



**HAL**  
open science

# Three-phonon and four-phonon interaction processes in a pair-condensed Fermi gas

Hadrien Kurkjian, Yvan Castin, Alice Sinatra

► **To cite this version:**

Hadrien Kurkjian, Yvan Castin, Alice Sinatra. Three-phonon and four-phonon interaction processes in a pair-condensed Fermi gas. 2016. hal-01392846v2

**HAL Id: hal-01392846**

**<https://hal.science/hal-01392846v2>**

Preprint submitted on 14 Nov 2016 (v2), last revised 18 Sep 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Three-phonon and four-phonon interaction processes in a pair-condensed Fermi gas

H. Kurkjian, Y. Castin, A. Sinatra

*Laboratoire Kastler Brossel, ENS-PSL, CNRS, UPMC-Sorbonne Universités and Collège de France, Paris, France*

We study the interactions among phonons and the phonon lifetime in a pair-condensed Fermi gas in the BEC-BCS crossover. To compute the phonon-phonon coupling amplitudes we use a microscopic model based on a generalized BCS Ansatz including moving pairs, which allows for a systematic expansion around the mean field BCS approximation of the ground state. We show that the quantum hydrodynamic expression of the amplitudes obtained by Landau and Khalatnikov apply only on the energy shell, that is for resonant processes that conserve energy. The microscopic model yields the same excitation spectrum as the Random Phase Approximation, with a linear (phononic) start and a concavity at low wave number that changes from upwards to downwards in the BEC-BCS crossover. When the concavity of the dispersion relation is upwards at low wave number, the leading damping mechanism at low temperature is the Beliaev-Landau process  $2 \text{ phonons} \leftrightarrow 1 \text{ phonon}$  while, when the concavity is downwards, it is the Landau-Khalatnikov process  $2 \text{ phonons} \leftrightarrow 2 \text{ phonons}$ . In both cases, by rescaling the wave vectors to absorb the dependence on the interaction strength, we obtain a universal formula for the damping rate. This universal formula corrects and extends the original analytic results of Landau and Khalatnikov [ZhETF **19**, 637 (1949)] for the  $2 \leftrightarrow 2$  processes in the downward concavity case. In the upward concavity case, for the Beliaev  $1 \leftrightarrow 2$  process for the unitary gas at zero temperature, we calculate the damping rate of an excitation with wave number  $q$  including the first correction proportional to  $q^7$  to the  $q^5$  hydrodynamic prediction, which was never done before in a systematic way.

## I. INTRODUCTION

In several many-body systems the low-lying collective excitations are phonons. At low temperature, interactions among phonons determine their lifetime, correlation time and mean-free path. Therefore, they play a central role in transport phenomena, such as thermal conduction in dielectric solids, in hydrodynamic properties, such as temperature dependent viscosity and attenuation of sound in liquid helium [1, 2], and in macroscopic coherence properties of degenerate gases, as they determine for example the intrinsic coherence time of the condensate both in bosonic and fermionic gases [3, 4]. The dominant phonon decay channel differs among physical systems, and depends in particular on the curvature of the phonon excitation branch [5–7]. For a convex dispersion relation at low wave-number, Beliaev-Landau  $2 \leftrightarrow 1$  processes involving three quasiparticle dominate [8, 9], while for a concave dispersion relation at low wave-number, Beliaev-Landau interactions are not resonant, and the Landau-Khalatnikov  $2 \leftrightarrow 2$  processes involving four quasiparticles dominate the dynamics.

Compared to other many-body systems, cold atomic gases offer the unique possibility to control and tune some microscopic parameters, in particular the interaction strength. For a cold atomic Fermi gas in two spin states  $\uparrow$  and  $\downarrow$ , interactions occur only in the  $s$ -wave between fermions of opposite spins and are fully characterized by a single parameter, the  $s$ -wave scattering length  $a$ , which can be adjusted with an external magnetic field close to a so-called Feshbach resonance. This feature has allowed cold Fermi gases experiments to study the crossover between the Bardeen-Cooper-Schrieffer (BCS) regime  $1/a \rightarrow -\infty$ , where the superfluid pairs  $\mathbf{k} \uparrow -\mathbf{k} \downarrow$  are localized in Fourier space close to the Fermi sur-

face, and the Bose-Einstein Condensate (BEC) regime  $1/a \rightarrow +\infty$ , of dilute tightly bound dimers that behave as bosonic particles [10–18]. For the unpolarized gas of spin-1/2 fermions, the excitation spectrum consists of two branches: a gapped fermionic branch describing the excitation of the internal degrees of freedom of the  $\uparrow\downarrow$  pairs, and a bosonic branch describing the collective motion of the pair center of mass, which has a phononic behavior at small wave vectors [7, 19–23]. If the density of excitations is sufficiently small, the elementary excitations are long-lived weakly interacting quasiparticles.

In cold gases of spin-1/2 fermions, the dispersion relation of phonons changes close to unitarity, from convex in the BEC regime to concave in the BCS regime [7, 22]. A complete experimental investigation of the interactions between phonons, in both the concave and convex cases, therefore seems possible within this same physical system. In particular, experiments should observe a sharp increase in the phonon lifetime by switching from the convex to the concave case through a tuning of the scattering length. Experimental studies of collective excitations in cold gases have been performed both in the spectral domain with Bragg scattering [24, 25] and in the time domain [26]. Those studies combined with the possibility to realize homogeneous systems in flat-bottom traps [27] makes the measurement of the decay caused by either the  $2 \leftrightarrow 1$  or the  $2 \leftrightarrow 2$  processes a concrete prospect.

To date, an exhaustive theoretical study of phonon interactions and a general expression of the phonon damping rates at low temperature for any interaction strength is missing. The most natural example of a convex dispersion relation is the Bogoliubov excitation branch of the weakly interacting Bose gas. In this case, Beliaev obtained the coupling amplitudes in a microscopic framework and, from those, the damping rate associated to

the  $1 \rightarrow 2$  processes [8]. The contribution of the  $2 \rightarrow 1$  Landau processes, which exist only at nonzero temperature, was obtained later in [28–30]. In superfluid spin-1/2 Fermi gases, Ref.[9] calculated the zero-temperature Beliaev  $1 \rightarrow 2$  damping rate by a phenomenological low-energy effective theory going one step beyond quantum hydrodynamics in order to include a curvature in the phonon dispersion relation. However, the authors omitted to include corrections to the phonon coupling amplitude, which makes their treatment inconsistent. In the bosonic case of phase II of  $^4\text{He}$ , whose dispersion relation was originally believed to be concave, Landau and Khalatnikov obtained the coupling amplitude for the  $2 \leftrightarrow 2$  processes, including the contribution of virtual off-resonant 3-phonon processes and introducing by hand corrections to hydrodynamics in the form of a cubic term in the excitation spectrum. However, the fact that they computed the damping rates only in the low and high wave-number limits makes this study incomplete and motivates us to revisit it. Studies of low-energy  $2 \leftrightarrow 2$  processes in  $^4\text{He}$  were put on hold after 1970, when it was discovered that the dispersion relation of  $^4\text{He}$  is in fact convex at low wave-number. Studies on processes in the wave-number ranges where the dispersion relation becomes concave again, particularly challenging due to the existence of a small convex region, are still an active research topic [6]. To our knowledge, the study of the  $2 \leftrightarrow 2$  processes in cold Fermi gases is mainly at the unitary limit and in the collisional hydrodynamic regime where the relevant measure for dissipation is the shear viscosity of the gas [31]. In this article, we present a complete study of the interaction processes between phonons in cold Fermi gases, at low enough temperature to be in the collisionless regime, for any interaction strength, therefore in both the concave and convex cases.

We describe the unpolarized pair-condensed Fermi gas using a microscopic semiclassical model, based on a variational state including moving pairs. This model allows for a systematic expansion of the Hamiltonian in terms of two canonically conjugate fields  $\beta$  and  $\beta^*$  that are weak if the density of excitations in the gas is low. The microscopic model and the principle of the method are exposed in section II.

In section III we consider the expansion of the Hamiltonian to the quadratic order in the fields  $\beta$  and  $\beta^*$ . By diagonalizing the quadratic problem, we find the spectrum of the excitations [7, 19–23]. Beyond the purely spectral results known in the literature, we derive here explicitly the quasiparticles modes and the annihilation/creation amplitudes, which we then quantify. This is rather elegantly done thanks to the Hamiltonian character of the equations in the semiclassical model.

Section IV is devoted to the calculation of the coupling amplitudes for three-phonon and four-phonon interaction processes. We give the quantum hydrodynamics [1] predictions for the amplitudes of the  $2 \leftrightarrow 1$  and the  $2 \leftrightarrow 2$  processes, and compare them on the energy shell to a microscopic derivation, based for  $2 \leftrightarrow 1$  on the fermionic

model of section III, and for  $2 \leftrightarrow 2$  on a model of bosons with finite range interactions designed to have a concave dispersion relation at low wavenumbers. Our microscopic test of the hydrodynamic predictions is particularly valuable for the  $2 \leftrightarrow 2$  processes, as (i) these processes involve non-resonant  $2 \leftrightarrow 1$  and  $3 \leftrightarrow 0$  processes, whose amplitudes are not correctly given by hydrodynamics, (ii) genuine hydrodynamics leads to a divergence in the  $2 \leftrightarrow 2$  amplitude that Landau and Khalatnikov had to regularize “by hand” by introducing a curvature in the phonon dispersion relation.

In section V we present a direct application of the previous results by calculating the damping rate of phonons in a Fermi gas in the whole BEC-BCS crossover. To this aim we use the effective on-shell coupling amplitudes for  $2 \leftrightarrow 1$  and  $2 \leftrightarrow 2$  processes calculated in section IV within a Master equation approach. The two main results of section V are that (i) both for a convex and concave dispersion, by introducing properly normalized dimensionless quantities, we obtain a universal curve, giving the damping rate as a function of the wave vector, for any interaction strength in the BEC-BCS crossover and (ii) we give all the analytic asymptotic behaviors of the phonon damping rates, for  $\hbar c q \ll k_B T$  and  $\hbar c q \gg k_B T$  where  $c$  is the speed of sound and  $T$  is the temperature, correcting in particular the result of Landau and Khalatnikov [1] in the concave case. Finally, for the unitary gas and at zero temperature only, we calculate the first correction to the hydrodynamic prediction for the phonon damping, using the effective field theory of Son and Wingate [32], and improving the result of [9].

## II. THE MICROSCOPIC APPROACH

### A. Interaction Hamiltonian in the $s$ -wave

We consider a gas of fermions in two internal states  $\uparrow$  and  $\downarrow$  on a cubic lattice of step  $l$  with periodic boundary conditions in the finite-size volume  $[0, L]^3$ . Opposite spin fermions have an on-site interaction characterized by a coupling constant  $g_0$

$$V(\mathbf{r}, \mathbf{r}') = g_0 \frac{\delta_{\mathbf{r}, \mathbf{r}'}}{l^3} \quad (1)$$

The Hamiltonian of the system in the grand canonical ensemble of chemical potential  $\mu$  is given by:

$$\hat{H} = l^3 \sum_{\mathbf{r}, \sigma=\uparrow/\downarrow} \hat{\psi}_\sigma^\dagger(\mathbf{r}) \left( -\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) + g_0 l^3 \sum_{\mathbf{r}} \hat{\psi}_\uparrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \quad (2)$$

where the discrete Laplacian operator  $\Delta_{\mathbf{r}}$  has the eigenfunctions  $e^{i\mathbf{k}\cdot\mathbf{r}}$  with eigenvalues  $-k^2$  and the field operator of the fermions has the discrete anticommutation relations  $\{\hat{\psi}_\sigma(\mathbf{r}), \hat{\psi}_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma\sigma'} \delta_{\mathbf{r}\mathbf{r}'}/l^3$ , with  $\sigma, \sigma' = \uparrow$  or  $\downarrow$ .

It is common to eliminate the coupling constant  $g_0$ , which depends on  $l$  and tends to 0 in the limit of a continuous space  $l \rightarrow 0$ , and to use the scattering length  $a$  as a parameter, which is fixed independently of  $l$  and is accessible in experiments. This is done using the following equation, derived from the scattering theory [33] applied to the potential (1)

$$\frac{1}{g_0} = \frac{m}{4\pi\hbar^2 a} - \int_{\text{FBZ}} \frac{d^3k}{(2\pi)^3} \frac{m}{\hbar^2 k^2} \quad (3)$$

where FBZ stands for the first Brillouin zone  $[-\pi/l, \pi/l]^3$  of the lattice.

### B. Ground state in the BCS approximation

The idea behind the theory of Bardeen-Cooper-Schrieffer (BCS) [34] is to search for an approximation of the ground state of the Hamiltonian (2) within the family of states

$$|\psi^{\text{BCS}}\rangle = \prod_{\mathbf{k} \in \mathcal{D}} \left( U_{\mathbf{k}} - V_{\mathbf{k}} \hat{a}_{\mathbf{k}\uparrow}^\dagger \hat{a}_{-\mathbf{k}\downarrow}^\dagger \right) |0\rangle \quad (4)$$

where  $\mathcal{D} = \frac{2\pi}{L}\mathbb{Z}^3 \cap [-\pi/l, \pi/l]^3$  is the ensemble of wavevectors of the first Brillouin zone satisfying the periodic boundary conditions, the operator  $\hat{a}_{\mathbf{k}\sigma}$  is a Fourier component of the field operator  $\hat{\psi}_\sigma(\mathbf{r})$  and annihilates a fermion of wavevector  $\mathbf{k}$  and spin  $\sigma$ ,  $V_{\mathbf{k}}$  is the probability amplitude of finding a  $\mathbf{k}\sigma$  fermions in  $|\psi^{\text{BCS}}\rangle$  and  $U_{\mathbf{k}} = \sqrt{1 - |V_{\mathbf{k}}|^2}$ . The locus of the minimizers of the classical energy functional  $\langle \psi^{\text{BCS}} | \hat{H} | \psi^{\text{BCS}} \rangle$  is the circle

$$U_{\mathbf{k}} = U_{\mathbf{k}}^0 \quad V_{\mathbf{k}} = V_{\mathbf{k}}^0 e^{i\phi} \quad (5)$$

where  $\phi \in [0, 2\pi[$  and the values of reference  $U_{\mathbf{k}}^0$  et  $V_{\mathbf{k}}^0$  are chosen reals.

To study the low-energy behavior of the system, we select on this circle the state of phase  $\phi = 0$

$$|\psi_0^{\text{BCS}}\rangle = \prod_{\mathbf{k}} \left( U_{\mathbf{k}}^0 - V_{\mathbf{k}}^0 \hat{a}_{\mathbf{k}\uparrow}^\dagger \hat{a}_{-\mathbf{k}\downarrow}^\dagger \right) |0\rangle \quad (6)$$

as the origin of our expansion. In this symmetry-breaking state, we define the *gap* of the BCS theory

$$\Delta \equiv g_0 \langle \psi_0^{\text{BCS}} | \hat{\psi}_\downarrow \hat{\psi}_\uparrow | \psi_0^{\text{BCS}} \rangle = -\frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} U_{\mathbf{k}}^0 V_{\mathbf{k}}^0, \quad (7)$$

This quantity replaces  $g_0$  or  $a$  as the most natural parameter characterizing the interaction strength in the BCS theory. It can be used to express the coefficients  $U_{\mathbf{k}}^0, V_{\mathbf{k}}^0$  in the form

$$V_{\mathbf{k}}^0 = \sqrt{\frac{1}{2} \left( 1 - \frac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \right)} \quad \text{and} \quad U_{\mathbf{k}}^0 = \sqrt{\frac{1}{2} \left( 1 + \frac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \right)} \quad (8)$$

with the energies

$$\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - \mu + \frac{g_0 \rho}{2} \quad (9)$$

$$\epsilon_{\mathbf{k}} = \sqrt{\Delta^2 + \xi_{\mathbf{k}}^2} \quad (10)$$

With the handful relation  $U_{\mathbf{k}}^0 V_{\mathbf{k}}^0 = \Delta/2\epsilon_{\mathbf{k}}$ , the gap equation (7) takes its more usual form:

$$\frac{1}{g_0} = -\frac{1}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} \frac{1}{2\epsilon_{\mathbf{k}}} \quad (11)$$

In the state (6), the average total density  $\rho$ , or average total number of particles  $\langle \hat{N} \rangle$  per unit volume, is given by

$$\rho \equiv \frac{\langle \hat{N} \rangle}{L^3} \equiv \frac{k_{\text{F}}^3}{3\pi^2} = \frac{2}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} (V_{\mathbf{k}}^0)^2 \quad (12)$$

where  $k_{\text{F}}$  is the Fermi wavenumber of the ideal gas of density  $\rho$ . Combined with Eq.(8) this leads to the BCS equation of state which relates  $\rho$  to  $\mu$  and  $\Delta$ .

Performing a Bogoliubov rotation on the particle creation and annihilation operators, one finally defines the creation and annihilation operators of fermionic excitations:

$$\hat{b}_{\mathbf{k}\uparrow} = U_{\mathbf{k}}^0 \hat{a}_{\mathbf{k}\uparrow} + V_{\mathbf{k}}^0 \hat{a}_{-\mathbf{k}\downarrow}^\dagger \quad (13)$$

$$\hat{b}_{-\mathbf{k}\downarrow} = -V_{\mathbf{k}}^0 \hat{a}_{\mathbf{k}\uparrow}^\dagger + U_{\mathbf{k}}^0 \hat{a}_{-\mathbf{k}\downarrow} \quad (14)$$

The BCS ground state (6) is the vacuum of these operators, that create  $\mathbf{k}\uparrow$  and  $-\mathbf{k}\downarrow$  quasiparticles of energy  $\epsilon_{\mathbf{k}}$ .

### C. Ansatz of moving pairs

We parameterize the fluctuations around the symmetry-breaking BCS ground state by the coherent state of quasiparticles:

$$|\psi\rangle = \mathcal{N}(t) \exp \left( \sum_{\mathbf{k}, \mathbf{q} \in \mathcal{D}} z_{\mathbf{k}+\mathbf{q}, \mathbf{k}} \hat{b}_{\mathbf{k}+\mathbf{q}\uparrow}^\dagger \hat{b}_{-\mathbf{k}\downarrow}^\dagger \right) |\psi_{\text{BCS}}^0\rangle \quad (15)$$

Contrarily to the BCS ground state (6) in which all the pairs are at rest this Ansatz includes pairs of quasiparticles with nonzero center-of-mass wavevector  $\mathbf{q}$ .

Refs.[35] and [36] explain how to apply the variational principle to the state (15). One cleverly introduces the variables

$$\beta_{\mathbf{k}, \mathbf{k}'} = - \left( z(1 + z^\dagger z)^{-1/2} \right)_{\mathbf{k}, \mathbf{k}'} \quad (16)$$

with  $z$  is the matrix  $(z_{\mathbf{k}, \mathbf{k}'} )_{\mathbf{k}, \mathbf{k}' \in \mathcal{D}}$ , so that the equations of motion take a Hamiltonian form

$$i\hbar \frac{d\beta_{\mathbf{k}', \mathbf{k}}}{dt} = \frac{\partial E}{\partial \beta_{\mathbf{k}', \mathbf{k}}^*} \quad (17)$$

$$-i\hbar \frac{d\beta_{\mathbf{k}', \mathbf{k}}^*}{dt} = \frac{\partial E}{\partial \beta_{\mathbf{k}', \mathbf{k}}} \quad (18)$$

One evaluates the associated classical Hamiltonian

$$E \equiv \langle \psi | \hat{H} | \psi \rangle \quad (19)$$

through Wick's theorem. Finally one obtains an elegant expression for the averages of the operators that are bilinear in  $\hat{b}_{\mathbf{k}\sigma}$  by performing a Schmidt decomposition of the matrix  $z$  and expressing the result in terms of the field  $\beta$

$$\kappa_{\mathbf{k}',\mathbf{k}} \equiv \langle \hat{b}_{-\mathbf{k}\downarrow} \hat{b}_{\mathbf{k}'\uparrow} \rangle = - \left( \beta(1 - \beta^\dagger \beta)^{1/2} \right)_{\mathbf{k}',\mathbf{k}} \quad (20)$$

$$\kappa_{\mathbf{k},\mathbf{k}'}^* \equiv \langle \hat{b}_{\mathbf{k}\uparrow}^\dagger \hat{b}_{-\mathbf{k}'\downarrow}^\dagger \rangle = - \left( \beta(1 - \beta^\dagger \beta)^{1/2} \right)_{\mathbf{k},\mathbf{k}'}^* \quad (21)$$

$$\rho_{\mathbf{k}',\mathbf{k}}^\uparrow \equiv \langle \hat{b}_{\mathbf{k}'\uparrow}^\dagger \hat{b}_{\mathbf{k}\uparrow} \rangle = (\beta \beta^\dagger)_{\mathbf{k},\mathbf{k}'} = \sum_{\mathbf{k}_1} \beta_{\mathbf{k}',\mathbf{k}_1}^* \beta_{\mathbf{k},\mathbf{k}_1} \quad (22)$$

$$\rho_{\mathbf{k},\mathbf{k}'}^\downarrow \equiv \langle \hat{b}_{-\mathbf{k}'\downarrow}^\dagger \hat{b}_{-\mathbf{k}\downarrow} \rangle = (\beta^\dagger \beta)_{\mathbf{k}',\mathbf{k}} = \sum_{\mathbf{k}_1} \beta_{\mathbf{k}_1,\mathbf{k}'}^* \beta_{\mathbf{k}_1,\mathbf{k}} \quad (23)$$

### III. QUADRATIC ORDER OF THE CLASSICAL HAMILTONIAN: NORMAL VARIABLES OF THE BOSONIC BRANCH AND PHONON OPERATORS

#### A. Linearized equations of motion

To linearize the equations of motion (17) and (18), we assume that the state (15) differs only slightly from the BCS ground state, so that

$$\forall \mathbf{k}, \mathbf{k}', z_{\mathbf{k},\mathbf{k}'} \ll 1 \quad \text{and} \quad \beta_{\mathbf{k},\mathbf{k}'} \ll 1, \quad (24)$$

We expand the energy functional in powers of the field  $\beta$

$$E = E_0 + E_2 + O(\beta^3) \quad (25)$$

where  $E_0 = \langle \psi_0^{\text{BCS}} | \hat{H} | \psi_0^{\text{BCS}} \rangle$  is a constant,  $E_2$  and  $E_3$  are respectively bilinear and trilinear in  $\beta$ . There are no linear terms in this expansion since the BCS ground state (corresponding to  $\beta = 0$ ) is a minimizer of  $E$ . As in the Random Phase Approximation (RPA) [19], the linearized equations of motion, which follow from (17) et (18) after replacement of  $E$  by  $E_2$ , are decoupled according to the total wavevector  $\mathbf{q}$ . For this reason, we rewrite the coordinates of the field  $\beta$  in the Anderson fashion, with the relative wavevector in the index and the center-of-mass one in the exponent

$$\beta_{\mathbf{k}}^{\mathbf{q}} \equiv \beta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} \quad (26)$$

In terms of the vectors  $\beta^{\mathbf{q}} = (\beta_{\mathbf{k}}^{\mathbf{q}})_{\mathbf{k} \in \mathcal{D}}$  and  $\bar{\beta}^{\mathbf{q}} = ((\beta_{\mathbf{k}}^{-\mathbf{q}})^*)_{\mathbf{k} \in \mathcal{D}}$ , the equations of motion take the matrix form

$$i\hbar \frac{d}{dt} \begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix} = \mathcal{L}^{\mathbf{q}} \begin{pmatrix} \beta^{\mathbf{q}} \\ \bar{\beta}^{\mathbf{q}} \end{pmatrix}, \quad (27)$$

with an evolution operator  $\mathcal{L}^{\mathbf{q}}$  that is symplectic

$$\sigma_z \mathcal{L}^{\mathbf{q}} \sigma_z = (\mathcal{L}^{\mathbf{q}})^\dagger \quad (28)$$

and particle-hole symmetric

$$\sigma_x \mathcal{L}^{\mathbf{q}} \sigma_x = -(\mathcal{L}^{\mathbf{q}})^* \quad (29)$$

We have introduced  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  in block notations.

We give an explicit expression of the evolution operator  $\mathcal{L}^{\mathbf{q}}$  in the sum-and-difference basis

$$y_{\mathbf{k}}^{\mathbf{q}} = \beta_{\mathbf{k}}^{\mathbf{q}} - \bar{\beta}_{\mathbf{k}}^{\mathbf{q}} \quad (30)$$

$$s_{\mathbf{k}}^{\mathbf{q}} = \beta_{\mathbf{k}}^{\mathbf{q}} + \bar{\beta}_{\mathbf{k}}^{\mathbf{q}} \quad (31)$$

We have

$$i\hbar \frac{dy_{\mathbf{k}}^{\mathbf{q}}}{dt} = \epsilon_{\mathbf{k}\mathbf{q}} s_{\mathbf{k}}^{\mathbf{q}} + \frac{g_0}{L^3} \sum_{\mathbf{k}' \in \mathcal{D}} \left( W_{\mathbf{k}\mathbf{q}}^- W_{\mathbf{k}'\mathbf{q}}^- + w_{\mathbf{k}\mathbf{q}}^+ w_{\mathbf{k}'\mathbf{q}}^+ \right) s_{\mathbf{k}'}^{\mathbf{q}} \quad (32)$$

$$i\hbar \frac{ds_{\mathbf{k}}^{\mathbf{q}}}{dt} = \epsilon_{\mathbf{k}\mathbf{q}} y_{\mathbf{k}}^{\mathbf{q}} + \frac{g_0}{L^3} \sum_{\mathbf{k}' \in \mathcal{D}} \left( W_{\mathbf{k}\mathbf{q}}^+ W_{\mathbf{k}'\mathbf{q}}^+ - w_{\mathbf{k}\mathbf{q}}^- w_{\mathbf{k}'\mathbf{q}}^- \right) y_{\mathbf{k}'}^{\mathbf{q}} \quad (33)$$

The coefficients  $W_{\mathbf{k},\mathbf{q}}^\pm$  and  $w_{\mathbf{k},\mathbf{q}}^\pm$  are ... combinations of the coefficients  $U_{\mathbf{k}}^0$  and  $V_{\mathbf{k}}^0$

$$W_{\mathbf{k}\mathbf{q}}^\pm = U_{\mathbf{k}+\mathbf{q}/2}^0 U_{\mathbf{k}-\mathbf{q}/2}^0 \pm V_{\mathbf{k}+\mathbf{q}/2}^0 V_{\mathbf{k}-\mathbf{q}/2}^0 \quad (34)$$

$$w_{\mathbf{k}\mathbf{q}}^\pm = U_{\mathbf{k}+\mathbf{q}/2}^0 V_{\mathbf{k}-\mathbf{q}/2}^0 \pm U_{\mathbf{k}-\mathbf{q}/2}^0 V_{\mathbf{k}+\mathbf{q}/2}^0 \quad (35)$$

and the energies  $\epsilon_{\mathbf{k}\mathbf{q}}$  are those of the continuum of two fermionic quasiparticles,

$$\epsilon_{\mathbf{k}\mathbf{q}} = \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2} \quad (36)$$

The equations of motion (32,33) contain two terms: first, an individual part that couples the amplitudes of same relative and center-of-mass wavevectors  $\mathbf{k}$  and  $\mathbf{q}$  and contains the trivial evolution of the operators  $\hat{b}_{\mathbf{k}\sigma}$  under the BCS Hamiltonian  $\hat{H}_{\text{BCS}} = E_0 + \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} \hat{b}_{\mathbf{k}\sigma}^\dagger \hat{b}_{\mathbf{k}\sigma}$ , and second, a collective part that couples the normal amplitudes  $y_{\mathbf{k}}^{\mathbf{q}}$  and  $s_{\mathbf{k}}^{\mathbf{q}}$  to collective amplitudes of same total wavevector  $\mathbf{q}$ . Our semi-classical equations of motion coincide with the quantum average in state (15) of the RPA equations [19]. This can be seen by using the definitions (13) and (14) to express Eqs.(78a-d) of Ref.[19] in terms of the quasiparticle operators and by neglecting the expectation values of the  $\hat{b}_{\mathbf{k}\sigma}^\dagger \hat{b}_{\mathbf{k}'\sigma}$  operators, which, by virtue of Eqs.(22) and (23), is justified at the linear order of the variational theory.

