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Convergence of approximations to stochastic scalar conservation laws

Sylvain Dotti* and Julien Vovelle†

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Abstract

We develop a general framework for the analysis of approximations to stochastic scalar conservation laws. Our aim is to prove, under minimal consistency properties and bounds, that such approximations are converging to the solution to a stochastic scalar conservation law. The weak probabilistic convergence mode is convergence in law, the most natural in this context. We use also a kinetic formulation and martingale methods. Our result is applied to the convergence of the Finite Volume Method in the companion paper [15].

Keywords: stochastic conservation laws, kinetic formulation, martingale methods

MSC Number: 60H15 (35L65 35R60)

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis and let $T > 0$. Consider the first-order scalar conservation law with stochastic forcing

$$du(x, t) + \operatorname{div}(A(u(x, t)))dt = \Phi(x, u(x, t))dW(t), \quad x \in \mathbb{T}^N, t \in (0, T). \quad (1)$$

Equation (1) is periodic in the space variable: $x \in \mathbb{T}^N$ where \mathbb{T}^N is the N -dimensional torus. The flux function A in (1) is supposed to be of class C^2 : $A \in C^2(\mathbb{R}; \mathbb{R}^N)$. We assume that A and its derivatives have at most polynomial growth. The right-hand side of (1) is a stochastic increment in infinite dimension. It is defined as follows (see [11] for the general theory): W is a cylindrical Wiener process, $W = \sum_{k \geq 1} \beta_k e_k$, where the coefficients β_k are independent Brownian processes and $(e_k)_{k \geq 1}$ is a complete orthonormal system in a Hilbert space H . For each $x \in \mathbb{T}^N$, $u \in \mathbb{R}$, $\Phi(x, u) \in L_2(H, \mathbb{R})$ is defined by $\Phi(x, u)e_k = g_k(x, u)$ where $g_k(\cdot, u)$ is a regular function on \mathbb{T}^N . Here, $L_2(H, K)$ denotes the set of Hilbert-Schmidt operator from the Hilbert space H to an other Hilbert space K . Since $K = \mathbb{R}$ in our case, this set is isomorphic to H , thus we may also define

$$\Phi(x, u) = \sum_{k \geq 1} g_k(x, u)e_k,$$

the action of $\Phi(x, u)$ on $e \in H$ being given by $\langle \Phi(x, u), e \rangle_H$. We assume $g_k \in C(\mathbb{T}^N \times \mathbb{R})$, with the bounds

$$\mathbf{G}^2(x, u) = \|\Phi(x, u)\|_H^2 = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0(1 + |u|^2), \quad (2)$$

$$\|\Phi(x, u) - \Phi(y, v)\|_H^2 = \sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1(|x - y|^2 + |u - v|h(|u - v|)), \quad (3)$$

where $x, y \in \mathbb{T}^N$, $u, v \in \mathbb{R}$, and h is a continuous non-decreasing function on \mathbb{R}_+ such that $h(0) = 0$. We assume also $0 \leq h(z) \leq 1$ for all $z \in \mathbb{R}_+$.

Notation: in what follows, we will use the convention of summation over repeated indices k . For example, we write $W = \beta_k e_k$ for the cylindrical Wiener process in (1).

This paper is a preliminary work to the analysis of convergence of the numerical approximation to (1) by the Finite Volume method with monotone fluxes, which is done in [15]. We give a general notion of family of approximate solutions, see Definition 23, and explain what kind of convergence of such family can be expected. Our main results in this regard are Theorem 29 about convergence to martingale solutions and Theorem 40 which gives criteria for convergence to path-wise solutions.

Problem (1) has already been studied in a series of papers. Like in the deterministic case, the approach to the existence of solutions has been the vanishing viscosity method, see [17, 31, 19, 41, 13, 9, 4, 5, 29] in particular. Approximation by the BGK method has been considered by M. Hofmanová in [25]. Some results of convergence of numerical approximations to (1) (by the Finite Volume method in particular) have also been obtained in [33, 2, 3, 1, 30, 32].

The main difference between this present paper and all the works cited above is in the way to answer to the following question: when considering the convergence of approximations to (32), which mode of convergence regarding the sample variable ω is used? Here, we develop an approach based on convergence in law, while in the work referred to¹, weak convergence (in Lebesgue spaces, or in the sense of Young measures, *cf.* Section 2.2) is considered. Convergence in law is the natural mode of convergence for the random variables which manifest in the approximation to (1). We are aware, however, that the difference between convergence in law and weak convergence is a particular technical issue, with no specific consequences on the kind of results that can be obtained. Nevertheless, it seemed important to us to develop fully the approach by convergence in law. To achieve this is also a way to expand the theory of stochastic first-order scalar conservation laws by kinetic formulation initiated in [13]. Indeed, in the companion paper [15], the convergence of The Finite Volume method is obtained after a kinetic re-formulation of the numerical scheme.

To complete this introduction, let us mention that the approximation of scalar conservation laws with stochastic flux has also been considered in [22] (time-discrete scheme) and [37] (space discrete scheme). For the corresponding Cauchy Problem, see [35, 34, 36, 21, 20, 26].

The plan of the paper is the following one: Section 2 to Section 4 are devoted to the analysis of the Cauchy Problem for (1): we introduce the kinetic formulation of the problem in Section 2, and prove a uniqueness result in Section 3. In Section 4, we

¹with the exception of [19], where quite a strong notion of solution is used however

develop a general approach to the analysis of convergence of approximate solutions to (1) based on martingale methods. Note that Section 2 and Section 3 are for a large part identical to Section 2 and Section 3 in [13]. There are however a lot of modifications, which were needed to prepare Section 4.

2 Kinetic solution

2.1 Definition

2.1.1 Predictable sets and functions

For $T > 0$, we denote by $\mathcal{B}([0, T])$ the Borel σ -algebra on $[0, T]$ and we denote by $\mathcal{P}_T \subset \mathcal{B}([0, T]) \otimes \mathcal{F}$ the predictable σ -algebra, [10, Section 2.2]. If E is a Banach space, a process $(f(t))$ with values in E is said to be weakly-predictable if the process $(\langle f(t), \varphi \rangle_{E, E'})$ is predictable for every φ in the topological dual E' . This is equivalent to say that f is weakly \mathcal{P}_T -measurable, in the sense of [42, Definition 1, p.130]. Similarly, we can define the notion of strong predictability: the process $(f(t))$ is said to be strongly predictable if there exists a sequence of E -valued, \mathcal{P}_T -measurable simple functions which converges to f at every point (t, ω) in a set of full measure in $[0, T] \times \Omega$. By Pettis' Theorem, [42, Theorem p.131], the two notions of measurability coincide if E is separable: in this case we say simply "predictable".

Let us assume that E is separable to introduce the following notations. Let $p \in [1, +\infty)$. The set $L^p([0, T] \times \Omega; E)$ is the set of E -valued, $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable, Bochner integrable functions f which satisfy

$$\iint_{[0, T] \times \Omega} \|f(t, \omega)\|_E^p d(\mathcal{L} \times \mathbb{P})(t, \omega) < +\infty,$$

where \mathcal{L} is the Lebesgue measure on $[0, T]$. Equivalently, by definition of the product measure $\mathcal{L} \times \mathbb{P}$,

$$\mathbb{E} \int_0^T \|f(t)\|_E^p dt < +\infty.$$

We denote by $L_{\mathcal{P}}^p([0, T] \times \Omega; E)$ the set of functions g in $L^p([0, T] \times \Omega; E)$ which are equal $\mathcal{L} \times \mathbb{P}$ -almost everywhere to a predictable function f . This is the case if, and only if, $\langle g, \varphi \rangle$ is equal $\mathcal{L} \times \mathbb{P}$ -almost everywhere to $\langle f, \varphi \rangle$ for all $\varphi \in E'$ (we use the fact that E' is separable since E is separable), so let us briefly consider the case $E = \mathbb{R}$. The class of processes in $L_{\mathcal{P}}^p([0, T] \times \Omega; \mathbb{R})$ is analysed in [10, p. 66] or [40, p. 172]. In particular, if $X(t)$ is an adapted process with

$$\mathbb{E} \int_0^T |X(t)|^p dt < +\infty,$$

then $X \in L_{\mathcal{P}}^p([0, T] \times \Omega; \mathbb{R})$. A progressively measurable process X in $L^p([0, T] \times \Omega; \mathbb{R})$ also is in $L_{\mathcal{P}}^p([0, T] \times \Omega; \mathbb{R})$.

Let $m \in \mathbb{N}^*$. In the case where E is itself a Lebesgue space $E = L^p(D)$, where D is an open subset of \mathbb{R}^m , we have $L^p([0, T] \times \Omega; L^p(D)) = L^p(D \times [0, T] \times \Omega)$, where $D \times [0, T] \times \Omega$ is endowed with the product measure $\mathcal{L}_{m+1} \times \mathbb{P}$ (\mathcal{L}_m being the m -dimensional Lebesgue measure), see [16, Section 1.8.1]. Similarly, we have

$$L^p_{\mathcal{P}}([0, T] \times \Omega; L^p(D)) = L^p_{\mathcal{P}}(D \times [0, T] \times \Omega),$$

where $L^p_{\mathcal{P}}(D \times [0, T] \times \Omega)$ is the set of functions in $L^p(D \times [0, T] \times \Omega)$ which are equal $\mathcal{L}_m \times \mathcal{L} \times \mathbb{P}$ -almost everywhere to a $\mathcal{B}(D) \times \mathcal{P}_T$ -measurable function (here $\mathcal{B}(D)$ is the Borel σ -algebra on D). We will apply these results with $D = (0, 1)^N$, in which case, by periodic extension, we obtain

$$L^p([0, T] \times \Omega; L^p(\mathbb{T}^N)) = L^p(\mathbb{T}^N \times [0, T] \times \Omega), \quad (4)$$

and similarly for spaces $L^p_{\mathcal{P}}$.

2.1.2 Kinetic measure, solution

Let $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ be the set of bounded Borel signed measures on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$. We denote by $\mathcal{M}_b^+(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ the subset of non-negative measures.

Definition 1 (Random measure). *A map m from Ω to $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ is said to be a random signed measure (on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$) if, for each $\phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, $\langle m, \phi \rangle: \Omega \rightarrow \mathbb{R}$ is a random variable. If almost-surely $m \in \mathcal{M}_b^+(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, we simply speak of random measure.*

If m is a random measure, then $\mathbb{E}m$ is well defined and this is a measure on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$. In particular, it satisfies the following tightness condition

$$\lim_{R \rightarrow +\infty} \mathbb{E}m(\mathbb{T}^N \times [0, T] \times B_R^c) = 0, \quad (5)$$

where $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$. We note this fact here, since uniform versions of (5) will be required when considering sequences of random measures, see (66).

To define a notion of solution, we introduce the following time averages (cf. Remark 3): let $T > 0$ and let $u \in L^p(\mathbb{T}^N \times [0, T] \times \Omega)$. For $\varepsilon \in (0, T/2)$, we set

$$M_\varepsilon^+ u(t) = \frac{1}{\varepsilon} \int_t^{T \wedge (t+\varepsilon)} u(s) ds, \quad M_\varepsilon^- u(t) = \frac{1}{\varepsilon} \int_{0 \vee (t-\varepsilon)}^t u(s) ds, \quad t \in (0, T). \quad (6)$$

We also fix a decreasing sequence (ε_n) such that $\varepsilon_0 < T/2$ and $\varepsilon_n \downarrow 0$.

Definition 2 (Solution). *Let $u_0 \in L^\infty(\mathbb{T}^N)$. A function $u \in L^1(\mathbb{T}^N \times [0, T] \times \Omega)$ is said to be a solution to (1) with initial datum u_0 if*

1. $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0, T] \times \Omega)$,

2. for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that

$$\mathbb{E} \left(\sup_n \sup_{t \in [0, T]} \|M_{\varepsilon_n}^\pm u(t)\|_{L^p(\mathbb{T}^N)}^p \right) \leq C_p, \quad (7)$$

3. there exists a random measure m such that $\mathbf{f} := \mathbf{1}_{u > \xi}$ satisfies: for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} & \int_0^T \langle \mathbf{f}(t), \partial_t \varphi(t) \rangle dt + \langle \mathbf{f}_0, \varphi(0) \rangle + \int_0^T \langle \mathbf{f}(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} g_k(x, u(x, t)) \varphi(x, t, u(x, t)) dx d\beta_k(t) \\ & \quad - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \partial_\xi \varphi(x, t, u(x, t)) \mathbf{G}^2(x, u(x, t)) dx dt + m(\partial_\xi \varphi), \quad (8) \end{aligned}$$

a.s., where $\mathbf{f}_0(x, \xi) = \mathbf{1}_{u_0(x) > \xi}$, $\mathbf{G}^2 := \sum_{k=1}^{\infty} |g_k|^2$ and $a(\xi) := A'(\xi)$.

Remark 3 (Time averages). A solution u in the sense of Definition 2 is not a process in the usual sense since it is only defined almost everywhere with respect to time. Part of our work below is to show that u has a natural representative which has almost-sure continuous trajectories with values in $L^p(\mathbb{T}^N)$ ($p \in [1, +\infty)$), see Corollary 21. If $u(t)$ was known to have some continuity properties from the start, then we would require simply that

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|u(t)\|_{L^p(\mathbb{T}^N)}^p \right) \leq C_p, \quad (9)$$

instead of (7).

In (8), we have used the brackets $\langle \cdot, \cdot \rangle$ to denote the duality between $C_c^\infty(\mathbb{T}^N \times \mathbb{R})$ and the space of distributions over $\mathbb{T}^N \times \mathbb{R}$. In what follows, we will denote similarly the integral

$$\langle F, G \rangle = \int_{\mathbb{T}^N} \int_{\mathbb{R}} F(x, \xi) G(x, \xi) dx d\xi, \quad F \in L^p(\mathbb{T}^N \times \mathbb{R}), G \in L^q(\mathbb{T}^N \times \mathbb{R}),$$

where $1 \leq p \leq +\infty$ and q is the conjugate exponent of p . In (8) also, we have used (with $\phi = \partial_\xi \varphi$) the shorthand $m(\phi)$ for

$$m(\phi) = \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \phi(x, t, \xi) dm(x, t, \xi), \quad \phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}).$$

2.2 Generalized solutions

With the purpose to prepare the proof of existence of solution, we introduce the following definitions.

Definition 4 (Young measure). *Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on \mathbb{R} . We say that a map $\nu: X \rightarrow \mathcal{P}_1(\mathbb{R})$ is a Young measure on X if, for all $\phi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\phi)$ from X to \mathbb{R} is measurable. We say that a Young measure ν vanishes at infinity if, for every $p \geq 1$,*

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) < +\infty. \quad (10)$$

Proposition 5 (An alternative definition of Young measures). *Let $(X, \mathcal{A}, \lambda)$ be a measure space with $\lambda(X) = 1$. Let \mathcal{L} be the Lebesgue measure on \mathbb{R} and let \mathcal{Y}^1 be the set of probability measures ν on $(X \times \mathbb{R}, \lambda \times \mathcal{L})$ such that $\pi_{\#}\nu = \lambda$, where $\pi_{\#}\nu$ is the push forward of ν by the projection $\pi: X \times \mathbb{R} \rightarrow X$. Then \mathcal{Y}^1 is the set of Young measures as defined in Definition 4.*

For the proof of this result, which uses the Disintegration Theorem, we refer to the discussion in [8, p.19-20] on the spaces \mathcal{Y}^1 and $\mathcal{Y}_{\text{dis}}^1$ (“dis” for “disintegration”: this corresponds to the Definition 4). Note that there is no loss in generality in assuming $\lambda(X) = 1$.

Definition 6 (Kinetic function). *Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. A measurable function $f: X \times \mathbb{R} \rightarrow [0, 1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for λ -a.e. $z \in X$, for all $\xi \in \mathbb{R}$,*

$$f(z, \xi) = \nu_z(\xi, +\infty).$$

We say that f is an equilibrium if there exists a measurable function $u: X \rightarrow \mathbb{R}$ such that $f(z, \xi) = \mathbf{f}(z, \xi) = \mathbf{1}_{u(z) > \xi}$ a.e., or, equivalently, $\nu_z = \delta_{\xi=u(z)}$ for a.e. $z \in X$.

Definition 7 (Conjugate function). *If $f: X \times \mathbb{R} \rightarrow [0, 1]$ is a kinetic function, we denote by \bar{f} the conjugate function $\bar{f} := 1 - f$.*

We also denote by χ_f the function defined by $\chi_f(z, \xi) = f(z, \xi) - \mathbf{1}_{0 > \xi}$. This correction to f is integrable on \mathbb{R} . Actually, it is decreasing faster than any power of $|\xi|$ at infinity. Indeed, we have $\chi_f(z, \xi) = -\nu_z(-\infty, \xi)$ when $\xi < 0$ and $\chi_f(z, \xi) = \nu_z(\xi, +\infty)$ when $\xi > 0$. Therefore

$$|\xi|^p \int_X |\chi_f(z, \xi)| d\lambda(z) \leq \int_X \int_{\mathbb{R}} |\zeta|^p d\nu_{x,t}(\zeta) d\lambda(z) < \infty, \quad (11)$$

for all $\xi \in \mathbb{R}$, $1 \leq p < +\infty$.

The so-called kinetic functions appear naturally when one examines the stability of a sequence of solutions to (1). We discuss this topic in details in Section 4, but let us already mention the following compactness results.

Theorem 8 (Compactness of Young measures). *Let $(X, \mathcal{A}, \lambda)$ be a finite measure space such that \mathcal{A} is countably generated. Let (ν^n) be a sequence of Young measures on X satisfying (10) uniformly for some $p \geq 1$:*

$$\sup_n \int_X \int_{\mathbb{R}} |\xi|^p d\nu_z^n(\xi) d\lambda(z) < +\infty. \quad (12)$$

Then there exists a Young measure ν on X and a subsequence still denoted (ν^n) such that, for all $h \in L^1(X)$, for all $\phi \in C_b(\mathbb{R})$,

$$\lim_{n \rightarrow +\infty} \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z^n(\xi) d\lambda(z) = \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z(\xi) d\lambda(z). \quad (13)$$

The convergence (13) is the convergence for the $\tau_{\mathcal{Y}^1}^W$ topology defined in [8, p.21]. By [8, Corollary 4.3.7], (12) implies that the set $\{\nu_n; n \in \mathbb{N}\}$ is $\tau_{\mathcal{Y}^1}^W$ -relatively compact, and for this result, it is not necessary to assume that \mathcal{A} is countably generated. This latter hypothesis is used as a criteria of metrizability of $\tau_{\mathcal{Y}^1}^W$, [8, Proposition 2.3.1]. A consequence of Theorem 8 is the following proposition.

