A classification of nodes for structural controllability

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Abstract

In this paper we consider (large and complex) interconnected networks. We assume that each state/node of the network can be selected to act as a steering node, meaning that such node then is influenced by its own individual control. This control may influence other nodes indirectly through steering node which it is controlling. The goal in this paper is to select steering nodes such that the overall system (in)directly becomes structurally controllable, where "structurally" means that the system is controllable for almost all of its numerical realizations. Obviously, it may happen that not every state needs to act as a steering node. In fact, in configurations where the overall system is required to be structurally controllable, some states may never need to be steering node, while some other states always have to be a steering node. In this paper, we aim to achieve structural controllability and we present a classification of the associated steering nodes as being essential (always required to be present), useful (present in certain configurations) and useless (never necessary in whatever configuration). The classification is based on two types of decomposition that naturally show up in the context of the two conditions (connection condition and rank condition) for structural controllability. The underlying methods are related to well-known and efficient network algorithms. The main result of the paper is the characterization of useless, useful and essential steering nodes in order to obtain a system that is structurally controlable. The results are illustrated by means of examples.

Keywords: Controllability, structured system theory, input connection condition, rank condition, steering node.

1. Introduction

The study and analysis of large complex networks has become extremely important as nowadays society becomes more and more interconnected. Examples of such complex systems appear in biology, genetics, social networks, large communication or energy networks, etc. After the analysis of the behavior of a particular complex network, one of the most relevant questions is whether its behavior can be controlled. Therefore, the controllability of complex networks has received a lot of attention in the recent years. Especially, the question of interest is where to put so-called driver nodes by which the behavior of the network can be controlled, see [10, 18, 25].

Here we call a state/node of the network a steering node if it can be influenced from the outside of the network (externally) by a control. Various types of control can be distinguished, like controls that influence possibly all nodes simultaneously, controls that each influence an individual node, or a mix of the previous two, see [4].

Next to the question of where to put the steering nodes, also their minimal number is a relevant issue. Clearly, having as few as possible steering nodes may be important in applications.

In this paper, we focus on the situation that each steering node is supposed to be controlled by precisely one control. In principle, all nodes of a network may act as steering nodes. However, it is clear that occasionally for some nodes it is useless to be a steering node because they can also be controlled indirectly by other nodes of the network. Therefore, a classification seems appropriate of which nodes, seen as steering nodes, are always useless, which nodes may play a role in certain steering node configurations, and which nodes are essential in the sense that they always have to be present in any appropriate steering node configuration.

The above classification can be compared with the component classification in terms of the impact in case of their failure. This is a classical field of reliability theory, see [2, 3]. Also in engineering, the classification of sensors in terms of their importance for preserving some property (such as observability) is an active research field, see [7]. In this paper we study the classification of steering nodes in terms of their importance for controllability. It should be noted that, although we use here some concepts and tools of [7], the present paper is much more than a dualisation of the results of [7]. The main difference is that our classification does not rely on a given set of inputs and corresponding actuators. Here the inputs have to be chosen in a given set and they impact only one state.

The steering node classification in this paper resembles a similar classification (in critical, redundant and intermittent nodes) in [16], see also [11] for a quantitative approach of the importance of nodes. Notice that an alternative way to classify the nodes is to compute for each of them the control centrality [19]. However, opposite to [16, 18], and as already indicated, we suppose that a control can only influence one steering node. The latter also seems to be much more realistic in applications. For this reason, we have to also take into account the input connection condition, apart from the rank condition that needs to be considered only when allowing controls that may influence several (or even all) steering nodes simultaneously.

The input connection condition can most easily be studied by decomposing the digraph associated to our networks into strongly connected components that are ordered themselves in an acyclic way. The rank condition that has to be checked can best be investigated by the so-called DM-decomposition. By using this decomposition the results in this paper can be presented in a more transparent and refined way than the results in [16], where such decomposition is not considered.

We extend our classification to the case where some nodes cannot be influenced by inputs. This situation, which is often met in practice, has also been considered in [22].

Our results are related with the recent literature about the Minimal Controllability Problem [4, 5, 21, 22, 1, 23].

The outline of this paper is as follows. In section 2 we introduce the type of systems that we consider and formulate the problem of steering node selection such that a system becomes controllable. We also introduce the notions of useless, useful and essential nodes for a specific property that we want the system to have. In section 3 we introduce structured systems, introduce their graph representation and recall some well-known conditions for structural controllability. The conditions consist of an input connection condition and a rank condition. In section 4 the two conditions are further analysed using connectivity aspects of the directed graph introduced in section 3, and for the rank condition by means of a DM-decomposition of the bipartite graph associated to the structured system. In section 5 we present conditions for a steering node to be useless, useful or essential for each of the two conditions for structural controllability. Our main result is the combination of all criteria to get characterizations of steering nodes that are useless, useful or essential for structural controllability as a whole. In section 6 we comment on the obtained results and illustrate them by means of two examples. In section 7 we extend the classification to the case where some nodes cannot be used as a steering node. In section 8 we pay some attention to the computational aspects of verifying the conditions obtained in this paper. We conclude by section 9 with a summary of the results of this paper and with some topics for future research.

2. Problem formulation

2.1. The controllability problem

In this paper, we consider a large scale system composed of n states interacting together with linear dynamics. We assume that we can represent the behavior of the whole system by the simple equation

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state vector and A is a real constant $n \times n$ matrix.

We will distinguish m states, called the steering nodes $\mathcal{S} = \{x_{i_1}, \ldots, x_{i_m}\}$, with $i_j \in \{1, \ldots, n\}$ and $i_1 < i_2 < \cdots < i_m$. To each steering node x_{i_j} we associate a control input u_j that acts only on this state node. In this way we obtain a system that can be represented as

$$\Sigma : \;\; \dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

where matrix B has m columns and its j-th column has all its entries equal to 0 except for $b_{i_j} (\neq 0)$, being the i_j-th component of column j of B. Hence, the node set $\mathcal{S}$ is in 1-1 correspondence with the (structure of) matrix B.

In the following we will be looking for a set of steering nodes such that the pair (A, B), as introduced above, is controllable. Here we understand by controllability of system Σ, see (2), the ability to steer Σ, in final time $\tau > 0$, from an arbitrary initial condition $x_0 = x(0)$ to an arbitrary final condition $x_1 = x(\tau)$, by the use of an appropriate control function $u(t)$ over the time interval $[0, \tau)$.

It is well-known that controllability of system Σ is equivalent, for instance, to rank $[B, AB, \cdots, A^{n-1}B] = n$. For this reason, the controllability of Σ is also referred to as the controllability of the pair (A, B).

