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Convergence of the Finite Volume Method for scalar conservation laws with multiplicative noise: an approach by kinetic formulation

Sylvain Dotti* and Julien Vovelle†

August 29, 2017

Abstract

We prove the convergence of the explicit-in-time Finite Volume method with monotone fluxes for the approximation of scalar first-order conservation laws with multiplicative, compactly supported noise.

Keywords: Finite Volume Method, stochastic scalar conservation law, kinetic formulation

MSC Number: 65M08 (35L60 35L65 35R60 60H15 65M12)

Contents

1	Introduction	2
2	Generalized solutions, approximate solutions	4
2.1	Solutions	4
2.2	Generalized solutions	6
2.3	Approximate solutions	7
3	The finite volume scheme	8
4	The kinetic formulation of the finite volume scheme	11
5	Energy estimates	14
5.1	Notations	14
5.2	Energy estimate and controls by the dissipation	16
5.3	Proof of Proposition 5.1, Proposition 5.2, Corollary 5.3	18

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6	Approximate kinetic equation	22
6.1	Discrete unknown	22
6.2	Space consistency	26
6.3	Stochastic terms	29
6.4	Conclusion	32
7	Convergence	32
7.1	Additional estimates	33
7.1.1	Tightness of (ν^δ)	33
7.1.2	Tightness of (m_δ)	36
7.2	Convergence	37

1 Introduction

Stochastic first-order scalar conservation law. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis and let $T > 0$. Consider the first-order scalar conservation law with stochastic forcing

$$du(x, t) + \operatorname{div}(A(u(x, t)))dt = \Phi(x, u(x, t))dW(t), \quad x \in \mathbb{T}^N, t \in (0, T). \quad (1.1)$$

Equation (1.1) is periodic in the space variable: $x \in \mathbb{T}^N$ where \mathbb{T}^N is the N -dimensional torus. The flux function A in (1.1) is supposed to be of class C^2 : $A \in C^2(\mathbb{R}; \mathbb{R}^N)$. We assume that A and its derivatives have at most polynomial growth. Without loss of generality, we will assume also that $A(0) = 0$. The right-hand side of (1.1) is a stochastic increment in infinite dimension. It is defined as follows (see [7] for the general theory): W is a cylindrical Wiener process, $W = \sum_{k \geq 1} \beta_k e_k$, where the coefficients β_k are independent Brownian processes and $(e_k)_{k \geq 1}$ is a complete orthonormal system in a Hilbert space H . For each $x \in \mathbb{T}^N$, $u \in \mathbb{R}$, $\Phi(x, u) \in L_2(H, \mathbb{R})$ is defined by $\Phi(x, u)e_k = g_k(x, u)$ where $g_k(\cdot, u)$ is a regular function on \mathbb{T}^N . Here, $L_2(H, K)$ denotes the set of Hilbert-Schmidt operator from the Hilbert space H to an other Hilbert space K . Since $K = \mathbb{R}$ in our case, this set is isomorphic to H , thus we may also define

$$\Phi(x, u) = \sum_{k \geq 1} g_k(x, u)e_k,$$

the action of $\Phi(x, u)$ on $e \in H$ being given by $\langle \Phi(x, u), e \rangle_H$. We assume $g_k \in C(\mathbb{T}^N \times \mathbb{R})$, with the bounds

$$\mathbf{G}^2(x, u) = \|\Phi(x, u)\|_H^2 = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0(1 + |u|^2), \quad (1.2)$$

$$\|\Phi(x, u) - \Phi(y, v)\|_H^2 = \sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1(|x - y|^2 + |u - v|h(|u - v|)), \quad (1.3)$$

where $x, y \in \mathbb{T}^N$, $u, v \in \mathbb{R}$, and h is a continuous non-decreasing function on \mathbb{R}_+ such that $h(0) = 0$. We assume also $0 \leq h(z) \leq 1$ for all $z \in \mathbb{R}_+$.

Notation: in what follows, we will use the convention of summation over repeated indices k . For example, we write $W = \beta_k e_k$.

Compactly supported multiplicative noise. In this paper, we study the numerical approximation of (1.1): our aim is to prove the convergence of the Finite Volume method with monotone fluxes, see Theorem 7.4. Our analysis will be restricted to the case of *multiplicative noise with compact support*. Indeed, from Section 3 to Section 7, we will work under the following hypothesis: there exists $a, b \in \mathbb{R}$, $a < b$, such that $g_k(x, u) = 0$ for all u outside the compact $[a, b]$, for all $x \in \mathbb{T}^N$, $k \geq 1$. For simplicity, we will take $a = -1$, $b = 1$. We will assume therefore that

$$\text{for all } u \in \mathbb{R}, |u| \geq 1 \Rightarrow g_k(x, u) = 0, \quad (1.4)$$

for all $x \in \mathbb{T}^N$, $k \geq 1$, and consider initial data with values in $[-1, 1]$. The solution of the continuous equation (1.1) then takes values in $[-1, 1]$ almost-surely (see [9, Theorem 22]). There is no loss in generality in considering that A is globally Lipschitz continuous then:

$$\text{Lip}(A) := \sup_{\xi \in \mathbb{R}} |A'(\xi)| < +\infty. \quad (1.5)$$

In that framework, we will build a stable and convergent approximation to (1.1) by an explicit-in-time Finite Volume method. Under (1.4), it is also natural to assume

$$\mathbf{G}^2(x, u) = \|\Phi(x, u)\|_H^2 = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0, \quad (1.6)$$

which is of course stronger than (1.2). We may also perform the analysis of convergence of the Finite Volume method under (1.2) instead of (1.6), but this puts exponential factors in various estimates, whereas these factors are close to 1 in the real implementation of the scheme.

Numerical approximation. Let us give a brief summary of the theory of (1.1) and of its approximation. Different approximation schemes to stochastically forced first-order conservation laws have already been analysed: time-discrete schemes, [17, 1, 18], space-discrete scheme [21], space-time Finite Volume discrete schemes:

- in space dimension 1, with strongly monotone fluxes, [22]
- in space dimension $N \geq 1$, by a flux-splitting scheme, [2],
- in space dimension $N \geq 1$, for general schemes with monotone fluxes, [3],

The Cauchy or the Cauchy-Dirichlet problem associated to (1.1) have been studied in [10, 20, 12, 33, 8, 6, 4, 5, 19].

The approximation of scalar conservation laws with stochastic flux has also been considered in [15] (time-discrete scheme) and [29] (space discrete scheme). For the corresponding Cauchy Problem, see [24, 23, 25, 14, 13, 16].

Kinetic formulation. To prove the convergence of the Finite Volume method with monotone fluxes, we will use the companion paper [9] and a kinetic formulation of the Finite Volume scheme. The subject of [9] is the convergence of approximations to (1.1) in the context of the kinetic formulation of scalar conservation laws. Such kinetic formulations have been developed in [26, 27, 28, 30, 31]. In [28], a kinetic formulation of Finite Volume E-schemes is given (and applied in particular to obtain sharp CFL criteria). For Finite Volume schemes with monotone fluxes, the kinetic formulation is simpler, we give it explicitly in Proposition 4.1. Based on the kinetic formulation and an energy estimate, we derive some a priori bounds on the numerical approximation (these are “weak BV estimates” in the terminology of [11, Lemma 25.2]), see Section 5. These estimates are used in the proof of consistency of the scheme when we show that it gives rise to an approximate solution to (1.1) in the sense of Definition 2.6. Our final result, *cf.* Theorem 7.4, should be compared to [3, Theorem 2]. This latter gives the convergence of the Finite Volume method with monotone fluxes in a very similar context, under the slightly stronger hypothesis that the ratio of the time step Δt with the spatial characteristic size h of the mesh tends to 0 when h tends to 0.

Plan of the paper. The plan of the paper is the following one. In the preliminary section 2, we give a brief summary of the notion of solution and approximate solution to (1.1) developed in [9]. In Section 3 is described the kind of approximation to (1.1) by the Finite Volume method which we consider here. In Section 4 we establish the kinetic formulation of the scheme. This numerical kinetic formulation is analysed as follows: energy estimates are derived in Section 5, then we show in Section 6 that this gives rise to an approximate generalized solution in the sense of Definition 2.6. We show some additional estimates and then conclude to the convergence of the scheme in Section 7. This result is stated in Theorem 7.4.

2 Generalized solutions, approximate solutions

The object of this section is to recall several results concerning the solutions to the Cauchy Problem associated to (1.1) and their approximations. We give the main statements, without much explanations or comments; those latter can be found in [9]: we give the precise references when needed.

2.1 Solutions

Definition 2.1 (Random measure). Let X be a topological space. If m is a map from Ω into the set of non-negative finite Borel measures on X such that, for each continuous and bounded function ϕ on X , $\langle m, \phi \rangle$ is a random variable, then we say that m is a random measure on X .

A random measure m is said to have a finite first moment if

$$\mathbb{E}m(\mathbb{T}^N \times [0, T] \times \mathbb{R}) < +\infty. \quad (2.1)$$

Definition 2.2 (Solution). Let $u_0 \in L^\infty(\mathbb{T}^N)$. An $L^1(\mathbb{T}^N)$ -valued stochastic process $(u(t))_{t \in [0, T]}$ is said to be a solution to (1.1) with initial datum u_0 if u and $\mathbf{f} := \mathbf{1}_{u > \xi}$ have the following properties:

1. $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0, T] \times \Omega)$,
2. for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$, almost-surely, $t \mapsto \langle \mathbf{f}(t), \varphi \rangle$ is càdlàg,
3. for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|u(t)\|_{L^p(\mathbb{T}^N)}^p \right) \leq C_p, \quad (2.2)$$

4. there exists a random measure m with first moment (2.1), such that for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0, T]$,

$$\begin{aligned} \langle \mathbf{f}(t), \varphi \rangle &= \langle \mathbf{f}_0, \varphi \rangle + \int_0^t \langle \mathbf{f}(s), a(\xi) \cdot \nabla \varphi \rangle ds \\ &\quad + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} g_k(x, u(x, s)) \varphi(x, u(x, s)) dx d\beta_k(s) \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_\xi \varphi(x, u(x, s)) \mathbf{G}^2(x, u(x, s)) dx ds - m(\partial_\xi \varphi)([0, t]), \end{aligned} \quad (2.3)$$

a.s., where $\mathbf{f}_0(x, \xi) = \mathbf{1}_{u_0(x) > \xi}$, $\mathbf{G}^2 := \sum_{k=1}^{\infty} |g_k|^2$ and $a(\xi) := A'(\xi)$.

In item 1, the index \mathcal{P} in $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0, T] \times \Omega)$ means that u is predictable. See [9, Section 2.1.1]. The function denoted $\mathbf{f} := \mathbf{1}_{u > \xi}$ is given more precisely by

$$(x, t, \xi) \mapsto \mathbf{1}_{u(x, t) > \xi}.$$

This is the characteristic function of the subgraph of u . To study the stability of solutions, or the convergence of approximate solutions (there are two similar problems), we have to consider the stability of this property, the fact of being the “characteristic function of the subgraph of a function”. If (u_n) , is a sequence of functions, say on a finite measure space X , $p \in (1, \infty)$ and (u_n) is bounded in $L^p(X)$, then there is a subsequence still denoted (u_n) which converges to a function u in $L^p(X)$ -weak. Up to a subsequence, the sequence of equilibrium functions $\mathbf{f}_n := \mathbf{1}_{u_n > \xi}$ is converging to a function f in $L^\infty(X \times \mathbb{R})$ -weak star. The limit f is equal to $\mathbf{f} := \mathbf{1}_{u > \xi}$ if, and only if, (u_n) is converging strongly, see [9, Lemma 2.6]. When strong convergence remains a priori unknown, the limit f still keeps some structural properties. This is a kinetic function in the sense of Definition 2.4 below, [9, Corollary 2.5]. Our notion of generalized solution is based on this notion.

2.2 Generalized solutions

Definition 2.3 (Young measure). Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on \mathbb{R} . We say that a map $\nu: X \rightarrow \mathcal{P}_1(\mathbb{R})$ is a Young measure on X if, for all $\phi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\phi)$ from X to \mathbb{R} is measurable. We say that a Young measure ν vanishes at infinity if, for every $p \geq 1$,

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) < +\infty. \quad (2.4)$$

Definition 2.4 (Kinetic function). Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. A measurable function $f: X \times \mathbb{R} \rightarrow [0, 1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for λ -a.e. $z \in X$, for all $\xi \in \mathbb{R}$,

$$f(z, \xi) = \nu_z(\xi, +\infty).$$

We say that f is an equilibrium if there exists a measurable function $u: X \rightarrow \mathbb{R}$ such that $f(z, \xi) = \mathbf{f}(z, \xi) = \mathbf{1}_{u(z) > \xi}$ a.e., or, equivalently, $\nu_z = \delta_{\xi=u(z)}$ for a.e. $z \in X$.

Definition 2.5 (Generalized solution). Let $f_0: \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1]$ be a kinetic function. An $L^\infty(\mathbb{T}^N \times \mathbb{R}; [0, 1])$ -valued process $(f(t))_{t \in [0, T]}$ is said to be a generalized solution to (1.1) with initial datum f_0 if $f(t)$ and $\nu_t := -\partial_\xi f(t)$ have the following properties:

1. for all $t \in [0, T]$, almost-surely, $f(t)$ is a kinetic function, and, for all $R > 0$, $f \in L^1_{\mathcal{P}}(\mathbb{T}^N \times (0, T) \times (-R, R) \times \Omega)$,
2. for all $\varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R})$, almost-surely, the map $t \mapsto \langle f(t), \varphi \rangle$ is càdlàg,
3. for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that

$$\mathbb{E} \left(\sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi) dx \right) \leq C_p, \quad (2.5)$$

4. there exists a random measure m with first moment (2.1), such that for all $\varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0, T]$, almost-surely,

$$\begin{aligned} \langle f(t), \varphi \rangle &= \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla_x \varphi \rangle ds \\ &+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{G}^2(x, \xi) \partial_\xi \varphi(x, \xi) d\nu_{x,s}(\xi) dx ds - m(\partial_\xi \varphi)([0, t]). \end{aligned} \quad (2.6)$$

The following statement is Theorem 3.2. in [9].

Theorem 2.1 (Uniqueness, Reduction). *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume (1.2)-(1.3). Then we have the following results:*

1. *there is at most one solution with initial datum u_0 to (1.1).*
2. *If f is a generalized solution to (1.1) with initial datum f_0 at equilibrium: $f_0 = \mathbf{1}_{u_0 > \xi}$, then there exists a solution u to (1.1) with initial datum u_0 such that $f(x, t, \xi) = \mathbf{1}_{u(x, t) > \xi}$ a.s., for a.e. (x, t, ξ) .*
3. *if u_1, u_2 are two solutions to (1.1) associated to the initial data $u_{1,0}, u_{2,0} \in L^\infty(\mathbb{T}^N)$ respectively, then*

$$\mathbb{E} \|(u_1(t) - u_2(t))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E} \|(u_{1,0} - u_{2,0})^+\|_{L^1(\mathbb{T}^N)}. \quad (2.7)$$

This implies the L^1 -contraction property, and comparison principle for solutions.

