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# Anomalies in local Weyl laws and applications to random topology at critical dimension

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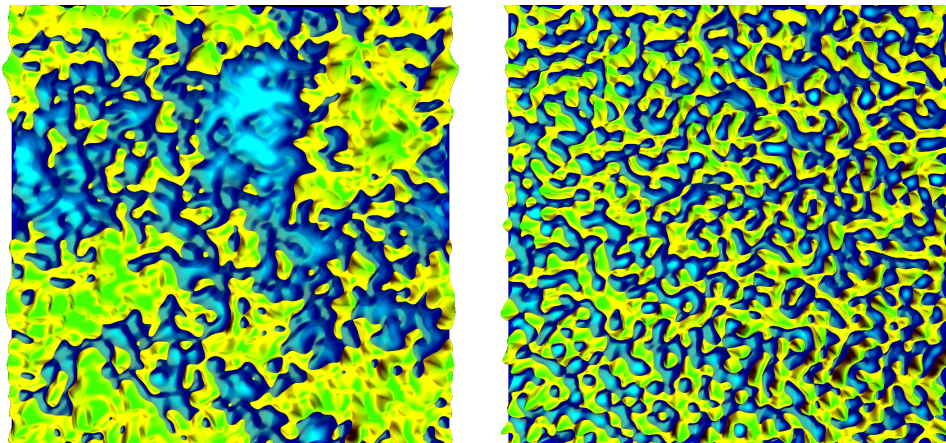
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## Abstract

Let  $\mathcal{M}$  be a smooth manifold of positive dimension  $n$  equipped with a smooth density  $d\mu_{\mathcal{M}}$ . Let  $A$  be a polyhomogeneous elliptic pseudo-differential operator of positive order  $m$  on  $\mathcal{M}$  which is symmetric for the  $L^2$  scalar product defined by  $d\mu_{\mathcal{M}}$ . For each  $L > 0$ , the space  $U_L = \bigoplus_{\lambda \leq L} \text{Ker}(A - \lambda Id)$  is a finite dimensional subspace of  $C^\infty(\mathcal{M})$ . Let  $\Pi_L$  be the spectral projector onto  $U_L$ . Given  $s \in \mathbb{R}$ , we compute the asymptotics of the integral kernel  $K_L$  of  $\Pi_L A^{-s}$  in the cases where  $n > ms$  and  $n = ms$  respectively. Next, assuming that  $\mathcal{M}$  is closed, let  $(e_n)_{n \in \mathbb{N}}$  and  $(\lambda_n)_{n \in \mathbb{N}}$  be the sequence of  $L^2$  normalized eigenfunctions and eigenvalues of  $A$  where the latter sequence organized in increasing order. Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent centered gaussians of variance 1. We fix a parameter  $s \in \mathbb{R}$  such that  $n \geq ms$  and consider the family  $(\phi_L)_{L > 0}$  of smooth random fields on  $\mathcal{M}$  defined by

$$\phi_L = \sum_{0 < \lambda_j \leq L} \lambda_j^{-\frac{s}{2}} \xi_j e_j$$

for each  $L > 0$ . It turns out that the covariance function of  $\phi_L$  is  $K_L$ . Using this information, we apply the derived asymptotics to study the zero set of  $\phi_L$ . If  $n > ms$  then the number of components of the zero set of  $\phi_L$  concentrates around  $aL^{\frac{n}{m}}$  for some positive constant  $a$ . On the other hand, if  $n = ms$ , each Betti number of the zero set has an expectation bounded by  $C \ln \left( L^{\frac{1}{m}} \right)^{-\frac{1}{2}} L^{\frac{n}{m}}$  where  $C$  is an explicit constant. When  $\mathcal{M}$  is a closed surface with a Riemmanian metric,  $A$  is the Laplacian and  $d\mu_{\mathcal{M}}$  is the Riemmanian volume,  $C$  equals  $\frac{1}{4\pi^2} \sqrt{\frac{3}{2}} \text{Vol}(\mathcal{M})$ .



*On the left (resp. right), an instance of the field  $\phi_{1000} = \sum_{0 < \lambda_j \leq 1000} \xi_j \lambda_j^{-\frac{s}{2}} e_j$  for the Laplacian on the flat torus with  $s = -1$  resp.  $s = 0$ . The green regions correspond to positive values of the field and the blue ones to negative values.*

## 1 Introduction

Over the past fifteen years, in the context of random topology, a lot of effort has been put in the study of the nodal set of certain smooth gaussian fields whose covariances turn out to be Schwartz kernels of spectral projectors for the Laplacian or other elliptic operators (see [2], [14], [22], [6], [13], [7], [17], [9], [15] and [20]). Several parallels have been drawn between these objects and the classical lattice models of statistical mechanics such as percolation and the discrete Gaussian Free Field (see [3] and [18]). These fields exhibit universal local behavior at a scale given by the smallest wavelength associated to the spectral window and decorrelate at large distances. This phenomenon ultimately stems from Lars Hörmander's estimate of the spectral function of an elliptic operator in [10].

In a recent paper, [18], we studied a random field on a surface defined also by a frequency cut-off but with weighted frequencies tuned so that the field would converge in distribution to the Gaussian Free Field. In this case, the covariance has a logarithmic singularity smoothed out at the scale of the smallest wavelength and does not vanish at large distances. In the present paper, we consider general polynomially weighted kernels in any dimension. We study their asymptotic behavior and show how the different behaviors observed affect the properties the nodal set. More precisely, let  $\mathcal{M}$  be a smooth closed manifold of dimension  $n$ , equipped with a smooth density  $d\mu_{\mathcal{M}}$  and a polyhomogeneous elliptic pseudo-differential operator  $A$  of positive order  $m$  self-adjoint with respect to the  $L^2 - d\mu_{\mathcal{M}}$  scalar product. Let  $(e_j)$  and  $(\lambda_j)$  be respectively the sequence of eigenfunctions and corresponding eigenvalues of  $A$ , where the latter are arranged in increasing order. In addition, let  $(\xi_j)$  be a sequence of independent, identically distributed centered gaussians

of variance 1. For each  $s \in \mathbb{R}$  such that  $0 \leq s \leq \frac{n}{m}$ , and each  $L \in \mathbb{R}$  such that  $L > 0$ , let

$$\phi_L = \sum_{0 < \lambda_j \leq L} \lambda_j^{-\frac{s}{2}} \xi_j e_j.$$

As explained later, for  $L$  large enough,  $\phi_L$  is almost surely regular on its zero set  $Z_L$  which is therefore a smooth closed hypersurface of  $\mathcal{M}$ . Let  $N_L$  be the number of connected components of  $Z_L$ . The number  $N_L$  behaves quite differently depending on the values of  $\frac{n}{m} - s$ . We will prove the following two results.

**Proposition 1.1.** *If  $s < \frac{n}{m}$  then there exists a constant  $a > 0$  such that*

$$\mathbb{E}[L^{-\frac{n}{m}} |N_L - a|] \rightarrow 0.$$

**Proposition 1.2.** *Suppose that  $s = \frac{n}{m}$ . Under some admissibility condition on the operator  $A$ , there exists an explicit constant  $C_0$  such that*

$$\limsup_{L \rightarrow +\infty} \frac{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}}{L^{\frac{n}{m}}} \mathbb{E}[N_L] \leq C_0.$$

If  $\mathcal{M}$  is a closed surface equipped with a riemannian metric  $g$ ,  $A$  is the Laplacian,  $d\mu_{\mathcal{M}}$  is the riemannian volume and  $s = 1$ , then  $A$  is admissible. Moreover, in this case,  $C_0 = \frac{1}{4\pi^2} \sqrt{\frac{3}{2}} \text{Vol}_g(\mathcal{M})$ .

These propositions are restated in full detail and proved in section 3 (see Corollaries 3.1 and 3.5 as well as equation (4)). The admissibility condition is defined in terms of the principal symbol of  $A$  (see Definition 2.5) and is generic in a sense that we will explain later (see Proposition 2.7). The reader familiar with the random topology literature will find these results reminiscent of previous work by Fedor Nazarov and Mikhail Sodin (see [15]) as well as Damien Gayet and Jean-Yves Welschinger (see [6]). The object of this article is to provide the analytical tools to apply their work to this more general setting. The main novelty is really the study of the asymptotics of the covariance function of a broad family of fields. These asymptotics build on the seminal result by Lars Hörmander (see [10]). This result gives an approximation of the Schwartz kernel  $E_L$  of the  $L^2$  orthogonal projector  $\Pi_L$  onto the space spanned by the functions  $e_j$  such that  $\lambda_j \leq L$  by an oscillatory integral. More precisely, let  $\sigma_A$  be the principal symbol of  $A$ . We have,

**Theorem 1.3.** *Around each  $x_0 \in \mathcal{M}$  there are local coordinates such that, in these coordinates, uniformly for each  $x, y \in \mathbb{R}^n$  with  $|x|, |y| \leq 1$  and for  $L \geq 1$ ,*

$$E_L\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbf{1}_{[\sigma_A(0, \xi) \leq 1]} e^{i\langle x-y, \xi \rangle} d\xi L^{\frac{n}{m}} + O\left(L^{\frac{n-1}{m}}\right).$$

Here and below, “in these coordinates” means that we implicitly compose  $E_L$  and  $\sigma_A$  by the coresponding charts. It turns out that if  $s = 0$ ,  $E_L$  is equal to the covariance function of the field  $\phi_L$ . In this paper we give a similar approximation for the covariance of  $\phi_L$  for any  $s \in [0, \frac{n}{m}]$ . This covariance is the Schwartz kernel  $K_L$  of  $\Pi_L A^{-s}$ .

**Theorem 1.4.** *Suppose that  $0 \leq s < \frac{n}{m}$ . Around each  $x_0 \in \mathcal{M}$  there are local coordinates such that, in these coordinates, uniformly for each  $x, y \in \mathbb{R}^n$  with  $|x|, |y| \leq 1$  and for  $L \geq 1$ ,*

$$K_L\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbf{1}_{[\sigma_A(0, \xi) \leq 1]} |\xi|^{-s} e^{i\langle x-y, \xi \rangle} d\xi L^{\frac{n}{m}-s} + O\left(\ln(L)^\epsilon L^{\frac{n-1}{m}}\right)$$

where  $\epsilon = 1$  if  $s = \frac{n-1}{2}$  and 0 otherwise.

When  $s = \frac{n}{m}$  the situation is quite different.

**Theorem 1.5.** *Suppose  $s = \frac{n}{m}$ . Suppose further that the symbol  $\sigma_A$  is admissible (see Definition 2.5). Then, around each  $x_0 \in \mathcal{M}$  there exist local coordinates such that, in these coordinates, uniformly for each  $x, y \in \mathbb{R}^n$ , with  $|x|, |y| \leq 1$  and  $L \geq 1$ ,*

$$K_L(x, y) = \frac{1}{(2\pi)^n} \delta(y) \left[ \ln\left(\sqrt{L}\right) - \ln_+\left(\sqrt{L}|x-y|\right) \right] + O(1).$$

Here,  $\delta(y) = \int_{\mathbb{R}^n} |\xi|^{1-n} \mathbf{1}_{[\sigma_A(y, \xi) \leq 1]} d\xi$  and  $\ln_+(t) = \max(\ln(t), 0)$ .

The covariance is therefore larger than what could be expected from the  $s < \frac{n}{m}$  case. The logarithmic factor later shows up in the topological results stated above. Here again, our results are stated below in full generality and precision (see Theorems 2.4 and 2.6).

The paper is organized as follows. In section 2 we introduce the central objects of our study and state precisely the corresponding results. In section 3 we present some applications to random topology. In section 4 we prove the results stated in section 2. More precisely, in section 5.1 we prove Theorem 2.4, in section 5.2 we prove the decay of certain oscillatory integrals using the results of section 6. Then, in section 5.3 we apply these results to the proof of Theorem 2.6. Lastly, in section 6, we show Propositions 2.7 and 6.2. Finally, in the appendix, we present a proof of a classical result (Theorem 2.2) on which we rely heavily.

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## 2 Kernel asymptotics

In this section we state our main analytical results in full generality. In order to do so we need to introduce some notation. Let  $n \geq 1$  and let  $dx$  denote the standard Lebesgue

measure on  $\mathbb{R}^n$ . We consider a polyhomogeneous elliptic pseudo-differential operator of positive order  $m$  acting on functions on a smooth manifold  $\mathcal{M}$  equipped with a smooth positive density  $d\mu_{\mathcal{M}}$ . We assume that  $A$  is symmetric for the  $L^2$  scalar product on  $(\mathcal{M}, d\mu_{\mathcal{M}})$ . In other words, that for any local coordinates on  $\mathcal{M}$ , there exists a sequence  $(\sigma_j)_{j \in \mathbb{N}}$  of complex valued smooth functions  $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$  such that

1. for each  $j \in \mathbb{N}$ ,  $\sigma_j$  is homogeneous of degree  $m - j$  in the second variable.
2. if the density  $d\mu_{\mathcal{M}}$  agrees with the Lebesgue measure in these coordinates, the function  $\sigma_A = \sigma_0$  is real-valued and positive.
3. the symbol of  $A$  read in these coordinates satisfies the asymptotic expansion  $\sum_{j \in \mathbb{N}} \sigma_j$  in the semi-classical sense (see for instance Proposition 18.1.3 of [12]).

Since  $A$  is elliptic symmetric, by Gårding's inequality (see Theorem 18.1.14 of [12]), it is bounded from below. This implies that its spectrum forms an increasing sequence of real numbers  $(\lambda_k)_{k \in \mathbb{N}}$  diverging to  $+\infty$ . The associated eigenfunctions  $e_k \in C^\infty(\mathcal{M})$ , when normalized adequately, form a Hilbert basis of  $L^2(\mathcal{M}, d\mu_{\mathcal{M}})$ . We introduce a parameter  $L > 0$  and let  $U_L$  be the vector space spanned by the eigenfunctions  $e_k$  whose corresponding eigenvalue is no greater than  $L$ . Let  $\Pi_L$  be the spectral projector on  $U_L$  and let  $E_L$  be its integral kernel. Since  $U_L$  is finite dimensional,  $E_L$  is smooth. The asymptotics of  $E_L$  as  $L \rightarrow +\infty$  are well known. Before giving more details, we introduce the notion of admissible phase function. This definition follows Hörmander (see [10]).

**Definition 2.1.** *Given an open subset  $U \subset \mathbb{R}^n$ , we will say that a function  $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$  is an admissible phase function if it satisfies the following conditions.*

1. *The function  $\psi$  is a symbol of order one in its third variable.*
2. *For each compact subset  $K \subset U \times U$  there exists  $C > 0$  such that for all  $(x, y, \xi) \in K \times \mathbb{R}^n$ ,*

$$\begin{aligned} |\xi|^2 &\leq C(|\partial_x \psi(x, y, \xi)|^2 + |\xi|^2 |\partial_\xi \psi(x, y, \xi)|^2); \\ |\xi|^2 &\leq C(|\partial_y \psi(x, y, \xi)|^2 + |\xi|^2 |\partial_\xi \psi(x, y, \xi)|^2). \end{aligned}$$

3. *For each  $(x, y, \xi) \in U \times U \times \mathbb{R}^n$ ,  $\langle x - y, \xi \rangle = 0$  implies that  $\psi(x, y, \xi) = 0$ .*
4. *For each  $x \in U$  and  $\xi \in \mathbb{R}^n$ ,  $\partial_x \psi(x, y, \xi)|_{y=x} = \xi$ .*
5. *There exists  $\psi_0 \in C^\infty(U \times U \times \mathbb{R}^n \setminus \{0\})$  satisfying all of the above properties and 1-homogeneous in  $\xi$  such that*

$$t^{-1} \psi(x, y, t\xi) \xrightarrow[t \rightarrow +\infty]{} \psi_0(x, y, \xi)$$

*where the convergence takes place in  $S^1(U \times U \times \mathbb{R}^n \setminus \{0\})$ .*

The following theorem describes the asymptotics of the Schwartz kernel of the spectral projector associated to  $A$  in terms of an oscillatory integral with an admissible phase.

**Theorem 2.2.** *Let  $(\mathcal{M}, d\mu_{\mathcal{M}})$  be a manifold equipped with a smooth positive density and  $A$  be an operator on  $\mathcal{M}$  as described above. Let  $P$  be a differential operator of order  $d$  acting on  $C^\infty(\mathcal{M} \times \mathcal{M})$ . Fix a point in  $\mathcal{M}$  and consider local coordinates  $(x_1, \dots, x_n)$  around it. Suppose further that the density  $d\mu_{\mathcal{M}}$  agrees with the Lebesgue measure in these coordinates. Let  $\sigma_A$  (resp.  $\sigma_P$ ) be the principal symbol of  $A$  (resp.  $P$ ) in these coordinates. Then, there exists an open neighborhood  $U$  of  $0 \in \mathbb{R}^n$ , an admissible phase function  $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$  and a constant  $C > 0$ , such that, in these coordinates, for each  $x, y \in U$  and  $L > 0$ ,*

$$\left| PE_L(x, y) - \frac{1}{(2\pi)^n} \int_{\sigma_A(x, \xi) \leq L} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) d\xi \right| \leq C(1 + L)^{\frac{n+d-1}{m}}.$$

Moreover, for each neighborhood  $W \subset U \times U$  of the diagonal there exists  $C > 0$  such that in local coordinates, for each  $(x, y) \in (U \times U) \setminus W$  and  $L > 0$ ,

$$\left| PE_L(x, y) \right| \leq C(1 + L)^{\frac{n+d-1}{m}}.$$

Here  $\partial_{x, y} \psi$  denotes the partial derivative of  $\psi$  with respect to the couple  $(x, y)$ . Note that this result is coordinate dependent since the notion of admissible phase function is not invariant. The phase  $\psi$  is constructed as the solution of the following equation, where  $\sigma \in S^1$  is such that  $\sigma^{\frac{1}{m}} - \sigma \in S^0$

$$\sigma(x, \partial_x \psi(x, y, \xi)) = \sigma(y, \xi) \tag{1}$$

with the boundary conditions dictated by the admissibility condition. The case where  $P = Id$  was proved by Lars Hörmander in [10]. The case where  $x = y$  was treated in [19] with some restrictions on  $P$ . Finally, Gayet and Welschinger extended this result to a general  $P$  (see Theorem 2.3 of [6]). While in their statement,  $x = y$ , their proof yields the off-diagonal case with only minor modifications. One recent result closely related to this theorem is Canzani and Hanin's asymptotics for the monochromatic spectral projector under some dynamical assumption on the geodesic flow (see [4] and [5]). For the convenience of the reader, we provide a proof of the full result relying on the wave kernel asymptotics provided in [10].

In this paper, we generalize this theorem by introducing a smooth function  $f \in C^\infty(\mathbb{R})$  and studying the asymptotics of the integral kernel of  $\Pi_L f(A)$ . Here  $f(A)$  is defined by functional calculus so that the kernel of  $\Pi_L f(A)$  is

$$K_L = \sum_{\lambda_k \leq L} f(\lambda_k) e_k \boxtimes \bar{e}_k.$$

This is again a smooth function. However, unless  $f$  has sufficient decay at  $+\infty$ , it will not converge as  $L \rightarrow +\infty$ . As explained in the introduction, we focus on the case where  $f$  decays polynomially with a small exponent. However, a partial result holds for a general  $f$  which might be useful for further generalizations. In section 4, we prove the following proposition.