A challenging problem is to find the minimal number of steering nodes such that (A, B) is controllable. This problem was tackled in [21], where it was proved to be NP-hard, see [14] for more on this notion. Moreover, it is proved in [21] that the minimum number of inputs needed for controllability in our context (one input acting on one state) is also the minimum number of states which are impacted by inputs when no structure is imposed to the B matrix. Minimizing the number of states that are impacted by inputs is certainly a more convincing way to approach the controllability of large scale systems. It is more appropriate than minimizing the number of inputs, without taking into account the number states impacted by these inputs as in [18, 25].

2.2. Steering node classification

A property $\mathcal{P}$ of the pair (A, B) can be seen as a mapping from the set $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ into the set $\{0, 1\}$ where the property $\mathcal{P}$ is true precisely when $\mathcal{P}(A, B) = 1$.

When a set of steerings nodes $\mathcal{S} \subseteq \mathcal{X}$, defining matrix B, and therefore also the pair (A, B), is such that a given property $\mathcal{P}$ is true, we call $\mathcal{S}$ an *admissible steering node set* for property $\mathcal{P}$.

Definition 1. *For a given property $\mathcal{P}$, a node x_i seen as steering node, can be classified as follows, see for instance [7, 8, 27].*

1. *Node x_i is called a* useless *steering node if for any admissible steering node set $\mathcal{S}$ containing x_i, $\mathcal{S}\backslash\{x_i\}$ is still an admissible steering node set for $\mathcal{P}$, where $\mathcal{S}\backslash\{x_i\}$ is the set $\mathcal{S}$ minus the node x_i.*
2. *A steering node which is not useless is called a* useful *steering node. Hence, node x_i is useful if there is an admissible steering node set $\mathcal{S}$ for $\mathcal{P}$ such that $x_i \in \mathcal{S}$, while $\mathcal{S}\backslash\{x_i\}$ is not admissible for $\mathcal{P}$.*
3. *Node x_i is called an* essential *steering node if x_i belongs to any admissible steering node set $\mathcal{S}$ for $\mathcal{P}$. Hence, x_i is an essential node if $\mathcal{S}\backslash\{x_i\}$ is not admissible for any admissible steering node set $\mathcal{S}$ for $\mathcal{P}$. The set of essential steering nodes is a subset of the set of useful steering nodes.*

In this paper we will focus our attention on the search and classification of steering nodes for the controllability in the context of structured systems.

3. Linear structured systems and structural controllability

In the remainder we assume that system (1) is structured, meaning that we assume that only the zero/nonzero of the entries in matrix A is known.

A structured system of type (1) can be associated with a directed graph $G(A) = (\mathcal{X}, \mathcal{W})$ as follows:

- the node set is $\mathcal{X}$, being the set of state nodes $\{x_1, x_2, \ldots, x_n\}$,
- the edge set is $\mathcal{W} = \{(x_i, x_j) | a_{ji} \neq 0\}$, where a_{ji} denotes the (j, i)th entry of matrix A and $a_{ji} \neq 0$ means that the (j, i)-th entry of A is a structural nonzero.

We define a path in $G(A)$ from a node x_{i_0} to a node x_{i_q} to be a sequence of edges, $(x_{i_0}, x_{i_1}), (x_{i_1}, x_{i_2})\ , \ldots, (x_{i_{q-1}}, x_{i_q})$, such that $x_{i_t} \in \mathcal{X}$ for $t = 0, 1, \ldots, q$, and $(x_{i_{t-1}}, x_{i_t}) \in \mathcal{W}$ for $t = 1, 2, \ldots, q$.

Let A be a structured matrix and let the total number of nonzero entries in A be given by l, then the structured system can be parameterized by a parameter vector $\lambda \in \mathbb{R}^l$, where each component of λ corresponds to precisely one nonzero. Hence, for each vector λ, a completely numerically specified system is obtained. The system is thus defined as

$$\dot{x}(t) = A_\lambda x(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and A_λ is the parameterized version of the structured matrix A.

The previous systems are called linear structured systems if only the zero/nonzero structure of the matrix A is known, or, equivalently, if the entries of the matrix A_λ are either fixed zeros or independent parameters (not related by algebraic equations), see [12, 20]. For such systems, we can study generic properties, *i.e.*, properties which are true for almost any value of the parameters.

One such property is, for instance, the generic controllability of a structured system. Another such property is the generic rank of a structured matrix. Given a structured matrix Q with k nonzeros and a parametrized version Q_μ, the generic rank of Q will be defined as the rank of Q_μ for almost all $\mu \in \mathbb{R}^k$ and will be denoted by g-rank Q. It can be shown that g-rank $Q = \max\limits_{\mu \in \mathbb{R}^k}$ rank Q_μ. Later in this paper the generic rank of matrix A will be used.

Example 1. *Consider the system defined by the structured matrix*

$$A = \begin{pmatrix} 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

or by a parametrized version with $\lambda \in \mathbb{R}^{11}$ *given by*

$$A_\Lambda = \begin{pmatrix} 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 & 0 & 0 & \lambda_5 & \lambda_6 \\ 0 & 0 & 0 & 0 & \lambda_7 & 0 & 0 & 0 \\ \lambda_8 & 0 & 0 & \lambda_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{10} & \lambda_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The associated digraph $G(A)$ is depicted in Figure 1. In the figure, for later use, also the strongly connected components are already indicated.

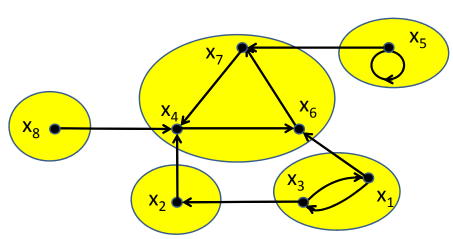

Figure 1: Digraph of Example 1

As in Section 2, we can select a set of m steering nodes in $\mathcal{X}$, to which we associate m control inputs. This induces an $n \times m$ matrix B with only m nonzero entries, one in each column and at most one in each row.

We can also parametrize the nonzeros in matrix B yielding a parameterized version B_λ where each nonzero entry in B is replaced by a parameter and in which the parameter vector λ is updated to also include the parameters in B_λ. Then λ includes the parameters of all nonzero entries in A and B.

We get then a controlled system Σ_λ, which is the parametrized counterpart of Σ defined in (2), described by

$$\Sigma_\lambda: \ \dot{x}(t) = A_\lambda x(t) + B_\lambda u(t). \tag{4}$$

The graph $G(\Sigma)$ can be obtained from $G(A)$ by adding the m input nodes through the set $\mathcal{U} = \{u_1, \ldots, u_m\}$ and m edges, one from each input node to the corresponding steering node. Hence, $G(\Sigma)$ has node set $\mathcal{X} \cup \mathcal{U}$, and the edge set is updated as $\mathcal{W} := \mathcal{W} \cup \{(u_j, x_{i_j}) | j = 1, 2, \ldots, m\}$.