#### B. Branch of collective excitations

We now look for the eigenmodes of the linear system (32,33) with a positive energy  $\hbar\omega_{\mathbf{q}}$  below the continuum  $k \mapsto \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}$  of two fermionic excitations, which is the spectrum we would obtain by restricting ourselves to the individual part of the system (32,33)

$$0 < \hbar\omega_{\mathbf{q}} < \inf_{\mathbf{k}} (\epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}) \quad (37)$$

The eigenvalue problem associated to (32,33) is solved in full generality in Ref.[36]. We give here the main steps of the solution. We introduce the collective amplitudes

$$Y_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} W_{\mathbf{kq}}^+ y_{\mathbf{k}}^{\mathbf{q}} \quad S_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} W_{\mathbf{kq}}^- s_{\mathbf{k}}^{\mathbf{q}} \quad (38)$$

$$y_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} w_{\mathbf{kq}}^- y_{\mathbf{k}}^{\mathbf{q}} \quad s_{\mathbf{q}} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} w_{\mathbf{kq}}^+ s_{\mathbf{k}}^{\mathbf{q}} \quad (39)$$

and we solve the  $2 \times 2$  linear system (32,33) to express the unknowns  $y_{\mathbf{k}}^{\mathbf{q}}$  et  $s_{\mathbf{k}}^{\mathbf{q}}$  in terms of the collective amplitudes. Next, we replace the result in (38,39) and come up with the homogeneous system

$$\begin{pmatrix} \Sigma_{W^+W^+}^{\epsilon} - 1 & \Sigma_{W^+W^-}^{\omega} & -\Sigma_{W^+w^-}^{\epsilon} & \Sigma_{W^+w^+}^{\omega} \\ \Sigma_{W^-W^+}^{\omega} & \Sigma_{W^-W^-}^{\epsilon} - 1 & -\Sigma_{W^-w^-}^{\omega} & \Sigma_{W^-w^+}^{\epsilon} \\ \Sigma_{w^-W^+}^{\epsilon} & \Sigma_{w^-W^-}^{\omega} & -\Sigma_{w^-w^-}^{\epsilon} - 1 & \Sigma_{w^-w^+}^{\omega} \\ \Sigma_{w^+W^+}^{\omega} & \Sigma_{w^+W^-}^{\epsilon} & -\Sigma_{w^+w^-}^{\omega} & \Sigma_{w^+w^+}^{\epsilon} - 1 \end{pmatrix} \times \begin{pmatrix} Y_{\mathbf{q}} \\ S_{\mathbf{q}} \\ y_{\mathbf{q}} \\ s_{\mathbf{q}} \end{pmatrix} = 0 \quad (40)$$

where we introduced the notations

$$\Sigma_{ab}^{\epsilon} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} \frac{\epsilon_{\mathbf{kq}} a_{\mathbf{kq}} b_{\mathbf{kq}}}{(\hbar\omega_{\mathbf{q}})^2 - (\epsilon_{\mathbf{kq}})^2} \quad (41)$$

$$\Sigma_{ab}^{\omega} = \frac{g_0}{L^3} \sum_{\mathbf{k} \in \mathcal{D}} \frac{\hbar\omega_{\mathbf{q}} a_{\mathbf{kq}} b_{\mathbf{kq}}}{(\hbar\omega_{\mathbf{q}})^2 - (\epsilon_{\mathbf{kq}})^2} \quad (42)$$

with  $a$  and  $b$  that can be any of  $W^+, W^-, w^+, w^-$ . The system (40) simplifies in the continuous limit  $l \rightarrow 0$ . Since  $g_0 \rightarrow 0$  and  $w_{\mathbf{kq}}^{\pm} \underset{k \rightarrow +\infty}{=} O(1/k^2)$ , all the  $\Sigma$  tend to 0 in the third and fourth lines, and therefore we must have

$$y_{\mathbf{q}} = s_{\mathbf{q}} = 0 \quad (43)$$

Next we divide the first two lines of (40) by  $g_0$ . The gap equation (11) ensures that all the divided matrix elements have a finite nonzero limit. The system therefore reduces to its  $2 \times 2$  upper left block, which, in the thermodynamic limit, we write as:

$$\begin{pmatrix} I_{++}(\omega_{\mathbf{q}}, q) & \hbar\omega_{\mathbf{q}} I_{+-}(\omega_{\mathbf{q}}, q) \\ \hbar\omega_{\mathbf{q}} I_{+-}(\omega_{\mathbf{q}}, q) & I_{--}(\omega_{\mathbf{q}}, q) \end{pmatrix} \begin{pmatrix} Y_{\mathbf{q}} \\ S_{\mathbf{q}} \end{pmatrix} = 0 \quad (44)$$

where the integrals  $I$  depend on the eigenfrequency  $\omega_{\mathbf{q}}$  and on the wavevector  $q$ ,

$$I_{++}(\omega, q) = \int_{\mathbb{R}^3} d^3k \left[ \frac{\epsilon_{\mathbf{kq}} (W_{\mathbf{kq}}^+)^2}{(\hbar\omega)^2 - (\epsilon_{\mathbf{kq}})^2} + \frac{1}{2\epsilon_{\mathbf{k}}} \right] \quad (45)$$

$$I_{--}(\omega, q) = \int_{\mathbb{R}^3} d^3k \left[ \frac{\epsilon_{\mathbf{kq}} (W_{\mathbf{kq}}^-)^2}{(\hbar\omega)^2 - (\epsilon_{\mathbf{kq}})^2} + \frac{1}{2\epsilon_{\mathbf{k}}} \right] \quad (46)$$

$$I_{+-}(\omega, q) = \int_{\mathbb{R}^3} d^3k \frac{W_{\mathbf{kq}}^+ W_{\mathbf{kq}}^-}{(\hbar\omega)^2 - (\epsilon_{\mathbf{kq}})^2} \quad (47)$$

The implicit equation on the eigenfrequency reads

$$I_{++}(\omega_{\mathbf{q}}, q) I_{--}(\omega_{\mathbf{q}}, q) = \hbar^2 \omega_{\mathbf{q}}^2 [I_{+-}(\omega_{\mathbf{q}}, q)]^2 \quad (48)$$

The same equation (48) is obtained by several other approaches: the RPA [19], a Gaussian approximation of the action in a path integral framework [20, 23], a Green's functions approach associated with a diagrammatic approximation [21]. The conditions on  $q$  for the existence of a solution  $\omega_{\mathbf{q}}$  are discussed in Ref.[21], while the concavity of the spectrum is studied in Ref.[7]. Beyond those previous works, we construct here in the next section the quantum operators associated to the collective modes.

### C. Construction of the normal variables of the collective branch

#### 1. General case

Using the symmetries (28) and (29) of the evolution operator, we obtain the normal amplitudes exactly as in the bosonic case with the Bogoliubov theory [37]. The first step is to derive the eigenvector  $\vec{e}_+$  of energy  $\hbar\omega_{\mathbf{q}} > 0$

$$\vec{e}_+(\mathbf{q}) = \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (49)$$

To obtain the analytic expression of the coefficients  $M$  et  $N$  in the continuous limit  $l \rightarrow 0$ , we solve the system (32,33) setting  $d/dt \rightarrow -i\omega_{\mathbf{q}}$  and we use Eqs.(43) and (44) to eliminate the collective amplitudes. We have

$$M_{\mathbf{k}}^{\mathbf{q}} - N_{\mathbf{k}}^{\mathbf{q}} = \frac{2\Delta \left( \epsilon_{\mathbf{kq}} W_{\mathbf{kq}}^+ - W_{\mathbf{kq}}^- \frac{I_{++}(\omega_{\mathbf{q}}, q)}{I_{+-}(\omega_{\mathbf{q}}, q)} \right)}{\sqrt{\mathcal{N}_{\mathbf{q}}} (\hbar^2 \omega_{\mathbf{q}}^2 - \epsilon_{\mathbf{kq}}^2)} \quad (50)$$

$$M_{\mathbf{k}}^{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}} = \frac{2\Delta \left( \hbar^2 \omega_{\mathbf{q}}^2 W_{\mathbf{kq}}^+ - \epsilon_{\mathbf{kq}} W_{\mathbf{kq}}^- \frac{I_{++}(\omega_{\mathbf{q}}, q)}{I_{+-}(\omega_{\mathbf{q}}, q)} \right)}{\hbar\omega_{\mathbf{q}} \sqrt{\mathcal{N}_{\mathbf{q}}} (\hbar^2 \omega_{\mathbf{q}}^2 - \epsilon_{\mathbf{kq}}^2)} \quad (51)$$

where the dimensionless normalization constant  $\mathcal{N}_{\mathbf{q}}$  will be determined by Eq.(57). Important properties of  $M$  and  $N$  are their invariance by internal ( $\mathbf{k} \rightarrow -\mathbf{k}$ ) and external ( $\mathbf{q} \rightarrow -\mathbf{q}$ ) parity, where internal and external refers to the structure of the pairs:

$$M_{\mathbf{k}}^{\mathbf{q}} = M_{-\mathbf{k}}^{\mathbf{q}} = M_{\mathbf{k}}^{-\mathbf{q}} \quad (52)$$

$$N_{\mathbf{k}}^{\mathbf{q}} = N_{-\mathbf{k}}^{\mathbf{q}} = N_{\mathbf{k}}^{-\mathbf{q}} \quad (53)$$

This is a consequence of the parity invariance of the problem and of the  $s$ -wave pairing. Due to the symmetry (29), one can associate to the eigenvector  $\vec{e}_+$  another eigenvector  $\vec{e}_-$  of energy  $-\hbar\omega_{\mathbf{q}}$  by a multiplication by  $\sigma_x$ :

$$\vec{e}_-(\mathbf{q}) = \sigma_x \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} = \begin{pmatrix} N_{\mathbf{k}}^{\mathbf{q}} \\ M_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (54)$$

We now derive the dual vectors with which we will project the field  $\beta$  on the eigenmodes  $\vec{e}_+$  and  $\vec{e}_-$ . The symplectic

symmetry (28) ensures that they are obtained by a mere multiplication by  $\sigma_z$

$$\vec{d}_+(\mathbf{q}) = \sigma_z \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} = \begin{pmatrix} M_{\mathbf{k}}^{\mathbf{q}} \\ -N_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (55)$$

$$\vec{d}_-(\mathbf{q}) = -\sigma_z \begin{pmatrix} N_{\mathbf{k}}^{\mathbf{q}} \\ M_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} = \begin{pmatrix} -N_{\mathbf{k}}^{\mathbf{q}} \\ M_{\mathbf{k}}^{\mathbf{q}} \end{pmatrix}_{\mathbf{k} \in \mathcal{D}} \quad (56)$$

Finally the eigenvectors and their dual vectors are normalized by choosing  $\mathcal{N}_{\mathbf{q}}$  so that

$$\left(\vec{d}_{\pm}\right)^* \cdot \vec{e}_{\pm} = \sum_{\mathbf{k} \in \mathcal{D}} (M_{\mathbf{k}}^{\mathbf{q}})^2 - (N_{\mathbf{k}}^{\mathbf{q}})^2 = 1 \quad (57)$$

To obtain the amplitudes  $b_{\mathbf{q}}$  of the collective modes, we project the field  $\beta$ :

$$b_{\mathbf{q}} = \vec{d}_+(\mathbf{q}) \cdot \begin{pmatrix} \beta_{\mathbf{q}} \\ \bar{\beta}_{\mathbf{q}} \end{pmatrix} = \sum_{\mathbf{k} \in \mathcal{D}} M_{\mathbf{k}}^{\mathbf{q}} \beta_{\mathbf{k}}^{\mathbf{q}} - N_{\mathbf{k}}^{\mathbf{q}} (\beta_{\mathbf{k}}^{-\mathbf{q}})^* \quad (58)$$

$$b_{-\mathbf{q}}^* = \vec{d}_-(\mathbf{q}) \cdot \begin{pmatrix} \beta_{\mathbf{q}} \\ \bar{\beta}_{\mathbf{q}} \end{pmatrix} = \sum_{\mathbf{k} \in \mathcal{D}} -N_{\mathbf{k}}^{\mathbf{q}} \beta_{\mathbf{k}}^{\mathbf{q}} + M_{\mathbf{k}}^{\mathbf{q}} (\beta_{\mathbf{k}}^{-\mathbf{q}})^* \quad (59)$$

This is the first central result of this paper. The equality  $b_{\mathbf{q}}^* = (b_{\mathbf{q}})^*$  suggested by our notation is a consequence of the invariance by external parity (52–53).

saut de ligne Conversely, to express the classical field in terms of the phonon amplitudes, we expand it on the eigenvectors:

$$\begin{pmatrix} \beta_{\mathbf{q}} \\ \bar{\beta}_{\mathbf{q}} \end{pmatrix} = b_{\mathbf{q}} \vec{e}_+(\mathbf{q}) + b_{-\mathbf{q}}^* \vec{e}_-(\mathbf{q}) + \dots \quad (60)$$

where in the ellipsis  $\dots$  we omitted the component of the field on the other eigenmodes of total wavevector  $\mathbf{q}$ . One can show [36] that in the continuous limit those omitted modes are nothing else than the continuum  $\mathbf{k} \mapsto \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}$  of biexcitations of fermionic quasiparticles with center-of-mass wavevector  $\mathbf{q}$ . Projecting the vectorial equation (60), we obtain:

$$\beta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} = M_{\mathbf{k}}^{\mathbf{q}} b_{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}} b_{-\mathbf{q}}^* + \dots \quad (61)$$

$$\beta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}+\mathbf{q}/2}^* = N_{\mathbf{k}}^{\mathbf{q}} b_{\mathbf{q}} + M_{\mathbf{k}}^{\mathbf{q}} b_{-\mathbf{q}}^* + \dots \quad (62)$$

After quantization (see below Sec.III D),  $M$  and  $N$  appear in these expressions as the coefficients of a new Bogoliubov rotation rearranging the fermionic bilinear operators (20,21) into bosonic quasiparticle operators. It comes on top of the rotation induced by the  $U_{\mathbf{k}}^0$  and  $V_{\mathbf{k}}^0$  coefficients rearranging the particle operators into fermionic quasiparticle operators. This new rotation acts on the pairs of fermions, hence the two indices of  $M$  and  $N$ .

## 2. Long-wave limit

In the long-wave limit  $q \rightarrow 0$ , we recall the expansion of the energy  $\hbar\omega_{\mathbf{q}}$  obtained by Ref.[7]:

$$\hbar\omega_{\mathbf{q}} \underset{q \rightarrow 0}{=} \hbar c q \left[ 1 + \frac{\gamma}{8} \left( \frac{\hbar q}{mc} \right)^2 + O \left( \frac{\hbar q}{mc} \right)^4 \right]. \quad (63)$$

In this expression,  $c$  is the speed of sound, derived, as in any superfluid, from the equation of state via the hydrodynamic formula

$$mc^2 = \rho \frac{d\mu}{d\rho}, \quad (64)$$

where the derivative with respect to  $\rho$  is taken for a fixed scattering length  $a$ . The coefficient  $\gamma$  of the cubic order of the expansion (63) is a rational fraction [7] of the variables

$$x = \frac{\Delta}{\mu} \quad \text{and} \quad y = \frac{d\Delta}{d\mu} = \frac{\int_{\mathbb{R}^3} d^3 k \frac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}}{\int_{\mathbb{R}^3} d^3 k \frac{\Delta}{\epsilon_{\mathbf{k}}^2}} \quad (65)$$

Note that  $y$  can be related to  $x$  by the use of the BCS equation of state (12) in the last equality of (65).

The expansion of the sum and the difference (50) and (51) of  $M_{\mathbf{k}}^{\mathbf{q}}$  and  $N_{\mathbf{k}}^{\mathbf{q}}$  then read:

$$\sqrt{\mathcal{N}_{\mathbf{q}}}(M_{\mathbf{k}}^{\mathbf{q}} - N_{\mathbf{k}}^{\mathbf{q}}) = -\frac{\Delta}{\epsilon_{\mathbf{k}}} + O(q^2) \quad (66)$$

$$\sqrt{\mathcal{N}_{\mathbf{q}}}(M_{\mathbf{k}}^{\mathbf{q}} + N_{\mathbf{k}}^{\mathbf{q}}) = \frac{\hbar c q \epsilon_{\mathbf{k}}}{2\Delta} \frac{dW_{\mathbf{k}\mathbf{0}}^-}{d\mu} + O(q^3) \quad (67)$$

$$\mathcal{N}_{\mathbf{q}}/L^3 = \frac{\hbar c q}{2} \frac{d\rho}{d\mu} + O(q^3) \quad (68)$$

Note that to lowest order these expressions coincide with the coefficients of the zero-energy mode  $\vec{e}_n$  and of the anomalous mode  $\vec{e}_a$  of the zero-momentum subspace evolution operator  $\mathcal{L}^0$  obtained in Ref.[38].

## D. Quantization of the normal variables

To quantize the amplitudes of the bosonic modes obtained in our semi-classical approach, we remember that  $\kappa$  represents a quasiparticle pair operator  $\kappa_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} = \langle \hat{b}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{b}_{\mathbf{k}+\mathbf{q}/2\uparrow} \rangle$ , and that the field  $\beta$  is equal to  $-\kappa$  at the linear order of the small amplitude approximation (24). We therefore perform the substitution

$$\beta_{\mathbf{k}}^{\mathbf{q}} = \beta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2} \rightarrow -\hat{b}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{b}_{\mathbf{k}+\mathbf{q}/2\uparrow}. \quad (69)$$

which transforms the amplitude  $b_{\mathbf{q}}$  in a quantum operator:

$$\hat{b}_{\mathbf{q}} = - \sum_{\mathbf{k} \in \mathcal{D}} \left( M_{\mathbf{k}}^{\mathbf{q}} \hat{b}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \hat{b}_{\mathbf{k}+\mathbf{q}/2\uparrow} - N_{\mathbf{k}}^{\mathbf{q}} \hat{b}_{\mathbf{k}-\mathbf{q}/2\uparrow}^{\dagger} \hat{b}_{-\mathbf{k}-\mathbf{q}/2\downarrow}^{\dagger} \right) \quad (70)$$

This intuitive action can in fact be justified by several arguments. First, the quantum operator (70) can also be obtained in the framework of the RPA by diagonalizing the homogeneous system for the quasiparticle pair creation and annihilation operators  $\hat{b}_{\mathbf{k}\uparrow}\hat{b}_{\mathbf{k}'\downarrow}$  and  $\hat{b}_{\mathbf{k}'\downarrow}\hat{b}_{\mathbf{k}\uparrow}^\dagger$ , and treating the operators  $\hat{b}_{\mathbf{k}\sigma}^\dagger\hat{b}_{\mathbf{k}'\sigma}$ , whose dynamics is trivial in the RPA and whose average value is sub-leading in the variational theory, as source terms. Second, one can apply the quantization procedure described in chapter 11 of Ref.[35] where the field  $\beta$  is mapped to a bosonic field operator  $\mathbb{B}$  (not to be confused with the bosonic excitation operators  $\hat{b}_{\mathbf{q}}$ ) called a *bosonic image*. The expression of the bosonic image of a two-body fermionic operator such as  $\hat{b}_{-\mathbf{k}+\mathbf{q}/2\downarrow}\hat{b}_{\mathbf{k}+\mathbf{q}/2\uparrow}$  in terms of the field  $\mathbb{B}$ , which would provide an exact version of the substitution (69), is not simple in the general case since it involves an infinite series expansion in powers of  $\mathbb{B}$ . Fortunately, in the limit of a weakly excited gas, one can neglect the operator population of the bosonic images  $\mathbb{B}\mathbb{B}^\dagger$  and thus justify the substitution (69), where we assimilate the two-body fermionic operators and their bosonic images. Last, we highlight the bosonic nature of the operator  $\hat{b}_{\mathbf{q}}$  (70) when the gas is weakly excited. The commutator of  $\hat{b}_{\mathbf{q}}$  and  $\hat{b}_{\mathbf{q}}^\dagger$  reads:

$$\begin{aligned} & [\hat{b}_{\mathbf{q}}, \hat{b}_{\mathbf{q}}^\dagger] - 1 = \\ & \sum_{\mathbf{k} \in \mathcal{D}} \left[ (N_{\mathbf{k}}^{\mathbf{q}})^2 \left( \hat{b}_{\mathbf{k}-\mathbf{q}/2\uparrow}^\dagger \hat{b}_{\mathbf{k}-\mathbf{q}/2\uparrow} + \hat{b}_{-\mathbf{k}-\mathbf{q}/2\downarrow}^\dagger \hat{b}_{-\mathbf{k}-\mathbf{q}/2\downarrow} \right) \right. \\ & \left. - (M_{\mathbf{k}}^{\mathbf{q}})^2 \left( \hat{b}_{\mathbf{k}+\mathbf{q}/2\uparrow}^\dagger \hat{b}_{\mathbf{k}+\mathbf{q}/2\uparrow} + \hat{b}_{-\mathbf{k}+\mathbf{q}/2\downarrow}^\dagger \hat{b}_{-\mathbf{k}+\mathbf{q}/2\downarrow} \right) \right] \quad (71) \end{aligned}$$

It differs from unity by fermionic quasiparticle population operators, that are exactly zero in the BCS ground state and of second order in the field  $z = O(\beta)$  in a quasiparticle coherent state such as (15).

#### IV. BEYOND THE QUADRATIC ORDER: INTERACTIONS BETWEEN PHONONS AND COMPARISON TO HYDRODYNAMICS

##### A. Three phonon processes

In this subsection, we study the processes involving three bosonic quasiparticles: both the  $2 \leftrightarrow 1$  Beliaev-Landau processes and the off-resonant  $3 \leftrightarrow 0$  processes. Our goal is to identify those processes in the expansion of the Hamiltonian in powers of the excitation field, and to extract the associated matrix elements. We shall use two independent theories and compare them. First, we will use our microscopic variational theory and expand the energy functional  $E$  (19) ... up to order three in  $\beta$ ,

$$E = E_0 + E_2 + E_3 + O(\beta^4) \quad (72)$$

and we will insert the expansion (60) of the field on the collective eigenmodes in the trilinear term  $E_3$ . We shall

perform this microscopic calculation in the case  $2 \leftrightarrow 1$  only. Second, we will use the quantum hydrodynamics of Landau and Khalatnikov [1]. This mesoscopic theory treats the pairs of fermions at large spatial scale as bosonic particles. It has the advantage of relying on the exact equation of state. It will be applied to the  $2 \leftrightarrow 1$  and  $3 \leftrightarrow 0$  processes. The comparison of the two theories will allow us to discuss the validity of quantum hydrodynamics.

##### 1. Microscopic approach

The idea behind the microscopic calculation of the three phonon coupling amplitudes is simple: we express the classical Hamiltonian (19) in terms of the amplitudes  $b_{\mathbf{q}}$  of the collective modes (see Subsec.III C), isolate the terms containing the creation and annihilation amplitudes  $b_{\mathbf{q}}^*$  and  $b_{\mathbf{q}}$  corresponding to the processes we study, and extract their coefficient. For the three phonon processes, we focus on the cubic terms gathered in  $E_3$ .

a. *General case* The cubic part of  $E$  can be written as

$$E_3 = \frac{g_0}{L^3} \sum_{\mathbf{k}, \mathbf{k}' \in \mathcal{D}} \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}} \left[ \left( \beta_{\mathbf{k}'+\mathbf{q}/2, \mathbf{k}'-\mathbf{q}/2} \rho_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}-\mathbf{q}/2}^\uparrow + \beta_{\mathbf{k}'-\mathbf{q}/2, \mathbf{k}'+\mathbf{q}/2} \rho_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}+\mathbf{q}/2}^\downarrow \right) + \text{c.c.} \right] \quad (73)$$

where we introduced the tensor

$$\mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}} = \frac{w_{\mathbf{k}\mathbf{q}}^+ W_{\mathbf{k}'\mathbf{q}}^- + w_{\mathbf{k}\mathbf{q}}^- W_{\mathbf{k}'\mathbf{q}}^+ + W_{\mathbf{k}\mathbf{q}}^+ w_{\mathbf{k}'\mathbf{q}}^- - W_{\mathbf{k}\mathbf{q}}^- w_{\mathbf{k}'\mathbf{q}}^+}{2} \quad (74)$$

We insert the expansion (60) in the expression (73) and obtain the following result [52]:

$$E_3 = \frac{mc^2}{(\rho L^3)^{1/2}} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathcal{D}} \delta_{\mathbf{q}_1+\mathbf{q}_2, \mathbf{q}_3} \times \left( \mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) b_{\mathbf{q}_1}^* b_{\mathbf{q}_2}^* b_{\mathbf{q}_3} + \text{c.c.} \right) + \dots \quad (75)$$

The vectors  $\mathbf{q}_1$ ,  $\mathbf{q}_2$  and  $\mathbf{q}_3$  are the three wavevectors involved in the Beliaev-Landau process, with  $\mathbf{q}_3$  the wavevector of the phonon which decays into phonons of wavevectors  $\mathbf{q}_1$  and  $\mathbf{q}_2$  or which originates from the merging of two such phonons. In the ellipsis ..., we omit the three-body processes involving non-bosonic excitations and the terms proportional to  $b_{\mathbf{q}_1} b_{\mathbf{q}_2} b_{\mathbf{q}_3}$  or  $b_{\mathbf{q}_1}^* b_{\mathbf{q}_2}^* b_{\mathbf{q}_3}^*$  describing the off-resonant  $3 \leftrightarrow 0$  processes which we do not study with the microscopic theory. We have factorized the quantity  $mc^2/(\rho L^3)^{1/2}$  and defined a dimensionless coupling amplitude  $\mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$  of the process  $b_{\mathbf{q}_1}^* b_{\mathbf{q}_2} b_{\mathbf{q}_3}$ , which is finite and nonzero in the thermody-



dynamic limit:

$$\begin{aligned} \frac{mc^2}{(\rho L^3)^{1/2}} \mathcal{A}_{\text{micro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) &= \frac{g_0}{L^3} \sum_{\mathbf{k}, \mathbf{k}' \in \mathcal{D}} \\ &\left[ \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}_1} M_{\mathbf{k}'}^{\mathbf{q}_1} \left( M_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_3} M_{\mathbf{k}-\mathbf{q}_3/2}^{\mathbf{q}_2} + N_{\mathbf{k}+\mathbf{q}_3/2}^{\mathbf{q}_2} N_{\mathbf{k}+\mathbf{q}_2/2}^{\mathbf{q}_3} \right) \right. \\ &+ \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}_1} N_{\mathbf{k}'}^{\mathbf{q}_1} \left( M_{\mathbf{k}+\mathbf{q}_3/2}^{\mathbf{q}_2} M_{\mathbf{k}+\mathbf{q}_2/2}^{\mathbf{q}_3} + N_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_3} N_{\mathbf{k}-\mathbf{q}_3/2}^{\mathbf{q}_2} \right) \\ &\left. + \mathcal{T}_{\mathbf{k}, \mathbf{k}'}^{\mathbf{q}_3} \left( N_{\mathbf{k}'}^{\mathbf{q}_3} N_{\mathbf{k}+\mathbf{q}_1/2}^{\mathbf{q}_2} M_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_1} + M_{\mathbf{k}'}^{\mathbf{q}_3} M_{\mathbf{k}+\mathbf{q}_1/2}^{\mathbf{q}_2} N_{\mathbf{k}-\mathbf{q}_2/2}^{\mathbf{q}_1} \right) \right. \\ &\quad \left. + \mathbf{q}_1 \leftrightarrow \mathbf{q}_2 \right] \quad (76) \end{aligned}$$

where the notation  $+\mathbf{q}_1 \leftrightarrow \mathbf{q}_2$  means that one must add to the terms already present in (76) those obtained by exchanging  $\mathbf{q}_1$  and  $\mathbf{q}_2$  while leaving  $\mathbf{q}_3$  unchanged.