**Proposition 2.3.** *We use the same notations as in Theorem 2.2. Let  $f \in C^\infty(\mathbb{R})$  with support in  $]0, +\infty[$ . Let  $K_L$  be the kernel of  $\Pi_L f(A)$ . Then, in local coordinates, uniformly for each  $x, y \in U$ , for each  $L > 0$ ,*

$$PK_L(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y, \xi) \leq L} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) f(\sigma_A(y, \xi)) d\xi \\ + O\left(f(L)L^{\frac{n+d-1}{m}}\right) + O\left(\int_0^L f'(\lambda) \lambda^{\frac{n+d-1}{m}} d\lambda\right).$$

*In addition, uniformly for any  $(x, y) \in (U \times U) \setminus W$ , for each  $L > 0$ ,*

$$PK_L(x, y) = O\left(f(L)L^{\frac{n+d-1}{m}}\right) + O\left(\int_0^L f'(\lambda) \lambda^{\frac{n+d-1}{m}} d\lambda\right).$$

*Finally, the constants implied by the  $O$ 's do not depend on  $f$ .*

Note that the restriction on the support of  $f$  is purely cosmetic since the spectrum of  $A$  is bounded from below. Of special interest to us is the case where  $f$  is of the form  $f(t) = \chi(t)t^z$  where  $z \in \mathbb{C}$  and  $\chi$  is some smooth function with support in  $]0, +\infty[$  equal to 1 for  $t$  large enough. In that case, Proposition 2.3 yields Theorem 2.4 as well as 2.6 below. The first deals with the case where  $n + d + m\Re(z) > 0$  and the second with the case where  $n + d + mz = 0$ .

**Theorem 2.4.** *We use the same notations as in Theorem 2.2 and fix  $z \in \mathbb{C}$ . Let  $f \in C^\infty(\mathbb{R})$  be supported in  $]0, +\infty[$  such that  $f(t) = t^z$  for  $t$  large enough. Let  $K_L$  be the Schwartz kernel of  $\Pi_L f(A)$ . Suppose that  $n + d + m\Re(z) > 0$ . For each  $x, y \in U$  and  $L \geq 1$ , let*

$$R_L^1(x, y) = L^{-z - \frac{n+d}{m}} \left[ PK_L(x, y) - \frac{1}{(2\pi)^n} \int_{\sigma_A(0, \xi) \leq 1} e^{i(\xi, x-y)L^{\frac{1}{m}}} \sigma_A(0, \xi)^z \sigma_P(0, 0, (\xi, -\xi)) d\xi \right].$$

*Then, there exists an open neighborhood  $V$  of  $0 \in U$  such that, uniformly for any  $L > 0$  and  $(x, y) \in V \times V$ ,  $R_L^1\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right)$  is  $O(L^{-\frac{1}{m}})$  if  $n + d + mz \neq 1$  and  $O(\ln(L)L^{-\frac{1}{m}})$  otherwise. In addition, uniformly for  $L > 0$  and  $(x, y) \in V \times V \setminus W$ ,  $PK_L(x, y)$  is  $O(L^{\frac{n+d-1}{m} + \Re(z)})$  if  $n + d + m\Re(z) \neq 1$  and  $O(\ln(L)L^{-\frac{1}{m}})$  otherwise.*

Thus, the  $A^z$  factor in  $\Pi_L A^z$  translates to a  $\sigma_A^z$  factor in the oscillatory integral approximating  $K_L$ . Note also that the second part of the theorem shows that  $K_L(x, y)$  becomes negligible at fixed distances compared to the case where  $|x - y| \leq L^{-\frac{1}{m}}$ . We prove Theorem 2.4 in section 5.1.

Before stating Theorem 2.6, we must introduce some more terminology. One key ingredient of the proof will be the decay of certain oscillatory integrals depending on the level



sets of  $\sigma_A$ . To observe this behavior we must impose certain condition on  $\sigma_A$ . This is the object of Definition 2.5. We will see, in Proposition 2.7 below, that it is almost always satisfied.

**Definition 2.5.** *We say that a positive  $m$ -homogeneous symbol  $\sigma$  on  $\mathcal{M}$  is admissible there exists  $k_0 \geq 2$  such that*

$$\forall (x, \xi) \in T'\mathcal{M} \exists k \in \{2, \dots, k_0\}, \sigma(x, \xi)^{k-1} \partial_\xi^k \sigma(x, \xi) \neq \frac{m(m-1) \dots (m-k+1)}{m^k} (\partial_\xi \sigma(x, \xi))^{\otimes k}. \quad (2)$$

Note that  $\partial_\xi^k \sigma$  is well defined because coordinate changes act linearly on the fibers of  $T^*\mathcal{M}$ . Next, since  $\sigma_A$  is homogeneous and positive, it defines a sphere bundle  $S^*U$  on the chart  $U$  of Theorem 2.2 by  $S_x^* = S_x^*U = \{\xi \in \mathbb{R}^n \mid \sigma_A(x, \xi) = 1\}$ . We denote by  $d\nu$  the smooth density on the bundle  $S^*U$  defined by the following equation

$$\forall x \in U, \forall u \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} u(\xi) d\xi = \int_0^{+\infty} \int_{S_x^*} u(t\xi) d_x \nu(\xi) t^{n-1} dt. \quad (3)$$

The principal symbol  $\sigma_A$  can be seen as a smooth function of the complement of the zero section in  $T^*\mathcal{M}$  (see Theorem 18.1.17 of [12]) which we henceforth denote by  $T'\mathcal{M}$ . Therefore, the  $S^*U$ 's given by different coordinates piece together to form a subset  $S^*\mathcal{M}$  of  $T'\mathcal{M}$ .

**Theorem 2.6.** *We use the same notations as in Theorem 2.4. Suppose that  $n+d+mz = 0$  and that either  $n = 1$  or  $\sigma_A$  is an admissible symbol. For each  $x, y \in U$  and  $L > 0$  let*

$$Y_P(x, y) = \int_{S_y^*} \sigma_P(x, y, \partial_{x,y}(\partial_\xi \psi(x, y, 0)\xi)) d_y \nu(\xi)$$

and

$$R_L^2(x, y) = PK_L(x, y) - \frac{1}{(2\pi)^n} Y_P(x, y) \left[ \ln \left( L^{\frac{1}{m}} \right) - \ln_+ \left( L^{\frac{1}{m}} |x - y| \right) \right].$$

Then, there exists  $V \subset U$  an open neighborhood of 0 such that, uniformly for  $L > 0$  and  $(x, y) \in U \times U$ ,  $R_L^2(x, y) = O(1)$ . In addition, there exists  $Q \in C^\infty(V \times V)$  and  $\alpha > 0$  depending only on  $\sigma_A$  such that, for each  $\kappa \geq 1$ , for  $L > 0$  large enough and  $(x, y) \in V \times V$  such that  $|x - y| \geq \kappa L^{-\frac{1}{m}}$ ,

$$PK_L(x, y) = -\frac{1}{(2\pi)^n} Y_P(x, y) \ln(|x - y|) + Q(x, y) + O(\kappa^{-\alpha}).$$

The  $\alpha$  appearing in the last equation is actually equal to 1 if  $n = 1$  or  $\frac{1}{k_0}$  where  $k_0 \in \mathbb{N}$ ,  $k_0 \geq 1$  is the integer appearing in the admissibility condition of  $\sigma_A$  (see Definition 2.5) if  $\sigma_A$  is indeed admissible. Note that, contrary to case where  $n + d + m\Re(z) > 0$ , the kernel  $K_L$  does not localize but rather spikes logarithmically around the diagonal. The singularity of the logarithm is then smoothed at scale  $L^{-\frac{1}{m}}$ . This theorem is a generalization of Theorem 3 of [18] which dealt only with the case where  $A$  was the Laplacian and  $\mathcal{M}$  was

two-dimensional. The main novelty of Theorem 2.6 is the assumption on the symbol and which was implicitly satisfied in the case of the Laplacian. We prove Theorem 2.6 in section 5.3. Finally, we prove that admissible symbols are generic in the following sense. Let  $m$  be a positive real number and let  $S_h^m(\mathcal{M})$  be the space of smooth functions on  $T^*\mathcal{M}$  that are homogeneous of degree  $m$  along the fibers. We endow this space with the restriction of the Whitney topology on  $C^\infty(T^*\mathcal{M})$  (see Definition 3.1 of [8]). Then,

**Proposition 2.7.** *Almost all symbols are admissible with  $k_0 = 5$  when  $n = 2$ ,  $k_0 = 3$  when  $n = 3$  or  $4$  and  $k_0 = 2$  when  $n \geq 5$  (see Definition 2.5). More precisely, for any  $m > 0$ , the set of admissible symbols in  $S_h^m(\mathcal{M})$  for the aforementioned  $k_0$  is a residual subset of  $S_h^m(\mathcal{M})$  for the restricted Whitney topology. Moreover, if  $\mathcal{M}$  is compact then it is open.*

We prove this proposition in section 6.

### 3 Applications to random topology

In this section, we use the theorems stated in 2 to apply or adapt previous results in random topology.

#### 3.1 Setting and results

We consider a smooth compact manifold  $\mathcal{M}$  equipped with a smooth positive density  $d\mu_{\mathcal{M}}$  and an operator  $A$  as before. We use the same notations as in the previous section. Moreover, we fix  $s \geq 0$  such that  $n \geq ms$  as well as a sequence of independent real centered gaussians of variance one  $(\xi_k)_{k \in \mathbb{N}}$ . In the case that  $n = ms$ , we assume that  $\sigma_A$  is admissible. We then define a family of gaussian fields  $\phi_L \in C^\infty(\mathcal{M})$  indexed by  $L > 0$  as

$$\phi_L = \sum_{0 < \lambda_j \leq L} \lambda_j^{-\frac{s}{2}} \xi_j e_j.$$

A simple calculation shows that the covariance function for  $\phi_L$  is

$$\mathbb{E}[\phi_L \boxtimes \overline{\phi_L}] = K_L = \sum_{0 < \lambda_j \leq L} \lambda_j^{-s} e_j \boxtimes \overline{e_j}$$

which we studied above. In this section, we will apply previous results from random topology to the field  $\phi_L$ , with some emphasis on the number  $N_L$  of connected components of the zero set  $Z_L$  of  $\phi_L$ . To begin with, it follows from sections 5.2 and 5.3 of [13] together with Theorems 2.4 and 2.6 that for  $L$  large enough,  $Z_L$  is almost surely smooth. Among the many available results regarding  $Z_L$  which could be adapted easily thanks to Theorems 2.4 and 2.6, we choose two which reveal the effect of the quantity  $n - ms$  on the topology of  $Z_L$ . To begin with, in the case where  $n > ms$ , the field  $\phi_L$  satisfies the conditions for Theorem 3 of [15].

**Corollary 3.1.** *Suppose that  $n > ms$ . Then, there exists a constant  $a > 0$  such that*

$$\mathbb{E}[L^{-\frac{n}{m}} |N_L - a|] \xrightarrow{L \rightarrow +\infty} 0.$$

**Proof.** By Theorem 2.4, around each point in  $\mathcal{M}$  there are local coordinates in which

$$L^{s-\frac{n}{m}} K_L \left( L^{-\frac{1}{m}} x, L^{-\frac{1}{m}} y \right) \xrightarrow{L \rightarrow +\infty} \frac{1}{(2\pi)^n} \int_{\sigma_A(0, \xi) \leq 1} e^{i\langle \xi, x-y \rangle} \sigma_A(0, \xi)^s d_0 \nu(\xi)$$

where the convergence takes place in  $C^\infty$  with respect to  $(x, y)$ . This shows that  $\left( L^{\frac{s}{2} - \frac{n}{2m}} \phi_L \right)_L$  is, in the terminology of [15], a tame parametric gaussian ensemble on  $\mathcal{M}$ . We can apply Theorem 3 of [15] and set  $a = \int_{\mathcal{M}} n_\infty$ . Moreover, the spectral measure of the limiting kernel charges a compact neighborhood of 0 so that, by section C.2 of [15], it satisfies condition  $(\rho 4)$  of [15] and the constant  $a$  is indeed positive.  $\square$

On the other hand, when  $n = ms$ ,  $N_L$  grows somewhat more slowly. In order to state a precise result, we first need to introduce some notation. Let  $f$  be a Morse function on  $\mathcal{M}$ . We denote by  $\text{Crit}(f)$  the set of its critical points and, for each  $x \in \mathcal{M} \setminus \text{Crit}(f)$ , we let  $H_x = \text{Ker}(d_x f)$ . The disjoint union  $\coprod_{x \in \mathcal{M}} H_x$  defines an integrable distribution, that is the tangent space of the smooth foliation  $\mathcal{H}$  of level sets of  $f$  on  $\mathcal{M} \setminus \text{Crit}(f) = \mathcal{M}'$ . By Lemma A.1 of [6] and Lemma 3.9, for  $L > 0$  large enough, the restriction of  $f$  to  $Z_L$  is almost surely a Morse function. For each  $i \in \{0, \dots, n-1\}$  and  $L > 0$  large enough, let

$$\nu_L^i = \sum_{x \in \text{Crit}_i(f|_{Z_L})} \delta_x.$$

Here  $\text{Crit}_i(f|_{Z_L})$  denotes the set of critical points of index  $i$  of the restriction of  $f$  to  $Z_L$ . For any vector space  $E$  we denote by  $\underline{E}$  the trivial bundle  $M \times E$  and for any  $k \in \mathbb{N}$ , by  $\text{Sym}^k(E)$  the  $k$ th symmetric power of  $E$ .

**Definition 3.2.** Let  $(E, g)$  be a Euclidean space. Then, the scalar product  $g$  on  $E$  defines an isomorphism  $\Xi : E \rightarrow E^*$ . From this isomorphism, we define a metric  $g^{-1}$  on  $E^*$  called the reciprocal product of  $g$  as

$$g^{-1}(w_1, w_2) := w_2(\Xi^{-1} w_1).$$

Similarly, we can define reciprocals of metrics on vector bundles.

**Definition 3.3.** Let  $E$  be a finite dimensional vector space and let  $C \in \text{Sym}^2(E)$  be a non-negative bilinear form on  $E^*$ . Let  $K = \{\xi \in E^* \mid C(\xi, \cdot) = 0\}$  and  $V = \{v \in E \mid \forall \xi \in K, \langle \xi, v \rangle = 0\}$ . Then,  $V \simeq (E^*/K)^*$ . By construction,  $C$  defines a scalar product on  $E^*/K$ . Let  $g$  be the reciprocal scalar product, on  $V$ . Consider the probability law on  $V$  with smooth density

$$\frac{1}{\sqrt{2\pi}^{\dim(V)}} e^{-\frac{1}{2}g(v,v)} dV_g(v).$$

Here  $dV_g$  is the Lebesgue measure defined by  $g$ . By the inclusion  $V \subset E$ , this defines a probability law on  $E$ . We call it the gaussian probability law with covariance  $C$ .

**Definition 3.4.** For each  $x \in \mathcal{M}$  and each  $k \geq 0$ , we define  $\beta_{k,k}(x)$  a scalar product on  $Sym^k(T_x\mathcal{M})$  by

$$\beta_{k,k}(v_1, v_2) = \frac{1}{(2\pi)^n} \int_{\sigma_A(x, \xi) \leq 1} \sigma_A^{-s}(x, \xi) \langle \xi \otimes \cdots \otimes \xi, v_1 \rangle \langle \xi \otimes \cdots \otimes \xi, v_2 \rangle d\xi.$$

This is always well defined unless  $s = \frac{n}{m}$  and  $k = 0$ . Let  $\varpi_x : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the scalar product

$$(t_1, t_2) \mapsto \frac{1}{(2\pi)^n} \left( \int_{S_x^* \mathcal{M}} d_x \nu \right) t_1 t_2.$$

We have the following result.

**Corollary 3.5.** Suppose that  $n = ms$ . For each  $x \in \mathcal{M}$ , let  $M_x$  be the matrix between orthonormal bases for  $\beta_{1,1}^{-1}(x)|_{H_x}$  of a random centered gaussian vector in  $Sym^2(H_x^*)$  of size  $n - 1$  whose covariance is the restriction of  $\beta_{2,2}(x)$  to  $Sym^2(H_x)$ . Then, for each  $i \in \{0, \dots, n - 1\}$ , as  $L \rightarrow +\infty$ ,  $\mathbb{E}[\nu_L^i]$  is equivalent to the smooth density

$$\sqrt{\frac{2}{\pi \text{Vol}(S_x^*)}} \mathbb{E}[|\det(M_x)| \mathbf{1}_{[\text{sgn}(M_x)=i]}] \frac{L^{\frac{n}{m}}}{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}} dV_{\beta_{1,1}}(x)$$

in the weak topology of measures. Here  $\text{sgn}(M_x)$  is the dimension of the largest subspace of  $H_x$  on which  $M_x$  is negative definite and  $\text{Vol}(S_x^*) = \int_{S_x^*} d_x \nu$ .

In particular, by applying the Morse inequalities we obtain following two corollaries. The first one follows by comparing Betti numbers with the number of critical points of  $f|_{Z_L}$ .

**Corollary 3.6.** Suppose that  $\mathcal{M}$  is a closed manifold. Then, for each  $i \in \{0, \dots, n - 1\}$ , the expectation of the  $i$ th Betti number  $b_i(Z_L)$  of  $Z_L$  satisfies

$$\limsup_{L \rightarrow \infty} \frac{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}}{L^{\frac{n}{m}}} \mathbb{E}[b_i(Z_L)] \leq \int_{\mathcal{M}} \sqrt{\frac{2}{\pi \text{Vol}(S_x^*)}} \mathbb{E}[|\det(M_x)| \mathbf{1}_{[\text{sgn}(M_x)=i]}] dV_{\beta_{1,1}}(x).$$

**Corollary 3.7.** Suppose that  $\mathcal{M}$  is a closed manifold. Then, the expectation of the Euler characteristic  $\chi(Z_L)$  of  $Z_L$  satisfies

$$\frac{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}}{L^{\frac{n}{m}}} \mathbb{E}[\chi(Z_L)] \xrightarrow{L \rightarrow +\infty} \int_{\mathcal{M}} \sqrt{\frac{2}{\pi \text{Vol}(S_x^*)}} \mathbb{E}[\det(M_x)] dV_{\beta_{1,1}}(x).$$

**Proof.** Let  $\nu_L^i(\mathcal{M}) = \int_{\mathcal{M}} \nu_L^i$  be the number of critical points of index  $i$  of  $f|_{Z_L}$ . By the Morse inequalities,  $\chi(Z_L) = \sum_{i=0}^{n-1} (-1)^i \nu_L^i(\mathcal{M})$ . Taking expectations, by Corollary 3.5,

$$\frac{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}}{L^{\frac{n}{m}}} \mathbb{E}[\chi(Z_L)] = \sum_{i=0}^{n-1} (-1)^i \int_{\mathcal{M}} \sqrt{\frac{2}{\pi \text{Vol}(S_x^*)}} \mathbb{E}[|\det(M_x)| \mathbf{1}_{[\text{sgn}(M_x)=i]}] dV_{\beta_{1,1}}(x) + o(1).$$

But  $\sum_{i=0}^{n-1} (-1)^i |\det(M_x)| \mathbf{1}_{[\text{sgn}(M_x)=i]} = \det(M_x)$  so

$$\frac{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}}{L^{\frac{n}{m}}} \mathbb{E}[\chi(Z_L)] = \int_{\mathcal{M}} \sqrt{\frac{2}{\pi \text{Vol}(S_x^*)}} \mathbb{E}[\det(M_x)] dV_{\beta_{1,1}}(x) + o(1).$$

□

### 3.2 Proof of Corollary 3.5

The proof of Corollary 3.5 will be based on Theorem 1.10 of [6]. We will also use the two following lemmas.

