_3/2}^{\mathbf{q}_2} M_{\mathbf{k}+\mathbf{q}_2/2}^{\mathbf{q}_3} + N_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_3} N_{\mathbf{k}-\mathbf{q}_3/2}^{\mathbf{q}_2} \right) + \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}_3} \left(N_{\mathbf{k}'}^{\mathbf{q}_3} N_{\mathbf{k}+\mathbf{q}_1/2}^{\mathbf{q}_2} M_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_1} + M_{\mathbf{k}'}^{\mathbf{q}_3} M_{\mathbf{k}+\mathbf{q}_1/2}^{\mathbf{q}_2} N_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_1} \right) + \mathbf{q}_1 \leftrightarrow \mathbf{q}_2 \right] \quad (80) \end{aligned}$$

where the notation $+\mathbf{q}_1 \leftrightarrow \mathbf{q}_2$ means that one must add to the terms already present in (80) those obtained by exchanging \mathbf{q}_1 and \mathbf{q}_2 while leaving \mathbf{q}_3 unchanged. Although the principle of our microscopic approach to calculate the $2 \leftrightarrow 1$ coupling amplitude is fairly simple, we obtain a relatively cumbersome expression (80). This is due to the resummation over the internal wave vectors \mathbf{k} and \mathbf{k}' of the colliding pairs. A much more tractable expression is obtained in the long-wave limit in the next subsection.

Long-wave limit. At low temperature, the low-wavevector phonons dominate the kinetics of the gas. This motivates a specific study of $\mathcal{A}^{2 \leftrightarrow 1}$ in the limit $q_1, q_2, q_3 \rightarrow 0$, where a comparison to hydrodynamics is meaningful. We perform the expansion of the expression (80) in this limit and for a continuous space, $l \rightarrow 0$, which allows us to integrate over the internal degrees of freedom \mathbf{k} and \mathbf{k}' of the pairs and to obtain an expression of the coupling amplitude that depends only of the external wavevectors \mathbf{q}_1 , \mathbf{q}_2 and \mathbf{q}_3 . The microscopic calculation is done in detail in reference [49]; the underlying fermionic nature of the problem makes it rather tedious. We give here only the final result, expressed in terms of the energies $\hbar\omega_{\mathbf{q}}$ instead of the wavevectors, with the shorthand notation $\omega_i \equiv \omega_{\mathbf{q}_i}$, $i = 1, 2, 3$. The two sets of variables are connected by the dispersion relation (57) and momentum conservation relations. In the limit $\omega_i \rightarrow 0$, we obtain:

$$\begin{aligned} \mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) &= \left(\frac{\hbar}{mc^2} \right)^{-1/2} \frac{1}{2^{3/2} \sqrt{\omega_1 \omega_2 \omega_3}} \left[2J(x, y) (\omega_1 + \omega_2 - \omega_3) \right. \\ &\left. + \left(\frac{\hbar}{mc^2} \right)^2 \left\{ A(x, y) (\omega_1 + \omega_2 - \omega_3) (\omega_1^2 + \omega_2^2 + \omega_3^2) + B(x, y) (\omega_3^3 - \omega_1^3 - \omega_2^3) + C(x, y) \omega_1 \omega_2 \omega_3 \right\} + O(\omega^5) \right] \quad (81) \end{aligned}$$

where the rational fractions A , B , C and J of the variables x and y defined in (59) are given in Appendix B. The denominator $(\omega_1 \omega_2 \omega_3)^{1/2}$ leads to a divergence of the coupling amplitude in the long-wave limit whenever the process is not on the energy shell, that is when it does not obey the energy conservation relation

$$\omega_3 = \omega_1 + \omega_2. \quad (82)$$

One can check that the Bose-Einstein Condensate (BEC) limit ($a \rightarrow 0^+$) of our result (81) coincides with the prediction of Bogoliubov theory [52, 7] for a weakly interacting gas of bosonic dimers².

Resonant processes. To conclude this microscopic study, we evaluate the coupling amplitude on the energy shell, that is for processes satisfying the energy conservation (82). Such processes are allowed in the limit $\omega_3 \rightarrow 0$ for a positive γ parameter only, that is when the dispersion relation $q \mapsto \omega_{\mathbf{q}}$ is convex at low q . We collect the rational fractions B and C to form the thermodynamic quantity

$$2B(x, y) + \frac{2}{3}C(x, y) = 1 + \frac{\rho}{3} \frac{d^2 \mu}{d\rho^2} \left(\frac{d\mu}{d\rho} \right)^{-1} \equiv 1 + \Lambda_{\text{F}} \quad (83)$$

2. The BEC limit corresponds to $x \rightarrow 0^+$ and $y \sim -4/x$ [11]; in Appendix B we give the expressions (B.12) of the rational fractions in this limit. It is then straightforward to check that our expression of $\mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}$ in the BEC limit coincides with equation (16) of reference [7] expanded to third order in q and expressed in terms of the Bogoliubov excitation energies of a gas of $\langle \hat{N}_{\text{B}} \rangle = \langle \hat{N} \rangle / 2$ bosonic dimers of mass $m_{\text{B}} = 2m$, and speed of sound c .

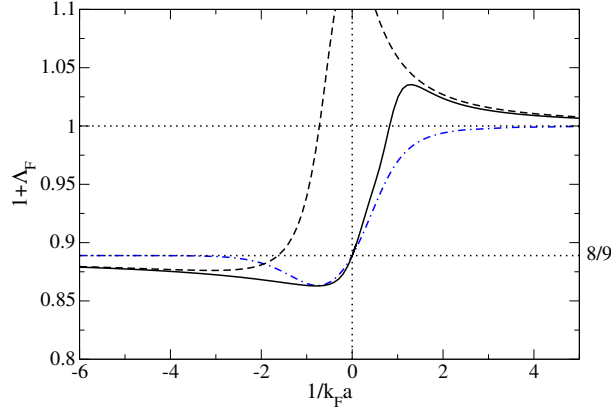


Figure 1: The factor $1 + \Lambda_F$, which is the Fermi gas equation-of-state dependent part in the $2 \leftrightarrow 1$ phonon coupling amplitude (see equation (83)), is plotted as a function of the interaction strength $1/k_F a$. The black solid line is obtained from the equation of state measured in [20]. In the BEC ($k_F a \rightarrow 0^+$) and BCS ($k_F a \rightarrow 0^-$) limits the dashed black lines are the asymptotic behaviors given by the Lee-Huang-Yang corrections [53, 54, 55] (that include the first correction to the mean field energy, scaling as $(k_F a)^2$ on the BCS side and as $\rho a \sqrt{\rho a^3}$ on the BEC side). For comparison the prediction of the BCS equation of state (12) is plotted as a dash-dotted blue line. Note that $1 + \Lambda_F$ tends to 1 in the BEC limit, as for a weakly interacting Bose gas. It is equal to $8/9$ in the unitary limit, and tends to the same value in the BCS limit, since in both cases $\mu \propto \rho^{2/3}$.

Its expression as a function of x and y is given in Appendix B and can also be obtained by differentiating twice the BCS equation of state (12) with respect to μ . This leads to an elegant expression of the on-shell coupling amplitude:

$$\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = 3(1 + \Lambda_F) \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32 m^3 c^3}} + O(q_3^{7/2}) \quad (84)$$

The thermodynamic quantity $1 + \Lambda_F$ is related to the so-called Grüneisen parameter $u = \frac{\partial \ln c}{\partial \ln \rho}$ by $1 + \Lambda_F = \frac{2}{3}(1 + u)$. Its occurrence as a factor in equation (84) is the only difference between this fermionic formula and its equivalent for a gas of weakly interacting bosons obtained by the Bogoliubov theory (see equations (D8) and (D9) of reference [7]). It is plotted as a function of the interaction strength in the BEC-BCS crossover in figure 1. It tends to 1 in the BEC limit ($x \rightarrow 0^+$, $y \sim -4/x$) like in a gas of weakly interacting bosons where $\mu \propto \rho$, and it is equal to $8/9$ at unitarity ($x = y$) and in the BCS limit ($x \rightarrow 0$, $y \rightarrow 0$) since in both cases $\mu \propto \rho^{2/3}$.

4.1.2. Hydrodynamic approach

We now compare our microscopic result (81) to the irrotational quantum hydrodynamics of Landau and Khalatnikov. This theory performs a large-scale description of the gas in terms of two hermitian quantum fields $\hat{\rho}(\mathbf{r}, t)$ and $\hat{\mathbf{v}}(\mathbf{r}, t)$, directly neglecting the exponentially weak density of fermionic quasiparticles at low T . The velocity field $\hat{\mathbf{v}}(\mathbf{r}, t)$ is assumed to be irrotational and is written as the (discrete) gradient of the phase field operator $\hat{\phi}(\mathbf{r}, t)$,

$$\hat{\mathbf{v}}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \hat{\phi}(\mathbf{r}, t), \quad (85)$$

canonically conjugated to the density field operator $\hat{\rho}(\mathbf{r}, t)$:

$$[\hat{\rho}(\mathbf{r}, t), \hat{\phi}(\mathbf{r}', t)] = i \frac{\delta_{\mathbf{r}, \mathbf{r}'}}{\beta}. \quad (86)$$

The dynamics of these fields is governed by the hydrodynamic Hamiltonian

$$\hat{H}_{\text{hydro}} = l^3 \sum_{\mathbf{r}} \left[\frac{\hbar^2}{2m} (\nabla \hat{\phi}) \cdot \hat{\rho} (\nabla \hat{\phi}) + e_{0,0}(\hat{\rho}) \right], \quad (87)$$

where $e_{0,0}$ is the bare energy density, that shall be renormalized by the eigenmodes zero-point energy as described in [8] to give rise to the true energy density e_0 in the ground state, related to the zero-temperature chemical potential by $\mu = de_0/d\rho$.

The procedure to follow is standard and similar to the microscopic approach. We linearize the equations of motion for weak spatial fluctuations of the density and phase fields:

$$\hat{\rho}(\mathbf{r}, t) = \hat{\rho}_0 + \delta\hat{\rho}(\mathbf{r}, t) \quad (88)$$

$$\hat{\phi}(\mathbf{r}, t) = \hat{\phi}_0(t) + \delta\hat{\phi}(\mathbf{r}, t) \quad (89)$$

where $\hat{\rho}_0$ will be replaced by the mean density ρ . We then expand the fields on the eigenmodes of the linearized dynamics:

$$\delta\hat{\rho}(\mathbf{r}, t) = \frac{\rho^{1/2}}{L^{3/2}} \sum_{\mathbf{q} \in \mathcal{D}^*} \left(\frac{\hbar \mathbf{q}}{2mc} \right)^{1/2} (\hat{b}_{\mathbf{q}} + \hat{b}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (90)$$

$$\delta\hat{\phi}(\mathbf{r}, t) = \frac{-i}{\rho^{1/2} L^{3/2}} \sum_{\mathbf{q} \in \mathcal{D}^*} \left(\frac{mc}{2\hbar \mathbf{q}} \right)^{1/2} (\hat{b}_{\mathbf{q}} - \hat{b}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (91)$$

where the bosonic operators $\hat{b}_{\mathbf{q}}$ are the hydrodynamic counterpart of those in equation (74). The corresponding bosonic excitations have a purely linear spectrum $\hbar\omega_{\mathbf{q}}^{\text{hydro}} = \hbar c q$ where the sound velocity c at density ρ is still given by the hydrodynamic expression (58). Next, we insert in the cubic part of the Hamiltonian (87)

$$\hat{H}_{\text{hydro}}^{(3)} = \int^3 \sum_{\mathbf{r}} \left[\frac{\hbar^2}{2m} \nabla \delta\hat{\phi} \cdot \delta\hat{\rho} \nabla \delta\hat{\phi} + \frac{1}{6} \frac{d^2\mu}{d\rho^2} (\delta\hat{\rho})^3 \right], \quad (92)$$

the modal expansions (90) and (91) to obtain

$$\begin{aligned} \hat{H}_{\text{hydro}}^{(3)} = \frac{mc^2}{(\rho L^3)^{1/2}} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathcal{D}^*} & \left[\delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3} \mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) (\hat{b}_{\mathbf{q}_1}^\dagger \hat{b}_{\mathbf{q}_2}^\dagger \hat{b}_{\mathbf{q}_3} + \text{h.c.}) \right. \\ & \left. + \delta_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3, \mathbf{0}} \mathcal{A}_{\text{hydro}}^{3 \leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) (\hat{b}_{\mathbf{q}_1}^\dagger \hat{b}_{\mathbf{q}_2}^\dagger \hat{b}_{\mathbf{q}_3}^\dagger + \text{h.c.}) \right] \quad (93) \end{aligned}$$

where the coupling amplitudes of the $2 \leftrightarrow 1$ and $3 \leftrightarrow 0$ processes are given by

$$\mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32 m^3 c^3}} (3\Lambda_F + u_{12} + u_{13} + u_{23}) \quad (94)$$

$$\mathcal{A}_{\text{hydro}}^{3 \leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{1}{3} \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32 m^3 c^3}} (3\Lambda_F + u_{12} + u_{13} + u_{23}) \quad (95)$$

in terms of the parameter Λ_F defined in equation (83) and of the cosine of the angle between the wave vectors \mathbf{q}_i and \mathbf{q}_j ,

$$u_{ij} = \frac{\mathbf{q}_i \cdot \mathbf{q}_j}{q_i q_j} \quad (96)$$

The $2 \leftrightarrow 1$ amplitude clearly differs from the amplitude (81) obtained by the microscopic approach (in particular it does not diverge in the large wavelength limit). Physically, this may be surprising at first sight. The collective-mode component of the field in the microscopic theory is indeed expected to correspond to the field of the quantum hydrodynamic theory: for example, the modal expansions of the density and phase fields (67,68) in the microscopic theory seem to match, after quantization of the normal modes $b_{\mathbf{q}}$, the modal expansions (90,91) of quantum hydrodynamics (apart from an overall factor 2 in equation (68) due to the fact that the phase $\theta(\mathbf{r}, t)$ of the pairing field is conjugated to the pair density, rather than to the particle density), so that the operators $\hat{b}_{\mathbf{q}}$ in equation (74) indeed seem to correspond to those in equations (90,91).

This discrepancy between the three-phonon coupling amplitudes can be attributed to two reasons: (i) the most obvious one is the difference of the Hamiltonians of the microscopic and hydrodynamic models, (ii) the somehow more hidden one is the fact that the modal expansions (67,68) in the microscopic model are valid only to first order in β , whereas the modal expansions (90,91) in quantum hydrodynamics are exact, with the consequence that the operators $\hat{b}_{\mathbf{q}}$ coincide in the two points of view only to first order in β . The first effect is due to the fact that quantum hydrodynamics is an effective field theory only valid at small wave numbers. The second effect is due to the fact that quantum hydrodynamics directly use the density field $\hat{\rho}(\mathbf{r}, t)$ and the phase field $\hat{\phi}(\mathbf{r}, t)$ as the canonically conjugated variables of the theory (this corresponds to a modulus-phase point of view), whereas the microscopy theory uses the pair fields $\beta_{\mathbf{k}, \mathbf{k}'}(t)$ and $\beta_{\mathbf{k}, \mathbf{k}'}^*(t)$ as conjugate variables, in which case the field density $\rho(\mathbf{r}, t)$ and the phase field $\theta(\mathbf{r}, t)$ are only specific subfields that are nonlinear functions of β and β^* , see equations (63,64). To qualitatively assess the relative importance of the two effects in the coupling amplitude discrepancies, the simplest way is to study the weakly interacting Bose gas case, which leads to less tedious calculations and reproduces the fermionic case in the BEC limit $k_F a \rightarrow 0^+$. This study is performed in Appendix C. We then conclude that the leading effect is effect (ii), that is the slight difference between the operators $\hat{b}_{\mathbf{q}}$ in the microscopic and hydrodynamic theories, as it explains the out-of-the-energy-shell divergence of the microscopic model coupling amplitude at vanishing wave numbers q_i . On the contrary, effect (i) only gives a negligible correction $O(q_i^{7/2})$ to the $\approx q_i^{3/2}$ hydrodynamic coupling amplitude.

One can check however that the microscopic-theory and the hydrodynamic results for the $2 \leftrightarrow 1$ coupling amplitude agree when the energy is conserved, see equation (84) and equation (94) written on the energy shell,

$$\mathcal{A}_{\text{hydroOnS}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = 3(1 + \Lambda_F) \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32m^3 c^3}} \quad (97)$$

In the hydrodynamic theory on the energy shell the three wave vectors \mathbf{q}_1 , \mathbf{q}_2 and \mathbf{q}_3 must indeed be colinear in the same direction in order to satisfy the equality $|\mathbf{q}_1 + \mathbf{q}_2| = q_1 + q_2$ imposed by energy conservation for a linear spectrum (this is the well-known equality case in the triangular inequality). This is a satisfactory and important conclusion as, we shall see it, the Beliaev-Landau phonon damping rates $\Gamma_{\mathbf{q}}$ of Sec.5.2 only involve the on-shell coupling amplitudes. In this respect, the microscopic and hydrodynamic theories lead as expected to the same low-energy physics.

4.2. Four-phonon processes

We now consider the $2 \leftrightarrow 2$ four-phonon process. When the excitation branch $q \mapsto \omega_{\mathbf{q}}$ is concave at low wave numbers, this is the resonant process involving the minimal number of phonons because the $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ processes are now forbidden by energy conservation. Also, this process is more intriguing on a theoretical point of view since it involves virtual nonresonant $1 \leftrightarrow 2$ or $3 \leftrightarrow 0$ intermediate processes. In this case, the equivalence of hydrodynamics with the microscopic approach is not obvious, as hydrodynamics does not correctly describe the $1 \leftrightarrow 2$ processes off-shell. In this section, we give the hydrodynamic prediction for the $2 \leftrightarrow 2$ effective coupling amplitude that includes the virtual processes, then we validate the result with a microscopic model. Since the fermionic microscopic model would be quite cumbersome, we use a boson model with finite range interactions, such that the excitation branch is concave at low q .

4.2.1. Transition amplitude

We need to calculate the transition amplitude between an initial state of energy E_i , which is an arbitrary Fock state of bosonic quasiparticles,

$$|i\rangle = |(n_{\mathbf{q}})_{\mathbf{q} \in \mathcal{D}}\rangle \quad (98)$$

and a final state of energy E_f where two phonons of wave vectors \mathbf{q}_1 and \mathbf{q}_2 were annihilated and replaced by phonons of wavevectors \mathbf{q}_3 and \mathbf{q}_4 :

$$|f\rangle = \frac{\hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2}}{\sqrt{n_{\mathbf{q}_1} n_{\mathbf{q}_2} (1 + n_{\mathbf{q}_3}) (1 + n_{\mathbf{q}_4})}} |i\rangle \quad (99)$$

Whatever the specific model, the Hamiltonian can be expanded as

$$\hat{H} = E_0 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \dots \quad (100)$$

where E_0 is a constant, \hat{H}_2 is the free quasiparticle Hamiltonian and \hat{H}_3, \hat{H}_4 are the third order and fourth order terms. \hat{H}_3 cannot directly couple $|i\rangle$ to $|f\rangle$, so we calculate the coupling to second order in perturbation theory, which amounts to treating \hat{H}_4 to first order and \hat{H}_3 to second order to construct an effective Hamiltonian [56]:

$$\langle f|\hat{H}^{2\leftrightarrow 2,\text{eff}}|i\rangle = \langle f|\hat{H}_4|i\rangle + \sum_{\lambda} \frac{\langle f|\hat{H}_3|\lambda\rangle\langle\lambda|\hat{H}_3|i\rangle}{E_i - E_{\lambda}} \equiv \mathcal{A}_{i\rightarrow f} \quad (101)$$

There exist 6 intermediate states $|\lambda\rangle$, labeled from I to VI, that can be accessed at zero temperature that is when all the modes $\mathbf{q} \neq \mathbf{q}_1, \mathbf{q}_2$ are initially empty. These states correspond to the creation and reabsorption of a virtual phonon by a three-phonon nonresonant process. They are represented by the diagrams on the left part of figure 2, with the virtual intermediate phonon plotted as a dashed line. To these six intermediate states $\lambda=\text{I-VI}$ correspond six other diagrams $\lambda=\text{I'-VI'}$, shown on the right part of figure 2, where the intermediate phonon has the same wave vector but is annihilated and recreated rather than being created then annihilated. These intermediate states exist only at nonzero temperature since the intermediate phonon must preexist in state $|i\rangle$.