*b. Long-wave limit* At low temperature, the low-wavevector phonons dominate the kinetics of the gas.

---


$$\begin{aligned} \mathcal{A}_{\text{micro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) &= \left( \frac{\hbar}{mc^2} \right)^{-1/2} \frac{1}{2^{3/2} \sqrt{\omega_1 \omega_2 \omega_3}} \left[ 2J(x, y) (\omega_1 + \omega_2 - \omega_3) \right. \\ &\left. + \left( \frac{\hbar}{mc^2} \right)^2 \{ A(x, y) (\omega_1 + \omega_2 - \omega_3) (\omega_1^2 + \omega_2^2 + \omega_3^2) + B(x, y) (\omega_3^3 - \omega_1^3 - \omega_2^3) + C(x, y) \omega_1 \omega_2 \omega_3 \} + O(\omega^5) \right] \quad (77) \end{aligned}$$

where the rational fractions  $A$ ,  $B$ ,  $C$  and  $J$  of the variables  $x$  and  $y$  defined in (65) are given in Appendix A. The denominator  $(\omega_1 \omega_2 \omega_3)^{1/2}$  leads to a divergence of the coupling amplitude in the long-wave limit whenever the process is not on the energy shell, that is when it does not obey the energy conservation relation

$$\omega_3 = \omega_1 + \omega_2. \quad (78)$$

One can check that the Bose-Einstein Condensate (BEC) limit ( $a \rightarrow 0^+$ ) of our result (77) coincides with the prediction of Bogoliubov theory [3, 39] for a weakly interacting gas of bosonic dimers [53].

*c. Resonant processes* To conclude this microscopic study, we evaluate the coupling amplitude on the energy shell, that is for processes satisfying the energy conservation (78). Such processes are allowed in the limit  $\omega_3 \rightarrow 0$  for a positive  $\gamma$  parameter only, that is when the dispersion relation  $q \mapsto \omega_{\mathbf{q}}$  is convex at low  $q$ . We collect the rational fractions  $B$  and  $C$  to form the thermodynamic quantity

$$2B(x, y) + \frac{2}{3}C(x, y) = 1 + \frac{\rho}{3} \frac{d^2 \mu}{d\rho^2} \left( \frac{d\mu}{d\rho} \right)^{-1} \equiv 1 + \Lambda_{\text{F}} \quad (79)$$

Its expression as a function of  $x$  and  $y$  is given in Appendix A and can also be obtained by differentiating twice the BCS equation of state (12) with respect to  $\mu$ .

This motivates a specific study of  $\mathcal{A}^{2\leftrightarrow 1}$  in the limit  $q_1, q_2, q_3 \rightarrow 0$ , where a comparison to hydrodynamics is meaningful. We perform the expansion of the expression (76) in this limit and for a continuous space,  $l \rightarrow 0$ , which allows us to integrate over the internal degrees of freedom  $\mathbf{k}$  and  $\mathbf{k}'$  of the pairs and to obtain an expression of the coupling amplitude that depends only of the external wavevectors  $\mathbf{q}_1$ ,  $\mathbf{q}_2$  and  $\mathbf{q}_3$ . The microscopic calculation is done in detail in Ref.[36]; the underlying fermionic nature of the problem makes it rather tedious. We give here only the final result, expressed in terms of the energies  $\hbar\omega_{\mathbf{q}}$  instead of the wavevectors, with the shorthand notation  $\omega_i \equiv \omega_{\mathbf{q}_i}$ ,  $i = 1, 2, 3$ . The two sets of variables are connected by the dispersion relation (63) and momentum conservation relations. In the limit  $\omega_i \rightarrow 0$ , we obtain:

---

This leads to an elegant expression of the on-shell coupling amplitude:

$$\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = 3(1 + \Lambda_{\text{F}}) \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32m^3 c^3}} + O(q_3^{7/2}) \quad (80)$$

The thermodynamic quantity  $1 + \Lambda_{\text{F}}$  is the only difference between this fermionic formula and its equivalent for a gas of weakly interacting bosons obtained by the Bogoliubov theory (see Eqs.(D8) and (D9) of Ref.[3]). It is plotted as a function of the interaction strength in the BEC-BCS crossover in Fig.1. It tends to 1 in the BEC limit ( $x \rightarrow 0^+$ ,  $y \sim -4/x$ ) like in a gas of weakly interacting bosons where  $\mu \propto \rho$ , and it is equal to 8/9 at unitarity ( $x = y$ ) and in the BCS limit ( $x \rightarrow 0$ ,  $y \rightarrow 0$ ) since in both cases  $\mu \propto \rho^{2/3}$ .

## 2. Hydrodynamic approach

We now compare our microscopic result (77) to the irrotational quantum hydrodynamics of Landau and Khalatnikov. This theory performs a large-scale description of the gas in terms of two hermitian quantum fields  $\hat{\rho}(\mathbf{r}, t)$  and  $\hat{\mathbf{v}}(\mathbf{r}, t)$ , directly neglecting the exponentially small density of fermionic quasi-particles at low  $T$ . The velocity field  $\hat{\mathbf{v}}(\mathbf{r}, t)$  is supposed to be irrotational and is written as the (discrete) gradient of the phase field oper-

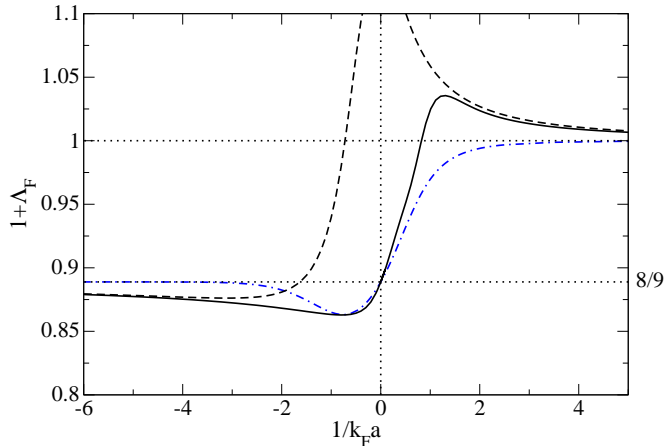


Figure 1: The factor  $1 + \Lambda_F$ , which is the Fermi gas equation-of-state dependent part in the  $2 \leftrightarrow 1$  phonon coupling amplitude (see equation (79)), is plotted as a function of the interaction strength  $1/k_F a$ . The black solid line is obtained from the equation of state measured in [16]. In the BEC ( $k_F a \rightarrow 0^+$ ) and BCS ( $k_F a \rightarrow 0^-$ ) limits the dashed black lines are the asymptotic behaviors given by the Lee-Huang-Yang corrections [40–42] (that include the first correction to the mean field energy, scaling as  $(k_F a)^2$  on the BCS side and as  $\rho a \sqrt{\rho a^3}$  on the BEC side). For comparison the prediction of the BCS equation of state (12) is plotted as a dash-dotted blue line. Note that  $1 + \Lambda_F$  tends to 1 in the BEC limit, as for a weakly interacting Bose gas. It is equal to  $8/9$  in the unitary limit, and tends to the same value in the BCS limit, since in both cases  $\mu \propto \rho^{2/3}$ .

ator  $\hat{\phi}(\mathbf{r}, t)$ ,

$$\hat{\mathbf{v}}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \hat{\phi}(\mathbf{r}, t), \quad (81)$$

canonically conjugated to the density field operator  $\hat{\rho}(\mathbf{r}, t)$ :

$$[\hat{\rho}(\mathbf{r}, t), \hat{\phi}(\mathbf{r}', t)] = i \frac{\delta_{\mathbf{r}, \mathbf{r}'}}{l^3}. \quad (82)$$

The dynamics of these fields is governed by the hydrodynamic Hamiltonian

$$\hat{H}_{\text{hydro}} = l^3 \sum_{\mathbf{r}} \left[ \frac{\hbar^2}{2m} \nabla \hat{\phi} \cdot \hat{\rho} \nabla \hat{\phi} + e_{0,0}(\hat{\rho}) \right], \quad (83)$$

where  $e_{0,0}$  is the bare energy density, that shall be renormalized by the eigenmodes zero-point energy as described in [4] to give rise to the true energy density  $e_0$  in the ground state, related to the zero-temperature chemical potential by  $\mu = de_0/d\rho$ .

The procedure to follow is standard and similar to the microscopic approach. We linearize the equations of motion for weak spatial fluctuations of the density and phase fields:

$$\hat{\rho}(\mathbf{r}, t) = \hat{\rho}_0 + \delta\hat{\rho}(\mathbf{r}, t) \quad (84)$$

$$\hat{\phi}(\mathbf{r}, t) = \hat{\phi}_0(t) + \delta\hat{\phi}(\mathbf{r}, t) \quad (85)$$

where  $\hat{\rho}_0$  will be replaced by the mean density  $\rho$ . We then expand the fields on the eigenmodes of the linearized dynamics:

$$\delta\hat{\rho}(\mathbf{r}, t) = \frac{\rho^{1/2}}{L^{3/2}} \sum_{\mathbf{q} \in \mathcal{D}^*} \left( \frac{\hbar \mathbf{q}}{2mc} \right)^{1/2} (\hat{b}_{\mathbf{q}} + \hat{b}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q} \cdot \mathbf{r}} \quad (86)$$

$$\delta\hat{\phi}(\mathbf{r}, t) = \frac{-i}{\rho^{1/2} L^{3/2}} \sum_{\mathbf{q} \in \mathcal{D}^*} \left( \frac{mc}{2\hbar \mathbf{q}} \right)^{1/2} (\hat{b}_{\mathbf{q}} - \hat{b}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q} \cdot \mathbf{r}} \quad (87)$$

where the bosonic operators  $\hat{b}_{\mathbf{q}}$  are the hydrodynamic counterpart of those in equation (70). The corresponding bosonic excitations have a purely linear spectrum  $\hbar\omega_{\mathbf{q}}^{\text{hydro}} = \hbar c q$  where the sound velocity  $c$  at density  $\rho$  is still given by the hydrodynamic expression (64). Next, we insert in the cubic part of the Hamiltonian (83)

$$\hat{H}_{\text{hydro}}^{(3)} = l^3 \sum_{\mathbf{r}} \left[ \frac{\hbar^2}{2m} \nabla \delta\hat{\phi} \cdot \delta\hat{\rho} \nabla \delta\hat{\phi} + \frac{1}{6} \frac{d^2 \mu}{d\rho^2} (\delta\hat{\rho})^3 \right], \quad (88)$$

the modal expansions (86) and (87) to obtain

$$\begin{aligned} \hat{H}_{\text{hydro}}^{(3)} = & \frac{mc^2}{(\rho L^3)^{1/2}} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathcal{D}^*} \left[ \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3} \mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) \left( \hat{b}_{\mathbf{q}_1}^\dagger \hat{b}_{\mathbf{q}_2}^\dagger \hat{b}_{\mathbf{q}_3} + \text{h.c.} \right) \right. \\ & \left. + \delta_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3, 0} \mathcal{A}_{\text{hydro}}^{3 \leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \left( \hat{b}_{\mathbf{q}_1}^\dagger \hat{b}_{\mathbf{q}_2}^\dagger \hat{b}_{\mathbf{q}_3}^\dagger + \text{h.c.} \right) \right] \quad (89) \end{aligned}$$

where the coupling amplitudes of the  $2 \leftrightarrow 1$  and  $3 \leftrightarrow 0$

processes are given by

$$\mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32m^3 c^3}} (3\Lambda_F + u_{12} + u_{13} + u_{23}) \quad (90)$$

$$\mathcal{A}_{\text{hydro}}^{3\leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{1}{3} \sqrt{\frac{\hbar^3 q_1 q_2 q_3}{32m^3 c^3}} (3\Lambda_F + u_{12} + u_{13} + u_{23}) \quad (91)$$

in terms of the parameter  $\Lambda_F$  defined in equation (79) and of the cosine of the angle between the wave vectors  $\mathbf{q}_i$  and  $\mathbf{q}_j$ ,

$$u_{ij} = \frac{\mathbf{q}_i \cdot \mathbf{q}_j}{q_i q_j} \quad (92)$$

The  $2 \leftrightarrow 1$  amplitude clearly differs from the amplitude (77) obtained by the microscopic approach (in particular it does not diverge in the large wavelength limit). One can check however that both results agree when the energy is conserved, see the equations (80) and (90), since in the hydrodynamic theory on the energy shell the three wave vectors  $\mathbf{q}_1$ ,  $\mathbf{q}_2$  and  $\mathbf{q}_3$  are colinear in the same direction in order to have the equality  $|\mathbf{q}_1 + \mathbf{q}_2| = q_1 + q_2$  in the triangular inequality.

## B. Four-phonon processes

We now consider the  $2 \leftrightarrow 2$  four-phonon process. When the excitation branch  $q \mapsto \omega_{\mathbf{q}}$  is concave at low wavenumbers, this is the resonant process involving the minimal number of phonons because the  $1 \leftrightarrow 2$  and  $1 \leftrightarrow 3$  processes are now forbidden by energy conservation. Also, this process is more intriguing on a theoretical point of view since it involves virtual non-resonant  $1 \leftrightarrow 2$  or  $3 \leftrightarrow 0$  intermediate processes. In this case, the equivalence of hydrodynamics with the microscopic approach is not obvious, as hydrodynamics does not correctly describe the  $1 \leftrightarrow 2$  processes off-shell. In this section, we give the hydrodynamic prediction for the  $2 \leftrightarrow 2$  effective coupling amplitude that includes the virtual processes, then we validate the result with a microscopic model. Since the fermionic microscopic model would be quite cumbersome, we use a boson model with finite range interactions, such that the excitation branch is concave at low  $q$ .

### 1. Transition amplitude

We need to calculate the transition amplitude between an initial state of energy  $E_i$ , which is an arbitrary Fock state of bosonic quasi-particles,

$$|i\rangle = |(n_{\mathbf{q}})_{\mathbf{q} \in \mathcal{D}}) \quad (93)$$

and a final state of energy  $E_f$  where two phonons of wave vectors  $\mathbf{q}_1$  and  $\mathbf{q}_2$  were annihilated and replaced by phonons of wavevectors  $\mathbf{q}_3$  and  $\mathbf{q}_4$ :

$$|f\rangle = \frac{\hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2}}{\sqrt{n_{\mathbf{q}_1} n_{\mathbf{q}_2} (1 + n_{\mathbf{q}_3}) (1 + n_{\mathbf{q}_4})}} |i\rangle \quad (94)$$

Whatever the specific model, the Hamiltonian can be expanded as

$$\hat{H} = E_0 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \dots \quad (95)$$

where  $E_0$  is a constant,  $\hat{H}_2$  is the quasi-particle Hamiltonian and  $\hat{H}_3$ ,  $\hat{H}_4$  are the third order and fourth order terms.  $\hat{H}_3$  cannot directly couple  $|i\rangle$  to  $|f\rangle$ , so we calculate the coupling to second order in perturbation theory, which amounts to treating  $\hat{H}_4$  to first order and  $\hat{H}_3$  to second order to construct an effective Hamiltonian [43]:

$$\langle f | \hat{H}^{2\leftrightarrow 2, \text{eff}} | i \rangle = \langle f | \hat{H}_4 | i \rangle + \sum_{\lambda} \frac{\langle f | \hat{H}_3 | \lambda \rangle \langle \lambda | \hat{H}_3 | i \rangle}{E_i - E_{\lambda}} \equiv \mathcal{A}_{i \rightarrow f} \quad (96)$$

There exist 6 intermediate states  $|\lambda\rangle$ , labeled from I to VI, that can be accessed at zero temperature that is when all the modes  $\mathbf{q} \neq \mathbf{q}_1, \mathbf{q}_2$  are initially empty. These states correspond to the creation and reabsorption of a virtual phonon by a three-phonon non-resonant process. They are represented by the diagrams on the left part of figure 2, with the virtual intermediate phonon plotted as a dashed line. To these six intermediate states  $\lambda = \text{I-VI}$  correspond six other diagrams  $\lambda = \text{I'-VI'}$ , shown on the right part of figure 2, where the intermediate phonon has the same wave vector but is annihilated and recreated rather than being created then annihilated. These intermediate states exist only at nonzero temperature since the intermediate phonon must preexist in state  $|i\rangle$ .

### 2. Effective $2 \leftrightarrow 2$ coupling amplitude

The effective coupling amplitude  $\mathcal{A}^{2\leftrightarrow 2, \text{eff}}$  of the  $2 \leftrightarrow 2$  process is defined by the following writing of the effective Hamiltonian:

$$\hat{H}^{2\leftrightarrow 2, \text{eff}} = \frac{mc^2}{\rho L^3} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \in \mathcal{D}} \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} \times \mathcal{A}^{2\leftrightarrow 2, \text{eff}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2} \quad (97)$$

By construction, the matrix element of  $\hat{H}^{2\leftrightarrow 2, \text{eff}}$  between  $|i\rangle$  and  $|f\rangle$  is the transition amplitude  $\mathcal{A}_{i \rightarrow f}$ . We thus have the relation

$$\mathcal{A}_{i \rightarrow f} = \sqrt{n_{\mathbf{q}_1} n_{\mathbf{q}_2} (1 + n_{\mathbf{q}_3}) (1 + n_{\mathbf{q}_4})} \frac{4mc^2}{\rho L^3} \mathcal{A}^{2\leftrightarrow 2, \text{eff}} \quad (98)$$

where the factor 4 is a counting factor.

From now on we restrict to on-shell processes, such that

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \quad (99)$$

with the shorthand notation  $\omega_i = \omega_{\mathbf{q}_i}$ ,  $i = 1, 2, 3, 4$ . In this case, a simplification occurs between each of the  $\lambda = \text{I, II, III, IV, V, VI}$  diagrams and its  $\lambda' = \text{I', II', III', IV', V', VI'}$  counterpart on the right column

of figure 2, which formally reduces the problem to zero temperature. Each diagram and its counterpart have indeed opposite energy denominators, with numerators that differ only through the factor involving the occupation number of the intermediate phonon  $\mathbf{q}$ ,  $(1 + n_{\mathbf{q}})$  in the left column (where the intermediate phonon is first created) and  $n_{\mathbf{q}}$  in the right column (where the intermediate phonon is first annihilated). Taking I and I' as an example, one has  $E_i - E_{I'} = \omega_1 + \omega_2 - \omega_{\mathbf{q}_1 + \mathbf{q}_2}$  and  $E_i - E_I = \omega_{\mathbf{q}_3 + \mathbf{q}_4} - \omega_3 - \omega_4 = -(E_i - E_{I'})$ ; in the matrix element  $\langle f | \hat{H}_3 | I \rangle \langle I | \hat{H}_3 | i \rangle$  one has the factor  $(1 + n_{\mathbf{q}_1 + \mathbf{q}_2})$

$$\begin{aligned} \mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 2, \text{eff}}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) &= \mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 2, \text{dir}}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \\ &+ \frac{\mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2) \mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_3, \mathbf{q}_4; \mathbf{q}_1 + \mathbf{q}_2)}{\tilde{\omega}_1 + \tilde{\omega}_2 - \tilde{\omega}_{1+2}} + \frac{9 \mathcal{A}^{3 \leftrightarrow 0}(\mathbf{q}_3, \mathbf{q}_4, -\mathbf{q}_1 - \mathbf{q}_2) \mathcal{A}^{3 \leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2)}{-(\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_{1+2})} \\ &+ \frac{\mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_3, \mathbf{q}_1 - \mathbf{q}_3; \mathbf{q}_1) \mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_1 - \mathbf{q}_3, \mathbf{q}_2; \mathbf{q}_4)}{\tilde{\omega}_1 - \tilde{\omega}_3 - \tilde{\omega}_{1-3}} + \frac{\mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_4, \mathbf{q}_3 - \mathbf{q}_1; \mathbf{q}_2) \mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_3 - \mathbf{q}_1, \mathbf{q}_1; \mathbf{q}_3)}{\tilde{\omega}_3 - \tilde{\omega}_1 - \tilde{\omega}_{3-1}} \\ &+ \frac{\mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_4, \mathbf{q}_1 - \mathbf{q}_4; \mathbf{q}_1) \mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_1 - \mathbf{q}_4, \mathbf{q}_2; \mathbf{q}_3)}{\tilde{\omega}_1 - \tilde{\omega}_4 - \tilde{\omega}_{1-4}} + \frac{\mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_3, \mathbf{q}_4 - \mathbf{q}_1; \mathbf{q}_2) \mathcal{A}^{2 \leftrightarrow 1}(\mathbf{q}_4 - \mathbf{q}_1, \mathbf{q}_1; \mathbf{q}_4)}{\tilde{\omega}_4 - \tilde{\omega}_1 - \tilde{\omega}_{4-1}} \quad (100) \end{aligned}$$

where the energies were rescaled by  $mc^2$ ,

$$\tilde{\omega} \equiv \frac{\hbar\omega}{mc^2} \quad \text{with the notation} \quad \omega_{i \pm j} \equiv \omega_{\mathbf{q}_i \pm \mathbf{q}_j}. \quad (101)$$

Note that the coupling due to diagram VI (whose contribution is written here immediately after the one of diagram I) has a counting factor equal to 9.