**Lemma 3.8.** *For  $L$  large enough and  $x \in \mathcal{M}$ , the symmetric bilinear form*

$$((t_1, v_1), (t_2, v_2)) \mapsto K_L(x, x) t_1 t_2 + d_x \otimes d_x K_L(v_1, v_2)$$

*defined on  $\mathbb{R}^* \oplus H_x$  is positive definite. Let  $g_L$  be its reciprocal metric. Similarly, the two tensor  $\varpi \oplus \beta_{1,1}|_H$  defines a metric on  $\mathbb{R}^* \oplus H$ . Let  $g_1$  be its reciprocal metric. Suppose that  $s = \frac{n}{m}$ . For each  $L > 0$ , let  $a_L : \mathbb{R} \oplus H^* \rightarrow \mathbb{R} \oplus H^*$  be the map of multiplication by  $\left(\ln\left(L^{\frac{1}{m}}\right)\right)^{\frac{1}{2}}$  along the fiber of  $\mathbb{R}$  and by  $L^{\frac{1}{m}}$  along the fiber of  $H^*$ . Then,*

$$a_L^* g_L \xrightarrow{L \rightarrow +\infty} g_1$$

*uniformly on compact sets.*

**Proof.** Let  $a_L^\dagger : \mathbb{R}^* \oplus H \rightarrow \mathbb{R}^* \oplus H$  be the adjoint of  $a_L$ . Then,  $(a_L^* g_L)^{-1} = ((a_L^\dagger)^{-1})^* g_L^{-1}$ . The map  $(a_L^\dagger)^{-1}$  acts by division by  $\left(\ln\left(L^{\frac{1}{m}}\right)\right)^{\frac{1}{2}}$  along the fiber of  $\mathbb{R}^*$  and by  $L^{\frac{1}{m}}$  along the fiber of  $H$ . Hence, for each  $x \in \mathcal{M}$ ,

$$((a_L^\dagger)^{-1})^* g_L^{-1}|_x = \left(\ln\left(L^{\frac{1}{m}}\right)\right)^{-1} K_L(x, x) dt^2 + L^{-\frac{2}{m}} d_x \otimes d_x K_L|_{H_x^2}.$$

By Theorem 2.6 for the first term and Theorem 2.4 for the second, this converges to  $\varpi \oplus \beta_{1,1}|_H$  uniformly for  $x$  in compact subsets of  $\mathcal{M}$ . □

For the second lemma, let us introduce some notation. For each  $k \in \mathbb{N}$ , let  $J_{\mathcal{H}}^k$  be the  $k$ th jet bundle over  $\mathcal{M}'$  of restrictions of smooth functions to the leaves of  $\mathcal{H}$ . For any  $l \in \mathbb{N}$  the kernel of the projection  $J_{\mathcal{H}}^{l+1} \rightarrow J_{\mathcal{H}}^l$  is canonically isomorphic to  $\text{Sym}^{l+1}(H^*)$ . Any metric on  $J_{\mathcal{H}}^2$  provides a complement to each of these kernels - the orthogonal - thus inducing an isomorphism  $\kappa_2 : J_{\mathcal{H}}^2 \rightarrow \bigoplus_{j=0}^2 \text{Sym}^j(H^*)$ . We denote by  $H^\perp$  the orthogonal of  $H$  with respect to the metric  $\beta_{1,1}$ .

**Lemma 3.9.** For each  $L > 0$ , let  $b_L : \bigoplus_{j=0}^2 \text{Sym}^j(H^*) \rightarrow \bigoplus_{j=0}^2 \text{Sym}^j(H^*)$  be defined by the following relation. For each  $(x, q_0, q_1, q_2) \in \bigoplus_{j=0}^2 \text{Sym}^j(H^*)$ ,

$$b_L(x, q_0, q_1, q_2) = \left( x, \left( \ln \left( L^{\frac{1}{m}} \right) \right)^{-\frac{1}{2}} q_0, L^{-\frac{1}{m}} q_1, L^{-\frac{2}{m}} q_2 \right).$$

For each  $x \in \mathcal{M}'$ , let  $(x, q_L^0(x), q_L^1(x), q_L^2(x)) = b_L \circ \kappa_2(j_{\mathcal{H}}^2(\phi_L))|_x$  and  $\tilde{q}_L^1(x) = d_x \phi_L|_{H^\perp}$ . Then, the random vector  $(x, q_L^0(x), \tilde{q}_L^1(x), q_L^2(x))$  converges in law to a random vector  $\mathcal{Q}_L(x) = (x, q^0(x), \tilde{q}^1(x), q^2(x)) \in \mathbb{R} \oplus T^* \mathcal{M} \oplus \text{Sym}^2(H^*)|_x$ , uniformly for  $x \in \mathcal{M}'$  in compact sets. The random vector  $\mathcal{Q}_L(x)$  is a centered gaussian whose covariance is  $\varpi(x) \oplus \beta_{1,1}(x) \oplus \beta_{2,2}(x)$  restricted to  $\mathbb{R} \oplus T^* \mathcal{M}' \oplus \text{Sym}^2(H^*)|_x$ . In particular, this is independent of the metric initially chosen on  $J_{\mathcal{H}}^2$ .

**Proof.** The covariance of  $(d_x \phi_L, \kappa_2(j_{\mathcal{H}}^2(\phi_L))|_x)$  is

$$\begin{pmatrix} d_x \otimes d_x K_L & (\kappa_2 \circ j_{\mathcal{H}}^2|_x) \otimes d_x K_L \\ d_x \otimes (\kappa_2 \circ j_{\mathcal{H}}^2|_x) K_L & (\kappa_2 \circ j_{\mathcal{H}}^2|_x) \otimes (\kappa_2 \circ j_{\mathcal{H}}^2|_x) K_L \end{pmatrix}$$

The lemma now follows by direct application of Theorems 2.4 and 2.6.  $\square$

Let  $s_L^1 = d_x \phi_L|_{H_x} \in \text{Hom}(H^\perp, \mathbb{R})$ . The last component  $s_L^2$  of  $\kappa_2(j_{\mathcal{H}}^2(\phi_L))$  is a section of  $\text{Sym}^2(H^*) \simeq \text{Hom}(H, H^*)$ . Consequently, the couple  $s_L = (s_L^1, s_L^2)$  defines an element of  $\text{Hom}(H^\perp, \mathbb{R}) \oplus \text{Hom}(H, H^*) \subset \text{Hom}(T\mathcal{M}', \mathbb{R} \oplus H^*)$ . We denote by  $\text{sgn}(s_L^2)$  the dimension of the largest subspace of  $H$  on which the bilinear form  $s_L^2$  is negative definite.

**Proof of Corollary 3.5.** With the above notations, Theorem 1.10 of [6] implies the following. For large enough values of  $L > 0$ ,  $\mathbb{E}[\nu_L^i]$  defines a smooth density such that for each  $x \in \mathcal{M}'$ ,

$$\mathbb{E}[\nu_L^i]_x = \frac{1}{\sqrt{2\pi}^n} \mathbb{E}[s_L^* dV_{g_L}(x) \mathbf{1}_{[\text{sgn}(s_L^2)=i]} | \phi_L(x) = 0, d_x \phi_L|_{H_x} = 0].$$

Rescaling  $s_L$  and  $g_L$  shows that  $\mathbb{E}[\nu_L^i]_x$  equals

$$\frac{1}{\sqrt{2\pi}^n} \frac{L^{\frac{n}{m}}}{\sqrt{\ln \left( L^{\frac{1}{m}} \right)}} \mathbb{E} \left[ \left( L^{-\frac{1}{m}} s_L^1, L^{-\frac{2}{m}} s_L^2 \right)^* dV_{g_L}(x) \mathbf{1}_{[\text{sgn}(L^{-\frac{2}{m}} s_L^2)=i]} | \phi_L(x) = 0, d_x \phi_L|_{H_x} = 0 \right].$$

In the right hand side, by Lemmas 3.8 and 3.9, the expectation defines a smooth density that converges to

$$\begin{aligned} & \mathbb{E}[(\tilde{q}^1|_{H^\perp}, q^2)^* dV_{g_1} \mathbf{1}_{[\text{sgn}(q^2)=i]} | q^0 = 0, \tilde{q}^1|_H = 0] \\ & = \mathbb{E}[\tilde{q}^1|_{H^\perp}^* dV_{\varpi^{-1}} | \tilde{q}^1|_H = 0] \otimes \mathbb{E}[q^{2*} dV_{\beta_{1,1}^{-1}|_H} \mathbf{1}_{[\text{sgn}(q^2)=i]}] \end{aligned}$$

uniformly on compact sets. Here we used that  $\mathbb{R}$  and  $H^*$  are perpendicular for  $g_1$  and that for each  $x$ ,  $q^0(x)$  is independent of  $\tilde{q}^1(x)$ . Since the covariance of  $\tilde{q}^1$  is  $\beta_{1,1}$ ,  $\tilde{q}^1|_{H^\perp}$  is

independent of  $\tilde{q}^1|_H$ . Therefore,

$$\begin{aligned}\mathbb{E}[\tilde{q}^1|_{H^\perp}^* dV_{\varpi^{-1}} \mid \tilde{q}^1|_H = 0] &= \mathbb{E}[\tilde{q}^1|_{H^\perp}^* dV_{\varpi^{-1}}(x)] \\ &= \sqrt{\frac{(2\pi)^n}{\text{Vol}(S_x^*)}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |t| e^{-\frac{1}{2}t^2} dt dV_{\beta_{1,1}|_{H^\perp}}(x) \\ &= \sqrt{\frac{2(2\pi)^n}{\pi \text{Vol}(S_x^*)}} dV_{\beta_{1,1}|_{H^\perp}}(x).\end{aligned}$$

Here  $\text{Vol}(S_x^*) = \int_{S_x^*} d_x \nu$ . Let  $M$  be the matrix of  $q^2$  in orthonormal coordinates for  $\beta_{1,1}$ . Then,

$$\mathbb{E}[q^{2*} dV_{\beta_{1,1}^{-1}|_H} \mathbf{1}_{[\text{sgn}(q^2)=i]}] = \mathbb{E}[|\det(M)| \mathbf{1}_{[\text{sgn}(M)=i]}] dV_{\beta_{1,1}|_H}.$$

To conclude,

$$\mathbb{E}[\nu_L^i|_x] \sim \sqrt{\frac{2}{\pi \text{Vol}(S_x^*)}} \mathbb{E}[|\det(M_x)| \mathbf{1}_{[\text{sgn}(M_x)=i]}] \frac{L^{\frac{n}{m}}}{\sqrt{\ln\left(L^{\frac{1}{m}}\right)}} dV_{\beta_{1,1}}(x).$$

□

We conclude this section with an example. Let  $\mathcal{M}$  be a closed manifold of dimension  $n \geq 2$  which we equip with a Riemmanian metric  $g$ . Let  $\Delta$  be the Laplace operator on  $(\mathcal{M}, g)$  and  $dV_g$  be the volume density defined by  $g$ . The density  $d\mu_{\mathcal{M}} = dV_g$  and the operator  $A = \Delta$  satisfy the hypotheses of Proposition 2.3 with  $m = 2$ . Indeed, the principal symbol of  $A$  is  $\sigma_A(x, \xi) = |\xi|_g^2$  so it is admissible. We set  $s = 1$  and consider the family of fields  $(\phi_L)_{L>0}$  associated to  $A$  and  $s$  as defined above. In this case, we call this family the cut-off Gaussian Free Field. This is because its covariance function is the kernel of  $\Pi_L \Delta^{-1}$  which converges in distribution to the covariance of the Gaussian Free Field when  $L \rightarrow \infty$ . Corollary 3.1 implies that in dimension 3 or more, the number  $N_L$  of connected components of the zero set of the cut-off Gaussian Free Field is equivalent to  $aL^{\frac{n}{2}}$  in probability for some positive constant  $a \in \mathbb{R}$ . Suppose now that  $n = 2$ . Then, we can apply Corollary 3.5. Let  $f$  be a Morse function on  $\mathcal{M}$ . Fix  $x \in \mathcal{M}$  and let  $\tau_1, \tau_2$  be an orthonormal basis of  $T_x \mathcal{M}$  such that  $\tau_1$  spans  $H_x$ . Let  $(\xi_1, \xi_2)$  be its dual basis. Then, in these coordinates, the metric  $\beta_{1,1}$  reads

$$\beta_{1,1}(v_1, v_2) = \frac{1}{4\pi^2} \int_{|\xi|^2 \leq 1} \frac{\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle}{|\xi|^2} d\xi.$$

This metric is clearly rotation invariant so it must be a multiple of  $g$ . Taking  $v_1 = v_2 = \tau_1$ , we deduce that  $\beta_{1,1} = c_1 g$  with

$$\begin{aligned}c_1 &= \frac{1}{4\pi^2} \int_{|\xi|^2 \leq 1} \frac{\xi_1^2}{|\xi|^2} d\xi \\ &= \frac{1}{4\pi^2} \int_0^1 t dt \int_0^{2\pi} \cos(\theta)^2 d\theta \\ &= \frac{1}{8\pi}.\end{aligned}$$

Consequently,  $dV_{\beta_{1,1}} = c_1 dV_g = \frac{1}{8\pi} dV_g$ . Moreover,  $Vol(S_x^*) = 2\pi$ . The metric  $\beta_{2,2}$  satisfies

$$\beta_{2,2}(v_1 \otimes w_1, v_2 \otimes w_2) = \frac{1}{4\pi^2} \int_{|\xi|^2 \leq 1} \frac{\langle \xi, v_1 \rangle \langle \xi, w_1 \rangle \langle \xi, v_2 \rangle \langle \xi, w_2 \rangle}{|\xi|^2} d\xi.$$

Let  $E$  be the matrix of a random gaussian tensor with covariance  $\beta_{2,2}$  in the coordinates  $(c_1^{-\frac{1}{2}}\tau_1, c_1^{-\frac{1}{2}}\tau_2)$ , which are orthonormal for the metric  $\beta_{1,1}$ . Then,

$$\mathbb{E}[E_{11}^2] = \frac{1}{4\pi^2 c_1} \int_{|\xi|^2 \leq 1} \frac{\xi_1^4}{|\xi|^2} d\xi = \frac{3}{2}.$$

Therefore the expected determinant in Corollary 3.5 is for  $i = 0$  or  $1$ ,

$$\frac{1}{2} \mathbb{E}[|E_{11}|] = \sqrt{\frac{3}{2}} \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} t e^{-\frac{1}{2}t^2} dt = \sqrt{\frac{3}{4\pi}}.$$

Hence, for  $i$  equal to either  $0$  or  $1$ ,

$$\mathbb{E}[\nu_L^i] \sim \frac{1}{4\pi^2} \sqrt{\frac{3}{2}} \frac{L}{\sqrt{\ln(\sqrt{L})}} dV_g.$$

In particular,

$$\limsup_{L \rightarrow +\infty} \frac{\sqrt{\ln(\sqrt{L})}}{L} N_L \leq \frac{1}{4\pi^2} \sqrt{\frac{3}{2}} Vol_g(\mathcal{M}). \quad (4)$$

## 4 Proof of the analytical results

In this section, we will prove the results of section 2. We will use the notations of that section. The overall proof follows that of Theorem 3 of [18]. However, in order to deal with this more general setting, we need to develop some additional tools. First, we will prove Proposition 2.3 and Theorem 2.4. To prove these results, we essentially apply integration by parts to the kernel  $E_L$  along the  $L$  variable and use Theorem 2.2. For the proof of Theorem 2.6, we need some additional tools which we develop in section 5.2. Most notably, we prove the decay of certain oscillatory integrals. In section 5.3 we prove Theorem 2.6 using the results of section 5.2. Some technical results are stated along the way but proved only later, in section 5.4.

## 5 Preliminaries

Let  $\mathcal{M}$  be a smooth manifold of dimension  $n \geq 2$  equipped with a smooth positive density  $d\mu_{\mathcal{M}}$ . Let  $A$  be an elliptic pseudo-differential operator on  $\mathcal{M}$  as in section 2. Let  $f \in C^\infty(\mathbb{R})$  be such that  $f(t)$  vanishes for  $-t$  large enough. Let  $K_L$  be the integral kernel of



$\Pi_L f(A)$ . Let  $P$  be a differential operator on  $\mathcal{M} \times \mathcal{M}$  of order  $d$  with principal symbol  $\sigma_P$ . Let us fix local coordinates  $x = (x_1, \dots, x_n)$  around a given point in  $\mathcal{M}$  such that  $d\mu_{\mathcal{M}}$  coincides with the Lebesgue measure in these coordinates. Let  $U, W \subset U \times U$  and  $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$  be as in Theorem 2.2. The following quantity will be central in our proofs. For any  $t > 0$ ,  $x, y \in U$  and  $\xi \in \mathbb{R}^n$  let

$$A_P(x, y, \xi, t) = e^{i\psi(x, y, t\xi)} \sigma_P(x, y, t^{-1} \partial_{x, y} \psi(x, y, t\xi)). \quad (5)$$

and

$$J_P(x, y, t) = \int_{S_y^*} A_P(x, y, \xi, t) d_y \nu(\xi). \quad (6)$$

Note that  $A_P$  satisfies the following equation. For any  $s, t > 0$ ,  $x, y \in U$  and  $\xi \in \mathbb{R}^n$ ,

$$A_P(x, y, s\xi, t) = s^d A_P(x, y, \xi, st). \quad (7)$$

At several points along the proof we will use properties of the phase  $\psi$  from Theorem 2.2 that follow from Definition 2.1. We gather these properties in the following lemma, which we prove in section 5.4 below.

**Lemma 5.1.** *Let  $U \subset \mathbb{R}^n$  and let  $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$  be an admissible phase function. For each  $s > 0$ , let  $\psi_t = t^{-1} \psi(\cdot, \cdot, t \cdot)$ . Then,*

1. *for each  $x, y \in U$  and each  $\xi \in \mathbb{R}^n$ ,  $\psi(x, y, 0) = \psi(x, x, \xi) = 0$  and  $\partial_x \psi(x, x, \xi) = \xi$ .*
2. *for any  $\alpha \in \mathbb{N}^n$  and for each compact subset  $K \subset U$  uniformly for  $t \geq 1$ , and for  $x, y \in K$  and  $\xi \in \mathbb{R}^n$  with  $|\xi| \geq 1$ ,*

$$\partial_\xi^\alpha \psi_t(x, y, \xi) = \partial_\xi^\alpha (\langle \xi, x - y \rangle) + O(|x - y|^2 |\xi|)$$

3. *for each compact subset  $K \subset U$ , there exists  $C > 0$  such that for each  $x, y \in K$  and each  $\xi \in \mathbb{R}^n$ ,*

$$\begin{aligned} |\partial_x \psi_t(x, y, \xi) - \xi| &\leq C|x - y| \\ |\partial_y \psi_t(x, y, \xi) + \xi| &\leq C|x - y|. \end{aligned}$$

4. *for each compact subset  $K \subset U$ , for each  $\alpha, \beta, \gamma \in \mathbb{N}^n$ , there exists  $C > 0$  such that for each  $s \geq 0$ , for each  $x, y \in K$  and  $\xi \in \mathbb{R}^n \setminus \{0\}$ ,*

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi)| \leq C(1 + |\xi|)^{1 - |\alpha|}$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

Before moving on, let us give some intuition about the proofs of Theorems 2.4 and 2.6 assuming Proposition 2.3. To make things simpler, we assume that  $P = Id$ , that  $\mathcal{M} = \mathbb{R}^n$ , that  $\sigma_A(x, \xi) = |\xi|$  and that  $\psi(x, y, \xi) = \langle \xi, x - y \rangle$ .

We start with Theorem 2.4. Then, taking  $f(t) = t^{-s}$  in Proposition tells us that, if  $s > -n$ ,  $K_L(x, y)$  is equal to  $\frac{1}{(2\pi)^n} \int_{|\xi| \leq L} |\xi|^{-s} e^{i\langle \xi, x-y \rangle} d\xi$  plus a negligible remainder term. Of course we should add a cut-off at 0 for  $f$  but this does not change the asymptotics. Setting  $\zeta = L\xi$ , we see that

$$K_L(L^{-1}x, L^{-1}y) = \frac{1}{(2\pi)^n} \int_{|\zeta| \leq 1} |\zeta|^{-s} e^{i\langle \zeta, x-y \rangle} d\zeta L^{n-s} (1 + o(1)).$$

This is essentially the claim made by Theorem 2.4.