4.2.2. Effective $2 \leftrightarrow 2$ coupling amplitude

The effective coupling amplitude $\mathcal{A}^{2\leftrightarrow 2,\text{eff}}$ of the $2 \leftrightarrow 2$ process is defined by the following writing of the effective Hamiltonian:

$$\hat{H}^{2\leftrightarrow 2,\text{eff}} = \frac{mc^2}{\rho L^3} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \in \mathcal{D}} \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} \mathcal{A}^{2\leftrightarrow 2,\text{eff}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \hat{b}_{\mathbf{q}_3}^{\dagger} \hat{b}_{\mathbf{q}_4}^{\dagger} \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2} \quad (102)$$

By construction, the matrix element of $\hat{H}^{2\leftrightarrow 2,\text{eff}}$ between $|i\rangle$ and $|f\rangle$ is the transition amplitude $\mathcal{A}_{i\rightarrow f}$. We thus have the relation

$$\mathcal{A}_{i\rightarrow f} = \sqrt{4 n_{\mathbf{q}_1} n_{\mathbf{q}_2} (1 + n_{\mathbf{q}_3}) (1 + n_{\mathbf{q}_4})} \frac{4mc^2}{\rho L^3} \mathcal{A}^{2\leftrightarrow 2,\text{eff}} \quad (103)$$

where the factor 4 is a counting factor.

From now on we restrict to on-shell processes, such that

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \quad (104)$$

with the shorthand notation $\omega_i = \omega_{\mathbf{q}_i}$, $i = 1, 2, 3, 4$. In this case, a simplification occurs between each of the $\lambda=\text{I,II,III,IV,V,VI}$ diagrams and its $\lambda'=\text{I',II',III',IV',V',VI'}$ counterpart on the right column of figure 2, which formally reduces the problem to zero temperature. Each diagram and its counterpart have indeed opposite energy denominators, with numerators that differ only through the factor involving the occupation number of the intermediate phonon \mathbf{q} , $(1 + n_{\mathbf{q}})$ in the left column (where the intermediate phonon is first created) and $n_{\mathbf{q}}$ in the right column (where the intermediate phonon is first annihilated). Taking I and I' as an example, one has $E_i - E_1 = \omega_1 + \omega_2 - \omega_{\mathbf{q}_1 + \mathbf{q}_2}$ and $E_i - E_{I'} = \omega_{\mathbf{q}_3 + \mathbf{q}_4} - \omega_3 - \omega_4 = -(E_i - E_1)$; in the matrix element $\langle f|\hat{H}_3|\text{I}\rangle\langle\text{I}|\hat{H}_3|i\rangle$ one has the factor $(1 + n_{\mathbf{q}_1 + \mathbf{q}_2})$ and in $\langle f|\hat{H}_3|\text{I}'\rangle\langle\text{I}'|\hat{H}_3|i\rangle$ one has the factor $n_{\mathbf{q}_1 + \mathbf{q}_2}$. Collecting the diagrams by pairs, we obtain an effective coupling amplitude identical to the zero-temperature one, that is with $n_{\mathbf{q}} = 0$ for all $\mathbf{q} \neq \mathbf{q}_1, \mathbf{q}_2$. In terms of the direct on-shell $2 \leftrightarrow 2$ coupling amplitude $\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2,\text{dir}}$ (that is related to \hat{H}_4 in the same way as $\mathcal{A}^{2\leftrightarrow 2,\text{eff}}$ is related to $\hat{H}^{2\leftrightarrow 2,\text{eff}}$) and of the amplitudes $\mathcal{A}^{2\leftrightarrow 1}$ and $\mathcal{A}^{3\leftrightarrow 0}$ introduced in section 4.1, the effective on-shell amplitude can finally be written as follows:

$$\begin{aligned} \mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2,\text{eff}}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) &= \mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2,\text{dir}}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \\ &+ \frac{\mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2) \mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_3, \mathbf{q}_4; \mathbf{q}_1 + \mathbf{q}_2)}{\check{\omega}_1 + \check{\omega}_2 - \check{\omega}_{1+2}} + \frac{9\mathcal{A}^{3\leftrightarrow 0}(\mathbf{q}_3, \mathbf{q}_4, -\mathbf{q}_1 - \mathbf{q}_2) \mathcal{A}^{3\leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2)}{-(\check{\omega}_1 + \check{\omega}_2 + \check{\omega}_{1+2})} \\ &+ \frac{\mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_3, \mathbf{q}_1 - \mathbf{q}_3; \mathbf{q}_1) \mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_1 - \mathbf{q}_3, \mathbf{q}_2; \mathbf{q}_4)}{\check{\omega}_1 - \check{\omega}_3 - \check{\omega}_{1-3}} + \frac{\mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_4, \mathbf{q}_3 - \mathbf{q}_1; \mathbf{q}_2) \mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_3 - \mathbf{q}_1, \mathbf{q}_1; \mathbf{q}_3)}{\check{\omega}_3 - \check{\omega}_1 - \check{\omega}_{3-1}} \\ &+ \frac{\mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_4, \mathbf{q}_1 - \mathbf{q}_4; \mathbf{q}_1) \mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_1 - \mathbf{q}_4, \mathbf{q}_2; \mathbf{q}_3)}{\check{\omega}_1 - \check{\omega}_4 - \check{\omega}_{1-4}} + \frac{\mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_3, \mathbf{q}_4 - \mathbf{q}_1; \mathbf{q}_2) \mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_4 - \mathbf{q}_1, \mathbf{q}_1; \mathbf{q}_4)}{\check{\omega}_4 - \check{\omega}_1 - \check{\omega}_{4-1}} \quad (105) \end{aligned}$$

where the energies were rescaled by mc^2 ,

$$\tilde{\omega} \equiv \frac{\hbar\omega}{mc^2} \quad \text{with the notation} \quad \omega_{i\pm j} \equiv \omega_{\mathbf{q}_i \pm \mathbf{q}_j}. \quad (106)$$

Note that the coupling due to diagram VI (whose contribution is written here immediately after the one of diagram I) has a counting factor equal to 9.

4.2.3. Effective amplitude in hydrodynamics

At low temperature, we need the large wavelength limit of the on-shell expression (105). If one wants to get it from quantum hydrodynamics, one must introduce in quantum hydrodynamics a correction grasping some element of microscopic physics. Indeed, the hydrodynamic excitation spectrum is purely linear, which causes the energy denominators in (105) to vanish, and the effective coupling amplitude to diverge, when the wave vectors are colinear. This is an artefact of hydrodynamics. In reality the spectrum has a nonzero γ curvature parameter, here $\gamma < 0$, so that the energy denominators do not vanish. For almost colinear wave vectors, it is then natural to regularize the coupling amplitude by replacing the hydrodynamic dispersion relation $\hbar\omega_{\mathbf{q}}^{\text{hydro}} = \hbar c q$ with the expansion (57) as done by Landau and Khalatnikov [1]. We shall bring a microscopic justification to this: for colinear wave vectors, our bosonic microscopic model (see subsection 4.2.4) gives a large wavelength equivalent of formula (105) that indeed agrees with the Landau-Khalatnikov modified hydrodynamics.

First we determine the direct $2 \leftrightarrow 2$ process amplitude from the quartic terms of the Hamiltonian:

$$\hat{H}_{\text{hydro}}^{(4)} = \frac{1}{24} \frac{d^3\mu}{d\rho^3} \int_{\mathbf{r}} \delta\hat{\rho}^4 \quad (107)$$

As we did previously, we insert the expansions (90) and (91) into $\hat{H}_{\text{hydro}}^{(4)}$ and we keep only the $2 \leftrightarrow 2$ terms $\hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2}$:

$$\hat{H}_{\text{hydro}}^{2\leftrightarrow 2, \text{dir}} = \frac{mc^2}{\rho L^3} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \in \mathcal{D}} \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} \mathcal{A}_{\text{hydro}}^{2\leftrightarrow 2, \text{dir}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2} \quad (108)$$

to obtain the rescaled direct $2 \leftrightarrow 2$ coupling amplitude

$$\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 2, \text{dir}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) = \frac{\Sigma_F}{16} \sqrt{\frac{\hbar^4 q_1 q_2 q_3 q_4}{m^4 c^4}} \quad (109)$$

where we introduced

$$\Sigma_F \equiv \frac{\rho^3}{mc^2} \frac{d^3\mu}{d\rho^3} \quad (110)$$

Second we combine the amplitude (109) with our previous expressions for the $2 \leftrightarrow 1$ (94) and $3 \leftrightarrow 0$ (95) amplitudes to obtain

$$\begin{aligned} \mathcal{A}_{\text{hydro corr, OnS}}^{2\leftrightarrow 2, \text{eff}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) = \frac{1}{16} \sqrt{\frac{\hbar^4 \omega_1 \omega_2 \omega_3 \omega_4}{m^4 c^8}} & \left(\Sigma_F + \frac{(\omega_1 + \omega_2)^2 A_{1234} + \omega_{1+2}^2 B_{1234}}{(\omega_1 + \omega_2)^2 - \omega_{1+2}^2} \right. \\ & \left. + \frac{(\omega_1 - \omega_3)^2 A_{1324} + \omega_{1-3}^2 B_{1324}}{(\omega_1 - \omega_3)^2 - \omega_{1-3}^2} + \frac{(\omega_1 - \omega_4)^2 A_{1423} + \omega_{1-4}^2 B_{1423}}{(\omega_1 - \omega_4)^2 - \omega_{1-4}^2} \right) \quad (111) \end{aligned}$$

where the index ‘‘corr’’ means that one goes beyond the hydrodynamic approximation for the dispersion relation in the denominators of expression (111) by using the cubic approximation (57), and we introduced the coefficients

$$A_{ijkl} = (3\Lambda_F + u_{ij})(1 + u_{kl}) + (3\Lambda_F + u_{kl})(1 + u_{ij}) + (1 + u_{ij})(1 + u_{kl}) \quad (112)$$

$$B_{ijkl} = (3\Lambda_F + u_{ij})(3\Lambda_F + u_{kl}) \quad (113)$$

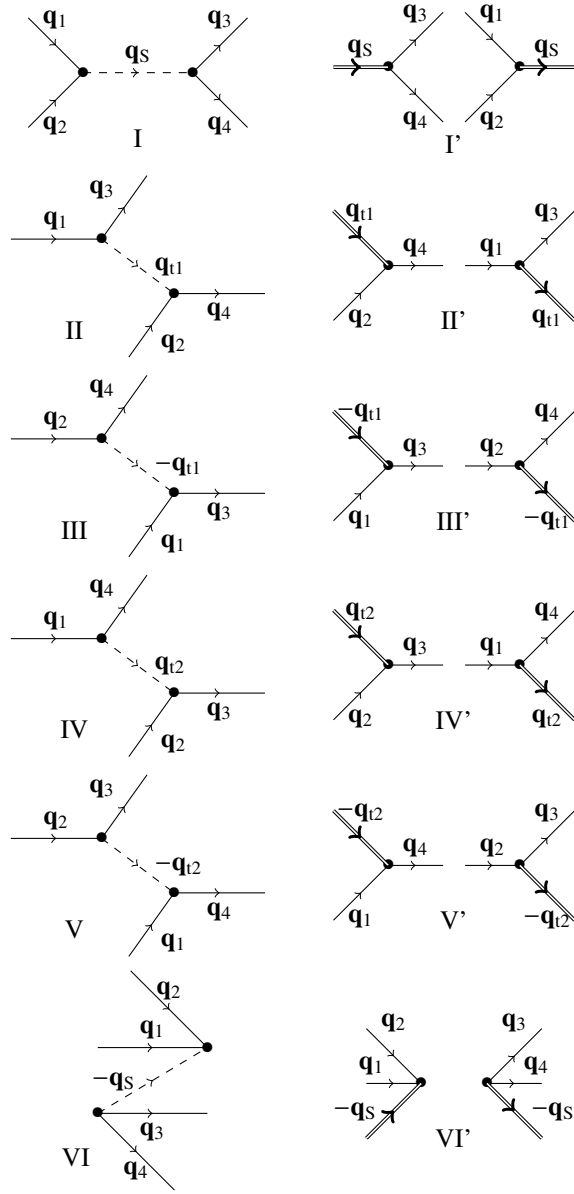


Figure 2: Second order diagrams for the 4-phonon process $(\mathbf{q}_1, \mathbf{q}_2) \rightarrow (\mathbf{q}_3, \mathbf{q}_4)$ with $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_3 + \mathbf{q}_4$, considered as two successive three-phonon processes. The incoming wave vectors \mathbf{q}_1 and \mathbf{q}_2 and the emergent ones \mathbf{q}_3 and \mathbf{q}_4 are plotted as a solid line with an arrow. On the left column the diagrams include a virtual intermediate phonon, plotted as a dashed line with an arrow. On the right column they include a real intermediate phonon, plotted as a double line with an arrow. On any given row, the two diagrams have the same intermediate phonon: in I and I' $\mathbf{q}_S = \mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_3 + \mathbf{q}_4$, in II and II' $\mathbf{q}_{t1} = \mathbf{q}_1 - \mathbf{q}_3 = \mathbf{q}_4 - \mathbf{q}_2$, in III and III' $-\mathbf{q}_{t1}$, in IV and IV' $\mathbf{q}_{t2} = \mathbf{q}_1 - \mathbf{q}_4 = \mathbf{q}_3 - \mathbf{q}_2$, in V and V' $-\mathbf{q}_{t2}$, in VI and VI' $-\mathbf{q}_S$.

4.2.4. Effective amplitude in a weakly interacting Bose gas with nonzero range interactions

To study the $2 \leftrightarrow 2$ process with a microscopic approach and understand how the divergence in hydrodynamics must be regularised, we shall not use the fermionic variational theory of section 2, which would be very heavy to manipulate due to the internal degrees of freedom of the pairs. We rather use a bosonic model with a large enough interaction range b so that the Bogoliubov excitation branch is concave at low q . In Appendix E, we describe this model in detail and we calculate the $2 \leftrightarrow 2$ effective coupling amplitude in the weakly interacting limit from Bogoliubov theory. On shell, in the limit of small wave vectors, we then recover the prediction (111) of corrected quantum hydrodynamics specialised to the bosonic equation of state (E.11), that is with $\Lambda_F = \Sigma_F = 0$ and $(m, c, \rho) \rightarrow (m_B, c_B, \rho_B)$, where m_B is the boson mass, c_B the sound velocity in the Bose gas and ρ_B the density of the bosons. The curvature of the spectrum, which eliminates the divergence of the $2 \leftrightarrow 2$ effective coupling amplitude, is present from the start in the microscopic model and does not need to be added by hand. This provides a microscopic justification to the prescription of Landau and Khalatnikov [1] described in the beginning of subsection 4.2.3. The result was not obvious from the start, and it is due to a subtle cancellation between the large wavelength divergences of the direct coupling term (E.20) and of the second-order-perturbation-theory virtual coupling term. The details of the microscopic calculations are thus very different from the hydrodynamic ones, where the direct coupling term is zero for the considered equation of state (E.11).

5. Application: phonon damping in the BEC-BCS crossover

5.1. A general master-equation expression of the damping rates

To calculate the damping rate, we view the phonon mode of wave vector \mathbf{q} as a harmonic oscillator coupled to the reservoir formed by the other quasiparticle modes [57], assumed to be at thermal equilibrium at temperature T . We thus split the low-energy effective Hamiltonian as

$$\hat{H} = \hbar\omega_{\mathbf{q}}\hat{b}_{\mathbf{q}}^{\dagger}\hat{b}_{\mathbf{q}} + \left(\sum_{\mathbf{q}' \neq \mathbf{q}} \hbar\omega_{\mathbf{q}'}\hat{b}_{\mathbf{q}'}^{\dagger}\hat{b}_{\mathbf{q}'} \right) + (\hat{R}^{\dagger}\hat{b}_{\mathbf{q}} + \hat{b}_{\mathbf{q}}^{\dagger}\hat{R}) + \dots \quad (114)$$

The first and the second terms, originating from \hat{H}_2 , describe the free evolution of the mode \mathbf{q} and of the reservoir, respectively. The third term, originating from the part of \hat{H}_3 or $\hat{H}^{2 \leftrightarrow 2, \text{eff}}$ involving $\hat{b}_{\mathbf{q}}$ or $\hat{b}_{\mathbf{q}}^{\dagger}$, gives the coupling between the reservoir and the mode \mathbf{q} . The ellipsis \dots includes (i) higher order nonlinear processes, (ii) processes that do not involve the mode \mathbf{q} , and (iii) terms of the form $\hat{R}'\hat{b}_{\mathbf{q}}^{\dagger}\hat{b}_{\mathbf{q}}$, where \hat{R}' is an operator of the reservoir, that shift the energy of the mode \mathbf{q} . In the Born-Markov approximation, valid in the weak coupling and collisionless limit of low temperature T and low wave number q at fixed nonzero $\hbar cq/k_B T$,³ one gets for the equation of motion of the mean number of excitations $\langle \hat{n}_{\mathbf{q}} \rangle$ in the mode \mathbf{q} [56]:

$$\frac{d}{dt}\langle \hat{n}_{\mathbf{q}} \rangle = -\Gamma_{\mathbf{q}}(\langle \hat{n}_{\mathbf{q}} \rangle - \bar{n}_{\mathbf{q}}) \quad (115)$$

where $\bar{n}_{\mathbf{q}}$ is the thermal equilibrium population of the mode:

$$\bar{n}_{\mathbf{q}} = \frac{1}{\exp\left(\frac{\hbar\omega_{\mathbf{q}}}{k_B T}\right) - 1} \quad (116)$$

The damping rate $\Gamma_{\mathbf{q}}$ is given by

$$\Gamma_{\mathbf{q}} = \int_{-\infty}^{+\infty} \frac{dt}{\hbar^2} e^{-i\omega_{\mathbf{q}}t} \text{Tr}_{\text{R}} \left([\hat{R}, \hat{R}^{\dagger}(t)] \sigma_{\text{R}}^{\text{eq}}(t) \right) \quad (117)$$

where Tr_{R} is the trace over the states of the reservoir, $\sigma_{\text{R}}^{\text{eq}}$ is the thermal equilibrium density operator of the reservoir, $[\hat{A}, \hat{B}]$ is the commutator of operators \hat{A} and \hat{B} , and the time evolution of the operator $\hat{R}^{\dagger}(t)$ of the reservoir is calculated in the interaction picture with the Hamiltonian $\sum_{\mathbf{q}' \neq \mathbf{q}} \hbar\omega_{\mathbf{q}'}\hat{b}_{\mathbf{q}'}^{\dagger}\hat{b}_{\mathbf{q}'}$.

3. When T and q tend to zero, the wave numbers q_i of the intermediate phonons contributing to $\Gamma_{\mathbf{q}}$ tend to zero, and so do in general the phonon coupling amplitudes appearing in the operator \hat{R} . Still a validity condition of the result (117) is that $\omega_{\mathbf{q}} \gg \Gamma_{\text{qth}}$, with $\hbar c q_{\text{th}} = k_B T$, that is the angular frequency $\omega_{\mathbf{q}}$ of the considered mode must greatly exceed the typical relaxation rate of the thermal phonons, which defines the so-called collisionless regime (as opposed to the hydrodynamic regime).