2.3 Approximate solutions

In [9], we prove the convergence of solutions to different problems, which are approximations of (1.1). The solutions of these approximate problems give rise to approximate solutions and, more precisely, of approximate generalized solutions, according to the following definition (see Definition 4.1 and Section 5 in [9]).

Definition 2.6 (Approximate generalized solutions). Let $f_0^n: \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1]$ be some kinetic functions. Let $(f^n(t))_{t \in [0, T]}$ be a sequence of $L^\infty(\mathbb{T}^N \times \mathbb{R}; [0, 1])$ -valued processes. Assume that the functions $f^n(t)$, and the associated Young measures $\nu_t^n = -\partial_\xi \varphi f^n(t)$ are satisfying item 1, 2, 3, in Definition 2.5 and Equation (2.6) up to an error term, i.e.: for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$, there exists an adapted process $\varepsilon^n(t, \varphi)$, with $t \mapsto \varepsilon^n(t, \varphi)$ almost-surely continuous such that

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, T]} |\varepsilon^n(t, \varphi)| = 0 \text{ in probability,} \quad (2.8)$$

and there exists some random measures m^n with first moment (2.1), such that, for all n , for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0, T]$, almost-surely,

$$\begin{aligned} \langle f^n(t), \varphi \rangle &= \varepsilon^n(t, \varphi) + \langle f_0^n, \varphi \rangle + \int_0^t \langle f^n(s), a(\xi) \cdot \nabla_x \varphi \rangle ds \\ &+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx d\beta_k(s) \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{G}^2(x, \xi) \partial_\xi \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx ds - m^n(\partial_\xi \varphi)([0, t]). \end{aligned} \quad (2.9)$$

Then we say that (f^n) is a sequence of approximate generalized solutions to (1.1) with initial datum f_0^n .

Consider a sequence (f_n) of approximate solutions to (1.1) satisfying the following (minimal) bounds.

1. There exists $C_p \geq 0$ independent on n such that $\nu^n := -\partial_\xi f^n$ satisfies

$$\mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x, t}^n(\xi) dx \right] \leq C_p, \quad (2.10)$$

2. the measures $\mathbb{E}m^n$ satisfy the bound

$$\sup_n \mathbb{E}m^n(\mathbb{T}^N \times [0, T] \times \mathbb{R}) < +\infty, \quad (2.11)$$

and the following tightness condition: if $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$, then

$$\lim_{R \rightarrow +\infty} \sup_n \mathbb{E}m^n(\mathbb{T}^N \times [0, T] \times B_R^c) = 0. \quad (2.12)$$

We give in [9] the proof of the following convergence result, see Theorem 40 in [9].

Theorem 2.2 (Convergence). *Suppose that there exists a sequence of approximate generalized solutions (f^n) to (1.1) with initial datum f_0^n satisfying (2.10), (2.11) and the tightness condition (2.12) and such that (f_0^n) converges to the equilibrium function $f_0(\xi) = \mathbf{1}_{u_0 > \xi}$ in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ -weak-*, where $u_0 \in L^\infty(\mathbb{T}^N)$. We have then*

1. *there exists a unique solution $u \in L^1(\mathbb{T}^N \times [0, T] \times \Omega)$ to (1.1) with initial datum u_0 ;*
2. *let*

$$u^n(x, t) = \int_{\mathbb{R}} \xi d\nu_{x, t}^n(\xi) = \int_{\mathbb{R}} (f^n(x, t, \xi) - \mathbf{1}_{0 > \xi}) d\xi.$$

Then, for all $p \in [1, \infty[$, (u^n) is converging to u with the following two different modes of convergence: $u_n \rightarrow u$ in $L^p(\mathbb{T}^N \times (0, T] \times \Omega)$ and almost surely, for all $t \in [0, T]$, $u_n(t) \rightarrow u(t)$ in $L^p(\mathbb{T}^N)$.

In the next section, we define the numerical approximation to (1.1) by the Finite Volume method. To prove the convergence of the method, we will show that the hypotheses of Theorem 2.2 are satisfied. The most difficult part in this programme is to prove that the numerical approximations generates a sequence of approximate generalized solutions, see Section 6.

3 The finite volume scheme

Mesh A mesh of \mathbb{T}^N is a family \mathcal{T}/\mathbb{Z}^N of disjoint connected open subsets $K \in (0, 1)^N$ which form a partition of $(0, 1)^N$ up to a negligible set. We denote by \mathcal{T} the mesh

$$\{K + l; l \in \mathbb{Z}^N, K \in \mathcal{T}/\mathbb{Z}^N\}$$

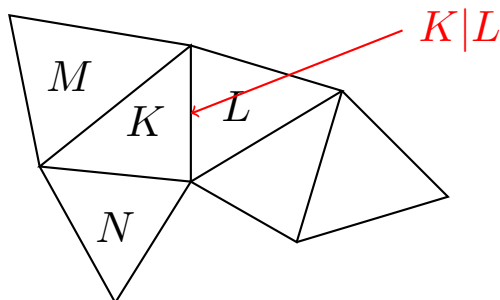
deduced on \mathbb{R}^N . For all distinct $K, L \in \mathcal{T}$, we assume that $\overline{K} \cap \overline{L}$ is contained in an hyperplane; the interface between K and L is denoted $K|L := \overline{K} \cap \overline{L}$. The set of neighbours of K is

$$\mathcal{N}(K) = \{L \in \mathcal{T}; L \neq K, K|L \neq \emptyset\}.$$

We use also the notation

$$\partial K = \bigcup_{L \in \mathcal{N}(K)} K|L.$$

In general, there should be no confusion between ∂K and the topological boundary $\overline{K} \setminus K$.



We also denote by $|K|$ the N -dimensional Lebesgue Measure of K and by $|\partial K|$ (respectively $|K|L|$) the $(N - 1)$ -dimensional Hausdorff measure of ∂K (respectively of $K|L$) (the $(N - 1)$ -dimensional Hausdorff measure is normalized to coincide with the $(N - 1)$ -dimensional Lebesgue measure on hyperplanes).

Scheme Let $(A_{K \rightarrow L})_{K \in \mathcal{T}, L \in \mathcal{N}(K)}$ be a family of monotone, Lipschitz continuous numerical flux, consistent with A : we assume that each function $A_{K \rightarrow L}$ satisfies the following properties.

- Monotony: $A_{K \rightarrow L}(v, w) \leq A_{K \rightarrow L}(v', w)$ for all $v, v', w \in \mathbb{R}$ with $v \leq v'$ and $A_{K \rightarrow L}(v, w) \geq A_{K \rightarrow L}(v, w')$ for all $v, w, w' \in \mathbb{R}$ with $w \leq w'$.
- Lipschitz regularity: there exists $L_A \geq 0$ such that

$$|A_{K \rightarrow L}(v, w) - A_{K \rightarrow L}(v', w')| \leq |K|L|L_A, \quad (3.1)$$

for all $v, v', w, w' \in \mathbb{R}$.

- Consistency:

$$A_{K \rightarrow L}(v, v) = \int_{K|L} A(v) \cdot n_{K,L} d\mathcal{H}^{N-1} = |K|L|A(v) \cdot n_{K,L}, \quad (3.2)$$

for all $v \in \mathbb{R}$, where $n_{K,L}$ is the outward unit normal to K on $K|L$.

- Conservative symmetry:

$$A_{K \rightarrow L}(v, w) = -A_{L \rightarrow K}(w, v), \quad (3.3)$$

for all $K, L \in \mathcal{T}$, $v, w \in \mathbb{R}$.

The conservative symmetry property ensures that the numerical flux $Q_{K \rightarrow L}^n$ defined below in (3.6) satisfies $Q_{K \rightarrow L}^n = -Q_{L \rightarrow K}^n$ for all K, L .

Let $t_n < t_{n+1}$ be two given discrete time. Let $\Delta t_n = t_{n+1} - t_n$. Knowing v_K^n , an approximation of the value of the solution u to (1.1) in the cell K at time t_n , we compute v_K^{n+1} , the approximation to the value of u in K at the next time step t_{n+1} , by the formula

$$|K|(v_K^{n+1} - v_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} Q_{K \rightarrow L}^n = |K|(\Delta t_n)^{1/2} g_{k,K}(v_K^n) X_k^{n+1}, \quad (3.4)$$

where $K \in \mathcal{T}$, with the initialization

$$v_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad K \in \mathcal{T}. \quad (3.5)$$

In (3.4), $\Delta t_n Q_{K \rightarrow L}^n$ is the numerical flux at the interface $K|L$ on the range of time $[t_n, t_{n+1}]$, where $Q_{K \rightarrow L}^n$ is given by

$$Q_{K \rightarrow L}^n = A_{K \rightarrow L}(v_K^n, v_L^n). \quad (3.6)$$

We have also defined

$$X_k^{n+1} = \frac{\beta_k(t_{n+1}) - \beta_k(t_n)}{(\Delta t_n)^{1/2}}. \quad (3.7)$$

Then, the $(X_k^{n+1})_{k \geq 1, n \in \mathbb{N}}$ are i.i.d. random variables with normalized centred normal law $\mathcal{N}(0, 1)$. Besides, for each $n \geq 1$, the sequence $(X_k^{n+1})_{k \geq 1}$ is independent on \mathcal{F}_n , the sigma-algebra generated by $\{X_k^{m+1}; k \geq 1, m < n\}$. The numerical functions $g_{k,K}$ are defined by the average

$$g_{k,K}(v) = \frac{1}{|K|} \int_K g_k(x, v) dx. \quad (3.8)$$

Then, in virtue of (1.6) we have

$$\mathbf{G}_K^2(v) := \sum_{k \geq 1} |g_{k,K}(v)|^2 \leq D_0, \quad (3.9)$$

where $v \in \mathbb{R}$, $K \in \mathcal{T}$. We deduce (3.9) from (1.6) and Jensen's Inequality. Similarly, we deduce from (1.3) and Jensen's Inequality that

$$\sum_{k \geq 1} |g_{k,K}(\xi) - g_k(y, \xi)|^2 \leq D_1 \frac{1}{|K|} \int_K |x - y|^2 dx,$$

for all $y \in \mathbb{T}^N$. In particular (switching from the variable y to the variable x), and assuming $\text{diam}(K) \leq h$ (this is the hypothesis (5.5), which we will make later), we have the following consistency estimate

$$\sum_{k \geq 1} |g_{k,K}(\xi) - g_k(x, \xi)|^2 \leq D_1 h^2, \quad (3.10)$$

for all $x \in K$, which will be used later (see (6.47) for example).

Remark 3.1 (Approximation in law). In effective computations, the random variables X_k^{n+1} are drawn at each time step. They are i.i.d. random variables with normalized centred normal law $\mathcal{N}(0, 1)$. In this situation, we will prove the convergence in law of the Finite Volume scheme to the solution to (1.1), see Remark 7.1 after Theorem 7.4.

Remark 3.2 (Global Lipschitz Numerical Flux). We assume in (3.1) that the numerical fluxes $A_{K \rightarrow L}$ are globally Lipschitz continuous. This is consistent with (1.5). Both (3.1) and (1.5) are strong hypotheses, except if a priori L^∞ -bounds are known on the solutions to (1.1), which is the case here, thanks to the hypothesis of compact support (1.4). Without loss of generality, we will assume that $\text{Lip}(A) \leq L_A$.

4 The kinetic formulation of the finite volume scheme

The kinetic formulation of the Finite Volume method has been introduced by Makridakis and Perthame in [28]. The principle is the following one. For linear transport equations, which corresponds to a linear flux function $A(u) = au$, $a \in \mathbb{R}^N$, the upwind numerical flux $A_{K \rightarrow L}$ in (3.6) is given by

$$A_{K \rightarrow L}(v, w) = [a_{K \rightarrow L}^*]^+ v - [a_{K \rightarrow L}^*]^- w, \quad (4.1)$$

where

$$a_{K \rightarrow L}^* := \int_{K|L} a \cdot n_{K|L} d\mathcal{H}^{N-1} = |K|L| a \cdot n_{K,L}, \quad (4.2)$$

with the usual notation $v^+ = \max(v, 0)$, $w^- = (-w)^+$. The discrete approximation of the transport equation

$$\partial_t f + a(\xi) \cdot \nabla_x f = 0$$

by the Finite Volume method is therefore

$$|K|(f_K^{n+1} - f_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) = 0, \quad (4.3)$$

where

$$a_{K \rightarrow L}^n(\xi) = [a_{K|L}^*(\xi)]^+ f_K^n - [a_{K|L}^*(\xi)]^- f_L^n.$$

Recall that \mathbf{G}_K is defined by (3.9). A kinetic formulation of (3.4) consistent with (4.3) would be

$$\begin{aligned} & |K|(\mathbf{f}_K^{n+1} - \mathbf{f}_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \\ &= |K|(\Delta t_n)^{1/2} \delta_{v_K^n = \xi} g_{k,K}(\xi) X_k^{n+1} + |K| \Delta t_n \partial_\xi \left(m_K^n(\xi) - \frac{1}{2} \mathbf{G}_K^2(\xi) \delta_{v_K^n = \xi} \right), \end{aligned} \quad (4.4)$$

where, for $K \in \mathcal{T}$, $n \in \mathbb{N}$, $\xi \in \mathbb{R}$, $m_K^n(\xi) \geq 0$, and where

$$\mathbf{f}_K^n(\xi) := \mathbf{1}_{v_K^n > \xi}. \quad (4.5)$$

This is not exactly the kinetic formulation that we will consider. See (6.9) for a correct version of (4.4). We will mainly work with a kinetic formulation (see (4.12)), obtained thanks to the following splitting method. For $K \in \mathcal{T}$ and $n \in \mathbb{N}$, let us define $v_K^{n+1/2}$ as the solution to

$$|K|(v_K^{n+1/2} - v_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} A_{K \rightarrow L}(v_K^n, v_L^n) = 0. \quad (4.6)$$

Then $v_K^{n+1/2}$ is the state reached after a step of deterministic evolution, by the discrete approximation of the equation $u_t + \operatorname{div}(A(u)) = 0$. To this step corresponds the kinetic formulation

$$|K|(\mathbf{f}_K^{n+1/2} - \mathbf{f}_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) = |K| \Delta t_n \partial_\xi m_K^n(\xi), \quad (4.7)$$

where $\mathbf{f}_K^m(\xi) = \mathbf{1}_{v_K^m > \xi}$, $m \in \{n, n + 1/2\}$ and

$$m_K^n \geq 0. \quad (4.8)$$

In (4.7), $a_{K \rightarrow L}^n(\xi)$ is a function

$$a_{K \rightarrow L}^n(\xi) = a_{K \rightarrow L}(\xi, v_K^n, v_L^n), \quad (4.9)$$

where $(\xi, v, w) \mapsto a_{K \rightarrow L}(\xi, v, w)$ satisfies the following consistency conditions:

$$\int_{\mathbb{R}} [a_{K \rightarrow L}(\xi, v, w) - a_{K \rightarrow L}^*(\xi) \mathbf{1}_{0 > \xi}] d\xi = A_{K \rightarrow L}(v, w), \quad (4.10)$$

$$a_{K \rightarrow L}(\xi, v, v) = a_{K \rightarrow L}^*(\xi) \mathbf{1}_{v > \xi}, \quad (4.11)$$

for all $\xi, v, w \in \mathbb{R}$, where $a_{K \rightarrow L}^*$ is defined by (4.2). Before we prove the existence of the kinetic formulation (4.7)-(4.8)-(4.10)-(4.11), see Proposition 4.1, let us first deduce from (4.7) the kinetic formulation of the whole scheme (3.4). This is the equation

$$\begin{aligned} & |K|(\mathbf{f}_K^{n+1}(\xi) - \mathbf{f}_K^n(\xi)) + \Delta t_n \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \\ &= |K| \Delta t_n \partial_\xi m_K^n(\xi) + |K| \left[\mathbf{f}_K^{n+1}(\xi) - \mathbf{f}_K^{n+1/2}(\xi) \right]. \end{aligned} \quad (4.12)$$

We may try to develop the term $\mathbf{f}_K^{n+1}(\xi) - \mathbf{f}_K^{n+1/2}(\xi)$ (this is done in (6.13) to obtain (6.9)), but (4.12) will be sufficient for the moment. It will be sufficient in particular to obtain the so-called energy estimates of Section 5.