For Theorem 2.6, we make the additional assumption that  $\dim(\mathcal{M}) = 1$ . This implies in particular that  $Y_P(x, y) = 2$ . In this case, Proposition 2.3 with  $f(t) = t^{-1} \mathbf{1}_{[t \geq 1]}$  tells us that

$$K_L(x, y) = \frac{1}{2\pi} \int_{1 \leq |\xi| \leq L} e^{i(x-y)\xi} \frac{d\xi}{|\xi|} + O(L^{-1}).$$

Since the cases where  $\xi > 0$  and  $\xi < 0$  are symmetric, we deal only with the first. Suppose that  $x > y$ , let  $r = x - y$  and let  $t = (x - y)\xi$ . Then,

$$\int_1^L e^{i(x-y)\xi} \frac{d\xi}{\xi} = \int_r^{rL} e^{it} \frac{dt}{t}.$$

Now, by integration by parts, it is easy to see that for all  $a \geq 1$ ,

$$\left| \int_1^a e^{it} \frac{dt}{t} \right| \leq 3. \quad (8)$$

Thus,

$$\int_1^L e^{i(x-y)\xi} \frac{d\xi}{\xi} = \int_r^{rL} e^{it} \mathbf{1}_{[t \leq 1]} \frac{dt}{t} + O(1).$$

Next, we compare  $e^{it}$  to 1. Since  $t^{-1}(e^{it} - 1)$  is bounded for  $t \in [0, 1]$ ,

$$\int_1^L e^{i(x-y)\xi} \frac{d\xi}{\xi} = \int_r^{rL} \mathbf{1}_{[t \leq 1]} \frac{dt}{t} + O(1).$$

Lastly for  $L \geq 1$ ,

$$\int_r^{rL} \mathbf{1}_{[t \leq 1]} \frac{dt}{t} = \ln(L) - \ln_+(rL).$$

Therefore,

$$K_L(x, y) = \frac{2}{2\pi} \left[ \ln(L) - \ln_+(rL) \right] + O(1).$$

This corresponds to the first statement of Theorem 2.4. The proof in dimension greater than one is somewhat different since the bound (8) is not valid anymore. We replace it by a generalized stationary decay formula. The rest of the arguments carry over to the general case with only technical adjustments.

## 5.1 Proof of Proposition 2.3 and Theorem 2.4

We begin by relating  $K_L$  to  $E_L$ .

**Lemma 5.2.** *For any  $L \in \mathbb{R}$ ,*

$$K_L = f(L)E_L - \int_0^L f'(\lambda)E_\lambda d\lambda.$$

This lemma generalizes Proposition 21 of [18].

**Proof.** The functions  $L \mapsto E_L$  and  $L \mapsto K_L$  are locally constant and define distributions on  $\mathbb{R}$  with values in  $C^\infty(\mathcal{M} \times \mathcal{M})$  supported on some interval  $[c, +\infty[$  with  $c \in \mathbb{R}$ . We denote by  $'$  the weak derivative with respect to  $L$  of these kernels. For all  $L > 0$ ,

$$E_L = \sum_{\lambda_k \leq L} e_k \boxtimes \bar{e}_k; \quad K_L = \sum_{\lambda_k \leq L} f(\lambda_k) e_k \boxtimes \bar{e}_k$$

so

$$K'_L = \sum_{k \in \mathbb{N}} \delta_{\lambda_k}(L) f(\lambda_k) e_k \boxtimes \bar{e}_k = f(L) \sum_{k \in \mathbb{N}} \delta_{\lambda_k}(L) e_k \boxtimes \bar{e}_k = f(L) E'_L$$

and

$$K_L = \int_0^L f(\lambda) E'_\lambda d\lambda.$$

The result follows by integration by parts.  $\square$

We can now prove the proposition using Theorem 2.2.

**Proof of Proposition 2.3.** First, let  $U$  be the open subset given by Theorem 2.2. By Theorem 2.2, for each uniformly for  $x, y \in U$  and  $L > 0$ ,

$$\begin{aligned} PE_L(x, y) &= \frac{1}{(2\pi)^n} \int_{\sigma_A(y, \xi) \leq L} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) d\xi + O\left(L^{\frac{n+d-1}{m}}\right) \\ &= \frac{1}{(2\pi)^n} \int_0^{L^{\frac{1}{m}}} J_P(x, y, t) t^{n+d-1} dt + O\left(L^{\frac{n+d-1}{m}}\right). \end{aligned}$$

In the second equality we used the definition of  $d\nu$  and  $J_P$  as well as the fact that  $\sigma_P$  is  $d$ -homogeneous along the fibers. Consequently, uniformly for any  $x, y \in U$  and  $L > 0$ ,

$$-\int_0^L f'(\lambda) PE_\lambda(x, y) d\lambda = -\frac{1}{(2\pi)^n} \int_0^L f'(\lambda) \int_0^{\lambda^{\frac{1}{m}}} J_P(x, y, t) t^{n+d-1} dt d\lambda + O\left(\int_{-\infty}^L f'(\lambda) \lambda^{\frac{n+d-1}{m}}\right).$$

Integrating by parts along  $\lambda$  the first term in the right hand side we get

$$-f(L)PE_L(x, y) + \frac{1}{(2\pi)^n} \int_0^L f(\lambda) \frac{1}{m} \lambda^{\frac{1}{m}-1} J_P(x, y, \lambda^{\frac{1}{m}}) \lambda^{\frac{n+d-1}{m}} d\lambda + O(f(L)L^{\frac{n+d-1}{m}}).$$

Setting  $u = \lambda^{\frac{1}{m}}$ , the integral term becomes

$$\begin{aligned} \int_0^L f(\lambda) \frac{1}{m} \lambda^{\frac{1}{m}-1} J_P(x, y, \lambda^{\frac{1}{m}}) \lambda^{\frac{n+d-1}{m}} d\lambda &= \int_0^{L^{\frac{1}{m}}} f(u^m) J_P(x, y, u) u^{n+d-1} du \\ &= \int_{\sigma_A(y, \xi) \leq L} e^{i\psi(x, y, \xi)} f(\sigma_A(y, \xi)) \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) d\xi. \end{aligned}$$

By Lemma 5.2,

$$PK_L = f(L)PE_L - \int_0^L f'(\lambda)PE_\lambda d\lambda.$$

Replacing the integral term by the expression derived above, we see that the  $f(L)PE_L$  terms cancel out and leave the first result of Proposition 2.3. For the case where  $(x, y) \in U \times U \setminus W$  just apply the corresponding estimate from Theorem 2.2 and proceed accordingly.  $\square$

To prove Theorem 2.4 we fix  $z \in \mathbb{C}$  and specialize to the case where  $f(t) = t^z$  for  $t > 0$  large enough.

**Proof of Theorem 2.4.** Let  $K$  be a compact neighborhood of  $0 \in U$ . By the second and third point of Lemma 5.1 (with  $\alpha = 0$ ) we have, uniformly for  $x, y \in K$ ,  $\xi \in S_0^*$ ,  $t > 0$  and  $L \geq 1$  large enough,

$$\begin{aligned} t^{-1}\psi\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, t\xi\right) &= L^{-\frac{1}{m}}\langle \xi, x - y \rangle + O\left(L^{-\frac{2}{m}}\right) \\ t^{-1}\partial_x\psi\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, t\xi\right) &= \xi + O\left(L^{-\frac{1}{m}}\right) \\ -t^{-1}\partial_y\psi\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, t\xi\right) &= \xi + O\left(L^{-\frac{1}{m}}\right). \end{aligned}$$

Let  $C_0 > 0$  be a constant to be fixed later. Since  $\sigma_P(x, y, \cdot)$ , depends smoothly on  $(x, y, \xi)$  and by the third point in Lemma 5.1, we have, uniformly for  $x, y \in K$ ,  $\xi \in S_0^*$ ,  $L \geq 1$  large enough and  $0 < t \leq C_0 L^{\frac{1}{m}}$ ,

$$\sigma_P\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, t^{-1}\partial_{x, y}\psi\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, t\xi\right)\right) = \sigma_P(0, 0, (\xi, -\xi)) + O\left(L^{-\frac{1}{m}}\right)$$

and

$$A_P\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, \xi, t\right) = A_P(0, 0, \xi, t) + O\left(L^{-\frac{1}{m}}\right) \quad (9)$$

$$= e^{iL^{-\frac{1}{m}}t\langle \xi, x - y \rangle} \sigma_P(0, 0, (\xi, -\xi)) + O\left(L^{-\frac{1}{m}}\right). \quad (10)$$

Recall that  $A_P$  was defined by equation (5). Here the  $O$ 's depend on  $C_0$ . Since  $\sigma_A$  is positive homogeneous in the second variable, we may choose  $C_0$  so that for each  $y \in K$ , each  $\xi \in \mathbb{R}^n$  and each  $L \geq 1$  large enough,  $\sigma_A\left(L^{-\frac{1}{m}}y, \xi\right) \leq L$  implies that  $\sigma_A(0, \xi) \leq C_0 L$ . Let  $C_1 > 0$  be such that  $f(t) = t^z$  for  $t \geq C_1$ . With this choice of  $C_0$ , equation (9) shows

that following integral is bounded uniformly for  $x, y \in K$  and  $L \geq 1$  large enough, where  $\tilde{f}(t)$  stands either for  $f(t)$  or for  $t^z$ ,

$$\int_{\sigma_A(L^{-\frac{1}{m}}y, \xi) \leq C_1} \tilde{f}(\sigma_A(L^{-\frac{1}{m}}y, L)) A_P\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, \sigma_A(0, \xi)^{-\frac{1}{m}}\xi, \sigma_A(0, \xi)^{\frac{1}{m}}\right) d\xi.$$

Therefore, by Proposition 2.3, since  $f(t) = t^z$  for  $t \geq C_1$  and since  $n + d + \Re(z) > 0$ ,  $PK_L\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right)$  equals

$$\frac{1}{(2\pi)^n} \int_{\sigma_A(L^{-\frac{1}{m}}y, \xi) \leq L} \sigma_A(0, \xi)^{\frac{d}{m}+z} A_P\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y, \sigma_A(0, \xi)^{-\frac{1}{m}}\xi, \sigma_A(0, \xi)^{\frac{1}{m}}\right) d\xi + O(g(L))$$

where  $g(L) = L^{\frac{n+d-1}{m}+\Re(z)}$  if  $n + d + mz \neq 1$  and  $\ln(L)L^{\frac{n+d-1}{m}+\Re(z)}$  otherwise. Since  $\sigma_A$  is positive and homogeneous of degree  $m$ , we have, uniformly in  $y \in K$  and  $L > 0$ ,

$$\int_{\sigma_A(L^{-\frac{1}{m}}y, \xi) \leq L} \sigma_A(0, \xi)^{\frac{d}{m}+z} d\xi = O(L^{n+d+m\Re(z)}).$$

Therefore, by equation (9),

$$PK_L\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right) = \frac{1}{(2\pi)^n} \int_{\sigma_A(L^{-\frac{1}{m}}y, \xi) \leq L} \sigma_A(0, \xi)^z A_P(0, 0, \sigma_A^{-\frac{1}{m}}(0, \xi)\xi, \sigma_A(0, \xi)^{\frac{1}{m}}) d\xi + O(g(L)).$$

Since  $\sigma_A(y, \xi)$  is both smooth in  $y$  and homogeneous of degree  $m$  in  $\xi$ , there exists a constant  $C_2$  such that for each  $y \in K$ , for each  $L \geq 1$  large enough, the symmetric difference of the sets  $\{\xi \in \mathbb{R}^n \mid \sigma_A(0, \xi) \leq L\}$  and  $\{\xi \in \mathbb{R}^n \mid \sigma_A(L^{-\frac{1}{m}}y, \xi) \leq L\}$  has volume at most  $C_2 L^{\frac{n-1}{m}}$ . Also, by construction of  $C_0$ , for any  $\xi$  in this symmetric difference, for any  $y \in K$  and for  $L \geq 1$  large enough,  $\sigma_A(0, \xi) \leq (1 + C_0)L$ . Consequently,

$$\begin{aligned} PK_L\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right) &= \frac{1}{(2\pi)^n} \int_{\sigma_A(0, \xi) \leq L} \sigma_A(0, \xi)^{\frac{d}{m}+z} A_P(0, 0, \sigma_A^{-\frac{1}{m}}(0, \xi)\xi, \sigma_A(0, \xi)^{\frac{1}{m}}) d\xi + O(g(L)) \\ &= \frac{1}{(2\pi)^n} \int_{\sigma_A(0, \xi) \leq L} \sigma_A(0, \xi)^z A_P(0, 0, \xi, 1) d\xi + O(g(L)) \text{ by equation (7)} \\ &= \frac{1}{(2\pi)^n} \int_{\sigma_A(0, \zeta) \leq 1} \sigma_A(0, \zeta)^z A_P\left(0, 0, L^{\frac{1}{m}}\zeta, 1\right) d\zeta L^{z+\frac{n}{m}} + O(g(L)) \text{ by setting } \xi = L\zeta. \end{aligned}$$

In conclusion, uniformly in  $L > 0$  and  $x, y \in K$ ,

$$PK_L\left(L^{-\frac{1}{m}}x, L^{-\frac{1}{m}}y\right) = \int_{\sigma_A(0, \xi) \leq 1} \sigma_A(0, \xi)^z e^{i\langle \xi, x-y \rangle} \sigma_P(0, 0, (\xi, -\xi)) d\xi L^{z+\frac{n+d}{m}} + O(g(L)).$$

This proves the first statement of the theorem for  $V = \mathring{K}$ . To prove the second statement, observe that by Lemma 5.2, for each  $L \geq C_1$  and  $x, y \in K$ ,

$$PK_L(x, y) = f(L)PE_L - \int_0^L f'(\lambda)PE_\lambda(x, y) d\lambda = L^z PE_L(x, y) - \int_{C_1}^L \lambda^{z-1} PE_\lambda(x, y) d\lambda + O(1)$$

where the  $O$  is uniform in  $x, y \in K$ . Next, by Theorem 2.2, there exists  $C_3 > 0$  such that for any  $(x, y) \in (K \times K) \setminus W$  and any  $L \geq C_1$ ,  $|PE_L(x, y)| \leq C_3 L^{\frac{n+d-1}{m}}$ . Therefore,

$$|PK_L(x, y)| \leq C_3 \left( L^{\frac{n+d-1}{m} + \Re(z)} + \int_{C^1}^L \lambda^{\frac{n+d-1}{m} + \Re(z) - 1} d\lambda \right).$$

Therefore there exists  $C_4 > 0$  such that for any  $L \geq C_1$  and any  $(x, y) \in (K \times K) \setminus W$ ,

$$|PK_L(x, y)| \leq C_4 L^{\frac{n+d-1}{m} + \Re(z)} \ln(L)^\epsilon$$

where  $\epsilon = 0$  if  $n + d + m\Re(z) = 1$ . This proves the second statement of Theorem 2.4.  $\square$

## 5.2 Oscillatory phase asymptotics

We keep the notations of the previous section. To prove in Theorem 2.6, we need three lemmas which we prove here. The first controls the behavior of the function  $A_P$  defined in equation (5).

**Lemma 5.3.** *The function  $A_P$  satisfies the following properties.*

1. *The function  $t \mapsto A_P(\cdot, \cdot, \cdot, t)$  extends continuously to  $t = 0$  as a function from  $\mathbb{R}_+$  to  $C^\infty(U \times U \times \mathbb{R}^n)$  and*

$$A_P(x, y, \xi, 0) = \sigma_P(x, y, \partial_{x,y}(\partial_\xi \psi(x, y, 0)\xi)).$$

2. *We have  $A_P(x, y, \xi, t) - A_P(x, y, \xi, 0) = O(t|x - y||\xi|^{d+1})$  uniformly for  $t \geq 0$  and  $x, y$  in compact subsets of  $U$  and  $\xi \in \mathbb{R}^n$ .*

Note that this lemma implies that the function  $t \mapsto J_P(\cdot, \cdot, t)$  extends continuously to  $t = 0$  as a function from  $\mathbb{R}_+$  to  $C^\infty(U \times U)$  and

$$J_P(x, y, 0) = \int_{S_y^*} \sigma_P(x, y, \partial_{x,y}(\partial_\xi \psi(x, y, 0)\xi)) d_y \nu(\xi). \quad (11)$$

**Proof.** The first statement follows from a Taylor expansion at  $t = 0$  of  $A_P$ . For the second statement, by equation (7), we may therefore restrict our attention to the case where  $\xi \in S_y^*$ . Next, we observe that by the first point of Lemma 5.1,  $A_P(y, y, \xi, t) = A_P(y, y, \xi, 0)$ . The function  $A_P$  is clearly  $C^1$  with respect to its first variable so that  $|A_P(x, y, \xi, t) - A_P(x, y, \xi, 0)|$  is no greater than

$$|x - y| \sup_{s \in [0,1]} |\partial_x A_P(sx + (1-s)y, y, \xi, t) - \partial_x A_P(sx + (1-s)y, y, \xi, 0)|.$$

Moreover, by applying Taylor estimates to  $\partial_x A_P$  with respect to  $t$  at  $t = 0$  and the fourth point of Lemma 5.1, we see that uniformly for  $x, y$  in compact subsets of  $U$  and  $\xi \in S_y^*$ ,

$$\partial_x A_P(x, y, \xi, t) = \partial_x A_P(x, y, \xi, 0) + O(t)$$

which completes the proof.  $\square$

The second lemma controls the behavior of  $J_P$  at  $t \rightarrow +\infty$ . This will be useful in dimension  $n \geq 2$ .

**Lemma 5.4.** *Suppose that the symbol  $\sigma_A$  has  $\varepsilon$ -non-degenerate energy levels (see Definition 6.1). Then, there exists  $V \subset U$  an open neighborhood of 0 and  $C > 0$  such that, uniformly for distinct  $x, y \in V$  and  $t > 0$*

$$|J_P(x, y, t)| \leq C(t|x - y|)^{-\varepsilon}.$$

This corresponds to Proposition 23 of [18] for  $\varepsilon = \frac{1}{2}$  although, in that setting, the non-degeneracy condition was always satisfied.