5.2. Convex case: Beliaev-Landau damping

For a convex dispersion relation, the bilinear terms in \hat{R} lead to resonant processes, so they give the leading contribution to $\Gamma_{\mathbf{q}}$ at low temperature. We thus insert in (117) the following expression for \hat{R}^\dagger :

$$\hat{R}^\dagger = \frac{mc^2}{(\rho L^3)^{1/2}} \left[\sum_{\mathbf{q}_2, \mathbf{q}_3} \mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_2, \mathbf{q}_3; \mathbf{q}) \hat{b}_{\mathbf{q}_2}^\dagger \hat{b}_{\mathbf{q}_3}^\dagger \delta_{\mathbf{q}, \mathbf{q}_2 + \mathbf{q}_3} + \sum_{\mathbf{q}_1, \mathbf{q}_2} 2\mathcal{A}^{2\leftrightarrow 1}(\mathbf{q}_2, \mathbf{q}; \mathbf{q}_1) \hat{b}_{\mathbf{q}_1}^\dagger \hat{b}_{\mathbf{q}_2}^\dagger \delta_{\mathbf{q}_2 + \mathbf{q}, \mathbf{q}_1} \right] \quad (118)$$

We split the contribution of the Beliaev process $\mathbf{q} \leftrightarrow (\mathbf{q}', (\mathbf{q} - \mathbf{q}'))$ (first contribution in (118)) from the one of the Landau process $(\mathbf{q}, \mathbf{q}') \leftrightarrow \mathbf{q} + \mathbf{q}'$ (second contribution in (118)):

$$\Gamma_q^{\text{Bel}} = \frac{(mc^2)^2}{2\pi^2 \hbar^2 \rho} \int_{\mathbb{R}^3} d^3 q' |\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(\mathbf{q}', \mathbf{q} - \mathbf{q}'; \mathbf{q})|^2 \delta(\omega_{\mathbf{q}'} + \omega_{\mathbf{q} - \mathbf{q}'} - \omega_{\mathbf{q}}) (1 + \bar{n}_{\mathbf{q} - \mathbf{q}'} + \bar{n}_{\mathbf{q}'}) \quad (119)$$

$$\Gamma_q^{\text{Lan}} = \frac{(mc^2)^2}{\pi^2 \hbar^2 \rho} \int_{\mathbb{R}^3} d^3 q' |\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(\mathbf{q}', \mathbf{q}; \mathbf{q}' + \mathbf{q})|^2 \delta(\omega_{\mathbf{q} + \mathbf{q}'} - \omega_{\mathbf{q}'} - \omega_{\mathbf{q}}) (\bar{n}_{\mathbf{q}'} - \bar{n}_{\mathbf{q}' + \mathbf{q}}) \quad (120)$$

and in the low temperature limit we use the leading order expression (84) of the on-shell coupling amplitude $2 \leftrightarrow 1$, or equivalently the hydrodynamic on-shell expression (97). We integrate over the wave vector \mathbf{q}' in spherical coordinates of polar axis \mathbf{q} to obtain the low temperature equivalents⁴

$$\Gamma_q^{\text{Bel}} \underset{T \rightarrow 0}{\sim} \frac{9(1 + \Lambda_F)^2}{32\pi} \frac{mc^2}{\hbar \rho} \left(\frac{mc}{\hbar} \right)^3 \left(\frac{k_B T}{mc^2} \right)^5 \tilde{\Gamma}^{\text{Bel}}(\tilde{q}) \quad (121)$$

$$\Gamma_q^{\text{Lan}} \underset{T \rightarrow 0}{\sim} \frac{9(1 + \Lambda_F)^2}{32\pi} \frac{mc^2}{\hbar \rho} \left(\frac{mc}{\hbar} \right)^3 \left(\frac{k_B T}{mc^2} \right)^5 \tilde{\Gamma}^{\text{Lan}}(\tilde{q}) \quad (122)$$

The wave numbers are here rescaled by the typical thermal wave number as follows:

$$\tilde{q} = \frac{\hbar c q}{k_B T} \quad (123)$$

and the $T \rightarrow 0$ limit is taken at fixed \tilde{q} . The functions $\tilde{\Gamma}^{\text{Bel}}$ and $\tilde{\Gamma}^{\text{Lan}}$ are universal functions of \tilde{q} , that can be expressed in terms of the Bose functions $g_\alpha(z) = \sum_{n=1}^{+\infty} z^n / n^\alpha$, also called polylogarithms $\text{Li}_\alpha(z)$, and of the Riemann ζ function, $\zeta(\alpha) = g_\alpha(1)$,⁵

$$\tilde{\Gamma}^{\text{Bel}}(\tilde{q}) = \frac{\tilde{q}^5}{30} - \frac{4\pi^4}{15} \tilde{q} + 48 [\zeta(5) - g_5(e^{-\tilde{q}})] - 24\tilde{q}g_4(e^{-\tilde{q}}) + 4\tilde{q}^2 [\zeta(3) - g_3(e^{-\tilde{q}})] \quad (124)$$

$$\tilde{\Gamma}^{\text{Lan}}(\tilde{q}) = \tilde{\Gamma}^{\text{Bel}}(\tilde{q}) - \frac{\tilde{q}^5}{30} + \frac{8\pi^4}{15} \tilde{q} \quad (125)$$

This leads to the following limiting behaviors:

$$\tilde{\Gamma}^{\text{Bel}} \underset{\tilde{q} \rightarrow 0}{=} \frac{\tilde{q}^4}{6} + \frac{\tilde{q}^6}{360} + O(\tilde{q}^8) \quad (126)$$

$$\tilde{\Gamma}^{\text{Bel}} \underset{\tilde{q} \rightarrow +\infty}{=} \frac{\tilde{q}^5}{30} + 4\zeta(3)\tilde{q}^2 - \frac{4\pi^4}{15} \tilde{q} + 48\zeta(5) + O(\tilde{q}^2 e^{-\tilde{q}}) \quad (127)$$

The corresponding ones for $\tilde{\Gamma}^{\text{Lan}}$ can be trivially deduced from equation (125). The two damping rates and their sum are plotted as functions of q in figure 3.

4. If one directly uses the quantum hydrodynamic linear dispersion relation $\omega_{\mathbf{q}} = cq$ in the argument of the Dirac distribution in equations (119,120), we face the polar integral $\int_{-1}^1 du \delta(u - 1)$ (u is the cosine of the angle between \mathbf{q} and \mathbf{q}' and the energy is conserved for $u = 1$ only), which seems to be equal to 1/2. If one correctly takes into account the strict convexity of the dispersion relation, through the curvature parameter $\gamma > 0$, we obtain the correct value $\int_{-1}^1 du \delta(u - u_0) = 1$ with the energy conserving root $u_0 \in]-1, 1[$ arbitrarily close to 1 in the low temperature limit. Obviously this value does not depend on γ because the Dirac distribution always has a unit mass, which explains why the results (121,122) are γ -independent, but in a subtle way it utilises the sign of γ : if γ was negative, there would be no energy conserving root u_0 in the interval of integration, and the resulting integral would vanish, indicating the absence of Beliaev-Landau damping for a concave dispersion relation.

5. One just has to series expand $1/(e^{\tilde{q}} - 1)$ and $1/(e^{\tilde{q} + \tilde{q}'} - 1)$ in powers of the variable $e^{-\tilde{q}}$ and to exchange summation and integration. We recall that $\zeta(4) = \pi^4/90$.

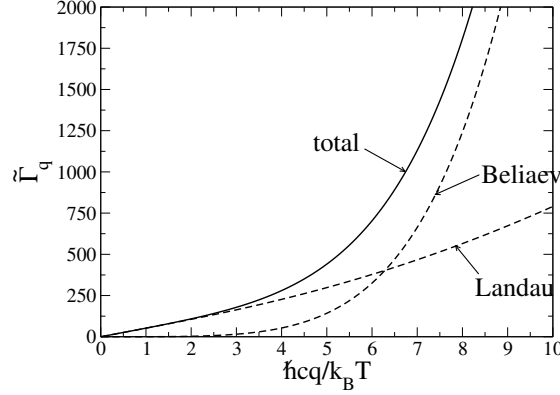


Figure 3: Low-temperature leading behavior of the total phonon damping rate (solid line) as a function of the rescaled wave number $\tilde{q} = \hbar c q / k_B T$ for a convex dispersion relation. This is the sum of the Beliaev (124) and Landau (125) rates (dashed lines). The rates are rescaled as in equations (121) and (122).

5.3. Concave case: Landau-Khalatnikov damping

For a concave dispersion relation, the mode \mathbf{q} is damped at sufficiently low temperature by the $2 \leftrightarrow 2$ process only; in equation (117), we thus keep in \hat{R}^\dagger the terms $\hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_2}$, that originate from the quartic Hamiltonian (102). More explicitly, we replace \hat{R}^\dagger in equation (117) by the following expression:

$$\hat{R}^\dagger = \frac{mc^2}{\rho L^3} \sum_{\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4} 2\mathcal{A}_{\text{eff}}^{2 \leftrightarrow 2}(\mathbf{q}, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_2} \delta_{\mathbf{q} + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} \quad (128)$$

This leads to

$$\Gamma_{\mathbf{q}}^{2 \leftrightarrow 2} = \frac{(mc^2)^2}{4\pi^5 \hbar^2 \rho^2} \int_{\mathbb{R}^6} d^3 q_2 d^3 q_3 |\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 2, \text{eff}}(\mathbf{q}, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4)|^2 \delta(\omega_{\mathbf{q}_3} + \omega_{\mathbf{q}_4} - \omega_{\mathbf{q}_2} - \omega_{\mathbf{q}}) \left[\bar{n}_{\mathbf{q}_2} (1 + \bar{n}_{\mathbf{q}_3}) (1 + \bar{n}_{\mathbf{q}_4}) - (1 + \bar{n}_{\mathbf{q}_2}) \bar{n}_{\mathbf{q}_3} \bar{n}_{\mathbf{q}_4} \right] \quad (129)$$

where the vector \mathbf{q}_4 is expressed in terms of the other wave vectors through momentum conservation:

$$\mathbf{q}_4 = \mathbf{q} + \mathbf{q}_2 - \mathbf{q}_3 \quad (130)$$

In what follows, we explain how to obtain a low-temperature equivalent for the rate $\Gamma_{\mathbf{q}}^{2 \leftrightarrow 2}$ at fixed $\tilde{q} = \hbar c q / k_B T$. We take as the polar z axis of the spherical coordinates the direction of \mathbf{q} ; the vectors \mathbf{q}_i then have coordinates (q_i, θ_i, ϕ_i) . We consider a temperature T , controlled by the small parameter

$$\epsilon \equiv \frac{k_B T}{mc^2} \ll 1, \quad (131)$$

so low that the typical wave numbers are much smaller than mc/\hbar and the bosonic branch is populated in its quasi-linear part only. In this case, the coupling amplitude $\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 2}$, that would be divergent at vanishing angles for a linear dispersion relation as already pointed out in section 4.2.3, is extremely peaked around $\theta_2 = \theta_3 = 0$ with a width of order ϵ in θ_2 and θ_3 ,⁶ and a height $1/\epsilon^2$ times larger than the typical amplitude at nonzero angles, as we shall see.

6. The peak width is of order ϵ because the intermediate-virtual-phonon energy change when the θ_i vary from 0 to ϵ is of the same order as the cubic correction to the hydrodynamic dispersion relation, knowing that the typical wave numbers are $\approx k_B T / \hbar c$. Taking as an example the intermediate phonon $\mathbf{q}_S = \mathbf{q} + \mathbf{q}_2$, one finds that $\hbar c[|\mathbf{q} + \mathbf{q}_2| - (q + q_2)] \sim -\frac{\hbar c q q_2}{q + q_2} \theta_2^2 \approx k_B T \theta_2^2$; this deviation is of the same order as the cubic term $\approx k_B T |\gamma| \epsilon^2$ in the equation (57) when $\theta_2^2 \approx |\gamma| \epsilon^2$, hence the change of variable (132).

Using conservation of energy and momentum (130), one also finds that $\theta_4 = O(\epsilon)$ over the width of the peak. We thus rescale the wave numbers as in equation (123) and the polar angles as

$$\tilde{\theta}_i = \frac{\theta_i}{\epsilon|\gamma|^{1/2}} \quad (132)$$

with $\gamma < 0$ the curvature parameter (57), then we perform a Taylor expansion of the coupling amplitude (111) for $\epsilon \rightarrow 0$ at fixed rescaled quantities:

$$\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2}(\mathbf{q}, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \underset{\epsilon \rightarrow 0}{=} \left(\frac{3(1 + \Lambda_F)}{4} \right)^2 \frac{(\tilde{q}\tilde{q}_2\tilde{q}_3\tilde{q}_4)^{1/2}}{|\gamma|} \mathcal{A}_{\text{red}}^{2\leftrightarrow 2}(\tilde{q}, \tilde{q}_2, \tilde{q}_3, \tilde{\theta}_2, \tilde{\theta}_3) + O(\epsilon^2) \quad (133)$$

The Σ_F -dependent term of the $\mathcal{A}^{2\leftrightarrow 2, \text{dir}}$ direct amplitude does not contribute at this order. This property, combined with the clever rescaling (132) of the polar angles, allowed us in equation (133) to pull out the factors $1 + \Lambda_F$ and γ that depend on the interaction strength. This leads to a universal $\mathcal{A}_{\text{red}}^{2\leftrightarrow 2}$ reduced amplitude:

$$\mathcal{A}_{\text{red}}^{2\leftrightarrow 2}(\tilde{q}, \tilde{q}_2, \tilde{q}_3, \tilde{\theta}_2, \tilde{\theta}_3) = \frac{1}{\tilde{q}\tilde{q}_2 \left(\frac{\tilde{\theta}_2^2}{(\tilde{q} + \tilde{q}_2)^2} + \frac{3}{4} \right)} - \frac{1}{\tilde{q}\tilde{q}_3 \left(\frac{\tilde{\theta}_3^2}{(\tilde{q} - \tilde{q}_3)^2} + \frac{3}{4} \right)} - \frac{1}{\tilde{q}(\tilde{q} + \tilde{q}_2 - \tilde{q}_3) \left(\frac{\tilde{\theta}_4^2}{(\tilde{q}_3 - \tilde{q}_2)^2} + \frac{3}{4} \right)} \quad (134)$$

The first, second and third terms in (134) originate from the second, third and fourth terms in (111). In the last two terms, we carefully distinguished the cases $q > q_3$ and $q < q_3$, $q > q_4$ and $q < q_4$ before taking the $\epsilon \rightarrow 0$ limit.

Next, the implicit relation issued from energy conservation,

$$\tilde{q}_4 = \tilde{q} + \tilde{q}_2 - \tilde{q}_3 - \frac{\epsilon^2|\gamma|}{8} (\tilde{q}^3 + \tilde{q}_2^3 - \tilde{q}_3^3 - \tilde{q}_4^3) + O(\epsilon^4) \quad (135)$$

is iterated once and combined with a spherical geometry calculus projecting relation (130) over \mathbf{q} . This gives the following expression for $\tilde{\theta}_4$:

$$\tilde{\theta}_4^2 = \frac{\tilde{q}_2\tilde{\theta}_2^2 - \tilde{q}_3\tilde{\theta}_3^2 - \frac{1}{4} [\tilde{q}^3 + \tilde{q}_2^3 - \tilde{q}_3^3 - (\tilde{q} + \tilde{q}_2 - \tilde{q}_3)^3]}{\tilde{q} + \tilde{q}_2 - \tilde{q}_3} + O(\epsilon^2) \quad (136)$$

Note that $\tilde{q}_3 < \tilde{q} + \tilde{q}_2$ according to (135). Also, due to the rotational invariance around \mathbf{q} , the integrand of (129) depends on the azimuthal angles only through their difference $\phi \equiv \phi_2 - \phi_3$.

The last step is to integrate the Dirac distribution ensuring energy conservation. This is conveniently done in a polar representation of the rescaled angles:

$$\tilde{\theta}_2 = R \cos \alpha \quad \tilde{\theta}_3 = R \sin \alpha \quad (137)$$

We also write the energy difference between the initial state and the final state as

$$\omega_{\mathbf{q}_3} + \omega_{\mathbf{q}_4} - \omega_{\mathbf{q}_2} - \omega_{\mathbf{q}} = \frac{mc^2}{\hbar} \frac{\epsilon^3|\gamma|}{2} (uR^2 + v) + O(\epsilon^5) \quad (138)$$

with

$$u = \frac{\tilde{q}(\tilde{q}_3 \sin^2 \alpha - \tilde{q}_2 \cos^2 \alpha) + \tilde{q}_2\tilde{q}_3(1 - \sin 2\alpha \cos \phi)}{\tilde{q} + \tilde{q}_2 - \tilde{q}_3} \quad (139)$$

$$v = \frac{1}{4} [\tilde{q}^3 + \tilde{q}_2^3 - \tilde{q}_3^3 - (\tilde{q} + \tilde{q}_2 - \tilde{q}_3)^3] \quad (140)$$

In the form (138), the Dirac distribution is readily integrated over R . We can finally express the $2 \leftrightarrow 2$ damping rate in terms of a universal function $\tilde{\Gamma}^{2\leftrightarrow 2}$ of the rescaled wave number \tilde{q} :

$$\frac{\hbar\Gamma_{\mathbf{q}}^{2\leftrightarrow 2}}{mc^2} \underset{\epsilon \rightarrow 0}{\sim} \frac{81(1 + \Lambda_F)^4}{256\pi^4|\gamma|} \left(\frac{k_B T}{mc^2} \right)^7 \left(\frac{mc}{\hbar\rho^{1/3}} \right)^6 \tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) \quad (141)$$

This is one of the central results of this paper. The function $\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q})$ is given by a quadruple integral ⁷

$$\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) = \int_0^\infty d\tilde{q}_2 \int_0^{\tilde{q}+\tilde{q}_2} d\tilde{q}_3 \frac{\tilde{q}\tilde{q}_2^3\tilde{q}_3^3(\tilde{q} + \tilde{q}_2 - \tilde{q}_3)}{|\nu|} \frac{[1 + f(\tilde{q}_2)]f(\tilde{q}_3)f(\tilde{q} + \tilde{q}_2 - \tilde{q}_3)}{f(\tilde{q})} \times \int_0^{\pi/2} d\alpha \int_0^\pi d\phi \sin \alpha \cos \alpha \Theta\left(-\frac{\nu}{u}\right) \left| \frac{\nu}{u} \right| \mathcal{A}_{\text{red}}^{2\leftrightarrow 2}\left(\tilde{q}, \tilde{q}_2, \tilde{q}_3, \left| \frac{\nu}{u} \right|^{1/2} \cos \alpha, \left| \frac{\nu}{u} \right|^{1/2} \sin \alpha\right)^2 \quad (142)$$

where we introduced the Heaviside function $\Theta(x \geq 0) = 1$, $\Theta(x < 0) = 0$ and the reduced Bose function $f(x) = 1/(e^x - 1)$. The occupation numbers in (129) were rewritten using the property $(1 + \bar{n}_{\mathbf{q}_i})/\bar{n}_{\mathbf{q}_i} = e^{\hbar\omega_{\mathbf{q}_i}/k_B T}$ and the conservation of energy. The function $\tilde{\Gamma}^{2\leftrightarrow 2}$ is plotted in figure 4. Its low- and high- \tilde{q} behaviors can be obtained analytically:

$$\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) \underset{\tilde{q} \rightarrow 0}{=} \frac{16\pi^5}{135} \tilde{q}^3 + O(\tilde{q}^4) \quad (143)$$

$$\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) \underset{\tilde{q} \rightarrow \infty}{=} \frac{16\pi\zeta(5)}{3} \tilde{q}^2 + O(\tilde{q}) \quad (144)$$

These limiting behaviors disagree with the results of Landau and Khalatnikov in reference [1] (see their equations (7.6) and (7.12) in the version [58]), even for the order in \tilde{q} of the leading terms. This discrepancy is due to the fact that these authors have neglected in the coupling amplitude at low \tilde{q} and at high \tilde{q} the contribution of the diagrams II to V of figure 2 (the diagram VI is nonresonant). This is not justified as already pointed out in reference [9], where it was observed numerically that the neglected diagrams are comparable in magnitude with the kept one. We find that at leading order, the neglected diagrams actually interfere destructively with the diagram I, which makes the exact results (143,144) sub-leading with respect to the Landau and Khalatnikov predictions by two orders in \tilde{q} .

5.4. Phonon damping beyond hydrodynamics

In this section we concentrate on the zero temperature unitary Fermi gas. Our goal is to calculate the phonon damping rates beyond the hydrodynamic result (127), which we rewrite in the form

$$\left(\Gamma_q^{\text{Bel}}\right)_{\text{hydro}}^{\text{unitary gas}} = \left(\frac{2}{9\pi}\right) \left(\frac{mc^2}{\hbar}\right) \left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \frac{\check{q}^5}{30} \quad (145)$$

where we introduced the notations

$$\check{q} \equiv \frac{\hbar q}{mc} \quad \text{and} \quad \check{\omega}_q \equiv \frac{\hbar\omega_{\mathbf{q}}}{mc^2} \quad (146)$$

To this aim, we first calculate the amplitude $\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3)$ to first order in the curvature parameter γ of the excitation spectrum, that is we go in expression (84) one step beyond the hydrodynamic order in the low wave number limit. Then we calculate the Beliaev rate from (119) including first order corrections in γ also in the excitation spectrum. Last, we question the accuracy of the Fermi golden rule delta distribution in (119) and we investigate processes of higher order than Beliaev.