Proposition 4.1 (Kinetic formulation of the Finite Volume method). *Set*

$$a_{K \rightarrow L}(\xi, v, w) = a_{K \rightarrow L}^*(\xi) \mathbf{1}_{\xi < v \wedge w} + [\partial_2 A_{K \rightarrow L}(v, \xi) \mathbf{1}_{v \leq \xi \leq w} + \partial_1 A_{K \rightarrow L}(\xi, w) \mathbf{1}_{w \leq \xi \leq v}] \quad (4.13)$$

and

$$m_K^n(\xi) = -\frac{1}{\Delta t_n} \left[(v_K^{n+1/2} - \xi)^+ - (v_K^n - \xi)^+ \right] - \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} \int_{\xi}^{+\infty} a_{K \rightarrow L}^n(\zeta) d\zeta. \quad (4.14)$$

Let us also assume that

$$\Delta t_n \frac{|\partial K|}{|K|} L_A \leq 1, \quad \forall K \in \mathcal{T}, \quad (4.15)$$

for all $n \in \mathbb{N}$, $K \in \mathcal{T}$. Then the equations (4.7)-(4.8)-(4.10)-(4.11) are satisfied and, besides, we have

$$a_{K \rightarrow L}(\xi, v, w) = 0, \quad \text{when } \xi \geq v \vee w, \quad (4.16)$$

for all $K, L \in \mathcal{T}$.

Remark 4.1 (Support of m_K^n). By (4.14), the definition (4.13) of $a_{K \rightarrow L}$ and the equation (4.6), $\xi \mapsto m_K^n(\xi)$ is compactly supported in the convex envelope of $v_K^{n+1/2}$, v_K^n , $\{v_L^n; L \in \mathcal{N}(K)\}$.

Proof of Proposition 4.1. We check at once (4.7) and (4.10), (4.11), (4.16). To show that $m_K^n(\xi) \geq 0$, let us introduce

$$\Phi_{K \rightarrow L}(\xi, v, w) = \int_{\xi}^{+\infty} a_{K \rightarrow L}(\zeta, v, w) d\zeta \quad (4.17)$$

$$\Phi_{K \rightarrow L}^n(\xi) = \Phi_{K \rightarrow L}(\xi, v_K^n, v_L^n) = \int_{\xi}^{+\infty} a_{K \rightarrow L}^n(\zeta) d\zeta. \quad (4.18)$$

A simple computation gives the formula

$$\Phi_{K \rightarrow L}^n(\xi) = A_{K \rightarrow L}(v_K^n, v_L^n) - A_{K \rightarrow L}(v_K^n \wedge \xi, v_L^n \wedge \xi). \quad (4.19)$$

By comparison with the identity $(v - \xi)^+ = v - v \wedge \xi$, the quantities $\Phi_{K \rightarrow L}^n(\xi)$ appears, in virtue of (4.19), as the numerical entropy fluxes associated to the entropy $\eta(v) := (v - \xi)^+$. Then $m_K^n(\xi) \geq 0$ is equivalent to the discrete entropy inequality

$$\frac{1}{\Delta t_n} \left[\eta(v_K^{n+1/2}) - \eta(v_K^n) \right] + \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} \Phi_{K \rightarrow L}^n(\xi) \leq 0. \quad (4.20)$$

It is a classical fact that, under the CFL condition (4.15), the deterministic Finite Volume scheme (4.6) has the following monotony property: $v_K^{n+1/2}$ in (4.6) is a non-decreasing function of each of the entries v_K^n , v_L^n , $L \in \mathcal{N}(K)$. This implies (4.20) then. See Lemma 25.1 and Lemma 27.1 in [11]. \square

5 Energy estimates

The Finite Volume scheme (3.4) may be compared to the stochastic parabolic equation

$$du^\varepsilon(x, t) + \operatorname{div}(A(u^\varepsilon(x, t)))dt - \varepsilon \Delta u^\varepsilon(x, t)dt = \Phi(x, u^\varepsilon(x, t))dW(t). \quad (5.1)$$

For (5.1), we have the energy estimate

$$\frac{1}{2} \frac{d}{dt} \mathbb{E} \|u^\varepsilon(t)\|_{L^2(\mathbb{T}^N)}^2 + \varepsilon \mathbb{E} \|\nabla u^\varepsilon(t)\|_{L^2(\mathbb{T}^N)}^2 = \frac{1}{2} \mathbb{E} \|\mathbf{G}(\cdot, u^\varepsilon(\cdot, t))\|_{L^2(\mathbb{T}^N)}^2. \quad (5.2)$$

(Recall that \mathbf{G} is defined by (1.2)). In the following Proposition 5.1, we obtain an analogous result for the Finite Volume scheme (3.4). To state Proposition 5.1, we need first some notations.

5.1 Notations

For a fixed final time $T > 0$, we denote by \mathfrak{d}_T the set of admissible space-step and time-steps, defined as follows: if $h > 0$ and $(\Delta t) = (\Delta t_0, \dots, \Delta t_{N_T-1})$, $N_T \in \mathbb{N}^*$, then we say that $\delta := (h, (\Delta t)) \in \mathfrak{d}_T$ if

$$\frac{1}{h} \in \mathbb{N}^*, \quad t_{N_T} := \sum_{n=0}^{N_T-1} \Delta t_n = T, \quad \sup_{0 \leq n < N_T} \Delta t_n \leq 1. \quad (5.3)$$

We say that $\delta \rightarrow 0$ if

$$|\delta| := h + \sup_{0 \leq n < N_T} \Delta t_n \rightarrow 0. \quad (5.4)$$

For a given mesh parameter $\delta = (h, (\Delta t)) \in \mathfrak{d}_T$, we assume that a mesh \mathcal{T} is given, with the following properties:

$$\operatorname{diam}(K) \leq h, \quad (5.5)$$

$$\alpha_N h^N \leq |K|, \quad (5.6)$$

$$|\partial K| \leq \frac{1}{\alpha_N} h^{N-1}, \quad (5.7)$$

for all $K \in \mathcal{T}$, where

$$\operatorname{diam}(K) = \max_{x, y \in K} |x - y|$$

is the diameter of K and α_N is a given positive absolute constant depending on the dimension N only. Note the following consequence of (5.6)-(5.7):

$$h|\partial K| \leq \frac{1}{\alpha_N^2} |K|, \quad (5.8)$$

for all $K \in \mathcal{T}$. We introduce then the discrete unknown $v_\delta(t)$ defined a.e. by

$$v_\delta(x, t) = v_K^n, \quad x \in K, t_n \leq t < t_{n+1}. \quad (5.9)$$

We will also need the intermediary discrete function

$$v_\delta^b(x, t_{n+1}) = v_K^{n+1/2}, \quad x \in K, \quad (5.10)$$

defined for $n \in \mathbb{N}$. Let us define the conjugate function $\bar{f} = 1 - f$. We introduce the following conjugate quantities:

$$\bar{a}_{K \rightarrow L}(\xi, v, w) = a_{K \rightarrow L}^*(\xi) - a_{K \rightarrow L}(\xi, v, w), \quad \bar{\Phi}_{K \rightarrow L}(\xi, v, w) = \int_{-\infty}^{\xi} \bar{a}_{K \rightarrow L}(\zeta, v, w) d\zeta. \quad (5.11)$$

We compute

$$\bar{\Phi}_{K \rightarrow L}(\xi, v, w) = A_{K \rightarrow L}(\xi, \xi) - A_{K \rightarrow L}(v \wedge \xi, w \wedge \xi). \quad (5.12)$$

We recognize in (5.12) a numerical flux associated to the entropy

$$v \mapsto (v - \xi)^- = \xi - v \wedge \xi.$$

From the explicit formula (4.13), we obtain the identity

$$\begin{aligned} \bar{a}_{K \rightarrow L}(\xi, v, w) &= a_{K \rightarrow L}^*(\xi) \mathbf{1}_{\xi > v \vee w} + (a_{K \rightarrow L}^*(\xi) - \partial_2 A_{K \rightarrow L}(v, \xi)) \mathbf{1}_{v \leq \xi \leq w} \\ &\quad + (a_{K \rightarrow L}^*(\xi) - \partial_1 A_{K \rightarrow L}(\xi, w)) \mathbf{1}_{w \leq \xi \leq v}. \end{aligned} \quad (5.13)$$

Note that, for $a_{K \rightarrow L}$ defined by (4.13), we have (using the fact that $\text{Lip}(A) \leq L_A$),

$$\sup\{|a_{K \rightarrow L}(\xi, v, w)|; \xi, v, w \in \mathbb{R}\} \leq L_A |K| |L|. \quad (5.14)$$

Formula (5.13) gives the estimate

$$\sup\{|\bar{a}_{K \rightarrow L}(\xi, v, w)|; \xi, v, w \in \mathbb{R}\} \leq 2L_A |K| |L|, \quad (5.15)$$

which is not optimal as (5.14) may be, since it has an additional factor 2. Consequently, we will use a slightly different formulation for $\bar{\Phi}_{K \rightarrow L}$:

$$\bar{\Phi}_{K \rightarrow L}(\xi, v, w) = \int_{-\infty}^{\xi} \bar{b}_{K \rightarrow L}(\zeta, \xi, v, w) d\zeta, \quad (5.16)$$

where

$$\bar{b}_{K \rightarrow L}(\zeta, \xi, v, w) := a_{K \rightarrow L}^*(\xi) \mathbf{1}_{\xi > v \vee w} + \partial_1 A_{K \rightarrow L}(\zeta, \xi) \mathbf{1}_{v \leq \xi \leq w} + \partial_2 A_{K \rightarrow L}(\xi, \zeta) \mathbf{1}_{w \leq \xi \leq v}. \quad (5.17)$$

We also introduce

$$\bar{b}_{K \rightarrow L}^n(\zeta, \xi) = \bar{b}_{K \rightarrow L}(\zeta, \xi, v_K^n, v_L^n). \quad (5.18)$$

Now for $\bar{b}_{K \rightarrow L}$, we have an estimate similar to (5.14):

$$\sup\{|\bar{b}_{K \rightarrow L}(\xi, v, w)|; \xi, \zeta, v, w \in \mathbb{R}\} \leq L_A |K| |L|. \quad (5.19)$$

5.2 Energy estimate and controls by the dissipation

Proposition 5.1 (Energy estimate for the Finite Volume Scheme). *Let $u_0 \in L^\infty(\mathbb{T})$, $T > 0$ and $\delta \in \mathfrak{D}_T$. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9). Set*

$$\mathcal{E}(T) = \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} m_K^n(\xi) d\xi. \quad (5.20)$$

Then, under the CFL condition (4.15), we have the energy estimate

$$\frac{1}{2} \mathbb{E} \|v_\delta(T)\|_{L^2(\mathbb{T}^N)}^2 + \mathbb{E} \mathcal{E}(T) = \frac{1}{2} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{1}{2} \mathbb{E} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \sum_{k \geq 1} |g_{k,K}(v_K^n)|^2. \quad (5.21)$$

In the following proposition we derive various estimates, where the right-hand side is controlled by the dissipation term $\mathcal{E}(T)$ introduced in (5.20).

Proposition 5.2 (Control by the dissipation). *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{D}_T$. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9). Let v_δ^b be defined by (5.10). Then, under the CFL condition*

$$2\Delta t_n \frac{|\partial K|}{|K|} \sup_{\xi \in \mathbb{R}} \frac{|a_{K \rightarrow L}^n(\xi)|}{|K|L} \leq (1 - \theta), \quad 0 \leq n < N, \quad K, L \in \mathcal{T}, \quad (5.22)$$

where $\theta \in (0, 1)$, we have the following control:

$$\begin{aligned} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} (\bar{\mathfrak{f}}_L^n - \bar{\mathfrak{f}}_K^n) \Phi_{K \rightarrow L}^n(\xi) d\xi \\ \leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \bar{\mathfrak{f}}_K^n(\xi) m_K^n(\xi) d\xi, \end{aligned} \quad (5.23)$$

and

$$\sum_{n=0}^{N_T-1} \left\| [v_\delta^b(t_{n+1}) - v_\delta(t_n)]_+ \right\|_{L^2(\mathbb{T}^N)}^2 \leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \bar{\mathfrak{f}}_K^n(\xi) m_K^n(\xi) d\xi. \quad (5.24)$$

Under the CFL condition

$$2\Delta t_n \frac{|\partial K|}{|K|} \sup_{\xi \in \mathbb{R}} \frac{|\bar{b}_{K \rightarrow L}^n(\xi, \xi)|}{|K|L} \leq (1 - \theta), \quad 0 \leq n < N, \quad K, L \in \mathcal{T}, \quad (5.25)$$

where $\theta \in (0, 1)$, (and where $\bar{b}_{K \rightarrow L}^n$ is defined by (5.18)) we have the following control:

$$\begin{aligned} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} (\mathbf{f}_L^n - \mathbf{f}_K^n) \bar{\Phi}_{K \rightarrow L}^n(\xi) d\xi \\ \leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \mathbf{f}_K^n(\xi) m_K^n(\xi) d\xi, \end{aligned} \quad (5.26)$$

and

$$\sum_{n=0}^{N_T-1} \left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_- \right\|_{L^2(\mathbb{T}^N)}^2 \leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \mathbf{f}_K^n(\xi) m_K^n(\xi) d\xi. \quad (5.27)$$

Eventually, as a corollary to Proposition 5.2, we obtain the following estimates.