Corollary 9 (Compactness of kinetic functions). *Let $(X, \mathcal{A}, \lambda)$ be a finite measure space such that \mathcal{A} is countably generated. Let (f_n) be a sequence of kinetic functions on $X \times \mathbb{R}$: $f_n(z, \xi) = \nu_z^n(\xi, +\infty)$ where ν^n are Young measures on X satisfying (12). Then there exists a kinetic function f on $X \times \mathbb{R}$ such that $f_n \rightharpoonup f$ in $L^\infty(X \times \mathbb{R})$ weak- $*$.*

We will also need the following result.

Lemma 10 (Convergence to an equilibrium). *Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. Let $p > 1$. Let (f_n) be a sequence of kinetic functions on $X \times \mathbb{R}$: $f_n(z, \xi) = \nu_z^n(\xi, +\infty)$ where ν^n are Young measures on X satisfying (12). Let f be a kinetic function on $X \times \mathbb{R}$ such that $f_n \rightharpoonup f$ in $L^\infty(X \times \mathbb{R})$ weak- $*$. Assume that f_n and f are equilibria:*

$$f_n(z, \xi) = \mathbf{1}_{u_n(z) > \xi}, \quad f(z, \xi) = \mathbf{1}_{u(z) > \xi}.$$

Then, for all $1 \leq q < p$, $u_n \rightarrow u$ in $L^q(X)$ strong.

Proof of Lemma 10. Let $r \in [1, +\infty]$. By choosing $\theta'(\xi)\gamma(z)$ as a test function, and by use of a standard approximation procedure, we obtain

$$\int_X \theta(u_n(z))\gamma(z) dz \rightarrow \int_X \theta(u(z))\gamma(z) dz \quad (14)$$

for all $\theta \in C^1(\mathbb{R})$, $\gamma \in L^r(X)$ such that $\sup_n \|\theta(u_n)\|_{L^{r'}(X)} < +\infty$, where r' is the conjugate exponent to r . Assume first $p > 2$. We show the strong convergence of (u_n) to u in $L^2(X)$, by developing the scalar product

$$\|u - u_n\|_{L^2(X)}^2 = \|u\|_{L^2(X)}^2 + \|u_n\|_{L^2(X)}^2 - 2\langle u, u_n \rangle_{L^2(X)}.$$

The convergence of the norms follows from (14) with $\theta(\xi) = \xi$, $\gamma(z) = 1$. The weak convergence $\langle u, u_n \rangle_{L^2(X)} \rightarrow \|u\|_{L^2(X)}^2$ follows from (14) with $\theta = 1$, $\gamma = u$. Still when $p > 2$, the remaining cases $1 \leq q < p$ are obtained by interpolation and by the uniform bound on $\|u_n\|_{L^p(X)}$. If $p \leq 2$ now, we notice that, for every $R > 0$, the truncate functions

$$T_R(u_n) := \min(R, \max(-R, u_n))$$

satisfy (14), and we can apply the reasoning above to show $T_R(u_n) \rightarrow T_R(u)$ in $L^r(X)$ strong for every $r < +\infty$. For $1 \leq q < p$, the uniform estimate

$$\|T_R(u_n) - u_n\|_{L^q(X)} \leq \frac{1}{R^{1/s}} \sup_n \|u_n\|_{L^p(X)}^{1+1/s}, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{s},$$

and similarly for u , then gives the result. \square

In the *deterministic setting*, if (u_n) is a sequence of solutions to (1), then, due to natural bounds and to Theorem 8, the sequence of Young measures δ_{u_n} on $X := \mathbb{T}^N \times (0, T)$ has, up to a subsequence, a limit ν . Then every non-linear expression $\phi(u_n)$ for $\phi \in C_b(\mathbb{R})$ will converge to $\langle \phi, \nu \rangle$ in the sense of (13). This is why it is natural (cf. [14]), for such non-linear problems as (1), to introduce the following generalization to Definition 2.

Definition 11 (Generalized solution). *Let $f_0: \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1]$ be a kinetic function. A measurable function $f: \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega \rightarrow [0, 1]$ is said to be a generalized solution to (1) with initial datum f_0 if*

1. almost-surely, f is a kinetic function,
2. for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that $\nu := -\partial_\xi f$ satisfies

$$\mathbb{E} \left(\sup_n \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p dM_{\varepsilon_n}^\pm \nu_{x,t}(\xi) dx \right) \leq C_p, \quad (15)$$

3. for all $\psi \in C_b(\mathbb{R})$, the random map $(x, t) \mapsto \langle \psi, \nu_{x,t} \rangle$ belongs to $L^2_p(\mathbb{T}^N \times [0, T] \times \Omega)$,
4. there exists a random measure m such that for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} & \int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx d\beta_k(t) \\ & - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, t, \xi) \mathbf{G}^2(x, \xi) d\nu_{(x,t)}(\xi) dx dt + m(\partial_\xi \varphi), \quad a.s. \end{aligned} \quad (16)$$

In (15), we have used the same notation as in (6): for $x \in \mathbb{T}^N$, $t \in (0, T)$,

$$M_\varepsilon^+ \nu_{x,t} = \frac{1}{\varepsilon} \int_t^{T \wedge (t+\varepsilon)} \nu_{x,s} ds, \quad M_\varepsilon^- \nu_{x,t} = \frac{1}{\varepsilon} \int_{0 \vee (t-\varepsilon)}^t \nu_{x,s} ds. \quad (17)$$

Note that the stochastic integral in (16) is well-defined. Indeed, the bound (15) implies

$$\mathbb{E} \left(\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p dM_{\varepsilon_n}^{\pm} \nu_{x,t}(\xi) dx \right) \leq C_p T. \quad (18)$$

Using successively Jensen's Inequality, the growth hypothesis (2), and (18) with $p = 2$, we obtain

$$\begin{aligned} \mathbb{E} \int_0^T \sum_{k \geq 1} \left| \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx \right|^2 \\ \leq \mathbb{E} \int_0^T \sum_{k \geq 1} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |g_k(x, \xi) \varphi(x, t, \xi)|^2 d\nu_{x,t}(\xi) dx \\ = \mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{G}^2(x, \xi) |\varphi(x, t, \xi)|^2 d\nu_{x,t}(\xi) dx \\ \leq \|\varphi\|_{L_{x,t,\xi}^\infty}^2 D_0 (1 + C_2 T). \end{aligned} \quad (19)$$

If $\varphi(x, t, \xi)$ is the tensor function $\psi(\xi)\theta(x, t)$, then

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx = \int_{\mathbb{T}^N} \langle \psi, \nu_{x,t} \rangle \theta(x, t) dx$$

is in $L_{\mathcal{P}}^2([0, T] \times \Omega)$ by item 3 in Definition 11. An argument of density then shows that

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx$$

is in $L_{\mathcal{P}}^2([0, T] \times \Omega)$. To sum up, we have proved the following result.

Lemma 12 (Admissible integrand). *Let $f_0: \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1]$ be a kinetic function. Let $f: \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega \rightarrow [0, 1]$ be a generalized solution to (1) with initial datum f_0 . Then, for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ the $l^2(\mathbb{N}^*)$ -valued process*

$$t \mapsto \left(\int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx \right)_{k \geq 1}$$

is in $L_{\mathcal{P}}^2([0, T] \times \Omega; l^2(\mathbb{N}^*))$.

Let us now state a simple result of reduction from generalized solution to mere solution.

Proposition 13. *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume that $f: \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega \rightarrow [0, 1]$ is a generalized solution to (1) with initial datum $\mathbf{1}_{u_0}$. If f is an equilibrium,*

$$f(x, t, \xi, \omega) = \mathbf{f}(x, t, \xi, \omega) = \mathbf{1}_{u(x,t,\omega) > \xi}, \quad (20)$$

for a.e. $(x, t, \xi, \omega) \in \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega$, then u is a solution to (1) with initial datum u_0 .

Proof of Proposition 13. If $f = \mathbf{1}_{u>\xi}$, then $u(t, x) = \int_{\mathbb{R}} \chi_f(x, t, \xi) d\xi$. Therefore $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0, T] \times \Omega)$. Moreover, $\nu = \delta_{\xi=u}$ and

$$\int_{\mathbb{T}^N} |u(t, x)|^p dx = \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi) dx.$$

Condition (7) is thus contained in the condition (15). It is also clear that, for $f = \mathbf{1}_{u>\xi}$ and $\nu = \delta_{\xi=u}$, Equation (16) is Equation (8). \square

We will show in Theorem 20 that (20), which we give as an hypothesis in Proposition 13, is automatically satisfied by any generalized solution starting from an equilibrium $f_0 = \mathbf{f}_0 = \mathbf{1}_{u_0>\xi}$.

We conclude this paragraph with two remarks. The first remark is the following

Lemma 14 (Distance to equilibrium). *Let (X, λ) be a finite measure space. Let $f: X \times \mathbb{R} \rightarrow [0, 1]$ be a kinetic function. Then*

$$m(\xi) := \int_{-\infty}^{\xi} (\mathbf{1}_{u>\zeta} - f(\zeta)) d\zeta, \quad \text{where } u := \int_{\mathbb{R}} \chi_f(\zeta) d\zeta,$$

is well defined and non-negative.

Note in particular that the difference $f(\xi) - \mathbf{1}_{u>\xi}$ writes $\partial_{\xi} m$ where $m \geq 0$.

Proof of Lemma 14. Let $\nu_z = -\partial_{\xi} f(z, \cdot)$, $z \in X$. By Jensen's Inequality, we have

$$H\left(\int_{\mathbb{R}} \zeta d\nu_z(\zeta)\right) \leq \int_{\mathbb{R}} H(\zeta) d\nu_z(\zeta) \tag{21}$$

for all convex sub-linear function $H: \mathbb{R} \rightarrow \mathbb{R}$. Note that

$$u(z) = \int_{\mathbb{R}} f(z, \zeta) - \mathbf{1}_{0>\zeta} d\zeta = \int_{\mathbb{R}} \zeta d\nu_z(\zeta)$$

by integration by parts. By integration by parts, we also have, for all sub-linear function $H \in C^1(\mathbb{R})$,

$$\int_{\mathbb{R}} H(\zeta) d\nu_z(\zeta) = H(0) + \int_{\mathbb{R}} H'(\zeta)(f(z, \zeta) - \mathbf{1}_{0>\zeta}) d\zeta$$

and

$$H(u(z)) = \int_{\mathbb{R}} H(\zeta) d\delta_{u(z)}(\zeta) = H(0) + \int_{\mathbb{R}} H'(\zeta)(\mathbf{1}_{u(z)>\zeta} - \mathbf{1}_{0>\zeta}) d\zeta.$$

By (21), it follows that

$$\int_{\mathbb{R}} H'(\zeta)(f(z, \zeta) - \mathbf{1}_{u(z)>\zeta}) d\zeta \geq 0$$

for all convex and sub-linear $H \in C^1(\mathbb{R})$. Approximating $\zeta \mapsto (\zeta - \xi)^-$ by such functions H , we obtain $m(\xi) \geq 0$. \square

2.3 Left and right limits of generalized solution

We show in the following proposition that, almost-surely, any generalized solution admits possibly different left and right weak limits at any point $t \in [0, T]$. This property is important to prove a comparison principle and then uniqueness. Also, it shows that the weak form (16) of the equation satisfied by a generalized solution can be strengthened: we write below (see (27)) a formulation which is weak only with respect to x and ξ .

Proposition 15 (Left and right weak limits). *Let f_0 be a kinetic initial datum. Let f be a generalized solution to (1) with initial datum f_0 . Then f admits almost-surely left and right limits at all point $t_* \in [0, T]$. More precisely, there exists a subset $\hat{\Omega} \subset \Omega$ of probability 1 such that, for all $\omega \in \hat{\Omega}$, for all $t_* \in [0, T]$, there exists some kinetic functions $f^{*,\pm}$ on $\mathbb{T}^N \times \mathbb{R}$ such that*

$$\langle f(t_* - \varepsilon), \varphi \rangle \rightarrow \langle f^{*, -}, \varphi \rangle$$

and

$$\langle f(t_* + \varepsilon), \varphi \rangle \rightarrow \langle f^{*, +}, \varphi \rangle$$

as $\varepsilon \rightarrow 0$ for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$. Moreover,

$$\langle f^{*, +} - f^{*, -}, \varphi \rangle = - \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{1}_{\{t_*\}}(t) dm(x, t, \xi). \quad (22)$$

Define f^+ on $[0, T] \times \hat{\Omega}$ and f^- on $(0, T] \times \hat{\Omega}$ by $f^\pm(t_*) := f^{*, \pm}$. Then

1. $f = f^+ = f^-$ almost everywhere in $\mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega$,
2. for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$, the process $(\langle f^+(t), \varphi \rangle)_{t \in [0, T]}$ is càdlàg (it is almost-surely continuous from the right with limits from the left) and $(\langle f^-(t), \varphi \rangle)_{t \in (0, T]}$ is càglàd.

Note that we obtain continuity with respect to time of solutions to (1) in Corollary 21 below.

Proof of Proposition 15. The set of test functions $C_c^1(\mathbb{T}^N \times \mathbb{R})$ (endowed with the topology of the uniform convergence of the functions and their first derivatives) is separable and we fix a dense countable subset \mathcal{D}_1 (see the argument about Γ in Section 4.5.1 for a proof of the existence of \mathcal{D}_1). For all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$, a.s., the map

$$\begin{aligned} J_\varphi: t \mapsto & \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{G}^2(x, \xi) d\nu_{x,s}(\xi) dx ds \end{aligned} \quad (23)$$

is continuous on $[0, T]$. Consequently: a.s., say for $\omega \in \Omega_1$ where Ω_1 is of full measure, for all $\varphi \in \mathcal{D}_1$, J_φ is continuous on $[0, T]$. For test functions of the form $(x, t, \xi) \mapsto$

$\varphi(x, \xi)\alpha(t)$, $\alpha \in C_c^1([0, T])$, $\varphi \in \mathcal{D}_1$, Fubini's Theorem and the weak formulation (16) give

$$\int_0^T g_\varphi(t)\alpha'(t)dt + \langle f_0, \varphi \rangle \alpha(0) = \langle m, \partial_\xi \varphi \rangle(\alpha), \quad (24)$$

where $g_\varphi(t) := \langle f(t), \varphi \rangle - J_\varphi(t)$. This shows that $\partial_t g_\varphi$ is a Radon measure on $(0, T)$, *i.e.* the function $g_\varphi \in BV(0, T)$. In particular it admits left and right limits at all points $t_* \in (0, T)$ and a right limit on $t_* = 0$, a left limit on $t_* = T$. Since J_φ is continuous, this also holds for $\langle f, \varphi \rangle$: for all $t_* \in (0, T)$, the limits

$$\langle f, \varphi \rangle(t_*+) := \lim_{t \downarrow t_*} \langle f, \varphi \rangle(t) \quad \text{and} \quad \langle f, \varphi \rangle(t_*-) := \lim_{t \uparrow t_*} \langle f, \varphi \rangle(t)$$

exist. Note that:

$$\langle f, \varphi \rangle(t_*+) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{t_*}^{t_*+\varepsilon} \langle f, \varphi \rangle(t)dt, \quad \langle f, \varphi \rangle(t_*-) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{t_*-\varepsilon}^{t_*} \langle f, \varphi \rangle(t)dt. \quad (25)$$

Now we use the estimate (15): there exists a set of full measure Ω_2 in Ω such that, for every $\omega \in \Omega_2$,

$$\sup_n \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d[\tilde{\nu}_{\varepsilon_n}^\pm]_{x,t}(\xi) dx \leq C_p(\omega) < +\infty. \quad (26)$$

Let $\omega \in \hat{\Omega} := \Omega_1 \cap \Omega_2$ be fixed. Assume n large enough to ensure $t_* + \varepsilon_n < T$. The function

$$f_n: \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1], \quad x \mapsto \frac{1}{\varepsilon_n} \int_{t_*}^{t_*+\varepsilon_n} f(x, t, \xi, \omega) dt$$

is a kinetic function. By (26), it satisfies the condition (12). Since $L^1(\mathbb{T}^N)$ is separable, and by Corollary 9, there exist a kinetic function $f^{*,+}$ on $\mathbb{T}^N \times \mathbb{R}$ and a subsequence (ε_{n_k}) such that

$$\frac{1}{\varepsilon_{n_k}} \int_{t_*}^{t_*+\varepsilon_{n_k}} f(t) dt \rightharpoonup f^{*,+}$$

weakly-* in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ as $k \rightarrow +\infty$. We obtain similar results for the left limit: up to a subsequence still denoted (ε_{n_k}) , we have

$$\frac{1}{\varepsilon_{n_k}} \int_{t_*-\varepsilon_{n_k}}^{t_*} f(t) dt \rightharpoonup f^{*,-},$$

weakly-* in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ as $k \rightarrow +\infty$, where $f^{*,-}$ is a kinetic function on $\mathbb{T}^N \times \mathbb{R}$. By (25), we deduce that

$$\langle f, \varphi \rangle(t_*+) = \langle f^{*,+}, \varphi \rangle \quad \text{and} \quad \langle f, \varphi \rangle(t_*-) = \langle f^{*,-}, \varphi \rangle.$$

The cases $t_* = 0$ and $t_* = T$ are treated similarly. To prove (22), we take for α the hat function $\alpha_k(t) = \frac{1}{\varepsilon_{n_k}} \min((t - t_* + \varepsilon_{n_k})^+, (t - t_* - \varepsilon_{n_k})^-)$ in (24) and get (22) at the limit $[k \rightarrow +\infty]$. Set then

$$A = \{t \in [0, T]; \mathbb{E}m(\mathbb{T}^N \times \{t\} \times \mathbb{R}) > 0\}.$$

Then A is countable. By (22), we have

$$\langle f^-, \varphi \rangle = \langle f^+, \varphi \rangle \quad \text{on} \quad ((0, T) \setminus A) \times \hat{\Omega},$$

for all $\varphi \in \mathcal{D}_1 \subset C_c^1(\mathbb{T}^N \times \mathbb{R})$. Since \mathcal{D}_1 is dense, this gives $f^+ = f^-$ almost everywhere in $\mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega$. To end the proof of item 1, let us consider first the applications

$$t \mapsto \langle f(\cdot, t, \cdot, \omega), \varphi \rangle, \quad t \mapsto \langle f^\pm(\cdot, t, \cdot, \omega), \varphi \rangle.$$