7. Let us give the explicit intervals of integration $[\alpha_{\min}, \alpha_{\max}]$ and $[\phi_{\min}, \phi_{\max}]$ imposed by the Heaviside function. We set $\chi = \frac{q_2 q_3 + q(q_3 \sin^2 \alpha - q_2 \cos^2 \alpha)}{q_2 q_3 \sin 2\alpha}$, $A = \frac{q}{2q_3} + \frac{q}{2q_2}$, $C = 1 - \frac{q}{2q_3} + \frac{q}{2q_2}$, $A' = \frac{A}{(1+A^2)^{1/2}}$ and $C' = \frac{C}{(1+C^2)^{1/2}}$. (i) If $\nu > 0$ and $q < q_3$, $2\alpha_{\min} = \arccos A' - \arccos C'$ and $2\alpha_{\max} = \arccos A' + \arccos C'$. (ii) If $\nu > 0$ and $q > q_3$, $\alpha_{\min} = 0$ and $2\alpha_{\max} = \arccos A' + \arccos C'$. (iii) If $\nu < 0$ and $q < q_3$, $\alpha_{\min} = 0$ and $\alpha_{\max} = \pi/2$. (iv) If $\nu < 0$ and $q > q_3$, $2\alpha_{\min} = \arccos C' - \arccos A'$ and $\alpha_{\max} = \pi/2$. In the cases (i) and (ii) ($\nu > 0$), $\phi_{\min} = 0$, $\phi_{\max} = \pi$ if $\chi < -1$ and $\phi_{\max} = \arccos \chi$ otherwise. In the cases (iii) and (iv) ($\nu < 0$), $\phi_{\max} = \pi$, $\phi_{\min} = 0$ if $\chi > 1$ and $\phi_{\min} = \arccos \chi$ otherwise. The integral over ϕ can be calculated analytically. One faces $\int_0^\Phi d\phi \left(\sum_i \frac{b_i}{a_i + \cos \phi}\right)^2$ that can be expressed in terms of the primitive $F(\Phi) = \int_0^\Phi d\phi \frac{1}{a + \cos \phi}$ and of its derivative with respect to a . If $a \in]-1, 1[$, $F(\Phi) = \frac{2}{(1-a^2)^{1/2}} \operatorname{argth} \left[\left(\frac{1-a}{1+a}\right)^{1/2} \tan \frac{\Phi}{2} \right]$. If $|a| > 1$, $F(\Phi) = \frac{2}{(a^2-1)^{1/2}} \operatorname{atan} \left[\frac{a-1}{(a^2-1)^{1/2}} \tan \frac{\Phi}{2} \right]$. If $a = 1$, $F(\Phi) = \tan \frac{\Phi}{2}$.

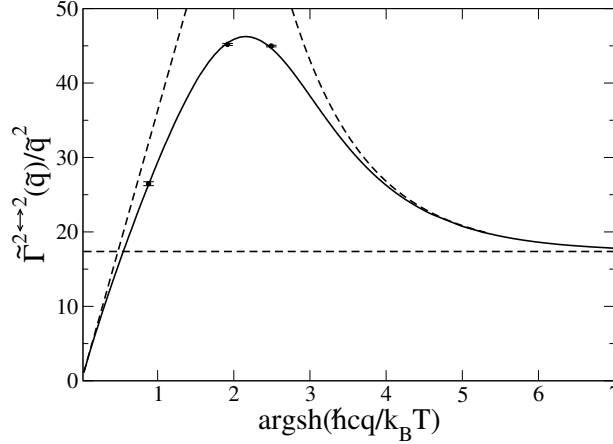


Figure 4: $2 \leftrightarrow 2$ damping rate as a function of the rescaled wave number $\tilde{q} = \hbar c q / k_B T$. In order to make the limiting behaviors more apparent, the rate is divided by \tilde{q}^2 and plotted as a function of $\text{argsh}(\tilde{q}) = \ln(\tilde{q} + \sqrt{1 + \tilde{q}^2})$. The oblique tangent at $\tilde{q} = 0$ corresponds to the low- \tilde{q} limiting behavior (see equation (143)). The horizontal straight line is the $\tilde{q} \rightarrow \infty$ limit of $\tilde{\Gamma}^{2 \leftrightarrow 2} / \tilde{q}^2$ (see equation (144)) and the asymptotic curve at large \tilde{q} is a fit of $\tilde{\Gamma}^{2 \leftrightarrow 2} / \tilde{q}^2$ by an affine function of $1/\tilde{q}$. The black symbols are numerical calculations of the integral (129). In the numerics, we used a dispersion relation of the Bogoliubov form (149) restricted to an interval $[0, q_{\text{max}}]$ over which its first order derivative is positive. The used dispersion relation coincides with (57) up to the third order in q . The discs are obtained by a linear extrapolation of the value of the integral for $\epsilon = 0$, and the error bars halfwidth is given by the difference with a quadratic extrapolation.

5.4.1. Amplitude of a Beliaev process $2 \leftrightarrow 1$ for the unitary gas

In this subsection we calculate the amplitude $\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3)$ including the first correction in γ from the results of Son and Wingate using conformal invariance in an effective field theory [43]. Son and Wingate constructed a Lagrangian that includes the first correction to hydrodynamics, expressed in terms of a phase field ϕ . For details on the formalism, we refer to their work [43], and to reference [44] where it was used to calculate nonzero-temperature corrections to the sound velocity.

In short, we first quadratize the Lagrangian to determine the phonon excitation spectrum $\omega_{\mathbf{q}}$ and the corresponding eigenmode amplitudes. The spectrum is of the cubic form (57) with the curvature parameter $\gamma = -\frac{64}{45c_0} (c_1 + \frac{3}{2}c_2)$, where the dimensionless constant c_0 appears in the leading (hydrodynamic) order of the Lagrangian, and the dimensionless constants c_1 and c_2 appear in the next-to-leading order and parameterize the first corrections beyond hydrodynamics. The constant c_0 is linked to the already measured [21] Bertsch parameter ξ_B relating the chemical potential μ to the Fermi wave number k_F or the total density ρ :

$$\mu = \xi_B \frac{\hbar^2 k_F^2}{2m} \quad \text{and} \quad \rho = \frac{k_F^3}{3\pi^2} \quad (147)$$

The constants c_1 and c_2 , on the contrary, have not been measured yet; they have not been calculated either with good precision from a microscopic theory, only estimates are available, see [42] and references therein. Second, we use the cubic terms in the Lagrangian to obtain the coupling between the phonon modes. This leads to the on-shell Beliaev coupling amplitude to first order beyond hydrodynamics:

$$\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(\mathbf{k}, \mathbf{k}'; \mathbf{q}) = -\frac{\sqrt{2}}{3} (\tilde{\omega}_q \tilde{\omega}_k \tilde{\omega}_{k'})^{1/2} \left[1 - \frac{7\gamma}{32} (\tilde{\omega}_q^2 + \tilde{\omega}_k^2 + \tilde{\omega}_{k'}^2) + o(\tilde{\omega}_q^2) \right] \quad (148)$$

Note that the same linear combination $\gamma = -\frac{64}{45c_0} (c_1 + \frac{3}{2}c_2)$ as in the spectrum appears in this result. Details of the calculation are given in Appendix D.

5.4.2. Phonon damping rate for the $T = 0$ unitary gas

Up to cubic order in q , we can rewrite the excitation spectrum in a Bogoliubov form

$$\tilde{\omega}_q = \tilde{q} \left(1 + \frac{\gamma}{4} \tilde{q}^2 \right)^{1/2} \quad (149)$$

This allows us to recycle the result (A14) in Appendix A of reference [7] to perform the angular integration in (119),

$$\left(\Gamma_q^{\text{Bel}}\right)^{\text{unitary gas}} = \left(\frac{mc^2}{\pi\hbar}\right)\left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \int_0^{\check{q}} d\check{k} |\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(k, k'; q)|^2 \frac{\check{k}}{\check{q}} \frac{\check{\omega}_q - \check{\omega}_k}{\left[1 + \gamma(\check{\omega}_q - \check{\omega}_k)^2\right]^{1/2}} \quad (150)$$

where we acknowledge in the notation the fact that the coupling amplitude (148), where $\check{\omega}_{k'} = \check{\omega}_q - \check{\omega}_k$, depends on the moduli of the wavevectors only. By performing the change of variable $\kappa \equiv \check{k}/\check{q}$, we express $\left[\left(\Gamma_q^{\text{Bel}}\right)^{\text{unitary gas}} \gamma^2/\check{q}\right]$ as an integral over κ between 0 and 1 of a function depending only on κ and on the small parameter $\gamma\check{q}^2$. By expanding this function to the sub-leading order, that is to third order in $\gamma\check{q}^2$, and performing the integral over κ , we obtain the provisional result

$$\left(\Gamma_q^{\text{Bel}}\right)_{\text{prov}}^{\text{unitary gas}} \underset{q \rightarrow 0}{=} \left(\frac{2}{9\pi}\right)\left(\frac{mc^2}{\hbar}\right)\left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \frac{\check{q}^5}{30} \left[1 - \frac{25}{112}\gamma\check{q}^2 + o(\check{q}^2)\right] \quad (151)$$

The sub-leading term in our provisional result (151) is qualitatively different from the one in [13], in particular it predicts a reduction of the damping rate with respect to the hydrodynamic prediction at low wavevectors rather than an increase. The disagreement is unexpected as the calculation of reference [13] is performed with the same methodology as ours and in the same spirit. We think it is due to fact that the dependence of $\mathcal{A}^{2\leftrightarrow 1}$ on γ was finally neglected in [13] while it gives a contribution of the same order as the dependence on γ in the spectrum.

Before accepting the result (151), we should ask ourselves which kind of correction originates from the fact that the considered one-phonon state is unstable, of width $\Gamma_q/2$, and as a consequence, that the free phonon energy is not exactly conserved in the one-to-two phonon decay process, contrarily to the constraint imposed by the Dirac delta function in equation (119). In order to estimate the order of magnitude of this effect, one can replace the Dirac delta function by a Lorentzian of half-width $\Gamma/2$,

$$\delta(\omega_{q'} + \omega_{q-q'} - \omega_q) \rightarrow \frac{\Gamma_q/2}{(\omega_{q'} + \omega_{q-q'} - \omega_q)^2 + \Gamma_q^2/4} \quad (152)$$

where Γ_q can be identified to its leading term in q^5 . One finds then that this effect introduces a correction to Γ_q that is of order q^7 in the limit $q \rightarrow 0$.⁸ To get the exact contribution of this broadening effect to Γ_q , we of course do not use the heuristic prescription (152) but we do a rigorous calculation in Appendix F. In short, using the resolvent of the Hamiltonian, we write at the Beliaev order a self consistent equation for the complex energy $z_q = \hbar\omega_q - i\hbar\Gamma_q/2$ of the \mathbf{q} phonon. If one replaces z_q by its zeroth order approximation $\hbar\omega_q + i\eta$ ($\eta \rightarrow 0^+$) in the implicit part of the equation, one recovers exactly (151). If one performs one self consistency iteration, that is if one replaces $-2\text{Im} z_q/\hbar$ by its usual hydrodynamic approximation (145), one obtains the final result at the Beliaev order:

$$\left(\Gamma_q^{\text{Bel}}\right)^{\text{unitary gas}} = \left(\frac{2}{9\pi}\right)\left(\frac{mc^2}{\hbar}\right)\left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \frac{\check{q}^5}{30} \left[1 - \frac{25}{112}\gamma\check{q}^2 + \frac{4\sqrt{3}\xi_B^{3/2}}{243\gamma}\check{q}^2 + o(\check{q}^2)\right] \quad (153)$$

where ξ_B is the Bertsch parameter (147) of the unitary gas.

In order to conclude, and to make the result (153) rigorous, we have to verify that no other process of higher order than the Beliaev $1 \rightarrow 2$ process, gives a contribution of order \check{q}^7 . A natural candidate is the cascade process represented in figure 5, which combines two $1 \rightarrow 2$ processes induced by the cubic Hamiltonian \hat{H}_3 , giving rise to an effective coupling amplitude $1 \rightarrow 3$, which is of second order with a virtual phonon⁹. We estimate the corresponding change in the complex energy of the phonon \mathbf{q} by treating the effective $1 \rightarrow 3$ coupling to second order in perturbation theory.

8. Using the note 4, we consider the polar integral $J = \int_{-1}^1 du \frac{\eta'/\pi}{\eta'^2 + (u-u_0)^2}$ where $\eta' \approx \Gamma_q/\check{q} \approx \check{q}^4$ and $1 - u_0 = 1 - \cos\theta_0 \approx \check{q}^2$ as in (F.26). Then $J - 1 \approx \check{q}^2$ gives a correction $O(\check{q}^7)$ to Γ_q^{Bel} coming from the finite energy width of the initial state.

9. The attentive reader has probably noticed that the quartic Hamiltonian \hat{H}_4 provides a direct $1 \rightarrow 3$ coupling to first order. The corresponding amplitude is in q^2 . This direct coupling contributes to Γ_q at the order q^9 , that is here negligible even by taking into account the effect of a small denominator in q^3 . For this reason we do not mention it in the main text.

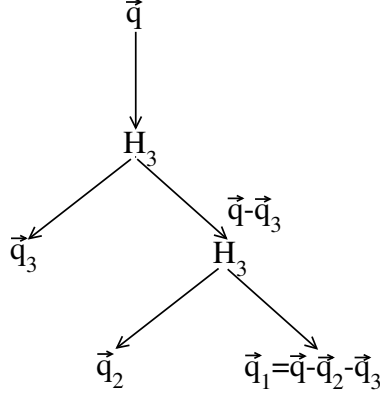


Figure 5: Effective coupling process, of second order in \hat{H}_3 involving the emission of a virtual phonon. As one can deduce from simple power counting (see the text), in the case of a convex dispersion relation and when the phonons \mathbf{q}_1 , \mathbf{q}_2 and \mathbf{q}_3 are emitted within a small angle $O(\hbar q/mc)$ with respect to \mathbf{q} , this process contributes to the order $(\hbar q/mc)^7$ to the $T = 0$ damping rate of a phonon of small wave vector \mathbf{q} . Its contribution has then to be added to the result (153) as we verify in Appendix F by an explicit calculation.

At first sight, the result is $O(\check{q}^9)$ hence negligible. One has to integrate over two independent emitted phonon wave vectors, for example \mathbf{q}_2 and \mathbf{q}_3 , the third one \mathbf{q}_1 being imposed by momentum conservation. As the wave numbers q_i are of order q , this gives a factor q^6 . It comes then the product of four matrix elements of \hat{H}_3 , as the effective $1 \rightarrow 3$ coupling contains two matrix elements and it is treated to the second order, which adds a global factor $(q^{3/2})^4 = q^6$. Finally, there are three energy denominators, one from second order perturbation theory and the other two from the effective $1 \rightarrow 3$ coupling. As the phonon energies are of order $\hbar cq$, this provides a factor q^3 in the denominator. The whole thing is $O(\check{q}^9)$ as announced.

The previous reasoning however neglects the enhancement effect of the small denominators of order q^3 , that occurs when the wave vectors \mathbf{q}_i are emitted forward, with small angles θ_i with respect to \mathbf{q} , which already played a crucial role in section 5.3. In the limiting case where \mathbf{q} and the \mathbf{q}_i are all aligned in the same direction, momentum conservation imposes

$$q = q_1 + q_2 + q_3 \quad (154)$$

so that the energy difference $\hbar\omega_{\mathbf{q}} - (\hbar\omega_{\mathbf{q}_1} + \hbar\omega_{\mathbf{q}_2} + \hbar\omega_{\mathbf{q}_3})$ is not of order q but rather of order q^3 , taking into account the cubic term in the dispersion relation (57). This conclusion extends to all the energy denominators as far as the emission angles are $O(\check{q})$. One can check indeed that $q_1 = |\mathbf{q} - \mathbf{q}_2 - \mathbf{q}_3|$ depends in relative value to second order in the emission angles θ_2 and θ_3 with coefficients of order \check{q}^0 , in the same way as the true dispersion relation deviates in relative value from that of hydrodynamics to second order in \check{q} .

Let us then refine the naive estimation $O(\check{q}^9)$ of the previous paragraph, by considering the integration over \mathbf{q}_2 and \mathbf{q}_3 within cones of angular aperture $O(\check{q})$ around \mathbf{q} . Each cone occupies a solid angle $O(\check{q}^2)$ so that we lose a factor q^4 in the integration over polar angles. On the other hand we gain a factor q^{-2} for each energy denominator, that is a global factor q^{-6} . We then predict a change in the complex energy of order \check{q}^7 , that is the same order as the correction to hydrodynamics appearing in equation (153).

In Appendix F, we then explicitly calculate the contribution to the damping rate $\Gamma_{\mathbf{q}}$ that comes from the effective coupling $1 \rightarrow 3$ treated to second order. We find that its expression, a rather tedious quintuple integral, leads to

$$(\Gamma_{\mathbf{q}}^{1 \rightarrow 3})^{\text{unitary gas}} = \left(\frac{2}{9\pi} \right) \left(\frac{mc^2}{\hbar} \right) \left(\frac{mc}{\hbar\rho^{1/3}} \right)^3 \frac{\check{q}^5}{30} \left[-\frac{2\sqrt{3}\xi_B^{3/2}}{567\gamma} \check{q}^2 + o(\check{q}^2) \right] \quad (155)$$

In the same appendix, we verify that no other process, of arbitrarily high order in \hat{H}_3 , \hat{H}_4 , etc., contributes to $\Gamma_{\mathbf{q}}$ to the order \check{q}^7 , even accounting for the enhancement due to small denominators in q^3 . We then can sum the contributions

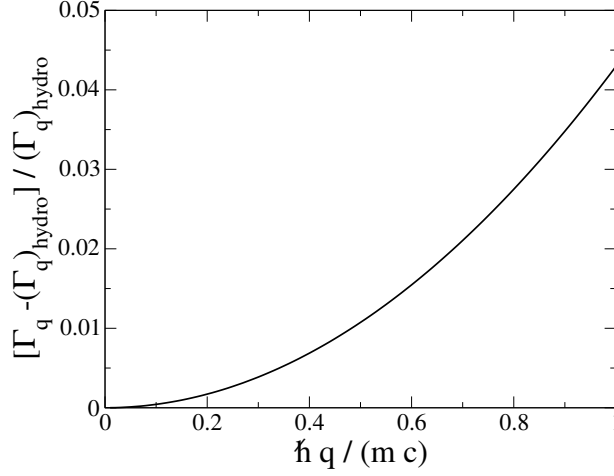


Figure 6: We show in relative value the first correction (156) to the hydrodynamic prediction for the phonon damping rate of the unitary gas at zero temperature. We took for the curvature parameter the RPA value $\gamma \approx 0.083769$ and the Bertsch parameter $\xi_B = 0.376$ experimentally measured [21].

(153) and (155) to obtain

$$(\Gamma_q)^{\text{unitary gas}} = \left(\frac{2}{9\pi}\right) \left(\frac{mc^2}{\hbar}\right) \left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \frac{\check{q}^5}{30} \left[1 - \frac{25}{112}\gamma\check{q}^2 + \frac{22\sqrt{3}\xi_B^{3/2}}{1701\gamma}\check{q}^2 + o(\check{q}^2)\right] \quad (156)$$

an exact expansion, plotted in figure 6, to be counted among the successes of this paper.

6. Conclusion

We presented a complete study of interaction processes between phonons in cold Fermi gases at low temperature, for any zero-range interaction strength between fermions, therefore in both the concave and convex cases for the phonon dispersion relation at low wave number q . We clarified the conditions of validity of a low-energy effective theory such as hydrodynamics by comparing it to a microscopic approach which takes into account the internal degrees of freedom of the pairs. Those effective theories correctly predict the phonon coupling amplitudes only on the energy shell.

One of the main contributions of this study is the microscopic derivation of the $2 \leftrightarrow 2$ coupling amplitude. Indeed, its expression in second order perturbation theory includes nonresonant $2 \leftrightarrow 1$ and $3 \leftrightarrow 0$ processes. Since these processes can be resonant in the quantum hydrodynamic treatment, where the excitation spectrum is linear, Landau and Khalatnikov had to introduce “by hand” a curvature to the dispersion relation to avoid the divergence of the coupling amplitude. At first sight this procedure is risky since, as we just said, quantum hydrodynamics does not predict correctly the coupling amplitudes for nonresonant (off-shell) processes. To provide a microscopic test of this procedure would be particularly cumbersome within our fermionic microscopic approach, so we rather considered a model of weakly interacting bosons with finite range interactions designed to have a concave dispersion relation. We found that although the microscopic expressions of the coupling amplitudes differ from that of hydrodynamics, when one sums up all the nonresonant three-phonon processes to second order and the direct $2 \leftrightarrow 2$ processes to first order, and one restricts on-shell with respect to the effective $2 \leftrightarrow 2$ interaction, the Landau-Khalatnikov prescription and the microscopic result for the effective $2 \leftrightarrow 2$ coupling amplitude agree.

As a second result of this paper, we gave universal formulas for the damping rates of both the $2 \leftrightarrow 1$ and $2 \leftrightarrow 2$ processes at low temperature, universal in the sense that the introduced reduced-rate functions $\check{q} \mapsto \check{\Gamma}(\check{q})$ do not depend on the atomic species, on the interaction strength, or even on the temperature. The most interesting result is the analytic derivation, as a function of q , of the phonon damping rate in the concave case, given to leading order in

temperature by the $2 \leftrightarrow 2$ processes, see (141) and figure 4, that is the subject of a Letter [59]. In the limiting cases $\hbar c q \ll k_B T$ and $\hbar c q \gg k_B T$, our result disagrees with the one of reference [1] and is sub-leading by two orders in $\hbar c q / k_B T$. This is due to the failed assumption in [1] that some interaction diagrams are negligible while in reality they destructively interfere with the supposedly leading diagram.