In $G(\Sigma)$, a path $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{q-1}}, v_{i_q})$, where $v_{i_0} \in \mathcal{U}$ and $v_{i_q} \in \mathcal{X}$, is called an input-state path. The system Σ is said to be *input-connected* if for any state node $x_i, i = 1, \ldots, n$, there exists an input-state path with end node x_i.

The notion of structural controllability was introduced and studied by Lin, who proved a necessary and sufficient condition for structural controllability in terms of graph theoretic objects called cacti, see [17]. The following result can proved to be equivalent to Lin's result, see for instance [24, 26].

Theorem 1. *Let Σ be the linear structured system defined by (2) with associated graph $G(\Sigma)$. The system is structurally controllable if and only if*

1. *the system Σ is input-connected,*
2. *g-rank$[A, B]$=n.*

In the following, the conditions 1 and 2 of Theorem 1 will be referred to as the input connection condition and the rank condition, respectively.

Given a structured system of type (1), with associated graph $G(A)$, the steering node selection problem then amounts to extend $G(A)$ with edges (input-steering node) in such a way that the conditions of Theorem 1 are fulfilled in the extended graph $G(\Sigma)$.

4. Structural controllability via steering node selection

We will first look at the two conditions for structural controllability individually.

4.1. Input connection condition

Consider the linear structured system defined by (1) with its associated graph $G(A)$. A strongly connected component $\mathcal{C}$ is defined to be a maximum set of nodes of $G(A)$ such that there exists a path, possibly of length zero, between any two nodes of $\mathcal{C}$. The graph can be partitioned into a set of strongly connected components and this set can be endowed with a partial order. We are then interested in strongly connected components which have no incoming edges from other strongly connected components.

Definition 2. *A strongly connected component of $G(A)$ with no incoming edge from another strongly connected component is called a* Critical Connection Component *(CCC). The number of Critical Connection Components is called the* connection defect *and is denoted by $d_c(A)$.*

From [4, 9], we can deduce a condition for a set of steering nodes to be admissible for the input connection condition.

Proposition 1. *Consider the linear structured system defined by (1) with associated graph $G(A)$. A steering node set $\mathcal{S}$ is admissible for the input connection condition if and only if there exists a node of $\mathcal{S}$ in any Critical Connection Component of $G(A)$.*

The previous notions and results can be illustrated on Example 1. The graph possesses five strongly connected components, namely $\{x_1, x_3\}$, $\{x_2\}$, $\{x_5\}$, $\{x_8\}$ and $\{x_4, x_6, x_7\}$, where $\{x_1, x_3\}$, $\{x_5\}$ and $\{x_8\}$ are the Critical Connection Components, so that $d_c(A) = 3$. It follows from Proposition 1, and it is clear from the graph, that input connection is verified if and only if x_5, x_8, and either x_1 or x_3 are steering nodes.

4.2. Rank condition

We next characterize the rank condition of Theorem 1, *i.e.*, g-rank$[A, B] = n$. We do this by first computing the generic rank of matrix A. Once this rank is known, the rank condition of Theorem 1 can be fullfilled by an appropriate choice of the steering nodes. The previous implies that the rank condition of Theorem 1 can be studied by looking at the generic rank of matrix A. The *rank defect* $n-$ g-rank (A), which is denoted $d_r(A)$, equals the number of steering nodes needed to make the rank condition of Theorem 1 become true.

Recall that in this paper matrix B is of special form as it consists of columns each having only one structural nonzero at distinct rows and therefore can be identified with a set of steering nodes.

The generic rank of A will be computed using a bipartite graph associated with the system (1). This bipartite graph $H(A)$ can be introduced as follows. The bipartite graph associated with system (1) is $H(A) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ where the sets $\mathcal{V}^+$ and $\mathcal{V}^-$ are two disjoint node sets, and $\mathcal{W}'$ is the edge set.

- The node set $\mathcal{V}^+$ is given by $\{x_1^+, \ldots, x_n^+\}$ and the node set $\mathcal{V}^-$ is given by $\{x_1^-, \ldots, x_n^-\}$. Notice that we have in fact split each state node x_i of $G(A)$ into two nodes x_i^+ and x_i^-.
- The edge set $\mathcal{W}'$ is described by $\mathcal{W}' = \{(x_i^+, x_j^-) | a_{ji} \neq 0\}$. In the latter, as before, $a_{ji} \neq 0$ means that the (j, i)-th entry of the matrix A is a structural nonzero.

A *matching* in the bipartite graph $H(A) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ is a set of edges $\mathcal{M} \subseteq \mathcal{W}'$ such that the edges in $\mathcal{M}$ have no common node. A node is covered by a matching if there exists an edge in the matching that is incident to the node. The cardinality of a matching, *i.e.*, the number of edges it consists of, is also called its size. A matching $\mathcal{M}$ is called maximum if its cardinality is maximum. The maximum matching problem consists of finding a matching of maximum cardinality. The maximum matching problem can be solved by using efficient combinatorial algorithms, see for example [15]. We recall the following proposition, see [9].

Proposition 2. *Consider the linear structured system defined by (1) with associated bipartite graph $H(A)$. The generic rank of A is equal to the size of a maximum matching in $H(A)$. In particular, $d_r(A) = n-$ g-rank(A) is the minimal number of steering nodes needed to make the rank condition of Theorem 1 become true.*

A useful tool to parameterize all maximum matchings in a bipartite graph is the Dulmage-Mendelsohn decomposition, see [13], abbreviated as DM-decomposition, which will be presented now, see also [20]. The DM-decomposition of the bipartite graph $H(A) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ is the uniquely defined family of bipartite subgraphs $H_i = (\mathcal{V}_i^+, \mathcal{V}_i^-; \mathcal{W}_i')$, called the DM-components, where $\{\mathcal{V}_0^+, \mathcal{V}_1^+, \cdots, \mathcal{V}_r^+, \mathcal{V}_\infty^+\}$ is a partition of $\mathcal{V}^+$, and likewise for $\mathcal{V}^-$ and $\mathcal{W}'$. In the decomposition, the bipartite subgraph H_0 is called *minimal inconsistent part*, the bipartite subgraph H_∞ is called *maximal inconsistent part*, and the other subgraphs $H_i, i = 1, \ldots, r$, are called *consistent parts*. The used names come from a partial ordering $\prec$ that can be assigned to the bipartite subgraphs, implying that $H_0 \prec H_i \prec H_\infty$, for $i = 1, \ldots, r$.