Here $\omega \in \hat{\Omega}$ and $\varphi \in \mathcal{D}_1$ are fixed. Let E_φ denote the set of Lebesgue points of $t \mapsto \langle f(\cdot, t, \cdot, \omega), \varphi \rangle$ on $(0, T)$, from which we remove the set A . Then E_φ is measurable and of full measure in $(0, T)$. If $t \in E_\varphi$, then

$$\frac{1}{2\varepsilon_n} \int_{t-\varepsilon_n}^{t+\varepsilon_n} \langle f(\cdot, s, \cdot, \omega), \varphi \rangle ds \rightarrow \langle f(\cdot, t, \cdot, \omega), \varphi \rangle,$$

but we have also (up to a subsequence possibly),

$$\begin{aligned} & \frac{1}{2\varepsilon_n} \int_{t-\varepsilon_n}^{t+\varepsilon_n} \langle f(\cdot, s, \cdot, \omega), \varphi \rangle ds \\ &= \frac{1}{2\varepsilon_n} \int_{t-\varepsilon_n}^t \langle f(\cdot, s, \cdot, \omega), \varphi \rangle ds + \frac{1}{2\varepsilon_n} \int_t^{t+\varepsilon_n} \langle f(\cdot, s, \cdot, \omega), \varphi \rangle ds \\ &\rightarrow \frac{1}{2} (\langle f^+(\cdot, t, \cdot, \omega), \varphi \rangle + \langle f^-(\cdot, t, \cdot, \omega), \varphi \rangle) = \langle f^+(\cdot, t, \cdot, \omega), \varphi \rangle = \langle f^-(\cdot, t, \cdot, \omega), \varphi \rangle. \end{aligned}$$

We set $E = \bigcap_{\varphi \in \mathcal{D}_1} E_\varphi$, then E is measurable and of full measure in $(0, T)$ and $f = f^+ = f^-$ in $\mathbb{T}^N \times E \times \mathbb{R} \times \hat{\Omega}$. For the proof of item 2, we take in (16) a test function of the form $(x, s, \xi) \mapsto \varphi(x, \xi) \alpha(s)$ where α is the following approximation of the characteristic function of the interval $[0, t]$:

$$\alpha(s) = \min \left(1, \left(1 - \frac{s-t}{\varepsilon} \right)_+ \right).$$

We obtain at the limit $[\varepsilon \rightarrow 0]$: for all $\omega \in \hat{\Omega}$, for all $t \in [0, T]$ and $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$,

$$\langle f^+(t), \varphi \rangle = \langle f_0, \varphi \rangle + J_\varphi(t) - \langle m, \partial_\xi \varphi \rangle([0, t]), \quad (27)$$

where $\langle m, \partial_\xi \varphi \rangle([0, t]) = \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) dm(x, s, \xi)$. This shows that the process $(\langle f^+(t), \varphi \rangle)_{t \in [0, T]}$ is càdlàg. We have, similarly,

$$\langle f^-(t), \varphi \rangle = \langle f_0, \varphi \rangle + J_\varphi(t) - \langle m, \partial_\xi \varphi \rangle([0, t]), \quad (28)$$

This shows that $(\langle f^-(t), \varphi \rangle)_{t \in (0, T]}$ is càglàd. \square

Remark 16 (Uniform bound). *Note that, by construction, f^\pm satisfies the following bound: for all $\omega \in \hat{\Omega}$, for all $t \in [0, T]$,*

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\pm(\xi) dx \leq C_p(\omega), \quad (29)$$

Using Fatou's lemma and the following bounds (see (15))

$$\mathbb{E} \frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\pm(\xi) dx \leq C_p, \quad \mathbb{E} \frac{1}{\varepsilon_n} \int_{t-\varepsilon_n}^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\pm(\xi) dx \leq C_p$$

we also have, for all $t \in [0, T]$,

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\pm(\xi) dx \leq C_p. \quad (30)$$

Proposition 17 (The case of equilibrium). *Assume that $f^{*, -}$ is at equilibrium in (22): there exists a random variable $u^* \in L^1(\mathbb{T}^d)$ such that $f^{*, -} = \mathbf{1}_{u^* > \xi}$ a.s. Then $f^{*, +} = f^{*, -}$.*

Proof of Proposition 17. Let m^* denote the restriction of m to $\mathbb{T}^N \times \{t_*\} \times \mathbb{R}$. We thus have

$$f^{*, +} - \mathbf{1}_{u^* > \xi} = \partial_\xi m^*. \quad (31)$$

Let (R_n) be a sequence increasing to $+\infty$. By the condition at infinity (5) on m , there exists a subset Ω_3 of Ω of probability 1 (independent on t_*) and a subsequence still denoted (R_n) such that, for all $\omega \in \Omega_3$, we have

$$\lim_{n \rightarrow +\infty} m(\mathbb{T}^N \times [0, T] \times B_{R_n}^c) = 0.$$

In particular, $\lim_{n \rightarrow +\infty} m^*(\mathbb{T}^N \times B_{R_n}^c) = 0$ and this implies that the right-hand side of (31) vanishes when we integrate it over \mathbb{R} . We have therefore: for all $\omega \in \hat{\Omega} \cap \Omega_3$, for a.e. $x \in \mathbb{T}^N$,

$$\int_{\mathbb{R}} (f^{*, +}(x, \xi) - \mathbf{1}_{0 > \xi}) d\xi = \int_{\mathbb{R}} (\mathbf{1}_{u^* > \xi} - \mathbf{1}_{0 > \xi}) d\xi = u^*.$$

Introduce now

$$p^* : \xi \mapsto \int_{-\infty}^{\xi} (\mathbf{1}_{u^* > \zeta} - f^{*, +}(\zeta)) d\zeta.$$

By Lemma 14, p^* is non-negative. Besides, $\partial_\xi(m^* + p^*) = 0$ due to (31) and the definition of p^* . Therefore $m^* + p^*$ is constant, and actually vanishes by the condition at infinity (5) and the obvious fact that p also vanishes when $|\xi| \rightarrow +\infty$. Since $m^*, p^* \geq 0$, we finally obtain $m^* = 0$ and $f^{*, +} = f^{*, -}$. \square

3 Comparison, uniqueness and reduction of generalized solutions

3.1 Doubling of variables

In this paragraph, we prove a technical proposition relating two generalized solutions f_i , $i = 1, 2$ of the equation

$$du_i + \operatorname{div}(A(u_i))dt = \Phi(u_i)dW. \quad (32)$$

Proposition 18. *Let f_i , $i = 1, 2$, be generalized solution to (32). Then, for $0 \leq t \leq T$, and non-negative test functions $\rho \in C^\infty(\mathbb{T}^N)$, $\psi \in C_c^\infty(\mathbb{R})$, we have*

$$\begin{aligned} & \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y) \psi(\xi-\zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy \\ & \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y) \psi(\xi-\zeta) f_{1,0}(x, \xi) \bar{f}_{2,0}(y, \zeta) d\xi d\zeta dx dy + \mathbf{I}_\rho + \mathbf{I}_\psi, \end{aligned} \quad (33)$$

where

$$\mathbf{I}_\rho = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (a(\xi) - a(\zeta)) \psi(\xi - \zeta) d\xi d\zeta \cdot \nabla_x \rho(x - y) dx dy ds$$

and

$$\mathbf{I}_\psi = \frac{1}{2} \int_{(\mathbb{T}^N)^2} \rho(x-y) \mathbb{E} \int_0^t \int_{\mathbb{R}^2} \psi(\xi-\zeta) \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi, \zeta) dx dy ds.$$

Remark 19. *Each term in (33) is finite. Let us for instance consider the first one on the right-hand side. Introduce the auxiliary functions*

$$\psi_1(\xi) = \int_{-\infty}^{\xi} \psi(s) ds, \quad \psi_2(\zeta) = \int_{-\infty}^{\zeta} \psi_1(\xi) d\xi,$$

and omit the index 0 in $f_{1,0}$, $\bar{f}_{2,0}$ for simplicity. Since ψ is compactly supported, both ψ_1 and ψ_2 vanish at $-\infty$. When $\xi \rightarrow +\infty$, ψ_1 remains bounded while ψ_2 has linear growth. Since

$$f_1(\xi) = \int_{(\xi, +\infty)} d\nu^1(\xi), \quad \bar{f}_2(\zeta) = \int_{(-\infty, \zeta)} d\nu^2(\zeta),$$

for a.e. $\xi, \zeta \in \mathbb{R}$ the Fubini Theorem gives us the formula

$$\int_{\mathbb{R}^2} \psi(\xi-\zeta) f_1(\xi) \bar{f}_2(\zeta) d\xi d\zeta = \int_{\mathbb{R}^2} \psi_2(u-v) d\nu^1(u) d\nu^2(v). \quad (34)$$

The function ψ_2 being sub-linear, and the measures ν^j having moments of arbitrary orders by (10), the right-hand side of (34) is finite.

Proof of Proposition 18. Set

$$G_i^2(x, \xi) = \sum_{k=1}^{\infty} |g_{k,i}(x, \xi)|^2, \quad i \in \{1, 2\}.$$

Let $\varphi_1 \in C_c^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi)$ and $\varphi_2 \in C_c^\infty(\mathbb{T}_y^N \times \mathbb{R}_\zeta)$ be some given test-functions. Equation (27) reads $\langle f_1^+(t), \varphi_1 \rangle = \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) + F_1(t)$, where F_1 is the stochastic integral

$$F_1(t) = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_{k,1} \varphi_1 d\nu_{x,s}^1(\xi) dx d\beta_k(s)$$

and $t \mapsto \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t])$ is the function with bounded variation defined by

$$\begin{aligned} \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) &= \langle f_{1,0}, \varphi_1 \rangle \delta_0([0, t]) + \int_0^t \langle f_1, a \cdot \nabla \varphi_1 \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi_1 \mathbf{G}_1^2 d\nu_{(x,s)}^1(\xi) dx ds - \langle m_1, \partial_\xi \varphi_1 \rangle([0, t]). \end{aligned}$$

Note that, by Remark 17, $\langle m_1, \partial_\xi \varphi_1 \rangle(\{0\}) = 0$ and thus the value of $\langle m_1^*, \partial_\xi \varphi_1 \rangle(\{0\})$ is $\langle f_{1,0}, \varphi_1 \rangle$. Similarly, we write a decomposition of $\langle \bar{f}_2^+(t), \varphi_2 \rangle$ as the sum of a stochastic integral

$$\bar{F}_2(t) = - \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_{k,2} \varphi_2 d\nu_{y,s}^2(\zeta) dy d\beta_k(s)$$

with a function with bounded variation given by

$$\begin{aligned} \langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle([0, t]) &= \langle \bar{f}_{2,0}, \varphi_2 \rangle \delta_0([0, t]) + \int_0^t \langle \bar{f}_2, a \cdot \nabla \varphi_2 \rangle ds \\ &\quad - \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\zeta \varphi_2 \mathbf{G}_1^2 d\nu_{(y,s)}^2(\zeta) dy ds + \langle m_2, \partial_\zeta \varphi_2 \rangle([0, t]). \end{aligned}$$

Again, we note that $\langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle(\{0\}) = \langle \bar{f}_{2,0}, \varphi_2 \rangle$. Let us define the test-function

$$\alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta).$$

We have then

$$\langle f_1^+(t), \varphi_1 \rangle \langle \bar{f}_2^+(t), \varphi_2 \rangle = \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle \quad (35)$$

There are four terms in the product in the left-hand side of (35). We use Itô's Formula to compute $F_1(t) \bar{F}_2(t)$, we use integration by parts for functions of finite variation (see for instance [40], chapter 0) for the term $\langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle([0, t])$, which gives

$$\begin{aligned} &\langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle([0, t]) \\ &= \langle m_1^*, \partial_\xi \varphi_1 \rangle(\{0\}) \langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle(\{0\}) + \int_{(0,t]} \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, s]) d\langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle(s) \\ &\quad + \int_{(0,t]} \langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle([0, s]) d\langle m_1^*, \partial_\xi \varphi_1 \rangle(s). \end{aligned}$$

We use also the following formula

$$\langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \bar{F}_2(t) = \int_0^t \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, s]) d\bar{F}_2(s) + \int_0^t \bar{F}_2(s) \langle m_1^*, \partial_\xi \varphi_1 \rangle(ds),$$

which is easy to obtain since \bar{F}_2 is continuous, and a similar formula for the product $\langle \bar{m}_2^*, \partial_\zeta \varphi_2 \rangle([0, t]) \bar{F}_1(t)$, to get, by (35), the following identity:

$$\begin{aligned} \mathbb{E} \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle &= \langle \langle f_{1,0} \bar{f}_{2,0}, \alpha \rangle \rangle \\ &+ \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) \cdot \nabla_x + a(\zeta) \cdot \nabla_y) \alpha d\xi d\zeta dx dy ds \\ &+ \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\xi \alpha \bar{f}_2(s) \mathbf{G}_1^2 d\nu_{(x,s)}^1(\xi) d\zeta dx dy ds \\ &- \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_\zeta \alpha f_1(s) \mathbf{G}_2^2 d\nu_{(y,s)}^2(\zeta) d\xi dy dx ds \\ &- \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \mathbf{G}_{1,2} \alpha d\nu_{x,s}^1(\xi) d\nu_{y,s}^2(\zeta) dx dy \\ &- \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s) \partial_\xi \alpha dm_1(x, s, \xi) d\zeta dy \\ &\quad + \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\zeta \alpha dm_2(y, s, \zeta) d\xi dx \quad (36) \end{aligned}$$

where $\mathbf{G}_{1,2}(x, y; \xi, \zeta) := \sum_{k \geq 1} g_{k,1}(x, \xi) g_{k,2}(y, \zeta)$ and $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the duality distribution over $\mathbb{T}_x^N \times \mathbb{R}_\xi \times \mathbb{T}_y^N \times \mathbb{R}_\zeta$. By a density argument, (36) remains true for any test-function $\alpha \in C_c^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi \times \mathbb{T}_y^N \times \mathbb{R}_\zeta)$. Using similar arguments as in Remark 19, the assumption that α is compactly supported can be relaxed thanks to the condition at infinity (5) on m_i and (10) on ν^i , $i = 1, 2$. Using truncates of α , we obtain that (36) remains true if $\alpha \in C_b^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi \times \mathbb{T}_y^N \times \mathbb{R}_\zeta)$ is compactly supported in a neighbourhood of the diagonal

$$\{(x, \xi, x, \xi); x \in \mathbb{T}^N, \xi \in \mathbb{R}\}.$$

We then take $\alpha = \rho\psi$ where $\rho = \rho(x - y)$, $\psi = \psi(\xi - \zeta)$. Note the remarkable identities

$$(\nabla_x + \nabla_y)\alpha = 0, \quad (\partial_\xi + \partial_\zeta)\alpha = 0. \quad (37)$$

In particular, the last term in (36) is

$$\begin{aligned} \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\zeta \alpha d\xi dx dm_2(y, s, \zeta) \\ = - \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\xi \alpha d\xi dx dm_2(y, s, \zeta) \\ = - \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu_{x,s}^{1,-}(\xi) dx dm_2(y, s, \zeta) \leq 0 \end{aligned}$$

since $\alpha \geq 0$. The symmetric term

$$\begin{aligned} -\mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s) \partial_\xi \alpha dm_1(x, s, \xi) d\zeta dy \\ = -\mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu_{y,s}^{2,+}(\zeta) dy dm_1(x, s, \xi) \end{aligned}$$

is, similarly, non-positive. Consequently, we have

$$\mathbb{E} \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle \leq \langle \langle f_{1,0} \bar{f}_{2,0}, \alpha \rangle \rangle + \mathbf{I}_\rho + \mathbf{I}_\psi, \quad (38)$$

where

$$\mathbf{I}_\rho := \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) \cdot \nabla_x + a(\zeta) \cdot \nabla_y) \alpha d\xi d\zeta dx dy ds$$

and

$$\begin{aligned} \mathbf{I}_\psi = \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\xi \alpha \bar{f}_2(s) \mathbf{G}_1^2 d\nu_{(x,s)}^1(\xi) d\zeta dx dy ds \\ - \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \partial_\zeta \alpha f_1(s) \mathbf{G}_2^2 d\nu_{(x,s)}^2(\zeta) d\xi dy dx ds \\ - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathbf{G}_{1,2} \alpha d\nu_{x,s}^1(\xi) d\nu_{y,s}^2(\zeta) dx dy. \end{aligned}$$

Equation (38) is indeed equation (33) for f_i^+ since, by (37),

$$\mathbf{I}_\rho = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) - a(\zeta)) \cdot \nabla_x \alpha d\xi d\zeta dx dy ds$$

and, by (37) also and integration by parts,

$$\begin{aligned} \mathbf{I}_\psi = \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha (\mathbf{G}_1^2 + \mathbf{G}_2^2 - 2\mathbf{G}_{1,2}) d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi, \zeta) dx dy ds \\ = \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 0} |g_k(x, \xi) - g_k(y, \zeta)|^2 d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi, \zeta) dx dy ds. \end{aligned}$$

To obtain the result for f_i^- , we take $t_n \uparrow t$, write (33) for $f_i^+(t_n)$ and let $n \rightarrow \infty$. \square

3.2 Uniqueness, reduction of generalized solution

In this section we use Proposition 18 above to deduce the uniqueness of solutions and the reduction of generalized solutions to solutions.

Theorem 20 (Uniqueness, Reduction). *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume (2)-(3). Then,*

- there is at most one solution with initial datum u_0 to (1). Besides, if f is a generalized solution to (1) with initial datum f_0 at equilibrium: $f_0 = \mathbf{1}_{u_0 > \xi}$, then there exists a solution u to (1) with initial datum u_0 such that $f(x, t, \xi) = \mathbf{1}_{u(x, t) > \xi}$ a.s., for a.e. (x, t, ξ) .
- if u_1, u_2 are two solutions to (1) associated to the initial data $u_{1,0}, u_{2,0} \in L^\infty(\mathbb{T}^N)$ respectively, then

$$\mathbb{E}\|(u_1(t) - u_2(t))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|(u_{1,0} - u_{2,0})^+\|_{L^1(\mathbb{T}^N)}. \quad (39)$$

This implies the L^1 -contraction property, and comparison principle for solutions.

