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Characterization of f -extremal disks

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Abstract

We show uniqueness for overdetermined elliptic problems defined on topological disks Ω with C^2 boundary, *i.e.*, positive solutions u to $\Delta u + f(u) = 0$ in $\Omega \subset (M^2, g)$ so that $u = 0$ and $\frac{\partial u}{\partial \vec{\eta}} = cte$ along $\partial\Omega$, $\vec{\eta}$ the unit outward normal along $\partial\Omega$ under the assumption of the existence of a candidate family. To do so, we adapt the Gálvez-Mira generalized Hopf-type Theorem [19] to the realm of overdetermined elliptic problem.

When (M^2, g) is the standard sphere \mathbb{S}^2 and f is a C^1 function so that $f(x) > 0$ and $f(x) \geq xf'(x)$ for any $x \in \mathbb{R}_+^*$, we construct such candidate family considering rotationally symmetric solutions. This proves the Berestycki-Caffarelli-Nirenberg conjecture in \mathbb{S}^2 for this choice of f . More precisely, this shows that if u is a positive solution to $\Delta u + f(u) = 0$ on a topological disk $\Omega \subset \mathbb{S}^2$ with C^2 boundary so that $u = 0$ and $\frac{\partial u}{\partial \vec{\eta}} = cte$ along $\partial\Omega$, then Ω must be a geodesic disk and u is rotationally symmetric. In particular, this gives a positive answer to the Schiffer conjecture D (cf. [33, 35]) for the first Dirichlet eigenvalue and classifies simply-connected harmonic domains (cf. [28], also called *Serrin Problem*) in \mathbb{S}^2 .

MSC 2010: 35Nxx; 53Cxx.

Key Words: Overdetermined Problems; Maximum principle; Neumann conditions; Index Theorem.

1 Introduction

Overdetermined elliptic problems (OEP), *i.e.*, finding a solution to an elliptic partial derivative equation constrained to both Dirichlet and Neumann conditions, appear frequently in physical models and free boundary problems.

Let Ω be an open connected domain of a complete connected Riemannian manifold (M, g) and consider the OEP given by

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle_g = \alpha & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\vec{\eta}$ is the unit outward normal vector along $\partial\Omega$, α a negative constant and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. A domain $\Omega \subset M$ that supports a solution to (1.1) is called an *f-extremal domain*. If $\Omega \subset \mathbb{R}^n$ (endowed with the standard Euclidean metric) is bounded and $f \equiv 1$, Serrin [34] proved that the ball is the only domain where the above problem admits a solution $u \in C^2(\Omega)$. This was generalized later to any Lipschitz function f by Pucci and Serrin [27]. Serrin's proof uses the moving plane method introduced by Alexandrov in [6] in order to prove that round spheres are the only constant mean curvature embedded hypersurfaces in \mathbb{R}^n . Right after, Weinberger [38] gave a short proof of the very same result.

In 1997, W. Reichel [29] extended Pucci-Serrin result for exterior domains, that is, connected smooth domains $\Omega \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is bounded, under the additional hypothesis that f is non-increasing and the solution u goes uniformly to a constant at infinity.

In the same year, Berestycki, Caffarelli and Nirenberg [7] considered the problem (1.1) when $\Omega \subset \mathbb{R}^n$ is an unbounded domain and also its complement, such problem appears naturally in the regularity of free boundary solutions at a boundary point. Under certain additional conditions they proved that the only *f*-extremal domain whose boundary is an epigraph over a hyperplane is a half-space. So, combining the results of Pucci-Serrin, Reichel and Berestycki-Caffarelli-Nirenberg, they formulated the following:

BCN conjecture: *If f is Lipschitz, $\Omega \subset \mathbb{R}^n$ is a smooth (in fact, Lipschitz) connected domain with $\mathbb{R}^n \setminus \Omega$ connected where the OEP (1.1) admits a bounded solution, then Ω is either a ball, a half-space, a cylinder $\mathbb{B}^k \times \mathbb{R}^{n-k}$ (\mathbb{B}^k is a ball of \mathbb{R}^n) or the complement of one of them.*

P. Sicbaldi [36] gave a counterexample of the BCN conjecture when $n \geq 3$. Nevertheless, the BCN conjecture motivated interesting works as, for example, those of Farina and collaborators ([15, 16, 17, 18] and references therein). Recently, important contributions have been made in dimension $n = 2$. First, Ros-Sicbaldi [30] exploited the analogy between OEPs and constant mean curvature surfaces (in short, CMC surfaces) which allowed them to prove the BCN conjecture in dimension 2 under some extra hypothesis. Second, Ros-Ruiz-Sicbaldi [31] proved that the BCN conjecture is true in dimension 2 for unbounded domains whose complement is unbounded, such domain must be a half-space. Also, Ros-Ruiz-Sicbaldi [32] constructed exterior domains different from the exterior of a geodesic ball in \mathbb{R}^2 for particular choices of the Lipschitz function f , this gives a counterexample to the BCN conjecture in \mathbb{R}^2 in all its generality. Hence, combining the works of Pucci-Serrin, Reichel and Ros-Ruiz-Sicbaldi we have