The DM-decomposition and the above components have the following properties, for details see [20].

Proposition 3. *Let $H(A) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ be a bipartite graph having a DM-decomposition, with $H_i = (\mathcal{V}_i^+, \mathcal{V}_i^-; \mathcal{W}_i')$, $i = 0, 1, \ldots, r, \infty$, as its DM-components. Then we have the following.*

1. *A maximum matching on $H(A)$ is a union of maximum matchings on the DM-components H_i, $i = 0, 1, \ldots, r, \infty$.*
2. *Every node of $\mathcal{V}_0^-$ (or $\mathcal{V}_\infty^+$,or $\mathcal{V}_i^+$, $\mathcal{V}_i^-$, $i = 1, \ldots, r$) is covered by any maximum matching on $H(A)$.*
3. *A node $v^+ \in \mathcal{V}^+$ belongs to the minimal inconsistent part H_0 if and only if there exists a maximum matching on $H(A)$ that does not cover node v^+, implying that $|\mathcal{V}_0^+| > |\mathcal{V}_0^-|$.*
4. *A node $v^- \in \mathcal{V}^-$ belongs to the maximal inconsistent part H_∞ if and only if there exists a maximum matching on $H(A)$ that does not cover node v^-, implying that $|\mathcal{V}_\infty^-| > |\mathcal{V}_\infty^+|$.*

Note that for the consistent parts there holds $|\mathcal{V}_i^+| = |\mathcal{V}_i^-|$ for $i = 1 \ldots, r$.

The rank condition for controllability can then be expressed using only the maximal inconsistent part of the DM-decomposition as follows, see [6].

Proposition 4. *Consider the linear structured system defined by (1) with associated bipartite graph $H(A)$ and the corresponding DM-decomposition. We have that g-rank$(A) = n - \big(|\mathcal{V}_\infty^-| - |\mathcal{V}_\infty^+|\big)$.*

The result follows from Proposition 3, especially point 4. Indeed, the size of a maximum matching equals $|\mathcal{V}_0^-| + |\mathcal{V}_\infty^+| + |\mathcal{V}_1^\pm| + \ldots + |\mathcal{V}_r^\pm|$, whereas $n = |\mathcal{V}_0^-| + |\mathcal{V}_\infty^-| + |\mathcal{V}_1^-| + \ldots + |\mathcal{V}_r^-|$, where $|\mathcal{V}_i^+| = |\mathcal{V}_i^-|$ for $i = 1 \ldots, r$.

Proposition 5. *Consider the linear structured system defined by (1) with associated bipartite graph $H(A)$ and the corresponding DM-decomposition. A steering node set $\mathcal{S}$ is admissible for the rank condition if and only if there exists a maximum matching of the bipartite subgraph H_∞ such that for every node x_i^- in $\mathcal{V}_\infty^-$ that is not covered by the matching, there holds $x_i \in \mathcal{S}$.*

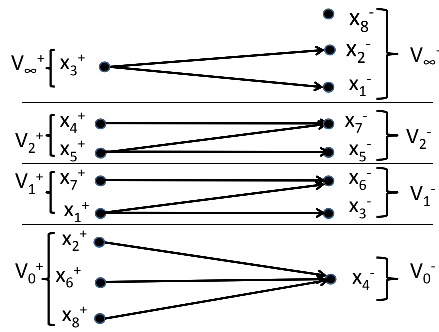

Figure 2: DM-decomposition of Example 1

The DM-decomposition corresponding to the bipartite graph associated with Example 1 is given in Figure 2. The maximum size of a matching in $H(A)$ is 6. Hence, the generic rank of A is equal to 6 and $d_r(A) = 2$. From Proposition 5 it follows that a maximum matching of H_∞ can be (x_3^+, x_1^-), which implies that a possible admissible steering node set for the rank condition is $\{x_2, x_8\}$, but also $\{x_1, x_8\}$ can act an admissible steering node set when (x_3^+, x_2^-) is choosen as maximum matching of H_∞.

5. Steering node classification for structural controllability

We start now with a structured system of type (1), hence only defined by the matrix A. As before, we denote the associated directed graph by $G(A)$ and the associated bipartite graph by $H(A)$. In this section we will give a classification of steering nodes according to the definitions of Section 2. We will provide this classification first for each condition (input connection condition and rank condition) and then for controllability.

5.1. Classification of steering nodes for input connection

For the input connection condition we have the following.

Proposition 6. *Consider a linear structured system of type (1) with associated graph $G(A)$. For the input connection property, node x_i, being element of an admissible steering node set $\mathcal{S}$ for the input connection condition, is*

1. *useless if and only if it belongs to no Critical Connection Component,*
2. *useful if and only if it belongs to a Critical Connection Component,*

3. *essential if and only if it belongs to a Critical Connection Component of cardinality one.*

Proof:

1. $\Rightarrow$ Assume that x_i belongs to a Critical Connection Component $\mathcal{C}_j$ and consider an admissible steering node set $\mathcal{S}$ composed of a steering node in each Critical Connection Component with x_i the only steering node in $\mathcal{C}_j$. Then $\mathcal{S}\backslash\{x_i\}$ is no longer an admissible steering node set, and consequently x_i is not useless.
   $\Leftarrow$ Conversely, assume now that the steering node x_i belongs to no Critical Connection Component, and consider any admissible steering node set $\mathcal{S}$ such that $x_i \in \mathcal{S}$. From Proposition 1, the set $\mathcal{S}$ must contain a node in each Critical Connection Component, therefore $\mathcal{S}\backslash\{x_i\}$ is still an admissible steering node set and x_i is useless.
2. Obvious from point 1.
3. $\Rightarrow$ Assume that x_i belongs to a Critical Connection Component $\mathcal{C}_j$. Hence, by the above point, x_i is usefull. Now assume that $\mathcal{C}_j$ of cardinality more than one. Denote by x_k another node of $\mathcal{C}_j$. We can construct an admissible steering node set containing x_k, but not x_i. Therefore, x_i is not essential.
   $\Leftarrow$ Conversely, if x_i belongs to a Critical Connection Component $\mathcal{C}_j$ of cardinality one, it follows from Proposition 1, that any admissible steering node set must contain x_i. Therefore x_i is essential.

Note that $\{x_i\}$ is a Critical Connection Component if and only if there is no edge (x_j, x_i) in $G(A)$ for $j \neq i$. Hence, node x_i is essential for the input connection condition if and only if there is no edge (x_j, x_i) in $G(A)$ for $j \neq i$.