Corollary 5.3 (Weak derivative estimates). *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{D}_T$. Assume that (1.4), (1.6), (3.1), (5.5), (5.6) and (5.7) are satisfied and that*

$$\Delta t_n \leq (1 - \theta) \frac{\alpha_N^2}{2L_A} h, \quad 0 \leq n < N_T, \quad (5.28)$$

where $\theta \in (0, 1)$. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9). Let v_δ^\flat be defined by (5.10). Then we have the spatial estimate

$$\begin{aligned} \mathbb{E} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} [(\bar{\mathbf{f}}_L^n - \bar{\mathbf{f}}_K^n) \Phi_{K \rightarrow L}^n(\xi) + (\mathbf{f}_L^n - \mathbf{f}_K^n) \bar{\Phi}_{K \rightarrow L}^n(\xi)] d\xi \\ \leq \frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{D_0 T}{\theta}, \end{aligned} \quad (5.29)$$

and the two following temporal estimates:

$$\mathbb{E} \sum_{n=0}^{N_T-1} \left\| v_\delta^\flat(t_{n+1}) - v_\delta(t_n) \right\|_{L^2(\mathbb{T}^N)}^2 \leq \frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{D_0 T}{\theta}, \quad (5.30)$$

and

$$\mathbb{E} \sum_{n=0}^{N_T-1} \|v_\delta(t_{n+1}) - v_\delta(t_n)\|_{L^2(\mathbb{T}^N)}^2 \leq \frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{2D_0 T}{\theta}. \quad (5.31)$$

5.3 Proof of Proposition 5.1, Proposition 5.2, Corollary 5.3

Proof of Proposition 5.1. We multiply first (4.7) by ξ and sum the result over $K \in \mathcal{T}/\mathbb{Z}^N$ and $\xi \in \mathbb{R}$ to get the following balance equation

$$\frac{1}{2} \|v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 + \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} m_K^n(\xi) d\xi = \frac{1}{2} \|v_\delta(t_n)\|_{L^2(\mathbb{T}^N)}^2. \quad (5.32)$$

We have used Remark 4.1 to justify the integration by parts in the term with the measure m_K^n . The term

$$\sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \quad (5.33)$$

related to the flux term in (4.7) has vanished. Indeed, (5.33) is equal to

$$\frac{1}{2} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) + a_{L \rightarrow K}^n(\xi) \quad (5.34)$$

by relabelling of the indexes of summation. All the arguments in (5.34) cancel individually in virtue of the conservative symmetry property (3.3) of $A_{K \rightarrow L}(v, w)$. Indeed, one can check that $a_{K \rightarrow L}$ inherits this property, *i.e.*

$$a_{K \rightarrow L}(\xi, v, w) = -a_{L \rightarrow K}(\xi, w, v), \quad K, L \in \mathcal{T}, v, w \in \mathbb{R}, \quad (5.35)$$

thanks to the explicit formula (4.13). To obtain the equation for the balance of energy corresponding to the stochastic forcing, we use the equation

$$v_K^{n+1} = v_K^{n+1/2} + (\Delta t_n)^{1/2} g_{k,K}(v_K^n) X_k^{n+1}, \quad (5.36)$$

which follows from the equation of the scheme (3.4) and the definition of $v_K^{n+1/2}$ by (4.6). Taking the square of both sides of (5.36) and using the independence of X_k^{n+1} and $v_i^{n+1/2}$, we obtain the identity

$$\frac{1}{2} \mathbb{E} \|v_\delta(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 = \frac{1}{2} \mathbb{E} \|v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 + \frac{\Delta t_n}{2} \mathbb{E} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \sum_{k \geq 1} |g_{k,K}(v_K^n)|^2. \quad (5.37)$$

Adding (5.32) to (5.37) gives (5.21). \square

Remark 5.1. Note that (5.36) also gives

$$\frac{1}{2} \mathbb{E} \|v_\delta(t_{n+1}) - v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 = \frac{\Delta t_n}{2} \mathbb{E} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \sum_{k \geq 1} |g_{k,K}(v_K^n)|^2, \quad (5.38)$$

for all $0 \leq n \leq N_T$.

Proof of Proposition 5.2. We begin with the proof of the estimates (5.23) and (5.24). Multiplying Equation (4.7) by $\bar{\mathbf{f}}_K^n := 1 - \mathbf{f}_K^n$, we obtain

$$|K| \bar{\mathbf{f}}_K^n(\xi) \mathbf{f}_K^{n+1/2}(\xi) + \Delta t_n \bar{\mathbf{f}}_K^n(\xi) \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) = |K| \Delta t_n \bar{\mathbf{f}}_K^n(\xi) \partial_\xi m_K^n(\xi). \quad (5.39)$$

Next, we multiply (5.39) by $(\xi - v_K^n)$ and sum the result over ξ, K . We use the first identity

$$\int_{\mathbb{R}} (\xi - v_K^n) \bar{\mathbf{f}}_K^n(\xi) \partial_\xi m_K^n(\xi) d\xi = \int_{\mathbb{R}} (\xi - v_K^n)_+ \partial_\xi m_K^n(\xi) d\xi = - \int_{\mathbb{R}} \bar{\mathbf{f}}_K^n(\xi) m_K^n(\xi) d\xi, \quad (5.40)$$

(once again, we use the fact that m_K^n is compactly supported to do the integration by parts in (5.40), cf. Remark 4.1) and the second identity

$$\int_{\mathbb{R}} (\xi - v_K^n) \bar{\mathbf{f}}_K^n(\xi) \mathbf{f}_K^{n+1/2}(\xi) d\xi = \frac{1}{2} (v_K^{n+1/2} - v_K^n)_+^2,$$

to obtain

$$\begin{aligned} \frac{1}{2} \left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_+ \right\|_{L^2(\mathbb{T}^N)}^2 + \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \bar{\mathbf{f}}_K^n(\xi) m_K^n(\xi) d\xi \\ = -\Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} (\xi - v_K^n)_+ a_{K \rightarrow L}^n(\xi) d\xi. \end{aligned} \quad (5.41)$$

We transform the right-hand side of (5.41) by integration by parts in ξ : this gives, thanks to (4.17)-(4.18), the term

$$-\Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} \bar{\mathbf{f}}_K^n(\xi) \Phi_{K \rightarrow L}^n(\xi) d\xi. \quad (5.42)$$

Then we can relabel the indices in (5.42) and use the conservative symmetry relation (consequence of (5.35))

$$\Phi_{K \rightarrow L}(\xi, v, w) = -\Phi_{L \rightarrow K}(\xi, w, v), \quad (5.43)$$

to see that

$$\begin{aligned} \frac{1}{2} \left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_+ \right\|_{L^2(\mathbb{T}^N)}^2 + \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \bar{\mathbf{f}}_K^n(\xi) m_K^n(\xi) d\xi \\ = \frac{1}{2} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} (\bar{\mathbf{f}}_L^n(\xi) - \bar{\mathbf{f}}_K^n(\xi)) \Phi_{K \rightarrow L}^n(\xi) d\xi. \end{aligned} \quad (5.44)$$

Note that the integrand $(\bar{\mathbf{f}}_L^n(\xi) - \bar{\mathbf{f}}_K^n(\xi)) \Phi_{K \rightarrow L}^n(\xi)$ is non-negative thanks to the monotony properties of $A_{K \rightarrow L}$ and (4.19). At this stage, in order to deduce (5.23) from (5.44), we

have to prove that, under the CFL condition (5.22), a fraction of the right-hand side of (5.44) controls the term

$$\frac{1}{2} \left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_+ \right\|_{L^2(\mathbb{T})}^2,$$

(see the estimate (5.49) below). To this end, we integrate Equation (5.39) over $\xi \in \mathbb{R}$. This gives

$$|K| [v_K^{n+1/2} - v_K^n]_+ + \Delta t_n \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} \bar{\mathbf{f}}_K^n(\xi) a_{K \rightarrow L}^n(\xi) d\xi \leq 0, \quad (5.45)$$

which reads also

$$|K| [v_K^{n+1/2} - v_K^n]_+ \leq -\Delta t_n \sum_{L \in \mathcal{N}(K)} \Phi_{K \rightarrow L}^n(v_K^n)$$

by (4.18) (note that it is also the discrete entropy inequality (4.20) with $\xi = v_K^n$). Taking the square, using the Cauchy-Schwarz Inequality and summing over $K \in \mathcal{T}/\mathbb{Z}^N$, we deduce that

$$\left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_+ \right\|_{L^2(\mathbb{T}^N)}^2 \leq \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \Delta t_n \frac{|\partial K|}{|K|} \sum_{L \in \mathcal{N}(K)} \frac{|\Phi_{K \rightarrow L}^n(v_K^n)|^2}{|K| |L|}. \quad (5.46)$$

Next, we note that $|\Phi_{K \rightarrow L}^n(v_K^n)|^2$ is non-trivial only if $v_K^n < v_L^n$. In that case, it can be decomposed as

$$|\Phi_{K \rightarrow L}^n(v_K^n)|^2 = -2 \int_{v_K^n}^{v_L^n} \Phi_{K \rightarrow L}^n(\xi) \partial_\xi \Phi_{K \rightarrow L}^n(\xi) d\xi = 2 \int_{v_K^n}^{v_L^n} \Phi_{K \rightarrow L}^n(\xi) a_{K \rightarrow L}^n(\xi) d\xi, \quad (5.47)$$

which is bounded by

$$2 \sup_{\xi \in \mathbb{R}} |a_{K \rightarrow L}^n(\xi)| \int_{v_K^n}^{v_L^n} |\Phi_{K \rightarrow L}^n(\xi)| d\xi = 2 \sup_{\xi \in \mathbb{R}} |a_{K \rightarrow L}^n(\xi)| \int_{\mathbb{R}} (\bar{\mathbf{f}}_L^n - \bar{\mathbf{f}}_K^n) \Phi_{K \rightarrow L}^n(\xi) d\xi. \quad (5.48)$$

Under the CFL condition (5.22), the estimate (5.46) can be completed into

$$\begin{aligned} & \frac{1}{2} \left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_+ \right\|_{L^2(\mathbb{T}^N)}^2 \\ & \leq (1 - \theta) \frac{1}{2} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} (\bar{\mathbf{f}}_L^n(\xi) - \bar{\mathbf{f}}_K^n(\xi)) \Phi_{K \rightarrow L}^n(\xi) d\xi. \end{aligned} \quad (5.49)$$

Using (5.44) then, we deduce the two estimates (5.23)-(5.24).

To prove the estimates (5.26) and (5.27), we proceed similarly: we start from the following equation on $\bar{\mathbf{f}}_K^n$, which is equivalent to (4.7):

$$|K| (\bar{\mathbf{f}}_K^{n+1/2}(\xi) - \bar{\mathbf{f}}_K^n(\xi)) + \Delta t_n \sum_{L \in \mathcal{N}(K)} \bar{a}_{K \rightarrow L}^n(\xi) = -|K| \Delta t_n \partial_\xi m_K^n(\xi). \quad (5.50)$$

Then we multiply Eq. (5.50) by \mathbf{f}_K^n , to obtain

$$|K|\mathbf{f}_K^n \bar{\mathbf{f}}_K^{n+1/2}(\xi) + \Delta t_n \sum_{L \in \mathcal{N}(K)} \mathbf{f}_K^n \bar{a}_{K \rightarrow L}^n(\xi) = -|K|\Delta t_n \mathbf{f}_K^n \partial_\xi m_K^n(\xi), \quad (5.51)$$

which is the analogue to (5.39). In a first step, we multiply (5.51) by $(v_K^n - \xi)$ and sum the result over $\xi \in \mathbb{R}$, $K \in \mathcal{T}/\mathbb{Z}^N$. This gives (compare to (5.41)-(5.44))

$$\begin{aligned} \frac{1}{2} \left\| [v_\delta^\flat(t_{n+1}) - v_\delta(t_n)]_- \right\|_{L^2(\mathbb{T}^N)}^2 + \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \mathbf{f}_K^n(\xi) m_K^n(\xi) d\xi \\ = -\Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \mathbf{f}_K^n \bar{\Phi}_{K \rightarrow L}^n(\xi) \\ = \frac{1}{2} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} (\mathbf{f}_L^n - \mathbf{f}_K^n) \bar{\Phi}_{K \rightarrow L}^n(\xi). \end{aligned} \quad (5.52)$$

To conclude to (5.26)-(5.27) under the CFL condition (5.25), we proceed as in (5.45)-(5.49) above, with the minor difference that, instead of the identity $\partial_\xi \bar{\Phi}_{K \rightarrow L}^n(\xi) = \bar{a}_{K \rightarrow L}^n(\xi)$, we use the formula $\partial_\xi \bar{\Phi}_{K \rightarrow L}^n(\xi) = \bar{b}_{K \rightarrow L}^n(\xi, \xi)$ (see (5.18)) when we develop $|\bar{\Phi}_{K \rightarrow L}^n(v_K^n)|^2$. \square

Remark 5.2. A slight modification of the lines (5.47)-(5.48) in the proof above shows that

$$|\Phi_{K \rightarrow L}^n(\xi \vee v_K^n)|^2 \leq 2 \sup_{\xi \in \mathbb{R}} |a_{K \rightarrow L}^n(\xi)| \int_{\mathbb{R}} (\bar{\mathbf{f}}_L^n - \bar{\mathbf{f}}_K^n) \Phi_{K \rightarrow L}^n(\xi) d\xi, \quad (5.53)$$

for all $\xi \in \mathbb{R}$. This estimate will be used in the proof of Lemma 6.4 below.