Corollary 21 (Continuity in time). *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume (2)-(3). Then, for every $p \in [1, +\infty)$, the solution u to (1) with initial datum u_0 has a representative in $L^p(\Omega; L^\infty(0, T; L^p(\mathbb{T}^N)))$ with almost-sure continuous trajectories in $L^p(\mathbb{T}^N)$.*

Proof of Theorem 20. Consider first the additive case: $\Phi(u)$ independent on u . Let $f_i, i = 1, 2$ be two generalized solutions to (1). Then, we use (33) with g_k independent on ξ and ζ . By (3), the last term I_ψ is bounded by

$$\frac{tD_1}{2} \|\psi\|_{L^\infty} \int_{(\mathbb{T}^N)^2} |x - y|^2 \rho(x - y) dx dy.$$

We then take $\psi := \psi_\delta$ and $\rho = \rho_\varepsilon$ where (ψ_δ) and (ρ_ε) are approximations to the identity on \mathbb{R} and \mathbb{T}^N respectively, i.e.

$$\psi_\delta(\xi) = \frac{1}{\delta} \psi\left(\frac{\xi}{\delta}\right), \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right),$$

where ψ and ρ are some given smooth probability densities on \mathbb{R} and \mathbb{T}^N respectively, to obtain

$$I_\psi \leq \frac{tD_1}{2} \varepsilon^2 \delta^{-1}. \quad (40)$$

Denote by $\nu_{x,t}^{i,\pm}$ the Young measure associated to f_i^\pm , $i \in \{1, 2\}$. By a computation similar to (34), we have, almost-surely, for almost all $x, y \in \mathbb{T}^N$,

$$\int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \xi) d\xi = \int_{\mathbb{R}^2} (u - v)^+ d\nu_{x,t}^{1,\pm}(u) d\nu_{y,t}^{2,\pm}(v). \quad (41)$$

By (34), we have also

$$\int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta = \int_{\mathbb{R}^2} \psi_{2,\delta}(u - v) d\nu_{x,t}^{1,\pm}(u) d\nu_{y,t}^{2,\pm}(v), \quad (42)$$

where

$$\psi_{2,\delta}(\xi) = \int_{-\infty}^{\xi} \psi_{1,\delta}(\zeta) d\zeta, \quad \psi_{1,\delta}(\xi) = \int_{-\infty}^{\xi} \psi_\delta(\zeta) d\zeta.$$

Assume that ψ is supported in $(0, +\infty)$. Then $\psi_{2,\delta}(\xi) = 0$ if $\xi \leq 0$ and, for $\xi > 0$,

$$\xi^+ - \psi_{2,\delta}(\xi) = \int_0^\xi \int_{\zeta/\delta}^{+\infty} \psi(u) du d\zeta = \int_0^{+\infty} \xi \wedge (\delta u) \psi(u) du. \quad (43)$$

Using (43) in (41), (42) gives

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \xi) d\xi - \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta \\ &\leq \int_{\mathbb{R}^2} \int_0^{+\infty} (u - v)^+ \wedge (\delta \zeta) \psi(\zeta) d\zeta d\nu_{x,t}^{1,\pm}(u) d\nu_{y,t}^{2,\pm}(v). \end{aligned}$$

Since $(u - v)^+ \wedge (\delta \zeta) \leq |u| \wedge (\delta \zeta) + |v| \wedge (\delta \zeta)$, we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \xi) d\xi - \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta \\ &\leq \int_0^{+\infty} \left(\int_{\mathbb{R}} |\xi| d\nu_{x,t}^{1,\pm}(\xi) + \int_{\mathbb{R}} |\xi| d\nu_{y,t}^{2,\pm}(\xi) \right) \wedge (\delta \zeta) \psi(\zeta) d\zeta. \end{aligned}$$

It follows that

$$\begin{aligned} &\left| \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}} \rho_\varepsilon(x - y) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \xi) d\xi dx dy \right. \\ &\quad \left. - \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x - y) \psi_\delta(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy \right| \\ &\leq \int_0^{+\infty} \left(\int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi| (d\nu_{x,t}^{1,\pm}(\xi) + d\nu_{x,t}^{2,\pm}(\xi)) dx \right) \wedge (\delta \zeta) \psi(\zeta) d\zeta \\ &\leq \int_0^{+\infty} (2C_1(\omega)) \wedge (\delta \zeta) \psi(\zeta) d\zeta. \quad (44) \end{aligned}$$

We have used (29) to obtain (44). When $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} &\left| \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}} \rho_\varepsilon(x - y) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \xi) d\xi dx dy - \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi dx \right| \\ &\leq \sup_{|z| < \varepsilon} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) |\bar{f}_2^\pm(x - z, t, \xi) - \bar{f}_2^\pm(x, t, \xi)| d\xi dx \\ &\leq \sup_{|z| < \varepsilon} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\chi_{f_2^\pm}(x - z, t, \xi) - \chi_{f_2^\pm}(x, t, \xi)| d\xi dx. \quad (45) \end{aligned}$$

Consequently (see (44), (45)),

$$\begin{aligned} \lim_{\varepsilon, \delta \rightarrow 0} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x - y) \psi_\delta(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy \\ = \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi dx, \end{aligned}$$

for all $\omega \in \hat{\Omega}$. Since

$$\begin{aligned} \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x-y) \psi_\delta(\xi-\zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy \\ \leq \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi dx, \end{aligned}$$

we conclude by Fatou's Lemma, that

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) dx d\xi \\ = \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x-y) \psi_\delta(\xi-\zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy + \eta_t(\varepsilon, \delta), \end{aligned}$$

where $\lim_{\varepsilon, \delta \rightarrow 0} \eta_t(\varepsilon, \delta) = 0$. We need now a bound on the term I_ρ . Since a has at most polynomial growth, there exists $C \geq 0$, $p > 1$, such that

$$|a(\xi) - a(\zeta)| \leq \Gamma(\xi, \zeta) |\xi - \zeta|, \quad \Gamma(\xi, \zeta) = C(1 + |\xi|^{p-1} + |\zeta|^{p-1}).$$

This gives

$$|I_\rho| \leq \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 \Gamma(\xi, \zeta) |\xi - \zeta| \psi_\delta(\xi - \zeta) |\nabla_x \rho_\varepsilon(x - y)| d\xi d\zeta dx dy d\sigma.$$

By integration by parts with respect to (ξ, ζ) , we deduce

$$|I_\rho| \leq \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \Upsilon(\xi, \zeta) d\nu_{x, \sigma}^1 \otimes \nu_{y, \sigma}^2 |\nabla_x \rho_\varepsilon(x - y)| dx dy d\sigma,$$

where

$$\Upsilon(\xi, \zeta) = \int_\zeta^{+\infty} \int_{-\infty}^\xi \Gamma(\xi', \zeta') |\xi' - \zeta'| \psi_\delta(\xi' - \zeta') d\xi' d\zeta'.$$

It is shown below that Υ admits the bound

$$\Upsilon(\xi, \zeta) \leq C(1 + |\xi|^p + |\zeta|^p) \delta. \quad (46)$$

Since ν^1 and ν^2 vanish at infinity, cf. (15), we then obtain, for a given constant C_p ,

$$|I_\rho| \leq t C_p \delta \left(\int_{\mathbb{T}^N} |\nabla_x \rho_\varepsilon(x)| dx \right).$$

It follows that, for possibly a different C_p ,

$$|I_\rho| \leq t C_p \delta \varepsilon^{-1}. \quad (47)$$

We then gather (40), (47) and (33) to deduce for $t \in [0, T]$

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(t) \bar{f}_2^\pm(t) dx d\xi \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_{1,0} \bar{f}_{2,0} dx d\xi + r(\varepsilon, \delta), \quad (48)$$

where the remainder $r(\varepsilon, \delta)$ is $r(\varepsilon, \delta) = TC_p \delta \varepsilon^{-1} + \frac{TD_1}{2} \varepsilon^2 \delta^{-1} + \eta_t(\varepsilon, \delta) + \eta_0(\varepsilon, \delta)$. Taking $\delta = \varepsilon^{4/3}$ and letting $\varepsilon \rightarrow 0$ gives

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(t) \bar{f}_2^\pm(t) dx d\xi \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_{1,0} \bar{f}_{2,0} dx d\xi. \quad (49)$$

Assume that f is a generalized solution to (1) with initial datum $\mathbf{1}_{u_0 > \xi}$. Since f_0 is the (translated) Heavyside function $\mathbf{1}_{u_0 > \xi}$, we have the identity $f_0 \bar{f}_0 = 0$. Taking $f_1 = f_2 = f$ in (49), we deduce $f^+(1 - f^+) = 0$ a.e., i.e. $f^+ \in \{0, 1\}$ a.e. The fact that $-\partial_\xi f^+$ is a Young measure then gives the conclusion: indeed, by Fubini's Theorem, for any $t \in [0, T]$, there is a set E_t of full measure in $\mathbb{T}^N \times \Omega$ such that, for $(x, \omega) \in E_t$, $f^+(x, t, \xi, \omega) \in \{0, 1\}$ for a.e. $\xi \in \mathbb{R}$. Let

$$\tilde{E}_t = E_t \cap (\mathbb{T}^N \times \hat{\Omega}).$$

The set \tilde{E}_t is of full measure in $\mathbb{T}^N \times \Omega$. For $(x, \omega) \in \tilde{E}_t$, $-\partial_\xi f^+(x, t, \cdot, \omega)$ is a probability measure on \mathbb{R} . Therefore $f^+(t, x, \xi, \omega) = \mathbf{1}_{u^+(x, t, \omega) > \xi}$ for a.e. $\xi \in \mathbb{R}$, where $u^+(x, t, \omega) = \int_{\mathbb{R}} (f^+(x, t, \xi, \omega) - \mathbf{1}_{\xi > 0}) d\xi$. We have a similar result for f^- . Proposition 13 implies that u^+ is a solution in the sense of Definition 2. Since $f = f^+$ a.e., this shows the reduction of generalized solutions to solutions. If now u_1 and u_2 are two solutions to (1), we deduce from (49) with $f_i = \mathbf{1}_{u_i > \xi}$ and from the identity

$$\int_{\mathbb{R}} \mathbf{1}_{u_1 > \xi} \overline{\mathbf{1}_{u_2 > \xi}} d\xi = (u_1 - u_2)^+,$$

the contraction property (39).

In the multiplicative case (Φ depending on u), the reasoning is similar, except that there is an additional term in the bound on I_ψ . More precisely, by Hypothesis (3) we obtain in place of (40) the estimate

$$I_\psi \leq \frac{TD_1}{2} \varepsilon^2 \delta^{-1} + \frac{D_1}{2} I_\psi^h,$$

where

$$I_\psi^h = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\varepsilon \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) |\xi - \zeta| h(|\xi - \zeta|) d\nu_{x,\sigma}^1 \otimes \nu_{y,\sigma}^2(\xi, \zeta) dx dy d\sigma.$$

Choosing $\psi_\delta(\xi) = \delta^{-1} \psi_1(\delta^{-1} \xi)$ with ψ_1 compactly supported gives

$$I_\psi \leq \frac{TD_1}{2} \varepsilon^2 \delta^{-1} + \frac{TD_1 C_\psi h(\delta)}{2}, \quad C_\psi := \sup_{\xi \in \mathbb{R}} \|\xi \psi_1(\xi)\|. \quad (50)$$

We deduce (48) with a remainder term $r'(\varepsilon, \delta) := r(\varepsilon, \delta) + \frac{TD_1 C_\psi h(\delta)}{2}$ and conclude the

proof as in the additive case. There remains to prove (46): setting $\xi'' = \xi' - \zeta'$, we have

$$\begin{aligned} \Upsilon(\xi, \zeta) &= \int_{\zeta}^{+\infty} \int_{|\xi''| < \delta, \xi'' < \xi - \zeta'} \Gamma(\xi'' + \zeta', \zeta') |\xi''| \psi_{\delta}(\xi'') d\xi'' d\zeta' \\ &\leq C \int_{\zeta}^{\xi + \delta} \max_{|\xi''| < \delta, \xi'' < \xi - \zeta'} \Gamma(\xi'' + \zeta', \zeta') d\zeta' \delta \\ &\leq C \int_{\zeta}^{\xi + \delta} (1 + |\xi|^{p-1} + |\zeta'|^{p-1}) d\zeta' \delta, \end{aligned}$$

which gives (46). \square

Proof of Corollary 21. We use the notations and the results of Proposition 15. We fix $p \in [1, +\infty)$. By Item 1. in Proposition 15, f^+ and f^- are generalized solutions to (1) associated to the initial datum $\mathbf{1}_{u_0 > \xi}$. By Theorem 20, they are at equilibrium: $f^{\pm} = \mathbf{1}_{u^{\pm} > \xi}$. It follows then from Item 2. in Proposition 15 and Lemma 10 that, almost-surely, u^+ is càdlàg as a function $[0, T] \rightarrow L^p(\mathbb{T}^N)$. Note that the limit from the left $u^+(t-0)$ of u^+ at $t \in (0, T]$ is $u^-(t)$ since the limit from the left of $\langle f^+, \varphi \rangle$ at $t \in (0, T]$ is $\langle f^-(t), \varphi \rangle$ (this is a consequence of the equations (27) and (28)). Furthermore, thanks to Proposition 17, the fact that f^- is at equilibrium has the following consequence: at every $t \in (0, T)$, $f^+(t) = f^-(t)$. In particular, we have $u^+ = u^-$ and thus, almost-surely, u^+ is continuous from $[0, T]$ into $L^p(\mathbb{T}^N)$. \square

We apply (39) to infer an L^∞ bound on solutions to (1) in the particular case of a multiplicative noise with compact support.

Theorem 22 (L^∞ bounds). *Assume (2)-(3) and*

$$g_k(x, -1) = g_k(x, 1) = 0, \quad (51)$$

for all $x \in \mathbb{T}^N$, $k \geq 1$. Let $u_0 \in L^\infty(\mathbb{T}^N)$ satisfy $-1 \leq u_0 \leq 1$ almost everywhere. Then, for all $t \geq 0$, the solution u to (1) with initial datum u_0 satisfies: almost-surely,

$$-1 \leq u(x, t) \leq 1, \quad (52)$$

a.e. in \mathbb{T}^N .

Proof of Theorem 22. We use (39) to compare u to the two particular constant solutions $(x, t) \mapsto -1$ and $(x, t) \mapsto 1$. \square

4 Convergence of approximate solutions

In this section, we develop the tools required for the proof of convergence of a certain type of approximate solutions to (1). The basic principle is to generalize the notion of solution introduced in Definition 2. Indeed, this facilitates the proof of existence/convergence. In a second step a result of reduction (or “rigidity result”), which asserts that a generalized

solution is a solution is used. This principle is of much use in the deterministic theory of conservation laws (cf. [14] with the use of “measure-valued entropy solutions”, [18] with the use of “entropy process solutions”, [39] with the use of kinetic solutions as defined here). We have already introduced a generalization of the notion of solution in Definition 11, and have proved a result of reduction in Theorem 20. Here we will work mainly on the probabilistic aspects of the questions. We will have to consider “solutions in law”, or “martingale solutions” (see the comment after Theorem 29 for more explanations about the terminology). The plan of this section is the following one: in Section 4.1, we define the notion of approximate generalized solution. In Section 4.2, we give a martingale characterization of the stochastic integral. In Section 4.3, we give some tightness results on sequence of approximate generalized solutions. The main result, Theorem 29, which shows the convergence of a sequence of approximate generalized solutions to a martingale generalized solution, is proved in Section 4.5. Eventually, we obtain a result of pathwise convergence in Section 4.6.

4.1 Approximate generalized solutions

Let d be an integer fixed once and for all.

Definition 23 (Approximate generalized solutions). *Let $f_0^n: \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1]$ be some kinetic functions. Let $f^n: \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega \rightarrow [0, 1]$ be some measurable kinetic functions. Assume that the functions f^n are satisfying item 1, 2, 3 in Definition 11 and Equation (16) up to an error term, i.e.: for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$, there exists an adapted process $\varepsilon^n(t, \varphi)$, with $t \mapsto \varepsilon^n(t, \varphi)$ almost-surely continuous such that*

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, T]} |\varepsilon^n(t, \varphi)| = 0 \text{ in probability,} \quad (53)$$

and there exists some random measures m^n , such that, for all n , for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0, T]$, almost-surely,

$$\begin{aligned} \langle f^n(t), \varphi \rangle &= \langle f^n(0), \varphi \rangle + \int_0^t \langle f^n(s), a(\xi) \cdot \nabla_x \varphi \rangle ds - \iiint_{\mathbb{T} \times [0, t] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) dm^n(x, s, \xi) \\ &\quad + \int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx d\beta_k(s) \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \mathbf{G}^2(x, \xi) \partial_\xi \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx ds + \varepsilon^n(t, \varphi). \end{aligned} \quad (54)$$

Then we say that (f_n) is a sequence of approximate generalized solutions to (1) with initial datum f_0^n .

Note that we use a formulation “at fixed given time t ” in (54), contrary to the formulation (16), which is weak in time. In particular, each function $t \mapsto \langle f^n(t), \varphi \rangle$ is càdlàg, and we will see that the accumulation point obtained in Theorem 29 has the same property (cf. Item 8 of Theorem 29).

4.2 Martingale characterization of the stochastic integral

In order to pass to the limit in an equation such as (54), we will first characterize (54) in terms of a martingale problem, and then we will use martingale methods to pass to the limit. In the present section, we give the characterization of (54) in terms of a martingale problem, see Proposition 24 and Proposition 25 below. We refer to [28, Example 1.4, p.143] for characterization of the standard Wiener Process in terms of a martingale problem. In the context of SDEs and SPDEs, such kind of characterizations have been applied in [38, 7, 27, 24, 12] in particular.

Let us define the stochastic integrands

$$h_{\varphi,k}^n(t) = \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,t}^n(\xi) dx, \quad h_{\varphi}^n(t) = (h_{\varphi,k}^n(t))_{k \geq 1}, \quad (55)$$

and the stochastic integrals

$$M_{\varphi}^n(t) = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx d\beta_k(s). \quad (56)$$

By Lemma 12, we have $h_{\varphi}^n \in L_{\mathcal{P}}^2([0, T] \times \Omega; l^2(\mathbb{N}^*))$ for all n, φ . Using Itô's Formula, we deduce from (56) the following statement.