5.2. Classification of steering nodes for the rank condition

For the rank condition we have the following.

Proposition 7. *Consider a linear structured system of type (1) with associated bipartite graph $H(A)$ and the corresponding DM-decomposition. For the rank condition property, node x_i, being element of an admissible steering node set $\mathcal{S}$ for the rank condition, is*

1. *useless if and only if x_i^- does not belong to the $\mathcal{V}_\infty^-$ set of the DM-decomposition,*
2. *useful if and only if x_i^- belongs to the $\mathcal{V}_\infty^-$ set of the DM-decomposition,*

3. *essential if and only if there exists no edge (x_j^+, x_i^-) in $H(A)$.*

Proof:

1. $\Rightarrow$ Assume that $x_i^- \in \mathcal{V}_\infty^-$. Then according to point 4 of Proposition 3, there exists a maximum matching on the bipartite subgraph H_∞ that does not cover x_i^-. Denote such matching by $\mathcal{M}_\infty$ and denote the nodes of $\mathcal{M}_\infty$ incident to $\mathcal{V}_\infty^-$ by $\mathcal{M}_\infty^-$. Define $\mathcal{S}^- = \mathcal{V}_\infty^- \backslash \mathcal{M}_\infty^-$ and let $\mathcal{S}$ denote the corresponding nodes in $\mathcal{X}$, i.e., $x_k^- \in \mathcal{S}^- \iff x_k \in \mathcal{S}$. Then, $\mathcal{M}_\infty^- \cup \mathcal{S}^- = \mathcal{V}_\infty^-$, $\mathcal{M}_\infty^- \cap \mathcal{S}^- = \emptyset$ and $|\mathcal{S}^-| = |\mathcal{V}_\infty^-| - |\mathcal{V}_\infty^+|$. According to Proposition 5 the set $\mathcal{S}$ is admissible. Clearly, $x_i \in \mathcal{S}^-$. Now leave x_i out of $\mathcal{S}$, or equivalently x_i^- out of $\mathcal{S}^-$, and consider $\mathcal{S}\backslash\{x_i\}$. Then according to Proposition 5, the set $\mathcal{S}\backslash\{x_i\}$ is not admissible. Indeed, any maximum matching on H_∞ has size $|\mathcal{V}_\infty^+|$. The number of nodes not covered by such matching is $|\mathcal{V}_\infty^-| - |\mathcal{V}_\infty^+|$, implying that at least one of the uncovered nodes can also not be covered by $\mathcal{S}^-\backslash\{x_i^-\}$, as $|\mathcal{S}^-\backslash\{x_i^-\}| < |\mathcal{V}_\infty^-| - |\mathcal{V}_\infty^+|$. With $\mathcal{S}\backslash\{x_i\}$ being not admissible, it follows that x_i is useful, i.e., is not useless.
   $\Leftarrow$ Assume that $x_i^- \notin \mathcal{V}_\infty^-$. Then $x_i^- \in \mathcal{V}^-\backslash\mathcal{V}_\infty^-$. Since $\mathcal{S}$ is admissible, by Proposition 5, there exists a maximum matching on the bipartite subgraph H_∞ such that for every $x_j^- \in \mathcal{V}_\infty^-$ not covered by this matching there holds that $x_j \in \mathcal{S}$. Now fix this matching. Recall that $x_i \in \mathcal{S}$ and leave x_i out of $\mathcal{S}$. Then still for every $x_j^- \in \mathcal{V}_\infty^-$ not covered by the matching there holds that $x_j \in \mathcal{S}\backslash\{x_i\}$. Hence, by Proposition 5, it follows that $\mathcal{S}\backslash\{x_i\}$ is also admissible and therefore x_i is useless.
2. Obvious from point 1.
3. $\Rightarrow$ Suppose that there exists an edge (x_j^+, x_i^-). Since x_i is at least useful, it follows from the ordering of the DM-components, see [20], that $x_j^+ \in \mathcal{V}_\infty^+$. Then there always exists a maximum matching of H_∞ that covers x_i^-. Indeed, if a given maximum matching does not cover x_i^-, we can exchange the edge (x_j^+, x_k^-), which belongs to the maximum matching, with (x_j^+, x_i^-) to get a new maximum matching of H_∞ that does cover x_i^-. The nodes in $\mathcal{V}_\infty^-$ not covered by this new matching induce an admissible steering node set that does not contain x_i. Therefore, x_i is not essential.
   $\Leftarrow$ Assume that $x_i^- \in \mathcal{V}_\infty^-$. If $H(A)$ has no edge (x_j^+, x_i^-), then no (maximum) matching on H_∞ will ever cover x_i^-. Hence, by Proposition 5, node x_i has to be in any admissible steering node set $\mathcal{S}$. Hence, x_i

is contained in any admissible steering node set and is consequently essential.

The previous results can be compared with those of [16] (more precisely with the Supplementary Information associated with this paper). Point 3 of our Proposition 7 corresponds to the Supplementary Note 1, but our proof is purely a graph theoretic one, instead of using integer programming arguments as in [16]. Point 1 of our Proposition 7 corresponds to the Supplementary Note 6, but our characterisation of useless nodes is much more explicit than in [16].

5.3. Classification of steering nodes for controllability

Next we combine the previous results to obtain a classification of steering nodes for structural controllability.

Theorem 2. *Consider a linear structured system of type (1) with associated graph $G(A)$, associated bipartite graph $H(A)$ and the corresponding DM-decomposition. For controllability, the steering node x_i, being an element of a steering node set $\mathcal{S}$ that is admissible for controllability, is*

1. *essential if and only if there exists no edge (x_j, x_i) for $j \neq i$ in $G(A)$.*
2. *useless if x_i belongs to no Critical Connection Component and x_i^- does not belong to the $\mathcal{V}_\infty^-$ set of the DM-decomposition.*

Proof:

1. $\Rightarrow$ Note that x_i is a node in the steering node set $\mathcal{S}$ that is admissible for controllability. Then $\mathcal{S}$ is a steering node set that is also admissible for both the input connection condition and the rank condition. Now assume that there is an edge (x_j, x_i) in $G(A)$, or equivalently (x_j^+, x_i^-) in $H(A)$. Then, by Propositions 6 and 7, node x_i is not essential for both the input connection condition and the rank condition. Hence, there exist steering node sets $\mathcal{S}_c$ and $\mathcal{S}_r$ that are admissible for the input connection and the rank condition, respectively, that both do not contain x_i. Then the union $\mathcal{S}_c \cup \mathcal{S}_r$ does not contain x_i too, and is an admissible steering node set for both the input connection condition and the rank condition, and consequently also for controllabiliy. Hence, node x_i is not essential for controllability.
   $\Leftarrow$ If there is no edge (x_j, x_i) for $i \neq j$ in $G(A)$, then either there is no edge (x_j, x_i) at all, or there is only an edge (x_i, x_i) in $G(A)$. In the

latter case, $\{x_i\}$ is a strongly connected component without incoming edges and it follows from Proposition 6 that x_i is essential for the input connection condition. In the former case, there is no edge (x_j^+, x_i^-) at all in H_∞, and it follows from Proposition 7 that x_i is essential for the rank condition. Being essential for one of the two conditions, it follows that x_i is essential for controllability.