Finally we also calculated, at zero temperature and in the unitary limit, the first correction $\propto q^7$ to the hydrodynamic prediction $\propto q^5$ for the single phonon decay rate. Our calculation allows to refine the prediction of [13] by (i) the actual inclusion of the beyond-hydrodynamics expression of the coupling amplitude [43], (ii) the inclusion of a finite width $\hbar \Gamma_{\mathbf{q}}$ in energy conservation in the Fermi golden rule, that is of a purely imaginary term of order q^5 in the energy denominator of perturbation theory and (iii) the inclusion of other processes, of higher order than the Beliaev process, in particular of the effective coupling $1 \rightarrow 3$ treated to second order.

All our predictions can be tested in state-of-art experiments on cold fermionic gases. In particular, a discussion of the observability of the Landau-Khalatnikov $2 \rightarrow 2$ damping, using cold atoms trapped in a flat-bottom potential, is done in [59], with a proposed experimental protocol for the selective excitation of a phonon standing wave in the box in a thermodynamic-limit regime, that is at a wave number q much larger than the inverse of the system size but still in the linear part of the excitation spectrum.

Appendix A. Implicit equation on the collective excitation spectrum

We explain here how to obtain from the linear system (33,34) the implicit equation (39) on the collective mode angular frequency $\omega_{\mathbf{q}}$ at wave vector \mathbf{q} , in the zero lattice spacing limit $l \rightarrow 0$. We introduce the collective amplitudes

$$Y_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} W_{\mathbf{kq}}^+ y_{\mathbf{k}}^{\mathbf{q}} \quad S_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} W_{\mathbf{kq}}^- s_{\mathbf{k}}^{\mathbf{q}} \quad (\text{A.1})$$

$$y_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} w_{\mathbf{kq}}^- y_{\mathbf{k}}^{\mathbf{q}} \quad s_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} w_{\mathbf{kq}}^+ s_{\mathbf{k}}^{\mathbf{q}} \quad (\text{A.2})$$

and we solve the 2×2 linear system (33,34) to express the unknowns $y_{\mathbf{k}}^{\mathbf{q}}$ and $s_{\mathbf{k}}^{\mathbf{q}}$ in terms of the collective amplitudes. Next, we replace the result in (A.1,A.2) and come up with the homogeneous system

$$\begin{pmatrix} \Sigma_{W^+ W^+}^{\epsilon} - 1 & \Sigma_{W^+ W^-}^{\omega} & -\Sigma_{W^+ w^-}^{\epsilon} & \Sigma_{W^+ w^+}^{\omega} \\ \Sigma_{W^- W^+}^{\omega} & \Sigma_{W^- W^-}^{\epsilon} - 1 & -\Sigma_{W^- w^-}^{\omega} & \Sigma_{W^- w^+}^{\epsilon} \\ \Sigma_{w^- W^+}^{\epsilon} & \Sigma_{w^- W^-}^{\omega} & -\Sigma_{w^- w^-}^{\epsilon} - 1 & \Sigma_{w^- w^+}^{\omega} \\ \Sigma_{w^+ W^+}^{\omega} & \Sigma_{w^+ W^-}^{\epsilon} & -\Sigma_{w^+ w^-}^{\omega} & \Sigma_{w^+ w^+}^{\epsilon} - 1 \end{pmatrix} \begin{pmatrix} Y_{\mathbf{q}} \\ S_{\mathbf{q}} \\ y_{\mathbf{q}} \\ s_{\mathbf{q}} \end{pmatrix} = 0 \quad (\text{A.3})$$

where we introduced the notations

$$\Sigma_{ab}^{\epsilon} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} \frac{\epsilon_{\mathbf{kq}} a_{\mathbf{kq}} b_{\mathbf{kq}}}{(\hbar \omega_{\mathbf{q}})^2 - (\epsilon_{\mathbf{kq}})^2} \quad (\text{A.4})$$

$$\Sigma_{ab}^{\omega} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} \frac{\hbar \omega_{\mathbf{q}} a_{\mathbf{kq}} b_{\mathbf{kq}}}{(\hbar \omega_{\mathbf{q}})^2 - (\epsilon_{\mathbf{kq}})^2} \quad (\text{A.5})$$

with a and b that can be any of W^+ , W^- , w^+ , w^- . The system (A.3) simplifies in the continuous limit $l \rightarrow 0$. Since $g_0 \rightarrow 0$ and $w_{\mathbf{kq}}^{\pm} = O(1/k^2)$, all the Σ tend to 0 in the third and fourth lines, and therefore we must have

$$y_{\mathbf{q}} = s_{\mathbf{q}} = 0 \quad (\text{A.6})$$

Next we divide the first two lines of (A.3) by g_0 . The gap equation (11) ensures that all the divided matrix elements have a finite nonzero limit. The system therefore reduces to its 2×2 upper left block, which, in the thermodynamic limit, we write as:

$$\begin{pmatrix} I_{++}(\omega_{\mathbf{q}}, q) & \hbar \omega_{\mathbf{q}} I_{+-}(\omega_{\mathbf{q}}, q) \\ \hbar \omega_{\mathbf{q}} I_{+-}(\omega_{\mathbf{q}}, q) & I_{--}(\omega_{\mathbf{q}}, q) \end{pmatrix} \begin{pmatrix} Y_{\mathbf{q}} \\ S_{\mathbf{q}} \end{pmatrix} = 0 \quad (\text{A.7})$$

where the integrals I depend on the eigenfrequency $\omega_{\mathbf{q}}$ and on the wavevector q , see equations (40,41,42). The sought implicit equation on the eigenfrequency (39) is the condition that the determinant of the 2×2 matrix in equation (A.7) is zero.

Appendix B. x and y rational fractions in the three-phonon process amplitudes

The rational fractions of the variables x and y that appear in the expression (81) of the microscopic phonon coupling amplitude are given by

$$J(x, y) = \frac{3xy}{2xy + 2} \quad (\text{B.1})$$

$$A(x, y) = \frac{\sum_{i=0}^3 A_i(y)x^i}{360(x^3 + x)(y^2 + 1)^3} \quad (\text{B.2})$$

$$B(x, y) = \frac{\sum_{i=0}^3 B_i(y)x^i}{18(x^3 + x)(y^2 + 1)^2} \quad (\text{B.3})$$

$$C(x, y) = \frac{y(xy + 1)}{6(x^3 + x)(y^2 + 1)^2} \left[x^2(5y^2 + 3) - xy(y^2 - 3) + 2(y^2 + 2) \right]$$

with

$$A_0(y) = -20y(5y^4 + 5y^2 + 2) \quad (\text{B.4})$$

$$A_1(y) = 2(10y^6 + 109y^4 + 63y^2 + 24) \quad (\text{B.5})$$

$$A_2(y) = -y(53y^4 + 186y^2 + 13) \quad (\text{B.6})$$

$$A_3(y) = 25y^6 + 116y^4 + 167y^2 + 36 \quad (\text{B.7})$$

$$B_0(y) = -2y(3y^2 + 2) \quad (\text{B.8})$$

$$B_1(y) = 3y^4 + 17y^2 + 8 \quad (\text{B.9})$$

$$B_2(y) = -2y(y^2 + 4) \quad (\text{B.10})$$

$$B_3(y) = 4y^4 + 11y^2 + 9 \quad (\text{B.11})$$

Their values in the BEC limit $x \rightarrow 0$, $y \sim -4/x$ are useful to connect the result (81) to its equivalent in the Bogoliubov theory:

$$A(x, y) \xrightarrow[\substack{x \rightarrow 0 \\ y \sim -4/x}]{\frac{1}{8}} \quad B(x, y) \xrightarrow[\substack{x \rightarrow 0 \\ y \sim -4/x}]{\frac{1}{4}} \quad (\text{B.12})$$

$$J(x, y) \xrightarrow[\substack{x \rightarrow 0 \\ y \sim -4/x}]{2} \quad C(x, y) \xrightarrow[\substack{x \rightarrow 0 \\ y \sim -4/x}]{\frac{3}{4}}$$

For resonant processes, these rational fractions add up to give the thermodynamic coefficient

$$1 + \frac{\rho}{3} \frac{d^2\mu}{d\rho^2} \left(\frac{d\mu}{d\rho} \right)^{-1} = \frac{\sum_{i=0}^3 D_i(y)x^i}{9(x^3 + x)(y^2 + 1)^2} \quad (\text{B.13})$$

with

$$D_0(y) = -4y^3 \quad (\text{B.14})$$

$$D_1(y) = 4(y^4 + 6y^2 + 2) \quad (\text{B.15})$$

$$D_2(y) = -y(y^4 - 6y^2 + 5) \quad (\text{B.16})$$

$$D_3(y) = 9y^4 + 14y^2 + 9 \quad (\text{B.17})$$

Appendix C. Low wave number out-of-the-energy-shell discrepancy between the microscopic and quantum hydrodynamic Beliaev-Landau coupling amplitudes

In this appendix, as announced in the penultimate paragraph of section 4.1, we clarify the physical origin of the discrepancy of the Beliaev-Landau 2 phonons \leftrightarrow 1 phonon coupling amplitudes of the microscopic model and of quantum hydrodynamics in the low wave number limit out of the energy shell: the former indeed diverges whereas the latter tends to zero. To avoid tedious calculations in the fermionic microscopic model, we rather use a bosonic microscopic model, the weakly interacting spatially homogeneous Bose gas, of grand canonical Hamiltonian \hat{H} at chemical potential μ , with

$$\hat{H} = l^3 \sum_{\mathbf{r}} \hat{\psi}^\dagger \left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} - \mu \right) \hat{\psi} + \frac{g_0}{2} \sum_{\mathbf{r}} l^3 \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \quad (\text{C.1})$$

We then easily make the link between this microscopic theory and quantum hydrodynamics through the modulus-phase representation of the bosonic field operator [60],

$$\hat{\psi}(\mathbf{r}) = e^{i\hat{\theta}(\mathbf{r})} \hat{\rho}^{1/2}(\mathbf{r}) \quad (\text{C.2})$$

where the phase field $\hat{\theta}(\mathbf{r})$ and the density field $\hat{\rho}(\mathbf{r})$ are canonically conjugated.

In the Bogoliubov $U(1)$ -symmetry breaking formalism, one splits the bosonic field as $\hat{\psi}(\mathbf{r}) = \psi_0 + \delta\hat{\psi}(\mathbf{r})$, where $\psi_0 > 0$ is the classical field energy minimiser, and one expands the Hamiltonian in powers of $\delta\hat{\psi}$, $\hat{H} = H_0 + \hat{H}_2 + \hat{H}_3 + \dots$. The field $\delta\hat{\psi}(\mathbf{r})$ is then expanded over the normal modes of \hat{H}_2 (Bogoliubov expansion):

$$\delta\hat{\psi}(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq \mathbf{0}} U_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \hat{b}_{\mathbf{q}}^{\delta\psi} + V_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}} (\hat{b}_{\mathbf{q}}^{\delta\psi})^\dagger \quad (\text{C.3})$$

where we omitted for simplicity the contribution of the anomalous modes associated to symmetry breaking and we explicitly indicated that the quasi-particle creation and annihilation operators are relative to a $\delta\hat{\psi}$ -expansion of \hat{H} . Insertion of the expansion (C.3) in \hat{H}_3 leads to the reduced Beliaev-Landau coupling amplitude in the Cartesian $\delta\hat{\psi}$ point of view:

$$\mathcal{A}_{\delta\psi}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = \frac{3}{4} s_1 s_2 s_3 + \frac{1}{4} (s_1 d_2 d_3 + s_2 d_1 d_3 - s_3 d_1 d_2) \quad (\text{C.4})$$

with $s_i = s_{\mathbf{q}_i}$, $s_{\mathbf{q}} = U_{\mathbf{q}} + V_{\mathbf{q}}$, $d_i = d_{\mathbf{q}_i}$ and $d_{\mathbf{q}} = U_{\mathbf{q}} - V_{\mathbf{q}}$.

In the modulus-phase point of view, adapting reference [60] to a symmetry breaking approach, we expand $\hat{\psi}(\mathbf{r})$ in powers of the density fluctuation $\delta\hat{\rho}(\mathbf{r}) = \hat{\rho}(\mathbf{r}) - \psi_0^2$ and of the phase $\hat{\theta}(\mathbf{r})$ so that $\delta\hat{\psi}(\mathbf{r}) = \delta\hat{\psi}_1(\mathbf{r}) + \delta\hat{\psi}_2(\mathbf{r}) + \dots$, with

$$\delta\hat{\psi}_1(\mathbf{r}) = \frac{\delta\hat{\rho}(\mathbf{r})}{2\psi_0} + i\psi_0 \hat{\theta}(\mathbf{r}) \quad (\text{C.5})$$

$$\delta\hat{\psi}_2(\mathbf{r}) = -\frac{1}{2} \psi_0 \hat{\theta}^2(\mathbf{r}) - \frac{\delta\hat{\rho}^2(\mathbf{r})}{8\psi_0^3} + i \frac{\hat{\theta}(\mathbf{r}) \delta\hat{\rho}(\mathbf{r})}{2\psi_0} \quad (\text{C.6})$$

To quadratic order in $\delta\hat{\rho}$ and $\hat{\theta}$, one replaces $\delta\hat{\psi}$ in \hat{H}_2 with $\delta\hat{\psi}_1$; the normal mode expansion of the corresponding quadratic Hamiltonian in $(\delta\hat{\rho}, \hat{\theta})$ gives (again neglecting for simplicity the symmetry-breaking related anomalous modes):

$$\delta\hat{\psi}_1(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq \mathbf{0}} U_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \hat{b}_{\mathbf{q}}^{\rho\theta} + V_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}} (\hat{b}_{\mathbf{q}}^{\rho\theta})^\dagger \quad (\text{C.7})$$

with the same amplitudes ($U_{\mathbf{q}}, V_{\mathbf{q}}$) as in equation (C.3) but with operator-valued coefficients $\hat{b}_{\mathbf{q}}^{\rho\theta}, (\hat{b}_{\mathbf{q}}^{\rho\theta})^\dagger$ that coincide with $\hat{b}_{\mathbf{q}}^{\delta\psi}, (\hat{b}_{\mathbf{q}}^{\delta\psi})^\dagger$, as $\delta\hat{\psi}_1$ and $\delta\hat{\psi}$ do, only to first order in the fluctuations. To cubic order in $\delta\hat{\rho}$ and $\hat{\theta}$, two different contributions are obtained: (i) the obvious one coming from the replacement of $\delta\hat{\psi}$ with $\delta\hat{\psi}_1$ in \hat{H}_3 , which leads to the Beliaev-Landau coupling amplitude $\mathcal{A}_{\delta\psi}^{2\leftrightarrow 1}$, this time among the $\hat{b}_{\mathbf{q}}^{\rho\theta}$, and (ii) the more indirect one, coming from the replacement of $\delta\hat{\psi}$ with $\delta\hat{\psi}_1 + \delta\hat{\psi}_2$ in \hat{H}_2 and extraction of the corresponding cubic contributions of the type $\delta\hat{\psi}_2^\dagger \delta\hat{\psi}_1$

and hermitian conjugate, which brings an extra term $\mathcal{A}_{\text{via } H_2}^{2\leftrightarrow 1}$ to the Beliaev-Landau coupling amplitude. At the end, the Beliaev-Landau coupling amplitude in the $(\hat{\rho}, \hat{\theta})$ point of view reads:

$$\mathcal{A}_{\rho\theta}^{2\leftrightarrow 1} = \mathcal{A}_{\delta\psi}^{2\leftrightarrow 1} + \mathcal{A}_{\text{via } H_2}^{2\leftrightarrow 1} \quad (\text{C.8})$$

with

$$\begin{aligned} \mathcal{A}_{\text{via } H_2}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) &= \frac{\check{\omega}_3 d_3}{8}(d_1 d_2 - s_1 s_2) + \frac{\check{\omega}_3 s_3}{8}(s_1 d_2 + s_2 d_1) \\ &\quad - \frac{\check{\omega}_2 d_2}{8}(d_1 d_3 + s_1 s_3) - \frac{\check{\omega}_1 d_1}{8}(d_2 d_3 + s_2 s_3) + \frac{d_3}{8}(s_1 s_2 \check{\omega}_2 + s_2 s_1 \check{\omega}_1) - \frac{s_3}{8}(d_2 s_1 \check{\omega}_1 + d_1 s_2 \check{\omega}_2) \end{aligned} \quad (\text{C.9})$$

with $\check{\omega}_i = \hbar\omega_{\mathbf{q}_i}/mc^2$, knowing that $mc^2 = \mu$ at this order. From the Bogoliubov expressions $\omega_i = [\check{E}_i(\check{E}_i + 2)]^{1/2}$ and $s_i = 1/d_i = [\check{E}_i/(\check{E}_i + 2)]^{1/4}$, one has $\check{\omega}_i d_i = (\check{E}_i + 2)s_i$ and $\check{\epsilon}_i s_i = \check{E}_i d_i$, where $\check{E}_i = E_{\mathbf{q}_i}/mc^2$ and $E_{\mathbf{q}} = \hbar^2 q^2/2m$. Using as well the conservation of momentum $\mathbf{q}_3 = \mathbf{q}_1 + \mathbf{q}_2$ and its squares $q_3^2 = (\mathbf{q}_1 + \mathbf{q}_2)^2$, $q_2^2 = (\mathbf{q}_3 - \mathbf{q}_1)^2$ and $q_1^2 = (\mathbf{q}_3 - \mathbf{q}_2)^2$, we obtain

$$\mathcal{A}_{\rho\theta}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = \frac{1}{8}(s_3 d_1 d_2 \check{\mathbf{q}}_1 \cdot \check{\mathbf{q}}_2 + s_2 d_1 d_3 \check{\mathbf{q}}_1 \cdot \check{\mathbf{q}}_3 + s_1 d_2 d_3 \check{\mathbf{q}}_2 \cdot \check{\mathbf{q}}_3) - \frac{1}{16}s_1 s_2 s_3(\check{q}_1^2 + \check{q}_2^2 + \check{q}_3^2) \quad (\text{C.10})$$

with $\check{\mathbf{q}}_i = \hbar\mathbf{q}_i/mc$.

We can now compare the Beliaev-Landau coupling amplitudes $\mathcal{A}_{\delta\psi}^{2\leftrightarrow 1}$ and $\mathcal{A}_{\rho\theta}^{2\leftrightarrow 1}$ among themselves and to the quantum hydrodynamic coupling amplitude. With the expressions of s_i and d_i given above, it is found from equation (C.4) that $\mathcal{A}_{\delta\psi}^{2\leftrightarrow 1}$ diverges when all the q_i tend to zero (out of the energy shell). On the contrary, even out of the energy shell, $\mathcal{A}_{\rho\theta}^{2\leftrightarrow 1}$ becomes equivalent to the quantum hydrodynamic result (94) (here $\Lambda_F = 0$) that tends to zero, but it does not coincide with it (it differs by relative corrections $O(q_i^2)$) due to the difference between the microscopic Hamiltonian (C.1) and the quantum hydrodynamic Hamiltonian.

As a final check, one can also directly insert the expansion (C.7) (turned into an expansion of $\delta\hat{\rho}$ and $\hat{\theta}$) in the terms of the Hamiltonian (40) of reference [60] that are cubic in $\delta\hat{\rho}$ and $\hat{\theta}$. One then directly recovers the Beliaev-Landau coupling amplitude (C.10) among the $\hat{b}_{\mathbf{q}}^{\rho\theta}$. Furthermore, if one neglects everywhere in this Hamiltonian (40) the kinetic energy stored in the density-variation $\delta\hat{\rho}$ of the field, at the origin of the so-called quantum pressure term, one recovers exactly the quantum hydrodynamic Beliaev-Landau coupling (94) (with here $\Lambda_F = 0$).

Appendix D. Beliaev coupling beyond hydrodynamics

We give here details of the calculation allowing one to obtain, for the unitary Fermi gas, the first beyond-hydrodynamic correction to the Beliaev phonon coupling amplitude from the Son and Wingate Lagrangian [43], see equation (148).