Proof of Corollary 5.3. Assume that (5.28) is satisfied. It is clear, in virtue of the estimate (5.8) and the bound (5.14) and (5.19) on $a_{K \rightarrow L}^n$ and $\bar{b}_{K \rightarrow L}^n$, that (5.28) implies the CFL conditions (5.22) and (5.25). Besides, due to (3.9), we have the bound

$$\sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \sum_{k \geq 1} |g_{k,K}(v_K^n)|^2 \leq D_0. \quad (5.54)$$

This gives

$$\sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \sum_{k \geq 1} |g_{k,K}(v_K^n)|^2 \leq D_0 T,$$

which, inserted in the energy estimate (5.21), shows that

$$\mathbb{E} \mathcal{E}(T) \leq \frac{1}{2} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{1}{2} D_0 T.$$

By addition of the estimates (5.23)-(5.26) and (5.24)-(5.27) respectively, we obtain therefore (5.29) and (5.30). There remains to prove (5.31). For that purpose, we use (5.38) and (5.54) to obtain

$$\mathbb{E} \|v_\delta(t_{n+1}) - v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 = \mathbb{E} \|v_\delta(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 - \mathbb{E} \|v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 \leq D_0 \Delta t_n. \quad (5.55)$$

Summing (5.55) over $0 \leq n < N_T$ and using (5.30) yields (5.31). \square

6 Approximate kinetic equation

6.1 Discrete unknown

In this section we will show that, when $\delta \rightarrow 0$, some discrete kinetic unknowns f_δ associated to the scheme (3.4) are approximate kinetic solutions. There may be several ways to define f_δ : it depends for example on the manner in which the discrete data f_K^n are assembled by interpolation. One of the constraints due to our definition of approximate kinetic solution is the formulation ‘‘at fixed t ’’ (2.9) (in opposition to a formulation which would be weak in time). To establish such a formulation in our context, a minimal amount of regularity of the function $t \mapsto f_\delta(t)$ is required (in particular, for all φ , $\langle f_\delta(t), \varphi \rangle$ should be a càdlàg process). For $t \in [t_n, t_{n+1})$, we will therefore consider the function $f_\delta(t)$ defined as the interpolation

$$f_\delta(x, t, \xi) = \frac{t - t_n}{\Delta t_n} \mathbf{1}_{v_\delta^\sharp(x, t) > \xi} + \frac{t_{n+1} - t}{\Delta t_n} \mathbf{1}_{v_\delta(x, t) > \xi}, \quad \xi \in \mathbb{R}, x \in \mathbb{T}^N, \quad (6.1)$$

where $v_\delta^\sharp(x, t)$ is given by

$$v_\delta^\sharp(x, t) = v_K^{n+1/2} + g_{k, K}(v_K^n)(\beta_k(t) - \beta_k(t_n)), \quad t_n \leq t < t_{n+1}, x \in K. \quad (6.2)$$

We set $v_K^\sharp(t) = v_\delta^\sharp(x, t)$, $x \in K$. Then, for $t \in [t_n, t_{n+1})$, $t \mapsto v_K^\sharp(t)$ is itself an interpolation between $v_K^{n+1/2}$ and v_K^{n+1} . We also denote by \mathbf{f}_δ the piecewise constant function

$$\mathbf{f}_\delta(x, t, \xi) = \mathbf{f}_K^n = \mathbf{1}_{v_\delta(x, t) > \xi}, \quad x \in K, t \in [t_n, t_{n+1}). \quad (6.3)$$

We check first that f_δ and \mathbf{f}_δ are close to each other.

Lemma 6.1. *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$. Assume that (1.3), (1.4), (1.6), (3.1), and (5.28) are satisfied. For $\delta \in \mathfrak{D}_T$, assume (5.6) and (5.7). Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9) and let $f_\delta, \mathbf{f}_\delta$ be defined by (6.1)-(6.3). Then*

$$\begin{aligned} \mathbb{E} \int_0^T \int_{\mathbb{T}^N} \left| \int_{\mathbb{R}} |f_\delta(x, t, \xi) - \mathbf{f}_\delta(x, t, \xi)| d\xi \right|^2 dx dt \\ \leq \left[\theta^{-1} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + D_0 T (1 + \theta^{-1}) \right] \left[\sup_{0 \leq n < N_T} \Delta t_n \right]. \end{aligned} \quad (6.4)$$

Proof of Lemma 6.1. Since

$$f_\delta(t) - \mathbf{f}_\delta(t) = \frac{t - t_n}{\Delta t_n} (\mathbf{1}_{v_\delta^\sharp(t) > \xi} - \mathbf{1}_{v_\delta(t) > \xi}),$$

for $t \in [t_n, t_{n+1})$ and since the factor $\frac{t - t_n}{\Delta t_n}$ is less than 1, the quantity we want to estimate is bounded by the following L^2 -norm:

$$\mathbb{E} \int_0^T \int_{\mathbb{T}^N} \left| \int_{\mathbb{R}} |f_\delta(x, t, \xi) - \mathbf{f}_\delta(x, t, \xi)| d\xi \right|^2 dx dt \leq \mathbb{E} \int_0^T \|v_\delta^\sharp(t) - v_\delta(t)\|_{L^2(\mathbb{T}^N)}^2 dt. \quad (6.5)$$

By definition of $v_\delta^\sharp(t)$ and independence and (3.9), we obtain

$$\begin{aligned} \mathbb{E} \int_0^T \int_{\mathbb{T}^N} \left| \int_{\mathbb{R}} |f_\delta(x, t, \xi) - \mathbf{f}_\delta(x, t, \xi)| d\xi \right|^2 dx dt &\leq D_0 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} |t - t_n| dt \\ &+ \mathbb{E} \sum_{n=0}^{N_T-1} \Delta t_n \left\| v_\delta^b(t_{n+1}) - v_\delta(t_n) \right\|_{L^2(\mathbb{T}^N)}^2. \end{aligned}$$

Using the temporal estimate (5.30), we deduce (6.4). \square

Remark 6.1. Note for a future use (cf. (6.45)) that we have just proved the estimate

$$\mathbb{E} \int_0^T \|v_\delta^\sharp(t) - v_\delta(t)\|_{L^2(\mathbb{T}^N)}^2 dt \leq \left[\theta^{-1} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + D_0 T (1 + \theta^{-1}) \right] \left[\sup_{0 \leq n < N_T} \Delta t_n \right] \quad (6.6)$$

To f_δ we will associate the Young measure

$$\nu_{x,t}^\delta(\xi) := -\partial_\xi f_\delta(x, t, \xi) = \frac{t - t_n}{\Delta t_n} \delta(\xi = v_\delta^\sharp(x, t)) + \frac{t_{n+1} - t}{\Delta t_n} \delta(\xi = v_\delta(x, t)), \quad (6.7)$$

We also denote by m_δ the discrete random measure given by

$$dm_\delta(x, t, \xi) = \sum_{n=0}^{N_T-1} \sum_{K \in \mathcal{T}} \mathbf{1}_{K \times [t_n, t_{n+1})}(x, t) m_K^n(\xi) dx dt d\xi. \quad (6.8)$$

Recall the definitions (5.3)-(5.4) (definition of the set of mesh parameter \mathfrak{D}_T in particular), that we will use in all the section.

Proposition 6.2 (Discrete kinetic equation). *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$. Assume that (1.3), (1.4), (1.6), (3.1) and (5.28) are satisfied. For $\delta \in \mathfrak{D}_T$, assume (5.6) and (5.7). Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9) and let f_δ , ν^δ , m_δ be defined by (6.1), (6.7), (6.8) respectively. Then f_δ satisfies the following discrete kinetic formulation: for all $t \in [t_n, t_{n+1}]$, $x \in K$, for all $\psi \in C_c^1(\mathbb{R})$,*

$$\begin{aligned} &\langle f_\delta(x, t), \psi \rangle - \langle f_\delta(x, t_n), \psi \rangle \\ &= -\frac{1}{|K|} \int_{t_n}^t \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \psi(\xi) d\xi ds - \int_{t_n}^t \int_{\mathbb{R}} \partial_\xi \psi(\xi) m_K^n(\xi) d\xi ds \\ &+ \frac{t - t_n}{\Delta t_n} \int_{t_n}^t g_{k,K}(v_K^n) \psi(v_\delta^\sharp(x, s)) d\beta_k(s) + \frac{1}{2} \frac{t - t_n}{\Delta t_n} \int_{t_n}^t \mathbf{G}_K^2(v_K^n) \partial_\xi \psi(v_\delta^\sharp(x, s)) ds. \end{aligned} \quad (6.9)$$

In (6.9), $\langle f_\delta(x, t), \psi \rangle$ stands for the product

$$\int_{\mathbb{R}} f_\delta(x, t, \xi) \psi(\xi) d\xi.$$

Proof of Proposition 6.2. Let Ψ be a primitive for ψ and let $x \in K$, $t \in [t_n, t_{n+1})$. By definition of f_δ , see Equation (6.1), we have

$$\langle f_\delta(x, t), \psi \rangle - \langle f_\delta(x, t_n), \psi \rangle = \frac{t - t_n}{\Delta t_n} \left[\Psi(v^\sharp(x, t)) - \Psi(v_\delta(x, t)) \right],$$

which we decompose as the sum of two terms:

$$\frac{t - t_n}{\Delta t_n} \left[\Psi(v^\sharp(x, t)) - \Psi(v^b(x, t_{n+1})) \right], \quad (6.10)$$

and

$$\frac{t - t_n}{\Delta t_n} \left[\Psi(v^b(x, t_{n+1})) - \Psi(v_\delta(x, t)) \right]. \quad (6.11)$$

We use first the deterministic kinetic formulation (4.7), which we multiply by $\psi(\xi)$. By integration over $\xi \in \mathbb{R}$, it gives

$$(6.11) = -\frac{1}{|K|} \int_{t_n}^t \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \psi(\xi) d\xi ds - \int_{t_n}^t \int_{\mathbb{R}} \partial_\xi \psi(\xi) m_K^n(\xi) d\xi ds. \quad (6.12)$$

By Itô's Formula on the other hand (cf. (6.2)), the term (6.10) is equal to

$$\frac{t - t_n}{\Delta t_n} \int_{t_n}^t g_{k,K}(v_K^n) \psi(v_\delta^\sharp(x, s)) d\beta_k(s) + \frac{1}{2} \frac{t - t_n}{\Delta t_n} \int_{t_n}^t \mathbf{G}_K^2(v_K^n) \partial_\xi \psi(v_\delta^\sharp(x, s)) ds. \quad (6.13)$$

Summing (6.12) and (6.13), we obtain (6.9). \square

We will prove now that the Finite Volume scheme (3.4) is consistent with (1.1). Indeed, we will show, thanks to the estimates obtained in Section 5, that an approximate kinetic equation for f_δ in the sense of (2.9) can be deduced from the discrete kinetic formulation (6.9).

Proposition 6.3 (Approximate kinetic equation). *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$. Assume that (1.3), (1.4), (1.6), (3.1) and (5.28) are satisfied. For $\delta \in \mathfrak{D}_T$, assume (5.6) and (5.7). Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9) and let f_δ , v^δ , m_δ be defined by (6.1), (6.7), (6.8) respectively. If (δ_m) is a sequence in \mathfrak{D}_T which tends to zero according to (5.4), then (f_{δ_m}) is a sequence of approximate solutions to (1.1) of order $d = 2$. Besides, $(f_{\delta_m}(0))$ converges to the equilibrium function $\mathbf{f}_0 = \mathbf{1}_{u_0 > \xi}$ in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ -weak- $*$.*

Proof of Proposition 6.3. The last assertion is clear: $(f_{\delta_m}(0))$ converges to the equilibrium function $\mathbf{f}_0 = \mathbf{1}_{u_0 > \xi}$ in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ -weak- $*$ since, by (3.5), $v^{\delta_m}(0) \rightarrow u_0$ a.e.

on \mathbb{T}^N . We will show that, for all $t \in [0, T]$, for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$,

$$\begin{aligned} \langle f_\delta(t), \varphi \rangle &= \langle f_\delta(0), \varphi \rangle - \iiint_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) dm_\delta(x, s, \xi) + \varepsilon^\delta(t, \varphi) \\ &\quad + \int_0^t \langle f_\delta(s), a(\xi) \nabla_x \varphi \rangle ds \end{aligned} \quad (6.14)$$

$$+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) dx d\beta_k(s) \quad (6.15)$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{G}^2(x, \xi) \partial_\xi \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) dx ds, \quad (6.16)$$

where the error term $\varepsilon^\delta(t, \varphi)$ satisfies

$$\lim_{\delta \rightarrow 0} \mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon^\delta(t, \varphi)|^2 \right] = 0, \quad (6.17)$$

for all $\varphi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$. Note that the convergence in probability (2.8) follows from (6.17). Given $\varphi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$, we introduce the averages over the cells $K \in \mathcal{T}$

$$\varphi_K(\xi) = \frac{1}{|K|} \int_{|K|} \varphi(x, \xi) dx, \quad \xi \in \mathbb{R}. \quad (6.18)$$

To prove (6.16), we apply the discrete kinetic equation (6.9) to $\xi \mapsto \varphi(x, \xi)$ for a fixed $x \in K$. Then we sum the result over $x \in \mathbb{T}^N$. By the telescopic formula

$$\langle f_\delta(x, t), \varphi \rangle - \langle f_\delta(x, 0), \varphi \rangle = \sum_{n=0}^{N_T-1} \langle f_\delta(x, t \wedge t_{n+1}), \varphi \rangle - \langle f_\delta(x, t \wedge t_n), \varphi \rangle,$$

we obtain

$$\begin{aligned} \langle f_\delta(t), \varphi \rangle &= \langle f_\delta(x, 0), \varphi \rangle - \iiint_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \partial_\xi \varphi(\xi) dm_\delta(x, s, \xi) \\ &\quad - \sum_{n=0}^{N_T-1} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \varphi_K(\xi) ds d\xi \end{aligned} \quad (6.19)$$

$$+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} g_{k,\delta}(x, \xi) \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) d\beta_k(s) \quad (6.20)$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} \mathbf{G}_\delta^2(x, \xi) \partial_\xi \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) ds, \quad (6.21)$$

where the measure $\mu_{x,s,t}^\delta$ on $\mathbb{R} \times \mathbb{R}$ is defined by

$$\langle \mu_{x,s,t}^\delta, \psi \rangle = \sum_{n=0}^{N_T-1} \frac{t \wedge t_{n+1} - t \wedge t_n}{\Delta t_n} \mathbf{1}_{[t_n, t_{n+1})}(s) \psi(v_\delta(x, s), v_\delta^\sharp(x, s)), \quad \psi \in C_b(\mathbb{R}^2), \quad (6.22)$$

and the discrete coefficient $g_{k,\delta}(x, \xi)$ is equal to $g_{k,K}(\xi)$ (cf. (3.8)) when $x \in K$ (similarly, $\mathbf{G}_\delta(x, \xi) := \mathbf{G}_K(\xi)$, $x \in K$). Note that $\mu_{x,s,t}^\delta$ is simply the Dirac mass at $(v_\delta(x, s), v_\delta^\sharp(x, s))$, except when $t_l \leq s \leq t$ (where l is the index such that $t_l \leq t < t_{l+1}$), in which case it is the same Dirac mass, with an additional multiplicative factor $\frac{t-t_l}{\Delta t_l}$.