**Proof.** To prove this lemma, we interpret  $J_P$  as an oscillatory integral whose phase is a deformation of  $(\omega \otimes \tau) \mapsto \langle \omega, \tau \rangle$ . First, fix  $K \subset U$  a compact neighborhood of 0 and for each  $s > 0$ , let  $\psi_t = t^{-1}\psi(\cdot, \cdot, t)$  and for each  $t, r > 0$ ,  $y \in U$ ,  $\xi, \tau \in \mathbb{R}^n$ , let

$$f_{t,r}(y, \xi, \tau) = r^{-1}\psi_t((y + r\tau), y, \xi).$$

Then, by the second point of Lemma 5.1, for every  $\alpha \in \mathbb{N}^n$ , uniformly for  $y \in K$  and  $\tau \in \mathbb{R}^n$  in compact sets, for  $\xi \in S_y^*$  and for  $t \geq 1$ ,

$$\partial_\xi^\alpha f_{t,r}(y, \xi, \tau) = \partial_\xi^\alpha \langle \xi, \tau \rangle + O(r).$$

In other words,  $f_{t,r}(y, \xi, \tau) \xrightarrow[r \rightarrow 0]{} \langle \xi, \tau \rangle$  smoothly in  $\xi$ , uniformly in  $0 \leq s \leq 1$  and  $(y, \tau)$  in compact subsets of  $K \times \mathbb{R}^n$ . Moreover, setting  $r = |x - y|$ , we get

$$\psi(x, y, t\xi) = t|x - y|f_{t,|x-y|}\left(y, \xi, \frac{x - y}{|x - y|}\right).$$

In addition, when  $x, y \in K$  are distinct, the vector  $\frac{x-y}{|x-y|}$  stays in a compact subset of a complement of  $\{0\}$  in  $\mathbb{R}^n$ . Moreover, by the fourth point of Lemma 5.1, the function

$$\xi \mapsto \sigma_P(x, y, t^{-1}\partial_{x,y}\psi(x, y, t\xi))$$

has bounded seminorms in  $C^\infty(\mathbb{R}^n)$  uniformly for  $x, y \in K$  and  $t \geq 1$ . Therefore, the fact that  $\sigma_A$  has  $\varepsilon$ -non-degenerate energy levels implies the existence of an open neighborhood  $V \subset U$  of 0, a constant  $t_0 > 0$  as well as a constant  $C > 0$  such that, uniformly for  $x, y \in V$  and  $t \geq t_0$ ,

$$\left| \int_{S_y^*} e^{i\psi(x,y,t\xi)} \sigma_P(x, y, t^{-1}\partial_{x,y}\psi(x, y, t\xi)) d_y \nu(\xi) \right| \leq C(t|x - y|)^{-\varepsilon}.$$

Here the parameter  $\eta$  from the definition of non-degenerate energy levels is the couple  $(t^{-1}, |x - y|) \in \mathbb{R}^2$ , the parameter  $\tau$  is  $t|x - y|$  and the function  $f_\eta$  is  $f_{t,|x-y|}$ . By the first statement of Lemma 5.3 the estimate is clear for  $t \leq t_0$ .  $\square$

In dimension  $n = 1$  however,  $\sigma_A$  never has non-degenerate energy levels. In this case we will use the following result.

**Lemma 5.5.** *For each compact subset  $K \subset U \times U$ , there exists  $C > 0$  such that for each  $0 < a \leq b$ , each  $\varepsilon \in \{-1, +1\}$  and each  $(x, y) \in K$*

$$\left| \int_{\varepsilon a}^{\varepsilon b} e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta)) |\eta|^{-d-1} d\eta \right| \leq Ca^{-1}.$$

To prove this, we will use the following technical lemma regarding  $\psi$ , which we prove in section 5.4 to avoid breaking the flow of the exposition. In addition to the properties of admissible phase functions, this Lemma uses the fact that  $\psi$  satisfies the eikonal equation associated to a symbol whose principal part is  $\sigma_A(x, \xi)^{\frac{1}{m}}$ .

**Lemma 5.6.** *Suppose that  $\mathcal{M}$  has dimension  $n = 1$  and let  $\psi \in S^1(U \times U \times \mathbb{R})$  be the admissible phase function from Definition 2.1. Then, for each compact subset  $K$  of  $U \times U$ , there exists  $c > 0$  such that, uniformly for  $(x, y) \in K$  and  $\xi \in \mathbb{R}$ ,  $\xi \neq 0$ ,  $|\partial_\xi^2 \psi(x, y, \xi)| \leq c \frac{(x-y)^2}{|\xi|}$ ,  $|\partial_\xi \partial_{x,y} \psi(x, y, \xi)| \leq c|x-y|$  and  $\frac{1}{c}|x-y| \leq |\partial_\xi \psi(x, y, \xi)| \leq c|x-y|$ .*

In dimension 1, the symbol  $\sigma$  appearing in equation (1) is “close” to  $\varrho(x)|\xi|$  for some positive function  $\varrho$  as  $\xi \rightarrow +\infty$ . If we replace  $\sigma(x, \xi)$  by  $\varrho(x)|\xi|$  in the equation, the solution is equal to  $\psi(x, y, \xi) = (x-y)\xi$  which satisfies the claims of both Lemma 5.6 and Lemma 5.5. Lemma 5.6 makes this approximation rigorous. We begin by proving Lemma 5.5 using Lemma 5.6.

**Proof of Lemma 5.5.** Since  $\sigma_P$  has order  $d$  in the third variable and by the second statement of Lemma 5.6, we have, uniformly for  $(x, y)$  in a compact subset of  $U \times U$  and for non-zero  $\eta \in \mathbb{R}$ ,

$$\begin{aligned} \sigma_P(x, y, |x-y|\partial_{x,y}\psi(x, y, |x-y|^{-1}\eta))|\eta|^{-d-1} &= O(|\eta|^{-1}) \\ \partial_\eta[\sigma_P(x, y, |x-y|\partial_{x,y}\psi(x, y, |x-y|^{-1}\eta))|\eta|^{-d-1}] &= O(|\eta|^{-2}). \end{aligned}$$

In addition, the first and third statement of Lemma 5.6, mean respectively that  $\partial_\eta^2 \psi(x, y, |x-y|^{-1}\eta) = O(|\eta|^{-1})$  and that  $\partial_\eta[\psi(x, y, |x-y|^{-1}\eta)]$  is bounded from above and below by a positive constant, both uniformly for  $(x, y)$  in compact subsets of  $U \times U$  and for  $\eta \in \mathbb{R}$ . Now, setting momentarily  $u(\eta) := \psi(x, y, |x-y|^{-1}\eta)$  and  $v(\eta) = \sigma_P(x, y, |x-y|\partial_{x,y}\psi(x, y, |x-y|^{-1}\eta))|\eta|^{-d-1}$ , we have, for any  $a, b > 0$  such that  $a \leq b$ ,

$$\int_a^b e^{iu(\eta)} v(\eta) d\eta = \left[ \frac{1}{i} e^{iu(\eta)} \frac{v(\eta)}{u'(\eta)} \right]_{\eta=a}^b - \int_a^b \frac{1}{i} e^{iu(\eta)} \left( \frac{v'(\eta)}{u'(\eta)} - \frac{v(\eta)u''(\eta)}{u'(\eta)^2} \right) d\eta.$$

The above observations show that, uniformly for  $(x, y)$  in compact subsets of  $U \times U$ , non-zero  $\eta \in [a, b]$  and  $a, b > 0$  such that  $a \leq b$ , we have  $\frac{v(a)}{u'(a)} = O(a^{-1})$ ,  $\frac{v(b)}{u'(b)} = O(b^{-1})$ ,  $\frac{v'(\eta)}{u'(\eta)} = O(\eta^{-2})$  and  $\frac{v(\eta)u''(\eta)}{u'(\eta)^2} = O(\eta^{-2})$ . Consequently, for any compact subset  $K$  of  $U \times U$ , there exists  $C > 0$  such that for any  $(x, y) \in K$  and any  $a, b > 0$  such that  $a \leq b$ ,

$$\left| \int_a^b e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P(x, y, |x-y|\partial_{x,y}\psi(x, y, |x-y|^{-1}\eta))|\eta|^{-d-1} d\eta \right| \leq Ca^{-1}.$$

The proof for  $\int_{-b}^{-a}$  is identical.  $\square$

### 5.3 Proof of Theorem 2.6

In this section we prove Theorem 2.6. We use the admissibility condition through Proposition 6.2 which is stated and proved in section 6. Suppose that  $n + d + mz = 0$ , so that



$z = -\frac{d+n}{m}$ . By Proposition 2.3, uniformly for  $x, y \in U$ ,

$$PK_L(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y, \xi) \leq L} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) f(\sigma_A(y, \xi)) d\xi + O\left(L^{-\frac{1}{m}}\right).$$

Since  $f(t) = t^z$  for  $t > 0$  large enough, there exists  $C > 0$  such that

$$PK_L(x, y) = \frac{1}{(2\pi)^n} \int_{C \leq f(y) |\xi|^m \leq L} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) \sigma_A(y, \xi)^{-\frac{d+1}{m}} d\xi \\ + Q_1(x, y) + O\left(L^{-\frac{1}{m}}\right)$$

where

$$Q_1(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y, \xi) \leq C} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) f(\sigma_A(y, \xi)) d\xi.$$

We will split the integral term in the last expression of  $PK_L$  as follows. Let  $\mathbf{1}_{[|\xi| \geq 1]}$  be equal to 0 if  $-1 \leq \xi \leq 1$  and 1 otherwise, let  $\mathbf{1}_{[|\xi| < 1]} = 1 - \mathbf{1}_{[|\xi| \geq 1]}$  and let, for any  $x, y \in U$ ,

$$I_L(x, y) = \frac{1}{(2\pi)^n} \int_{C \leq \sigma_A(y, \xi) \leq L} \mathbf{1}_{[\sigma_A(y, \xi) |x-y| \geq 1]} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) \sigma_A(y, \xi)^{-\frac{d+1}{m}} d\xi \\ II_L(x, y) = \frac{1}{(2\pi)^n} \int_{C \leq \sigma_A(y, \xi) \leq L} \mathbf{1}_{[\sigma_A(y, \xi) |x-y| < 1]} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) \sigma_A(y, \xi)^{-\frac{d+1}{m}} d\xi.$$

Then,

$$PK_L(x, y) = I_L(x, y) + II_L(x, y) + Q_1(x, y) + O\left(L^{-\frac{1}{m}}\right).$$

Theorem 2.6 is an immediate consequence of the following two lemmas.

**Lemma 5.7.** *Suppose that either  $n = 1$  or  $\sigma_A$  is admissible. There exist an open neighborhood  $V \subset U$  of  $0 \in \mathbb{R}^n$ , a function  $Q_2 \in L^\infty(V \times V)$  and a constant  $C > 0$  such that for any  $x, y \in V$  and  $L > 0$ ,*

$$\left| I_L(x, y) - Q_2(x, y) \right| \leq C \min\left(L^{-\frac{1}{k_0 m}} |x - y|^{-\frac{1}{k_0}}, 1\right)$$

where  $k_0 = 1$  if  $\dim(\mathcal{M}) = 1$  and  $k_0$  is the order of admissibility of  $\sigma_A$  if it is admissible.

In dimension one, we prove the lemma using Lemma 5.5 while in the case of admissible symbols we use Proposition 6.2. It is the only place where we use this theorem.

**Lemma 5.8.** *There exist an open neighborhood  $V \subset U$  of 0 and a constant  $C > 0$  such that for all  $x, y \in V$  and  $L > 0$ ,*

$$\left| II_L(x, y) - \frac{1}{(2\pi)^n} Y_P(x, y) \left[ \ln\left(L^{\frac{1}{m}}\right) - \ln_+\left(L^{\frac{1}{m}} |x - y|\right) \right] \right| \leq C.$$

Moreover  $II_L(x, y)$  is independent of  $L$  as long as  $|x - y| \geq L$ .

In particular,  $II_L(x, y) - \ln\left(L^{\frac{1}{m}}\right) + \ln_+\left(L^{\frac{1}{m}}|x - y|\right)$  is also independent of  $L$  for  $|x - y| \geq L$ . In the proof of Lemma 5.7 we will repeatedly use the following equality, which appears by changing of variables  $\eta = |x - y|\xi$ . For any  $a, b > 0$

$$\int_{a|x-y|^{-m} \leq \sigma_A(y, \xi) \leq b|x-y|^{-m}} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) \sigma_A(y, \xi)^{-\frac{d+1}{m}} d\xi = \quad (12)$$

$$\int_{a \leq \sigma_A(y, \eta) \leq b} e^{i\psi(x, y, |x-y|^{-1}\eta)} \sigma_P(x, y, |x-y| \partial_{x, y} \psi(x, y, |x-y|^{-1}\eta)) \sigma_A(y, \eta)^{-\frac{d+1}{m}} d\eta. \quad (13)$$

Here we used that  $\sigma_P$  has order  $d$  in the third variable.

**Proof of Lemma 5.7.** Suppose first that  $\mathcal{M}$  has dimension one and fix  $K \subset U$  a compact neighborhood of 0. By Lemma 5.5, the integral

$$\int_{C|x-y|^m \leq \sigma_A(y, \eta) \leq |x-y|^m L} \mathbf{1}_{[f(y)^{-\frac{1}{m}}|\eta| \geq 1]} e^{i\psi(x, y, |x-y|^{-1}\eta)} \sigma_P(x, y, |x-y| \partial_{x, y} \psi(x, y, |x-y|^{-1}\eta)) \sigma_A(y, |\eta|^{-1}\eta)^{-\frac{d+1}{m}} |\eta|^{-d-1} d\eta$$

converges for fixed  $x \neq y$  as  $L \rightarrow +\infty$  and the remainder term is  $O\left(\min\left(|x-y|^{-1}L^{-\frac{1}{m}}, 1\right)\right)$  uniformly for distinct  $x, y \in K$ . Here, we use that, since  $\dim(\mathcal{M}) = 1$ ,  $\sigma_A(y, |\eta|^{-1}\eta)$  depends only on the sign of  $\eta$ . By equation (12), for  $x \neq y$ , the integral  $I_L(x, y)$  converges to some limit  $Q_2(x, y)$  as  $L \rightarrow +\infty$  in such a way that the remainder term  $R_{2,L}(x, y)$  is  $O\left(\min\left(|x-y|^{-1}L^{-\frac{1}{m}}, 1\right)\right)$ . This ends the proof of the one-dimensional case with  $V = \overset{\circ}{K}$ . Suppose now that  $n \geq 2$  and  $\sigma_A$  is admissible for some integer  $k_0$ . Then, for any  $L > 0$  and  $x, y \in U$ ,

$$\begin{aligned} I_L(x, y) &= \frac{1}{(2\pi)^n} \int_{C^{\frac{1}{m}}}^{L^{\frac{1}{m}}} \mathbf{1}_{[|x-y|t \geq 1]} J_P(x, y, t) \frac{dt}{t} \\ &= \frac{1}{(2\pi)^n} \int_{C^{\frac{1}{m}|x-y|}}^{L^{\frac{1}{m}|x-y|}} \mathbf{1}_{[s \geq 1]} J_P(x, y, |x-y|^{-1}s) \frac{ds}{s} \text{ setting } s = |x-y|t. \end{aligned}$$

Here we used equation (3). By Proposition 6.2, it has  $\frac{1}{k_0}$ -non-degenerate energy levels. Therefore, by Lemma 5.4, there exist an open neighborhood  $V \subset U$  of 0 and a constant  $C > 0$  such that, uniformly for distinct  $x, y \in V$  and  $t > 0$ ,  $|J_P(x, y, t)| \leq C(|x-y|t)^{-\frac{1}{k_0}}$ . Therefore, the function  $s \mapsto s^{-1} J_P(x, y, |x-y|^{-1}s)$  is uniformly integrable for any distinct  $x, y \in V$  and

$$\begin{aligned} \left| I_L(x, y) - \int_{C^{\frac{1}{m}|x-y|}}^{+\infty} \mathbf{1}_{s \geq 1} J_P(x, y, |x-y|^{-1}s) \frac{ds}{s} \right| &\leq C' \int_{\max\left(|x-y|L^{\frac{1}{m}}, 1\right)}^{+\infty} s^{-1-\frac{1}{k_0}} ds \\ &= \frac{C''}{k_0} \min\left(1, L^{-\frac{1}{k_0 m}} |x-y|^{-\frac{1}{k_0}}\right). \end{aligned}$$

By continuity, this stays true for  $x = y$ . This proves the lemma for  $\sigma_A$  admissible with

$$Q_2(x, y) = \int_{C^{\frac{1}{m}}|x-y|}^{+\infty} \mathbf{1}_{s \geq 1} J_P(x, y, |x-y|^{-1}s) \frac{ds}{s}.$$

□

**Proof of Lemma 5.8.** The proof of the second statement is obvious from the definition of  $II_L$  and the expression  $\ln\left(L^{\frac{1}{m}}\right) - \ln_+\left(L^{\frac{1}{m}}|x-y|\right)$ . We now prove the first statement. Recall that

$$II_L(x, y) = \frac{1}{(2\pi)^n} \int_{C \leq \sigma_A(y, \xi) \leq L} \mathbf{1}_{[\sigma_A(y, \xi)|x-y| < 1]} e^{i\psi(x, y, \xi)} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) \sigma_A(y, \xi)^{-\frac{d+1}{m}} d\xi.$$

Since the integrand equals  $\mathbf{1}_{[\sigma_A(y, \xi)|x-y| < 1]} A_P(x, y, \xi, \sigma_A(y, \xi)^{\frac{1}{m}}) \sigma_A(y, \xi)^{-\frac{1}{m}}$ , by equation (12),  $II_L(x, y)$  equals

$$\frac{1}{(2\pi)^n} \int_{C|x-y|^m \leq \sigma_A(y, \eta) \leq L|x-y|^m} \mathbf{1}_{[\sigma_A(y, \eta) < 1]} A_P(x, y, |x-y|\eta, |x-y|\sigma_A(y, \eta)^{\frac{1}{m}}) \sigma_A(y, \eta)^{-\frac{1}{m}} d\eta.$$

According to the first point of Lemma 5.3, there exist  $V \subset U$  a neighborhood of 0 and a constant  $C' > 0$  such that for any  $x, y \in V$  and any  $\eta \in \mathbb{R}^n$ ,

$$\left| A_P(x, y, |x-y|\eta, |x-y|\sigma_A(y, \eta)^{\frac{1}{m}}) - A_P(x, y, |x-y|\eta, 0) \right| \leq C' \sigma_A(y, \eta)^{\frac{1}{m}} |x-y|^d |\eta|^d$$

so that there exists  $C'' > 0$  for which

$$\begin{aligned} \left| II_L(x, y) - \frac{1}{(2\pi)^n} \int_{C|x-y|^m \leq \sigma_A(y, \eta) \leq L|x-y|^m} \mathbf{1}_{[\sigma_A(y, \eta) < 1]} A_P(x, y, |x-y|\eta, 0) \sigma_A(y, \eta)^{-\frac{1}{m}} d\eta \right| \\ \leq C'' \min(L^{\frac{1}{m}}|x-y|^{d+1}, |x-y|^d). \end{aligned}$$

In particular, this is uniformly bounded for  $x, y \in V$ . Now, by definition of  $J_P$  (see equation (6)),

$$\int_{C|x-y|^m \leq \sigma_A(y, \eta) \leq L|x-y|^m} \mathbf{1}_{[\sigma_A(y, \eta) < 1]} A_P(x, y, |x-y|\eta, 0) \sigma_A(y, \eta)^{-\frac{1}{m}} d\eta$$

equals

$$J_P(x, y, 0) \int_{C^{\frac{1}{m}}|x-y|}^{L^{\frac{1}{m}}|x-y|} \mathbf{1}_{[|x-y|s < 1]} \frac{ds}{s}.$$

Finally

$$\begin{aligned} \int_{C^{\frac{1}{m}}}^{L^{\frac{1}{m}}} \mathbf{1}_{[|x-y|s < 1]} \frac{ds}{s} &= \int_{C^{\frac{1}{m}}|x-y|}^{L^{\frac{1}{m}}|x-y|} \mathbf{1}_{[t < 1]} \frac{dt}{t} \\ &= \ln\left(L^{\frac{1}{m}}\right) - \ln_+\left(L^{\frac{1}{m}}|x-y|\right) - \ln\left(C^{\frac{1}{m}}\right) \end{aligned}$$

so that, uniformly for any  $(x, y) \in V \times V$  and  $L > 0$ ,

$$II_L(x, y) = \frac{1}{(2\pi)^n} J_P(x, y, 0) \left[ \ln\left(L^{\frac{1}{m}}\right) - \ln_+\left(L^{\frac{1}{m}}|x-y|\right) - \ln\left(C^{\frac{1}{m}}\right) \right] + O(1).$$

Finally, by equation (11)  $J_P(x, y, 0) = Y_P(x, y)$ . □

## 5.4 Proof of Lemmas 5.1 and 5.6

In this section we prove Lemmas 5.1 and 5.6 used in the proof of Theorems 2.4 and 2.6. These lemmas follow from the properties of the phase  $\psi$  and their proof does not rely on any other results proved in this article.