For convenience we move to a Hamiltonian formalism, introducing the field Π that is canonically conjugate to the phase field ϕ and that represents (up to a sign) density fluctuations, to obtain (here $\hbar = m = 1$)

$$\mathcal{H}_2^{(0)} = \frac{2\mu^{-1/2}}{15c_0}\Pi^2 + \frac{5}{2}c_0\mu^{3/2}\frac{1}{2}(\mathbf{grad}\phi)^2 \quad (\text{D.1})$$

$$\mathcal{H}_3^{(0)} = \frac{4\mu^{-2}}{3(15c_0)^2}\Pi^3 - \frac{1}{2}\Pi(\mathbf{grad}\phi)^2 \quad (\text{D.2})$$

$$\mathcal{H}_2^{(2)} = -\frac{16c_1\mu^{-3/2}}{(15c_0)^2}(\mathbf{grad}\Pi)^2 - c_2\mu^{1/2}(\Delta\phi)^2 \quad (\text{D.3})$$

$$\mathcal{H}_3^{(2)} = -\frac{96c_1\mu^{-3}}{(15c_0)^3}\Pi(\mathbf{grad}\Pi)^2 + \frac{2c_2\mu^{-1}}{15c_0}\Pi(\Delta\phi)^2 \quad (\text{D.4})$$

where the index 2 or 3 refers to the expansion order of the Hamiltonian in Π and ϕ while the exponent (0) or (2) refers to the expansion order in powers of spatial gradients, the zeroth order being the standard hydrodynamics. Note that

Π and $\mathbf{grad} \phi$ are of the same order. At the hydrodynamic level, the Hamiltonian depends on a single constant c_0 that must be determined from a microscopic theory. It is linked as follows to the Bertsch parameter (147):

$$c_0 = \frac{2^{5/2}}{15\pi^2 \xi_B^{3/2}} \quad (\text{D.5})$$

The first correction to hydrodynamics involves two other dimensionless constants c_1 and c_2 . By following the procedure already used in this paper, we first use the quadratized Hamiltonian $\mathcal{H}_2^{(0)} + \mathcal{H}_2^{(2)}$ to determine the excitation spectrum as in [43]:

$$\omega_{\mathbf{q}} = \left(\frac{2\mu}{3}\right)^{1/2} q \left[1 - \pi^2 (2\xi_B)^{1/2} \left(c_1 + \frac{3}{2}c_2 \right) \left(\frac{q}{k_F} \right)^2 + o(q^2) \right] \equiv cq \left[1 + \frac{\gamma}{8} \left(\frac{\hbar q}{mc} \right)^2 + o(q^2) \right] \quad (\text{D.6})$$

and the modal expansion of the fields in the quantization volume L^3 :

$$\Pi(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq 0} \Pi_{\mathbf{q}} (b_{\mathbf{q}} + b_{-\mathbf{q}}^*) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (\text{D.7})$$

$$\phi(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq 0} \phi_{\mathbf{q}} (b_{\mathbf{q}} - b_{-\mathbf{q}}^*) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (\text{D.8})$$

with the amplitudes

$$\Pi_{\mathbf{q}} = \frac{1}{\sqrt{2}} \left(\frac{A_{\mathbf{q}}}{B_{\mathbf{q}}} \right)^{1/4} \quad (\text{D.9})$$

$$\phi_{\mathbf{q}} = \frac{i}{\sqrt{2}} \left(\frac{A_{\mathbf{q}}}{B_{\mathbf{q}}} \right)^{-1/4} \quad (\text{D.10})$$

$$\frac{A_{\mathbf{q}}}{B_{\mathbf{q}}} = \frac{75}{8} (c_0 \mu q)^2 \left[1 + \frac{8\mu^{-1}}{15c_0} \left(c_1 - \frac{3}{2}c_2 \right) q^2 + o(q^2) \right] \quad (\text{D.11})$$

Indeed $\begin{pmatrix} \Pi_{\mathbf{q}} \\ \phi_{\mathbf{q}} \end{pmatrix}$ is an eigenvector of eigenvalue $-i\omega_{\mathbf{q}}$ of the matrix $\begin{pmatrix} 0 & -A_{\mathbf{q}} \\ B_{\mathbf{q}} & 0 \end{pmatrix}$ that appears in the Hamiltonian equations of motion. Note that different linear combinations of the constants c_1 and c_2 appear in the spectrum (D.6) and in the modal amplitudes (D.11). By inserting the modal decomposition in the cubic Hamiltonian $\mathcal{H}_3^{(0)} + \mathcal{H}_3^{(2)}$, and isolating the Beliaev terms $2 \leftrightarrow 1$ as we did in (93), we obtain the on-shell Beliaev coupling amplitude to first order beyond hydrodynamics as given in (148).

Appendix E. Three-phonon and four-phonon processes in a bosonic model

Here we describe and we implement the microscopic bosonic model used in section 4.2.4 to test the $2 \leftrightarrow 2$ effective phonon coupling predicted by quantum hydrodynamics. We consider spinless bosons on a cubic lattice of lattice constant l , with a large enough interaction range b so that their excitation spectrum is concave at low wave number q . The Hamiltonian of the lattice model reads

$$\hat{H}_B = l^3 \sum_{\mathbf{r}} \hat{\psi}^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m_B} \Delta_{\mathbf{r}} \right) \hat{\psi}(\mathbf{r}) + \frac{l^6}{2} \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) \quad (\text{E.1})$$

with the interaction potential

$$V(\mathbf{r}) = V_0 e^{-r^2/2b^2} \quad (\text{E.2})$$

of Fourier transform

$$\tilde{V}(\mathbf{q}) = \tilde{V}_0 e^{-q^2 b^2/2} \quad \text{with} \quad \tilde{V}_0 = (2\pi)^{3/2} b^3 V_0 \quad (\text{E.3})$$

The Bose gas is in the weakly interacting regime $(\rho_B a_B^3)^{1/2} \ll 1$, where $\rho_B = N_B/L^3$ is the density of the bosons and the s -wave scattering length a_B is given by $4\pi\hbar^2 a_B/m_B = \tilde{V}_0$ in the Born approximation. Following the Bogoliubov theory [61] in its $U(1)$ symmetry preserving version [62, 50, 63, 7], we split the bosonic field operator as

$$\hat{\psi}(\mathbf{r}) = e^{i\hat{\theta}_0} \left[\hat{n}_0^{1/2} \phi_0(\mathbf{r}) + \hat{\Lambda}(\mathbf{r}) \right] \quad (\text{E.4})$$

where $\hat{\theta}_0$ is the condensate phase operator, \hat{n}_0 is the number of bosons in the condensate mode $\phi_0(\mathbf{r}) = 1/L^{3/2}$ and the non-condensed field operator $\hat{\Lambda}(\mathbf{r})$, orthogonal to the condensate mode, conserves the number of particles. Within the subspace with fixed total number of bosons N_B , we eliminate \hat{n}_0 through the relation

$$\hat{n}_0 = N_B - l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}) \quad (\text{E.5})$$

To describe the $2 \leftrightarrow 2$ processes, we expand the Hamiltonian in powers of $\hat{\Lambda}$ up to order 4:

$$\hat{H}_B = \hat{H}_{B0} + \hat{H}_{B2} + \hat{H}_{B3} + \hat{H}_{B4} + \dots \quad (\text{E.6})$$

We obtain¹⁰

$$\hat{H}_{B0} = \tilde{V}_0 \frac{N_B^2}{2L^3} \quad (\text{E.7})$$

$$\hat{H}_{B2} = l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m_B} \Delta_{\mathbf{r}} \right) \hat{\Lambda}(\mathbf{r}) + \rho_B l^6 \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \left(\hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') + \frac{1}{2} \left[\hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}^\dagger(\mathbf{r}') + \hat{\Lambda}(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') \right] \right) \quad (\text{E.8})$$

$$\hat{H}_{B3} = \frac{\rho_B^{1/2}}{2} l^6 \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \left(\left[\hat{\Lambda}^\dagger(\mathbf{r}) + \hat{\Lambda}^\dagger(\mathbf{r}') \right] \hat{\Lambda}(\mathbf{r}') \hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}^\dagger(\mathbf{r}') \hat{\Lambda}^\dagger(\mathbf{r}) \left[\hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}(\mathbf{r}') \right] \right) \quad (\text{E.9})$$

$$\begin{aligned} \hat{H}_{B4} = & \frac{l^6}{2} \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}^\dagger(\mathbf{r}') \hat{\Lambda}(\mathbf{r}') \hat{\Lambda}(\mathbf{r}) - \frac{\tilde{V}_0}{2L^3} \left(l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}) \right)^2 \\ & - \frac{1}{L^3} \left(l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}) \right) \left(l^6 \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \left[\hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') + \frac{1}{2} \left[\hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}^\dagger(\mathbf{r}') + \hat{\Lambda}(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') \right] \right] \right) \end{aligned} \quad (\text{E.10})$$

0th order. From \hat{H}_{B0} we get the equation of state to leading order:

$$\mu_B = \rho_B \tilde{V}_0 \equiv \frac{\hbar^2}{2m\xi^2} \quad (\text{E.11})$$

where we introduced the healing length ξ and the chemical potential μ_B of the Bose gas. We then apply relation (58) to obtain the sound velocity

$$m_B c_B^2 = \mu_B \quad (\text{E.12})$$

2nd order. \hat{H}_{B2} can be diagonalised by a Bogoliubov transformation:

$$\hat{\Lambda}(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq 0} \left(U_{\mathbf{q}}^B \hat{b}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} + V_{\mathbf{q}}^B \hat{b}_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{r}} \right) \quad (\text{E.13})$$

where the $\hat{b}_{\mathbf{q}}$ are annihilation operators of bosonic quasiparticles and the amplitudes $U_{\mathbf{q}}^B$ and $V_{\mathbf{q}}^B$ are given by

$$U_{\mathbf{q}}^B + V_{\mathbf{q}}^B \equiv S_{\mathbf{q}} = \left(\frac{\frac{\hbar^2 q^2}{2m_B}}{\frac{\hbar^2 q^2}{2m_B} + 2\rho_B \tilde{V}(\mathbf{q})} \right)^{1/4} \quad (\text{E.14})$$

$$U_{\mathbf{q}}^B - V_{\mathbf{q}}^B \equiv D_{\mathbf{q}} = \frac{1}{S_{\mathbf{q}}} \quad (\text{E.15})$$

10. We are here in the large N_B limit and we neglect 1 as compared to \hat{n}_0 and to N_B . In the equations (E.8) and (E.10), there was a cancellation of the Hartree contribution with a chemical potential-type contribution originating from the expansion of the condensate interaction energy $\tilde{V}_0 \hat{n}_0^2 / 2L^3$ in powers of the number of non-condensed particles.

We have introduced the angular eigenfrequencies of the Bogoliubov quasiparticles

$$\hbar\omega_{\mathbf{q}}^{\text{B}} = \left[\frac{\hbar^2 q^2}{2m_{\text{B}}} \left(\frac{\hbar^2 q^2}{2m_{\text{B}}} + 2\rho_{\text{B}} \tilde{V}(\mathbf{q}) \right) \right]^{1/2} \quad (\text{E.16})$$

The dispersion relation $q \mapsto \omega_{\mathbf{q}}$ is then concave in the vicinity of $q = 0$ under the condition, assumed to be satisfied in what follows,

$$b > \xi \quad (\text{E.17})$$

3rd order. We insert the modal expansion (E.13) in the cubic Hamiltonian (E.9) that we write in the form (93) with the constants m, c and ρ replaced with $m_{\text{B}}, c_{\text{B}}$ and ρ_{B} respectively, and with coupling amplitudes now given by

$$\mathcal{A}_{\text{B}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = \frac{1}{4\tilde{V}_0} \left[\tilde{V}(\mathbf{q}_3) S_3 (S_1 S_2 - D_1 D_2) + \tilde{V}(\mathbf{q}_1) S_1 (S_2 S_3 + D_2 D_3) + \tilde{V}(\mathbf{q}_2) S_2 (S_1 S_3 + D_1 D_3) \right] \quad (\text{E.18})$$

$$\mathcal{A}_{\text{B}}^{3 \leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{1}{12\tilde{V}_0} \left[\tilde{V}(\mathbf{q}_1) S_1 (S_2 S_3 - D_2 D_3) + \tilde{V}(\mathbf{q}_2) S_2 (S_1 S_3 - D_1 D_3) + \tilde{V}(\mathbf{q}_3) S_3 (S_1 S_2 - D_1 D_2) \right] \quad (\text{E.19})$$

with the notations $S_i \equiv S_{\mathbf{q}_i}$ and $D_i \equiv D_{\mathbf{q}_i}$.

4th order. We insert the modal expansion (E.13) in the quartic Hamiltonian (E.10) that we write in the form (108) with $(m, c, \rho) \rightarrow (m_{\text{B}}, c_{\text{B}}, \rho_{\text{B}})$ and a direct $2 \leftrightarrow 2$ coupling amplitude given by

$$\begin{aligned} \mathcal{A}_{\text{B}}^{2 \leftrightarrow 2, \text{dir}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) = & \frac{1}{32\tilde{V}_0} \left[\left(\tilde{V}(\mathbf{q}_1 + \mathbf{q}_2) + \tilde{V}(\mathbf{q}_3 + \mathbf{q}_4) \right) (S_1 S_2 - D_1 D_2) (S_3 S_4 - D_3 D_4) + \left(\tilde{V}(\mathbf{q}_3 - \mathbf{q}_1) + \tilde{V}(\mathbf{q}_2 - \mathbf{q}_4) \right) \right. \\ & \left. \times (S_1 S_3 + D_1 D_3) (S_2 S_4 + D_2 D_4) + \left(\tilde{V}(\mathbf{q}_4 - \mathbf{q}_1) + \tilde{V}(\mathbf{q}_2 - \mathbf{q}_3) \right) (S_1 S_4 + D_1 D_4) (S_2 S_3 + D_2 D_3) \right] \quad (\text{E.20}) \end{aligned}$$

We considered here the general case where the \mathbf{q}_i and their opposite are two-by-two distinct, in which case only the first term of $\hat{H}_{\text{B}4}$ contributes. The other terms of $\hat{H}_{\text{B}4}$, that originate from the expansion (E.5) of \hat{n}_0 in powers of the number of non-condensed particles, contribute to the equation of state beyond Bogoliubov theory [60].

Effective coupling amplitude. We obtain the on-shell effective coupling amplitude of the bosonic model $\mathcal{A}_{\text{B,OnS}}^{2 \leftrightarrow 2, \text{eff}}$ from the amplitudes (E.18, E.19, E.20) as prescribed by the equation (105). We then expand it in the limit of small wave vectors and express the result in terms of the angular frequencies $\omega_{\mathbf{q}_i}^{\text{B}}, i = 1, 2, 3, 4$ and $\omega_{\mathbf{q}_1 + \mathbf{q}_2}^{\text{B}}, \omega_{\mathbf{q}_1 - \mathbf{q}_3}^{\text{B}}, \omega_{\mathbf{q}_1 - \mathbf{q}_4}^{\text{B}}$. We then recover the hydrodynamic prediction corrected by the Landau and Khalatnikov prescription, as explained in section 4.2.4.

Appendix F. Contribution of the Beliaev and higher order processes to the $T = 0$ decay of the single phonon state

Appendix F.1. Presentation of the problem and link with the resolvent of the Hamiltonian

We consider the damping rate $\Gamma_{\mathbf{q}}$ of a phonon with wave vector \mathbf{q} prepared at zero temperature in a spatially homogeneous gas of fermions. We assume that the phononic excitation branch $q \mapsto \omega_{\mathbf{q}}$ is convex in vicinity of $q = 0$. The general problem is to determine the behavior of $\Gamma_{\mathbf{q}}$ in the limit $q \rightarrow 0$, where one can use an effective low energy theory to describe the coupling among phonons. The dominant process is of course the $1 \rightarrow 2$ Beliaev one. By introducing a dimensionless parameter $\gamma > 0$ describing both the correction in q^3 to the hydrodynamic linear excitation spectrum, and the first correction to the hydrodynamic coupling amplitude $1 \rightarrow 2$, we have obtained the provisional result (151) up to the sub-leading order q^7 . In this appendix, we give some details on the obtention of the final result at the Beliaev order (153), and we examine the possible contributions of order q^7 of all the higher order processes, such as the cascade process $1 \rightarrow 2 \rightarrow 3$ in figure 5, leading to the correction (155) and to the final result (156).

In the subspace with a fixed total momentum $\hbar\mathbf{q}$, the one-phonon state $|\mathbf{q}\rangle$ is the only discrete state, as it is the only one completely characterized by a single wave vector value. It is an eigenstate of \hat{H}_2 , the part of the Hamiltonian

that is quadratic in the phonon creation \hat{b}^\dagger and annihilation \hat{b} operators. It is nevertheless coupled to the two-phonon, three-phonon, etc. continua by the rest of the Hamiltonian $\hat{V} = \hat{H}_3 + \hat{H}_4 + \dots$, that contains cubic, quartic, etc. terms, when written in the normal order for the \hat{b} .

As a consequence, the discrete state will in general get diluted in the continua, giving rise to a complex pole $z_{\mathbf{q}}$ in the analytic continuation of the resolvent $\hat{G}(z) = (z - \hat{H})^{-1}$ of the full Hamiltonian [56]. This pole can be written as

$$z_{\mathbf{q}} = \hbar\omega_{\mathbf{q}} - i\frac{\hbar\Gamma_{\mathbf{q}}}{2} \quad (\text{F.1})$$

where $\omega_{\mathbf{q}}$ is the angular eigenfrequency of the phonon and $\Gamma_{\mathbf{q}}$ its damping rate¹¹.

Appendix F.2. Perturbative calculation and power counting

The $q \rightarrow 0$ limit corresponds to the weak coupling limit. This is apparent in the quantum hydrodynamic theory as the modal expansion of the velocity and density fluctuations $\hat{\mathbf{v}}$ and $\delta\hat{\rho}$ involves coefficients that tend to zero as $q^{1/2}$, so that the matrix elements of \hat{H}_p between phonon Fock states behave as $q^{p/2}$:

$$\hat{H}_p \underset{q \rightarrow 0}{\approx} q^{p/2} \quad (\text{F.2})$$

One can then attempt a perturbative calculation of $\Gamma_{\mathbf{q}}$ starting from the exact expression obtained by the projector method [56]

$$\langle \mathbf{q} | \hat{G}(z) | \mathbf{q} \rangle = \frac{1}{z - \langle \mathbf{q} | \hat{H}_{\text{eff}}(z) | \mathbf{q} \rangle} \quad (\text{F.3})$$

then expanding in powers of \hat{V} the matrix element of the effective Hamiltonian in the state $|\mathbf{q}\rangle$:

$$\langle \mathbf{q} | H_{\text{eff}}(z) | \mathbf{q} \rangle = \langle \mathbf{q} | \hat{H}_2 | \mathbf{q} \rangle + \langle \mathbf{q} | \hat{V} \hat{Q} \frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}_2\hat{Q}} \hat{Q} \hat{V} | \mathbf{q} \rangle + \langle \mathbf{q} | \hat{V} \hat{Q} \frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}_2\hat{Q}} \hat{Q} \hat{V} \hat{Q} \frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}_2\hat{Q}} \hat{Q} \hat{V} | \mathbf{q} \rangle + \dots \quad (\text{F.4})$$

where $\hat{Q} = 1 - |\mathbf{q}\rangle\langle \mathbf{q}|$ projects orthogonally to $|\mathbf{q}\rangle$. The result (153) corresponds to the second term on the right-hand side of the equation, the Beliaev term, in which the contribution of \hat{V} reduces to that of \hat{H}_3 . In all the following terms of order 3 in \hat{V} or higher, z can be approximated in the denominator by

$$z = \hbar\omega_{\mathbf{q}} + i\eta, \quad \eta \rightarrow 0^+, \quad (\text{F.5})$$

the displacement of z by an imaginary part that is $O(q^5)$ in these terms does not contribute to $\Gamma_{\mathbf{q}}$ at the order q^7 . The same conclusion does not hold for the Beliaev term, as we shall see in subsection Appendix F.3.

Let us consider in (F.4) the term \mathcal{T}_n of order n in \hat{V} and let us try to give an upper bound to its order in q as done in the simple reasoning below equation (153).