The term (6.19) is a discrete space derivative: we will show that it is an approximation of the term (6.14). The two terms (6.20) and (6.21) are close to (6.15) and (6.16) respectively. We analyse those terms separately (see Section 6.2, Section 6.3). The conclusion of the proof of Proposition 6.3 is given in Section 6.4.

6.2 Space consistency

Lemma 6.4. *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{D}_T$. Assume that (1.4), (1.6), (3.1), (5.6), (5.7) and (5.28) are satisfied. Then, for all $\varphi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$, we have*

$$\begin{aligned} & \int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} a(\xi) \cdot \nabla_x \varphi f_\delta(s) dx ds d\xi \\ &= - \sum_{n=0}^{N_T-1} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \varphi_K(\xi) d\xi + \varepsilon_{\text{space},0}^\delta(t, \varphi) + \varepsilon_{\text{space},1}^\delta(t, \varphi), \end{aligned} \quad (6.23)$$

for all $t \in [0, T]$, with the estimates

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} |\varepsilon_{\text{space},0}^\delta(t, \varphi)|^2 \\ & \leq T |L_A|^2 \|\nabla_x \varphi\|_{L_{x,\xi}^\infty}^2 \left[\frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + D_0 T \left(1 + \frac{1}{\theta} \right) \right] \sup_{0 \leq n < N_T} \Delta t_n, \end{aligned} \quad (6.24)$$

and, for all compact $\Lambda \subset \mathbb{R}$, for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$ supported in $\mathbb{T}^N \times \Lambda$,

$$\mathbb{E} \sup_{t \in [0, T]} |\varepsilon_{\text{space},1}^\delta(t, \varphi)|^2 \leq \frac{16L_A T}{\alpha_N^2} |\Lambda|^2 \|\partial_\xi \nabla_x \varphi\|_{L_{x,\xi}^\infty}^2 \left[\frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{2D_0 T}{\theta} \right] h. \quad (6.25)$$

Proof of Lemma 6.4. To begin with, we replace f_δ by \mathbf{f}_δ in the left-hand side of (6.23). This accounts for the first error term

$$\varepsilon_{\text{space},0}^\delta(t, \varphi) = \int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} a(\xi) \cdot \nabla_x \varphi (f_\delta(s) - \mathbf{f}_\delta(s)) dx ds d\xi.$$

Thanks to Lemma 6.1, we have the estimate (6.24) for $\varepsilon_{\text{space},0}^\delta(t, \varphi)$. Then, we use the following development:

$$\begin{aligned} & \int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} a(\xi) \cdot \nabla_x \varphi \mathbf{f}_\delta(s) dx ds d\xi \\ &= \sum_{n=0}^{N_T-1} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{K \in \mathcal{N}(K)} \int_K a(\xi) \cdot \nabla_x \varphi \mathbf{f}_\delta(s) dx ds d\xi, \end{aligned} \quad (6.26)$$

Since $\mathbf{f}_\delta(s)$ has a constant value \mathbf{f}_K^n in $K \times [t_n, t_{n+1})$, we obtain, thanks to the Stokes formula,

$$\begin{aligned} & \int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} a(\xi) \cdot \nabla_x \varphi \mathbf{f}_\delta(s) dx ds d\xi \\ &= \sum_{n=0}^{N_T-1} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{K \in \mathcal{N}(K)} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^*(\xi) \varphi_{K|L} \mathbf{f}_K^n ds d\xi, \end{aligned} \quad (6.27)$$

where $a_{K \rightarrow L}^*(\xi)$ is defined by (4.2) and $\varphi_{K|L}$ is the mean-value of φ over $K|L$:

$$\varphi_{K|L}(\xi) = \frac{1}{|K|L|} \int_{K|L} \varphi(x, \xi) d\mathcal{H}^{N-1}(x).$$

We add a corrective term to (6.27) to obtain

$$\begin{aligned} & \int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} a(\xi) \cdot \nabla_x \varphi \mathbf{f}_\delta(s) dx ds d\xi \\ &= \sum_{n=0}^{N_T-1} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{K \in \mathcal{N}(K)} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^*(\xi) (\varphi_{K|L} - \varphi_K) \mathbf{f}_K^n ds d\xi. \end{aligned} \quad (6.28)$$

Equation (6.28) follows indeed from (6.27) by the anti-symmetry property (5.35) of $a_{K \rightarrow L}$. Note that Equation (6.28) is more natural than Equation (6.27) (when one has in mind the decomposition of a volume integral over each cells K), by use of the correspondence

$$a(\xi) \cdot \nabla_x \varphi \simeq \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^*(\xi) (\varphi_{K|L} - \varphi_K) \text{ in } K.$$

By (5.35), the discrete convective term in (6.19) is

$$\sum_{n=0}^{N_T-1} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) (\varphi_{K|L} - \varphi_K) ds d\xi. \quad (6.29)$$

To estimate how close is the right-hand side of (6.28) to (6.29), we have to compare $a_{K \rightarrow L}^n(\xi)$ and $a_{K \rightarrow L}^*(\xi) \mathbf{f}_K^n(\xi)$. Let $\gamma \in W^{1,1}(\mathbb{R}_\xi)$. If $v_K^n \leq v_L^n$, then

$$\int_{\mathbb{R}} \gamma(\xi) [a_{K \rightarrow L}^*(\xi) \mathbf{f}_K^n(\xi) - a_{K \rightarrow L}^n(\xi)] d\xi = - \int_{\mathbb{R}} \gamma(\xi) \bar{\mathbf{f}}_K^n(\xi) a_{K \rightarrow L}^n(\xi) d\xi \quad (6.30)$$

by the consistency hypothesis (4.11) and the support condition (4.16). Using an integration by parts and (4.17), we obtain

$$\int_{\mathbb{R}} \gamma(\xi) [a_{K \rightarrow L}^*(\xi) \mathbf{f}_K^n(\xi) - a_{K \rightarrow L}^n(\xi)] d\xi = \int_{\mathbb{R}} \gamma'(\xi) \Phi_{K \rightarrow L}^n(\xi \vee v_K^n) d\xi. \quad (6.31)$$

Similarly, if $v_L^n \leq v_K^n$, then

$$\int_{\mathbb{R}} \gamma(\xi) [a_{K \rightarrow L}^*(\xi) \mathbf{f}_K^n(\xi) - a_{K \rightarrow L}^n(\xi)] d\xi = - \int_{\mathbb{R}} \gamma'(\xi) \bar{\Phi}_{K \rightarrow L}^n(\xi \wedge v_K^n) d\xi. \quad (6.32)$$

We deduce that (6.23) is satisfied with an error term

$$\begin{aligned} \varepsilon_{\text{space},1}^\delta(t, \varphi) &:= \int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} a(\xi) \cdot \nabla_x \varphi \mathbf{f}_\delta(s) dx ds d\xi \\ &\quad + \sum_{n=0}^{N_T-1} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^n(\xi) \varphi_K(\xi) d\xi, \end{aligned} \quad (6.33)$$

which is bounded as follows:

$$\begin{aligned} |\varepsilon_{\text{space},1}^\delta(t, \varphi)| &\leq \sum_{n=0}^{N_T-1} \int_{t \wedge t_n}^{t \wedge t_{n+1}} \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} |\partial_\xi \varphi_{K|L} - \partial_\xi \varphi_K| \\ &\quad \times \left[\mathbf{1}_{v_K^n \leq v_L^n} |\Phi_{K \rightarrow L}^n(\xi \vee v_K^n)| + \mathbf{1}_{v_L^n < v_K^n} |\bar{\Phi}_{K \rightarrow L}^n(\xi \wedge v_K^n)| \right] d\xi, \end{aligned} \quad (6.34)$$

By (5.5), we have

$$|\partial_\xi \varphi_{K|L}(\xi) - \partial_\xi \varphi_K(\xi)| \leq \|\partial_\xi \nabla_x \varphi(\cdot, \xi)\|_{L_x^\infty} h,$$

for all $\xi \in \mathbb{R}$. If φ is compactly supported in $\mathbb{T}^N \times \Lambda$, we obtain thus the bound

$$|\varepsilon_{\text{space},1}^\delta(t, \varphi)| \leq \|\partial_\xi \nabla_x \varphi\|_{L_{x,\xi}^\infty} |\Lambda| B_{\text{space}} h, \quad (6.35)$$

where B_{space} is equal to

$$\sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \left[\mathbf{1}_{v_K^n \leq v_L^n} \sup_{\xi \in \mathbb{R}} |\Phi_{K \rightarrow L}^n(\xi \vee v_K^n)| + \mathbf{1}_{v_L^n < v_K^n} \sup_{\xi \in \mathbb{R}} |\bar{\Phi}_{K \rightarrow L}^n(\xi \wedge v_K^n)| \right].$$

We seek for a bound of order $h^{-1/2}$ on B_{space} . For notational convenience we will estimate only the first part

$$B_{\text{space}}^1 := \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \mathbf{1}_{v_K^n \leq v_L^n} \sup_{\xi \in \mathbb{R}} |\Phi_{K \rightarrow L}^n(\xi \vee v_K^n)|,$$

since the bound on the second part in B_{space} will be similar. By the Cauchy Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}|B_{\text{space}}^1|^2 &\leq \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |\partial K| \\ &\quad \times \mathbb{E} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \frac{\mathbf{1}_{v_K^n \leq v_L^n}}{|K|L|} \sup_{\xi \in \mathbb{R}} |\Phi_{K \rightarrow L}^n(\xi \vee v_K^n)|^2. \end{aligned}$$

We use the estimate (5.53), which gives

$$|\Phi_{K \rightarrow L}^n(\xi \vee v_K^n)|^2 \leq 2L_A |K| |L| \int_{\mathbb{R}} (\bar{\mathbf{f}}_L^n - \bar{\mathbf{f}}_K^n) \Phi_{K \rightarrow L}^n(\xi) d\xi,$$

thanks to (5.14). We also use (5.8), and get

$$\mathbb{E}|B_{\text{space}}^1|^2 \leq \frac{2L_A T}{\alpha_N^2 h} \mathbb{E} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} \sum_{L \in \mathcal{N}(K)} \int_{\mathbb{R}} (\bar{\mathbf{f}}_L^n - \bar{\mathbf{f}}_K^n) \Phi_{K \rightarrow L}^n(\xi) d\xi.$$

With (5.29) and (6.35), we conclude to (6.25). \square

6.3 Stochastic terms

Lemma 6.5. *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1) and (5.28) are satisfied. Then, for all $\varphi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$, we have*

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} g_{k,\delta}(x, \xi) \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) d\beta_k(s) \\ &= \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) dx d\beta_k(s) + \varepsilon_{\mathbb{W},1}^\delta(t, \varphi) + \varepsilon_{\mathbb{W},2}^\delta(t, \varphi), \end{aligned} \quad (6.36)$$

and

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} \mathbf{G}_\delta^2(x, \xi) \partial_\xi \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) ds \\ &= \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{G}^2(x, \xi) \partial_\xi \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) dx ds + \varepsilon_{\mathbb{W},3}^\delta(t, \varphi) + \varepsilon_{\mathbb{W},4}^\delta(t, \varphi), \end{aligned} \quad (6.37)$$

where

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_{\mathbb{W},1}^\delta(t, \varphi)|^2 \right] \leq 4D_1 T \|\varphi\|_{L_{x,\xi}^\infty}^2 h^2 + 2D_0 \|\varphi\|_{L_{x,\xi}^\infty}^2 \left[\sup_{0 \leq n < N_T} \Delta t_n \right], \quad (6.38)$$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_{\mathbb{W},3}^\delta(t, \varphi)|^2 \right] \leq 4D_1 T \|\partial_\xi \varphi\|_{L_{x,\xi}^\infty}^2 h^2 + 2D_0 \|\partial_\xi \varphi\|_{L_{x,\xi}^\infty}^2 \left[\sup_{0 \leq n < N_T} \Delta t_n \right], \quad (6.39)$$

and

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_{\mathbb{W},2}^\delta(t, \varphi)|^2 \right] &\leq 2D_0 \|\varphi\|_{L_{x,\xi}^\infty}^2 \left[\sup_{0 \leq n < N_T} \Delta t_n \right] \\ &\quad + 8 \left\{ D_1 \|\varphi\|_{L_{x,v}^\infty}^2 + D_0 \|\partial_\xi \varphi\|_{L_{x,\xi}^\infty}^2 \right\} \\ &\quad \times \left[\frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T})}^2 + \frac{3D_0 T}{\theta} \right] \left[\sup_{0 \leq n < N_T} \Delta t_n \right]^{1/2}. \end{aligned} \quad (6.40)$$

Eventually, $\varepsilon_{\mathbb{W},4}^\delta(t, \varphi)$ satisfies the same estimate as $\varepsilon_{\mathbb{W},2}^\delta(t, \varphi)$ with $\partial_\xi \varphi$ instead of φ in the right-hand side of (6.40).