**Proof of Lemma 5.1.** The first point follows by the third and fourth point in Definition 2.1. The last point is just a restatement of the first point in Definition 2.1. To prove the second point first we fix a compact subset  $K \subset U$  as well as a multiindex  $\alpha \in \mathbb{N}^n$ . Given  $\xi \in \mathbb{R}^n$  such that  $|\xi| \geq 1$ , set  $u = |\xi|$  and  $\eta = |\xi|^{-1}\xi$ . By the fifth point of Definition 2.1, the family  $(\psi_t)_{t \geq 1} = (t^{-1}\psi(\cdot, \cdot, t))$  is bounded in  $C^\infty$ . Hence, uniformly for  $x, y \in K$ ,  $t \geq 1$ , and  $\xi, u$  and  $\eta$  as before,

$$\partial_\eta^\alpha \psi_{ut}(x, y, \eta) = \partial_\eta^\alpha \psi_{ut}(y, y, \eta) + \partial_x \partial_\eta^\alpha \psi_{ut}(y, y, \eta)(x - y) + O(|x - y|^2).$$

Now, by the first point of Lemma 5.1,  $\partial_\eta^\alpha \psi_{ut}(y, y, \eta) = 0$  and by the fourth point in Definition 2.1,  $\partial_\eta^\alpha \psi_{ut}(y, y, \eta)(x - y) = \langle \eta, x - y \rangle$ . Multiplying the above expansion by  $u$  and replacing  $u$  and  $\eta$  by their definitions, one obtains, uniformly in  $x, y \in K$ ,  $t \geq 1$  and  $\xi \in \mathbb{R}^n$  with  $|\xi| \geq 1$ ,

$$\partial_\xi^\alpha \psi_t(x, y, \xi) = \langle \xi, x - y \rangle + O(|x - y|^2 |\xi|).$$

The proof of the third point of Lemma 5.1 is analogous to that of the second point, but with  $\partial_\xi^\alpha$  replaced by  $\partial_x$  or  $\partial_y$ . We mention only that by combining the third and fourth points of Definition 2.1, one obtains that  $\partial_y \psi(x, y, \xi) \Big|_{x=y} = -\xi$ .  $\square$

Now we prove Lemma 5.6.

**Proof of Lemma 5.6.** Let us fix a compact subset  $K \subset U \times U$ . Since  $\dim(\mathcal{M}) = 1$ , any such compact subset is contained in a finite union of closed rectangular subsets of  $U \times U$ . Thus, we assume  $K = I \times I$  where  $I$  is a segment contained in  $U$ . By the fourth point of Lemma 5.1 there exists  $C > 0$  such that for each  $(x, y) \in K$  and each non-zero  $\xi \in \mathbb{R}$ ,  $|\partial_\xi^2 \partial_x^2 \psi(x, y, \xi)| \leq C |\xi|^{-1}$ . Moreover, by the first statement of Lemma 5.1,  $\psi(x, x, \xi) = 0$  and  $\partial_x \psi(x, x, \xi) = \xi$  for any  $x \in U$  and any  $\xi \in \mathbb{R}$ . Therefore, for any  $(x, y) \in K$ , and any non-zero  $\xi \in \mathbb{R}$ ,

$$|\partial_\xi^2 \psi(x, y, \xi)| = \left| \int_y^x \int_y^z \partial_\xi^2 \partial_x^2 \psi(w, y, \xi) dw dz \right| \leq \frac{c}{2} (x - y)^2 |\xi|^{-1}.$$

This proves the first statement of the lemma. The proof of the second statement is analogous to the first. Now, since the symbol  $\sigma_A$  of  $A$  is polyhomogeneous and  $\dim(\mathcal{M}) = 1$  there exist  $c > 0$ ,  $\varrho \in C^\infty(U)$  a positive function as well as  $\tilde{\tau} \in S^{m-1}(U \times \mathbb{R})$  such that  $\sigma_A(x, \xi) = \varrho(x)^m |\xi|^m + \tilde{\tau}(x, \xi)$  for  $|\xi| \geq c$  and  $x \in U$ . By construction of  $\psi$  as explained in Lemma 7.1, there exists another symbol  $\tau \in S^0(U \times \mathbb{R})$  such that if  $\sigma(x, \xi) = \varrho(x) |\xi| + \tau(x, \xi)$  for  $|\xi| \geq c$  and  $x \in U$ , then

$$\forall \xi \in \mathbb{R} \setminus [-c, c], \forall x, y \in U, \sigma(x, \partial_x \psi(x, y, \xi)) = \sigma(y, \xi).$$

Since  $\tau \in S^0$ , there exists a constant larger than  $\max(c, 1)$  which we will also call  $c$  such that for any  $x \in I$  and  $\xi \in \mathbb{R}$  such that  $|\xi| \geq c$ ,

$$\begin{aligned} \frac{1}{c}|\xi| &\leq \sigma(x, \xi) \leq c|\xi| \\ \frac{1}{c} &\leq \partial_\xi \sigma(x, \xi) \leq c. \end{aligned}$$

Let  $(\sigma^{-1})(x, \cdot)$  be the inverse of  $\sigma(x, \cdot) : [c, +\infty[ \rightarrow [\sigma(x, c), +\infty[$ . Let us fix  $x_0 \in I$ . Then, for any  $x \in I$ ,

$$\partial_x \psi(x, x_0, \xi) = (\sigma^{-1})(x, \sigma(x_0, \xi)).$$

Differentiating this equation we obtain the following expression for  $\partial_\xi \partial_x \psi$ .

$$\partial_\xi \partial_x \psi(x, x_0, \xi) = \partial_\xi (\sigma^{-1})(x, \sigma(x_0, \xi)) \partial_\xi \sigma(x_0, \xi).$$

Now, by definition of  $\sigma^{-1}$ , we have, for  $x \in I$  and  $\xi \in \mathbb{R}$  such that  $\xi \geq c' = \max_{y \in I} \sigma(y, c)$ ,

$$\partial_\xi (\sigma^{-1})(x, \xi) = \left( \partial_\xi \sigma(x, \sigma^{-1}(x, \xi)) \right)^{-1} = \left( \varrho(x) + \partial_\xi \tau(x, \sigma^{-1}(x, \xi)) \right)^{-1}$$

where  $\varrho(x)$  is bounded on  $I$  from above and below by positive constants and  $\partial_\xi \tau(x, \sigma^{-1}(x, \xi))$  is  $O(|\sigma^{-1}(x, \xi)|^{-1})$  uniformly for  $x \in I$ . Since  $\sigma^{-1}(x, \xi) \xrightarrow[\xi \rightarrow +\infty]{} +\infty$  then

$$\varrho(x) + \partial_\xi \tau(x, \sigma^{-1}(x, \xi)) \rightarrow \varrho(x)$$

so that  $\partial_\xi (\varrho(x) \sigma^{-1}(x, \xi)) \xrightarrow[\xi \rightarrow +\infty]{} 1$  where both convergences are uniform for  $x \in I$ . As a consequence, uniformly for  $x \in I$  we have  $\sigma^{-1}(x, \xi) \sim \varrho(x)^{-1} \xi$  as  $\xi \rightarrow +\infty$ . Therefore,

$$\partial_x \partial_\xi \psi(x, x_0, \xi) = \partial_\xi (\sigma^{-1})(x, \sigma(x_0, \xi)) \partial_\xi \sigma(x_0, \xi) \rightarrow \frac{\varrho(x_0)}{\varrho(x)}.$$

Since  $\varrho$  is bounded from above and below by positive constants, there exists  $C > 0$  such that for any  $x \in I$  and any  $\xi \in \mathbb{R}$  such that  $\xi \geq C$ ,

$$C^{-1} \leq \partial_x \partial_\xi \psi(x, x_0, \xi) \leq C.$$

Recall that, by the first point of Lemma 5.1,  $\psi(x, x, \xi) = 0$  for any  $x \in U$  and any  $\xi \in \mathbb{R}$ . Thus, for any  $\xi \geq C$ ,

$$|\partial_\xi \psi(x, x_0, \xi)| = \left| \int_{x_0}^x \partial_\xi \partial_x \psi(y, x_0, \xi) dy \right| \in [C^{-1}|x - x_0|, C|x - x_0|]$$

where  $C$  is independent of  $x \in I$  and  $x_0 \in I$ . The case where  $\xi < 0$  is symmetric. This proves the third statement of the lemma.  $\square$

## 6 Non-degeneracy conditions for $\sigma_A$

As announced in the introduction the admissibility assumption for the symbol implies the decay of certain oscillatory integrals. To make this more precise, let us introduce the following terminology. For any subset  $E \subset \mathbb{R}^k$  we endow  $C^0(E)$ , the space of continuous functions on  $E$ , with the topology of uniform convergence on compact sets. For any open subset  $F \subset \mathbb{R}^k$  we endow  $C^\infty(F)$  with the topology of uniform convergence of derivatives on compact subsets.

**Definition 6.1.** *Let  $\varepsilon > 0$ ,  $m > 0$  and let  $U \subset \mathbb{R}^n$  be an open subset. Let  $\sigma \in C^\infty(U \times \mathbb{R}^n \setminus \{0\})$  be homogeneous of degree  $m$  in the second variable. For each  $x \in U$ , let  $S_x^* = \{\xi \in \mathbb{R}^n \setminus \{0\} \mid \sigma(x, \xi) = 1\}$ .*

1. *Given a compact subset  $K \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$  let us call a deformation of the height function for  $\sigma$  over  $K$  any family of continuous functions*

$$f_\eta : \{(x, \tau, \xi) \in K \times \mathbb{R}^n \mid \sigma(x, \xi) = 1\} \rightarrow \mathbb{R}$$

*indexed by  $\eta \in \mathbb{R}^p$  such that*

- *the function  $f_0(x, \tau, \xi) = \langle \tau, \xi \rangle$*
- *the following two maps are continuous*

$$\begin{aligned} \mathbb{R}^p &\rightarrow C^0(K \times \mathbb{R}^n) \\ \eta &\mapsto f_\eta \\ \mathbb{R}^p \times K &\rightarrow C^\infty(\mathbb{R}^n) \\ (\eta, x, \tau) &\mapsto f_\eta(x, \tau, \cdot). \end{aligned}$$

2. *We say that  $\sigma$  has  $\varepsilon$ -non-degenerate energy levels if, for any compact subset  $K$  of  $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$  and any deformation of the height function  $(f_\eta)_\eta$  for  $\sigma$  over  $K$  there exist  $C > 0$ ,  $V \subset \mathbb{R}^p$  a neighborhood of 0 and  $k \in \mathbb{N}$  depending only on  $K$  and  $\alpha$  such that for each  $(x, \tau) \in K$ , each  $t > 0$ , each  $\eta \in V$  and each  $u \in C^\infty(\mathbb{R}^n)$ ,*

$$\left| \int_{S_x^*} e^{itf_\eta(x, \tau, \xi)} u(\xi) d_x \nu(\xi) \right| \leq C \|u\|_{C^k} t^{-\varepsilon}. \quad (14)$$

*Here  $\|\cdot\|_{C^k}$  denotes the  $C^k$  norm.*

3. *We say that a homogeneous symbol on a manifold has non-degenerate energy levels if it has this property when written in any local coordinate system.*

Note that for the case of symbols on a manifold, since coordinate changes act linearly on the fibers of  $T^*\mathcal{M}$  and since linear transformations do not affect the decay of the above integrals, it is enough to check the criterion for one atlas. We prove the following proposition, which may be of independent interest to some readers.

**Proposition 6.2.** *Homogeneous positive symbols of degree  $m > 0$  on  $\mathcal{M}$  admissible for some integer  $k_0 \geq 2$  have  $\frac{1}{k_0}$ -non-degenerate energy levels.*

In this section, we provide a proof of Propositions 6.2 and 2.7. For this we need three lemmas which we state here and prove at the end of the section.

**Lemma 6.3.** *Let  $m$  be a positive real number,  $U$  be an open set of  $\mathbb{R}^p$ ,  $f \in C^\infty(U)$  be a positive function,  $b \in \mathbb{R}$  and  $v \in \mathbb{R}^p$ . Then, for all  $k \geq 2$  and for all  $x \in U$ ,*

$$\begin{aligned} \forall h \in \{1, \dots, k-1\}, \quad d_x^h \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] &= 0 \Rightarrow \\ d_x^k \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] &= -\frac{1}{m} (\langle x, v \rangle + b) f(x)^{-\frac{1}{m}-k} \\ &\quad \left[ f(x)^{k-1} d_x^k f - \frac{m(m-1) \dots (m-k+1)}{m^k} (d_x f)^{\otimes k} \right]. \end{aligned}$$

**Lemma 6.4.** *Let  $m$  be a positive real number and let  $f \in C^\infty(\mathbb{R}^p \setminus \{0\})$  be an  $m$ -homogeneous function. Then, for each  $x \in \mathbb{R}^p \setminus \{0\}$ , each hyperplane  $H \subset \mathbb{R}^p$  not containing  $x$  and each  $k_0 \geq 2$ ,*

$$\forall k \in \{2, \dots, k_0\}, \quad f(x)^{k-1} d_x^k f = \frac{m(m-1) \dots (m-k+1)}{m^k} (d_x f)^{\otimes k} \quad (15)$$

is equivalent to

$$\forall k \in \{2, \dots, k_0\}, \quad f(x)^{k-1} d_x^k f|_H = \frac{m(m-1) \dots (m-k+1)}{m^k} (d_x f)^{\otimes k}|_H. \quad (16)$$

The following lemma is a generalization of the classical stationary phase formula. There are many generalizations of this formula (see for instance [1] as well as section 7.7 of [11]). However, we were unable to find this particular result in the literature. Essentially, we apply Malgrange's preparation theorem to reduce the problem to the case of polynomial phases which is dealt with in [1].

**Lemma 6.5.** *Let  $p, q \in \mathbb{N}$ ,  $q \geq 1$ . We denote by  $\eta$  the variables in  $\mathbb{R}^p$  and  $x$  the variables in  $\mathbb{R}^q$ . Let  $(f_\eta)_{\eta \in \mathbb{R}^p}$  be a family of smooth functions on  $\mathbb{R}^q$  whose derivatives depend smoothly on the parameter  $\eta$  for uniform convergence on compact sets. Suppose that there exists  $k \geq 1$  such that  $d_0^k f_0 \neq 0$ . Then, there exist neighborhoods  $U$  of  $0 \in \mathbb{R}^p$  and  $V$  of  $0 \in \mathbb{R}^q$ , an integer  $l \in \mathbb{N}$  and a constant  $C > 0$  such that we have the following estimate. For each  $u \in C_c^\infty(V)$ , each  $\eta \in U$  and each  $a > 0$ ,*

$$\left| \int_{\mathbb{R}^q} e^{iaf_\eta(x)} u(x) dx \right| \leq C \|u\|_{C^l} a^{-\frac{1}{k}}. \quad (17)$$

Here, if  $u \in C^\infty(\mathbb{R}^q)$ ,  $\|u\|_{C^l}$  denotes  $\sum_{|\alpha| \leq l} \sup_{x \in \mathbb{R}^q} |\partial^\alpha u(x)|$ .

We now prove the proposition using these three results.

**Proof of Proposition 6.2.** Let us fix some local coordinates around a point in  $\mathcal{M}$ . Recall that for any  $x \in \mathbb{R}^n$ ,  $S_x^* = \{\xi \in \mathbb{R}^n, \sigma(x, \xi) = 1\}$ . Fix a compact subset

$K \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$  and let  $(f_\eta)_{\eta \in \mathbb{R}^p}$  be a deformation of the height function of  $\sigma_A$  over  $K$  (see Definition 6.1). We wish to control the behavior of the following integral

$$\int_{S_x^*} e^{itf_\eta(x, \tau, \xi)} u(\xi) d_x \nu(\xi)$$

where  $x \in \mathbb{R}^n$ ,  $\eta \in \mathbb{R}^p$ ,  $t > 0$  and  $u \in C^\infty(\mathbb{R}^n)$ . Let  $x_0 \in \mathbb{R}^n$  and  $\xi_0 \in S_{x_0}^*$ . Let  $(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^n$  be such that the family  $(\xi_0, \xi_1, \dots, \xi_{n-1})$  is independent. For any  $x \in \mathbb{R}^n$ , let

$$\begin{aligned} \beta_x : \mathbb{R}^{n-1} &\rightarrow S_x^* \\ (q_1, \dots, q_{n-1}) &\mapsto \sigma(x, \xi_0 + \sum_{j=1}^{n-1} q_j \xi_j)^{-\frac{1}{m}} \left( \xi_0 + \sum_{j=1}^{n-1} q_j \xi_j \right). \end{aligned}$$

The condition on the  $\xi_j$  is open so there exist an open neighborhood  $D$  of  $0 \in \mathbb{R}^{n-1}$  an open neighborhood  $A \times B$  of  $(x_0, \xi_0)$  such that for each  $(x, \xi) \in A \times B$  with  $\sigma(x, \xi) = 1$ ,  $\beta_x : D \rightarrow S_x^*$  defines local coordinates near  $\xi$ . By partition of unity, we may restrict our attention to the case where  $u \in C^\infty(\mathbb{R}^n)$  is supported in  $B$ . Notice that for any  $x \in A$ , for any  $q = (q_1, \dots, q_{n-1}) \in \beta_x^{-1}(B)$  and for any  $v \in \mathbb{R}^n$ ,

$$f_0(x, \tau, \beta_x(q)) = \sigma(x, \xi_0 + \sum_{j=1}^{n-1} q_j \xi_j)^{-\frac{1}{m}} \left( \langle \xi_0, \tau \rangle + \sum_{j=1}^{n-1} q_j \langle \xi_j, \tau \rangle \right).$$

Let  $\theta_\eta(x, \tau, q) := f_\eta(x, \tau, \beta_x(q))$ . Then,  $\eta \mapsto \theta_\eta$  is continuous from  $\mathbb{R}^p$  into  $C^0(A \times B \times D)$  for the topology of uniform convergence on compact sets. Moreover,  $(\eta, x, \tau) \mapsto \theta_\eta(x, \tau, \cdot)$  is continuous from  $\mathbb{R}^p \times A \times B$  into  $C^\infty(D)$  for the topology of uniform convergence of derivatives on compact subsets. Let  $f : \mathbb{R}^n \times D \rightarrow \mathbb{R}$  be defined by  $f(x, q) = \sigma(x, \xi_0 + \sum_{j=1}^{n-1} q_j \xi_j)$ . Then,  $\theta_0(x, \tau, q) = f(x, q)^{-\frac{1}{m}} \left( \langle \xi_0, \tau \rangle + \sum_{j=1}^{n-1} q_j \langle \xi_j, \tau \rangle \right)$ . Changing variables by  $\beta_x$  in the integral, it is enough to prove that for each  $x_0 \in \mathbb{R}^n$  and  $\tau_0 \in \mathbb{R}^n \setminus \{0\}$ , there exists a constant  $C > 0$ , an integer  $l \in \mathbb{N}$  and a neighborhood  $V$  of  $(x_0, \tau_0, 0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$  such that for any  $(x, \tau, \eta) \in V$ , any  $u \in C^\infty(D)$  and any  $t > 0$ ,

$$\left| \int_{\mathbb{R}^{n-1}} e^{it\theta_\eta(x, \tau, q)} u(q) dq \right| \leq C \|u\|_{C^l} t^{-\frac{1}{k_0}}.$$

By Lemma 6.5 it suffices to show that for each  $x \in A$  and each  $v \in \mathbb{R}^n \setminus 0$ , there exists  $1 \leq k \leq k_0$  such that  $d_q^k \theta_0(x, v, 0) \neq 0$ . Suppose that for each  $1 \leq k \leq k_0$ ,  $d_q^k \theta_0(x, v, 0) = 0$ . Then,  $k = 1$  implies, for each  $1 \leq j \leq n-1$ ,  $\langle \xi_j, v \rangle = \frac{1}{m} \langle \xi_0, v \rangle f(x, 0)^{-1} \partial_{q_j} f(x, 0)$ . In particular,  $\langle \xi_0, v \rangle \neq 0$ . Indeed, otherwise, for each  $j$ ,  $\langle \xi_j, v \rangle = 0$  and  $v = 0$ . Hence, by Lemma 6.3, for each  $2 \leq k \leq k_0$ ,  $d_q^k f(x, 0) = \frac{m(m-1)\dots(m-k+1)}{m^k} f^{k-1}(x, 0) (d_q f(x, 0))^{\otimes k}$ . Since the chart  $\phi$  acts linearly on the fibers of the cotangent bundle, this implies that there exists  $(x, \xi) \in T^* \phi^{-1}(A)$  such that for all  $2 \leq k \leq k_0$ ,  $\sigma(x, \xi)^{k-1} \partial_\xi^k \sigma(x, \xi) = \frac{m(m-1)\dots(m-k+1)}{m^k} (\partial_\xi \sigma(x, \xi))^{\otimes k}$  when restricted to the hyperplane  $H \subset T_\xi^*(T_z^* M) \simeq T_z^* M$



spanned by  $((d_z\phi)^*\xi_j)_{j=1,\dots,n-1}$ . But the  $w_j$  are such that  $\xi \notin H$ . Since  $\sigma$  is  $m$  homogeneous, then, by Lemma 6.6, for each  $2 \leq k \leq k_0$ ,

$$\sigma(x, \xi)^{k-1} \partial_\xi^k \sigma(x, \xi) = \frac{m(m-1) \dots (m-k+1)}{m^k} (\partial_\xi \sigma(x, \xi))^{\otimes k}.$$

This contradicts our initial assumption (see equation (2)).  $\square$

We still need to prove the three lemmas.