Taking (F.2) into account, we must retain in the \hat{V} factors as many as possible contributions of \hat{H}_3 and, if not, those of \hat{H}_4 . As \hat{H}_3 changes the parity of the phonon number, while \hat{H}_4 preserves it, we will retain only \hat{H}_3 if n is even, but it will be necessary to keep at least one factor \hat{H}_4 if n is odd. The minimal number of independent wave vectors of the virtual phonons is $n/2$ for n even, $(n+1)/2$ for n odd, and the integration over each independent wave vector provides a factor q^3 . Finally, one has to count a factor $1/q$ for each of the $n-1$ energy denominators. All together, this leads to

$$\mathcal{T}_n = O(q^{2n+1}) \quad \text{if } n \text{ even} \quad (\text{F.6})$$

$$\mathcal{T}_n = O(q^{2n+3}) \quad \text{if } n \text{ odd} \quad (\text{F.7})$$

11. We cannot calculate here the corrections to $\omega_{\mathbf{q}}$ due to the coupling with the continua, even at the order 2 in \hat{V} , because we rely on an effective Hamiltonian. If we try, we would encounter principal part integrals with ultraviolet divergences that, once inserted in the unperturbed value in \hat{H}_2 would give, following the ideas of renormalisation, the true value, which would however remain uncalculated and unknown in the absence of a microscopic model [8]. Within this context, we do not find it useful to distinguish between the non-perturbed phonon eigenenergy for \hat{H}_2 , which we should rigorously note $\epsilon_{\mathbf{q}}^{(2)}$, and the true energy $\epsilon_{\mathbf{q}}$. By the way, in order to obtain the scaling laws with q of the perturbative terms of order n in \hat{V} , we assume that the wave numbers of the virtual phonons are $O(q)$, hence that the ultraviolet cutoff is set at a wave number Aq , $A \gg 1$. This has the big advantage that the real correction to the angular eigenfrequency $\epsilon_{\mathbf{q}}^{(2)}$ due to \hat{H}_3 to the leading order is $O(q^5)$ and it affects neither the γ parameter in equation (57) nor the damping rate $\Gamma_{\mathbf{q}}$ to the order q^7 .

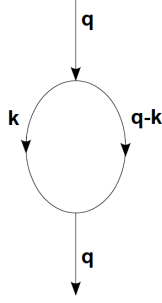


Figure F.7: Diagram of order 2 in \hat{V} that contributes to the damping rate $\Gamma_{\mathbf{q}}$ of the phonon \mathbf{q} to the order q^7 . It represents the Beliaev process whose leading order is q^5 . On the figure, each vertex represents an action of $\hat{H}_3^{(\pm)}$.

One would then conclude with no surprise, that the processes of higher order than Beliaev, that is of order $n \geq 3$ in \hat{V} , give a contribution to $\Gamma_{\mathbf{q}}$ that is $O(q^9)$ hence negligible.

The simple reasoning above forgets the possibility that the energy denominators are of order q^3 . For this case to occur, the contribution to the energy denominator of the linear part of the spectrum must have a chance to cancel out so the considered processes must have the possibility to be resonant. One must then keep in \hat{H}_3 only the terms in $\hat{b}^\dagger \hat{b}^\dagger \hat{b}$, that constitute the raising part $\hat{H}_3^{(+)}$ of the cubic Hamiltonian, or the terms in $\hat{b}^\dagger \hat{b} \hat{b}$, that constitute its lowering part $\hat{H}_3^{(-)}$. The other terms in $\hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger$ and $\hat{b} \hat{b} \hat{b}$ induce nonresonant processes. In \hat{H}_4 , we shall only keep terms $\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}$, that are the most favorable potentially resonant terms, as they add a single independent wave vector.

In order to have all the energy denominators of order q^3 , all the independent wave vectors must be emitted forward, within a narrow cone of angular aperture $O(q)$ with respect to the direction of \mathbf{q} . The angular integration then brings a factor q^2 for each independent wave vector, that is q^n for n even and q^{n+1} for n odd, but we gain a factor $q^{-2(n-1)}$ thanks to $n-1$ small denominators.

This leads to the refined upper bounds

$$\mathcal{T}_n = O(q^{n+3}) \quad \text{if } n \text{ even} \quad (\text{F.8})$$

$$\mathcal{T}_n = O(q^{n+6}) \quad \text{if } n \text{ odd} \quad (\text{F.9})$$

The conclusion of this discussion is that only the order 4 in \hat{V} has a chance to produce a correction in q^7 to the result $\Gamma_{\mathbf{q}}^{\text{Bel}}$ in equation (153). We analyze it in more detail in subsection Appendix F.4. We can also confirm that the order 2 in \hat{V} gives the leading contribution to $\Gamma_{\mathbf{q}}$ in q^5 . This however does not exonerate us from a rigorous study of its sub-leading contributions up to the order q^7 , which is done in subsection Appendix F.3.

Appendix F.3. Study to second order in \hat{V}

In this subsection we restrict the effective Hamiltonian in equation (F.4) to second order in \hat{V} and calculate the resulting value of $\Gamma_{\mathbf{q}}$ up to the order q^7 .

To this order in \hat{V} several different diagrams actually contribute, as $\hat{V} = \hat{H}_3 + \hat{H}_4 + \hat{H}_5 + \dots$. The leading order in q can be estimated by power counting. If the two factors \hat{V} are both equal to \hat{H}_3 , one has to integrate over at least one independent wave vector and the leading order is q^5 as in equation (F.8). If they are both equal to \hat{H}_4 , one has to integrate over at least two independent wave vectors and the leading order is q^9 without the ‘‘small denominators’’ effect, but not any better with the ‘‘small denominators’’. If one of the factors \hat{V} involves \hat{H}_p , with $p \geq 5$, the contribution is even smaller and negligible. We can then restrict to $\hat{V} = \hat{H}_3$, and furthermore to $\hat{V} = \hat{H}_3^{(\pm)}$ as the non resonant terms in $\hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger$ and $\hat{b} \hat{b} \hat{b}$ of \hat{H}_3 involve two independent wave vectors and contribute in $O(q^8)$.

There remains then a single diagram, the Beliaev one, represented in figure F.7. Let us introduce the corresponding Beliaev self-energy, for any complex number z with $\text{Im } z > 0$:

$$\Delta \epsilon_{\mathbf{q}}^{\text{Bel}}(z) = \frac{1}{2} \sum_{\mathbf{k}} \frac{|\langle \mathbf{k}, \mathbf{q} - \mathbf{k} | \hat{H}_3^{(+)} | \mathbf{q} \rangle|^2}{z - (\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{q}-\mathbf{k}})} = \frac{2(mc^2)^2}{\rho} \int \frac{d^3 k}{(2\pi)^3} \frac{[\mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 1}(\mathbf{k}, \mathbf{q} - \mathbf{k}; \mathbf{q})]^2}{z - (\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{q}-\mathbf{k}})} \quad (\text{F.10})$$

where we considered the thermodynamic limit and we have used the form (93) of the hydrodynamic Hamiltonian. To this order of approximation, the pole $z_{\mathbf{q}}$ associated to the phonon \mathbf{q} is the solution of the implicit equation

$$z_{\mathbf{q}}^{\text{Bel}} = \epsilon_{\mathbf{q}} + \Delta\epsilon_{\mathbf{q}}^{\text{Bel}\downarrow}(z_{\mathbf{q}}^{\text{Bel}}) \quad (\text{F.11})$$

where the downward arrow denotes the analytic continuation of the self-energy from the upper half-plane $\text{Im } z > 0$ to the lower one $\text{Im } z < 0$ through the branch cut on the real positive axis. Let us separate $z_{\mathbf{q}}^{\text{Bel}}$ into the real and imaginary parts as in (F.1) and let us expand the right-hand side in powers of $\Gamma_{\mathbf{q}}$:

$$z_{\mathbf{q}}^{\text{Bel}} = \epsilon_{\mathbf{q}} + \Delta\epsilon_{\mathbf{q}}^{\text{Bel}\downarrow}(\epsilon_{\mathbf{q}}) - \frac{i\hbar\Gamma_{\mathbf{q}}}{2} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}\downarrow}(\epsilon_{\mathbf{q}}) + O(q^9) \quad (\text{F.12})$$

It is enough here to truncate the Taylor expansion to the order one; on the other hand the zeroth order is not enough¹². The values of the analytic continuation and of its derivative in $\epsilon_{\mathbf{q}}$ are obtained as the limits when $\eta \rightarrow 0^+$ of the noncontinued functions in $z = \epsilon_{\mathbf{q}} + i\eta$:

$$z_{\mathbf{q}}^{\text{Bel}} = \epsilon_{\mathbf{q}} + \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) - \frac{i\hbar\Gamma_{\mathbf{q}}}{2} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) + O(q^9) \quad (\text{F.13})$$

The second term is the ordinary perturbative result. It leads to the damping rate $\Gamma_{\mathbf{q},\text{pert}}^{\text{Bel}}$ given by the zero temperature limit of equation (119),

$$\frac{\hbar\Gamma_{\mathbf{q},\text{pert}}^{\text{Bel}}}{2} = -\text{Im} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) \quad (\text{F.14})$$

and its value for the unitary gas has been calculated up to the order q^7 in equation (151). Once the imaginary part of equation (F.13) has been taken, we find a nonnegligible correction to this order, resulting from a first self consistent iteration of $z_{\mathbf{q}}$ in the implicit equation (F.11):

$$\Gamma_{\mathbf{q}}^{\text{Bel}} = \Gamma_{\mathbf{q},\text{pert}}^{\text{Bel}} \left[1 + \text{Re} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) + O(q^4) \right] \quad (\text{F.15})$$

with

$$\frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) = -\frac{2(mc^2)^2}{\rho} \int \frac{d^3k}{(2\pi)^3} \frac{[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{k}, \mathbf{q} - \mathbf{k}; \mathbf{q})]^2}{(\epsilon_{\mathbf{q}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}-\mathbf{k}} + i\eta)^2} \quad (\text{F.16})$$

In order to obtain an equivalent of this derivative when $\check{q} \equiv \hbar q/mc \rightarrow 0$, we apply the same technique as in section 5.3. We use spherical coordinates of polar axis \mathbf{q} and perform the rescalings $k = \bar{k}q$ and $\theta = \gamma^{1/2}\check{\theta}$ on the modulus and polar angle of the vector \mathbf{k} , with $\bar{k} < 1$ to profit from the ‘‘small denominators’’ effect. In the integrand and hence in the coupling amplitude (94), we take the limit $\check{q} \rightarrow 0$ with \bar{k} and $\check{\theta}$ fixed, with the intermediate results

$$|\mathbf{q} - \mathbf{k}| = q(1 - \bar{k}) \left[1 + \frac{\gamma\bar{k}\check{\theta}^2\check{q}^2}{2(1 - \bar{k})^2} + O(\check{q}^4) \right] \quad (\text{F.17})$$

$$\epsilon_{\mathbf{q}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}-\mathbf{k}} \sim \frac{\gamma\check{q}^3 mc^2}{8} \left[3\bar{k}(1 - \bar{k}) - \frac{4\bar{k}\check{\theta}^2}{1 - \bar{k}} \right] \quad (\text{F.18})$$

$$[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{k}, \mathbf{q} - \mathbf{k}; \mathbf{q})]^2 \sim \frac{2}{9}\check{q}^3\bar{k}(1 - \bar{k}) \quad (\text{F.19})$$

taking into account the fact that $1 + \Lambda_{\text{F}} = 8/9$ at unitarity. The integration with respect to $\check{\theta}$ over $[0, +\infty[$ is simple after the change of variable $X = \check{\theta}^2$, and we are left with

$$\frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) \sim \frac{8\check{q}^2}{9\pi^2\gamma} \left(\frac{mc}{\hbar\rho^{1/3}} \right)^3 \int_0^1 d\bar{k} \frac{\bar{k}^2(1 - \bar{k})^2}{3\bar{k}(1 - \bar{k}) + i\eta} \quad (\text{F.20})$$

The dominated convergence theorem allows us to take the $\eta \rightarrow 0$ limit in the integrand and to obtain the result (153), knowing that $\rho(\hbar/mc)^3 = \sqrt{3}/(\pi^2\xi_B^3)^{3/2}$. As a side remark, the fact that the derivative with respect to z (F.20) is real in the limit $\eta \rightarrow 0^+$, means that a change in the real dispersion relation $q \mapsto \omega_{\mathbf{q}}$ of order q^5 does not affect $\Gamma_{\mathbf{q}}$ to the order q^7 , which the calculation technique of subsection 5.4.2 also shows.

12. The Taylor contribution of order n is $O(q^{2n+5})$ as the n th order derivative of the self-energy, that involves an energy denominator to the power $n + 1$, is of order $q^8/q^{3(n+1)}$ accounting for the ‘‘small denominators’’ effect.

Appendix F.4. Study to the order 4 in \hat{V}

We have seen in subsection [Appendix F.2](#) that the order 4 in \hat{V} may contribute to $\Gamma_{\mathbf{q}}$ to the order q^7 . Let us write the corresponding correction $\delta\Gamma_{\mathbf{q}}$ restricting to the leading order, that is keeping $\hat{H}_3^{(-)}$ in the first two factors \hat{V} and $\hat{H}_3^{(+)}$ in the last two, this sequence being imposed by the presence of the projectors \hat{Q} :

$$-\frac{\hbar\delta\Gamma_{\mathbf{q}}}{2} = \text{Im}\langle\mathbf{q}|\hat{W}^{(-)}\frac{\hat{Q}}{\epsilon_{\mathbf{q}} + i\eta - \hat{H}_2}\hat{W}^{(+)}|\mathbf{q}\rangle \quad (\text{F.21})$$

where we introduced, for $\varepsilon = \pm$, the effective coupling operator $1 \leftrightarrow 3$ to second order

$$\hat{W}^{(\varepsilon)} \equiv \hat{H}_3^{(\varepsilon)}\frac{\hat{Q}}{\epsilon_{\mathbf{q}} + i\eta - \hat{H}_2}\hat{H}_3^{(\varepsilon)} \quad (\text{F.22})$$

In these expressions, one can use for \hat{H}_3 the quantum hydrodynamic approximation (93), provided one includes in $\epsilon_{\mathbf{q}} = \hbar\omega_{\mathbf{q}}$ the cubic correction (57) to the linear hydrodynamic spectrum. As the matrix elements of \hat{H}_3 are real in the Fock basis, the matrices representing $\hat{W}^{(\pm)}$ are obtained one from the other by transposition. After the insertion of a closure relation in the three-phonon subspace, one obtains

$$-\frac{\hbar\delta\Gamma_{\mathbf{q}}}{2} = \text{Im} \frac{1}{3!} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3} \delta_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3, \mathbf{q}} \frac{(\langle\mathbf{q}|\hat{W}^{(-)}|\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\rangle)^2}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})} \quad (\text{F.23})$$

where, by virtue of equation (93),

$$\begin{aligned} \langle\mathbf{q}|\hat{W}^{(-)}|\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\rangle = & \frac{4(mc^2)^2}{\rho L^3} \left[\frac{\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3; \mathbf{q})}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})} \right. \\ & \left. + \frac{\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_2, \mathbf{q}_3; \mathbf{q}_2 + \mathbf{q}_3)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_2 + \mathbf{q}_3, \mathbf{q}_1; \mathbf{q})}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3} + \epsilon_{\mathbf{q}_1})} + \frac{\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_3; \mathbf{q}_1 + \mathbf{q}_3)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_3, \mathbf{q}_2; \mathbf{q})}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_3} + \epsilon_{\mathbf{q}_2})} \right] \quad (\text{F.24}) \end{aligned}$$

The contribution to the matrix element of the first term between square brackets, that we note $P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)$, corresponds to the process in figure 5. The other two terms, $P(\mathbf{q}_2, \mathbf{q}_3|\mathbf{q}_1)$ and $P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)$, are deduced from the first one by circular permutation. As the rest of the summand in (F.23) is invariant by permutation of the three wave vectors, we can replace its numerator $[P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3) + P(\mathbf{q}_2, \mathbf{q}_3|\mathbf{q}_1) + P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)]^2$ by $3[P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)]^2 + 6P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)$, which amounts to considering the two diagrams in figure F.8 and the integral expression

$$\begin{aligned} -\frac{\hbar\delta\Gamma_{\mathbf{q}}}{2} = & \lim_{\eta \rightarrow 0^+} \text{Im} \frac{8(mc^2)^4}{\rho^2} \int \frac{d^3q_2 d^3q_3}{(2\pi)^6} \left\{ \frac{[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3; \mathbf{q})]^2}{[\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})][\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})]^2} \right. \\ & \left. + \frac{2\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3; \mathbf{q})\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_3; \mathbf{q}_1 + \mathbf{q}_3)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_3, \mathbf{q}_2; \mathbf{q})}{[\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})][\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})][\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_3} + \epsilon_{\mathbf{q}_2})]} \right\} \quad (\text{F.25}) \end{aligned}$$

where we eliminated the sum over \mathbf{q}_1 thanks to momentum conservation, $\mathbf{q}_1 = \mathbf{q} - (\mathbf{q}_2 + \mathbf{q}_3)$, and took the thermodynamic limit.

In practice, the integration over \mathbf{q}_2 and \mathbf{q}_3 is performed in spherical coordinates of polar axis \mathbf{q} . To evaluate the contribution to $\delta\Gamma_{\mathbf{q}}$ of the emission cones of \mathbf{q}_2 and \mathbf{q}_3 of angular aperture $O(q)$ around \mathbf{q} , we rescale the polar angles θ_i as follows:

$$\check{\theta}_i = \frac{\theta_i}{\gamma^{1/2}\check{q}} \quad (\text{F.26})$$

then we let $\check{q} = \hbar q/mc$ tend to zero with $\check{\theta}_i$ fixed. The calculation is similar to the one in section 5.3. One introduces the polar representation as in (137):

$$\check{\theta}_2 = R \cos \alpha \quad \check{\theta}_3 = R \sin \alpha \quad (\text{F.27})$$

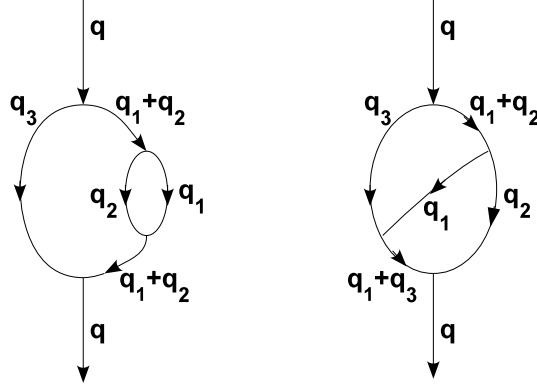


Figure F.8: The two diagrams of order 4 in \hat{V} that may contribute to the damping rate $\Gamma_{\mathbf{q}}$ of the phonon \mathbf{q} to the order q^7 thanks to the “small denominator” effect. The one on the left (I) is a Beliaev process with a single loop correction (itself of the Beliaev nature) to the virtual phonons angular eigenfrequency. The one on the right (II) is a Beliaev process enriched by an interaction between the virtual phonons. Each vertex corresponds to an action of $\hat{H}_3^{(\pm)}$.

The wave numbers q_i must be rescaled by q , meaning that $\bar{q}_i \equiv \frac{q_i}{q}$ are kept fixed when taking the $\check{q} \rightarrow 0$ limit, with the constraint

$$\bar{q}_2 + \bar{q}_3 \leq 1 \quad (\text{F.28})$$

following from (154) and from the positivity of q_1 .