2. Assume that x_i is useless for both properties and consider any steering node set $\mathcal{S}$ containing x_i that is admissible for controllability. The two properties remain satisfied with $\mathcal{S}\backslash\{x_i\}$, therefore controllability is preserved too and consequently x_i is useless for controllability.

6. Some remarks and examples

Some remarkable points follow from Theorem 2.

Remark 1. *From the set of essential steering nodes for controllability, being the union of the essential steering nodes for the two subproperties, it follows that an essential steering node is a node with no incoming edge from another state (but may have a self-loop). The latter can be related with the interesting discussion in [10, 29] on the importance of self-loops in applications.*

Remark 2. *In Theorem 2, we only characterize a subset of the useless steering nodes for controllability (namely those which are useless for both subproperties). Indeed, as will be seen in a next example, some steering nodes may be useless for controllability, while being useful for one of the subproperties.*

Example 1 (cont.) *First we illustrate our main result on Example 1 whose graph $G(A)$ is depicted in Figure 1, while the bipartite graph $H(A)$ and its DM-decomposition are shown in Figure 2.*

From Proposition 6 it follows that nodes x_2, x_4, x_6, x_7 are useless for input connection, while nodes x_1, x_3, x_5, x_8 are useful, with nodes x_5, x_8 being essential, because they correspond to Critical Connection Components of cardinality one.

From Proposition 7 it follows that nodes x_3, x_4, x_5, x_6, x_7 are useless for the rank condition, nodes x_1, x_2, x_8 are useful, with node x_8 being essential, because it has no incoming edge. From Theorem 2 it follows that nodes x_5, x_8 are essential steering nodes for controllability, while nodes x_4, x_6, x_7 are useless for controllability. It can be checked by inspection that besides *nodes x_5, x_8 also the nodes x_1, x_2, x_3 are useful for controllability. For example, node x_2 cannot be discarded from the admissible steering node set $\{x_2, x_3, x_5, x_8\}$ because of the rank condition.*

We give now an example to illustrate Remark 2.

Example 2. *Let A given below be a (3×3) structured matrix, whose digraph $G(A)$ is depicted in Figure 3.*

$$A = \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}.$$

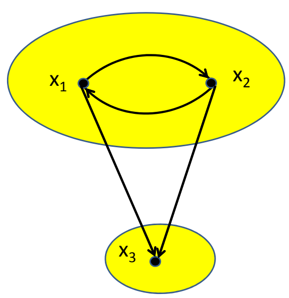

Figure 3: Digraph of Example 2 with strongly connected components

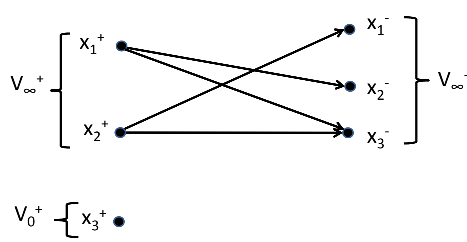

Figure 4: DM-decomposition of Example 2

The graph $G(A)$ possesses two strongly connected components, being $\{x_1, x_2\}$ and $\{x_3\}$, where $\{x_1, x_2\}$ is the Critical Connection Component. Hence, $d_c(A) = 1$.

The DM-decomposition of $H(A)$ is shown in Figure 4. Since the maximum size of a matching in $H(A)$ is 2, the generic rank of A is equal to 2 and $d_r(A) = 1$.

From Proposition 6 it follows that node x_3 is useless for input connection, while nodes x_1 and x_2 are useful.

From Proposition 7 it follows that there are neither useless nor essential nodes for the rank condition, so that nodes x_1, x_2 and x_3 are just useful for the rank condition. From Theorem 2 we get no information about useless steering nodes for controllability because there are no useless steering nodes for both subproperties. However, it is clear that node x_3 is useless for controllability. Indeed, the input connection implies that nodes x_1 or x_2 should be steering nodes, which is also enough to ensure the rank condition.

7. Extension to the case of forbidden nodes

In this Section, as in [22], we assume that there is a set of forbidden nodes, which cannot be used as steering nodes. This situation is frequently met in applications.

Let us denote by $\mathcal{F} \subseteq \mathcal{X}$ the set of forbidden nodes, and the complementary subset by $\mathcal{E} = \mathcal{X} \backslash \mathcal{F}$. The nodes of $\mathcal{E}$ will be called *effective nodes*. These effective nodes $\mathcal{E} = \{x_{i_1}, \ldots, x_{i_p}\}$, with $i_j \in \{1, \ldots, n\}$ and $i_1 < i_2 < \cdots < i_p$, can be associated with control inputs in order to define a steering node set $\mathcal{S} \subseteq \mathcal{E}$ as in Section 2.

We extend the notions and definitions of Section 2, by restricting the possible sets of steering nodes $\mathcal{S}$ to be subsets of $\mathcal{E}$. It is clear that, depending on $\mathcal{E}$, an admissible steering node set for controllability does not necessarily exist. Let us first examine this question.

As was done for the steering node set in Section 2, we can associate a matrix B_E with the effective node set $\mathcal{E}$. The condition of existence of an admissible steering node set for controllability is stated in the following, rather obvious, lemma, which is given without proof.

Lemma 1. *There exists an admissible effective steering node set for controllability for the system (1) if and only if the pair (A, B_E) is controllable.*

If considering the problem in the structured system framework, the controllability of (A, B_E) can be tested using a version of Lin's theorem as in Theorem 1. First define the bipartite graph $H(A, B_E)$ by adding to $\mathcal{V}^+$, on $H(A)$, a set of inputs vertices $u_1, \ldots, u_p$ and edges $(u_j, x_{i_j}^-)$, for $j = i, \ldots, p$.

Using refinements of the two conditions of Theorem 1, see for example [4], the result can be reformulated as follows.