**Proof of Lemma 6.3.** First, for all  $x \in \mathbb{R}^p$ ,

$$d_x \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] = -\frac{1}{m} f(x)^{-\frac{1}{m}-1} (\langle x, v \rangle + b) d_x f + f(x)^{-\frac{1}{m}} \langle \cdot, v \rangle.$$

Therefore,  $d_x \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] = 0$  implies that  $\langle \cdot, v \rangle = \frac{1}{m} (\langle x, v \rangle + b) f(x)^{-1} d_x f$ . We will first prove that there exists a sequence of real numbers  $(a_k)_{k \geq 2}$  depending only on  $m$  such that for all  $k \geq 2$  and all  $x \in U$ ,

$$\begin{aligned} \forall 1 \leq h \leq k-1, \quad d_x^h \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] = 0 &\Rightarrow \\ d_x^k \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] &= -\frac{1}{m} (\langle x, v \rangle + b) f(x)^{-\frac{1}{m}-k} \left[ f(x)^{k-1} d_x^k f - a_k (d_x f)^{\otimes k} \right]. \end{aligned}$$

Afterwards, we will identify  $(a_k)$  using a suitable choice for  $f$ . To prove the existence of the  $(a_k)$ , we will consider the successive  $d_x^h f$  as elements of the commutative algebra of symmetric multilinear forms on  $\mathbb{R}^p$ . By the Leibniz rule, for each  $k$ , there exist two polynomials  $A_k, B_k \in \mathbb{R}[X_0, \dots, X_{k-1}]$  such that

$$\begin{aligned} d_x^k \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] &= -\frac{1}{m} f(x)^{-\frac{1}{m}-k} \left[ f(x)^{k-1} (\langle x, v \rangle + b) d_x^k f \right. \\ &\quad \left. + A_k(f(x), d_x f, \dots, d_x^{k-1} f) (\langle x, v \rangle + b) + B_k(f(x), d_x f, \dots, d_x^{k-1} f) \langle \cdot, v \rangle \right]. \end{aligned}$$

Moreover,  $A_k$  and  $B_k$  are such that  $A_k(f(x), d_x f, \dots, d_x^{k-1} f)$  and  $B_k(f(x), d_x f, \dots, d_x^{k-1} f)$  are  $k$ -linear forms. The observation made above shows that there exists  $C_k \in \mathbb{R}[X_0, \dots, X_{k-1}]$  of degree  $k$  such that if we assume that  $d_x \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] = 0$  then

$$d_x^k \left[ f(x)^{-\frac{1}{m}} (\langle x, v \rangle + b) \right] = -\frac{1}{m} (\langle x, v \rangle + b) f(x)^{-\frac{1}{m}-k} \left[ f(x)^{k-1} d_x^k f - C_k(f(x), d_x f, \dots, d_x^{k-1} f) \right].$$

Moreover,  $C_k$  is such that  $C_k(f(x), d_x f, \dots, d_x^{k-1} f)$  is a  $k$ -linear form. Now, we work by induction. For  $k = 2$ ,  $C_k$  must be a multiple of  $X_1^2$ . Let  $k > 2$  and suppose that the lemma is true for all  $h \leq k-1$ . If  $(\langle x, v \rangle + b) = 0$  we are done. Otherwise, for each  $2 \leq h \leq k-1$ ,  $d_x^h f = m a_h f(x)^{1-h} (d_x f)^{\otimes h}$ . Replacing each  $d_x^h f$  by this expression in  $C_k(f(x), d_x f, \dots, d_x^{k-1} f)$  results in a homogeneous polynomial of degree  $k$  in  $f(x)$  and  $d_x f$  such that each term is a  $k$ -linear form. But these constraints imply that it is a real multiple of  $(d_x f)^{\otimes k}$ .

Now that we have proved the existence of the sequence  $(a_k)$  we choose  $f(x) = x^m$  defined

on  $]0, +\infty[$ ,  $b = 0$  and  $v = 1 \in \mathbb{R}$  such that for any  $x \in ]0, +\infty[$ ,  $f(x)^{-\frac{1}{m}} \langle x, v \rangle = 1$ . This function is constant so all of its derivatives vanish. Therefore, taking for instance  $x = 1$ , for all  $k \geq 2$ ,

$$m(m-1) \dots (m-k+1) = \left[ (x^m)^{k-1} \frac{d^k}{dx^k} (x^m) \right] \Big|_{x=1} = a_k \left( \frac{d}{dx} \Big|_{x=1} (x^m) \right)^k = a_k m^k.$$

□

**Proof of Lemma 6.4.** Equation (15) implies (16) by restriction to  $H$ . Let us assume (16) and prove the converse. Since  $x \notin H$ ,  $\mathbb{R}x \oplus H$  generate  $\mathbb{R}^p$ . By multilinearity, it is enough to prove (15) when the  $k$  forms are evaluated on families of the form  $(x, \dots, x, y_1, \dots, y_h)$  where  $y_1, \dots, y_h \in H$  and  $h \in \{1, \dots, k\}$ . Now, since  $f$  is homogeneous, by Euler's equation, for any  $h \leq k$ , and for any  $y_1, \dots, y_h \in H$ ,

$$d_x^k f(x, \dots, x, y_1, \dots, y_h) = \underbrace{(m-h) \dots (m-k+1)}_{1 \text{ if } k=h} d_x^h f(y_1, \dots, y_h)$$

$$\text{and } (d_x f)^{\otimes k}(x, \dots, x, y_1, \dots, y_h) = m^{k-h} f^{k-h}(x) (d_x f)^{\otimes h}(y_1, \dots, y_h).$$

Applying (16) to compare the right hand sides of each line we get equation (15). □

In order to prove Lemma 6.5, we need the following multilinear algebra result.

**Lemma 6.6.** *Let  $T$  be a symmetric  $k$ -linear form on  $\mathbb{R}^l$ . For each  $v \in \mathbb{R}^l$ , let  $\Theta(v) = T(v, v, \dots, v)$ . Then, for all  $v_1, \dots, v_k \in \mathbb{R}^l$ ,*

$$T(v_1, \dots, v_k) = 2^{-k} \sum_{\epsilon \in \{-1, 1\}^k} \prod_{i=1}^k \epsilon_i \Theta \left( \sum_{j=1}^k \epsilon_j v_j \right).$$

In particular,  $\Theta = 0 \Rightarrow T = 0$ .

**Proof of Lemma 6.6.** Given any multiindex  $p = (p_1, \dots, p_k) \in \mathbb{N}^k$  and any  $v_1, \dots, v_k \in \mathbb{R}^l$ , we will denote by  $T[v_1^{p_1} \dots v_k^{p_k}]$  the form  $T$  evaluated in the  $v_j$ 's where the  $j$ th term appears  $p_j$  times. This is well defined because  $T$  is symmetric. Then, for each  $v_1, \dots, v_k \in \mathbb{R}^l$ ,

$$\begin{aligned} \sum_{\epsilon \in \{-1, 1\}^k} \prod_{i=1}^k \epsilon_i \Theta \left( \sum_{j=1}^k \epsilon_j v_j \right) &= \sum_{\epsilon \in \{-1, 1\}^k} \prod_{i=1}^k \epsilon_i \sum_{p_1 + \dots + p_k = k} \binom{k}{p_1, \dots, p_k} T[(\epsilon_1 v_1)^{p_1} \dots (\epsilon_k v_k)^{p_k}] \\ &= \sum_{p_1 + \dots + p_k = k} \binom{k}{p_1, \dots, p_k} T[v_1^{p_1} \dots v_k^{p_k}] \sum_{\epsilon \in \{-1, 1\}^k} \prod_{j=1}^k \epsilon_j^{p_j+1}. \end{aligned}$$

Given  $j \in \{1, \dots, k\}$  and  $(p_1, \dots, p_k)$  such that  $p_j = 0$ , applying the bijection  $(\epsilon_1, \dots, \epsilon_k) \mapsto (\epsilon_1, \dots, -\epsilon_j, \dots, \epsilon_k)$  shows that  $\sum_{\epsilon \in \{-1, 1\}^k} \prod_{j=1}^k \epsilon_j^{p_j+1} = 0$ . Thus, the only remaining term is the one corresponding to  $p_1 = \dots = p_k = 1$ . Therefore,

$$\sum_{\epsilon \in \{-1, 1\}^k} \prod_{i=1}^k \epsilon_i \Theta \left( \sum_{i=1}^k \epsilon_i v_i \right) = 2^k T(v_1, \dots, v_k).$$

□

**Proof of Lemma 6.5.** We start with the case where  $q = 1$ . For each  $\eta \in \mathbb{R}^p$ , we denote by  $d_x^j f_\eta \in \mathbb{R}$  the  $j$ th derivative of  $f_\eta$ . Let  $k$  be the smallest positive integer such that  $d_0^k f_0 \neq 0$ . Without loss of generality, we may assume that  $d_0^k f_0 > 0$  and that for all  $\eta \in \mathbb{R}^q$ ,  $f_\eta(0) = 0$ . We now apply Theorem 7.5.13 of [11], a variant of the Malgrange preparation theorem, to the function  $x \mapsto f_\eta(x)$ . According to this theorem, there exist smooth functions  $y_\eta \in C^\infty(\mathbb{R})$  and  $b_1, \dots, b_{k-2} \in C^0(\mathbb{R}^p)$  such that

$$\begin{aligned} y_\eta(0) &= 0 \\ y'_\eta(0) &> 0 \\ \forall j \in \{1, \dots, k-2\}, b_j(0) &= 0 \\ \forall \eta \in \mathbb{R}^p, f_\eta(x) &= \frac{y_\eta(x)^k}{k} + b_{k-2}(\eta)y_\eta(x)^{k-2} + \dots + b_1(\eta)y_\eta(x) \text{ if } k \geq 2 \\ \forall \eta \in \mathbb{R}^p, f_\eta(x) &= y_\eta(x) \text{ if } k = 1. \end{aligned}$$

Moreover, since the derivatives of  $f_\eta$  depend continuously on  $\eta$ , the proof of Theorem 7.5.13 shows that  $(y_\eta)_\eta$  can be chosen to have the same property. In particular there exists an open neighborhood  $W$  of 0 in  $\mathbb{R}^p$  and an  $\varepsilon > 0$  such that for  $\eta \in W$ ,  $x \mapsto y_\eta(x)$  is a local diffeomorphism from  $I = ]-\varepsilon, \varepsilon[$  onto its image. By imposing that  $u$  in equation (17) must be supported in  $I$  and changing variables in the integral through this diffeomorphism, we reduce the problem to the case where  $f$  is a polynomial in the integration variable with coefficients depending continuously on  $\eta$ . But this case is well known, see for instance Theorems 7.5 and 7.7 of [1].

For the multidimensional case, let  $q \geq 2$  and let  $k$  be the smallest positive integer such that  $d_0^k f_0 \neq 0$ . By Lemma 6.6 there exists  $v \in \mathbb{R}^q$  such that  $\left. \frac{d^k}{dt^k} f_0(tv) \right|_{t=0} \neq 0$ . Note that  $v$  must be non-zero. By applying a linear isomorphism to  $\mathbb{R}^q$  we may, without loss of generality, assume that  $\partial_{x_q}^k f_0(0) \neq 0$ . Set  $\tilde{x} = (x_1, \dots, x_{q-1})$  and consider  $U$  a bounded open neighborhood of  $(0, 0) \in \mathbb{R}^{p+q-1}$  and an open interval  $I \subset \mathbb{R}$  containing 0, an integer  $l \in \mathbb{N}$  and a constant  $C > 0$  given by the one-dimensional case. Let  $\tilde{U}$  be the projection of  $U$  onto the first  $q-1$  coordinates. Then, given  $u \in C^\infty(\mathbb{R}^q)$  compactly supported in  $\tilde{U} \times I$ ,  $a > 0$  and  $\eta \in \mathbb{R}^p$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^q} e^{iaf_\eta(\tilde{x}, x_q)} u(\tilde{x}, x_q) dx \right| &\leq \int_{\mathbb{R}^{q-1}} \left| \int_{\mathbb{R}} e^{iaf_\eta(\tilde{x}, x_q)} u(\tilde{x}, x_q) dx_q \right| d\tilde{x} \\ &\leq \text{Vol}(\tilde{U}) \sup_{\tilde{x} \in \tilde{U}} \left| \int_{\mathbb{R}} e^{iaf_\eta(\tilde{x}, x_q)} u(\tilde{x}, x_q) dx_q \right|. \end{aligned}$$

Now, setting  $\tilde{\eta} = (\eta, \tilde{x})$  as a new parameter,  $f_{\tilde{\eta}} = f_\eta(\tilde{x}, \cdot)$  satisfies the hypotheses of Lemma 6.5 so we may apply the one-dimensional case. There exists  $C > 0$ , independent of  $u$  and  $a$  as before, such that

$$\begin{aligned} &\leq C \sup_{\tilde{x} \in \tilde{U}} \|u(\tilde{x}, \cdot)\|_{C^l} a^{-\frac{1}{k}} \\ &\leq C \|u\|_{C^l} a^{-\frac{1}{k}}. \end{aligned}$$

□

We now finish off the section by proving Proposition 2.7. Recall that  $S_h^m(\mathcal{M})$  is the space of smooth  $m$ -homogeneous symbols on  $\mathcal{M}$ . We endow  $S_h^m(\mathcal{M})$  with the restriction of the Whitney topology (see Definition 3.1 of [8]) on  $C^\infty(\mathcal{M})$ . Moreover, we denote by  $J^k$  the jet bundles and by  $j^k$  the jet maps as in Definition of 2.1 of [8]. Here we introduce an auxiliary metric  $g$  on  $\mathcal{M}$  and consider the cosphere bundle  $\Sigma_g^*\mathcal{M} = \{(x, \xi) \in T^*\mathcal{M} \mid \|\xi\|_g = 1\}$ . If we equip  $C^\infty(\Sigma_g^*\mathcal{M})$  with the Whitney topology, the restriction map  $\rho : S_h^m(\mathcal{M}) \rightarrow C^\infty(\Sigma_g^*\mathcal{M})$  is a homeomorphism. For each  $k_0 \geq 2$ , let  $X_{k_0}$  be the set of  $((x, \xi), f(x, \xi), d_{(x, \xi)}f, d_{(x, \xi)}^2f, \dots, d_{(x, \xi)}^{k_0}f) \in J^{k_0}(\Sigma_g^*\mathcal{M})$  such that  $f(x, \xi) > 0$  and

$$\forall 2 \leq k \leq k_0, f(x, \xi)^{k-1} \partial_\xi^k f(x, \xi) = \frac{m(m-1) \dots (m-k+1)}{m^k} (\partial_\xi f(x, \xi))^{\otimes k} \in (T_\xi^*(T_x^*\mathcal{M}))^{\otimes k}.$$

This system of equations is clearly regular provided  $f(x, \xi)^{k-1} > 0$ , in which case  $X_{k_0}$  is a submanifold of  $J^{k_0}(\Sigma_g^*\mathcal{M})$  of codimension

$$\sum_{k=2}^{k_0} \binom{n+k-2}{k}.$$

Here,  $\binom{n+k-2}{k}$  is the dimension of the space of symmetric homogeneous polynomials in  $n-1$  variables of degree  $k$ . Note that  $\dim(\Sigma_g^*M) = 2n-1$  and

$$2 \times 2 - 1 < 1 + 1 + 1 + 1 \quad (n = 2, k = 5)$$

$$2 \times 3 - 1 < 3 + 4 \quad (n = 3, k = 3)$$

$$2 \times 4 - 1 < 6 + 10 \quad (n = 4, k = 3)$$

$$\forall n \geq 5, 2 \times n - 1 < \binom{n}{2}.$$

By Thom's transversality Theorem 4.9 of [8], if  $k_0 = 5$  when  $n = 2$ ,  $k_0 = 3$  when  $n = 3$  or  $4$  and  $k_0 = 2$  when  $n \geq 5$ , then the set of functions  $f \in C^\infty(\Sigma_g^*\mathcal{M})$  such that  $(j^{k_0(n)}f)(\mathcal{M}) \cap X_{k_0(n)} = \emptyset$  is a residual set for the Whitney topology. Moreover, if  $\mathcal{M}$  is compact, it is open. By Lemma 6.4, for each  $k_0 \geq 2$  and  $(x, \xi) \in \Sigma_g^*\mathcal{M}$ , a symbol  $\sigma \in S_h^m(\mathcal{M})$  satisfies

$$\forall 2 \leq k \leq k_0, \sigma(x, \xi)^{k-1} \partial_\xi^k \sigma(x, \xi) = \frac{m(m-1) \dots (m-k+1)}{m^k} (\partial_\xi \sigma(x, \xi))^{\otimes k}$$

if and only if  $j^{k_0}(\rho(\sigma))(x, \xi) \in X_{k_0}$  and the proof is over. Here, the hyperplane  $H$  in Lemma 6.4 is  $T_\xi^*(\Sigma_x^*\mathcal{M}) \subset T_\xi^*(T_x^*\mathcal{M})$ .

## 7 Appendix : Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by following the approach used in [10] and in [6]. As explained above, [6] contains all the essential arguments for Theorem 2.2 despite the focus

on the case where  $x = y$ . In this section we merely wish to confirm this by revisiting the proof. We will use the notations of the introduction and make the additional assumption that  $A$  is of order  $m = 1$ . The general case will easily follow, as will be explained at the end.

## 7.1 Preliminaries

We call these functions symbols. Let  $\mathcal{M}$  be a smooth manifold of positive dimension  $n > 0$  endowed with a volume density  $d\mu_{\mathcal{M}}$ . Let  $A$  be a pseudo-differential operator on  $\mathcal{M}$  as in section 2 of order  $m = 1$ . Let  $\sigma_A$  be its principal symbol. The following lemma is a summary of the results proved in section 4 of [10].