Proof of Lemma 6.5. Define

$$\varepsilon_{\mathbb{W},1}^\delta(t, \varphi) = \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} [g_{k,\delta}(x, \xi) - g_k(x, \xi)] \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) dx d\beta_k(s),$$

and let $\varepsilon_{\mathbb{W},2}^\delta(t, \varphi)$ be equal to

$$\int_0^t \int_{\mathbb{T}^N} \left[\int_{\mathbb{R} \times \mathbb{R}} g_k(x, \xi) \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) - \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) \right] dx d\beta_k(s).$$

Then (6.36) is satisfied. Note that $n \mapsto \varepsilon_{\mathbb{W},1}^\delta(t_n, \varphi)$ is a (\mathcal{F}_{t_n}) -martingale. By Doob's Inequality, Jensen's Inequality (note that $\mu_{x,s,t}^\delta(\mathbb{R} \times \mathbb{R}) \leq 1$) and (3.10), we deduce

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq n < N_T} |\varepsilon_{\mathbb{W},1}^\delta(t_n, \varphi)|^2 \right] \\ & \leq 4\mathbb{E} \int_0^{t_{N_T}} \left| \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} [g_{k,\delta}(x, \xi) - g_k(x, \xi)] \varphi(x, \zeta) d\mu_{x,s,t_{N_T}}^\delta(\xi, \zeta) \right|^2 dx ds \\ & \leq 4\mathbb{E} \int_0^{t_{N_T}} \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} |g_{k,\delta}(x, \xi) - g_k(x, \xi)|^2 d\mu_{x,s,t}^\delta(\xi, \zeta) dx ds \|\varphi\|_{L_{x,\xi}^\infty}^2 \\ & \leq 4D_1 T \|\varphi\|_{L_{x,\xi}^\infty}^2 h^2. \end{aligned}$$

Besides, we see, using Itô's Isometry, and (1.2), (3.9), that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [t_n, t_{n+1})} |\varepsilon_{\mathbb{W},1}^\delta(t, \varphi) - \varepsilon_{\mathbb{W},1}^\delta(t_n, \varphi)|^2 \right] \\ & \leq \mathbb{E} \int_{t_n}^{t_{n+1}} \left| \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} [g_{k,\delta}(x, v_\delta(s, x)) - g_k(x, v_\delta(s, x))] \varphi(x, v_\delta^\sharp(s, x)) \right|^2 dx ds \\ & \leq 2D_0 \|\varphi\|_{L_{x,\xi}^\infty}^2 \left[\sup_{0 \leq n < N_T} \Delta t_n \right]. \end{aligned}$$

Similarly, we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_{\mathbb{W},2}^\delta(t, \varphi)|^2 \right] \leq 2D_0 \|\varphi\|_{L_{x,\xi}^\infty}^2 \left[\sup_{0 \leq n < N_T} \Delta t_n \right] + \mathbb{E} \left[\sup_{0 \leq n < N_T} |\varepsilon_{\mathbb{W},2}^\delta(t_n, \varphi)|^2 \right].$$

Using Doob's Inequality, we obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_{\mathbb{W},2}^\delta(t, \varphi)|^2 \right] \leq 2D_0 \|\varphi\|_{L_{x,\xi}^\infty}^2 \left[\sup_{0 \leq n < N_T} \Delta t_n \right] + 4\mathbb{E} |\varepsilon_{\mathbb{W},2}^\delta(t_{N_T}, \varphi)|^2.$$

By Itô's Formula, $\mathbb{E} |\varepsilon_{\mathbb{W},2}^\delta(t_{N_T}, \varphi)|^2$ is bounded from above by

$$\mathbb{E} \int_0^{t_{N_T}} \int_{\mathbb{T}^N} \sum_{k \geq 1} \left| \int_{\mathbb{R} \times \mathbb{R}} g_k(x, \xi) \varphi(x, \zeta) d\mu_{x,s,t_{N_T}}^\delta(\xi, \zeta) - \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) \right|^2 dx ds. \quad (6.41)$$

We have, for $t \in [0, T)$, $t \in [t_n, t_{n+1})$, $n < N_T$, and $\psi \in C_b(\mathbb{R} \times \mathbb{R})$,

$$\begin{aligned} \langle \mu_{x,t,t_{N_T}}^\delta, \psi \rangle &= \langle \nu_{x,t} \otimes \nu_{x,t}, \psi \rangle - \frac{t - t_n}{\Delta t_n} \left[\psi(v_\delta^\#(x, t), v_\delta^\#(x, t)) - \psi(v_\delta(x, t), v_\delta^\#(x, t)) \right] \\ &\quad - \frac{t_{n+1} - t}{\Delta t_n} \left[\psi(v_\delta(x, t), v_\delta(x, t)) - \psi(v_\delta(x, t), v_\delta^\#(x, t)) \right]. \end{aligned}$$

We estimate therefore (6.41) by the two terms

$$2\mathbb{E} \sum_{0 \leq n < N_T} \int_{t_n}^{t_{n+1}} \int_{\mathbb{T}^N} \sum_{k \geq 1} \left| g_k(x, v_\delta^\#(x, t)) - g_k(x, v_\delta(x, t)) \right|^2 |\varphi(x, \xi)|^2 dx dt, \quad (6.42)$$

and

$$2\mathbb{E} \sum_{0 \leq n < N_T} \int_{t_n}^{t_{n+1}} \int_{\mathbb{T}^N} \sum_{k \geq 1} \left| \varphi(x, v_\delta^\#(x, t)) - \varphi(x, v_\delta(x, t)) \right|^2 |g_k(x, v_\delta(x, t))|^2 dx dt. \quad (6.43)$$

Note that (1.3) gives, for all $\eta > 0$, and $\bar{v}, v \in \mathbb{R}$,

$$\sum_{k \geq 1} |g_k(x, \bar{v}) - g_k(x, v)|^2 \leq D_1 |\bar{v} - v| \leq D_1 \left(\eta + \frac{1}{\eta} |\bar{v} - v|^2 \right). \quad (6.44)$$

In virtue of (6.44), we can bound (6.42) by

$$2D_1 \|\varphi\|_{L_{x,v}^\infty}^2 \left[\eta + \frac{1}{\eta} \mathbb{E} \int_0^T \|v_\delta^\#(t) - v_\delta(t)\|_{L^2(\mathbb{T}^N)_2}^2 dt \right].$$

Using (6.6) and taking $\eta = [\sup_{0 \leq n < N_T} \Delta t_n]^{1/2}$, we deduce that (6.42) is bounded by

$$2D_1 \|\varphi\|_{L_{x,\xi}^\infty}^2 \left[\frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{3D_0 T}{\theta} \right] \left[\sup_{0 \leq n < N_T} \Delta t_n \right]^{1/2}. \quad (6.45)$$

An estimate on (6.43) is obtained as follows: (6.43) is bounded by

$$2D_0 \|\partial_\xi \varphi\|_{L_{x,\xi}^\infty}^2 \mathbb{E} \int_0^T \|v_\delta^\#(t) - v_\delta(t)\|_{L^2(\mathbb{T}^N)_2}^2 dt.$$

Using (6.6) gives an estimate on (6.43) from above by

$$2D_0 \|\partial_\xi \varphi\|_{L_{x,\xi}^\infty}^2 \left[\frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{3D_0 T}{\theta} \right] \left[\sup_{0 \leq n < N_T} \Delta t_n \right]. \quad (6.46)$$

Next, we denote by $\varepsilon_{W,3}^\delta(t, \varphi)$ and $\varepsilon_{W,4}^\delta(t, \varphi)$ the error terms

$$\begin{aligned} &\int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R} \times \mathbb{R}} [\mathbf{G}_{k,\delta}^2(x, \xi) - \mathbf{G}_k^2(x, \xi)] \partial_\xi \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) dx ds, \\ &\int_0^t \int_{\mathbb{T}^N} \left[\int_{\mathbb{R} \times \mathbb{R}} \mathbf{G}_k^2(x, \xi) \partial_\xi \varphi(x, \zeta) d\mu_{x,s,t}^\delta(\xi, \zeta) - \int_{\mathbb{R}} \mathbf{G}_k^2(x, \xi) \partial_\xi \varphi(x, \xi) d\nu_{x,s}^\delta(\xi) \right] dx ds. \end{aligned}$$

We have, for $x \in K$, $\eta > 0$,

$$\begin{aligned} |\mathbf{G}_{k,\delta}^2(x, \xi) - \mathbf{G}_k^2(x, \xi)| &= \left| \sum_{k \geq 1} (g_{k,K}(\xi) - g_k(x, \xi))(g_{k,K}(\xi) + g_k(x, \xi)) \right| \\ &\leq \frac{1}{2\eta} \sum_{k \geq 1} |g_{k,K}(\xi) - g_k(x, \xi)|^2 + \eta \sum_{k \geq 1} |g_{k,K}(\xi)|^2 + |g_k(x, \xi)|^2. \end{aligned}$$

Using (3.10), (1.2) and (3.9) and taking $\eta = h$, we see that

$$|\mathbf{G}_{k,\delta}^2(x, \xi) - \mathbf{G}_k^2(x, \xi)| \leq (D_0 + D_1)h. \quad (6.47)$$

This is sufficient to obtain (6.39) and the last statement of the lemma (estimate on $\varepsilon_{W,4}^\delta(t, \varphi)$). \square

6.4 Conclusion

To conclude, let us set

$$\varepsilon_\delta(\varphi) = \sum_{j=1}^4 \varepsilon_{W,j}^\delta(t, \varphi) - \varepsilon_{\text{space},0}^\delta(t, \varphi) - \varepsilon_{\text{space},1}^\delta(t, \varphi).$$

Then the approximate kinetic equation (6.16) follows from the discrete kinetic equation (6.21) and from the consistency estimates (6.23)-(6.36)-(6.37). Since $\|v_\delta(0)\|_{L^2(\mathbb{T}^N)} \leq \|u_0\|_{L^2(\mathbb{T}^N)}$ (the projection (3.5) onto piecewise-constant functions is an orthogonal projection in $L^2(\mathbb{T}^N)$), it follows from the error estimates (6.24), (6.25), (6.38), (6.39), (6.40) and from the CFL condition (5.28) that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon^\delta(t, \varphi)|^2 \right] \leq C(\varphi) |\delta|^{1/2},$$

where $C(\varphi)$ is a constant that depends on $\|u_0\|_{L^2(\mathbb{T}^N)}$, on D_0, D_1, L_A , on the parameter θ in (5.28), on T , on $|\Lambda|$, where Λ is the support of φ , and on the norms $\|\partial_{x_i}^{j_i} \partial_\xi^k \varphi\|_{L^\infty(\mathbb{T}^N \times \mathbb{R})}$ with $j_i + k \leq 2$.

7 Convergence

To apply Theorem 2.2 on the basis of Proposition 6.3, we need to establish some additional estimates on the numerical Young measure ν^δ and on the numerical random measure m^δ . This is done in Section 7.1. We conclude to the convergence of the Finite Volume method in Section 7, Theorem 7.4.

7.1 Additional estimates

7.1.1 Tightness of (ν^δ)

Lemma 7.1 (Tightness of (ν^δ)). *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1) and (5.28) are satisfied. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9) and let ν^δ be defined by (6.7). Let $p \in [1, +\infty)$. We have*

$$\mathbb{E} \left(\sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} (1 + |\xi|^p) d\nu_{x,t}^\delta(\xi) dx \right) \leq C_p, \quad (7.1)$$

where C_p is a constant depending on D_0 , p , T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$ only.

Proof of Lemma 7.1. It is sufficient to do the proof for $p \in 2\mathbb{N}^*$ since $1 + |\xi|^p \leq 2(1 + |\xi|^q)$ for all $\xi \in \mathbb{R}$ if $q \geq p$. Note that

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\delta(\xi) dx = \frac{t - t_n}{\Delta t_n} \|v_\delta^\sharp(t)\|_{L^p(\mathbb{T}^N)}^p + \frac{t_{n+1} - t}{\Delta t_n} \|v_\delta(t)\|_{L^p(\mathbb{T}^N)}^p,$$

for $t \in [t_n, t_{n+1})$. Recall also that v_δ^\sharp is defined by (6.2). Let

$$\varphi_p(\xi) = p\xi^{p-1} = \partial_\xi \xi^p \quad \xi \in \mathbb{R}.$$

We multiply Equation (4.7) by $\varphi_p(\xi)$ and sum the result over K , ξ . We obtain then, thanks to (5.35),

$$\|v_\delta^\flat(t_{n+1})\|_{L^p(\mathbb{T}^N)}^p + p(p-1)\Delta t_n \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \int_{\mathbb{R}} \xi^{p-2} m_K^n(\xi) d\xi = \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p. \quad (7.2)$$

In particular, we have the L^p estimate

$$\|v_\delta^\flat(t_{n+1})\|_{L^p(\mathbb{T}^N)}^p \leq \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p. \quad (7.3)$$

Let us now estimate the increase of L^p -norm due to the stochastic evolution. By Itô's Formula and (6.2), we have

$$\begin{aligned} & |v_K^{n+1}|^p \\ &= |v_K^{n+1/2}|^p + p \int_{t_n}^{t_{n+1}} v_K^\sharp(t)^{p-1} g_{k,K}(v_K^n) d\beta_k(t) + \frac{1}{2} p(p-1) \int_{t_n}^{t_{n+1}} v_K^\sharp(t)^{p-2} \mathbf{G}_K^2(v_K^n) dt, \end{aligned}$$

and thus

$$\begin{aligned} \|v_\delta(t_{n+1})\|_{L^p(\mathbb{T}^N)}^p &= \|v_\delta^\flat(t_{n+1})\|_{L^p(\mathbb{T}^N)}^p + p \int_{t_n}^{t_{n+1}} \langle v_\delta^\sharp(t)^{p-1}, \gamma_k^n \rangle_{L^2(\mathbb{T}^N)} d\beta_k(t) \\ &\quad + \frac{1}{2} p(p-1) \int_{t_n}^{t_{n+1}} \langle v_\delta^\sharp(t)^{p-2}, \Gamma^n \rangle_{L^2(\mathbb{T}^N)} dt, \quad (7.4) \end{aligned}$$

where

$$\gamma_k^n(x) = g_{k,K}(v_K^n), \quad \Gamma^n(x) = \mathbf{G}_K^2(v_K^n), \quad x \in K.$$

Using (7.3) and induction, we obtain

$$\|v_\delta(T)\|_{L^p(\mathbb{T}^N)}^p \leq \|v_\delta(0)\|_{L^p(\mathbb{T}^N)}^p + M_{N_T} + B_{N_T}, \quad (7.5)$$

where (M_N) is the martingale

$$M_N = p \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle v_\delta^\sharp(t)^{p-1}, \gamma_k^n \rangle_{L^2(\mathbb{T}^N)} d\beta_k(t)$$

and

$$B_N = \frac{1}{2} p(p-1) \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle v_\delta^\sharp(t)^{p-2}, \Gamma^n \rangle_{L^2(\mathbb{T}^N)} dt.$$

Note that the argument $\langle v_\delta^\sharp(t)^{p-2}, \Gamma^n \rangle_{L^2(\mathbb{T}^N)}$ in B_N is non-negative since $\Gamma^n \geq 0$ and $p-2 \in 2\mathbb{N}$. Consequently, $\mathbb{E} \sup_{0 \leq n \leq N_T} B_n = \mathbb{E} B_{N_T}$ and we deduce the following bound

$$\mathbb{E} \sup_{0 \leq n \leq N_T} B_n \leq \frac{1}{2} p(p-1) D_0 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} dt. \quad (7.6)$$