Proposition 24. *Let (f^n) be a sequence of approximate generalized solutions to (1) with initial datum f_0^n . Let $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$. Let $M_{\varphi}^n(t)$ be defined by (56) and $h_{\varphi,k}^n(t)$ by (55). Then the processes*

$$M_{\varphi}^n(t), \quad M_{\varphi}^n(t) \beta_k(t) - \int_0^t h_{\varphi,k}^n(s) ds, \quad |M_{\varphi}^n(t)|^2 - \int_0^t \|h_{\varphi}^n(s)\|_{l^2(\mathbb{N}^*)}^2 ds, \quad (57)$$

are (\mathcal{F}_t) -martingales.

What will interest us is the reciprocal statement.

Proposition 25. *Let $h \in L_{\mathcal{P}}^2([0, T] \times \Omega; l^2(\mathbb{N}^*))$. Let $X(t)$ be a stochastic process starting from 0 such that the processes*

$$X(t), \quad X(t) \beta_k(t) - \int_0^t h_k(s) ds, \quad |X(t)|^2 - \int_0^t \|h(s)\|_{l^2(\mathbb{N}^*)}^2 ds \quad (58)$$

are (\mathcal{F}_t) -martingales. Then

$$X(t) = \sum_{k \geq 1} \int_0^t h_k(s) d\beta_k(s), \quad (59)$$

for all $t \in [0, T]$.

Proof of Proposition 25. The proof can be found in [24, Proposition A.1]. Let us give some details about it. We first claim that the following identity is satisfied:

$$\mathbb{E} \left[(X(t) - X(s)) \int_s^t \theta(\sigma) d\beta_k(\sigma) d\sigma - \int_s^t h_k(\sigma) \theta(\sigma) d\sigma \middle| \mathcal{F}_s \right] = 0 \quad (60)$$

for all $0 \leq s \leq t \leq T$, all $k \geq 1$ and all $\theta \in L^2_{\mathcal{P}}([0, T] \times \Omega)$. The proof consists in approximating θ on the interval $[s, t]$ by predictable simple functions. It is similar to a computation of quadratic variation. Note that (60) uses only the fact that

$$X(t), \quad X(t)\beta_k(t) - \int_0^t h_k(s) ds$$

are (\mathcal{F}_t) -martingales. We apply (60) with $s = 0$ and $\theta = h_k$ and sum over k to obtain

$$\mathbb{E}[X(t)\bar{X}(t)] = \mathbb{E} \int_0^t \|h(s)\|_{l^2(\mathbb{N}^*)}^2 ds, \quad \bar{X}(t) := \sum_{k \geq 1} \int_0^t h_k(s) d\beta_k(s). \quad (61)$$

This gives the expression of the cross-product when we expand the term $\mathbb{E}|X(t) - \bar{X}(t)|^2$. Using the fact that

$$|X(t)|^2 - \int_0^t \|h(s)\|_{l^2(\mathbb{N}^*)}^2 ds$$

is a (\mathcal{F}_t) -martingale and applying Itô's Isometry to $\mathbb{E}|\bar{X}(t)|^2$ shows that the square terms are also given by

$$\mathbb{E}|X(t)|^2 = \mathbb{E}|\bar{X}(t)|^2 = \int_0^t \|h(s)\|_{l^2(\mathbb{N}^*)}^2 ds.$$

It follows that $X(t) = \bar{X}(t)$. □

4.3 Tightness

Let (f^n) be a sequence of approximate generalized solutions, in the sense of Definition 23. Recall that \mathcal{Y}^1 is the notation for the set of Young measures on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$ (cf. Proposition 5) and that $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ is the notation for the set of bounded Borel measures on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$ while $\mathcal{M}_b^+(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ is the subset of non-negative measures. Let ν^n be the Young measure associated to f^n ($\nu^n = -\partial_\xi f^n$). The law of ν^n is a probability measure on the space \mathcal{Y}^1 . We will see in Section 4.3.1 that, under a natural a priori bound, see (62), the sequence $(\text{Law}(\nu^n))$ is tight in \mathcal{Y}^1 . In Section 4.3.2, this is the sequence $(\text{Law}(m^n))$ that we will analyse. We show under (65) and (66) that it is tight in $\mathcal{M}_b^+(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ (see, more specifically, Proposition 27).

We also need to analyse the tightness of $(\langle f_n(t), \varphi \rangle)$ in the Skorokhod space $D([0, T])$: this is done in Section 4.3.3.

4.3.1 Compactness of the Young measures

In this section, we will use the following notions: we say that a sequence (ν^n) of \mathcal{Y}^1 converges to ν in \mathcal{Y}^1 if (13) is satisfied. A **random Young measure** is by definition a \mathcal{Y}^1 -valued random variable.

Proposition 26. *Let (f^n) be a sequence of approximate generalized solutions to (1) with initial datum f_0^n . Assume that the following bound is satisfied: for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ independent on n such that $\nu^n := -\partial_\xi f^n$ satisfies*

$$\mathbb{E} \left(\sup_{l \in \mathbb{N}} \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p dM_{\varepsilon_l}^\pm \nu_{x,t}^n(\xi) dx \right) \leq C_p, \quad (62)$$

Then, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some random Young measures $\tilde{\nu}^n$, $\tilde{\nu}$, such that

1. $\tilde{\nu}^n$ has the same law as ν^n ,

2. $\tilde{\nu}$ satisfies

$$\tilde{\mathbb{E}} \left(\sup_{l \in \mathbb{N}} \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p dM_{\varepsilon_l}^\pm \tilde{\nu}_{x,t}(\xi) dx \right) \leq C_p, \quad (63)$$

3. up to a subsequence still denoted $(\tilde{\nu}^n)$, there is $\tilde{\mathbb{P}}$ -almost-sure convergence of $(\tilde{\nu}^n)$ to $\tilde{\nu}$ in \mathcal{Y}^1 .

Furthermore, if $\tilde{f}^n, \tilde{f}: \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \tilde{\Omega} \rightarrow [0, 1]$ are defined by

$$\tilde{f}^n(x, t, \xi) = \tilde{\nu}_{x,t}^n(\xi, +\infty), \quad \tilde{f}(x, t, \xi) = \tilde{\nu}_{x,t}(\xi, +\infty),$$

then $\tilde{f}^n \rightarrow \tilde{f}$ in $L^\infty(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak- $*$ $\tilde{\mathbb{P}}$ -almost-surely, \tilde{f} being a kinetic function.

Proof of Proposition 26. Note first that (62) yields

$$\mathbb{E} \left(\int_{\mathbb{T}^N} \int_0^T \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^n(\xi) dx dt \right) \leq C_p. \quad (64)$$

For $R > 0$, let us denote by K_R the set of Young measures $\nu \in \mathcal{Y}^1$ such that

$$\int_{\mathbb{T}^N} \int_0^T \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi) dx dt \leq R.$$

By [8, Theorem 4.3.2, Theorem 4.3.8, Theorem 2.1.3], the set K_R is compact in \mathcal{Y}^1 for the $\tau_{\mathcal{Y}^1}^W$ -topology, which is metrizable, [8, Theorem 2.3.1] and corresponds to the convergence (13). By (64), we have

$$\mathbb{P}(\nu^n \notin K_R) \leq \frac{C_p}{R},$$

which shows that the sequence (ν^n) of \mathcal{Y}^1 -valued random variables is tight. The set \mathcal{Y}^1 endowed with the $\tau_{\mathcal{Y}^1}^W$ -topology is Polish, [8, Theorem 2.3.3]: we can use the Prokhorov's metric, [6, p. 72]. By Prokhorov's Theorem, [6, Theorem 5.1], there exists a \mathcal{Y}^1 -valued random variable ν and a subsequence still denoted (ν^n) such that (ν^n) converges in probability to ν . Since the maps

$$\psi_p^\pm : \mathcal{Y}^1 \rightarrow [0, +\infty], \quad \nu \mapsto \sup_{l \in \mathbb{N}} \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p dM_{\varepsilon_l}^\pm \nu_{x,t}(\xi) dx,$$

are lower semi-continuous, we have

$$\mathbb{E}\psi_p^\pm(\nu) \leq \liminf_{n \rightarrow +\infty} \mathbb{E}\psi_p^\pm(\nu^n) \leq C_p$$

by (62) and, consequently, ν satisfies the condition (15). Let us now apply the Skorokhod Theorem [6, p. 70]: there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some random variables $\tilde{\nu}^n, \tilde{\nu}$, such that

1. $\tilde{\nu}^n$ and $\tilde{\nu}$ have the same laws as ν^n and ν respectively,
2. up to a subsequence still denoted $(\tilde{\nu}^n)$, there is $\tilde{\mathbb{P}}$ -almost-sure convergence of $(\tilde{\nu}^n)$ to $\tilde{\nu}$ in \mathcal{Y}^1 .

Since $\tilde{\nu}$ and ν have same laws, $\tilde{\nu}$ satisfies the bound (63). If we apply Corollary 9, we obtain that $\tilde{f}^n \rightarrow \tilde{f}$ in $L^\infty(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak-* $\tilde{\mathbb{P}}$ -almost-surely, \tilde{f} being a kinetic function. \square

4.3.2 Compactness of the random measures

Proposition 27. *Let (f^n) be a sequence of approximate generalized solutions to (1) with initial datum f_0^n . Assume that*

$$\mathbb{E}m^n(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \text{ is uniformly bounded,} \tag{65}$$

and that m^n vanishes for large ξ uniformly in n : if $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$, then

$$\lim_{R \rightarrow +\infty} \mathbb{E}m^n(\mathbb{T}^N \times [0, T] \times B_R^c) = 0, \tag{66}$$

uniformly in n . Then, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some random measures $\tilde{m}^n, \tilde{m}: \tilde{\Omega} \rightarrow \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ such that

1. \tilde{m}^n has the same law as m^n ,
2. up to a subsequence still denoted (\tilde{m}^n) , there is $\tilde{\mathbb{P}}$ -almost-sure convergence of (\tilde{m}^n) to \tilde{m} in $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak-*.

Proof of Proposition 27. Let $\eta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$\eta(R) = \sup_{n \in \mathbb{N}} \mathbb{E} m^n(\mathbb{T}^N \times [0, T] \times B_R^c).$$

Let h be a fixed function on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$, h continuous, positive, integrable. Proving the statement for the sequence of measures

$$B \mapsto m^n(B) + \int_B h(x, t, \xi) dx dt d\xi$$

is equivalent to prove the statement for the original sequence (m^n) . We will assume therefore that $\eta(R) > 0$ for all $R \geq 0$ and that

$$\|m^n\| := m^n(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \geq \delta > 0,$$

where δ is independent on n . Let $\mu^n := \frac{m^n}{\|m^n\|}$. We consider the random variables $X^n = (\mu^n, \|m^n\|)$, taking values in $\mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \times \mathbb{R}_+$, where $\mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ is the set of probability measures on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$. For $A > 0$, let K_A be the set of probability measures $\mu \in \mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ such that

$$\sup_{R > 1} \frac{\mu(\mathbb{T}^N \times [0, T] \times B_R^c)}{\eta(R)} \leq A.$$

Then K_A is compact in $\mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak-* by Prokhorov's Theorem and (66). Using the Markov Inequality, and the definition of $\eta(R)$, we obtain

$$\mathbb{P}(\mu^n \notin K_A) \leq \frac{C}{A},$$

where C is independent on n : this shows that (μ^n) is tight in $\mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ endowed with the topology of the weak convergence of probability measures. Similarly, using (65) and the Markov Inequality, we have

$$\mathbb{P}(\|m^n\| > A) \leq \frac{C}{A},$$

where C is independent on n ; therefore $(\|m^n\|)$ is tight in \mathbb{R} . It follows that (X^n) is tight in $\mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \times \mathbb{R}_+$ endowed with the product topology. This topology is separable, metrizable and there exists a compatible metric which turns the space into a complete space (we can take the sum of the Prokhorov's metric and of the usual metric on \mathbb{R}_+). Therefore we can apply the Skorokhod Theorem: there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some random variables $\tilde{X}^n = (\tilde{\mu}^n, \tilde{\alpha}^n)$, $\tilde{X} = (\tilde{\mu}, \tilde{\alpha})$ such that \tilde{X}^n has same law as X^n and, $\tilde{\mathbb{P}}$ -almost-surely, $\tilde{X}^n \rightarrow \tilde{X}$ in $\mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \times \mathbb{R}_+$. Set $\tilde{m}^n = \tilde{\alpha}^n \tilde{\mu}^n$ and $\tilde{m} = \tilde{\alpha} \tilde{\mu}$. Then \tilde{m}^n has the same law as m^n and there is $\tilde{\mathbb{P}}$ -almost-sure convergence of (\tilde{m}^n) to \tilde{m} in $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak-*. \square

4.3.3 Tightness in the Skorokhod space

Let $D([0, T])$ denote the space of càdlàg functions on $[0, T]$. See [28, VI.1] and [6, Chapter 3] for the definition of $D([0, T])$. Let (f^n) be a sequence of approximate generalized solutions to (1) with initial datum f_0^n . In Section 4.4 below, where we analyse the convergence of (f^n) , it would be desirable to have a result of tightness of the processes $t \mapsto \langle f^n(t), \varphi \rangle$ since as random variables in $D([0, T])$ (here, φ is a given test-function). It seems difficult to get such a result. The only fact which we can infer naturally from (62), (65), (66), is that the sequence of processes

$$t \mapsto \langle f^n(t), \varphi \rangle + A_\varphi^n(t), \quad A_\varphi^n(t) := \langle m^n, \partial_\xi \varphi \rangle([0, t]),$$

is tight in $D([0, T])$, see Proposition 28 below. Showing additionally that (A_φ^n) is tight in $D([0, T])$ seems impossible, however, if no additional properties of (m^n) are known. Indeed, the weak convergence of $\mu^n := \langle m^n, \partial_\xi \varphi \rangle$ to a measure μ on $[0, T]$ is not a sufficient condition for the convergence of A_φ^n to $A(t) = \mu([0, t])$ in $D([0, T])$. Consider for example the case

$$\mu^n = \delta_{t_* - s_n} + \delta_{t_* - \sigma_n},$$

where $t_* \in (0, T)$ and $(s_n) \downarrow 0$, $(\sigma_n) \downarrow 0$ with $s_n < \sigma_n$ for all n . Then (μ_n) converges weakly to $\mu = 2\delta_{t_*}$, we have

$$\alpha_n(t) := \mu_n([0, t]) \rightarrow \alpha(t) := \mu([0, t])$$

for every $t \in [0, T]$, but (α_n) , or any subsequence of (α_n) , does not converge to α in $D([0, T])$. This example should be compared to [28, Example 1.20, p.329]. See also Theorem 2.15, p.342 in [28].

As asserted above, we will show that the sequence of processes

$$t \mapsto \langle f^n(t), \varphi \rangle + A_\varphi^n(t), \quad A_\varphi^n(t) := \langle m^n, \partial_\xi \varphi \rangle([0, t]),$$

where

$$\langle m^n, \partial_\xi \varphi \rangle([0, t]) := \iiint_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \partial_\xi \varphi(x, s, \xi) dm^n(x, s, \xi),$$

is tight in $D([0, T])$. It is sufficient to show that

$$t \mapsto \langle f^n(t), \varphi \rangle + B_\varphi^n(t), \quad B_\varphi^n(t) := \langle m^n, \partial_\xi \varphi \rangle([0, t]) - \varepsilon^n(t, \varphi) \quad (67)$$

is tight in $D([0, T])$ since each function $t \mapsto \varepsilon^n(t, \varphi)$ converges in probability to 0 in $C([0, T])$ by (53). We have

$$\langle f^n(t), \varphi \rangle + B_\varphi^n(t) = \langle f^n(0), \varphi \rangle + J_\varphi^n(t), \quad (68)$$

\mathbb{P} -almost-surely, where

$$\begin{aligned} J_\varphi^n: t \mapsto & \int_0^t \langle f^n(s), a(\xi) \cdot \nabla \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx d\beta_k(s) \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{G}^2(x, \xi) d\nu_{x,s}^n(\xi) dx ds. \end{aligned} \quad (69)$$

We will show that $(J_\varphi^n(t))$ is tight in $C([0, T])$.

Proposition 28. Let (f^n) be a sequence of approximate generalized solutions to (1) with initial datum f_0^n . For $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$, set

$$\begin{aligned} D_\varphi^n(t) &= \int_0^t \langle f^n(s), a(\xi) \cdot \nabla \varphi \rangle ds, \\ M_\varphi^n(t) &= \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx d\beta_k(s), \\ I_\varphi^n(t) &= \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{G}^2(x, \xi) d\nu_{x,s}^n(\xi) dx ds. \end{aligned}$$

Then each sequence (D_φ^n) , (M_φ^n) , (I_φ^n) is tight in $C([0, T])$. In particular, the sequence (J_φ^n) defined by (69) is tight in $C([0, T])$.

Proof of Proposition 28. Note first the trivial uniform bounds

$$\mathbb{E}|D_\varphi^n(t)|, \mathbb{E}|M_\varphi^n(t)|, \mathbb{E}|I_\varphi^n(t)| = \mathcal{O}(1),$$

obtained for $t = 0$ since all three terms vanish. We then use the Kolmogorov's criterion to obtain some bounds in some Hölder space $C^\alpha([0, T])$. We have the following estimate on the square of the increments of D_φ^n :

$$\mathbb{E}|D_\varphi^n(t) - D_\varphi^n(\sigma)|^2 \leq \|a \cdot \nabla \varphi\|_{L^1(\mathbb{T}^N \times \mathbb{R})}^2 |t - \sigma|^2, \quad (70)$$

since $|f_n| \leq 1$ almost-surely. Similarly, we have

$$\mathbb{E}|I_\varphi^n(t) - I_\varphi^n(\sigma)|^2 \leq D_0 \|\partial_\xi \varphi\|_{L^\infty(\mathbb{T}^N \times \mathbb{R})}^2 |t - \sigma|^2. \quad (71)$$

The estimates (70) and (71) give some bounds on $\mathbb{E}\|D_\varphi^n\|_{C^\alpha([0, T])}$ and $\mathbb{E}\|I_\varphi^n\|_{C^\alpha([0, T])}$ respectively, for $\alpha < \frac{1}{2}$. Furthermore, the Burkholder - Davis - Gundy Inequality gives, for $p > 2$,

$$\begin{aligned} \mathbb{E}|M_\varphi^n(t) - M_\varphi^n(\sigma)|^p &\leq \mathbb{E} \left[\sup_{\sigma \leq r \leq t} |M_\varphi^n(r) - M_\varphi^n(\sigma)| \right]^p \\ &\leq C_p \mathbb{E} \left[\sum_{k \geq 1} \int_\sigma^t \left| \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}^n(\xi) dx \right|^2 ds \right]^{p/2}. \end{aligned}$$

By Jensen's Inequality, and a bound analogous to (19), we obtain

$$\mathbb{E}|M_\varphi^n(t) - M_\varphi^n(\sigma)|^p \leq C_p \|\varphi\|_{L^\infty(\mathbb{T}^N \times \mathbb{R})}^p D_0^{p/2} |t - \sigma|^{p/2}, \quad (72)$$

and (72) gives a bound on $\mathbb{E}\|M_\varphi^n\|_{C^\alpha([0, T])}$ for $\alpha < \frac{1}{2} - \frac{1}{p}$. We obtain in this way some tightness conditions on the laws of D^n , M^n , I^n respectively on $C([0, T])$. \square

4.4 Convergence of approximate generalized solutions

We conclude here this section about the stability of generalized solutions by the following statement.