Proposition 8. *There exists an admissible steering node set for controllability of the structured system (1) if and only if*

1. *for any Critical Connection Component $\mathcal{C}_i$ of $G(A)$ we have $\mathcal{C}_i \cap \mathcal{E} \neq \emptyset$,*
2. *the size of a maximum matching in $H(A, B_E)$ is n.*

Definition 1 can readily be extended to the case of forbidden nodes by simply adding that $\mathcal{S} \subseteq \mathcal{E}$, i.e., admissible steering nodes should be effective.

For a given property $\mathcal{P}$, we denote by Es the set of essential nodes and by Us the set of useless nodes when there is no restriction on the set of possible steering nodes, *i.e.*, when $\mathcal{F} = \emptyset$. Denote by $Es_{\mathcal{E}}$ the set of essential nodes and $Us_{\mathcal{E}}$ the set of useless nodes when the set of possible steering nodes is $\mathcal{E}$. Then we have the following simple general result.

Lemma 2. *With the previous definitions, we have:*

1. $Es \subset Es_{\mathcal{E}}$,
2. $Us \cap \mathcal{E} \subset Us_{\mathcal{E}}$.

Proof:

Consider again the definitions of essential and useless nodes in Definition 1. It is obvious that the restriction of the set of possible nodes do not change their properties, and the result follows.

In simple terms, the previous means that the introduction of forbidden nodes can only increase the sets of essential and useless nodes.

Notice that in point 1 of Lemma 2, the intersection with $\mathcal{E}$ need not be explicited, since the essential nodes of the problem without forbidden nodes must belong to any set of effective nodes.

7.1. Classification of steering nodes for input connection

For the input connection condition we have the following.

Proposition 9. *Consider a linear structured system of type (1) with associated graph $G(A)$ and effective node set $\mathcal{E}$. For the input connection property, node x_i, being element of an admissible steering node set $\mathcal{S} \subseteq \mathcal{E}$ for the input connection condition, is*

1. *useless if and only if it belongs to no Critical Connection Component,*
2. *useful if and only if it belongs to a Critical Connection Component,*
3. *essential if and only it is the unique effective node in a Critical Connection Component.*

Proof:

1. The arguments are exactly the same as those of point 1 in the proof of Proposition 6.
2. Obvious from point 1.
3. $\Rightarrow$ Assume that x_i is effective and belongs to a Critical Connection Component $\mathcal{C}_j$. Hence, by the above point, x_i is usefull. Now assume that there exists another effective node $x_k \in \mathcal{C}_j$. We can construct an admissible steering node set containing x_k, but not x_i. Therefore, x_i is not essential.
   $\Leftarrow$ Conversely, if x_i is the unique effective node in a Critical Connection Component $\mathcal{C}_j$, it follows from Proposition 1, that any admissible steering node set must contain x_i. Therefore x_i is essential as effective node

It is immediate to check that Proposition 9 reduces to Proposition 6 when $\mathcal{F} = \emptyset$.

7.2. Classification of steering nodes for the rank condition

Proposition 10. *Consider a linear structured system of type (1) with associated graph $G(A)$, associated bipartite graph $H(A)$, and the corresponding DM-decomposition. Consider a set of effective nodes $\mathcal{E}$, B_E the corresponding input matrix, and $H(A, B_E)$ the associated bipartite graph with its DM-decomposition. For the rank condition property, node x_j, being element of an admissible steering node set $\mathcal{S} \subseteq \mathcal{E}$ for the rank condition, is*

1. *useless if and only if x_i^- does not belong to the $\mathcal{V}_\infty^-$ set of the DM-decomposition of $H(A)$,*
2. *useful if and only if x_i^- belongs to the $\mathcal{V}_\infty^-$ set of the DM-decomposition of $H(A)$,*
3. *essential if and only if the corresponding input node u_{i_j} does not belong to the $\mathcal{V}_0^+$ set in $H(A, B_E)$.*

Proof:

1. $\Rightarrow$ We have at our disposal p effective basis vectors $\{v_{i_1}, \ldots, v_{i_p}\}$ such that rank $[A, v_{i_1}, \ldots, v_{i_p}] = n$. If v_{i_k} is such that rank $[A, v_{i_k}] = \text{rank}A$, or $v_{i_k} \in \text{Span}A$, this node is clearly useless for the rank condition. In graph terms this corresponds to the fact that x_k^- does not belong to the $\mathcal{V}_\infty^-$ set of $H(A)$.
   $\Leftarrow$ The arguments are the same as in the proof of Proposition 7.
2. Obvious from point 1.
3. $\Rightarrow$ Notice first that, since the size of a maximum matching in $H(A, B_E)$ is n, the H_∞ part of $H(A, B_E)$ is empty. Suppose now that u_{i_j} belongs to the $\mathcal{V}_0^+$ set in $H(A, B_E)$. Then there exists a maximum matching in $H(A, B_E)$ which does not cover u_{i_j}. Therefore one can build an admissible steering node set which does not contain x_j, so x_j is not essential.
   $\Leftarrow$ Assume that u_{i_j} does not belong to the $\mathcal{V}_0^+$ set in $H(A, B_E)$. Then, any maximum matching in $H(A, B_E)$ contains u_{i_j} and discarding u_{i_j} (which is equivalent to discard the corresponding effective node x_j from the set $\mathcal{S}$), would decrease the size of a maximum matching. Then $\mathcal{S} \backslash \{x_j\}$ is not admissible and x_j is essential.

7.3. Classification of steering nodes for controllability

The two previous propositions can be combined to get the general classification of nodes for controllability.

Theorem 3. *Consider a linear structured system of type (1) with associated graph $G(A)$, associated bipartite graph $H(A)$, and the corresponding DM-decomposition. Consider a set of effective nodes $\mathcal{E}$, B_E the corresponding input matrix, and $H(A, B_E)$ the associated bipartite graph with its DM-decomposition. For controllability, the steering node x_j, being an element of a steering node set $\mathcal{S} \subset \mathcal{E}$ that is admissible for controllability, is*

1. *essential if and only x_j is the unique effective node in a Critical Connection Component or the corresponding input node u_{i_j} does not belong to the $\mathcal{V}_0^+$ set in $H(A, B_E)$.*
2. *useless if x_j belongs to no Critical Connection Component and x_j^- does not belong to the $\mathcal{V}_\infty^-$ set of the DM-decomposition of $H(A)$.*

Proof: Follows the same lines as the proof of Theorem 2.