We have used (3.9) to obtain (7.6). If $p = 2$, then $\mathbb{E} \|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} = 1$ and is therefore bounded. To estimate $\mathbb{E} \|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2}$ when $p \geq 4$, note that $v_\delta^\sharp(t) = v_\delta^b(t_{n+1}) + z_\delta^n(t)$ for $t \in (t_n, t_{n+1})$, where $z_\delta^n(x, t) := \gamma_k^n(x)(\beta_k(t) - \beta_k(t_n))$ is, conditionally to \mathcal{F}_n , a Gaussian random variable with variance, for $x \in K$,

$$\mathbb{E} [|z_K^n(t)|^2 | \mathcal{F}_n] = (t - t_n) \mathbf{G}_K^2(v_K^n) \leq D_0 \Delta t_n$$

by (3.9). In particular, we have the bound

$$\begin{aligned} \mathbb{E} \|z_\delta^n(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} &= \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \mathbb{E} (\mathbb{E} [|z_K^n(t)|^{p-2} | \mathcal{F}_n]) \\ &= C(p) \sum_{K \in \mathcal{T}/\mathbb{Z}^N} |K| \mathbb{E} (\mathbb{E} [|z_K^n(t)|^2 | \mathcal{F}_n])^{(p-2)/2} \leq C(p) (D_0 \Delta t_n)^{(p-2)/2}, \end{aligned}$$

where $C(p)$ is a constant depending on p . It follows, using (7.3), that we have the estimate

$$\begin{aligned} \mathbb{E} \|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} &\leq C(p, D_0) \left(1 + \mathbb{E} \|v_\delta^b(t_{n+1})\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \right) \\ &\leq C(p, D_0) \left(1 + \mathbb{E} \|v_\delta(t_n)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \right), \end{aligned}$$

where $C(p, D_0)$ is a constant depending on p and D_0 . In particular, we have

$$\sup_{0 \leq n < N_T} \sup_{t \in (t_n, t_{n+1})} \mathbb{E} \|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \leq C(p, D_0) \left(1 + \sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \right). \quad (7.7)$$

By (7.6), we conclude that

$$\mathbb{E} \sup_{0 \leq n \leq N_T} B_n \leq C(p, D_0)T \left(1 + \sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \right), \quad (7.8)$$

for possibly a different constant $C(p, D_0)$. Let us now turn to the estimate of the quantity $\mathbb{E} \sup_{0 \leq n \leq N_T} |M_n|$. The martingale (M_N) can be rewritten as a stochastic integral (with an integrand which is a simple function). Consequently, the quadratic variation of M_{N_T} is

$$\begin{aligned} \langle M_{N_T} \rangle &= p^2 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \sum_k |\langle |v_\delta^\#(t)|^{p-1}, \gamma_k^n \rangle_{L^2(\mathbb{T}^N)}|^2 dt \\ &\leq p^2 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \sum_k \| |v_\delta^\#(t)|^{p-1} \|_{L^2(\mathbb{T}^N)}^2 \|\gamma_k^n\|_{L^2(\mathbb{T}^N)}^2 dt \\ &= p^2 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \|v_\delta^\#(t)\|_{L^{2(p-1)}(\mathbb{T}^N)}^{2(p-1)} \|\Gamma^n\|_{L^1(\mathbb{T}^N)} dt \\ &\leq p^2 D_0 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \|v_\delta^\#(t)\|_{L^{2(p-1)}(\mathbb{T}^N)}^{2(p-1)} dt, \end{aligned}$$

by (3.9). Using (7.7) (with $2p$ instead of p) gives thus

$$\mathbb{E} \langle M_{N_T} \rangle \leq p^2 D_0 T C(2p, D_0) \left(1 + \sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^{2p-2}(\mathbb{T}^N)}^{2p-2} \right). \quad (7.9)$$

By Burkholder - Davis - Gundy's Inequality, there exists a constant C_{BDG} such that

$$\mathbb{E} \sup_{0 \leq n \leq N_T} |M_n| \leq C_{\text{BDG}} \mathbb{E} \langle M_{N_T} \rangle^{1/2}.$$

By Jensen's Inequality and the estimate (7.9), we obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \leq n \leq N_T} |M_n| &\leq C_{\text{BDG}} (\mathbb{E} \langle M_{N_T} \rangle)^{1/2} \\ &\leq C_{\text{BDG}} p (D_0 T C(2p, D_0))^{1/2} \left(1 + \sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^{2p-2}(\mathbb{T}^N)}^{2p-2} \right)^{1/2}. \end{aligned} \quad (7.10)$$

We can conclude now. Since $\mathbb{E} M_{N_T} = 0$, taking expectation in (7.5) (where we replace N_T by n) gives

$$\mathbb{E} \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p \leq \|v_\delta(0)\|_{L^p(\mathbb{T}^N)}^p + \mathbb{E} B_n.$$

Note (see Section 6.4) that

$$\|v_\delta(0)\|_{L^p(\mathbb{T}^N)} \leq \|u_0\|_{L^p(\mathbb{T}^N)} \leq \|u_0\|_{L^\infty(\mathbb{T}^N)}.$$

By (7.8), this gives

$$\sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p \leq \|u_0\|_{L^\infty(\mathbb{T}^N)}^p + C(p, D_0)T \left(1 + \sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \right).$$

By iteration on $p \in 2\mathbb{N}^*$, we deduce, for every such p , that

$$\sup_{0 \leq n < N_T} \mathbb{E} \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p \leq C_p, \quad (7.11)$$

where the constant C_p depends on p , D_0 , T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$. Denote generally by C_p any such constant, possibly different from line to line, depending only on p , D_0 , T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$. By (7.11) with $2p - 2$ instead of p , we have $\mathbb{E} \sup_{0 \leq n \leq N_T} |M_n| \leq C_p$. Then we use (7.8) with $p - 2$ instead of p to obtain $\mathbb{E} \sup_{0 \leq n \leq N_T} B_n \leq C_p$. By (7.5), we deduce

$$\mathbb{E} \sup_{0 \leq n < N_T} \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p \leq C_p, \quad (7.12)$$

which concludes the proof of the lemma. \square

7.1.2 Tightness of (m_δ)

Lemma 7.2 (Tightness of (m_δ)). *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{D}_T$. Assume that (1.4), (1.6), (3.1) and (5.28) are satisfied. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(5.9) and let m_δ be defined by (6.8). Then, for all $p \geq 1$, we have*

$$\mathbb{E} \left| \iiint_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} (1 + |\xi|^p) dm_\delta(x, t, \xi) \right|^2 \leq C_p, \quad (7.13)$$

where C_p is a constant depending on D_0 , p , T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$ only.

Proof of Lemma 7.2. Let $p \in 2\mathbb{N}^*$. By (7.2), we have

$$\sum_{K \in \mathcal{T}/\mathbb{Z}^N} \int_{\mathbb{R}} \xi^{p-2} m_K^n(\xi) d\xi \leq \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p.$$

We multiply this inequality by Δt_n . Summing over $n \in \{0, \dots, N_T - 1\}$, we obtain

$$\iiint_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \xi^{p-2} dm_\delta(x, t, \xi) \leq T \sup_{0 \leq n < N_T} \|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p.$$

Since $\|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^{2p} \leq \|v_\delta(t_n)\|_{L^{2p}(\mathbb{T}^N)}^{2p}$, we have

$$\mathbb{E} \left| \iiint_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \xi^{p-2} dm_\delta(x, t, \xi) \right|^2 \leq T^2 \mathbb{E} \sup_{0 \leq n < N_T} \|v_\delta(t_n)\|_{L^{2p}(\mathbb{T}^N)}^{2p} \leq T^2 C_p$$

by (7.12). Taking first $p = 2$, then $p \in 2\mathbb{N}^*$ arbitrary, we obtain (7.13). \square

7.2 Convergence

We may now apply the theorem 2.2, to obtain the following results.

Theorem 7.3. *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$. Assume that the hypotheses (1.3), (1.4), (1.6), are satisfied. Then there exists a unique solution u to (1.1) with initial datum u_0 , in the sense of Definition 2.2. Besides, for all $1 \leq p < +\infty$, almost-surely, $u \in C([0, T]; L^p(\mathbb{T}^N))$.*

Theorem 7.4. *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$. Assume that the hypotheses (1.3), (1.4), (1.6), (3.1), (5.6), (5.7) and (5.28) are satisfied. Let u be the solution to (1.1) with initial datum u_0 and let v_δ be the solution to the Finite Volume scheme (3.4)-(3.5)-(3.6)-(3.7). Then we have the convergence*

$$\lim_{\delta \rightarrow 0} \mathbb{E} \|v_\delta - u\|_{L^p(\mathbb{T}^N \times (0, T))}^p = 0, \quad (7.14)$$

for all $p \in [1, \infty)$.

Remark 7.1. If the X_k^{n+1} are merely i.i.d. random variables with normalized centred normal law $\mathcal{N}(0, 1)$, then (v_δ) is converging to u in $L^p(\mathbb{T}^N \times (0, T))$ in law when $\delta \rightarrow 0$. Indeed, the identity (3.7) is only satisfied in law now, hence v_δ has the same law as the function \tilde{v}_δ defined by (3.4)-(3.5)-(3.6), with X_k^{n+1} replaced by the right-hand side of (3.7). We apply the conclusion of Theorem 7.4 to \tilde{v}_δ . As a corollary, we obtain the convergence in law of (\tilde{v}_δ) to u in $L^p(\mathbb{T}^N \times (0, T))$. A slightly different manner of expressing the same thing is to notice that, when the discrete increments (X_k^{n+1}) are some given normal law $\mathcal{N}(0, 1)$, then we can construct a set of Brownian motions $\tilde{\beta}_k(t)$ such that

$$X_k^{n+1} = \frac{\tilde{\beta}_k(t_{n+1}) - \tilde{\beta}_k(t_n)}{(\Delta t_n)^{1/2}}. \quad (7.15)$$

Indeed, without loss of generality, we can restrict ourselves to the case $\Delta t_n = 1$ in (7.15) and use the Lévy-Ciesielski construction of the Brownian motion, [32, Section 3.2] on $[0, 1]$ as follows: we define (cf. [32, Formula (3.1)])

$$G_0 = X_k^{n+1}, G_1 = X_1^{n+1}, \dots, G_{k-1} = X_{k-1}^{n+1}, G_k = X_{k+1}^{n+1}, G_{k+1} = X_{k+2}^{n+1}, \dots$$

and we set

$$\tilde{\beta}(t) = \sum_{p=0}^{\infty} G_p \langle \mathbf{1}_{[0, t]}, H_p \rangle,$$

where the H_p 's are the Haar basis of $L^2(0, 1)$. Then (7.15) follows from the fact that

$$\int_0^1 H_p(t) dt = \langle H_p, H_0 \rangle = \delta_{p0}.$$

Remark 7.2. Theorem 7.3 has already been proved in [8] (see also Section 5 in [9]) under less restrictive hypotheses (having a compactly supported noise is unnecessary).

We give the statement together with Theorem 7.4, however, to emphasize the fact that the convergence of the Finite Volume method, in the framework which we use, give both the existence-uniqueness of the solution to the limit continuous problem, and the convergence of the numerical method to this solution. It is not necessary to provide the existence of the solution u to (1.1) by an external means.

Proof of Theorem 7.3 and Theorem 7.4. Let us first prove the theorem 7.4. We take the existence of u , solution to (1.1) with initial datum u_0 for granted. By Proposition 6.3, Lemma 7.1 and Lemma 7.2, we may apply Theorem 2.2 to f_δ : we obtain (7.14) with z_δ instead of v_δ , where

$$z_\delta(x, t) := \int_{\mathbb{R}} \xi d\nu_{x,t}^\delta(\xi) = \frac{t - t_n}{\Delta t_n} \bar{v}_\delta(x, t) + \frac{t_{n+1} - t}{\Delta t_n} v_\delta(x, t),$$

for $t \in [t_n, t_{n+1}]$. By (6.6), we have the estimate

$$\mathbb{E} \int_0^T \|z_\delta - v_\delta\|_{L^2(\mathbb{T}^N)}^2 dt = \mathcal{O}(|\delta|)$$

on the difference between z_δ and v_δ . This gives (7.14) for $p \leq 2$. If $p > 2$, we use the inequality

$$\begin{aligned} \mathbb{E} \|v_\delta - u\|_{L^p(\mathbb{T}^N \times (0, T))}^p &\leq \|v_\delta - u\|_{L^2(\Omega \times \mathbb{T}^N \times (0, T))} \|v_\delta - u\|_{L^{2(p-1)}(\Omega \times \mathbb{T}^N \times (0, T))}^{p-1} \\ &\leq \frac{1}{\eta} \mathbb{E} \|v_\delta - u\|_{L^2(\mathbb{T}^N \times (0, T))}^2 + \eta \mathbb{E} \|v_\delta - u\|_{L^{2(p-1)}(\mathbb{T}^N \times (0, T))}^{2(p-1)}, \end{aligned} \quad (7.16)$$

where η is a positive parameter. Thanks to the uniform bounds (2.2)-(7.12), we can choose η independent on δ to have the second term in (7.16) smaller than an arbitrary threshold. By the convergence result for $p = 2$ the first term in (7.16) is then also small for δ close to 0. This concludes the proof of theorem 7.4. To prove 7.3, we just need to construct an approximation scheme satisfying (3.1), (5.6), (5.7) and (5.28) and to compute v_δ by (3.4)-(3.5)-(3.6)-(3.7). We can use a cartesian grid for this purpose: let $h_m = \frac{1}{m}$, where $m \in \mathbb{N}^*$. Let \mathcal{T}/\mathbb{Z}^N be the set of open hypercubes of length h_m obtained by translates of the original hypercube $(0, h_m)^N$ by vectors $h_m x$, x having components in $\{0, 1, \dots, m-1\}$. Then (5.6), (5.7) are satisfied with $\alpha_N = 2^{-N}$ since an hypercube has 2^N sides. We can choose the Godunov numerical fluxes, defined as follows:

$$A_{K \rightarrow L}(v, w) = \begin{cases} |K|L| \min_{v \leq \xi \leq w} A(\xi) \cdot n_{K,L} & \text{if } v \leq w, \\ |K|L| \max_{w \leq \xi \leq v} A(\xi) \cdot n_{K,L} & \text{if } w \leq v. \end{cases}$$

These fluxes $A_{K \rightarrow L}$ are monotone (non-increasing in the first variable, non-increasing in the second variable). They satisfy the hypotheses of regularity, consistency (3.1) and (3.2) respectively with $L_A = \text{Lip}(A)$. The conservative symmetry property (3.3) is also satisfied. At last, to ensure (5.28), we just need to take a uniform time step Δt like

$$\Delta t = \frac{1}{2} \frac{\alpha_N^2}{2L_A} h_m.$$

We have then (5.28) with $\theta = \frac{1}{2}$. At this stage, Proposition 6.3 provides a sequence of approximate solutions (f_m) . Thanks to the uniform bounds established in Section 7.1, we can apply Theorem 2.2: this gives the existence of a unique solution u to (1.1) with initial datum u_0 . By Corollary 3.3 in [9], we have $u \in C([0, T]; L^p(\mathbb{T}^N))$ almost-surely. \square

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