**Lemma 7.1.** *Firstly, the spectral function of  $A$  defines a tempered distribution of the  $L$  variable with values in  $C^\infty(\mathcal{M} \times \mathcal{M})$ . In addition, for each set of local coordinates in which  $d\mu_{\mathcal{M}}$  coincides with the Lebesgue measure on  $\mathbb{R}^n$ , there is an open neighborhood  $U$  of  $0 \in \mathbb{R}^n$  such that there exist  $\varepsilon > 0$ , an admissible phase function  $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ , a symbol  $\sigma \in S^1(U, \mathbb{R}^n)$ , a function  $k \in C^\infty(U \times U \times ]-\varepsilon, \varepsilon[)$  and a symbol  $q \in S^0(U \times ]-\varepsilon, \varepsilon[ \times U, \mathbb{R}^n)$ , for which*

$$\mathcal{F}_L[E'_L(x, y)](t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} q(x, t, y, \xi) e^{i(\psi(x, y, \xi) - t\sigma(y, \xi))} d\xi + k(x, y, t).$$

Here  $\mathcal{F}_L$  (resp.  $'$ ) denotes the Fourier transform (resp. the derivative) with respect to the variable  $L$ , in the sense of temperate distributions, and the integral is to be understood in the sense of Fourier integral operators (see Theorem 2.4 of [10]). We have

1. The function  $\psi$  satisfies the equation

$$\forall x, y \in U, \xi \in \mathbb{R}^n, \sigma(x, \partial_x \psi(x, y, \xi)) = \sigma(y, \xi).$$

2. For each  $t \in ]-\varepsilon, \varepsilon[$  and  $\xi \in \mathbb{R}^n$ , the function  $q(\cdot, t, \cdot, \xi)$  has compact support in  $U \times U$  uniformly in  $(t, \xi)$  and  $q(x, 0, y, \xi) - 1$  is a symbol of order  $-1$  as long as  $x, y$  belong to some open neighborhood  $U_0$  of  $0$  in  $U$ .
3.  $\sigma - \sigma_A^{\frac{1}{m}} \in S^0$ .

We will also need the following classical lemma. Here and below,  $\mathcal{S}(\mathbb{R})$  will denote the space of Schwartz functions.

**Lemma 7.2.** *For each  $\varepsilon > 0$  there is a function  $\rho \in \mathcal{S}(\mathbb{R})$  such that  $\mathcal{F}(\rho)$  has compact support contained in  $]-\varepsilon, \varepsilon[$ ,  $\rho > 0$  and  $\mathcal{F}(\rho)(0) = 1$ .*

**Proof.** Choose  $f \in \mathcal{S}(\mathbb{R})$  whose Fourier transform has support in  $]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$ . Then it is easy to see that  $\rho = f^2 * f^2$  satisfies the required properties.  $\square$

Before we proceed, let us fix  $U$ ,  $\psi$ ,  $q$ ,  $k$  and  $\rho$  as in Lemmas 7.1 and 7.2 as well as a differential operator  $P$  on  $\mathcal{M} \times \mathcal{M}$  of order  $d$  with principal symbol  $\sigma_P$ . Let  $E_{L,P} = PE_L$ .

In order to estimate this  $E_{L,P}$ , we will first convolve it with  $\rho$  in order to estimate it using Lemma 7.1. Then, we will compare  $E_{L,P}$  to its convolution with  $\rho$  which we denote - somewhat liberally - by

$$\rho * E_{L,P} = \int_{\mathbb{R}} \rho(\lambda) E_{L-\lambda,P} d\lambda.$$

## 7.2 Estimating the convoluted kernel

In this section we provide the following expression for  $\rho * E_{L,P}$  in the local coordinates chosen in Lemma 7.1.

**Lemma 7.3.** *There is an open set  $V \subset U$  containing 0 such that, as  $L \rightarrow \infty$  and uniformly for  $(x, y) \in V \times V$ ,*

$$\rho * E_{L,P}(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma(y,\xi) \leq L} \sigma_P(x, y, \partial_{x,y} \psi(x, y, \xi)) e^{i\psi(x,y,\xi)} d\xi + O(L^{n+d-1}).$$

In order to do so we use the three lemmas stated below, whose proofs are given at the end of the section. To begin with, we use the information of Lemma 7.1 to give a first expression for  $\rho * E_{L,P}$ .

**Lemma 7.4.** *The quantity*

$$\rho * E_{L,P}(x, y) - \int_{-\infty}^L \frac{1}{(2\pi)^n} \int_{T_y^* M} \mathcal{F}_t^{-1} \left[ \mathcal{F}(\rho) P(q(x, t, y, \xi)) e^{i(\psi(x,y,t,\xi) - t\sigma(y,\xi))} \right] (\lambda) d\xi d\lambda$$

*is bounded uniformly for  $(x, y) \in U \times U$ .*

Here and below  $\mathcal{F}$  is the Fourier transform and the occasional subscript indicates the variable on which the transform is taken. Let us now investigate the effect of the differential operator  $P$  on the right hand side of this expression. By the Leibniz rule, there is a finite family of symbols  $(\sigma_j)_{0 \leq j \leq d} \in C^\infty(U \times \mathbb{R}^n)^{d+1}$  such that for each  $j$ ,  $\sigma_j$  is homogeneous of degree  $j$ , such that

$$P \left[ q(x, t, y, \xi) e^{i(\psi(x,y,\xi) - t\sigma(y,\xi))} \right] = \left[ \sum_{j=0}^d \sigma_j(x, t, y, \xi) \right] e^{i(\psi(x,y,\xi) - t\sigma(y,\xi))}$$

and such that

$$\sigma_d(x, t, y, \xi) = q(x, t, y, \xi) \sigma_P(x, y, \partial_{x,y}(\psi(x, y, \xi) - t\sigma(y, \xi))).$$

Now, for each  $j$ , let

$$R_j(x, y, L, \xi) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \mathcal{F}(\rho)(t) \sigma_j(x, t, y, \xi) e^{itL} dt$$

and

$$S_j(x, y, L) = \int_{-\infty}^L \int_{\mathbb{R}^n} R_j(x, y, \lambda - \sigma(y, \xi), \xi) e^{i\psi(x,y,\xi)} d\xi d\lambda.$$

Then,

$$\int_{-\infty}^L \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_t^{-1} \left[ \mathcal{F}(\rho) P(q(x, t, y, \xi) e^{i(\psi(x, y, \xi) - t\sigma(y, \xi))}) \right] (\lambda) d\xi d\lambda = \sum_{j=0}^d S_j(x, y, L).$$

Each  $S_j$  will grow at an order corresponding to the degree of the associated symbol. This is shown in the following lemma.

**Lemma 7.5.** *There is an open set  $V \subset U$  containing 0 such that, as  $L \rightarrow \infty$  and uniformly for  $(x, y) \in V \times V$ ,*

$$S_j(x, y, L) = \frac{1}{(2\pi)^n} \int_{\sigma(y, \xi) \leq L} \sigma_j(x, 0, y, \xi) e^{i\psi(x, y, \xi)} d\xi + O(L^{n+j-1}).$$

Similarly since  $q(x, 0, y, \xi) - 1 \in S^{-1}(U_0 \times U_0, \mathbb{R}^n)$ , from a computation analogous to the proof of Lemma 7.5 and left to the reader, replacing  $\sigma_d$  by

$$(q(x, 0, y, \xi) - 1) \sigma_P(x, y, \partial_{x, y}(\psi(x, y, \xi) - t\sigma(y, \xi))) \in S^{d-1}$$

one can remove  $q$  from the main term, which results in the following.

**Lemma 7.6.** *There is an open set  $V \subset U$  containing 0 such that, as  $L \rightarrow \infty$  and uniformly for  $(x, y) \in V \times V$ ,*

$$S_d(x, y, L) = \frac{1}{(2\pi)^n} \int_{\sigma(y, \xi) \leq L} \sigma_P(x, y, \partial_{x, y}(\psi(x, y, \xi) - t\sigma(y, \xi))) e^{i\psi(x, y, \xi)} d\xi + O(L^{n+d-1}).$$

The juxtaposition of these results yields Lemma 7.3.

**Proof of Lemma 7.4.** First of all,

$$\frac{d}{d\lambda} (\rho * e_{\lambda, P}(x, y))|_{\lambda=L} - \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_t^{-1} \left[ \mathcal{F}(\rho)(t) P(q(x, t, y, \xi) e^{i(\psi(x, t, y, \xi) - t\sigma(y, \xi))}) \right] (L) d\xi \quad (18)$$

$$= \mathcal{F}_t^{-1} [\mathcal{F}(\rho)(t) Pk(x, t, y)] (L). \quad (19)$$

Since  $k \in C^\infty(U \times U \times ] - \varepsilon, \varepsilon[)$  and  $\mathcal{F}(\rho)$  is supported in  $] - \varepsilon, \varepsilon[$ ,

$$\mathcal{F}_t^{-1} [\mathcal{F}(\rho)(t) Pk(x, t, y)] (L) \in \mathcal{S}(\mathbb{R}).$$

Therefore,

$$\rho * E_{L, P}(x, y) - \int_{-\infty}^L \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_t^{-1} \left[ \mathcal{F}(\rho) P(q(x, t, y, \xi) e^{i(\psi(x, y, \xi) - t\sigma(y, \xi))}) \right] (\lambda) d\xi d\lambda$$

is bounded.  $\square$

**Proof of Lemma 7.5.** In this proof, all generic constants will be implicitly uniform with respect to  $(x, y) \in V \times V$ . Let us fix  $y \in V$  and define the following three domains of integration.

$$\begin{aligned} E_1 &= \{(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda \leq L, \sigma(y, \xi) \leq L\} \\ E_2 &= \{(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda \leq L, \sigma(y, \xi) > L\} \\ E_3 &= \{(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda > L, \sigma(y, \xi) \leq L\}. \end{aligned}$$

Moreover, for  $l = 1, 2, 3$ , let  $I_l = \int_{E_l} R_j(x, y, \lambda - \sigma(y, \xi), \xi) e^{i\psi(x, y, \xi)} d\xi d\lambda$ . We will prove that  $I_2$  and  $I_3$  are  $O(L^{n+j-1})$ . The following calculation will then yield the desired identity. Here we use Fubini's theorem and the fact that  $\mathcal{F}(\rho)(0) = \int_{\mathbb{R}} \rho(\lambda) d\lambda = 1$ .

$$\begin{aligned} S_j(x, y, L) &= I_1 + I_2 = I_1 + I_3 + O(L^{n+j-1}) \\ &= \int_{\sigma(y, \xi) \leq L} \left[ \int_{\mathbb{R}} R_j(x, y, s, \xi) ds \right] e^{i\psi(x, y, \xi)} d\xi + O(L^{n+j-1}) \\ &= \frac{1}{(2\pi)^n} \int_{\sigma(y, \xi) \leq L} \sigma_j(x, 0, y, \xi) e^{i\psi(x, y, \xi)} d\xi + O(L^{n+j-1}). \end{aligned}$$

First of all,  $R_j$  is rapidly decreasing in the third variable and, since  $\sigma$  is elliptic of degree 1, bounded by  $\sigma(y, \xi)^j$  with respect to the last variable,  $\xi$ . Therefore, for each  $N > 0$  there is a constant  $C > 0$  such that

$$|R_j(x, y, \lambda, \xi)| \leq \frac{C\sigma(y, \xi)^j}{(1 + |\lambda|)^N}.$$

Since  $\sigma$  is elliptic of order 1, the hypersurface  $L^{-1}\{\sigma(y, \xi) = L\} \subset \mathbb{R}^n$  converges smoothly for  $L \rightarrow \infty$  uniformly in  $y$  to  $S_y^* = \{\sigma_A(y, \xi) = 1\}$  and the volume of  $\{\sigma(x, \xi) = \beta\} \subset \mathbb{R}^n$  is  $O(\beta^{n-1})$ . Taking  $N = 2n + j + 1$ , we deduce that

$$\begin{aligned} |I_2| &\leq C \int_{-\infty}^L \int_{\sigma(y, \xi) > L} \frac{\sigma(y, \xi)^j}{(1 + |\lambda - \sigma(y, \xi)|)^{2n+j+1}} d\xi d\lambda \leq C \int_{-\infty}^L \int_L^{+\infty} \frac{\beta^{n+j-1}}{(1 + |\lambda - \beta|)^{2n+j+1}} d\beta d\lambda \\ &\leq C \int_L^{+\infty} \int_{-\infty}^{L-\beta} \frac{\beta^{n+j-1}}{(1 + |s|)^{2n+j+1}} ds d\beta \leq C \int_L^{+\infty} \frac{\beta^{n+j-1}}{(1 + \beta - L)^{2n+j}} d\beta \\ &\leq C \int_0^{+\infty} \frac{(\gamma + L)^{n+j-1}}{(1 + \gamma)^{2n+j}} d\gamma \leq CL^{n+j-1}. \end{aligned}$$

Here we applied first the change of variables  $s = \lambda - \beta$  and then  $\gamma = \beta - L$ . The case of  $I_3$  is analogous and by a similar calculation we deduce that  $I_1$  is well defined.  $\square$

### 7.3 Comparison of the kernel and its convolution

In this section we set about proving that  $E_{L,P}$  is close enough to its convolution with  $\rho$ . This is encapsulated in the following lemma.



**Lemma 7.7.** *There is an open set  $V \subset U$  containing 0 such that, as  $L \rightarrow \infty$  and uniformly for  $(x, y) \in V \times V$ ,*

$$\rho * E_{L,P}(x, y) - E_{L,P}(x, y) = O(L^{n+d-1}).$$

As before, the proofs are relegated to the end of the section. In order to prove Lemma 7.7 we first estimate the growth of the  $R_j$  as follows.

**Lemma 7.8.** *There is an open set  $V \subset U$  containing 0 such that, as  $L \rightarrow \infty$  and uniformly for  $(x, y) \in V \times V$ ,*

$$\int_{\mathbb{R}^n} R_j(x, y, L - \sigma(y, \xi), \xi) e^{i\psi(x, y, \xi)} d\xi = O(L^{n+j-1}).$$

This lemma follows from a computation analogous to the bound on  $I_2$  and  $I_3$  given in the proof of Lemma 7.5 above and the details are left to the reader. It allows us to prove a second intermediate result from which we obtain Lemma 7.7 directly.

**Lemma 7.9.** *There is an open set  $V \subset U$  containing 0 such that, as  $L \rightarrow \infty$  and uniformly for  $(x, y) \in V \times V$ ,*

$$E_{L+1,P}(x, y) - E_{L,P}(x, y) = O(L^{n+d-1}).$$

**Proof of Lemma 7.9.** We begin with the case where  $x = y$  and  $P$  is of the form  $P_1 \otimes P_1$ . For brevity we define

$$u(L) = E_{L,P}(x, x) = \sum_{\lambda_k \leq L} |(P_1 e_k)(x)|^2.$$

Recall  $\rho > 0$  so it stays greater than some constant  $a > 0$  on the interval  $[-1, 0]$ . Moreover  $u$  is increasing so by equation (18) and Lemma 7.8,

$$\begin{aligned} 0 \leq u(L+1) - u(L) &= \int_L^{L+1} u'(\lambda) d\lambda \leq \frac{1}{a} \int_L^{L+1} \rho(L - \lambda) u'(\lambda) d\lambda \\ &\leq \frac{1}{a} \frac{d}{dL} (\rho * u) \leq \frac{1}{a} \sum_{j=0}^d \int_{\mathbb{R}^n} R_j(x, y, L - \sigma(y, \xi)) d\xi + O(L^{n+d-1}) = O(L^{n+d-1}). \end{aligned}$$

Now if  $P$  is of the form  $P_1 \otimes P_2$ , and for any  $x$  and  $y$ , let  $X = (P_1 e_k)_{L < \lambda_k \leq L+1}$  and  $Y = (P_2 e_k)_{L < \lambda_k \leq L+1}$  be two vectors in some  $\mathbb{C}^q$  which we equip with the standard hermitian product “ $\star$ ”. Then,  $E_{L+1,P}(x, y) - E_{L,P}(x, y) = X \star \bar{Y}$  so

$$\begin{aligned} |E_{L+1,P}(x, y) - E_{L,P}(x, y)|^2 &\leq |X|^2 |Y|^2 \\ &= |E_{L+1, P_1 \otimes P_1}(x, y) - E_{L, P_1 \otimes P_1}(x, y)| |E_{L+1, P_2 \otimes P_2}(x, y) - E_{L, P_2 \otimes P_2}(x, y)| \\ &\leq \frac{1}{4} \left( E_{L+1, P_1 \otimes P_1}(x, x) - E_{L, P_1 \otimes P_1}(x, x) + E_{L+1, P_1 \otimes P_1}(y, y) - E_{L, P_1 \otimes P_1}(y, y) \right) \\ &\quad \times \left( E_{L+1, P_2 \otimes P_2}(x, x) - E_{L, P_2 \otimes P_2}(x, x) + E_{L+1, P_2 \otimes P_2}(y, y) - E_{L, P_2 \otimes P_2}(y, y) \right) \\ &\leq CL^{2n+2d-2}. \end{aligned}$$

Here we used first the Cauchy-Schwarz inequality, then the mean value inequality, then on each factor,

$$2|P_1 e_k(x) \overline{P_1 e_k(y)}| \leq |P_1 e_k(x)|^2 + |P_1 e_k(y)|^2$$

and finally the above estimate. In general  $P$  is a locally finite sum of operators of the form  $P_1 \otimes P_2$ .  $\square$

**Proof of Lemma 7.7.** First of all, according to Lemma 7.9 there is a constant  $C$  such that for all  $L \geq 0$  and  $\lambda$ ,

$$|E_{L+\lambda, P}(x, y) - E_{L, P}(x, y)| \leq C(1 + |\lambda| + L)^{n+d-1}(1 + |\lambda|).$$

Consequently

$$\begin{aligned} |(\rho * E_{L, P}(x, y) - E_{L, P}(x, y))| &\leq \left| \int \rho(\lambda) E_{L+\lambda, P}(x, y) d\lambda - E_{L, P}(x, y) \right| \\ &\leq \int \rho(\lambda) |E_{L+\lambda, P}(x, y) - E_{L, P}(x, y)| d\lambda \\ &\leq C \int \rho(\lambda) (1 + |\lambda| + L)^{n+d-1} (1 + |\lambda|) d\lambda \\ &\leq C' L^{n+d-1} \end{aligned}$$

for some  $C' > 0$ . Here we used that  $\rho > 0$ ,  $\rho$  is rapidly decreasing and  $\int_{\mathbb{R}} \rho(\lambda) d\lambda = \mathcal{F}(\rho)(0) = 1$ .  $\square$

## 7.4 Conclusion

Combining Lemmas 7.3 and 7.7 we obtain the following:

$$E_{L, P}(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma(y, \xi) \leq L} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) e^{i\psi(x, y, \xi)} d\xi + O(L^{n+d-1}).$$

Since  $\sigma - \sigma_A^{\frac{1}{m}} \in S^0$ , replacing one by the other adds only a  $O(L^{n+d-1})$  term. Therefore,

$$E_{L, P}(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y, \xi) \leq L} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) e^{i\psi(x, y, \xi)} d\xi + O(L^{n+d-1}).$$

This estimate is valid and uniform for  $x, y \in V$ .

So far we have worked under the assumption that  $A$  is of order 1. For the general case, that is,  $A$  of order  $m > 0$ , just apply the previous result to  $A^{\frac{1}{m}}$  which is also an elliptic positive self-adjoint pseudo-differential operator (see [21]), with the same eigenfunctions as  $A$  and whose corresponding eigenvalues are just those of  $A$  to the power  $\frac{1}{m}$ . Moreover, if  $\sigma_A$  is the principal symbol of  $P$ , then  $(\sigma_A)^{\frac{1}{m}}$  is equal to the principal symbol of  $P^{\frac{1}{m}}$ . Here we use that  $A$  is polyhomogeneous. In that case,

$$E_{L, P}(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y, \xi) \leq L} \sigma_P(x, y, \partial_{x, y} \psi(x, y, \xi)) e^{i\psi(x, y, \xi)} d\xi + O(L^{\frac{n+d-1}{m}}).$$

This concludes the proof of Theorem 2.2.

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