We give as an intermediate result the expression of an energy denominator in the $\hat{W}^{(-)}$ matrix elements,

$$\frac{\epsilon_{\mathbf{q}} - (\epsilon_{\mathbf{q}-\mathbf{q}_3} + \epsilon_{\mathbf{q}_3})}{mc^2} \underset{\check{q} \rightarrow 0}{=} \frac{\gamma \check{q}^3}{2} \left[\frac{3}{4} \bar{q}_3 (1 - \bar{q}_3) - \frac{\bar{q}_3 R^2 \sin^2 \alpha}{1 - \bar{q}_3} \right] + O(\check{q}^5) \quad (\text{F.29})$$

and the energy denominator in the 3 phonon subspace,

$$\frac{\epsilon_{\mathbf{q}} - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})}{mc^2} \underset{\check{q} \rightarrow 0}{=} \frac{\gamma \check{q}^3}{2} (v - uR^2) + O(\check{q}^5) \quad (\text{F.30})$$

whose manifestly positive coefficients (once they are written in the proper form) look like those of equations (139) and (140):

$$u \equiv \frac{\bar{q}_2 \cos^2 \alpha + \bar{q}_3 \sin^2 \alpha - \bar{q}_2 \bar{q}_3 (1 - \cos \phi \sin 2\alpha)}{1 - \bar{q}_2 - \bar{q}_3} = \frac{(\bar{q}_2 \cos \alpha - \bar{q}_3 \sin \alpha)^2 + \bar{q}_2 \bar{q}_3 (1 + \cos \phi) \sin 2\alpha}{1 - \bar{q}_2 - \bar{q}_3} + \bar{q}_2 \cos^2 \alpha + \bar{q}_3 \sin^2 \alpha \quad (\text{F.31})$$

and

$$v \equiv \frac{1}{4} [1 - \bar{q}_2^3 - \bar{q}_3^3 - (1 - \bar{q}_2 - \bar{q}_3)^3] = \frac{3}{4} (1 - \bar{q}_2)(1 - \bar{q}_3)(\bar{q}_2 + \bar{q}_3) \quad (\text{F.32})$$

One finally obtains

$$-\frac{\hbar \delta \Gamma_{\mathbf{q}}}{2} \underset{\check{q} \rightarrow 0}{=} mc^2 \frac{[3(1 + \Lambda_F)]^4}{3 \times 2\pi} \frac{\check{q}^7}{2^8 \pi^4 \gamma} \left(\frac{mc}{\hbar \rho^{1/3}} \right)^6 (\text{Im } I^{(I)} + \text{Im } I^{(II)}) + o(\check{q}^7) \quad (\text{F.33})$$

with the parameter Λ_F defined in equation (83) and the quintuple integrals, coming from the left (type I) diagram and the right (type II) diagram of figure F.8, that should be evaluated in the limit $\eta \rightarrow 0^+$:

$$I^{(I)} = \lim_{\eta \rightarrow 0^+} \int_0^1 d\bar{q}_2 \int_0^{1-\bar{q}_2} d\bar{q}_3 \int_{-\pi}^{\pi} d\phi \int_0^{\pi/2} d\alpha \int_0^{A^2/\check{q}^2} \frac{dX}{2} \frac{3X \sin \alpha \cos \alpha (1 - \bar{q}_3)^2 (1 - \bar{q}_2 - \bar{q}_3) \bar{q}_2^3 \bar{q}_3^3}{(v - uX + i\eta) \left[\frac{3}{4} \bar{q}_3 (1 - \bar{q}_3) - \frac{\bar{q}_3 X \sin^2 \alpha}{1 - \bar{q}_3} + i\eta \right]^2} \quad (\text{F.34})$$

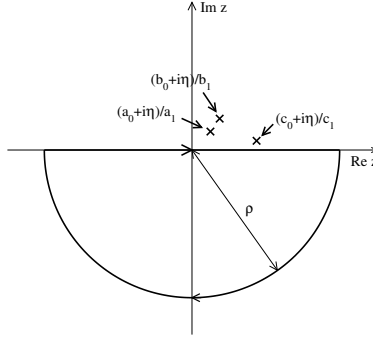


Figure F.9: Thick line: contour integration in the complex plane that should be used to apply the residue theorem, in order to obtain the identity (F.39). The real numbers $a_0, a_1, b_0, b_1, c_0, c_1$ and η are positive. The radius ρ of the half-circle tends to infinity. The crosses correspond to the poles of the integrand in the case they are all simple, but the result holds also in the general case.

$$I^{(\text{II})} = \lim_{\eta \rightarrow 0^+} \int_0^1 d\bar{q}_2 \int_0^{1-\bar{q}_2} d\bar{q}_3 \int_{-\pi}^{\pi} d\phi \int_0^{\pi/2} d\alpha \times \int_0^{+\infty} \frac{dX}{2} \frac{6X \sin \alpha \cos \alpha (1 - \bar{q}_2)(1 - \bar{q}_3)(1 - \bar{q}_2 - \bar{q}_3) \bar{q}_2^3 \bar{q}_3^3}{(v - uX + i\eta) \left[\frac{3}{4} \bar{q}_2 (1 - \bar{q}_2) - \frac{\bar{q}_2 X \cos^2 \alpha}{1 - \bar{q}_2} + i\eta \right] \left[\frac{3}{4} \bar{q}_3 (1 - \bar{q}_3) - \frac{\bar{q}_3 X \sin^2 \alpha}{1 - \bar{q}_3} + i\eta \right]} \quad (\text{F.35})$$

In the integration over the radius R we have performed the change of variables $X = R^2$. This allows to see that the integral over R , that is over X , in (F.35) has a zero imaginary part in the limit $\eta \rightarrow 0^+$, and hence

$$\text{Im } I^{(\text{II})} = 0 \quad (\text{F.36})$$

Indeed all the factors in the integrand denominator in (F.35) have the form $\alpha - \beta X + i\eta$, with $\alpha > 0$ and $\beta > 0$. As a consequence, the integral over $X \in [0, +\infty[$ can be evaluated using the following elementary theorem:

Theorem: We define $a(X) = a_0 - a_1 X$, $b(X) = b_0 - b_1 X$ and $c(X) = c_0 - c_1 X$ for all $X \in \mathbb{R}$. If all the coefficients $a_0, a_1, b_0, b_1, c_0, c_1$ are > 0 and $\eta \in \mathbb{R}$, then

$$\lim_{\eta \rightarrow 0^+} \int_{\mathbb{R}^+} dX \text{Im} \frac{X}{[a(X) + i\eta][b(X) + i\eta][c(X) + i\eta]} = 0 \quad (\text{F.37})$$

Proof: We first note that the theorem would be trivially proved if the integral over X was taken over \mathbb{R}^- . Indeed, for all $\eta > 0$ and for all $X < 0$,

$$\left| \frac{X}{[a(X) + i\eta][b(X) + i\eta][c(X) + i\eta]} \right| < \frac{X}{a(X)b(X)c(X)} \quad (\text{F.38})$$

where the upper bound has a finite integral over \mathbb{R}^- , so that the dominated convergence theorem can be used to exchange the order of the integral and limit $\eta \rightarrow 0^+$. We can then extend the integration domain in (F.37) to the whole \mathbb{R} without changing the result. On the other hand we can use the following identity valid for any $\eta > 0$ not necessarily infinitesimal, concerning both the real and the imaginary parts:

$$\int_{\mathbb{R}} dX \frac{X}{[a(X) + i\eta][b(X) + i\eta][c(X) + i\eta]} = 0 \quad (\text{F.39})$$

To show it, we use the residue theorem by closing downwards the integration contour following a half-circle of radius $\rho \rightarrow +\infty$ in the lower half-plane; the behavior of the integrand as $1/X^2$ for large $|X|$ allows for it, which leads to a vanishing $O(1/\rho)$ contribution of the half-circle. As the integrand poles are all in the upper half-plane, they are not

enclosed by the contour, see the figure F.9, hence (F.39) and (F.37)¹³. \square

One might think that the same reasoning applies to the contribution (F.34) and that $\text{Im } I^{(1)} = 0$, in which case $\delta\Gamma_{\mathbf{q}} = o(\check{q}^7)$ and there would be no correction to add to the result (153). This is not the case because, in order to obtain a finite value of $I^{(1)}$, we have this time to keep in the integral over X a finite value A^2/\check{q}^2 of the upper bound, which prevents the application of the theorem. Here, $A > 0$ is a cut-off constant whose precise value is not relevant, and the power law in \check{q}^{-2} comes from the fact that $\theta_i \leq \pi$ and then $\check{\theta}_i \leq \pi/(\gamma^{1/2}\check{q})$ in (F.26), which implies $X = R^2 = \check{\theta}_2^2 + \check{\theta}_3^2 = O(\check{q}^{-2})$. If one simply replaces A^2/\check{q}^2 by $+\infty$ in equation (F.34), one finds indeed that the integral over X diverges as α^{-2} when α tends to zero, which leads to an integral over α that diverges logarithmically for $\alpha = 0$.¹⁴ To progress, we cut the integration interval over α into two, a sub-interval $[0, \nu]$ for which the cut-off A^2/\check{q}^2 in the integration over X is necessary, and a sub-interval $[\nu, \pi/2]$ for which the integration over X can be extended to $+\infty$:

$$\int_0^{\pi/2} d\alpha \int_0^{A^2/\check{q}^2} dX f(\alpha, X) \rightarrow \int_0^{\nu} d\alpha \int_0^{A^2/\check{q}^2} dX f(\alpha, X) + \int_{\nu}^{\pi/2} d\alpha \int_0^{+\infty} dX f(\alpha, X) \quad (\text{F.40})$$

The contribution of the second bit to $\text{Im } I^{(1)}$ is zero, because of the theorem. The contribution of the first bit does not depend on the value of ν . We can then calculate it for a value of ν sufficiently small to approximate each contribution depending on α in the integrand by its leading order in α , that is $\sin \alpha \simeq \alpha$, $\cos \alpha \simeq 1$, $u \simeq u_0 = \bar{q}_2(1 - \bar{q}_3)/(1 - \bar{q}_2 - \bar{q}_3)$. The dependence in ϕ then disappears and the integration over ϕ simply gives a factor 2π . After a simple calculation of the integral over α :

$$\int_0^{\nu} \frac{\alpha d\alpha}{\left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) - \frac{\bar{q}_3 X \alpha^2}{1 - \bar{q}_3} + i\eta\right]^2} = \left[\frac{1 - \bar{q}_3}{2\bar{q}_3 X \left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) - \frac{\bar{q}_3 X \alpha^2}{1 - \bar{q}_3} + i\eta\right]} \right]_{\alpha=0}^{\alpha=\nu} \quad (\text{F.41})$$

we shall concentrate on the integration over X . The fully integrated term $\alpha = \nu$ of equation (F.41), after a multiplication by the factor X in the numerator of (F.34) and a division by the denominator $(\nu - u_0 X + i\eta)$, gives an integrand that is $O(1/X^2)$; one can in this case replace the upper bound A^2/\check{q}^2 in the integration over X by $+\infty$ and use the following variant of the theorem, in order to show that its contribution to $\text{Im } I^{(1)}$ is exactly zero:

$$\lim_{\eta \rightarrow 0^+} \int_{\mathbb{R}^+} dX \text{Im} \frac{1}{[a(X) + i\eta][b(X) + i\eta]} = 0 \quad (\text{F.42})$$

The fully integrated term $\alpha = 0$ of equation (F.41) leads on the contrary to an integral over X of nonzero imaginary

13. We supposed here that the leading contribution to $\delta\Gamma_{\mathbf{q}}$ comes from the ‘‘bicone’’ configuration in which both \mathbf{q}_2 and \mathbf{q}_3 are in the forward emission cone of angular aperture $O(\check{q})$ with respect to the direction of \mathbf{q} . One can imagine a more subtle scenario, called ‘‘unicone’’, in which only the vector \mathbf{q}_3 would be in this cone, while \mathbf{q}_2 would be at an angle $\approx \check{q}^0$ with \mathbf{q} . In this case, only the denominator of $P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)$ is $\approx \check{q}^3$, while that of $P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)$ is $\approx \check{q}$. The crossed term $P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)$ is then negligible with respect to $[P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)]^2$. The global energy denominator $\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})$ of (F.23) is also $\approx \check{q}$, which makes us lose a factor \check{q}^2 , but this is exactly compensated by the loss of a factor \check{q}^2 in the numerator in the polar integral $\int d\theta_2 \sin \theta_2$. The ensemble seems then to contribute to the same order \check{q}^7 as the bicone configuration. The integration over the polar angle θ_3 rescaled as in (F.26), with the inclusion of the three-dimensional Jacobian \check{q}_3^2 from the integration over \check{q}_3 , however leads to

$$\int_0^{+\infty} \frac{\check{q}_3^2 \check{\theta}_3 d\check{\theta}_3}{\left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) - \frac{\bar{q}_3 \check{\theta}_3^2}{1 - \bar{q}_3} + i\eta\right]^2} = -\frac{\bar{q}_3(1 - \bar{q}_3)/2}{\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) + i\eta}$$

which has a real limit $-\frac{2}{3}$ when $\eta \rightarrow 0^+$. Hence the sought imaginary part in $\delta\Gamma_{\mathbf{q}}$ may only come from the global energy denominator, which would provide a Dirac $\delta[\epsilon_{\mathbf{q}} - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})]$; however, to the leading order in \check{q} that we consider here, $\mathbf{q}_3 = \bar{q}_3 \mathbf{q}$, with $0 < \bar{q}_3 < 1$, and the dispersion relation is linear, so that the argument of the Dirac distribution cannot be zero unless \mathbf{q}_2 and \mathbf{q} are collinear and of the same direction, which is in contradiction with the hypothesis of having a \mathbf{q}_2 outside the forward emission cone. The same arguments hold for the damping rate $\Gamma_{\mathbf{q}}^{2 \leftrightarrow 2}$ of section 5.3, and justifies that we only have considered there the ‘‘bicone’’ configuration.

14. This phenomenon does not occur in the contribution (F.35) because neither the factor containing the term $X \sin^2 \alpha$, dangerous when $\alpha \rightarrow 0$, nor the one containing the term $X \cos^2 \alpha$, dangerous when $\alpha \rightarrow \pi/2$, are taken to the square. As a consequence, the integral over X from 0 to $+\infty$ diverges only as $\ln \alpha$ or $\ln(\frac{\pi}{2} - \alpha)$ when $\alpha \rightarrow 0$ or $\alpha \rightarrow \pi/2$. Mathematically, by replacing in equation (F.35) the integral $\int_0^{+\infty} dX f(X)$ by $-\int_{-\infty}^0 dX f(X)$ from the identity (F.39), then by using the dominated convergence theorem, we can justify the exchange of the limit $\eta \rightarrow 0^+$ and the integration over $\check{q}_2, \bar{q}_3, \phi$ and α , and prove the result (F.36).

part in the limit $\eta \rightarrow 0^+$:

$$\text{Im} \int_0^{A^2/\check{q}^2} \frac{(1 - \bar{q}_3)dX}{2\bar{q}_3(v - u_0X + i\eta)[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) + i\eta]} = -\frac{1 - \bar{q}_3}{2\bar{q}_3u_0} \text{Im} \frac{[\ln(v - u_0A^2\check{q}^{-2} + i\eta) - \ln(v + i\eta)]}{\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) + i\eta} \xrightarrow{\eta \rightarrow 0^+} -\frac{\pi}{u_0} \frac{\Theta(u_0A^2\check{q}^{-2} - v)}{\frac{3}{2}\bar{q}_3^2} \quad (\text{F.43})$$

where Θ is the Heaviside function and \ln is the usual branch of the complex logarithm, with a branch cut along the real negative axis. We are then left with

$$\text{Im} I^{(1)} = 2\pi^2 \int_0^1 d\bar{q}_2 \int_0^{1-\bar{q}_2} d\bar{q}_3 (1 - \bar{q}_3)(1 - \bar{q}_2 - \bar{q}_3)^2 \bar{q}_2^2 \bar{q}_3 \Theta(u_0A^2\check{q}^{-2} - v) \xrightarrow{\check{q} \rightarrow 0} \frac{\pi^2}{840} \quad (\text{F.44})$$

hence the correction (155) to be added to the result (153).

To conclude, one might ask whether there is a physical interpretation to the contributing diagram of order 4 in \hat{V} , which is the left one (I) in figure F.8. In the calculation of the complex energy shift $\Delta\epsilon_{\mathbf{q}}^{\text{Bel}}$ of a phonon \mathbf{q} induced by a Beliaev process, this diagram takes into account the effect of a modification of the angular eigenfrequency of the virtual phonons, modification itself induced by a Beliaev process at this order. More quantitatively, let us introduce the Beliaev self-energy (F.10), where we note now more rigorously $\epsilon_{\mathbf{k}}^{(2)}$ the unperturbed eigenenergy of a phonon with wave vector \mathbf{k} in \hat{H}_2 , to distinguish it from the exact energy $\epsilon_{\mathbf{q}}$ (see the note 11). We verified that the change in the complex energy of the phonon \mathbf{q} originating from the left diagram (I) is exactly

$$\delta z_{\mathbf{q}}^{(1)} = \frac{1}{2} \sum_{\mathbf{q}_3} \frac{(\langle \mathbf{q}_3, \mathbf{q} - \mathbf{q}_3 | \hat{H}_3 | \mathbf{q} \rangle)^2}{[\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_3}^{(2)} + \epsilon_{\mathbf{q}-\mathbf{q}_3}^{(2)})]^2} [\Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}-\mathbf{q}_3}^{(2)}) + \Delta\epsilon_{\mathbf{q}-\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}_3}^{(2)})] \quad (\text{F.45})$$

The reader will notice that the argument z of $\Delta\epsilon_{\mathbf{q}-\mathbf{q}_3}^{\text{Bel}}$ is here the exact energy $\epsilon_{\mathbf{q}}$ minus the non perturbed energy of the virtual phonon that is spectator in the process of diagram (I), that is the one that does not participate to the loop. The sum of the correction (I) and of the usual perturbative Beliaev shift of the \mathbf{q} phonon energy can then be written, to the order 4 in \hat{H}_3 , as a Beliaev shift for a renormalized complex dispersion relation:

$$\Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) + \delta z_{\mathbf{q}}^{(1)} \simeq \frac{1}{2} \sum_{\mathbf{q}_3, \mathbf{q}_4} \frac{(\langle \mathbf{q}_3, \mathbf{q}_4 | \hat{H}_3 | \mathbf{q} \rangle)^2 \delta_{\mathbf{q}_3 + \mathbf{q}_4, \mathbf{q}}}{\epsilon_{\mathbf{q}} + i\eta - [\epsilon_{\mathbf{q}_3}^{(2)} + \Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}_4}^{(2)}) + \epsilon_{\mathbf{q}_4}^{(2)} + \Delta\epsilon_{\mathbf{q}_4}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}_3}^{(2)})]} \quad (\text{F.46})$$

By the way, the formulation (F.45) provides a second calculation method, alternative to the one of equation (F.34). With the rescalings of subsection 5.3 suitable to the ‘‘small denominators’’, and by using (F.18) with (\mathbf{q}, \mathbf{k}) successively equal to $(\mathbf{q}_3, \mathbf{k})$ and $(\mathbf{q}, \mathbf{q}_3)$, where \mathbf{k} is the integration variable appearing in the expression taken from (F.10) of $\Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}-\mathbf{q}_3}^{(2)})$, we find

$$\frac{\Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}-\mathbf{q}_3}^{(2)})}{mc^2} = -\frac{9(1 + \Lambda_{\text{F}})^2}{64\pi^2} \left(\frac{mc}{\hbar\rho^{1/3}} \right)^3 \bar{q}_3^5 \check{q}^5 \times \int_0^1 d\bar{k} \bar{k}^2 (1 - \bar{k})^2 \left[i\pi + \ln \left(\frac{4\bar{q}_3 \bar{k} A^2}{(1 - \bar{k}) \check{q}^2} \right) - \ln \left(i\eta + 3\bar{q}_3(1 - \bar{q}_3) + 3\bar{q}_3^3 \bar{k}(1 - \bar{k}) - \frac{4\bar{q}_3 \check{\theta}_3^2}{1 - \bar{q}_3} \right) \right] \quad (\text{F.47})$$

Here $\bar{q}_3 = q_3/q$, $\bar{k} = k/q_3$, $\check{\theta}_3 = \theta_3/(\gamma^{1/2}\check{q})$, θ_3 is the non oriented angle between the vectors \mathbf{q} and \mathbf{q}_3 . The integral over $\check{\theta} = \theta/(\gamma^{1/2}\check{q}_3)$, where θ is the angle between \mathbf{k} and \mathbf{q} , has been explicitly performed over the interval $[0, A/\check{q}_3]$ with the same cut-off parameter A as in equation (F.34). After the insertion of (F.47) in equation (F.45), we obtain

$$\frac{\text{Im} \delta z_{\mathbf{q}}^{(1)}}{mc^2} = -\frac{9^2(1 + \Lambda_{\text{F}})^4}{64\pi^4 \gamma} \left(\frac{mc}{\hbar\rho^{1/3}} \right)^6 \check{q}^7 \int_0^1 d\bar{q}_3 \bar{q}_3^8 (1 - \bar{q}_3) \int_0^1 d\bar{k} \bar{k}^2 (1 - \bar{k})^2 \times \text{Im} \int_0^{+\infty} dX_3 \frac{i\pi - \ln \left(i\eta + 3\bar{q}_3(1 - \bar{q}_3) + 3\bar{q}_3^3 \bar{k}(1 - \bar{k}) - \frac{4\bar{q}_3 X_3}{1 - \bar{q}_3} \right)}{\left(i\eta + 3\bar{q}_3(1 - \bar{q}_3) - \frac{4\bar{q}_3 X_3}{1 - \bar{q}_3} \right)^2} \quad (\text{F.48})$$

We omitted here the $\check{\theta}_3$ -independent logarithmic real term, in between square brackets in equation (F.47), because its contribution to the final result is clearly real in the limit $\eta \rightarrow 0^+$. The dependence on the cut-off parameter A then disappears. The integral over $X_3 = \check{\theta}_3^2$ of the bit with the logarithm is calculated by parts, by taking the derivative of the logarithm; one finds that its contribution is zero in the limit $\eta \rightarrow 0^+$ by using the identity (F.42). Finally, only the term $i\pi$ matters, and it leads exactly to the result (155).

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