Theorem 29 (Convergence to martingale solutions). *Let (f^n) be a sequence of approximate generalized solutions to (1) with initial datum f_0^n satisfying (62), (65) and (66). We suppose that there exists a kinetic function f_0 on \mathbb{T}^N such that $f_0^n \rightarrow f_0$ in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ -weak-*. Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a filtration $\tilde{\mathcal{F}}_t$, some $\tilde{\mathcal{F}}_t$ -adapted independent Brownian motions $(\tilde{\beta}_k)_{k \geq 1}$, some random Young measures $\tilde{\nu}^n$, $\tilde{\nu}$, some random measures \tilde{m}^n , \tilde{m} on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$ such that*

1. $\tilde{\nu}^n$ has the same law as ν^n ,
2. $\tilde{\nu}$ satisfies (63),
3. up to a subsequence still denoted $(\tilde{\nu}^n)$, there is $\tilde{\mathbb{P}}$ -almost-sure convergence of $(\tilde{\nu}^n)$ to $\tilde{\nu}$ in \mathcal{Y}^1 .
4. for all $\psi \in C_b(\mathbb{R})$, the random map $(x, t) \mapsto \langle \psi, \tilde{\nu}_{x,t} \rangle$ belongs to $L^2_{\tilde{\mathbb{P}}}(\mathbb{T}^N \times [0, T] \times \tilde{\Omega})$,
5. \tilde{m}^n has the same law as m^n ,
6. up to a subsequence still denoted (\tilde{m}^n) , there is $\tilde{\mathbb{P}}$ -almost-sure convergence of (\tilde{m}^n) to \tilde{m} in $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak-*.

Let \tilde{f} be defined by $\tilde{f}(x, t, \xi) = \tilde{\nu}_{x,t}(\xi, +\infty)$, then, $\tilde{\mathbb{P}}$ -almost-surely, \tilde{f} is a kinetic function and

7. up to a subsequence, and $\tilde{\mathbb{P}}$ -almost-surely, \tilde{f}^n converges in $L^\infty(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ -weak-* to \tilde{f}
8. for all φ in $C_c^d(\mathbb{T}^N \times \mathbb{R})$, $\tilde{\mathbb{P}}$ -almost-surely, $t \mapsto \langle \tilde{f}(t), \varphi \rangle$ is càdlàg,
9. for all $\varphi \in C_c^d(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, \tilde{f} satisfies

$$\begin{aligned}
& \int_0^T \langle \tilde{f}(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle \tilde{f}(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\
&= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\tilde{\nu}_{x,t}(\xi) dx d\tilde{\beta}_k(t) \\
&- \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, t, \xi) \mathbf{G}^2(x, \xi) d\tilde{\nu}_{(x,t)}(\xi) dx dt + \tilde{m}(\partial_\xi \varphi), \tilde{\mathbb{P}} - a.s.
\end{aligned} \tag{73}$$

After one does the substitution

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \beta_k(t)) \leftarrow (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}_t, \tilde{\beta}_k(t)),$$

which is a substitution of the probabilistic data in the Cauchy Problem for Equation (1), the points 2, 4, 9 in Theorem 29 show that \tilde{f} is a generalized solution associated to the initial datum f_0 . Such a function \tilde{f} , which turns out to be a generalized solution to (1) after a substitution of the probabilistic data, is called a martingale generalized solution. The term martingale refers to the martingale characterization of (73), cf. Proposition 24 and Proposition 25, which we will use to prove Theorem 29.

4.5 Proof of Theorem 29

In this section, we will give the proof of Theorem 29. We will use the results (and the proofs) of Proposition 26, Proposition 27, see Section 4.3.1 and Section 4.3.2 respectively.

4.5.1 State space and Skorokhod's Theorem

Recall that

$$W(t) = \sum_{k \geq 1} \beta_k(t) e_k,$$

where $(e_k)_{k \geq 1}$ is the orthonormal basis of the Hilbert space H . Let \mathfrak{U} be an other separable Hilbert space such that $H \hookrightarrow \mathfrak{U}$ with Hilbert-Schmidt injection. Then the trajectories of W are \mathbb{P} -a.s. in the path-space $\mathcal{X}_W = C([0, T]; \mathfrak{U})$ (see [11, Theorem 4.3]). We consider the C^d -norm

$$\|\varphi\|_{C^d} = \sup\{\|D^m \varphi\|_{L^\infty(\mathbb{T}^N \times \mathbb{R})}; m \in \{0, \dots, d\}^{N+1}\}$$

on $C_c^d(\mathbb{T}^N \times \mathbb{R})$. Let

$$\Gamma = \{\varphi_1, \varphi_2, \dots\}$$

be a dense countable subset of $C_c^d(\mathbb{T}^N \times \mathbb{R})$ for this norm. We can construct Γ as follows: let

$$\rho_\varepsilon(x, \xi) := \frac{1}{\varepsilon^{N+1}} \rho(\varepsilon^{-N} x, \varepsilon^{-1} \xi)$$

be a compactly supported approximation of the unit on $\mathbb{T}^N \times \mathbb{R}$. Let $\{\theta_p; p \in \mathbb{N}\}$ be a dense subset of $L^1(\mathbb{T}^N \times \mathbb{R})$. We can assume that all the functions θ_p are compactly supported (otherwise, we use a process of truncation). Then any function in $C_c^d(\mathbb{T}^N \times \mathbb{R})$ can be approximated by functions in

$$\Gamma := \{\rho_{k^{-1}} * \theta_p; p \in \mathbb{N}, k \in \mathbb{N}^*\} \subset C_c^d(\mathbb{T}^N \times \mathbb{R})$$

for the convergence measured by the C^d -norm. Indeed, given $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$, $a > 0$, and $m \in \{0, \dots, d\}^{N+1}$, we have, by the triangular inequality,

$$\begin{aligned} \|D^m \varphi - D^m \rho_\varepsilon * \theta_p\|_{L^\infty} &\leq \|D^m \varphi - D^m \rho_\varepsilon * \varphi\|_{L^\infty} + \|D^m \rho_\varepsilon * (\varphi - \theta_p)\|_{L^\infty} \\ &\leq \omega_{D^m \varphi}(\varepsilon) + \frac{\|\rho\|_{L^\infty}}{\varepsilon^{N+1+|m|}} \|\varphi - \theta_p\|_{L^1}, \end{aligned} \quad (74)$$

since the norm of $D^m \rho_\varepsilon$ in L^∞ is bounded by $\frac{\|\rho\|_{L^\infty}}{\varepsilon^{N+1+|m|}}$. In (74), $\omega_{D^m \varphi}$ denotes the modulus of continuity of $D^m \varphi$. We choose $\varepsilon = k^{-1}$ with k large enough to ensure $\omega_{D^m \varphi}(\varepsilon) < a$ for all $m \in \{0, \dots, d\}^{N+1}$. Taking then $p \in \mathbb{N}$ such that $\|\varphi - \theta_p\|_{L^1} < a\varepsilon^{(d+1)(N+1)}$, we obtain $\|\varphi - \rho_{k^{-1}} * \theta_p\|_{C^d} < 2a$.

Let also \mathbb{R}^∞ denote the product space $\prod_{\varphi \in \Gamma} \mathbb{R}$ endowed with the topology of point-wise convergence. As such, \mathbb{R}^∞ is separable, complete and admits a compatible metric. Define the Polish space

$$\mathcal{E} := C([0, T]; \mathbb{R}^\infty) \times C([0, T]; \mathbb{R}^\infty) \times C([0, T]; \mathbb{R}^\infty),$$

and

$$\varepsilon_\varphi^n(t) = \varepsilon^n(t, \varphi), \quad \langle f_\leftarrow^n, \varphi \rangle(t) = \langle f^n(t), \varphi \rangle + \langle m^n, \partial_\xi \varphi \rangle([0, t]) - \varepsilon_\varphi^n(t), \quad (75)$$

for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$. Note that, as a consequence of Equation (68) and Proposition 28, we know that, for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$,

$$(\langle f_\leftarrow^n, \varphi \rangle) \text{ is tight in } C([0, T]). \quad (76)$$

By (53), we also have $\varepsilon_\varphi^n \rightarrow 0$ in $C([0, T])$ for all $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$. We introduce the three following sequences:

$$\{f_\leftarrow^n(t)\} := (\langle f_\leftarrow^n(t), \varphi_j \rangle)_{j \in \mathbb{N}}, \quad \{M^n(t)\} := (M_{\varphi_j}^n(t))_{j \in \mathbb{N}}, \quad \{\varepsilon^n(t)\} := (\varepsilon_{\varphi_j}^n(t))_{j \in \mathbb{N}},$$

where M_φ^n is defined by (56). We will consider the multiplet

$$Z^n = (\nu^n, \{f_\leftarrow^n\}, \{M^n\}, \{\varepsilon^n\}, \mu^n, \|m^n\|, W) \in \mathcal{X},$$

where the state space \mathcal{X} is

$$\mathcal{X} := \mathcal{Y}^1 \times \mathcal{E} \times \mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \times \mathbb{R}_+ \times \mathcal{X}_W.$$

Let $\varepsilon > 0$. By (76), there exists for each $j \in \mathbb{N}$ a compact K_j in $C([0, T])$ such that

$$\inf_{n \in \mathbb{N}} \mathbb{P}(\langle f_\leftarrow^n, \varphi_j \rangle \in K_j) \geq 1 - \frac{\varepsilon}{2^j}.$$

Let $K = \prod_{j \in \mathbb{N}} K_j$. Then K is compact² in $C([0, T]; \mathbb{R}^\infty)$ and

$$\mathbb{P}(\{f_\leftarrow^n\} \in K^c) \leq \sum_{j \in \mathbb{N}} \mathbb{P}(\langle f_\leftarrow^n, \varphi_j \rangle \in K_j^c) \leq \sum_{j \in \mathbb{N}} \frac{\varepsilon}{2^j} = \varepsilon,$$

for all $n \in \mathbb{N}$. This shows that $(\{f_\leftarrow^n\})$ is tight in $C([0, T]; \mathbb{R}^\infty)$. We have similar results about $(\{M^n\})$ and $(\{\varepsilon^n\})$ thanks to Proposition 28. On \mathcal{X}_W we consider the topology induced by the norm

$$\|v\| = \sup_{t \in [0, T]} \|v(t)\|_{\mathfrak{U}}$$

²since $C([0, T]; \mathbb{R}^\infty)$ is homeomorphic to the countable product, over Γ , of copies of $C([0, T]; \mathbb{R})$

Then \mathcal{X}_W is separable and complete. A first consequence of this is the fact that the law of the single random variable W is tight in \mathcal{X}_W . A second consequence is the fact that \mathcal{X} is a separable completely-metrizable space. By Section 4.3.1 and Section 4.3.2, we conclude that (Z^n) is tight in the Polish space \mathcal{X} . We may thus apply Skorokhod's Theorem to (Z^n) : there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some random variable \tilde{Z}^n, \tilde{Z} such that \tilde{Z}^n has the same law as Z^n and, up to a subsequence, $\tilde{\mathbb{P}}$ -almost-surely, \tilde{Z}^n converges to \tilde{Z} in \mathcal{X} .

4.5.2 Identification of the limit

Let us denote the component of \tilde{Z} as follows:

$$\tilde{Z} = (\tilde{\nu}, \{\tilde{f}_{\leftarrow}\}, \{\tilde{M}\}, \{\tilde{\varepsilon}\}, \tilde{\mu}, \tilde{\alpha}, \tilde{W}).$$

Note first that $\tilde{\varepsilon} = 0$ by (53). Let $f^*(x, t, \xi) = \tilde{\nu}_{x,t}(\xi, +\infty)$ and $m^* = \tilde{\alpha}\tilde{\mu}$. It was shown in the proof of Proposition 26 that item 1, 2, 3, 7 of Theorem 29 are satisfied. By the proof of Proposition 27, we have also 5, 6 of Theorem 29. For each $t \in [0, T]$, define the distribution $f_{\leftarrow}^*(t)$ by

$$\langle f_{\leftarrow}^*(t), \varphi \rangle = \langle f^*(t), \varphi \rangle + \langle m^*, \partial_{\xi}\varphi \rangle([0, t]). \quad (77)$$

We have the following identification result:

Lemma 30. *For $j \in \mathbb{N}$, the j -th component of $\{\tilde{f}_{\leftarrow}\}$ is $\langle f_{\leftarrow}^*, \varphi_j \rangle$, i.e.*

$$\{\tilde{f}_{\leftarrow}\} = \{f_{\leftarrow}^*\} \quad \tilde{\mathbb{P}} - a.s.$$

Proof of Lemma 30. Let

$$\tilde{A}_j^n(t) = \langle \tilde{m}^n, \partial_{\xi}\varphi_j \rangle([0, t]), \quad A_j^*(t) = \langle m^*, \partial_{\xi}\varphi_j \rangle([0, t]). \quad (78)$$

The sequence of processes $t \mapsto \tilde{A}_j^n(t)$ is bounded by

$$\|\partial_{\xi}\varphi_j\|_{L^{\infty}(\mathbb{T}^N \times \mathbb{R})} \sup_n \tilde{m}^n(\mathbb{T}^N \times [0, T] \times \mathbb{R})$$

and $\sup_n \tilde{m}^n(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ is $\tilde{\mathbb{P}}$ -almost-surely finite by (65). We also have, $\tilde{\mathbb{P}}$ -almost-surely, $\tilde{A}_j^n(t) \rightarrow A_j^*(t)$ for all $t \in [0, T]$, except for the set N_j^* of atomic points t such that A_j^* has a jump at t , which form a countable subset of $[0, T]$. In particular, $\tilde{\mathbb{P}}$ -almost-surely, $\tilde{A}_j^n(t) \rightarrow A_j^*(t)$ for a.e. $t \in [0, T]$. By the dominated convergence theorem, it follows that, $\tilde{\mathbb{P}}$ -almost-surely,

$$\tilde{A}_j^n \rightarrow A_j^* \text{ in } L^{\infty}(0, T) \text{ weak} - *. \quad (79)$$

The convergence of $(\tilde{\nu}^n)$, which means that, $\tilde{\mathbb{P}}$ -almost-surely,

$$((x, t) \mapsto \langle \tilde{\nu}_{(x,t)}^n, \phi \rangle) \rightarrow ((x, t) \mapsto \langle \tilde{\nu}_{x,t}, \phi \rangle) \text{ in } L^{\infty}(\mathbb{T}^N \times (0, T)) \text{ weak} - *,$$

has the consequence that, $\tilde{\mathbb{P}}$ -almost-surely,

$$\iint_{\mathbb{T}^N \times (0, T)} \int_{\mathbb{R}} \alpha(x, t, \xi) d\tilde{\nu}_{x, t}^n(\xi) dx dt \rightarrow \iint_{\mathbb{T}^N \times (0, T)} \int_{\mathbb{R}} \alpha(x, t, \xi) d\tilde{\nu}_{x, t}(\xi) dx dt, \quad (80)$$

for every bounded Carathéodory integrand α . This is a consequence of the identity $\tau_{\mathcal{Y}^1}^M = \tau_{\mathcal{Y}^1}^W$ in the Portmanteau Theorem [8, Theorem 2.1.3] (see also [8, Lemma 1.2.3] about Carathéodory integrands). We deduce from (80) that, $\tilde{\mathbb{P}}$ -almost-surely,

$$\left(t \mapsto \int_{\mathbb{T}^N} \int_{\mathbb{R}} \alpha(x, \xi) d\tilde{\nu}_{x, t}^n(\xi) dx \right) \rightarrow \left(t \mapsto \iint_{\mathbb{T}^N \times (0, T)} \int_{\mathbb{R}} \alpha(x, \xi) d\tilde{\nu}_{x, t}(\xi) dx \right) \quad (81)$$

in $L^\infty(0, T)$ -weak-*, for every bounded Carathéodory integrand $\alpha(x, \xi)$. We apply (81) to

$$\alpha(x, \xi) = \int_{-\infty}^{\xi} \varphi_j(x, \zeta) d\zeta,$$

to obtain the convergence, $\tilde{\mathbb{P}}$ -almost-surely, $\langle \tilde{f}^n, \varphi_j \rangle \rightarrow \langle f^*, \varphi_j \rangle$ in $L^\infty(0, T)$ weak-*. By (75), (77) and (79), we deduce that, $\tilde{\mathbb{P}}$ -almost-surely, $\langle \tilde{f}_{\leftarrow}^n, \varphi_j \rangle \rightarrow \langle f_{\leftarrow}^*, \varphi_j \rangle$ in $L^\infty(0, T)$ weak-*. We also know that, $\tilde{\mathbb{P}}$ -almost-surely, $\langle \tilde{f}_{\leftarrow}^n, \varphi_j \rangle$ converges to the j -th component of $\{\tilde{f}_{\leftarrow}\}$ in $C([0, T])$. Since convergence in $C([0, T])$ implies convergence in $L^\infty(0, T)$ weak-*, and since the limit in $L^\infty(0, T)$ weak-* is unique, we have $\{f_{\leftarrow}^*\} = \{\tilde{f}_{\leftarrow}^*\}$, $\tilde{\mathbb{P}}$ -almost-surely. \square

Remark 31. *we have seen, in the proof of Lemma 30, that, $\tilde{\mathbb{P}}$ -almost-surely, $\tilde{A}_j^n(t) \rightarrow A_j^*(t)$ for all t in $(0, T)$, outside of a countable set N_j^* . In Equation (75), the terms $\langle \tilde{f}_{\leftarrow}^n(t), \varphi \rangle$ and $\tilde{\varepsilon}_n(t, \varphi)$ are converging. Therefore, $\tilde{\mathbb{P}}$ -almost-surely,*

$$\left\{ \tilde{f}^n(t) \right\} \rightarrow \left\{ f^*(t) \right\} \text{ in } \mathbb{R}^\infty, \quad (82)$$

for a.e. $t \in (0, T)$.