### 3. Effective amplitude in hydrodynamics

At low temperature, we need the large wavelength limit of expression (100). If one wants to get it from quantum hydrodynamics, one must introduce in quantum hydrodynamics a correction grasping some element of microscopic physics. As we shall indeed explain in subsection IV B 4, the hydrodynamic excitation spectrum is purely linear, which leads to a vanishing of the energy denominators in (100), when the wave vectors are colinear, and to a divergent effective coupling amplitude. This is an artefact of hydrodynamics. In reality the spectrum has a nonzero  $\gamma$  curvature parameter, here  $\gamma < 0$ , so that the energy denominators do not vanish. For almost colinear wave vectors, it is then natural to regularize the coupling amplitude by replacing the hydrodynamic dispersion relation  $\hbar\omega_{\mathbf{q}}^{\text{hydro}} = \hbar c q$  with the expansion (63) as done by Landau and Khalatnikov [1]. We shall bring a microscopic justification to this: for colinear wave vectors, our bosonic microscopic model (see subsection IV B 4) gives a large wavelength equivalent of formula (100) that indeed agrees with the Landau-Khalatnikov modified hydrodynamics.

and in  $\langle f | \hat{H}_3 | I \rangle \langle I | \hat{H}_3 | i \rangle$  one has the factor  $n_{\mathbf{q}_1 + \mathbf{q}_2}$ . Collecting the diagrams by pairs, we obtain an effective coupling amplitude identical to the zero-temperature one, that is with  $n_{\mathbf{q}} = 0$  for all  $\mathbf{q} \neq \mathbf{q}_1, \mathbf{q}_2$ . In terms of the direct on-shell  $2 \leftrightarrow 2$  coupling amplitude  $\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 2, \text{dir}}$  (that is related to  $\hat{H}_4$  in the same way as  $\mathcal{A}^{2 \leftrightarrow 2, \text{eff}}$  is related to  $\hat{H}^{2 \leftrightarrow 2, \text{eff}}$ ) and of the amplitudes  $\mathcal{A}^{2 \leftrightarrow 1}$  and  $\mathcal{A}^{3 \leftrightarrow 0}$  introduced in section IV A, the effective on-shell amplitude can finally be written as follows:

First we determine the direct  $2 \leftrightarrow 2$  process amplitude from the quartic terms of the Hamiltonian:

$$\hat{H}_{\text{hydro}}^{(4)} = \frac{1}{24} \frac{d^3 \mu}{d\rho^3} l^3 \sum_{\mathbf{r}} \delta \rho^4 \quad (102)$$

As we did previously, we insert the expansions (86) and (87) into  $\hat{H}_{\text{hydro}}^{(4)}$  and we keep only the  $2 \leftrightarrow 2$  terms  $\hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2}$ :

$$\begin{aligned} \hat{H}_{\text{hydro}}^{2 \leftrightarrow 2, \text{dir}} &= \frac{mc^2}{\rho L^3} \sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \in \mathcal{D}} \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} \\ &\times \mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 2, \text{dir}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_1} \hat{b}_{\mathbf{q}_2} \quad (103) \end{aligned}$$

to obtain the rescaled direct  $2 \leftrightarrow 2$  coupling amplitude

$$\mathcal{A}_{\text{hydro}}^{2 \leftrightarrow 2, \text{dir}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) = \frac{\Sigma_{\text{F}}}{16} \sqrt{\frac{\hbar^4 q_1 q_2 q_3 q_4}{m^4 c^4}} \quad (104)$$

where we introduced

$$\Sigma_{\text{F}} \equiv \frac{\rho^3}{mc^2} \frac{d^3 \mu}{d\rho^3} \quad (105)$$

Second we combine the amplitude (104) with our previous expressions for the  $2 \leftrightarrow 1$  (90) and  $3 \leftrightarrow 0$  (91)

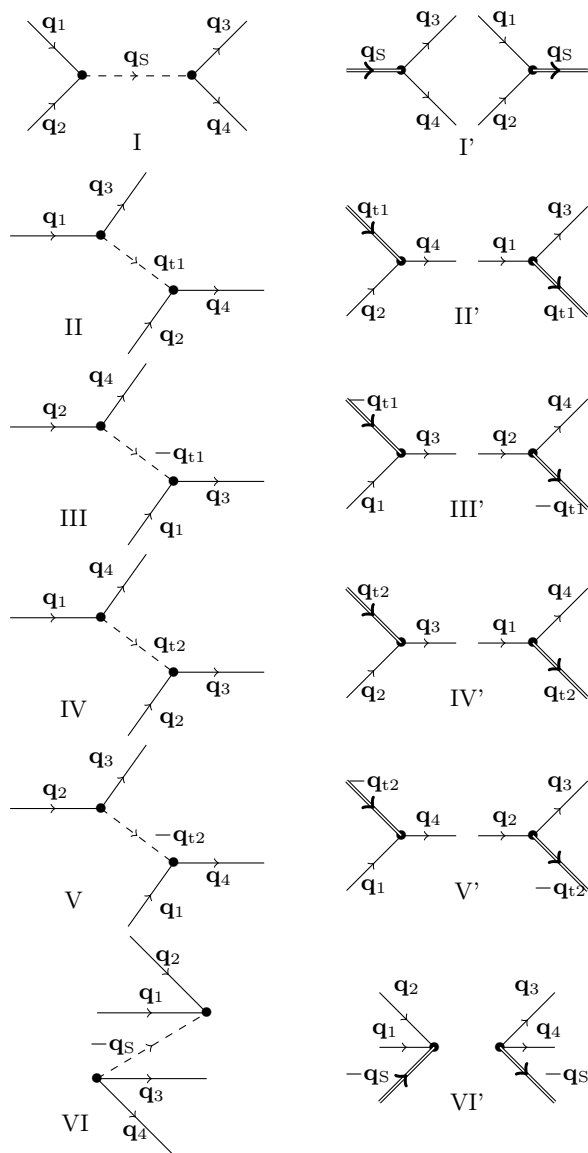


Figure 2: Second order diagrams for the 4-phonon process  $(\mathbf{q}_1, \mathbf{q}_2) \rightarrow (\mathbf{q}_3, \mathbf{q}_4)$  with  $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_3 + \mathbf{q}_4$ , considered as two successive three-phonon processes. The incoming wave vectors  $\mathbf{q}_1$  and  $\mathbf{q}_2$  and the emergent ones  $\mathbf{q}_3$  and  $\mathbf{q}_4$  are plotted as a solid line with an arrow. On the left column the diagrams include a virtual intermediate phonon, plotted as a dashed line with an arrow. On the right column they include a real intermediate phonon, plotted as a double line with an arrow. On any given row, the two diagrams have the same intermediate phonon: in I and I'  $\mathbf{q}_s = \mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_3 + \mathbf{q}_4$ , in II and II'  $\mathbf{q}_{t1} = \mathbf{q}_1 - \mathbf{q}_3 = \mathbf{q}_4 - \mathbf{q}_2$ , in III and III'  $-\mathbf{q}_{t1}$ , in IV and IV'  $\mathbf{q}_{t2} = \mathbf{q}_1 - \mathbf{q}_4 = \mathbf{q}_3 - \mathbf{q}_2$ , in V and V'  $-\mathbf{q}_{t2}$ , in VI and VI'  $-\mathbf{q}_s$ .

amplitudes to obtain

$$\mathcal{A}_{\text{hydro corr, OnS}}^{2 \leftrightarrow 2, \text{eff}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) = \frac{1}{16} \sqrt{\frac{\hbar^4 \omega_1 \omega_2 \omega_3 \omega_4}{m^4 c^8}} \left( \Sigma_{\text{F}} + \frac{(\omega_1 + \omega_2)^2 A_{1234} + \omega_{1+2}^2 B_{1234}}{(\omega_1 + \omega_2)^2 - \omega_{1+2}^2} + \frac{(\omega_1 - \omega_3)^2 A_{1324} + \omega_{1-3}^2 B_{1324}}{(\omega_1 - \omega_3)^2 - \omega_{1-3}^2} + \frac{(\omega_1 - \omega_4)^2 A_{1423} + \omega_{1-4}^2 B_{1423}}{(\omega_1 - \omega_4)^2 - \omega_{1-4}^2} \right) \quad (106)$$

where the index ‘‘corr’’ means that one goes beyond the hydrodynamic approximation for the dispersion relation in the denominators of expression (106) by using the cubic approximation (63), and we introduced the coefficients

$$A_{ijkl} = (3\Lambda_{\text{F}} + u_{ij})(1 + u_{kl}) + (3\Lambda_{\text{F}} + u_{kl})(1 + u_{ij}) + (1 + u_{ij})(1 + u_{kl}) \quad (107)$$

$$B_{ijkl} = (3\Lambda_{\text{F}} + u_{ij})(3\Lambda_{\text{F}} + u_{kl}) \quad (108)$$

#### 4. Effective amplitude in a weakly interacting Bose gas with nonzero range interactions

To study the  $2 \leftrightarrow 2$  process with a microscopic approach and understand how the divergence in hydrodynamics must be regularised, we shall not use the fermionic variational theory of section II, which would be very heavy to manipulate due to the internal degrees of freedom of the pairs. We rather use a bosonic model with a large enough interaction range  $b$  so that the Bogoliubov excitation branch is concave at low  $q$ . The Hamiltonian of the lattice model reads

$$\hat{H}_{\text{B}} = l^3 \sum_{\mathbf{r}} \hat{\psi}^\dagger(\mathbf{r}) \left( -\frac{\hbar^2}{2m_{\text{B}}} \Delta_{\mathbf{r}} \right) \hat{\psi}(\mathbf{r}) + \frac{l^6}{2} \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) \quad (109)$$

with the interaction potential

$$V(\mathbf{r}) = V_0 e^{-r^2/2b^2} \quad (110)$$

of Fourier transform

$$\tilde{V}(\mathbf{q}) = \tilde{V}_0 e^{-q^2 b^2/2} \quad \text{with} \quad \tilde{V}_0 = (2\pi)^{3/2} b^3 V_0 \quad (111)$$

The Bose gas is in the weakly interacting regime  $(\rho_{\text{B}} a_{\text{B}}^3)^{1/2} \ll 1$ , where  $\rho_{\text{B}} = N_{\text{B}}/L^3$  is the density of the bosons and the  $s$ -wave scattering length  $a_{\text{B}}$  is given by  $4\pi\hbar^2 a_{\text{B}}/m_{\text{B}} = \tilde{V}_0$  in the Born approximation. Following the Bogoliubov theory [44] in its  $U(1)$  symmetry preserving version [3, 37, 45, 46], we split the bosonic field operator as

$$\hat{\psi}(\mathbf{r}) = e^{i\hat{\theta}_0} \left[ \hat{n}_0^{1/2} \phi_0(\mathbf{r}) + \hat{\Lambda}(\mathbf{r}) \right] \quad (112)$$

where  $\hat{\theta}_0$  is the condensate phase operator,  $\hat{n}_0$  is the number of bosons in the condensate mode  $\phi_0(\mathbf{r}) = 1/L^{3/2}$  and the non-condensed field operator  $\hat{\Lambda}(\mathbf{r})$ , orthogonal to the condensate mode, conserves the number of particles. Within the subspace with fixed total number of bosons  $N_B$ , we eliminate  $\hat{n}_0$  through the relation

$$\hat{n}_0 = N_B - l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}) \quad (113)$$

To describe the  $2 \leftrightarrow 2$  processes, we expand the Hamiltonian in powers of  $\hat{\Lambda}$  up to order 4:

$$\hat{H}_B = \hat{H}_{B0} + \hat{H}_{B2} + \hat{H}_{B3} + \hat{H}_{B4} + \dots \quad (114)$$

We obtain [54]

$$\hat{H}_{B0} = \tilde{V}_0 \frac{N_B^2}{2L^3} \quad (115)$$

$$\hat{H}_{B2} = l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \left( -\frac{\hbar^2}{2m_B} \Delta_{\mathbf{r}} \right) \hat{\Lambda}(\mathbf{r}) + \rho_B l^6 \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \left( \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') + \frac{1}{2} \left[ \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}^\dagger(\mathbf{r}') + \hat{\Lambda}(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') \right] \right) \quad (116)$$

$$\hat{H}_{B3} = \frac{\rho_B}{2} l^6 \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \left( \left[ \hat{\Lambda}^\dagger(\mathbf{r}) + \hat{\Lambda}^\dagger(\mathbf{r}') \right] \hat{\Lambda}(\mathbf{r}') \hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}^\dagger(\mathbf{r}') \hat{\Lambda}^\dagger(\mathbf{r}) \left[ \hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}(\mathbf{r}') \right] \right) \quad (117)$$

$$\hat{H}_{B4} = \frac{l^6}{2} \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}^\dagger(\mathbf{r}') \hat{\Lambda}(\mathbf{r}') \hat{\Lambda}(\mathbf{r}) - \frac{\tilde{V}_0}{2L^3} \left( l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}) \right)^2 \quad (118)$$

$$- \frac{1}{L^3} \left( l^3 \sum_{\mathbf{r}} \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}) \right) \left( l^6 \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \left[ \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') + \frac{1}{2} \left[ \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}^\dagger(\mathbf{r}') + \hat{\Lambda}(\mathbf{r}) \hat{\Lambda}(\mathbf{r}') \right] \right] \right)$$

*a. 0th order* From  $\hat{H}_{B0}$  we get the equation of state to leading order:

$$\mu_B = \rho_B \tilde{V}_0 \equiv \frac{\hbar^2}{2m_B \xi^2} \quad (119)$$

where we introduced the healing length  $\xi$  and the chemical potential  $\mu_B$  of the Bose gas. We then apply relation (64) to obtain the sound velocity

$$m_B c_B^2 = \mu_B \quad (120)$$

*b. 2nd order*  $\hat{H}_{B2}$  can be diagonalised by a Bogoliubov transformation:

$$\hat{\Lambda}(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq 0} \left( U_{\mathbf{q}}^B \hat{b}_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{r}} + V_{\mathbf{q}}^B \hat{b}_{\mathbf{q}}^\dagger e^{-i\mathbf{q} \cdot \mathbf{r}} \right) \quad (121)$$

where the  $\hat{b}_{\mathbf{q}}$  are annihilation operators of bosonic quasi-particles and the amplitudes  $U_{\mathbf{q}}^B$  and  $V_{\mathbf{q}}^B$  are given by

$$U_{\mathbf{q}}^B + V_{\mathbf{q}}^B \equiv S_{\mathbf{q}} = \left( \frac{\frac{\hbar^2 q^2}{2m_B}}{\frac{\hbar^2 q^2}{2m_B} + 2\rho_B \tilde{V}(\mathbf{q})} \right)^{1/4} \quad (122)$$

$$U_{\mathbf{q}}^B - V_{\mathbf{q}}^B \equiv D_{\mathbf{q}} = \frac{1}{S_{\mathbf{q}}} \quad (123)$$

We have introduced the angular eigenfrequencies of the Bogoliubov quasi-particles

$$\hbar \omega_{\mathbf{q}}^B = \left[ \frac{\hbar^2 q^2}{2m_B} \left( \frac{\hbar^2 q^2}{2m_B} + 2\rho_B \tilde{V}(\mathbf{q}) \right) \right]^{1/2} \quad (124)$$

The dispersion relation  $q \mapsto \omega_{\mathbf{q}}$  is then concave in the vicinity of  $q = 0$  under the condition, assumed to be satisfied in what follows,

$$b > \xi \quad (125)$$

*c. 3rd order* We insert the modal expansion (121) in the cubic Hamiltonian (117) that we write in the form (89) with the constants  $m$ ,  $c$  and  $\rho$  replaced with  $m_B$ ,  $c_B$  and  $\rho_B$  respectively, and with coupling amplitudes now given by

$$\mathcal{A}_B^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3) = \frac{1}{4\tilde{V}_0} \left[ \tilde{V}(\mathbf{q}_3) S_3 (S_1 S_2 - D_1 D_2) \right. \\ \left. + \tilde{V}(\mathbf{q}_1) S_1 (S_2 S_3 + D_2 D_3) + \tilde{V}(\mathbf{q}_2) S_2 (S_1 S_3 + D_1 D_3) \right] \quad (126)$$

$$\mathcal{A}_B^{3 \leftrightarrow 0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{1}{12\tilde{V}_0} \left[ \tilde{V}(\mathbf{q}_1) S_1 (S_2 S_3 - D_2 D_3) \right. \\ \left. + \tilde{V}(\mathbf{q}_2) S_2 (S_1 S_3 - D_1 D_3) + \tilde{V}(\mathbf{q}_3) S_3 (S_1 S_2 - D_1 D_2) \right] \quad (127)$$

with the notations  $S_i \equiv S_{\mathbf{q}_i}$  and  $D_i \equiv D_{\mathbf{q}_i}$ .

*d. 4th order* We insert the modal expansion (121) in the quartic Hamiltonian (118) that we write in the form (103) with  $(m, c, \rho) \rightarrow (m_B, c_B, \rho_B)$  and a direct  $2 \leftrightarrow 2$

coupling amplitude given by

$$\begin{aligned} \mathcal{A}_B^{2\leftrightarrow 2,\text{dir}}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) &= \frac{1}{32\tilde{V}_0} \times \\ &\left[ \left( \tilde{V}(\mathbf{q}_1 + \mathbf{q}_2) + \tilde{V}(\mathbf{q}_3 + \mathbf{q}_4) \right) (S_1 S_2 - D_1 D_2) (S_3 S_4 - D_3 D_4) \right. \\ &+ \left( \tilde{V}(\mathbf{q}_3 - \mathbf{q}_1) + \tilde{V}(\mathbf{q}_2 - \mathbf{q}_4) \right) (S_1 S_3 + D_1 D_3) (S_2 S_4 + D_2 D_4) \\ &\left. + \left( \tilde{V}(\mathbf{q}_4 - \mathbf{q}_1) + \tilde{V}(\mathbf{q}_2 - \mathbf{q}_3) \right) (S_1 S_4 + D_1 D_4) (S_2 S_3 + D_2 D_3) \right] \end{aligned} \quad (128)$$

We considered here the general case where the  $\mathbf{q}_i$  and their opposite are two-by-two distinct, in which case only the first term of  $\hat{H}_{B4}$  contributes. The other terms of  $\hat{H}_{B4}$ , that originate from the expansion (113) of  $\hat{n}_0$  in powers of the number of non-condensed particles, contribute to the equation of state beyond Bogoliubov theory [47].

*e. Effective coupling amplitude* We obtain the on-shell effective coupling amplitude of the bosonic model  $\mathcal{A}_{B,\text{OnS}}^{2\leftrightarrow 2,\text{eff}}$  from the amplitudes (126,127,128) as prescribed by the equation (100). We then expand it in the limit of small wave vectors and express the result in terms of the angular frequencies  $\omega_{\mathbf{q}_i}^B$ ,  $i = 1, 2, 3, 4$  and  $\omega_{\mathbf{q}_1 + \mathbf{q}_2}^B$ ,  $\omega_{\mathbf{q}_1 - \mathbf{q}_3}^B$ ,  $\omega_{\mathbf{q}_1 - \mathbf{q}_4}^B$ . We then recover the hydrodynamic prediction (106) specialised to the bosonic equation of state (119), with  $\Lambda_F = \Sigma_F = 0$  and  $(m, c, \rho) \rightarrow (m_B, c_B, \rho_B)$ , provided that we use, in the case of colinear wave vectors, the prescription of Landau and Khalatnikov [1] described in the beginning of subsection IV B 3: this provides a microscopic justification to this prescription. The result was not obvious from the start, and it is due to a subtle cancellation between the large wavelength divergences of the direct coupling term (128) and of the second-order-perturbation-theory virtual coupling term. The details of the microscopic calculations are thus very different from the hydrodynamic ones, where the direct coupling term is zero for the considered equation of state (119).

## V. APPLICATION: PHONON DAMPING IN THE BEC-BCS CROSSOVER

### A. A general master-equation expression of the damping rates

To calculate the damping rate, we view the phonon mode of wave vector  $\mathbf{q}$  as a harmonic oscillator coupled to the reservoir formed by the other quasi-particle modes [48], assumed to be at thermal equilibrium at temperature  $T$ . We thus split the low-energy effective Hamiltonian as

$$\hat{H} = \hbar\omega_{\mathbf{q}} \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}} + \left( \sum_{\mathbf{q}' \neq \mathbf{q}} \hbar\omega_{\mathbf{q}'} \hat{b}_{\mathbf{q}'}^\dagger \hat{b}_{\mathbf{q}'} \right) + (\hat{R}^\dagger \hat{b}_{\mathbf{q}} + \hat{b}_{\mathbf{q}}^\dagger \hat{R}) + \dots \quad (129)$$

The first and the second terms, originating from  $\hat{H}_2$ , describe the free evolution of the mode  $\mathbf{q}$  and of the reservoir, respectively. The third term, originating from the part of  $\hat{H}_3$  or  $\hat{H}^{2\leftrightarrow 2,\text{eff}}$  involving  $\hat{b}_{\mathbf{q}}$  or  $\hat{b}_{\mathbf{q}}^\dagger$ , gives the coupling between the reservoir and the mode  $\mathbf{q}$ . The ellipsis ... includes higher order non-linear processes and the processes that do not involve the mode  $\mathbf{q}$ , as well as terms shifting the energy of the mode  $\mathbf{q}$  of the form  $\hat{R}' \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}}$  where  $\hat{R}'$  is an operator of the reservoir. In the Born-Markov approximation, one gets for the equation of motion of the mean number of excitations  $\langle \hat{n}_{\mathbf{q}} \rangle$  in the mode  $\mathbf{q}$  [43]:

$$\frac{d}{dt} \langle \hat{n}_{\mathbf{q}} \rangle = -\Gamma_{\mathbf{q}} (\langle \hat{n}_{\mathbf{q}} \rangle - \bar{n}_{\mathbf{q}}) \quad (130)$$

where  $\bar{n}_{\mathbf{q}}$  is the thermal equilibrium population of the mode:

$$\bar{n}_{\mathbf{q}} = \frac{1}{\exp\left(\frac{\hbar\omega_{\mathbf{q}}}{k_B T}\right) - 1} \quad (131)$$

The damping rate  $\Gamma_{\mathbf{q}}$  is given by

$$\Gamma_{\mathbf{q}} = \int_{-\infty}^{+\infty} \frac{dt}{\hbar^2} e^{-i\omega_{\mathbf{q}} t} \text{Tr}_R \left( [\hat{R}, \hat{R}^\dagger(t)] \sigma_R^{\text{eq}}(t) \right) \quad (132)$$

where  $\text{Tr}_R$  is the trace over the states of the reservoir,  $\sigma_R^{\text{eq}}$  is the thermal equilibrium density operator of the reservoir,  $[\hat{A}, \hat{B}]$  is the commutator of operators  $\hat{A}$  and  $\hat{B}$ , and the time evolution of the operator  $\hat{R}^\dagger(t)$  of the reservoir is calculated in the interaction picture with the Hamiltonian  $\sum_{\mathbf{q}' \neq \mathbf{q}} \hbar\omega_{\mathbf{q}'} \hat{b}_{\mathbf{q}'}^\dagger \hat{b}_{\mathbf{q}'}$ .

### B. Convex case: Beliaev-Landau damping

For a convex dispersion relation, the bilinear terms in  $\hat{R}$  lead to resonant processes, so they give the leading contribution to  $\Gamma_{\mathbf{q}}$  at low temperature. We split the contribution of the Beliaev process  $\mathbf{q} \leftrightarrow (\mathbf{q}', (\mathbf{q} - \mathbf{q}'))$  from the one of the Landau process  $(\mathbf{q}, \mathbf{q}') \leftrightarrow \mathbf{q} + \mathbf{q}'$ :

$$\begin{aligned} \Gamma_q^{\text{Bel}} &= \frac{(mc^2)^2}{2\pi^2 \hbar^2 \rho} \int_{\mathbb{R}^3} d^3 q' |\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(\mathbf{q}', \mathbf{q} - \mathbf{q}'; \mathbf{q})|^2 \\ &\times \delta(\omega_{\mathbf{q}'} + \omega_{\mathbf{q} - \mathbf{q}'} - \omega_{\mathbf{q}}) (1 + \bar{n}_{\mathbf{q} - \mathbf{q}'} + \bar{n}_{\mathbf{q}'}) \end{aligned} \quad (133)$$

$$\begin{aligned} \Gamma_q^{\text{Lan}} &= \frac{(mc^2)^2}{\pi^2 \hbar^2 \rho} \int_{\mathbb{R}^3} d^3 q' |\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 1}(\mathbf{q}', \mathbf{q}; \mathbf{q}' + \mathbf{q})|^2 \\ &\times \delta(\omega_{\mathbf{q} + \mathbf{q}'} - \omega_{\mathbf{q}'} - \omega_{\mathbf{q}}) (\bar{n}_{\mathbf{q}'} - \bar{n}_{\mathbf{q}' + \mathbf{q}}) \end{aligned} \quad (134)$$

and we use the expression (80) of the on-shell coupling amplitude  $2 \leftrightarrow 1$ . We integrate over the wave vector  $\mathbf{q}'$  in spherical coordinates of polar axis  $\mathbf{q}$  [55] to obtain the low temperature equivalents

$$\Gamma_q^{\text{Bel}} \underset{T \rightarrow 0}{\sim} \frac{9(1 + \Lambda_F)^2}{32\pi} \frac{mc^2}{\hbar\rho} \left(\frac{mc}{\hbar}\right)^3 \left(\frac{k_B T}{mc^2}\right)^5 \tilde{\Gamma}^{\text{Bel}}(\tilde{q}) \quad (135)$$

$$\Gamma_q^{\text{Lan}} \underset{T \rightarrow 0}{\sim} \frac{9(1 + \Lambda_F)^2}{32\pi} \frac{mc^2}{\hbar\rho} \left(\frac{mc}{\hbar}\right)^3 \left(\frac{k_B T}{mc^2}\right)^5 \tilde{\Gamma}^{\text{Lan}}(\tilde{q}) \quad (136)$$

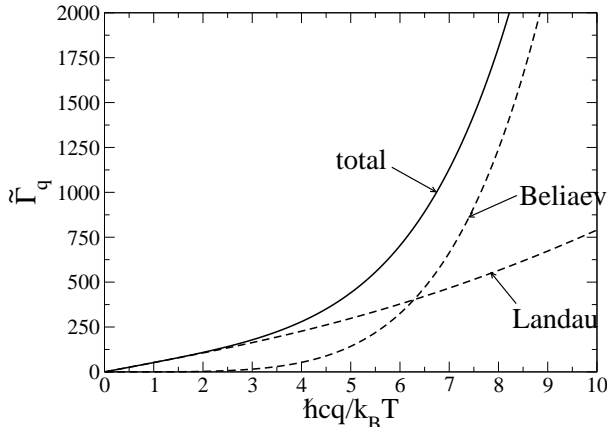


Figure 3: Low-temperature leading behavior of the total phonon damping rate as a function of the rescaled wave number  $\tilde{q} = \hbar cq/k_B T$  for a convex dispersion relation (solid line). This is the sum of the Beliaev (138) and Landau (139) rates (dashed lines). The rates are rescaled as in equations (135) and (136).

The wave numbers are here rescaled by the typical thermal wave number as follows:

$$\tilde{q} = \frac{\hbar cq}{k_B T} \quad (137)$$

and the  $T \rightarrow 0$  limit is taken at fixed  $\tilde{q}$ . The functions  $\tilde{\Gamma}^{\text{Bel}}$  and  $\tilde{\Gamma}^{\text{Lan}}$  are universal functions of  $\tilde{q}$ , that

$$\Gamma_{\mathbf{q}}^{2\leftrightarrow 2} = \frac{(mc^2)^2}{4\pi^5 \hbar^2 \rho^2} \int_{\mathbb{R}^6} d^3 q_2 d^3 q_3 |\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2, \text{eff}}(\mathbf{q}, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4)|^2 \delta(\omega_{\mathbf{q}_3} + \omega_{\mathbf{q}_4} - \omega_{\mathbf{q}_2} - \omega_{\mathbf{q}}) [\bar{n}_{\mathbf{q}_2}(1 + \bar{n}_{\mathbf{q}_3})(1 + \bar{n}_{\mathbf{q}_4}) - (1 + \bar{n}_{\mathbf{q}_2})\bar{n}_{\mathbf{q}_3}\bar{n}_{\mathbf{q}_4}] \quad (142)$$

where the vector  $\mathbf{q}_4$  is expressed in terms of the other wave vectors through momentum conservation:

$$\mathbf{q}_4 = \mathbf{q} + \mathbf{q}_2 - \mathbf{q}_3 \quad (143)$$

In what follows, we explain how to obtain a low-temperature equivalent for the rate  $\Gamma_{\mathbf{q}}^{2\leftrightarrow 2}$  at fixed  $\tilde{q} = \hbar cq/k_B T$ . We take as the polar  $z$  axis of the spherical coordinates the direction of  $\mathbf{q}$ ; the vectors  $\mathbf{q}_i$  then have coordinates  $(q_i, \theta_i, \phi_i)$ . We consider a temperature  $T$ , controlled by the small parameter

$$\epsilon \equiv \frac{k_B T}{mc^2} \ll 1, \quad (144)$$

so low that the typical wave numbers are much smaller than  $mc/\hbar$  and the bosonic branch is populated in its

can be expressed in terms of the Bose functions  $g_\alpha(z) = \sum_{n=1}^{+\infty} z^n/n^\alpha$ , also called polylogarithms  $\text{Li}_\alpha(z)$ , and of the Riemann  $\zeta$  function,  $\zeta(\alpha) = g_\alpha(1)$  [56],

$$\tilde{\Gamma}^{\text{Bel}}(\tilde{q}) = \frac{\tilde{q}^5}{30} - \frac{4\pi^4}{15}\tilde{q} + 48[\zeta(5) - g_5(e^{-\tilde{q}})] - 24\tilde{q}g_4(e^{-\tilde{q}}) + 4\tilde{q}^2[\zeta(3) - g_3(e^{-\tilde{q}})] \quad (138)$$

$$\tilde{\Gamma}^{\text{Lan}}(\tilde{q}) = \tilde{\Gamma}^{\text{Bel}}(\tilde{q}) - \frac{\tilde{q}^5}{30} + \frac{8\pi^4}{15}\tilde{q} \quad (139)$$

This leads to the following limiting behaviors:

$$\tilde{\Gamma}^{\text{Bel}} \underset{\tilde{q} \rightarrow 0}{=} \frac{\tilde{q}^4}{6} + \frac{\tilde{q}^6}{360} + O(\tilde{q}^8) \quad (140)$$

$$\tilde{\Gamma}^{\text{Bel}} \underset{\tilde{q} \rightarrow +\infty}{=} \frac{\tilde{q}^5}{30} + 4\zeta(3)\tilde{q}^2 - \frac{4\pi^4}{15}\tilde{q} + 48\zeta(5) + O(\tilde{q}^2 e^{-\tilde{q}}) \quad (141)$$

The corresponding ones for  $\tilde{\Gamma}^{\text{Lan}}$  can be trivially deduced from equation (139). The two damping rates and their sum are plotted as functions of  $q$  in figure 3.