Example 1 (cont.) *Consider again Example 1, assume that the set of forbidden nodes is $\mathcal{F} = \{x_1, x_4, x_7\}$ and then that $\mathcal{E} = \{x_2, x_3, x_5, x_6, x_8\}$. As previously, nodes x_5 and x_8 are essential for input connection, but now x_3 is essential too, since it is the only effective node in the Critical Connection Component $\{x_1, x_3\}$. Concerning the rank condition, nodes x_3, x_5 and x_6 in $\mathcal{E}$ are useless since x_3^-, x_5^- and x_6^- do not belong to the $\mathcal{V}_\infty^-$ set of the DM-decomposition of $H(A)$. The DM-Decomposition of the graph $H(A, B_E)$, see Figure 5, shows that the input nodes u_2 and u_8 do not belong to the $\mathcal{V}_0^+$ set*

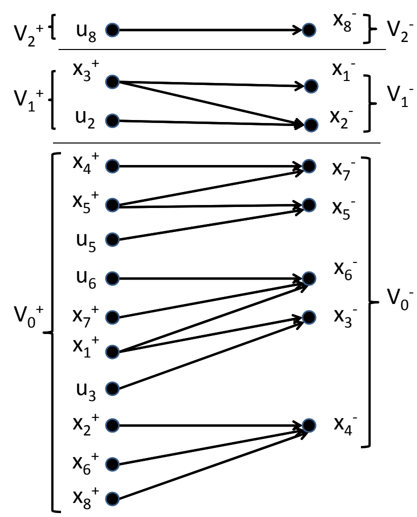

Figure 5: Digraph of bipartite graph of $H(A, B_E)$ in Example 1

in $H(A, B_E)$, so x_2 and x_8 are essential nodes for the rank condition.

In summary, for controllability, as nodes in $\mathcal{E}$ nodes x_2, x_3, x_5 and x_8 are essential, and node x_6 is useless because it does not belong to a Critial Connection Component.

Notice that in this case, $\{x_2, x_3, x_5, x_8\}$ is also a solution for the minimal controllability problem, see [22, 1, 23]. This solution contains 4 *nodes, while when $\mathcal{F} = \emptyset$, it can be checked that $\{x_1, x_5, x_8\}$ is a solution for the minimal controllability problem with only* 3 *nodes.*

Example 3. *In this example the use of forbidden and effective nodes will be illustrated in more details since the example will show that the inclusions in*

Lemma 2 can be strict. Therefore, consider the system described by the 4×4 *structured matrix*

$$A = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & 0 \end{pmatrix}.$$

The digraph $G(A)$ *of the system is displayed in Figure 6. The DM-*

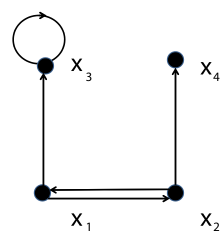

Figure 6: Digraph of Example 3

decomposition of the bipartite graph $H(A)$ *is displayed in Figure 7.*

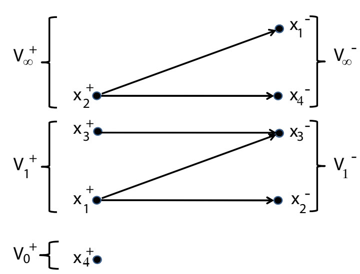

Figure 7: DM-decomposition of bipartite graph of Example 3

From the results of Section 4, we can compute the sets of useless and essential nodes. In summary, in the context of Lemma 2, we have the following sets Es *and* Us *with the associated properties.*

	Es	Us
input connection	$\emptyset$	$\{x_3, x_4\}$
rank	$\emptyset$	$\{x_2, x_3\}$
controllability	$\emptyset$	$\{x_3\}$

Now consider $\mathcal{E} = \{x_1, x_3, x_4\}$ as a set of effective nodes, so that x_2 is a forbidden node.

From the results of this section, and using the DM-decomposition of the bipartite graph $H(A, B_E)$, which is displayed Figure 8, one can compute the sets of useless and essential nodes.

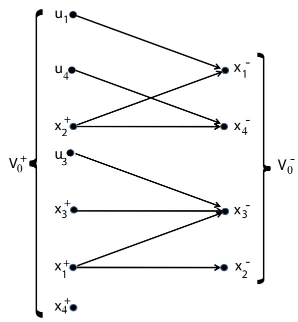

Figure 8: DM-decomposition of bipartite graph of $H(A, B_E)$ in Example 3

In summary, in terms of Lemma 2 and the effective node set $\mathcal{E} = \{x_1, x_3, x_4\}$, we have the following sets $Es, Us, Es_{\mathcal{E}}$ and $Us_{\mathcal{E}}$ with the associated properties.

	Es	Us	$Es_{\mathcal{E}}$	$Us_{\mathcal{E}}$
connection	$\emptyset$	$\{x_3, x_4\}$	$\{x_1\}$	$\{x_3, x_4\}$
rank	$\emptyset$	$\{x_2, x_3\}$	$\emptyset$	$\{x_3\}$
controllability	$\emptyset$	$\{x_3\}$	$\{x_1\}$	$\{x_3, x_4\}$

So, in the context of Lemma 2, we have a strict inclusion $Es \subset Es_{\mathcal{E}}$ for the input connection condition and the controllability condition, and a strict inclusion $Us \cap \mathcal{E} \subset Us_{\mathcal{E}}$ for the controllability condition.

Conversely, it can be seen from the comparison of Propositions 6 and 9, that for input connection, in point 2 of Lemma 2 (concerning useless steering nodes) we have indeed an equality. The same can be said for the rank condition by comparing Propositions 7 and 10.

8. Complexity analysis

The classification of steering nodes mainly implies the computation of the Critical Connection Components of $G(A)$ and of the DM-decomposition of $H(A)$.

The decomposition of a graph into strongly connected components is a standard combinatorial problem for which very efficient polynomial algorithms are available [28]. In particular, Kosaraju's algorithm gives the decomposition in strongly connected components with a complexity $O(n + k)$, where k is the number of non-zero entries in A.

The DM-decomposition implies first the determination of a particular maximum matching, which is then completed by an alternate chain technique, see the details in [20]. Since the alternate chain technique is of a lower complexity than the maximum matching search, the complexity of the DM-decomposition is the same as, for example, in [15].

It follows that steering nodes classification for structured systems is a polynomial problem.

9. Conclusions and outlook

In this paper we studied a classification of steering nodes into useless, usefull or essential ones in order for a large complex structured system to become structually controllable. For the individual conditions for structural controllability, being the input connection condition and the rank condition, this classification could be given completely. However, for their combination, culminating in a classification of steering nodes for structural controllabilty, this still is not settled completely as far as useless steering nodes are concerned. This will remain a topic for further research. The methods underlying the obtained classifications are based on well-understood algorithms coming from the theory of flows in networks. A generalization of the results is given in case the nodes of the system can be divided into forbidden and effective (=non-forbidden) nodes.

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