As a corollary to Lemma 30, we obtain item 8 in Theorem 29.

Corollary 32. *For all $\varphi \in \Gamma$, $\tilde{\mathbb{P}}$ -almost-surely, $t \mapsto \langle f^*(t), \varphi \rangle$ is càdlàg.*

Proof of Corollary 32. Indeed $\tilde{\mathbb{P}}$ -almost-surely, $t \mapsto \langle \tilde{f}_{\leftarrow}^n(t), \varphi \rangle$ is continuous, hence càdlàg, and $t \mapsto \langle m, \partial_\xi \varphi \rangle([0, t])$ is càdlàg. \square

Let us set

$$M_\varphi^*(t) = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\tilde{\nu}_{x, s}(\xi) dx d\tilde{\beta}_k(s), \quad (83)$$

($\tilde{\beta}_k$ is defined in Lemma 36 below). Our aim is to prove the identification $\{\tilde{M}\} = \{M^*\}$. To do this, we will use the martingale characterization developed in Section 4. The proof is decomposed in several steps.

Step 1. Filtration The approximation procedures to (1) (vanishing viscosity method, Finite Volume method as here) construct approximate solutions on arbitrary time intervals $[0, T]$. We will therefore consider the functions as defined on the whole time interval \mathbb{R}_+ . This is simply to avoid the special case of the final time in the definition of the Skorokhod space $D([0, T])$, cf. [6], [28, Remark 1.10]. Let E be a Polish space. Let us introduce the following notations (see [28, Definition 1.1 p.325] in the case $E = \mathbb{R}^m$): on the space $D(\mathbb{R}_+; E)$, $\mathcal{D}_t^0(E)$ is the σ -algebra generated by the maps $\alpha \mapsto \alpha(s)$, $s \leq t$;

$$\mathcal{D}_t(E) = \bigcap_{t < s} \mathcal{D}_s^0(E), \quad \mathcal{D}_{t-}(E) = \bigvee_{s < t} \mathcal{D}_s(E).$$

Note that $\mathcal{D}_t(E) \neq \mathcal{D}_t^0(E)$: the time of entrance in an open subset U of E ,

$$\tau_U(\alpha) = \inf \{t \geq 0; \alpha(t) \in U\}$$

is a stopping time with respect to $(\mathcal{D}_t(E))$, but not with respect to $(\mathcal{D}_t^0(E))$.

Proposition 33. *Let $t > 0$. Given a continuous bounded function $\theta: E \rightarrow \mathbb{R}$, $s \in [0, t]$ and $\varepsilon > 0$, let $\theta_{\#s}$ denote the evaluation map $\alpha \mapsto \theta(\alpha(s))$ on $D(\mathbb{R}_+; E)$, and let $\theta_{\#s}^\varepsilon$ denote the regularization*

$$\theta_{\#s}^\varepsilon: \alpha \rightarrow \frac{1}{\varepsilon} \int_s^{t \wedge (s+\varepsilon)} \theta(\alpha(\sigma)) d\sigma \quad (84)$$

of $\theta_{\#s}$. Then $\theta_{\#s}^\varepsilon$ is a $\mathcal{D}_{t-}(E)$ -measurable bounded function, continuous for the Skorokhod topology, and $\mathcal{D}_{t-}(E)$ is generated by the finite products $\theta_{\#s_1}^{1, \varepsilon_1} \cdots \theta_{\#s_k}^{k, \varepsilon_k}$, $k \geq 1$, $0 \leq s_1 < \cdots < s_k < t$, $0 < \varepsilon_1, \dots, \varepsilon_k$, $\theta^1, \dots, \theta^k \in C_b(E)$.

Proof of Proposition 33. This is essentially the proof of [28, Lemma 1.45 p.335]. Let $\alpha \in D(\mathbb{R}_+; E)$ and let (α_n) be a sequence in $D(\mathbb{R}_+; E)$ such that $\alpha_n \rightarrow \alpha$ a.e. on $[0, t]$: this is the case if $\alpha_n \rightarrow \alpha$ in $D(\mathbb{R}_+; E)$ since $\alpha_n(\sigma) \rightarrow \alpha(\sigma)$ for every σ not in the (countable) jump set of α . Then, by the dominated convergence theorem, $\theta_{\#s}^\varepsilon(\alpha_n) \rightarrow \theta_{\#s}^\varepsilon(\alpha)$. Therefore $\theta_{\#s}^\varepsilon$ is a bounded function, continuous for the Skorokhod topology. The σ -algebra $\mathcal{D}_{t-}(E)$ is generated by the characteristic functions $\mathbf{1}_A$ of cylindrical sets

$$A = \{\alpha \in D(\mathbb{R}_+; E); \alpha(s_1) \in B_1, \dots, \alpha(s_k) \in B_k\},$$

for B_1, \dots, B_k closed subsets of E and $0 \leq s_1 < \cdots < s_k < t$. We can choose some sequences of continuous bounded functions $\theta_1^n, \dots, \theta_k^n: E \rightarrow \mathbb{R}$ converging simply to the characteristic functions $\mathbf{1}_{B_1}, \dots, \mathbf{1}_{B_k}$ (by considering, for example, the function distance to B_j , which is continuous). It is then clear that $\theta_{\#s}^\varepsilon$ is $\mathcal{D}_{t-}(E)$ -measurable and that $\mathcal{D}_{t-}(E)$ is generated by the finite products $\theta_{\#s_1}^1 \cdots \theta_{\#s_k}^k$. Since

$$\theta_{\#s_j}^{j, \varepsilon_j}(\alpha) \rightarrow \theta_{\#s_j}^j(\alpha),$$

when $\varepsilon_j \rightarrow 0$ for all $j \in \{1, \dots, k\}$, $\alpha \in D(\mathbb{R}_+; E)$, we conclude that $\mathcal{D}_{t-}(E)$ is generated by the finite products $\theta_{\#s_1}^{1, \varepsilon_1} \cdots \theta_{\#s_k}^{k, \varepsilon_k}$. \square

Remark 34. Note that the function

$$H = \theta_{\#s_1}^{1,\varepsilon_1} \cdots \theta_{\#s_k}^{k,\varepsilon_k}$$

is more than merely continuous for the Skorokhod topology. Indeed, what we have seen in the proof of Proposition 33 is that, for any $\alpha \in D(\mathbb{R}_+; E)$ and any sequence (α_n) in $D(\mathbb{R}_+; E)$ such that $\alpha_n \rightarrow \alpha$ a.e. on $[0, t]$, we have $H(\alpha_n) \rightarrow H(\alpha)$.

Define

$$\{f^*\} = (\langle f^*, \varphi_j \rangle)_{j \in \mathbb{N}}, \quad E = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathfrak{U}.$$

Recall that \mathbb{R}^∞ is the product space $\prod_{\varphi \in \Gamma} \mathbb{R}$ endowed with the topology of point-wise convergence. Since E is a product of Polish spaces, it is a Polish space. Since the product of $D(\mathbb{R}_+; \mathbb{R}^\infty)$ with $C(\mathbb{R}_+; \mathbb{R}^\infty \times \mathfrak{U})$ is, topologically, a subset of $D(\mathbb{R}_+; E)$, the triplet $(\{f^*\}, \{\tilde{M}\}, \tilde{W})$ is an element of $D(\mathbb{R}_+; E)$.

Definition 35. The filtration $(\tilde{\mathcal{F}}_t)$ is the completion of the filtration generated by the triplet $(\{f^*\}, \{\tilde{M}\}, \tilde{W})$:

$$\tilde{\mathcal{F}}_t = \sigma((\{f^*\}, \{\tilde{M}\}, \tilde{W})^{-1}(\mathcal{D}_t(E)) \cup \{N \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(N) = 0\}), \quad t \in [0, T]. \quad (85)$$

Note that $(\tilde{\mathcal{F}}_t)$ is right-continuous since $(\mathcal{D}_t(E))$ is, and complete by definition.

Step 2. Wiener process Let $j: H \rightarrow \mathfrak{U}$ denote the injection of H into \mathfrak{U} . Note that $j \circ j^*$ is a Trace-class operator on \mathfrak{U} . The Brownian motions $\tilde{\beta}_k(t)$ are the components of $\tilde{W}(t)$ on the orthonormal basis (e_k) :

Lemma 36. The process \tilde{W} has a modification which is a $(\tilde{\mathcal{F}}_t)$ -adapted $j \circ j^*$ -Wiener process, and there exists a collection of mutually independent real-valued $(\tilde{\mathcal{F}}_t)$ -Brownian motions $\{\tilde{\beta}_k\}_{k \geq 1}$ such that

$$\tilde{W} = \sum_{k \geq 1} \tilde{\beta}_k e_k \quad (86)$$

in $C([0, T]; \mathfrak{U})$.

Note: see [11, Paragraph 4.1] for the definition of a Q -Wiener process.

Proof of Lemma 36. It is clear that \tilde{W} is a $j \circ j^*$ -cylindrical Wiener process (this notion is stable by convergence in law; actually it can be characterized in terms of the law of \tilde{W} uniquely if we drop the usual hypothesis of a.s. continuity of the trajectories. This latter property of continuity can be recovered, after a possible modification of the process, by using Kolmogorov's Theorem). Also \tilde{W} is $(\tilde{\mathcal{F}}_t)$ -adapted by definition of the filtration $(\tilde{\mathcal{F}}_t)$. By [11, Proposition 4.1], we obtain the decomposition (86). The $\tilde{\mathbb{P}}$ -a.s. convergence of the sum in (86) in the space $C([0, T]; \mathfrak{U})$ is proved as in [11, Theorem 4.3]. \square

Step 3. Martingales

Proposition 37. *Let $\varphi_j \in \Gamma$. Let $\tilde{h}_{j,k}(t)$ be defined by*

$$\tilde{h}_{j,k}(t) = \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi_j(x, \xi) d\tilde{\nu}_{x,t}(\xi) dx.$$

Then, for $j \in \mathbb{N}$, $k \geq 1$, the processes

$$\tilde{M}_j(t), \quad \tilde{M}_j(t) \tilde{\beta}_k(t) - \int_0^t \tilde{h}_{j,k}(s) ds, \quad |\tilde{M}_j(t)|^2 - \int_0^t \|\tilde{h}_j(s)\|_{l^2(\mathbb{N}^*)}^2 ds, \quad (87)$$

are $(\tilde{\mathcal{F}}_t)$ -martingales.

Proof of Proposition 37. The proof is similar to the proof of [28, Proposition 1.1 p.522], except that we do not use any hypothesis of boundedness here since we use the $\tilde{\mathbb{P}}$ -almost-sure convergence and the Vitali Theorem to pass to the limit in the expectation of the quantities of interest (an other minor difference with the proof of [28, Proposition 1.1 p.522] is that \tilde{M} is known to be continuous $\tilde{\mathbb{P}}$ -a.s., not only càdlàg).

Let $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$ and let H be a $\mathcal{D}_{t_1-}(E)$ -measurable bounded function. By Proposition 33 and Remark 34, we can assume without loss of generality that H is continuous for the Skorokhod topology, and, even more, that $H(\alpha_n) \rightarrow H(\alpha)$ for each sequence (α_n) of $D(\mathbb{R}_+; E)$ and each càdlàg function $\alpha \in D(\mathbb{R}_+; E)$ such that $\alpha_n \rightarrow \alpha$ a.e. By identities of the laws of M_φ^n and \tilde{M}_φ^n , we have

$$\tilde{\mathbb{E}} |\tilde{M}_{\varphi_j}^n(t_2) - \tilde{M}_{\varphi_j}^n(t_1)|^2 = \mathbb{E} |M_{\varphi_j}^n(t_2) - M_{\varphi_j}^n(t_1)|^2.$$

Using (72), it follows that

$$\sup_n \tilde{\mathbb{E}} \left| H \left(\{\tilde{f}^n\}, \{\tilde{M}^n\}, \tilde{W} \right) \left[\tilde{M}_{\varphi_j}^n(t_2) - \tilde{M}_{\varphi_j}^n(t_1) \right] \right|^2 < +\infty,$$

since H is bounded. We have besides

$$\left(\{\tilde{f}^n\}, \{\tilde{M}^n\}, \tilde{W} \right) \rightarrow \left(\{f^*\}, \{\tilde{M}\}, \tilde{W} \right) \quad (88)$$

a.e., $\tilde{\mathbb{P}}$ -almost-surely by (82) and thus,

$$H \left(\{\tilde{f}^n\}, \{\tilde{M}^n\}, \tilde{W} \right) \rightarrow H \left(\{f^*\}, \{\tilde{M}\}, \tilde{W} \right),$$

$\tilde{\mathbb{P}}$ -almost-surely. There is also convergence

$$\tilde{M}_{\varphi_j}^n(t_2) - \tilde{M}_{\varphi_j}^n(t_1) \rightarrow \tilde{M}_j(t_2) - \tilde{M}_j(t_1)$$

$\tilde{\mathbb{P}}$ -almost-surely. By Vitali's Theorem, we obtain

$$\begin{aligned} \tilde{\mathbb{E}} \left[H \left(\{\tilde{f}^n\}, \{\tilde{M}^n\}, \tilde{W} \right) \left(\tilde{M}_{\varphi_j}^n(t_2) - \tilde{M}_{\varphi_j}^n(t_1) \right) \right] \\ \rightarrow \tilde{\mathbb{E}} \left[H \left(\{f^*\}, \{\tilde{M}\}, \tilde{W} \right) \left(\tilde{M}_j(t_2) - \tilde{M}_j(t_1) \right) \right]. \quad (89) \end{aligned}$$

By identities of the laws, the left-hand side of (89) is

$$\mathbb{E} \left[H(\{f^n\}, \{M^n\}, W) \left(M_{\varphi_j}^n(t_2) - M_{\varphi_j}^n(t_1) \right) \right] = 0,$$

since M_{φ}^n is a (\mathcal{F}_t) -martingale. We deduce from (89) thus that

$$\tilde{\mathbb{E}} \left[H(\{f^*\}, \{\tilde{M}\}, \tilde{W}) \left(\tilde{M}_j(t_2) - \tilde{M}_j(t_1) \right) \right] = 0. \quad (90)$$

Let now $s, t \in [0, T)$ with $s < t$. Let (s_n) and (t_n) be some decreasing sequences in \mathbb{R}_+ , converging to s and t respectively. Let H be a $\mathcal{D}_s(E)$ -measurable bounded function. Then H is a $\mathcal{D}_{s_n-}(\mathbb{R}^{2+m})$ -measurable bounded function since $s < s_n$. By passing to the limit in (90) written with $t_1 = s_n$, $t_2 = t_n$ (we use the right-continuity of the processes here), we obtain

$$\tilde{\mathbb{E}} \left[H(\{f^*\}, \{\tilde{M}\}, \tilde{W}) \left(\tilde{M}_j(t) - \tilde{M}_j(s) \right) \right] = 0. \quad (91)$$

This shows that $(\tilde{M}_j(t))$ is a $\tilde{\mathcal{F}}_t$ -martingale. To go on, let us define now the increasing processes

$$\tilde{H}_{j,k}^n(t) = \int_0^t \tilde{h}_{j,k}^n(s) ds, \quad \tilde{H}_{j,k}(t) = \int_0^t \tilde{h}_{j,k}(s) ds,$$

and

$$\tilde{\mathcal{H}}_j^n(t) = \int_0^t \|\tilde{h}_j^n(s)\|_{l^2(\mathbb{N}^*)}^2 ds, \quad \tilde{\mathcal{H}}_j(t) = \int_0^t \|\tilde{h}_j(s)\|_{l^2(\mathbb{N}^*)}^2 ds,$$

and the processes

$$\begin{cases} \tilde{Y}_k^n(t) = \tilde{M}_j^n(t) \tilde{\beta}_k^n(t) - \tilde{H}_{j,k}^n(t), & \tilde{Y}_k(t) = \tilde{M}_j(t) \tilde{\beta}_k(t) - \tilde{H}_{j,k}(t), \\ \tilde{V}^n(t) = |\tilde{M}_j^n(t)|^2 - \tilde{\mathcal{H}}_j^n(t), & \tilde{V}(t) = |\tilde{M}_j(t)|^2 - \tilde{\mathcal{H}}_j(t). \end{cases}$$

To complete the proof of Proposition 37, we have to prove that $(\tilde{Y}_k(t))$ and $(\tilde{V}(t))$ are $\tilde{\mathcal{F}}_t$ -martingale. We will use the following result.