### C. Concave case: Landau-Khalatnikov damping

For a concave dispersion relation, the mode  $\mathbf{q}$  is damped at sufficiently low temperature by the  $2 \leftrightarrow 2$  process only; in equation (132), we thus keep in  $\hat{R}$  the terms  $\hat{b}_{\mathbf{q}_3}^\dagger \hat{b}_{\mathbf{q}_4}^\dagger \hat{b}_{\mathbf{q}_2}$ , that originate from the quartic Hamiltonian (97). This leads to

quasi-linear part only. In this case, the coupling amplitude  $\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2}$ , that would be divergent at vanishing angles for a linear dispersion relation as already pointed out in section IV B 3, is extremely peaked around  $\theta_2 = \theta_3 = 0$  with a width of order  $\epsilon$  in  $\theta_2$  and  $\theta_3$  [57] and a height  $1/\epsilon^2$  times larger than the typical amplitude at nonzero angles, as we shall see. Using conservation of energy and momentum (143), one also finds that  $\theta_4 = O(\epsilon)$  over the width of the peak. We thus rescale the wave numbers as in equation (137) and the polar angles as

$$\tilde{\theta}_i = \frac{\theta_i}{\epsilon|\gamma|^{1/2}} \quad (145)$$

with  $\gamma < 0$  the curvature parameter (63), then we perform a Taylor expansion of the coupling amplitude (106)



for  $\epsilon \rightarrow 0$  at fixed rescaled quantities:

$$\mathcal{A}_{\text{OnS}}^{2\leftrightarrow 2}(\mathbf{q}, \mathbf{q}_2; \mathbf{q}_3, \mathbf{q}_4) \underset{\epsilon \rightarrow 0}{=} \left( \frac{3(1 + \Lambda_F)}{4} \right)^2 \frac{(\tilde{q}\tilde{q}_2\tilde{q}_3\tilde{q}_4)^{1/2}}{|\gamma|} \times \mathcal{A}_{\text{red}}^{2\leftrightarrow 2}(\tilde{q}, \tilde{q}_2, \tilde{q}_3, \tilde{\theta}_2, \tilde{\theta}_3) + O(\epsilon^2) \quad (146)$$

The  $\Sigma_F$ -dependent term of the  $\mathcal{A}^{2\leftrightarrow 2, \text{dir}}$  direct amplitude does not contribute at this order. This property, combined with the clever rescaling (145) of the polar angles, allowed us in equation (146) to pull out the factors  $1 + \Lambda_F$  and  $\gamma$  that depend on the interaction strength. This leads to a universal  $\mathcal{A}_{\text{red}}^{2\leftrightarrow 2}$  reduced amplitude:

$$\mathcal{A}_{\text{red}}^{2\leftrightarrow 2}(\tilde{q}, \tilde{q}_2, \tilde{q}_3, \tilde{\theta}_2, \tilde{\theta}_3) = \frac{1}{\tilde{q}\tilde{q}_2 \left( \frac{\tilde{\theta}_2^2}{(\tilde{q} + \tilde{q}_2)^2} + \frac{3}{4} \right)} - \frac{1}{\tilde{q}\tilde{q}_3 \left( \frac{\tilde{\theta}_3^2}{(\tilde{q} - \tilde{q}_3)^2} + \frac{3}{4} \right)} - \frac{1}{\tilde{q}(\tilde{q} + \tilde{q}_2 - \tilde{q}_3) \left( \frac{\tilde{\theta}_4^2}{(\tilde{q}_3 - \tilde{q}_2)^2} + \frac{3}{4} \right)} \quad (147)$$

The first, second and third terms in (147) originate from the second, third and fourth terms in (106). In the last two terms, we carefully distinguished the cases  $q > q_3$  and  $q < q_3$ ,  $q > q_4$  and  $q < q_4$ . Next, the implicit relation issued from energy conservation,

$$\tilde{q}_4 = \tilde{q} + \tilde{q}_2 - \tilde{q}_3 - \frac{\epsilon^2 |\gamma|}{8} (\tilde{q}^3 + \tilde{q}_2^3 - \tilde{q}_3^3 - \tilde{q}_4^3) + O(\epsilon^4) \quad (148)$$

is iterated once and combined with a spherical geometry calculus projecting relation (143) over  $\mathbf{q}$ . This gives the following expression for  $\tilde{\theta}_4$ :

$$\tilde{\theta}_4^2 = \frac{\tilde{q}_2 \tilde{\theta}_2^2 - \tilde{q}_3 \tilde{\theta}_3^2 - \frac{1}{4} [\tilde{q}^3 + \tilde{q}_2^3 - \tilde{q}_3^3 - (\tilde{q} + \tilde{q}_2 - \tilde{q}_3)^3]}{\tilde{q} + \tilde{q}_2 - \tilde{q}_3} + O(\epsilon^2) \quad (149)$$

Note that  $\tilde{q}_3 < \tilde{q} + \tilde{q}_2$  according to (148). Also, due to the rotational invariance around  $\mathbf{q}$ , the integrand of (142) depends on the azimuthal angles only through their difference  $\phi \equiv \phi_2 - \phi_3$ .

The last step is to integrate the Dirac distribution ensuring energy conservation. This is conveniently done in a polar representation of the rescaled angles:

$$\tilde{\theta}_2 = R \cos \alpha \quad \tilde{\theta}_3 = R \sin \alpha \quad (150)$$

We also write the energy difference between the initial state and the final state as

$$\omega_{\mathbf{q}_3} + \omega_{\mathbf{q}_4} - \omega_{\mathbf{q}_2} - \omega_{\mathbf{q}} = \frac{mc^2}{\hbar} \frac{\epsilon^3 |\gamma|}{2} (uR^2 + v) + O(\epsilon^5) \quad (151)$$

with

$$u = \frac{\tilde{q}(\tilde{q}_3 \sin^2 \alpha - \tilde{q}_2 \cos^2 \alpha) + \tilde{q}_2 \tilde{q}_3 (1 - \sin 2\alpha \cos \phi)}{\tilde{q} + \tilde{q}_2 - \tilde{q}_3} \quad (152)$$

$$v = \frac{1}{4} [\tilde{q}^3 + \tilde{q}_2^3 - \tilde{q}_3^3 - (\tilde{q} + \tilde{q}_2 - \tilde{q}_3)^3] \quad (153)$$

In the form (151), the Dirac distribution is readily integrated over  $R$ . We can finally express the  $2 \leftrightarrow 2$  damping rate in terms of a universal function  $\tilde{\Gamma}^{2\leftrightarrow 2}$  of the rescaled wave number  $\tilde{q}$ :

$$\frac{\hbar \Gamma_{\mathbf{q}}^{2\leftrightarrow 2}}{mc^2} \underset{\epsilon \rightarrow 0}{\sim} \frac{81(1 + \Lambda_F)^4}{256\pi^4 |\gamma|} \left( \frac{k_B T}{mc^2} \right)^7 \left( \frac{mc}{\hbar \rho^{1/3}} \right)^6 \tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) \quad (154)$$

This is one of the central results of this paper. The function  $\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q})$  is given by a quadruple integral [58]

$$\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) = \int_0^\infty d\tilde{q}_2 \int_0^{\tilde{q} + \tilde{q}_2} d\tilde{q}_3 \frac{\tilde{q}\tilde{q}_2^3\tilde{q}_3^3(\tilde{q} + \tilde{q}_2 - \tilde{q}_3)}{|\gamma|} \frac{[1 + f(\tilde{q}_2)]f(\tilde{q}_3)f(\tilde{q} + \tilde{q}_2 - \tilde{q}_3)}{f(\tilde{q})} \times \int_0^{\pi/2} d\alpha \int_0^\pi d\phi \sin \alpha \cos \alpha Y\left(-\frac{v}{u}\right) \left| \frac{v}{u} \mathcal{A}_{\text{red}}^{2\leftrightarrow 2}\left(\tilde{q}, \tilde{q}_2, \tilde{q}_3, \left|\frac{v}{u}\right|^{1/2} \cos \alpha, \left|\frac{v}{u}\right|^{1/2} \sin \alpha\right) \right|^2 \quad (155)$$

where we introduced the Heaviside function  $Y(x \geq 0) = 1$ ,  $Y(x < 0) = 0$  and the reduced Bose function  $f(x) = 1/(e^x - 1)$ . The occupation numbers in (142) were rewritten using the property  $(1 + \tilde{n}_{\mathbf{q}_i})/\tilde{n}_{\mathbf{q}_i} = e^{\hbar\omega_{\mathbf{q}_i}/k_B T}$  and the conservation of energy. The function  $\tilde{\Gamma}^{2\leftrightarrow 2}$  is plotted in figure 4. Its low- and high- $\tilde{q}$  behaviors can be obtained

analytically:

$$\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) \underset{\tilde{q} \rightarrow 0}{=} \frac{16\pi^5}{135} \tilde{q}^3 + O(\tilde{q}^4) \quad (156)$$

$$\tilde{\Gamma}^{2\leftrightarrow 2}(\tilde{q}) \underset{\tilde{q} \rightarrow \infty}{=} \frac{16\pi\zeta(5)}{3} \tilde{q}^2 + O(\tilde{q}) \quad (157)$$

These limiting behaviors disagree with the results of Landau and Khalatnikov in reference [1] (see their equations

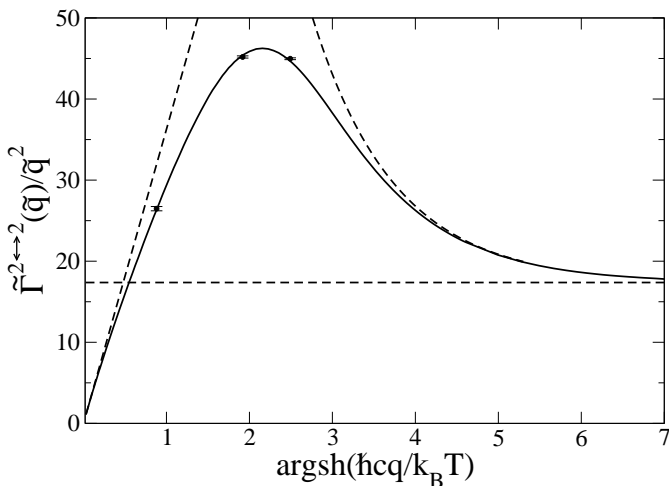


Figure 4:  $2 \leftrightarrow 2$  damping rate as a function of the rescaled wave number  $\tilde{q}$ . In order to make the limiting behaviors more apparent, the rate is divided by  $\tilde{q}^2$  and plotted as a function of  $\text{argsh}(\tilde{q}) = \ln(\tilde{q} + \sqrt{1 + \tilde{q}^2})$ . The oblique tangent at  $\tilde{q} = 0$  corresponds to the low- $\tilde{q}$  limiting behavior (see equation (157)). The horizontal straight line is the  $\tilde{q} \rightarrow \infty$  limit of  $\tilde{\Gamma}^{2 \leftrightarrow 2}/\tilde{q}^2$  (see equation (157)) and the asymptotic curve at large  $\tilde{q}$  is a fit of  $\tilde{\Gamma}^{2 \leftrightarrow 2}/\tilde{q}^2$  by an affine function of  $1/\tilde{q}$ . The black symbols are numerical calculations of the integral (142). We used a dispersion relation of the Bogoliubov form (172) restricted to an interval  $[0, q_{\text{max}}]$  over which its first order derivative is positive. The used dispersion relation coincides with (63) up to the third order in  $q$ . The discs are obtained by a linear extrapolation of the value of the integral for  $\epsilon = 0$ , and the error bars halfwidth is given by the difference with a quadratic extrapolation.

(7.6) et (7.12) in the version [49]), even for the order in  $\tilde{q}$  of the leading terms. This discrepancy is due to the fact that these authors have neglected in the coupling amplitude at low  $\tilde{q}$  and at high  $\tilde{q}$  the contribution of the diagrams II to V of figure 2. This is not justified as already noticed in [5]. We find that at leading order, the neglected diagrams actually interfere destructively with the diagram I, which makes the exact results (156,157) sub-leading with respect to the Landau and Khalatnikov predictions by two orders in  $\tilde{q}$ .

#### D. Phonon damping beyond hydrodynamics

In this section we concentrate on the zero temperature Unitary Fermi gas. Our goal is to calculate the phonon damping rates beyond the hydrodynamic result (141), which we rewrite in the form

$$\left(\Gamma_q^{\text{Bel}}\right)_{\text{hydro}}^{\text{Unitary gas}} = \left(\frac{2}{9\pi}\right) \left(\frac{mc^2}{\hbar}\right) \left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \frac{\tilde{q}^5}{30} \quad (158)$$

where we introduced the notations

$$\tilde{q} \equiv \frac{\hbar q}{mc} \quad \text{and} \quad \tilde{\omega}_q \equiv \frac{\hbar\omega_q}{mc^2} \quad (159)$$

To this aim, we first calculate the amplitude  $\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3)$  to first order in the curvature parameter  $\gamma$  of the excitation spectrum, and then we calculate the Beliaev rate from (133) including first order corrections in  $\gamma$  also in the excitation spectrum.

##### 1. Amplitude of a Beliaev process $2 \leftrightarrow 1$ for the unitary gas

In this subsection we calculate the amplitude  $\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_3)$  including the first correction in  $\gamma$  from the results of Son and Wingate using conformal invariance in an effective field theory [32]. Son and Wingate build a Lagrangian that includes the first correction to hydrodynamics, expressed in terms of a “phase field”  $\phi$ . For convenience we move to a Hamiltonian formalism, introducing the canonically conjugate field  $\Pi$ , representing (up to a sign) density fluctuations to obtain (here  $\hbar = m = 1$ )

$$\mathcal{H}_2^{(0)} = \frac{2\mu^{-1/2}}{15c_0}\Pi^2 + \frac{5}{2}c_0\mu^{3/2}\frac{1}{2}(\mathbf{grad}\phi)^2 \quad (160)$$

$$\mathcal{H}_3^{(0)} = \frac{4\mu^{-2}}{3(15c_0)^2}\Pi^3 - \frac{1}{2}\Pi(\mathbf{grad}\phi)^2 \quad (161)$$

$$\mathcal{H}_2^{(2)} = -\frac{16c_1\mu^{-3/2}}{(15c_0)^2}(\mathbf{grad}\Pi)^2 - c_2\mu^{1/2}(\Delta\phi)^2 \quad (162)$$

$$\mathcal{H}_3^{(2)} = -\frac{96c_1\mu^{-3}}{(15c_0)^3}\Pi(\mathbf{grad}\Pi)^2 + \frac{2c_2\mu^{-1}}{15c_0}\Pi(\Delta\phi)^2 \quad (163)$$

where the index 2 or 3 refers to the expansion order of the Hamiltonian in  $\Pi$  and  $\phi$  while the exponent (0) or (2) refers to the expansion order in powers of spatial gradients, the zeroth order being the standard hydrodynamics. Note that  $\Pi$  and  $\mathbf{grad}\phi$  are of the same order. At the hydrodynamic level, the Hamiltonian depends on a single constant  $c_0$  that must be determined from a microscopic theory. It is linked as follows to the Bertsch parameter  $\xi_B$  relating the chemical potential  $\mu$  to the Fermi wavenumber  $k_F$  or the total density  $\rho$ :

$$c_0 = \frac{2^{5/2}}{15\pi^2\xi_B^3} \quad \text{where} \quad \mu = \xi_B \frac{k_F^2}{2} \quad \text{and} \quad \rho = \frac{k_F^3}{3\pi^2} \quad (164)$$

The first correction to hydrodynamics involves two other constants  $c_1$  and  $c_2$  that, contrarily to  $c_0$ , have not been measured yet or even determined from a microscopic theory. By following the procedure already used in this paper, we first use the quadratized Hamiltonian  $\mathcal{H}_2^{(0)} + \mathcal{H}_2^{(2)}$  to determine the excitation spectrum as in [32]:

$$\begin{aligned} \omega_{\mathbf{q}} &= \left(\frac{2\mu}{3}\right)^{1/2} q \left[ 1 - \pi^2(2\xi_B)^{1/2} \left(c_1 + \frac{3}{2}c_2\right) \left(\frac{q}{k_F}\right)^2 \right. \\ &\quad \left. + o(q^2) \right] \\ &\equiv cq \left[ 1 + \frac{\gamma}{8} \left(\frac{\hbar q}{mc}\right)^2 + o(q^2) \right] \end{aligned} \quad (165)$$

and the modal expansion of the fields in the quantization volume  $L^3$ :

$$\Pi(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq 0} \Pi_{\mathbf{q}}(b_{\mathbf{q}} + b_{-\mathbf{q}}^*) e^{i\mathbf{q} \cdot \mathbf{r}} \quad (166)$$

$$\phi(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{q} \neq 0} \phi_{\mathbf{q}}(b_{\mathbf{q}} - b_{-\mathbf{q}}^*) e^{i\mathbf{q} \cdot \mathbf{r}} \quad (167)$$

with the amplitudes

$$\Pi_{\mathbf{q}} = \frac{1}{\sqrt{2}} \left( \frac{A_{\mathbf{q}}}{B_{\mathbf{q}}} \right)^{1/4} \quad (168)$$

$$\phi_{\mathbf{q}} = \frac{i}{\sqrt{2}} \left( \frac{A_{\mathbf{q}}}{B_{\mathbf{q}}} \right)^{-1/4} \quad (169)$$

$$\frac{A_{\mathbf{q}}}{B_{\mathbf{q}}} = \frac{75}{8} (c_0 \mu q)^2 \left[ 1 + \frac{8\mu^{-1}}{15c_0} \left( c_1 - \frac{3}{2} c_2 \right) q^2 + o(q^2) \right] \quad (170)$$

Indeed  $\begin{pmatrix} \Pi_{\mathbf{q}} \\ \phi_{\mathbf{q}} \end{pmatrix}$  is an eigenvector of a matrix  $\begin{pmatrix} 0 & -A_{\mathbf{q}} \\ B_{\mathbf{q}} & 0 \end{pmatrix}$  with the eigenvalue  $-i\omega_{\mathbf{q}}$ . Note that different linear combinations of the constants  $c_1$  and  $c_2$  appear in the spectrum (165) and in the modal amplitudes (170). By inserting the modal decomposition in the cubic Hamiltonian  $\mathcal{H}_3^{(0)} + \mathcal{H}_3^{(2)}$ , and isolating the Beliaev terms  $2 \leftrightarrow 1$  as we did in (89), we obtain the on-shell Beliaev coupling amplitude to first order beyond hydrodynamics:

$$\begin{aligned} \mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(\mathbf{k}, \mathbf{k}'; \mathbf{q}) &= -\frac{\sqrt{2}}{3} (\tilde{\omega}_q \tilde{\omega}_k \tilde{\omega}_{k'})^{1/2} \\ &\times \left[ 1 - \frac{7\gamma}{32} (\tilde{\omega}_q^2 + \tilde{\omega}_k^2 + \tilde{\omega}_{k'}^2) + o(\tilde{\omega}_q^2) \right] \end{aligned} \quad (171)$$

Remarkably the same linear combination  $\gamma = -\frac{64}{45c_0} (c_1 + \frac{3}{2}c_2)$  as in the spectrum appears in this result.

## 2. Phonon damping rate for the $T = 0$ unitary gas

Up to cubic order in  $q$ , we can rewrite the excitation spectrum in a Bogoliubov form

$$\tilde{\omega}_q = \tilde{q} \left( 1 + \frac{\gamma \tilde{q}^2}{4} \right)^{1/2} \quad (172)$$

This allows us to recycle the result (A14) in Appendix A of reference [50] to perform the angular integration in (133),

$$\begin{aligned} (\Gamma_{\mathbf{q}}^{\text{Bel}})^{\text{Unitary gas}} &= \left( \frac{mc^2}{\pi \hbar} \right) \left( \frac{mc}{\hbar \rho^{1/3}} \right)^3 \\ &\int_0^{\tilde{q}} d\tilde{k} |\mathcal{A}_{\text{OnS}}^{2 \leftrightarrow 1}(k, k'; q)|^2 \frac{\tilde{k}}{\tilde{q}} \frac{\tilde{\omega}_q - \tilde{\omega}_k}{[1 + \gamma(\tilde{\omega}_q - \tilde{\omega}_k)^2]^{1/2}} \end{aligned} \quad (173)$$

where we acknowledge the fact that the coupling amplitude (171), where  $\tilde{\omega}_{k'} = \tilde{\omega}_q - \tilde{\omega}_k$ , depends on the moduli of the wavevectors only. By performing the change of variable  $\kappa \equiv \tilde{k}/\tilde{q}$ , we express  $[(\Gamma_{\mathbf{q}}^{\text{Bel}})^{\text{Unitary gas}} \gamma^2/\tilde{q}]$  as an integral over  $\kappa$  between 0 and 1 of a function depending only on  $\kappa$  and on the small parameter  $\gamma \tilde{q}^2$ . By expanding this function to the sub-leading order, that is to third order in  $\gamma \tilde{q}^2$ , and performing the integral over  $\kappa$ , we obtain the provisional result

$$\begin{aligned} (\Gamma_{\mathbf{q}}^{\text{Bel}})_{\text{prov}}^{\text{Unitary gas}} &\stackrel{q \rightarrow 0}{=} \left( \frac{2}{9\pi} \right) \left( \frac{mc^2}{\hbar} \right) \left( \frac{mc}{\hbar \rho^{1/3}} \right)^3 \\ &\times \frac{\tilde{q}^5}{30} \left[ 1 - \frac{25}{112} \gamma \tilde{q}^2 + o(\tilde{q}^2) \right] \end{aligned} \quad (174)$$

The sub-leading term in our provisional result (174) is qualitatively different from the one in [9], in particular it predicts a reduction of the damping rate with respect to the hydrodynamic prediction at low wavevectors rather than an increase. The disagreement is unexpected as the calculation of reference [9] is performed with the same methodology as ours and in the same spirit. We think it is due to fact that the dependence of  $\mathcal{A}^{2 \leftrightarrow 1}$  on  $\gamma$  was finally neglected in [9] while it gives a contribution of the same order as the dependence on  $\gamma$  in the spectrum.

Before accepting the result (174), we should ask ourselves which kind of correction originates from the fact that the considered one-phonon state is unstable, of width  $\Gamma_{\mathbf{q}}/2$ , and as a consequence, that the energy is not exactly conserved in the one-to-two phonon decay process, contrarily to the constraint imposed by the Dirac delta function in equation (133). In order to estimate the order of magnitude of this effect, one can replace the Dirac delta function by a Lorentzian of half-width  $\Gamma/2$ ,

$$\delta(\omega_{\mathbf{q}'} + \omega_{\mathbf{q}-\mathbf{q}'} - \omega_{\mathbf{q}}) \rightarrow \frac{\Gamma_{\mathbf{q}}/2}{(\omega_{\mathbf{q}'} + \omega_{\mathbf{q}-\mathbf{q}'} - \omega_{\mathbf{q}})^2 + \Gamma_{\mathbf{q}}^2/4} \quad (175)$$

where  $\Gamma_{\mathbf{q}}$  can be identified to its leading term in  $q^5$ . One finds then that this effect introduces a correction to  $\Gamma_{\mathbf{q}}$  that is of order  $q^7$  in the limit  $q \rightarrow 0$  [59]. We go beyond this heuristic calculation in the appendix B. Using the resolvent of the Hamiltonian, we write at the Beliaev order a self consistent equation for the complex energy  $z_{\mathbf{q}} = \hbar\omega_{\mathbf{q}} - i\hbar\Gamma_{\mathbf{q}}/2$  of the  $\mathbf{q}$  phonon. If one replaces  $z_{\mathbf{q}}$  by its zeroth order approximation  $\hbar\omega_{\mathbf{q}} + i\eta$  in the implicit part of the equation, one recovers exactly (174). If one performs one self consistence iteration, that is if one replaces  $-2 \text{Im} z_{\mathbf{q}}/\hbar$  by its usual hydrodynamic approximation (158), one obtains the final result at the Beliaev order:

$$\begin{aligned} (\hbar\Gamma_{\mathbf{q}}^{\text{Bel}})^{\text{Unitary gas}} &= \left( \frac{2}{9\pi} \right) \left( \frac{mc^2}{\hbar} \right) \left( \frac{mc}{\hbar \rho^{1/3}} \right)^3 \\ &\times \frac{\tilde{q}^5}{30} \left[ 1 - \frac{25}{112} \gamma \tilde{q}^2 + \frac{4\sqrt{3}\zeta_B^{3/2}}{243\gamma} \tilde{q}^2 + o(\tilde{q}^2) \right] \end{aligned} \quad (176)$$

In order to conclude, and to make the result (176) rigorous, we have to verify that no other process of higher order than the Beliaev  $1 \rightarrow 2$  process, gives a contribution of order  $\tilde{q}^7$ . A natural candidate is the cascade process represented in figure 6, which combines two  $1 \rightarrow 2$  processes induced by the cubic Hamiltonian  $\hat{H}_3$ , giving rise to an effective coupling amplitude  $1 \rightarrow 3$ , which is of second order with a virtual phonon [60]. We estimate the corresponding change in the complex energy of the phonon  $\mathbf{q}$  by treating the effective  $1 \rightarrow 3$  coupling to second order in perturbation theory.

At first sight, the result is  $O(\tilde{q}^9)$  hence negligible. One has to integrate over two independent emitted phonon wave vectors, for example  $\mathbf{q}_2$  and  $\mathbf{q}_3$ , the third  $\mathbf{q}_1$  being imposed by momentum conservation. As the wave numbers  $q_i$  are of order  $q$ , this gives a factor  $q^6$ . It comes then the product of four matrix elements of  $\hat{H}_3$ , as the effective  $1 \rightarrow 3$  coupling contains two matrix elements and it is treated to the second order, which adds a global factor  $(q^{3/2})^4 = q^6$ . Finally, there are three energy denominators, one from second order perturbation theory and the other two from the effective  $1 \rightarrow 3$  coupling. As the phonon energies are of order  $\hbar c q$ , this provides a factor  $q^3$  in the denominator. The whole thing is  $O(\tilde{q}^9)$  as announced.

The previous reasoning however neglects the enhancement effect of the small denominators of order  $q^3$ , that occurs when the wave vectors  $\mathbf{q}_i$  are emitted forward, with small angles  $\theta_i$  with respect to  $\mathbf{q}$ , which already played a crucial role in section VC. In the limiting case where  $\mathbf{q}$  and the  $\mathbf{q}_i$  are all aligned in the same direction, momentum conservation imposes

$$q = q_1 + q_2 + q_3 \quad (177)$$

so that the energy difference  $\hbar\omega_{\mathbf{q}} - (\hbar\omega_{\mathbf{q}_1} + \hbar\omega_{\mathbf{q}_2} + \hbar\omega_{\mathbf{q}_3})$  is not of order  $q$  but rather of order  $q^3$ , taking into account the cubic term in the dispersion relation (63). This conclusion extends to all the energy denominators as far as the emission angles are  $O(\tilde{q})$ . One can check indeed that  $q_1 = |\mathbf{q} - \mathbf{q}_2 - \mathbf{q}_3|$  depends in relative value to second order in the emission angles  $\theta_2$  and  $\theta_3$  with coefficients of order  $\tilde{q}^0$ , in the same way as the true dispersion relation deviates in relative value from that of hydrodynamics to second order in  $\tilde{q}$ .

Let us then refine the naive estimation  $O(\tilde{q}^9)$  of the previous paragraph, by considering the integration over  $\mathbf{q}_2$  and  $\mathbf{q}_3$  within cones of angular aperture  $O(\tilde{q})$  around  $\mathbf{q}$ . Each cone occupies a solid angle  $O(\tilde{q}^2)$  so that we lose a factor  $q^4$  in the integration over polar angles. On the other hand we gain a factor  $q^{-2}$  for each energy denominator, that is a global factor  $q^{-6}$ . We then predict a change in the complex energy of order  $\tilde{q}^7$ , that is the same order as the correction to hydrodynamics appearing in equation (176).

In the appendix B, we then explicitly calculate the contribution to the damping rate  $\Gamma_{\mathbf{q}}$  that comes from the effective couplings  $1 \rightarrow 3$  treated to second order. We find that its expression, a rather tedious quintuple integral,

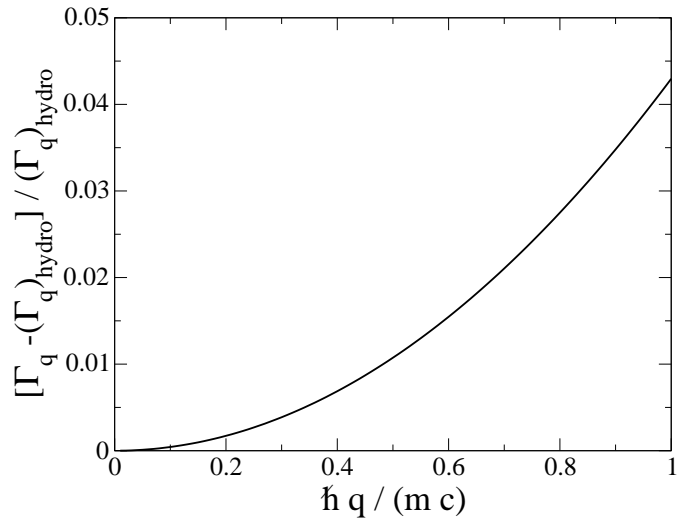


Figure 5: We show in relative value the first correction (179) to the hydrodynamic prediction for the phonon damping rate of the unitary gas at zero temperature. We took for the curvature parameter the RPA value  $\gamma \simeq 0,083769$  and the Bertsch parameter  $\xi_B = 0.376$  experimentally measured [17].

leads to

$$(\hbar\Gamma_{\mathbf{q}}^{1 \rightarrow 3})^{\text{unitary gas}} = \left(\frac{2}{9\pi}\right) \left(\frac{mc^2}{\hbar}\right) \left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \times \frac{\tilde{q}^5}{30} \left[ -\frac{2\sqrt{3}\xi_B^{3/2}}{567\gamma} \tilde{q}^2 + o(\tilde{q}^2) \right] \quad (178)$$

In the same appendix, we verify that no other process, of arbitrarily high order in  $\hat{H}_3$ ,  $\hat{H}_4$ , etc, contributes to the order  $\tilde{q}^7$  to  $\Gamma_{\mathbf{q}}$ , even accounting for the enhancement due to small denominators in  $q^3$ . We then can sum the contributions (176) and (178) to obtain

$$(\hbar\Gamma_{\mathbf{q}})^{\text{unitary gas}} = \left(\frac{2}{9\pi}\right) \left(\frac{mc^2}{\hbar}\right) \left(\frac{mc}{\hbar\rho^{1/3}}\right)^3 \times \frac{\tilde{q}^5}{30} \left[ 1 - \frac{25}{112}\gamma\tilde{q}^2 + \frac{22\sqrt{3}\xi_B^{3/2}}{1701\gamma} \tilde{q}^2 + o(\tilde{q}^2) \right] \quad (179)$$

an exact result to be counted among the successes of this paper.

## VI. CONCLUSION

We presented a complete study of interaction processes between phonons in cold Fermi gases at low temperature, for any interaction strength between fermions, therefore in both the concave and convex cases for the phonon dispersion relation at low wave number  $q$ . We clarified the conditions of validity of low-energy effective theories such as hydrodynamics by comparing them to a microscopic

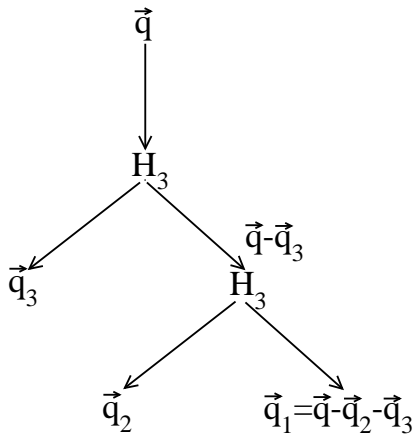


Figure 6: Effective coupling process, of second order in  $\hat{H}_3$  involving the emission of a virtual phonon. As one can deduce from simple power counting (see the text), in the case of a convex dispersion relation and when the phonons  $\mathbf{q}_1$ ,  $\mathbf{q}_2$  and  $\mathbf{q}_3$  are emitted within a small angle  $O(\hbar q/mc)$  with respect to  $\mathbf{q}$ , this process contributes to the order  $(\hbar q/mc)^7$  to the  $T = 0$  damping rate of a phonon of small wave vector  $\mathbf{q}$ . Its contribution has then to be added to the result (176) as we verify in the appendix B by an explicit calculation.

approach which takes into account the internal degrees of freedom of the pairs. Those effective theories correctly predict the phonon coupling amplitudes only on the energy shell.

One of the main contributions of this study is the microscopic derivation of the  $2 \leftrightarrow 2$  coupling amplitude. Indeed, its expression in second order perturbation theory includes non-resonant  $2 \leftrightarrow 1$  and  $3 \leftrightarrow 0$  processes. Since these processes can be resonant in the hydrodynamic treatment, where the excitation spectrum is linear, Landau and Khalatnikov had to introduce “by hand” a curvature to the dispersion relation to avoid the divergence of the coupling amplitude. At first sight this procedure is risky since, as we just said, quantum hydrodynamics does not predict correctly the coupling amplitudes for non-resonant (off-shell) processes. To provide a microscopic test of this procedure would be particularly cumbersome within our fermionic microscopic approach, so we rather considered a model of weakly interacting bosons with finite range interactions designed to have a concave dispersion relation. We found that although the microscopic expressions of the coupling amplitudes differ from that of hydrodynamics, when one sums up all the non-resonant three-phonon processes to the second order and the direct  $2 \leftrightarrow 2$  processes to first order, and one restricts on-shell with respect to the effective  $2 \leftrightarrow 2$  interaction, the Landau-Khalatnikov prescription and the microscopic result for the effective  $2 \leftrightarrow 2$  coupling amplitude agree.

As a second result of this paper, we gave universal formulas for the damping rates of both the  $2 \leftrightarrow 1$  and  $2 \leftrightarrow 2$  processes at low temperature. The most interesting

result is the analytic derivation, as a function of  $q$ , of the phonon damping rate in the concave case, given to leading order in temperature by the  $2 \leftrightarrow 2$  processes, see (154) and Fig. 4, that shall be the subject of a Letter [51]. In the limiting cases  $\hbar cq \ll k_B T$  and  $\hbar cq \gg k_B T$ , our result disagrees with the one of Ref. [1] and is sub-leading by two orders in  $\hbar cq/k_B T$ . This is due to the failed assumption in [1] that some interaction diagrams are negligible while in reality they destructively interfere with the supposedly leading diagram.

Finally we also calculated, at zero temperature and in the unitary limit, the first correction  $\propto q^7$  to the hydrodynamic prediction  $\propto q^5$  for the single phonon decay rate. Our calculation allows to refine the prediction of [9] by (i) the actual inclusion of the beyond-hydrodynamics expression of the coupling amplitude [32], (ii) the inclusion of a finite width  $\hbar\Gamma_{\mathbf{q}}$  in energy conservation in the Fermi golden rule, that is of a purely imaginary term of order  $q^5$  in the energy denominator of perturbation theory and (iii) the inclusion of other processes, of higher order than the Beliaev process, in particular of the effective coupling  $1 \rightarrow 3$  treated to second order.

All our predictions can be tested in state-of-art experiments on cold fermionic gases. In particular, a discussion of the observability of the Landau-Khalatnikov  $2 \rightarrow 2$  damping, using cold atoms trapped in a flat-bottom potential, is done in [51].

### Appendix A: $x$ and $y$ rational fractions in the three-phonon process amplitudes

The rational fractions of the variables  $x$  and  $y$  that appear in the expression (77) of the microscopic scattering amplitude are given by

$$J(x, y) = \frac{3xy}{2xy + 2} \quad (\text{A1})$$

$$A(x, y) = \frac{\sum_{i=0}^3 A_i(y)x^i}{360(x^3 + x)(y^2 + 1)^3} \quad (\text{A2})$$

$$B(x, y) = \frac{\sum_{i=0}^3 B_i(y)x^i}{18(x^3 + x)(y^2 + 1)^2} \quad (\text{A3})$$

$$C(x, y) = \frac{y(xy + 1)}{6(x^3 + x)(y^2 + 1)^2} \times [x^2(5y^2 + 3) - xy(y^2 - 3) + 2(y^2 + 2)] \quad (\text{A4})$$

with

$$A_0(y) = -20y(5y^4 + 5y^2 + 2) \quad (\text{A5})$$

$$A_1(y) = 2(10y^6 + 109y^4 + 63y^2 + 24) \quad (\text{A6})$$

$$A_2(y) = -y(53y^4 + 186y^2 + 13) \quad (\text{A7})$$

$$A_3(y) = 25y^6 + 116y^4 + 167y^2 + 36 \quad (\text{A8})$$

$$B_0(y) = -2y(3y^2 + 2) \quad (\text{A9})$$

$$B_1(y) = 3y^4 + 17y^2 + 8 \quad (\text{A10})$$

$$B_2(y) = -2y(y^2 + 4) \quad (\text{A11})$$

$$B_3(y) = 4y^4 + 11y^2 + 9 \quad (\text{A12})$$

Their values in the BEC limit  $x \rightarrow 0, y \sim -4/x$  are useful to connect the result (77) to its equivalent in the Bogoliubov theory:

$$A(x, y) \xrightarrow[y \sim -4/x]{x \rightarrow 0} \frac{1}{8} \quad B(x, y) \xrightarrow[y \sim -4/x]{x \rightarrow 0} \frac{1}{4} \quad (\text{A13})$$

$$J(x, y) \xrightarrow[y \sim -4/x]{x \rightarrow 0} 2 \quad C(x, y) \xrightarrow[y \sim -4/x]{x \rightarrow 0} \frac{3}{4}$$

For resonant processes, these rational fractions add up to give the thermodynamic coefficient

$$1 + \frac{\rho}{3} \frac{d^2 \mu}{d\rho^2} \left( \frac{d\mu}{d\rho} \right)^{-1} = \frac{\sum_{i=0}^3 D_i(y) x^i}{9(x^3 + x)(y^2 + 1)^2} \quad (\text{A14})$$

with

$$D_0(y) = -4y^3 \quad (\text{A15})$$

$$D_1(y) = 4(y^4 + 6y^2 + 2) \quad (\text{A16})$$

$$D_2(y) = -y(y^4 - 6y^2 + 5) \quad (\text{A17})$$

$$D_3(y) = 9y^4 + 14y^2 + 9 \quad (\text{A18})$$

## Appendix B: Contribution of the Beliaev and higher order processes to the $T = 0$ decay of the single phonon state

### 1. Presentation of the problem and link with the resolvent of the Hamiltonian

We consider the damping rate  $\Gamma_{\mathbf{q}}$  of a phonon with wave vector  $\mathbf{q}$  prepared at zero temperature in a spatially homogeneous gas of fermions. We assume that the phononic excitation branch  $q \mapsto \omega_{\mathbf{q}}$  is convex in vicinity of  $q = 0$ . The general problem is to determine the behavior of  $\Gamma_{\mathbf{q}}$  in the limit  $q \rightarrow 0$ , where one can use an effective low energy theory to describe the coupling among phonons. The dominant process is of course the  $1 \rightarrow 2$  Beliaev one. By introducing a dimensionless parameter  $\gamma > 0$  describing both the correction in  $q^3$  to the hydrodynamic linear excitation spectrum, and the first correction to the hydrodynamic coupling amplitude  $1 \rightarrow 2$ , we have obtained the provisional result (174) up to the sub-leading order  $q^7$ . In this appendix, we give

some details on the obtention of the final result at the Beliaev order (176), and we examine the possible contributions of order  $q^7$  of all the higher order processes, such as the cascade process  $1 \rightarrow 2 \rightarrow 3$  in figure 6, leading to the correction (178) and to the final result (179).

In the subspace with a fixed total momentum  $\hbar\mathbf{q}$ , the one-phonon state  $|\mathbf{q}\rangle$  is the only discrete state, as it is the only one completely characterized by a single wave vector value. It is an eigenstate of  $\hat{H}_2$ , the part of the Hamiltonian that is quadratic in the phonon creation  $\hat{b}^\dagger$  and annihilation  $\hat{b}$  operators. It is nevertheless coupled to the two-phonon, three-phonon, etc. continua by the rest of the Hamiltonian  $\hat{V} = \hat{H}_3 + \hat{H}_4 + \dots$ , that contains cubic, quartic, etc. terms, when written in the normal order for the  $\hat{b}$ .

As a consequence, the discrete state will in general get diluted in the continua, giving rise to a complex pole  $z_{\mathbf{q}}$  in the analytic continuation of the resolvent  $\hat{G}(z) = (z - \hat{H})^{-1}$  of the full Hamiltonian [43]. This pole can be written as

$$z_{\mathbf{q}} = \hbar\omega_{\mathbf{q}} - i \frac{\hbar\Gamma_{\mathbf{q}}}{2} \quad (\text{B1})$$

where  $\omega_{\mathbf{q}}$  is the angular eigenfrequency of the phonon and  $\Gamma_{\mathbf{q}}$  its damping rate [61].

### 2. Perturbative calculation and power counting

The  $q \rightarrow 0$  limit corresponds to the weak coupling limit. This is apparent in the quantum hydrodynamic theory as the modal expansion of the velocity and density fluctuations  $\hat{\mathbf{v}}$  and  $\delta\hat{\rho}$  involves coefficients that tend to zero as  $q^{1/2}$ , so that the matrix elements of  $\hat{H}_p$  between phonon Fock states behave as  $q^{p/2}$ :

$$\hat{H}_p \underset{q \rightarrow 0}{\approx} q^{p/2} \quad (\text{B2})$$

One can then attempt a perturbative calculation of  $\Gamma_{\mathbf{q}}$  by starting from the exact expression obtained by the projector method [43]

$$\langle \mathbf{q} | \hat{G}(z) | \mathbf{q} \rangle = \frac{1}{z - \langle \mathbf{q} | \hat{H}_{\text{eff}}(z) | \mathbf{q} \rangle} \quad (\text{B3})$$

then expanding in powers of  $\hat{V}$  the matrix element of the effective Hamiltonian in the state  $|\mathbf{q}\rangle$ :

$$\begin{aligned} \langle \mathbf{q} | H_{\text{eff}}(z) | \mathbf{q} \rangle &= \langle \mathbf{q} | \hat{H}_2 | \mathbf{q} \rangle + \langle \mathbf{q} | \hat{V} \hat{Q} \frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}_2\hat{Q}} \hat{Q} \hat{V} | \mathbf{q} \rangle \\ &+ \langle \mathbf{q} | \hat{V} \hat{Q} \frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}_2\hat{Q}} \hat{Q} \hat{V} \hat{Q} \frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}_2\hat{Q}} \hat{Q} \hat{V} | \mathbf{q} \rangle + \dots \end{aligned} \quad (\text{B4})$$

where  $\hat{Q} = 1 - |\mathbf{q}\rangle\langle\mathbf{q}|$  projects orthogonally to  $|\mathbf{q}\rangle$ . The result (176) corresponds to the second term on the right-hand side of the equation, the Beliaev term, in which

the contribution of  $\hat{V}$  reduces to that of  $\hat{H}_3$ . In all the following terms of order 3 in  $\hat{V}$  or higher,  $z$  can be approximated in the denominator by

$$z = \hbar\omega_{\mathbf{q}} + i\eta, \quad \eta \rightarrow 0^+, \quad (\text{B5})$$

the displacement of  $z$  by an imaginary part that is  $O(q^5)$  in these terms does not contribute to  $\Gamma_{\mathbf{q}}$  at the order  $q^7$ . The same conclusion does not hold for the Beliaev term, as we shall see in the sub-section B3.

Let us consider in (B4) the term  $\mathcal{T}_n$  of order  $n$  in  $\hat{V}$  and let us try to give an upper bound to its order in  $q$  as done in the simple reasoning below equation (176).

Taking (B2) into account, we must retain in the  $\hat{V}$  factors as many as possible contributions of  $\hat{H}_3$  and, if not, those of  $\hat{H}_4$ . As  $\hat{H}_3$  changes the parity of the phonon number, while  $\hat{H}_4$  preserves it, we will retain only  $\hat{H}_3$  if  $n$  is even, but it will be necessary to keep at least one factor  $\hat{H}_4$  if  $n$  is odd. The minimal number of independent wave vectors of the virtual phonons is  $n/2$  for  $n$  even,  $(n+1)/2$  for  $n$  odd, and the integration over each independent wave vector provides a factor  $q^3$ . Finally, one has to count a factor  $1/q$  for each of the  $n-1$  energy denominators. All together, this leads to

$$\mathcal{T}_n = O(q^{2n+1}) \quad \text{if } n \text{ even} \quad (\text{B6})$$

$$\mathcal{T}_n = O(q^{2n+3}) \quad \text{if } n \text{ odd} \quad (\text{B7})$$

One would then conclude with no surprise, that the processes of higher order than Beliaev, that is of order  $n \geq 3$  in  $\hat{V}$ , give a contribution to  $\Gamma_{\mathbf{q}}$  that is  $O(q^9)$  hence negligible.

The simple reasoning above however neglects the possibility for the energy denominators of being of order  $q^3$ . For this case to occur, it is necessary for the considered processes to be resonant. One must then keep in  $\hat{H}_3$  only the terms in  $\hat{b}^\dagger \hat{b}^\dagger \hat{b}$ , that form the raising part  $\hat{H}_3^{(+)}$  of the cubic Hamiltonian, or the terms in  $\hat{b}^\dagger \hat{b} \hat{b}$ , that form its lowering part  $\hat{H}_3^{(-)}$ . The other terms in  $\hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger$  and  $\hat{b} \hat{b} \hat{b}$  induce non-resonant processes.

In  $\hat{H}_4$ , we shall only keep terms  $\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}$ , that are the most favorable potentially resonant terms, as they add a single independent wave vector. In order that all the energy denominators are  $q^3$ , all the independent wave vectors must be emitted forward, within a narrow cone of angular aperture  $O(q)$  with respect to the direction of  $\mathbf{q}$ . The angular integration then pulls out a factor  $q^2$  for each independent wave vector, that is  $q^n$  for  $n$  even and  $q^{n+1}$  for  $n$  odd, but we gain a factor  $q^{-2(n-1)}$  thanks to  $n-1$  small denominators.

This leads to the refined upper bounds

$$\mathcal{T}_n = O(q^{n+3}) \quad \text{si } n \text{ even} \quad (\text{B8})$$

$$\mathcal{T}_n = O(q^{n+6}) \quad \text{si } n \text{ odd} \quad (\text{B9})$$

The conclusion from this discussion is that only the order 4 in  $\hat{V}$  has a chance to provide a correction in  $q^7$  to the result  $\Gamma_{\mathbf{q}}^{\text{Bel}}$  in equation (176). We analyze it in more detail

in the subsection B4. We can also confirm that the the order 2 in  $\hat{V}$  gives the leading contribution to  $\Gamma_{\mathbf{q}}$  in  $q^5$ . This however does not exonerate us from a rigorous study of its sub-leading contributions up to the order  $q^7$ , which is done in the sub-section B3.

### 3. Study to second order in $\hat{V}$

In this subsection we restrict the effective Hamiltonian in equation (B4) to second order in  $\hat{V}$  and calculate the resulting value of  $\Gamma_{\mathbf{q}}$  up to the order  $q^7$ .

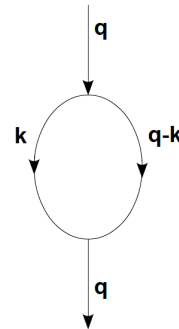


Figure 7: Diagram of order 2 in  $\hat{V}$  that contributes to the damping rate  $\Gamma_{\mathbf{q}}$  of the phonon  $\mathbf{q}$  to the order  $q^7$ . It represents the Beliaev process whose leading order is  $q^5$ . On the figure, each vertex represents an action of  $\hat{H}_3^{(\pm)}$ .

To this order in  $\hat{V}$  several different diagrams actually contribute, as  $\hat{V} = \hat{H}_3 + \hat{H}_4 + \hat{H}_5 + \dots$ . The leading order in  $q$  can be estimated by power counting. If the two factors  $\hat{V}$  are both equal to  $\hat{H}_3$ , one has to integrate over at least one independent wave vector and the leading order is  $q^5$  as in equation (B8). If they are both equal to  $\hat{H}_4$ , one has to integrate over at least two independent wave vectors and the leading order is  $q^9$  without the “small denominators” effect, but not any better with the “small denominators”. If one of the factors  $\hat{V}$  involves  $\hat{H}_p$ , with  $p \geq 5$ , the contribution is even smaller and negligible. We can then restrict to  $\hat{V} = \hat{H}_3$ , and furthermore to  $\hat{V} = \hat{H}_3^{(\pm)}$  as the non resonant terms in  $\hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger$  et  $\hat{b} \hat{b} \hat{b}$  of  $\hat{H}_3$  involve two independent wave vectors and contribute in  $O(q^8)$ .

There remains then a single diagram, the Beliaev one, represented in figure 7. Let us introduce the corresponding Beliaev self-energy, for any complex number  $z$  with  $\text{Im } z > 0$ :

$$\begin{aligned} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(z) &= \frac{1}{2} \sum_{\mathbf{k}} \frac{|\langle \mathbf{k}, \mathbf{q} - \mathbf{k} | \hat{H}_3^{(+)} | \mathbf{q} \rangle|^2}{z - (\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{q}-\mathbf{k}})} \\ &= \frac{2(mc^2)^2}{\rho} \int \frac{d^3k}{(2\pi)^3} \frac{[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow+1}(\mathbf{k}, \mathbf{q} - \mathbf{k}; \mathbf{q})]^2}{z - (\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{q}-\mathbf{k}})} \end{aligned} \quad (\text{B10})$$

where we considered the thermodynamic limit and we have used the form (89) of the hydrodynamic Hamiltonian

nian. To this order of approximation, the pole  $z_{\mathbf{q}}$  associated to the phonon  $\mathbf{q}$  is the solution of the implicit equation

$$z_{\mathbf{q}}^{\text{Bel}} = \epsilon_{\mathbf{q}} + \Delta\epsilon_{\mathbf{q}}^{\text{Bel}\downarrow}(z_{\mathbf{q}}^{\text{Bel}}) \quad (\text{B11})$$

where the downward arrow denotes the analytic continuation of the self-energy from the upper half-plane  $\text{Im } z > 0$  to the lower one  $\text{Im } z < 0$  through the branch cut on the real positive axis. Let us separate  $z_{\mathbf{q}}^{\text{Bel}}$  into the real and imaginary parts as in (B1) and let us expand the right-hand side in powers of  $\Gamma_{\mathbf{q}}$ :

$$z_{\mathbf{q}}^{\text{Bel}} = \epsilon_{\mathbf{q}} + \Delta\epsilon_{\mathbf{q}}^{\text{Bel}\downarrow}(\epsilon_{\mathbf{q}}) - \frac{i\hbar\Gamma_{\mathbf{q}}}{2} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}\downarrow}(\epsilon_{\mathbf{q}}) + O(\tilde{q}^9) \quad (\text{B12})$$

It's enough here to truncate the Taylor expansion to the order one; on the other hand the zeroth order is not enough [62]. The values of the analytic continuation and of its derivative in  $\epsilon_{\mathbf{q}}$  are obtained as the limits when  $\eta \rightarrow 0^+$  of the non-continued functions in  $z = \epsilon_{\mathbf{q}} + i\eta$ :

$$z_{\mathbf{q}}^{\text{Bel}} = \epsilon_{\mathbf{q}} + \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) - \frac{i\hbar\Gamma_{\mathbf{q}}}{2} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) + O(\tilde{q}^9) \quad (\text{B13})$$

The second term is the ordinary perturbative result. It leads to the damping rate  $\Gamma_{\mathbf{q},\text{pert}}^{\text{Bel}}$  given by the zero temperature limit of equation (133),

$$\frac{\hbar\Gamma_{\mathbf{q},\text{pert}}^{\text{Bel}}}{2} = -\text{Im} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) \quad (\text{B14})$$

and its value for the unitary gas has been calculated up to the order  $q^7$  in equation (174). Once the imaginary part of equation (B13) has been taken, we find however a non negligible correction to this order, resulting from a first self consistent iteration of  $z_{\mathbf{q}}$  in the implicit equation (B11):

$$\Gamma_{\mathbf{q}}^{\text{Bel}} = \Gamma_{\mathbf{q},\text{pert}}^{\text{Bel}} \left[ 1 + \text{Re} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) + O(\tilde{q}^4) \right] \quad (\text{B15})$$

with

$$\begin{aligned} \frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) &= -\frac{2(mc^2)^2}{\rho} \\ &\times \int \frac{d^3k}{(2\pi)^3} \frac{[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{k}, \mathbf{q} - \mathbf{k}; \mathbf{q})]^2}{(\epsilon_{\mathbf{q}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}-\mathbf{k}} + i\eta)^2} \end{aligned} \quad (\text{B16})$$

In order to obtain an equivalent of this derivative when  $\tilde{q} \rightarrow 0$ , we apply the same technique as in section VC. We use spherical coordinates of polar axis  $\mathbf{q}$  and perform the rescalings  $k = \bar{k}q$  and  $\theta = \gamma^{1/2}\tilde{q}\tilde{\theta}$  on the modulus and polar angle of the vector  $\mathbf{k}$ , with  $\bar{k} < 1$  to profit from the ‘‘small denominators’’ effect. In the integrand and hence in the coupling amplitude (90), we take the limit  $\tilde{q} \rightarrow 0$

with  $\bar{k}$  and  $\tilde{\theta}$  fixed, with the intermediate results

$$|\mathbf{q} - \mathbf{k}| = q(1 - \bar{k}) \left[ 1 + \frac{\gamma\bar{k}\tilde{\theta}^2\tilde{q}^2}{2(1 - \bar{k})^2} + O(\tilde{q}^4) \right] \quad (\text{B17})$$

$$\epsilon_{\mathbf{q}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}-\mathbf{k}} \sim \frac{\gamma\tilde{q}^3 mc^2}{8} \left[ 3\bar{k}(1 - \bar{k}) - \frac{4\bar{k}\tilde{\theta}^2}{1 - \bar{k}} \right] \quad (\text{B18})$$

$$[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{k}, \mathbf{q} - \mathbf{k}; \mathbf{q})]^2 \sim \frac{2}{9}\tilde{q}^3\bar{k}(1 - \bar{k}) \quad (\text{B19})$$

taking into account the fact that  $1 + \Lambda_{\text{F}} = 8/9$  at unitarity. The integration with respect to  $\tilde{\theta}$  over  $[0, +\infty[$  is simple after the change of variable  $X = \tilde{\theta}^2$ , and we are left with

$$\frac{d}{dz} \Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) \sim \frac{8\tilde{q}^2}{9\pi^2\gamma} \left( \frac{mc}{\hbar\rho^{1/3}} \right)^3 \int_0^1 d\bar{k} \frac{\bar{k}^2(1 - \bar{k})^2}{3\bar{k}(1 - \bar{k}) + i\eta} \quad (\text{B20})$$

The dominated convergence theorem allows us to take the  $\eta \rightarrow 0$  limit in the integrand and to obtain the result (176), knowing that  $\rho(\hbar/mc)^3 = \sqrt{3}/(\pi^2\xi_{\text{B}}^{3/2})$ . As a side remark, the fact that the derivative with respect to  $z$  (B20) is real in the limit  $\eta \rightarrow 0^+$ , means that a change in the real dispersion relation  $q \mapsto \omega_{\mathbf{q}}$  of order  $q^5$  does not affect  $\Gamma_{\mathbf{q}}$  to the order  $q^7$ , which we have already shown in another way in sub-section VD2.