Lemma 38. *Let $T > 0$. Then, for all $j \in \mathbb{N}$, $k \in \mathbb{N}^*$, $\tilde{\mathbb{P}}$ -almost-surely, $\tilde{h}_{j,k}^n \rightarrow \tilde{h}_{j,k}$ and $\|\tilde{h}_j^n(\cdot)\|_{l^2(\mathbb{N}^*)}^2 \rightarrow \|\tilde{h}_j(\cdot)\|_{l^2(\mathbb{N}^*)}^2$ in $L^1(0, T)$, when $n \rightarrow +\infty$.*

Lemma 38 implies that, $\tilde{\mathbb{P}}$ -almost-surely, for every $t \in [0, T]$, $\tilde{H}_{j,k}^n(t)$ and $\tilde{\mathcal{H}}_j^n(t)$ are converging to $\tilde{H}_{j,k}(t)$ and $\tilde{\mathcal{H}}_j(t)$ respectively. We have also $\tilde{M}_j^n \rightarrow \tilde{M}_j$ in $C(\mathbb{R}_+)$, from which follows the convergences $\tilde{M}_j^n \tilde{\beta}_k \rightarrow \tilde{M}_j \tilde{\beta}_k$ and $|\tilde{M}_j^n|^2 \rightarrow |\tilde{M}_j|^2$ in $C(\mathbb{R}_+)$, $\tilde{\mathbb{P}}$ -almost-surely. We deduce that, $\tilde{\mathbb{P}}$ -almost-surely,

$$\tilde{Y}_k^n(t) \rightarrow \tilde{Y}_k(t), \quad \tilde{V}^n(t) \rightarrow \tilde{V}(t), \quad (92)$$

for all $t \geq 0$. With the estimate (72), it is easy to obtain the bounds

$$\tilde{\mathbb{E}} |\tilde{Y}_k^n(t) - \tilde{Y}_k^n(s)|^2 \leq C, \quad \tilde{\mathbb{E}} |\tilde{V}^n(t) - \tilde{V}^n(s)|^2 \leq C, \quad (93)$$

where the constant C depend on $s, t \in [0, T]$, k , but not on n . By (92) and (93) (this last condition shows the equi-integrability of $(\tilde{Y}_k^n(t) - \tilde{Y}_k^n(s))$ and $(\tilde{V}^n(t) - \tilde{V}^n(s))$ respectively), we can use the arguments applied to the martingale $\tilde{M}_\varphi^n(t)$ in the first part of the proof: it will establish that $\tilde{Y}_k(t)$ and $\tilde{V}(t)$ are $(\tilde{\mathcal{F}}_t)$ -martingales. \square

Let us now give the

Proof of Lemma 38. Let $\pi: \mathbb{T}^N \times [0, T] \times \mathbb{R} \rightarrow [0, T]$ denote the projection $(x, t, \xi) \mapsto t$. Let $\pi_{\#}m^*$ denote the push-forward of m^* by π : $\tilde{\mathbb{P}}$ -almost-surely, this is a non-negative measure on $[0, T]$. Let

$$N^* = \left\{ t \in [0, T]; \tilde{\mathbb{P}}(\pi_{\#}m^*(\{t\}) > 0) > 0 \right\}. \quad (94)$$

Note that N^* is at most countable. Indeed, if $t \in N^*$, then there exists $p, q \in \mathbb{N}$ such that

$$\tilde{\mathbb{P}}\left(\pi_{\#}m^*(\{t\}) > \frac{1}{q}\right) > \frac{1}{2p}.$$

By the Markov Inequality,

$$\tilde{\mathbb{E}}\pi_{\#}m^*(\{t\}) > \frac{1}{2pq}, \quad (95)$$

and there can be only a finite number of $t \in [0, T]$ satisfying (95) since $\tilde{\mathbb{E}}\pi_{\#}m^*$ is a finite measure on $[0, T]$. Define, for every $\psi \in C_b(\mathbb{T}^N \times \mathbb{R})$,

$$\tilde{h}_\psi^n(t) = \int_{\mathbb{T}^N} \int_{\mathbb{R}} \psi(x, \xi) d\tilde{\nu}_{x,t}^n(\xi) dx, \quad \tilde{h}_\psi(t) = \int_{\mathbb{T}^N} \int_{\mathbb{R}} \psi(x, \xi) d\tilde{\nu}_{x,t}(\xi) dx. \quad (96)$$

We have seen in the proof of Lemma 30 that $\tilde{h}_\psi^n \rightarrow \tilde{h}_\psi$ in $L^\infty(0, T)$ -weak-*. We will show also the following convergence result:

$$\text{for all } t \in [0, T] \setminus N^*, \tilde{h}_\psi^n(t) \rightarrow \tilde{h}_\psi(t). \quad (97)$$

Assume

$$\sup_n \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} (1 + |\xi|^2) d\tilde{\nu}_{x,t}^n(\xi) dx dt < +\infty. \quad (98)$$

By (64), we know that (98) is satisfied $\tilde{\mathbb{P}}$ -almost-surely. By an argument of density, and by the tightness condition (98), it is sufficient to prove (97) in the case where

$$\psi(x, \xi) = \int_{-\infty}^{\xi} \varphi(x, \zeta) d\zeta, \quad \varphi \in \Gamma. \quad (99)$$

We have then the expression

$$\tilde{h}_\psi^n(t) = \langle \tilde{f}^n(t), \varphi \rangle = \langle \tilde{f}_\leftarrow^n(t), \varphi \rangle - \langle \tilde{m}^n, \psi \rangle([0, t]) - \tilde{\varepsilon}_\varphi^n(t),$$

by (75), which gives the desired convergence. Let us apply then (97) with $\psi = g_k \varphi_j$. The growth hypothesis (2) shows that

$$\left| \tilde{h}_{j,k}^n(t) \right|^2 \leq D_0 \|\varphi_j\|_{L^\infty(\mathbb{T}^N \times \mathbb{R})}^2 \int_{\mathbb{T}^N} \int_{\mathbb{R}} (1 + |\xi|^2) d\tilde{\nu}_{x,t}^n(\xi) dx.$$

By (98) and the dominated convergence theorem, it follows that, $\tilde{\mathbb{P}}$ -almost-surely, $\tilde{h}_{j,k}^n \rightarrow \tilde{h}_{j,k}$ and $\|\tilde{h}_j^n\|_{l^2(\mathbb{N}^*)}^2 \rightarrow \|\tilde{h}_j\|_{l^2(\mathbb{N}^*)}^2$ in $L^1(0, T)$. \square

Step 4. Conclusion of the martingale method Let us first prove that $M_\varphi^*(t)$ given in (83) is well-defined.

Lemma 39. *Item 4 in Theorem 29 is satisfied, i.e.: for all $\psi \in C_b(\mathbb{R})$, $(x, t) \mapsto \langle \psi, \tilde{\nu}_{x,t} \rangle$ belongs to $L^2_{\tilde{\mathbb{P}}}(\mathbb{T}^N \times [0, T] \times \tilde{\Omega})$.*

Proof of Lemma 39. Let $\varphi \in \Gamma$. By Corollary 32, the process $X_\varphi^*: t \mapsto \langle \tilde{f}^*(t), \varphi \rangle$ is càdlàg. Since $(X_\varphi^*(t))$ is adapted by definition of $(\tilde{\mathcal{F}}_t)$, it is an optional process [40, p. 172]. In particular, X_φ^* is progressively measurable [40, Proposition 4.8], hence $X_\varphi^* \in L^2_{\tilde{\mathbb{P}}}([0, T] \times \tilde{\Omega})$. This holds true when φ is any element of $C_c^d(\mathbb{T}^N \times \mathbb{R})$. Indeed, as $L^2_{\tilde{\mathbb{P}}}([0, T] \times \tilde{\Omega})$ is closed in $L^2([0, T] \times \tilde{\Omega})$ and Γ dense for the C^d -norm, it is sufficient to show that

$$X_{\varphi_n}^* \rightarrow X_\varphi^* \text{ in } L^2([0, T] \times \tilde{\Omega}) \quad (100)$$

if $\varphi_n \rightarrow \varphi$ and $\partial_\xi \varphi_n \rightarrow \partial_\xi \varphi$ for the C^0 -norm on $\mathbb{T}^N \times \mathbb{R}$. This is clearly the case since

$$|X_{\varphi_n}^*(t) - X_\varphi^*(t)| = \left| \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi(\varphi_n - \varphi) d\tilde{\nu}_{x,t}(\xi) dx \right| \leq \sup_{x \in \mathbb{T}^N, \xi \in \mathbb{R}} |\partial_\xi \varphi_n(x, \xi) - \partial_\xi \varphi(x, \xi)|.$$

We have used the identity

$$X_\varphi^* = \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi d\tilde{\nu}_{x,t}(\xi) dx,$$

which shows that

$$t \mapsto \int_{\mathbb{T}^N} \int_{\mathbb{R}} \phi(x, \xi) d\tilde{\nu}_{x,t}(\xi) dx \in L^2_{\tilde{\mathbb{P}}}([0, T] \times \tilde{\Omega}) \quad (101)$$

for every ϕ of the form $\phi(x, \xi) = \int_{-\infty}^{\xi} \varphi(x, \zeta) d\zeta$. By a new argument of approximation and truncation, we see that (101) is satisfied for every Carathéodory function $\phi(x, \xi)$. In particular (take $\phi(x, \xi) = \theta(x)\psi(\xi)$ with $\theta \in L^2(\mathbb{T}^N)$), for all $\psi \in C_b(\mathbb{R})$, $(x, t) \mapsto \langle \psi, \tilde{\nu}_{x,t} \rangle$ belongs to $L^2_{\tilde{\mathbb{P}}}([0, T] \times \tilde{\Omega}; L^2(\mathbb{T}^N))$ (here we use the fact that being weakly or strongly $\tilde{\mathbb{P}}$ -measurable is the same thing, cf. Section 2.1.1). By (4), we obtain the result. \square

We can apply now Proposition 25, thanks to Proposition 37, to obtain $\tilde{M}_\varphi(t) = M_\varphi^*(t)$, with $M_\varphi^*(t)$ defined by (83), for every $\varphi \in \Gamma$.

Step 5. Equation Let $\varphi \in \Gamma$. From (68), we deduce

$$\langle \tilde{f}^n(t), \varphi \rangle + \tilde{A}_\varphi^n(t) = \langle f^n(0), \varphi \rangle + \tilde{J}_\varphi^n(t) + \tilde{\varepsilon}_\varphi^n(t), \quad (102)$$

$\tilde{\mathbb{P}}$ -almost-surely, where $\tilde{A}_\varphi^n(t) = \langle \tilde{m}^n, \partial_\xi \varphi \rangle([0, t])$. Since

$$\tilde{A}_\varphi^n(t) \rightarrow \langle m^*, \partial_\xi \varphi \rangle([0, t]),$$

$\tilde{\mathbb{P}}$ -almost-surely, for all t outside a countable set N_φ^* (cf. Remark 31), we can pass to the limit in (102) to obtain

$$\langle f^*(t), \varphi \rangle + \langle m^*, \partial_\xi \varphi \rangle([0, t]) = \langle f(0), \varphi \rangle + \tilde{J}_\varphi(t), \quad (103)$$

$\tilde{\mathbb{P}}$ -almost-surely, for all $t \in [0, T] \setminus N_\varphi^*$, where $\tilde{J}_\varphi(t)$ is given by

$$\begin{aligned} \tilde{J}_\varphi: t \mapsto & \int_0^t \langle f^*(s), a(\xi) \cdot \nabla \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\tilde{\nu}_{x,s}(\xi) dx d\tilde{\beta}_k(s) \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{G}^2(x, \xi) d\tilde{\nu}_{x,s}(\xi) dx ds. \end{aligned}$$

Since $t \mapsto \langle f^*(t), \varphi \rangle$ is càdlàg by Corollary 32, $\tilde{\mathbb{P}}$ -almost-surely, (103) is satisfied for every $t \in [0, T]$. By a density argument (see the properties of Γ in Section 4.5.1), the result holds true for any $\varphi \in C_c^d(\mathbb{T}^N \times \mathbb{R})$. This proves Item 8 of Theorem 29. This formulation at fixed t also implies the weak formulation (73). \square

4.6 Pathwise solutions and almost-sure convergence

If f_0 is at equilibrium in Theorem 29, then we have seen in Theorem 20 that (1) admits a unique solution for a given initial datum. We can use this uniqueness result to obtain existence of pathwise solution and almost-sure convergence of the sequence of approximate solutions in that case.

Theorem 40 (Path-wise solution). *Suppose that there exists a sequence of approximate generalized solutions (f^n) to (1) with initial datum f_0^n satisfying (62), (65) and the tightness condition (66) and such that (f_0^n) converges to the equilibrium function $\mathbf{f}_0(\xi) = \mathbf{1}_{u_0 > \xi}$ in $L^\infty(\mathbb{T}^N \times \mathbb{R})$ -weak-*, where $u_0 \in L^\infty(\mathbb{T}^N)$. We have then*

1. *there exists a unique solution $u \in L^1(\mathbb{T}^N \times [0, T] \times \Omega)$ to (1) with initial datum u_0 ;*
2. *let*

$$u^n(x, t) = \int_{\mathbb{R}} \xi d\nu_{x,t}^n(\xi) = \int_{\mathbb{R}} (f^n(x, t, \xi) - \mathbf{1}_{0 > \xi}) d\xi.$$

Then, for all $p \in [1, \infty[$, (u^n) is almost-surely converging to u in $L^p(\mathbb{T}^N \times (0, T))$.

Proof of Theorem 40. We use the Gyöngy-Krylov argument, [23, Lemma 1.1] (the basis of the Gyöngy-Krylov argument is this simple fact: if a couple (X_n, Y_n) of random variables converges in law to a random variable written (Z, Z) , i.e. concentrated on the diagonal, then $X_n - Y_n$ converges to 0 in probability). Let us go back to Section 4.5.1. We introduce the random variable

$$Z^{n,q} = (\nu^n, \{f_{\leftarrow}^n\}, \{M^n\}, \{\varepsilon^n\}, \mu^n, \|m^n\|, \nu^q, \{f_{\leftarrow}^q\}, \{M^q\}, \{\varepsilon^q\}, \mu^q, \|m^q\|, W)$$

in the state space \mathcal{Z} equal to

$$\mathcal{Y}^1 \times \mathcal{E} \times \mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \times \mathbb{R}_+ \times \mathcal{Y}^1 \times \mathcal{E} \times \mathcal{P}^1(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \times \mathbb{R}_+ \times \mathcal{X}_W.$$

We repeat the arguments used in Section 4.5 to show that $Z^{n,q}$ is tight in \mathcal{Z} and that there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a new random variable $\tilde{Z}^{n,q}$ with the same law as $Z^{n,q}$, such that a subsequence $(\tilde{Z}^{n_i, q_i})_i$ is converging $\tilde{\mathbb{P}}$ -almost-surely in \mathcal{Z} to a random variable \tilde{Z} . Let $\tilde{\nu}$ be the first component of \tilde{Z} and $\tilde{\check{\nu}}$ be the seventh component of \tilde{Z} . Repeating the steps 1-5 in Section 4.5.2 we obtain two generalized solutions

$$\tilde{f}(x, t, \xi) = \tilde{\nu}_{(x,t)}(\xi, +\infty), \quad \check{f}(x, t, \xi) = \tilde{\check{\nu}}_{(x,t)}(\xi, +\infty),$$

to Equation (1) with probabilistic data $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t), \tilde{W})$, where $(\tilde{\mathcal{F}}_t)$ is the completion of the filtration generated by the five-uplet $(\{\tilde{f}\}, \{\tilde{M}\}, \{\check{f}\}, \{\check{M}\}, \tilde{W})$:

$$\tilde{\mathcal{F}}_t = \sigma((\{\tilde{f}\}, \{\tilde{M}\}, \{\check{f}\}, \{\check{M}\}, \tilde{W})^{-1} (\mathcal{D}_t(\check{E}_1) \times \mathcal{D}_t(\check{E}_2)) \cup \{N \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(N) = 0\}),$$

for $t \in [0, T]$, with

$$\check{E}_1 := \mathbb{R}^\infty \times \mathbb{R}^\infty, \quad \check{E}_2 := \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathfrak{U}.$$

Note that $\mathcal{D}_t(\check{E}_1) \times \mathcal{D}_t(\check{E}_2) \neq \mathcal{D}_t(\check{E}_1 \times \check{E}_2)$ since the natural topologies of $D(\mathbb{R}_+; \check{E}_1) \times D(\mathbb{R}_+; \check{E}_2)$ and $D(\mathbb{R}_+; \check{E}_1 \times \check{E}_2)$ are different (the topology of the former is the product topology of the Skorokhod topologies on each space: this authorizes two changes of times, one for each coordinate; for the Skorokhod topology on $D(\mathbb{R}_+; \check{E}_1 \times \check{E}_2)$, only one change of time is admissible). The solutions \tilde{f} and \check{f} have the same initial condition f_0 , which is an equilibrium function \mathbf{f}_0 . By Theorem 20, we have

$$\tilde{f} = \check{f} = \mathbf{f}, \tag{104}$$

where \mathbf{f} is the equilibrium function $\mathbf{1}_{\tilde{u} > \xi}$, where

$$\tilde{u}(x, t) := \int_{\mathbb{R}} \xi d\tilde{\nu}_{(x,t)}(\xi).$$

A first consequence of (104) is that $\tilde{\nu} = \check{\nu}$. By the Gyöngy-Krylov argument hence, we obtain that (ν^n) is converging in probability in \mathcal{Y}^1 . Extracting an additional subsequence if necessary, we can assume that (ν^n) is converging almost-surely in \mathcal{Y}^1 . Note that Theorem 20 does not give the uniqueness of the random measure m , but it clearly gives the uniqueness of $\partial_\xi m$ (use simply Equation (16)). Therefore we can use the Gyöngy-Krylov argument (and extraction of subsequence with a diagonal argument) to obtain the almost-sure convergence of $m^n(\partial_\xi \varphi)([0, t])$ for all $\varphi \in \Gamma$ and $t \notin N^*$, where N^* is the jump set defined in (94). By the arguments of Section 4.5.2 (except for Step 1., since the filtration (\mathcal{F}_t) is already known here), it follows that $f(t, x, \xi) := \nu_{x,t}(\xi, +\infty)$ is a generalized solution to (1). We use the second identity in (104) now. It says equivalently

that $\tilde{\mathbb{P}}$ -almost-surely, for a.e. (x, t) , $\tilde{\nu}_{(x,t)} = \delta_{\tilde{u}(x,t)}$. The fact that $\tilde{\nu}$ is a Dirac mass can be characterized in terms of equality in the Jensen Inequality:

$$\tilde{\mathbb{E}} \iint_{\mathbb{T}^N \times (0,T)} \Phi \left(\int_{\mathbb{R}} \xi d\tilde{\nu}_{(x,t)}(\xi) \right) = \tilde{\mathbb{E}} \iint_{\mathbb{T}^N \times (0,T)} \int_{\mathbb{R}} \Phi(\xi) d\tilde{\nu}_{(x,t)}(\xi), \quad (105)$$

where Φ is a strictly convex, polynomially bounded function, like $\Phi(\xi) = \xi^2$ for example. The identity (105) depends on $\text{Law}(\tilde{\nu}) = \text{Law}(\nu)$ uniquely. Therefore ν also is almost-surely a Dirac mass: \mathbb{P} -almost-surely, for a.e. (x, t) , $\nu_{(x,t)} = \delta_{u(x,t)}$, where

$$u(x, t) := \int_{\mathbb{R}} \xi d\nu_{(x,t)}(\xi).$$

By Proposition 13, u is a solution to (1): it is the unique solution by Theorem 20. Using Lemma 10, we obtain the second point of Theorem 40. \square

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