#### 4. Study to the order 4 in $\hat{V}$

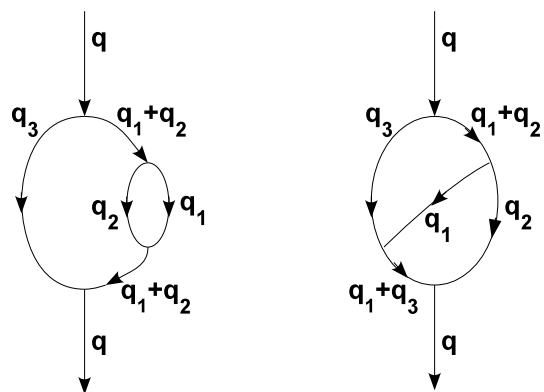


Figure 8: The two diagrams of order 4 in  $\hat{V}$  that may contribute to the damping rate  $\Gamma_{\mathbf{q}}$  of the phonon  $\mathbf{q}$  to the order  $q^7$  thanks to the ‘‘small denominator’’ effect. The one on the left (I) is a Beliaev process with a single loop correction (itself of the Beliaev nature) to the virtual phonons angular eigenfrequency. The one on the right (II) is a Beliaev process enriched by an interaction between the virtual phonons. Each vertex corresponds to an action of  $\hat{H}_3^{(\pm)}$ .

We have seen in the sub-section B2 that the order 4 in  $\hat{V}$  may contribute to  $\Gamma_{\mathbf{q}}$  to the order  $q^7$ . Let us write the corresponding correction  $\delta\Gamma_{\mathbf{q}}$  restricting to the leading order, that is keeping  $\hat{H}_3^{(-)}$  in the first two factors  $\hat{V}$  and



$\hat{H}_3^{(+)}$  in the last two, this sequence being imposed by the presence of the projectors  $\hat{Q}$ :

$$-\frac{\hbar\delta\Gamma_{\mathbf{q}}}{2} = \text{Im}\langle\mathbf{q}|\hat{W}^{(-)}\frac{\hat{Q}}{\epsilon_{\mathbf{q}} + i\eta - \hat{H}_2}\hat{W}^{(+)}|\mathbf{q}\rangle \quad (\text{B21})$$

where we introduced, for  $\varepsilon = \pm$ , the effective coupling operator  $1 \leftrightarrow 3$  to second order

$$\hat{W}^{(\varepsilon)} \equiv \hat{H}_3^{(\varepsilon)}\frac{\hat{Q}}{\epsilon_{\mathbf{q}} + i\eta - \hat{H}_2}\hat{H}_3^{(\varepsilon)} \quad (\text{B22})$$

In these expressions, one can use for  $\hat{H}_3$  the quantum hydrodynamic approximation (89), provided one includes in

$\epsilon_{\mathbf{q}} = \hbar\omega_{\mathbf{q}}$  the cubic correction (63) to the linear hydrodynamical spectrum. As the matrix elements of  $\hat{H}_3$  are real in the Fock basis, the matrices representing  $\hat{W}^{(\pm)}$  are obtained one from the other by transposition. After the insertion of a closure relation in the three-phonon subspace, one obtains

$$-\frac{\hbar\delta\Gamma_{\mathbf{q}}}{2} = \text{Im}\frac{1}{3!}\sum_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3}\delta_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3, \mathbf{q}} \times \frac{|\langle\mathbf{q}|\hat{W}^{(-)}|\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\rangle|^2}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})} \quad (\text{B23})$$

where, accounting for equation (89),

$$\langle\mathbf{q}|\hat{W}^{(-)}|\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\rangle = \frac{4(mc^2)^2}{\rho L^3}\left[\frac{\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3; \mathbf{q})}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1 + \mathbf{q}_2} + \epsilon_{\mathbf{q}_3})} + \frac{\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_2, \mathbf{q}_3; \mathbf{q}_2 + \mathbf{q}_3)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_2 + \mathbf{q}_3, \mathbf{q}_1; \mathbf{q})}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_2 + \mathbf{q}_3} + \epsilon_{\mathbf{q}_1})} + \frac{\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_3; \mathbf{q}_1 + \mathbf{q}_3)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_3, \mathbf{q}_2; \mathbf{q})}{\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1 + \mathbf{q}_3} + \epsilon_{\mathbf{q}_2})}\right] \quad (\text{B24})$$

The contribution to the matrix element of the first term between square brackets, that is  $P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)$ , corresponds to the process in figure 6. The other two,  $P(\mathbf{q}_2, \mathbf{q}_3|\mathbf{q}_1)$  and  $P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)$ , are deduced from the first one by circular permutation. As the rest of the summand in (B23) is invariant by permutation of the three wave vectors, we can replace its numerator  $[P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3) + P(\mathbf{q}_2, \mathbf{q}_3|\mathbf{q}_1) + P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)]^2$  by  $3[P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)]^2 + 6P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)$ , which amounts to considering the two diagrams in figure 8 and the integral expression

$$-\frac{\hbar\delta\Gamma_{\mathbf{q}}}{2} = \lim_{\eta \rightarrow 0^+} \text{Im}\frac{8(mc^2)^4}{\rho^2}\int\frac{d^3q_2 d^3q_3}{(2\pi)^6}\left\{\frac{[\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3; \mathbf{q})]^2}{[\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})][\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1 + \mathbf{q}_2} + \epsilon_{\mathbf{q}_3})]^2} + \frac{2\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1 + \mathbf{q}_2)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3; \mathbf{q})\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1, \mathbf{q}_3; \mathbf{q}_1 + \mathbf{q}_3)\mathcal{A}_{\text{hydro}}^{2\leftrightarrow 1}(\mathbf{q}_1 + \mathbf{q}_3, \mathbf{q}_2; \mathbf{q})}{[\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})][\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1 + \mathbf{q}_2} + \epsilon_{\mathbf{q}_3})][\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1 + \mathbf{q}_3} + \epsilon_{\mathbf{q}_2})]}\right\} \quad (\text{B25})$$

where we eliminated the sum over  $\mathbf{q}_1$  thanks to momentum conservation,  $\mathbf{q}_1 = \mathbf{q} - (\mathbf{q}_2 + \mathbf{q}_3)$ , and took the thermodynamic limit.

In practice, the integration over  $\mathbf{q}_2$  et  $\mathbf{q}_3$  is performed in spherical coordinates of polar axis  $\mathbf{q}$ . To evaluate the contribution to  $\delta\Gamma_{\mathbf{q}}$  of the forward emission cones of  $\mathbf{q}_2$  and  $\mathbf{q}_3$  within an angle  $O(q)$ , we rescale the polar angles  $\theta_i$  as follows:

$$\check{\theta}_i = \frac{\theta_i}{\gamma^{1/2}\check{q}} \quad (\text{B26})$$

then we let  $\check{q} = \hbar q/mc$  tend to zero with  $\check{\theta}_i$  fixed. The calculation is similar to the one in section VC. One uses the polar representation (150). The wave numbers  $q_i$  must be rescaled by  $q$ , meaning that  $\bar{q}_i \equiv \frac{q_i}{q}$  are kept fixed when taking the limit, with the constraint

$$\bar{q}_2 + \bar{q}_3 \leq 1 \quad (\text{B27})$$

coming from (177) and of the positivity of  $q_1$ .

We give as an intermediate result the expression of an energy denominator in the  $\hat{W}^{(-)}$  matrix elements,

$$\frac{\epsilon_{\mathbf{q}} - (\epsilon_{\mathbf{q} - \mathbf{q}_3} + \epsilon_{\mathbf{q}_3})}{mc^2} \underset{\check{q} \rightarrow 0}{=} \frac{\gamma\check{q}^3}{2}\left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) - \frac{\bar{q}_3 R^2 \sin^2 \alpha}{1 - \bar{q}_3}\right] + O(\check{q}^5) \quad (\text{B28})$$

and the energy denominator in the 3 phonon subspace,

$$\frac{\epsilon_{\mathbf{q}} - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})}{mc^2} \underset{\check{q} \rightarrow 0}{=} \frac{\gamma\check{q}^3}{2}(v - uR^2) + O(\check{q}^5) \quad (\text{B29})$$

whose manifestly positive coefficients (once they are written in the proper form) look like those of equations (152)

and (153):

$$u \equiv \frac{\bar{q}_2 \cos^2 \alpha + \bar{q}_3 \sin^2 \alpha - \bar{q}_2 \bar{q}_3 (1 - \cos \phi \sin 2\alpha)}{1 - \bar{q}_2 - \bar{q}_3} \\ = \frac{(\bar{q}_2 \cos \alpha - \bar{q}_3 \sin \alpha)^2 + \bar{q}_2 \bar{q}_3 (1 + \cos \phi) \sin 2\alpha}{1 - \bar{q}_2 - \bar{q}_3} \\ + \bar{q}_2 \cos^2 \alpha + \bar{q}_3 \sin^2 \alpha \quad (\text{B30})$$

and

$$v \equiv \frac{1}{4} [1 - \bar{q}_2^3 - \bar{q}_3^3 - (1 - \bar{q}_2 - \bar{q}_3)^3] = \frac{3}{4} (1 - \bar{q}_2)(1 - \bar{q}_3)(\bar{q}_2 + \bar{q}_3) \quad (\text{B31})$$

One finally obtains

$$- \frac{\hbar \delta \Gamma_{\mathbf{q}}}{2} \underset{\tilde{q} \rightarrow 0}{=} m c^2 \frac{[3(1 + \Lambda_{\text{F}})]^4}{3 \times 2\pi} \frac{\tilde{q}^7}{2^8 \pi^4 \gamma} \left( \frac{m c}{\hbar \rho^{1/3}} \right)^6 \\ \times (\text{Im } I^{(\text{I})} + \text{Im } I^{(\text{II})}) + o(\tilde{q}^7) \quad (\text{B32})$$

with the parameter  $\Lambda_{\text{F}}$  defined in equation (79) and the quintuple integrals, coming from the left (type I) diagram and the right (type II) diagram of figure 8, that should be evaluated in the limit  $\eta \rightarrow 0^+$ :

$$I^{(\text{I})} = \lim_{\eta \rightarrow 0^+} \int_0^1 d\bar{q}_2 \int_0^{1-\bar{q}_2} d\bar{q}_3 \int_{-\pi}^{\pi} d\phi \int_0^{\pi/2} d\alpha \int_0^{A^2/\tilde{q}^2} dX \frac{3X \sin \alpha \cos \alpha (1 - \bar{q}_3)^2 (1 - \bar{q}_2 - \bar{q}_3) \bar{q}_2^3 \bar{q}_3^3}{2 (v - uX + i\eta) [\frac{3}{4} \bar{q}_3 (1 - \bar{q}_3) - \frac{\bar{q}_3 X \sin^2 \alpha}{1 - \bar{q}_3} + i\eta]^2} \quad (\text{B33})$$

$$I^{(\text{II})} = \lim_{\eta \rightarrow 0^+} \int_0^1 d\bar{q}_2 \int_0^{1-\bar{q}_2} d\bar{q}_3 \int_{-\pi}^{\pi} d\phi \int_0^{\pi/2} d\alpha \\ \times \int_0^{+\infty} \frac{dX}{2} \frac{6X \sin \alpha \cos \alpha (1 - \bar{q}_2)(1 - \bar{q}_3)(1 - \bar{q}_2 - \bar{q}_3) \bar{q}_2^3 \bar{q}_3^3}{(v - uX + i\eta) [\frac{3}{4} \bar{q}_2 (1 - \bar{q}_2) - \frac{\bar{q}_2 X \cos^2 \alpha}{1 - \bar{q}_2} + i\eta] [\frac{3}{4} \bar{q}_3 (1 - \bar{q}_3) - \frac{\bar{q}_3 X \sin^2 \alpha}{1 - \bar{q}_3} + i\eta]} \quad (\text{B34})$$

In the integration over the radius  $R$  we have performed the change of variables  $X = R^2$ . This allows to see that the integral over  $R$ , that is over  $X$ , in (B34) has a zero imaginary part in the limit  $\eta \rightarrow 0^+$ , and hence

$$\text{Im } I^{(\text{II})} = 0 \quad (\text{B35})$$

Indeed all the factors in the integrand denominator in (B34) have the form  $\alpha - \beta X + i\eta$ , with  $\alpha > 0$  and  $\beta > 0$ . As a consequence, the integral over  $X \in [0, +\infty[$  can be evaluated using the following elementary theorem:

**Theorem:** We define  $a(X) = a_0 - a_1 X$ ,  $b(X) = b_0 - b_1 X$  and  $c(X) = c_0 - c_1 X$  for all  $X \in \mathbb{R}$ . If all the coefficients  $a_0, a_1, b_0, b_1, c_0, c_1$  are  $> 0$  and  $\eta \in \mathbb{R}$ , then

$$\lim_{\eta \rightarrow 0^+} \int_{\mathbb{R}^+} dX \text{Im} \frac{X}{[a(X) + i\eta][b(X) + i\eta][c(X) + i\eta]} = 0 \quad (\text{B36})$$

**Proof:** We first note that the theorem would be trivially proved if the integral over  $X$  was taken over  $\mathbb{R}^-$ . Indeed, for all  $\eta > 0$  and for all  $X < 0$ ,

$$\left| \frac{X}{[a(X) + i\eta][b(X) + i\eta][c(X) + i\eta]} \right| < \frac{X}{a(X)b(X)c(X)} \quad (\text{B37})$$

where the upper bound has a finite integral over  $\mathbb{R}^-$ , so that the dominated convergence theorem to exchange the order of the integral and limit  $\eta \rightarrow 0^+$  can be used. We can then extend the integration domain in (B36) to the

whole  $\mathbb{R}$  without changing the result. On the other hand we can use the following identity valid for any  $\eta > 0$  not necessarily infinitesimal, concerning both the real and the imaginary parts:

$$\int_{\mathbb{R}} dX \frac{X}{[a(X) + i\eta][b(X) + i\eta][c(X) + i\eta]} = 0 \quad (\text{B38})$$

To show it, it's enough to use the residue theorem by closing downwards the integration contour following a half-circle of radius  $\rho \rightarrow +\infty$  in the lower half-plane; the behavior of the integrand as  $1/X^2$  for large  $|X|$  allows for it, which leads to a contribution to the half-circle in  $O(1/\rho)$ . As the integrand poles are all in the upper half-plane, they are not enclosed by the contour, see the figure 9, hence (B38) and (B36) [63]. ■

One might think that the same reasoning applies to the contribution (B33) and that  $\text{Im } I^{(\text{I})} = 0$ , in which case  $\delta \Gamma_{\mathbf{q}} = o(\tilde{q}^7)$  and there would be no correction to add to the result (176). This is not the case because, in order to obtain a finite value of  $I^{(\text{I})}$ , we are this time obliged to keep in the integral over  $X$  a finite value  $A^2/\tilde{q}^2$  of the upper bound, which prevents from the application of the theorem. Here,  $A > 0$  is a cut-off constant whose precise value is not relevant, and the power law in  $\tilde{q}^{-2}$  comes from the fact that  $\theta_i \leq \pi$  and then  $\theta_i \leq \pi/(\gamma^{1/2} \tilde{q})$  in (B26), which implies  $X = R^2 = \theta_2^2 + \theta_3^2 = O(\tilde{q}^{-2})$ . If one simply replaces  $A^2/\tilde{q}^2$  by  $+\infty$  in equation (B33), one finds indeed that the integral over  $X$  diverges as  $\alpha^{-2}$

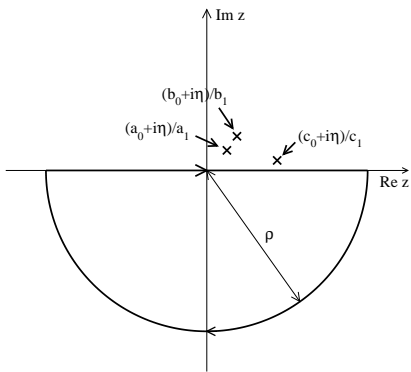


Figure 9: Thick line: contour integration in the complex plane that should be used to apply the residue theorem, in order to obtain the identity (B38). The real numbers  $a_0, a_1, b_0, b_1, c_0, c_1$  and  $\eta$  are positive. The radius  $\rho$  of the half-circle tends to infinity. The crosses correspond to the poles of the integrand in the case they are all simple, but the result holds also in the general case.

when  $\alpha$  tends to zero, which leads to an integral over  $\alpha$  that diverges logarithmically for  $\alpha = 0$  [64]. To progress, we cut the integration interval over  $\alpha$  into two, a sub-interval  $[0, \nu]$  for which the cut-off  $A^2/\tilde{q}^2$  in the integration over  $X$  is necessary, and a sub-integral  $[\nu, \pi/2]$  for which the integration over  $X$  can be extended to  $+\infty$ :

$$\int_0^{\pi/2} d\alpha \int_0^{A^2/\tilde{q}^2} dX f(\alpha, X) \rightarrow \int_0^\nu d\alpha \int_0^{A^2/\tilde{q}^2} dX f(\alpha, X) + \int_\nu^{\pi/2} d\alpha \int_0^{+\infty} dX f(\alpha, X) \quad (\text{B39})$$

The contribution of the second bit to  $\text{Im } I^{(1)}$  is zero, because of the theorem. The contribution of the first bit does not depend on the value of  $\nu$ . We can then calculate it for a value of  $\nu$  sufficiently small to approximate each contribution depending on  $\alpha$  in the integrand by its leading order in  $\alpha$ , that is  $\sin \alpha \simeq \alpha$ ,  $\cos \alpha \simeq 1$ ,  $u \simeq u_0 = \bar{q}_2(1 - \bar{q}_3)/(1 - \bar{q}_2 - \bar{q}_3)$ . The dependence in  $\phi$  then disappears and the integration over  $\phi$  simply gives a factor  $2\pi$ . After a simple calculation of the integral over  $\alpha$ :

$$\int_0^\nu \frac{\alpha d\alpha}{\left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) - \frac{\bar{q}_3 X \alpha^2}{1 - \bar{q}_3} + i\eta\right]^2} = \left[ \frac{1 - \bar{q}_3}{2\bar{q}_3 X \left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) - \frac{\bar{q}_3 X \alpha^2}{1 - \bar{q}_3} + i\eta\right]} \right]_{\alpha=0}^{\alpha=\nu} \quad (\text{B40})$$

we shall concentrate on the integration over  $X$ . The fully integrated term  $\alpha = \nu$  of equation (B40), after a multiplication by the factor  $X$  in the numerator of (B33) and a division by the denominator  $(v - u_0 X + i\eta)$ , gives an integrand in  $O(1/X^2)$ ; one can in this case replace the upper integration bound  $A^2/\tilde{q}^2$  for  $X$  by  $+\infty$  and use the following variant of the theorem, in order to show that

its contribution to  $\text{Im } I^{(1)}$  is exactly zero:

$$\lim_{\eta \rightarrow 0^+} \int_{\mathbb{R}^+} dX \text{Im} \frac{1}{[a(X) + i\eta][b(X) + i\eta]} = 0 \quad (\text{B41})$$

The fully integrated term  $\alpha = 0$  of equation (B40) leads on the contrary to an integral over  $X$  of nonzero imaginary part in the limit  $\eta \rightarrow 0^+$ :

$$\text{Im} \int_0^{A^2/\tilde{q}^2} \frac{(1 - \bar{q}_3)dX}{2\bar{q}_3(v - u_0 X + i\eta) \left[\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) + i\eta\right]} = -\frac{1 - \bar{q}_3}{2\bar{q}_3 u_0} \text{Im} \frac{[\ln(v - u_0 A^2 \tilde{q}^{-2} + i\eta) - \ln(v + i\eta)]}{\frac{3}{4}\bar{q}_3(1 - \bar{q}_3) + i\eta} \xrightarrow{\eta \rightarrow 0^+} -\frac{\pi}{u_0} \frac{Y(u_0 A^2 \tilde{q}^{-2} - v)}{\frac{3}{2}\bar{q}_3^2} \quad (\text{B42})$$

where  $\ln$  is the usual branch of the complex logarithm, with a branch cut along the real negative axis. We are then left with

$$\text{Im } I^{(1)} = 2\pi^2 \int_0^1 d\bar{q}_2 \int_0^{1-\bar{q}_2} d\bar{q}_3 (1 - \bar{q}_3)(1 - \bar{q}_2 - \bar{q}_3)^2 \bar{q}_2^2 \bar{q}_3 \times Y(u_0 A^2 \tilde{q}^{-2} - v) \xrightarrow{\tilde{q} \rightarrow 0} \frac{\pi^2}{840} \quad (\text{B43})$$

hence the correction (178) to be added to the result (176).

To conclude, one might ask whether there is a physical interpretation to the contributing diagram of order 4 in  $\hat{V}$ , which is the left one (I) in figure 8. In the calculation of the complex energy shift  $\Delta \epsilon_{\mathbf{q}}^{\text{Bel}}$  of a phonon  $\mathbf{q}$  induced by a Beliaev process, this diagram takes into account the effect of a modification of the angular eigenfrequency of the virtual phonons, modification itself induced by a Beliaev process at this order. More quantitatively, let us introduce the Beliaev self-energy (B10), where we note now more rigorously  $\epsilon_{\mathbf{k}}^{(2)}$  the unperturbed eigenenergy of a phonon with wave vector  $\mathbf{k}$  in  $\hat{H}_2$ , to distinguish it from the exact energy  $\epsilon_{\mathbf{q}}$  (see the note [61]). We verified that the change in the complex energy of the phonon  $\mathbf{q}$  originating from the left diagram (I) is exactly

$$\delta z_{\mathbf{q}}^{(1)} = \frac{1}{2} \sum_{\mathbf{q}_3} \frac{(\langle \mathbf{q}_3, \mathbf{q} - \mathbf{q}_3 | \hat{H}_3 | \mathbf{q} \rangle)^2}{[\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_3}^{(2)} + \epsilon_{\mathbf{q} - \mathbf{q}_3}^{(2)})]^2} \times \left[ \Delta \epsilon_{\mathbf{q}_3}^{\text{Bel}} (\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q} - \mathbf{q}_3}^{(2)}) + \Delta \epsilon_{\mathbf{q} - \mathbf{q}_3}^{\text{Bel}} (\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}_3}^{(2)}) \right] \quad (\text{B44})$$

The reader will notice that the argument  $z$  of  $\Delta \epsilon_{\mathbf{q} - \mathbf{q}_3}^{\text{Bel}}$  is here the exact energy  $\epsilon_{\mathbf{q}}$  minus the non perturbed energy of the virtual phonon that is spectator in the process of diagram (I), that is the one that does not participate to the loop. The sum of the correction (I) and of the usual perturbative Beliaev shift of the  $\mathbf{q}$  phonon energy can then be written, to the order 4 in  $\hat{H}_3$ , as a Beliaev shift for a renormalized complex dispersion relation:

$$\Delta\epsilon_{\mathbf{q}}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta) + \delta z_{\mathbf{q}}^{(1)} \simeq \frac{1}{2} \sum_{\mathbf{q}_3, \mathbf{q}_4} \frac{(\langle \mathbf{q}_3, \mathbf{q}_4 | \hat{H}_3 | \mathbf{q} \rangle)^2 \delta_{\mathbf{q}_3 + \mathbf{q}_4, \mathbf{q}}}{\epsilon_{\mathbf{q}} + i\eta - [\epsilon_{\mathbf{q}_3}^{(2)} + \Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}_4}^{(2)}) + \epsilon_{\mathbf{q}_4}^{(2)} + \Delta\epsilon_{\mathbf{q}_4}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}_3}^{(2)})]} \quad (\text{B45})$$

By the way, the formulation (B44) provides a second calculation method, alternative to the one of equation (B33). With the rescalings of sub-section VC suitable to the “small denominators”, and by using (B18) with  $(\mathbf{q}, \mathbf{k})$

that is successively equal to  $(\mathbf{q}_3, \mathbf{k})$  and  $(\mathbf{q}, \mathbf{q}_3)$ , where  $\mathbf{k}$  is the integration variable appearing in the expression taken from (B10) of  $\Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}-\mathbf{q}_3}^{(2)})$ , we find

$$\frac{\Delta\epsilon_{\mathbf{q}_3}^{\text{Bel}}(\epsilon_{\mathbf{q}} + i\eta - \epsilon_{\mathbf{q}-\mathbf{q}_3}^{(2)})}{mc^2} = -\frac{9(1 + \Lambda_{\text{F}})^2}{64\pi^2} \left( \frac{mc}{\hbar\rho^{1/3}} \right)^3 \bar{q}_3^5 \check{q}^5 \times \int_0^1 d\bar{k} \bar{k}^2 (1 - \bar{k})^2 \left[ i\pi + \ln \left( \frac{4\bar{q}_3 \bar{k} A^2}{(1 - \bar{k}) \check{q}^2} \right) - \ln \left( i\eta + 3\bar{q}_3(1 - \bar{q}_3) + 3\bar{q}_3^3 \bar{k}(1 - \bar{k}) - \frac{4\bar{q}_3 \check{\theta}_3^2}{1 - \bar{q}_3} \right) \right] \quad (\text{B46})$$

Here  $\bar{q}_3 = q_3/q$ ,  $\bar{k} = k/q_3$ ,  $\check{\theta}_3 = \theta_3/(\gamma^{1/2}\check{q})$ ,  $\theta_3$  is the non oriented angle between the vectors  $\mathbf{q}$  and  $\mathbf{q}_3$ . The integral over  $\check{\theta} = \theta/(\gamma^{1/2}\check{q}_3)$ , where  $\theta$  is the angle between  $\mathbf{k}$  and  $\mathbf{q}$ , has been explicitly performed over the interval  $[0, A/\check{q}_3]$  with the same cut-off parameter  $A$  as in equation (B33). After the insertion of (B46) in the equation (B44), we obtain

$$\frac{\text{Im} \delta z_{\mathbf{q}}^{(1)}}{mc^2} = -\frac{9^2(1 + \Lambda_{\text{F}})^4}{64\pi^4\gamma} \left( \frac{mc}{\hbar\rho^{1/3}} \right)^6 \check{q}^7 \int_0^1 d\bar{q}_3 \bar{q}_3^8 (1 - \bar{q}_3) \int_0^1 d\bar{k} \bar{k}^2 (1 - \bar{k})^2 \times \text{Im} \int_0^{+\infty} dX_3 \frac{i\pi - \ln \left( i\eta + 3\bar{q}_3(1 - \bar{q}_3) + 3\bar{q}_3^3 \bar{k}(1 - \bar{k}) - \frac{4\bar{q}_3 X_3}{1 - \bar{q}_3} \right)}{\left( i\eta + 3\bar{q}_3(1 - \bar{q}_3) - \frac{4\bar{q}_3 X_3}{1 - \bar{q}_3} \right)^2} \quad (\text{B47})$$

We omitted here the  $\check{\theta}_3$ -independent logarithmic real term, in between square brackets in equation (B46), because its contribution to the final result is clearly real in the limit  $\eta \rightarrow 0^+$ . The dependence on the cut-off parameter  $A$  then vanishes. The integral over  $X_3 = \check{\theta}_3^2$  of the bit with the logarithm is calculated by parts, by taking the derivative of the logarithm; one finds that its contribution is zero in the limit  $\eta \rightarrow 0^+$  by using the identity (B41). Finally, only the term  $i\pi$  matters, and it leads exactly to the result (178).

- 
- [1] Lev Landau and Isaak Khalatnikov. Teoriya vyazkosti Geliya-II. *Zh. Eksp. Teor. Fiz.*, 19:637, 1949.
  - [2] I.M. Khalatnikov and D.M. Chernikova. Relaxation Phenomena in Superfluid Helium. *ZhETF*, 49:1957, June 1966.
  - [3] A. Sinatra, Y. Castin, and E. Witkowska. Coherence time of a Bose-Einstein condensate. *Phys. Rev. A*, 80:033614, Sep 2009.
  - [4] H. Kurkjian, Y. Castin, and A. Sinatra. Brouillage thermique d’un gaz cohérent de fermions. *C. R. Phys.*, 17:789, 2016.
  - [5] M A H Tucker and A F G Wyatt. Four-phonon scattering in superfluid 4 He. *Journal of Physics: Condensed Matter*, 4(38):7745, 1992.
  - [6] I. N. Adamenko, Yu. A. Kitsenko, K. E. Nemchenko, and A. F. G. Wyatt. Theory of scattering between two phonon beams in superfluid helium. *Phys. Rev. B*, 80:014509, July 2009.
  - [7] H. Kurkjian, Y. Castin, and A. Sinatra. Concavity of the collective excitation branch of a Fermi gas in the BEC-BCS crossover. *Phys. Rev. A*, 93:013623, January 2016.
  - [8] S.T. Beliaev. Application of the Methods of Quantum Field Theory to a System of Bosons. *Zh. Eksp. Teor. Fiz.*, 34:417, August 1958.
  - [9] G. Bighin, L. Salasnich, P. A. Marchetti, and F. Toigo. Beliaev damping of the Goldstone mode in atomic Fermi superfluids. *Phys. Rev. A*, 92:023638, August 2015.
  - [10] K. M. O’Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas. Observation of a Strongly Interacting Degenerate Fermi Gas of Atoms. *Science*, 298(5601):2179–2182, 2002.
  - [11] T. Bourdel, J. Cubizolles, L. Khaykovich, K. M. F. Ma-

- galhães, S. J. J. M. F. Kokkelmans, G. V. Shlyapnikov, and C. Salomon. Measurement of the Interaction Energy near a Feshbach Resonance in a  ${}^6\text{Li}$  Fermi Gas. *Phys. Rev. Lett.*, 91:020402, July 2003.
- [12] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm. Crossover from a Molecular Bose-Einstein Condensate to a Degenerate Fermi Gas. *Phys. Rev. Lett.*, 92:120401, March 2004.
- [13] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm. Collective Excitations of a Degenerate Gas at the BEC-BCS Crossover. *Phys. Rev. Lett.*, 92:203201, May 2004.
- [14] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle. Condensation of Pairs of Fermionic Atoms near a Feshbach Resonance. *Phys. Rev. Lett.*, 92:120403, March 2004.
- [15] M. W. Zwierlein, J. R. Abo-Shaer, A. Schirotzek, C. H. Schunck, and W. Ketterle. Vortices and superfluidity in a strongly interacting Fermi gas. *Nature*, 435(7045):1047–1051, June 2005.
- [16] S. Nascimbène, N. Navon, K. J. Jiang, F. Chevy, and C. Salomon. Exploring the thermodynamics of a universal Fermi gas. *Nature*, 463(7284):1057–1060, February 2010.
- [17] Mark J. H. Ku, Ariel T. Sommer, Lawrence W. Cheuk, and Martin W. Zwierlein. Revealing the Superfluid Lambda Transition in the Universal Thermodynamics of a Unitary Fermi Gas. *Science*, 335(6068):563–567, 2012.
- [18] Leonid A. Sidorenkov, Meng Khoon Tey, Rudolf Grimm, Yan-Hua Hou, Lev Pitaevskii, and Sandro Stringari. Second sound and the superfluid fraction in a Fermi gas with resonant interactions. *Nature*, 498(7452):78–81, June 2013.
- [19] P.W. Anderson. Random-Phase Approximation in the Theory of Superconductivity. *Phys. Rev.*, 112:1900–1916, 1958.
- [20] M. Marini, F. Pistolesi, and G.C. Strinati. Evolution from BCS superconductivity to Bose condensation: analytic results for the crossover in three dimensions. *European Physical Journal B*, 1:151–159, 1998.
- [21] R. Combescot, M. Yu. Kagan, and S. Stringari. Collective mode of homogeneous superfluid Fermi gases in the BEC-BCS crossover. *Phys. Rev. A*, 74:042717, October 2006.
- [22] S. N. Klimin, J. Tempere, and Jeroen P. A. Devreese. Pair Excitations and Parameters of State of Imbalanced Fermi Gases at Finite Temperatures. *Journal of Low Temperature Physics*, 165(5):261–274, 2011.
- [23] Mohit Randeria and Edward Taylor. Crossover from Bardeen-Cooper-Schrieffer to Bose-Einstein Condensation and the Unitary Fermi Gas. *Annual Review of Condensed Matter Physics*, 5:209–232, 2014.
- [24] J. Steinhauer, R. Ozeri, N. Katz, and N. Davidson. Excitation Spectrum of a Bose-Einstein Condensate. *Phys. Rev. Lett.*, 88:120407, March 2002.
- [25] N. Katz, J. Steinhauer, R. Ozeri, and N. Davidson. Beliaev Damping of Quasiparticles in a Bose-Einstein Condensate. *Phys. Rev. Lett.*, 89:220401, Nov 2002.
- [26] F. Chevy, V. Bretin, P. Rosenbusch, K. W. Madison, and J. Dalibard. Transverse Breathing Mode of an Elongated Bose-Einstein Condensate. *Phys. Rev. Lett.*, 88:250402, June 2002.
- [27] Alexander L. Gaunt, Tobias F. Schmidutz, Igor Gotlibovych, Robert P. Smith, and Zoran Hadzibabic. Bose-Einstein Condensation of Atoms in a Uniform Potential. *Phys. Rev. Lett.*, 110:200406, May 2013.
- [28] P.C Hohenberg and P.C Martin. Microscopic theory of superfluid helium. *Annals of Physics*, 34(2):291 – 359, 1965.
- [29] P Szépfalussy and I Kondor. On the dynamics of continuous phase transitions. *Annals of Physics*, 82(1):1 – 53, 1974.
- [30] L.P. Pitaevskii and S. Stringari. Landau damping in dilute Bose gases. *Physics Letters A*, 235(4):398 – 402, 1997.
- [31] Gautam Rupak and Thomas Schäfer. Shear viscosity of a superfluid Fermi gas in the unitarity limit. *Phys. Rev. A*, 76:053607, November 2007.
- [32] D.T. Son and M. Wingate. General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas. *Ann. Physics*, 321:197–224, 2006.
- [33] Yvan Castin. Basic Theory Tools for Degenerate Fermi Gases. In M. Inguscio, W.Ketterle, and C. Salomon, editors, *Ultra-cold Fermi Gases*. Società Italiana di Fisica, Bologna, 2007.
- [34] J. Bardeen, L. N. Cooper, and J. R. Schrieffer. Theory of Superconductivity. *Phys. Rev.*, 108:1175–1204, December 1957.
- [35] Jean-Paul Blaizot and Georges Ripka. *Quantum Theory of Finite Systems*. MIT Press, Cambridge, Massachusetts, 1985.
- [36] H. Kurkjian. *Cohérence, brouillage et dynamique de phase dans un condensat de paires de fermions*. PhD thesis, École Normale Supérieure, Paris, 2016.
- [37] Yvan Castin and Ralph Dum. Low-temperature Bose-Einstein Condensates in Time-dependent traps. *Phys. Rev. A*, 57:3008–3021, 1998.
- [38] H. Kurkjian, Y. Castin, and A. Sinatra. Phase operators and blurring time of a pair-condensed Fermi gas. *Phys. Rev. A*, 88:063623, December 2013.
- [39] S. Giorgini. Damping in dilute Bose gases: A mean-field approach. *Phys. Rev. A*, 57:2949–2957, April 1998.
- [40] T. D. Lee and C. N. Yang. Many-Body Problem in Quantum Mechanics and Quantum Statistical Mechanics. *Phys. Rev.*, 105:1119–1120, February 1957.
- [41] A.A. Abrikosov and I.M. Khalatnikov. Concerning a Model for a Non-Ideal Fermi Gas. *Sov. Phys. JETP*, 6:888, 1958 [ZhETF, **33**, 1154].
- [42] V.M. Galitskii. The Energy Spectrum of a Non-ideal Fermi Gas. *Sov. Phys. JETP*, 7:104, 1958 [ZhETF, **34**, 151].
- [43] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Processus d’interaction entre photons et atomes*. InterEditions et Éditions du CNRS, Paris, 1988.
- [44] N.N. Bogoliubov. A new method in the theory of superconductivity I. *Soviet Physics JETP*, 34(7)(1), October 1958.
- [45] C. W. Gardiner. Particle-number-conserving Bogoliubov method which demonstrates the validity of the time-dependent Gross-Pitaevskii equation for a highly condensed Bose gas. *Phys. Rev. A*, 56:1414–1423, August 1997.
- [46] Y. Castin. Bose-Einstein Condensates in Atomic Gases: Simple Theoretical Results. In R. Kaiser, C. Westbrook, and F. David, editors, *Coherent atomic matter waves: 27 July–27 August 1999*, pages 1–136. EDP Sciences, Les Ulis, Springer, Berlin, 2001.
- [47] Christophe Mora and Yvan Castin. Extension of Bo-

- goliubov theory to quasicondensates. *Phys. Rev. A*, 67:053615, May 2003.
- [48] A. Sinatra, Y. Castin, and E. Witkowska. Nondiffusive phase spreading of a Bose-Einstein condensate at finite temperature. *Phys. Rev. A*, 75:033616, March 2007.
- [49] L. D. Landau. The theory of the viscosity of Helium II: II. Calculation of the viscosity coefficient. In D. Ter Haar, editor, *Collected Papers of LD Landau*, page 511. Pergamon Press, Oxford, 1965.
- [50] A. Sinatra, Y. Castin, and E. Witkowska. Coherence time of a Bose-Einstein condensate. *Phys. Rev. A*, 80:033614, September 2009.
- [51] H. Kurkjian, Y. Castin, and A. Sinatra. Landau-Khalatnikov phonon damping in strongly interacting Fermi gases. *arXiv:1608.00402*, 2016.
- [52] Everywhere in this article the coupling amplitudes  $A^{m \leftrightarrow n}$  are symmetric functions of their  $m$  first arguments, the wavevectors  $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$  of the incoming phonons, as well as of their  $n$  last arguments, the wavevectors  $\mathbf{q}_{m+1}, \mathbf{q}_{m+2}, \dots, \mathbf{q}_{m+n}$  of the outgoing phonons. With this property the amplitudes are uniquely determined.
- [53] The BEC limit corresponds to  $x \rightarrow 0^+$  and  $y \sim -4/x$  [7]; in appendix A we give the expressions (A13) of the rational fractions in this limit. It is then straightforward to check that our expression of  $\mathcal{A}_{\text{micro}}^{2 \leftrightarrow 1}$  in the BEC limit coincides with Eq.(16) of Ref.[3] expanded to third order in  $q$  and expressed in terms of the Bogoliubov excitation energies of a gas of  $\langle \hat{N}_B \rangle = \langle \hat{N} \rangle / 2$  bosonic dimers of mass  $m_B = 2m$ , and speed of sound  $c$ .
- [54] We are here in the large  $N_B$  limit and we neglect 1 as compared to  $\hat{n}_0$  and to  $N_B$ . In the equations (116) and (118), there was a cancellation of the Hartree contribution with a chemical potential-type contribution originating from the expansion of the condensate interaction energy  $\hat{V}_0 \hat{n}_0^2 / 2L^3$  in powers of the number of non-condensed particles.
- [55] If one really uses a linear dispersion relation  $q \mapsto \omega_{\mathbf{q}}$  in the argument of the Dirac distribution, we face the polar integral  $\int_{-1}^1 du \delta(u-1)$ , which seems to be equal to 1/2. If one correctly takes into account the strict convexity of the dispersion relation, we obtain the correct value  $\int_{-1}^1 du \delta(u-u_0) = 1$  with the root  $u_0 \in ]-1, 1[$ . Here  $u$  is the cosine of the angle between  $\mathbf{q}$  and  $\mathbf{q}'$ .
- [56] One just has to series expand  $1/(e^{\hat{a}'} - 1)$  and  $1/(e^{\hat{a}'+\hat{a}} - 1)$  in powers of the variable  $e^{-\hat{a}'}$  and to exchange summation and integration. We recall that  $\zeta(4) = \pi^4/90$ .
- [57] The peak width is of order  $\epsilon$  because the intermediate-virtual-phonon energy change when the  $\theta_i$  vary from 0 to  $\epsilon$  is of the same order as the cubic correction to the hydrodynamic dispersion relation, knowing that the typical wave numbers are  $\approx k_B T / \hbar c$ . Taking as an example the intermediate phonon  $\mathbf{q}_S = \mathbf{q} + \mathbf{q}_2$ , one finds that  $\hbar c[|\mathbf{q} + \mathbf{q}_2| - (q + q_2)] \sim -\frac{\hbar c q q_2}{q + q_2} \theta_2^2 \approx k_B T \theta_2^2$ ; this deviation is of the same order as the cubic term  $\approx k_B T |\gamma| \epsilon^2$  in the equation (63) when  $\theta_2^2 \approx |\gamma| \epsilon^2$ , hence the change of variable (145).
- [58] Let us give the explicit intervals of integration  $[\alpha_{\min}, \alpha_{\max}]$  and  $[\phi_{\min}, \phi_{\max}]$  imposed by the Heaviside function. We set  $\chi = \frac{q_2 q_3 + q(q_3 \sin^2 \alpha - q_2 \cos^2 \alpha)}{q_2 q_3 \sin 2\alpha}$ ,  $A = \frac{q}{2q_3} + \frac{q}{2q_2}$ ,  $C = 1 - \frac{q}{2q_3} + \frac{q}{2q_2}$ ,  $A' = \frac{A}{(1+A^2)^{1/2}}$  and  $C' = \frac{C}{(1+A^2)^{1/2}}$ . (i) If  $v > 0$  and  $q < q_3$ ,  $2\alpha_{\min} = \arccos A' - \arccos C'$  and  $2\alpha_{\max} = \arccos A' + \arccos C'$ . (ii) If  $v > 0$  and  $q > q_3$ ,  $\alpha_{\min} = 0$  and  $2\alpha_{\max} = \arccos A' + \arccos C'$ . (iii) If  $v < 0$  and  $q < q_3$ ,  $\alpha_{\min} = 0$  and  $\alpha_{\max} = \pi/2$ . (iv) If  $v < 0$  and  $q > q_3$ ,  $2\alpha_{\min} = \arccos C' - \arccos A'$  and  $\alpha_{\max} = \pi/2$ . In the cases (i) and (ii) ( $v > 0$ ),  $\phi_{\min} = 0$ ,  $\phi_{\max} = \pi$  if  $\chi < -1$  and  $\phi_{\max} = \arccos \chi$  otherwise. In the cases (iii) and (iv) ( $v < 0$ ),  $\phi_{\max} = \pi$ ,  $\phi_{\min} = 0$  if  $\chi > 1$  and  $\phi_{\min} = \arccos \chi$  otherwise. The integral over  $\phi$  can be calculated analytically. One faces  $\int_0^\Phi d\phi \left( \sum_i \frac{b_i}{a_i + \cos \phi} \right)^2$  that can be expressed in terms of the primitive  $F(\Phi) = \int_0^\Phi d\phi \frac{1}{a + \cos \phi}$  and of its derivative with respect to  $a$ . If  $a \in ]-1, 1[$ ,  $F(\Phi) = \frac{2}{(1-a^2)^{1/2}} \operatorname{argth} \left[ \left( \frac{1-a}{1+a} \right)^{1/2} \tan \frac{\Phi}{2} \right]$ . If  $|a| > 1$ ,  $F(\Phi) = \frac{2}{(a^2-1)^{1/2}} \operatorname{atan} \left[ \frac{a-1}{(a^2-1)^{1/2}} \tan \frac{\Phi}{2} \right]$ . If  $a = 1$ ,  $F(\Phi) = \tan \frac{\Phi}{2}$ .
- [59] Using the note [55], we consider the polar integral  $J = \int_{-1}^1 du \frac{\eta'/\pi}{\eta'^2 + (u-u_0)^2}$  where  $\eta' \approx \Gamma_{\mathbf{q}}/\tilde{q} \approx \tilde{q}^4$  and  $1 - u_0 = 1 - \cos \theta_0 \approx \tilde{q}^2$  as in (B26). Then  $J - 1 \approx \tilde{q}^2$  gives a correction  $O(\tilde{q}^7)$  to  $\Gamma_{\mathbf{q}}^{\text{Bel}}$  coming from the finite energy width of the initial state.
- [60] The attentive reader has probably noticed that the quartic Hamiltonian  $\hat{H}_4$  provides a direct  $1 \rightarrow 3$  coupling to first order. The corresponding amplitude is in  $\tilde{q}^2$ . This direct coupling contributes to  $\Gamma_{\mathbf{q}}$  at the order  $\tilde{q}^9$ , that is here negligible even by taking into account the effect of a small denominator in  $\tilde{q}^3$ . For this reason we do not mention it in the main text.
- [61] We cannot calculate here the corrections to  $\omega_{\mathbf{q}}$  due to the coupling with the continua, even at the order 2 in  $\hat{V}$ , because we rely on an effective Hamiltonian. If we try, we would encounter principal part integrals with ultraviolet divergences that, once inserted in the unperturbed value in  $\hat{H}_2$  would give, following the ideas of renormalisation, the true value, which would however remain uncalculated and unknown in the absence of a microscopic model [4]. Within this context, we don't find it useful to distinguish between the non perturbed phonon eigenenergy for  $\hat{H}_2$ , which we should rigorously note  $\epsilon_{\mathbf{q}}^{(2)}$ , and the true energy  $\epsilon_{\mathbf{q}}$ . By the way, in order to obtain the scaling laws with  $q$  of the perturbative terms of order  $n$  in  $\hat{V}$ , we assume that the wave numbers of the virtual phonons are  $O(q)$ , hence that the ultraviolet cutoff is set at a wave number  $Aq$ ,  $A \gg 1$ . This has the big advantage that the real correction to the angular eigenfrequency  $\epsilon_{\mathbf{q}}^{(2)}$  due to  $\hat{H}_3$  to the leading order is  $O(q^5)$  and it affects neither the  $\gamma$  parameter in equation (63) nor the damping rate  $\Gamma_{\mathbf{q}}$  to the order  $\tilde{q}^7$ .
- [62] The Taylor contribution of order  $n$  is  $O(q^{2n+5})$  as the  $n$ th order derivative of the self-energy, that involves an energy denominator to the power  $n+1$ , is of order  $q^8/q^{3(n+1)}$  accounting for the "small denominators" effect.
- [63] We supposed here that the leading contribution to  $\delta\Gamma_{\mathbf{q}}$  comes from the configuration "bicone" in which both  $\mathbf{q}_2$  and  $\mathbf{q}_3$  are in the forward emission cone of angular aperture  $O(\tilde{q})$  with respect to the direction of  $\mathbf{q}$ . One can imagine a more subtle scenario, called "unicone", in which only the vector  $\mathbf{q}_3$  would be in this cone, while  $\mathbf{q}_2$  would be at an angle  $\approx \tilde{q}^0$  with  $\mathbf{q}$ . In this case, only the denominator of  $P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)$  is  $\approx \tilde{q}^3$ , while that of  $P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)$  is  $\approx \tilde{q}$ . The crossed term  $P(\mathbf{q}_1, \mathbf{q}_3|\mathbf{q}_2)P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)$  is then negligible with respect to  $[P(\mathbf{q}_1, \mathbf{q}_2|\mathbf{q}_3)]^2$ . The global energy denominator  $\epsilon_{\mathbf{q}} + i\eta - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})$  of (B23) is also

$\approx \tilde{q}$ , which makes us lose a factor  $\tilde{q}^2$ , but this is exactly compensated by the loss of a factor  $\tilde{q}^2$  in the numerator in the polar integral  $\int d\theta_2 \sin \theta_2$ . The ensemble seems then to contribute to the same order  $\tilde{q}^7$  as the bicone configuration. The integration over the polar angle  $\theta_3$  rescaled as in (B26), with the inclusion of the three-dimensional Jacobian  $\tilde{q}_3^2$  from the integration over  $\bar{q}_3$ , however leads to

$$\int_0^{+\infty} \frac{\tilde{q}_3^2 \tilde{\theta}_3 d\tilde{\theta}_3}{\left[ \frac{3}{4} \tilde{q}_3 (1 - \bar{q}_3) - \frac{\tilde{q}_3 \tilde{\theta}_3^2}{1 - \bar{q}_3} + i\eta \right]^2} = -\frac{\bar{q}_3 (1 - \bar{q}_3) / 2}{\frac{3}{4} \tilde{q}_3 (1 - \bar{q}_3) + i\eta}$$

which has a real limit  $-\frac{2}{3}$  when  $\eta \rightarrow 0^+$ . Hence the sought imaginary part in  $\delta\Gamma_{\mathbf{q}}$  may only come from the global energy denominator, which would provide a Dirac  $\delta[\epsilon_{\mathbf{q}} - (\epsilon_{\mathbf{q}_1} + \epsilon_{\mathbf{q}_2} + \epsilon_{\mathbf{q}_3})]$ ; however, to the leading order in  $\tilde{q}$  that we consider here,  $\mathbf{q}_3 = \bar{q}_3 \mathbf{q}$ , with  $0 < \bar{q}_3 < 1$ , and the dispersion relation is linear, so that the argument of the Dirac can't be zero unless  $\mathbf{q}_2$  and  $\mathbf{q}$  are collinear in

the same direction, which is in contradiction with the hypothesis of having a  $\mathbf{q}_2$  outside the forward emission cone. The same arguments hold for the damping rate  $\Gamma_{\mathbf{q}}^{2\leftrightarrow 2}$  of section VC, and justifies that we only have considered there the ‘‘bicone’’ configuration.

[64] This phenomenon does not occur in the contribution (B34) because neither the factor containing the term  $X \sin^2 \alpha$ , dangerous when  $\alpha \rightarrow 0$ , nor the one containing the term  $X \cos^2 \alpha$ , dangerous when  $\alpha \rightarrow \pi/2$ , are taken to the square. As a consequence, the integral over  $X$  from 0 to  $+\infty$  diverges only as  $\ln \alpha$  or  $\ln(\frac{\pi}{2} - \alpha)$  when  $\alpha \rightarrow 0$  or  $\alpha \rightarrow \pi/2$ . Mathematically, by replacing in equation (B34) the integral  $\int_0^{+\infty} dX f(X)$  by  $-\int_{-\infty}^0 dX f(X)$  from the identity (B38), then by using the dominated convergence theorem, we can justify the exchange of the limit  $\eta \rightarrow 0^+$  and the integration over  $\bar{q}_2, \bar{q}_3, \phi$  and  $\alpha$ , and prove the result (B35).