Theorem [27, 29, 31]. *Let f be a non-increasing Lipschitz function and $\Omega \subset \mathbb{R}^2$ a $C^{2,\alpha}$ connected domain whose complement is connected. Assume that the OEP (1.1) admits a bounded solution u that goes uniformly to a constant at infinity, then $\partial\Omega$ has constant curvature.*

Observe that $\partial\Omega$ has constant curvature if, and only if, Ω is either a ball, the exterior of a ball or a half-space. Recall that the hypothesis that f is non-increasing and $u \rightarrow C$ uniformly at infinity are only needed in Reichel's Theorem. About the regularity of $\partial\Omega$, Pucci-Serrin and Reichel assumed $C^{2,\alpha}$ and Ros-Ruiz-Sicbaldi only Lipschitz. The above result is the best one can expect in this situation since the BCN conjecture is not true for any Lipschitz function f (cf. [32]). In other

words, we must assume additional conditions on f and/or u (see [3, 37] for related conditions on f and more general operators).

Regarding other Space Forms, combining the works of Molzon [25], Espinar-Mao [14] and Espinar-Farina-Mazet [12], we can prove the BCN conjecture for domains in the Hyperbolic space \mathbb{H}^2 under similar hypothesis than the Euclidean case, specifically:

Theorem [12, 14, 25]. *Let f be a non-increasing Lipschitz function and $\Omega \subset \mathbb{H}^2$ a $C^{2,\alpha}$ connected domain whose complement is connected. Assume that the OEP (1.1) admits a bounded solution u that goes uniformly to a constant at infinity, then $\partial\Omega$ has constant curvature.*

Note that $\partial\Omega$ has constant curvature in \mathbb{H}^2 if, and only if, Ω is either a geodesic disk, a horodisk, a half-space determined by a complete geodesic or equidistant curve, or the complement of one of them. It would be interesting to construct a counterexample for exterior domains in \mathbb{H}^2 in the spirit of [32].

In the two dimensional sphere, the BCN conjecture reads as

BCN conjecture in \mathbb{S}^2 : *If f is Lipschitz, $\Omega \subset \mathbb{S}^2$ is a topological disk with C^2 boundary where the OEP (1.1) admits a solution, then Ω is a geodesic ball.*

Let us point out a couple of remarks. If we assume that $\Omega \subset \mathbb{S}^2$ is connected and its complement also is connected then Ω is simply connected, in other words, Ω is a topological disk.

The previous mentioned works in \mathbb{R}^2 and \mathbb{H}^2 rely heavily in some variant of the Alexandrov moving plane method introduced by Serrin in the context of OEP. We could also use this technique for domains $\Omega \subset \mathbb{S}^2$ to prove the BCN conjecture in \mathbb{S}^2 as far as Ω is contained in some hemisphere of \mathbb{S}^2 (see the analogy with embedded CMC surfaces in \mathbb{S}^3).

Another capital result on CMC surfaces in \mathbb{R}^3 is Hopf's Theorem that states that the only compact immersed CMC surface of genus zero (a topological sphere) in \mathbb{R}^3 is a round sphere. H. Hopf gave two proofs of this result, one of them, the most interesting for us, is based on the fact that for any CMC surface either the surface is totally umbilic or there exists a line field with isolated singularities of negative index. So, the Poincaré-Hopf index theorem eliminates the second possibility in a topological sphere, thus the CMC surface must be a round sphere. Later, J. Nistche [26] extended Hopf's Theorem to compact immersed disks of constant mean curvature in \mathbb{R}^3 assuming that the boundary is a line of curvature, that is, it must be totally umbilic. If it were not totally umbilic, in the interior of the disk, he can define the same line field as Hopf and the condition on the boundary implies that the line field can be extended continuously across the boundary after symmetrization of the domain, hence, Nistche ended up with a line field in a sphere with isolated singularities of negative index, which is impossible. Such technique depends heavily on the dimension but it has been widely used in different geometric situations in order to classify topological spheres (and compact disks under assumptions on the boundary) without assuming embeddedness, a crucial hypothesis in the Alexandrov moving plane method (cf. [1, 2, 4, 5, 10, 13, 20, 23] and references therein).

In a recent paper, J.A. Gálvez and P. Mira [19] proved an extremely general version of Hopf's Theorem. Such version contains all the previous results mentioned here among others, new ones and also applications to other problems as the Alexandrov conjecture on the uniqueness of immersed spheres with prescribed curvature in \mathbb{R}^3 . So, as Serrin did with the Alexandrov moving

plane method, the main point of this paper is to adapt the Gálvez-Mira method to OEP, this can be seen as a Nistche type result for OEP.

More precisely, the idea is to associate to each solution of the OEP (1.1) a traceless symmetric bilinear form. This bilinear form is defined in the support of the solution and it either vanishes identically or has isolated zeroes. In the case it has isolated zeroes, a traceless symmetric bilinear form defines a Lorentzian metric away from its zeroes and, it is well-known, induces two line fields with singularities at the zeroes. First, we show that the singularities of the line fields are of negative order and second, the boundary is a "line of curvature" of this Lorentzian metric. This impose restrictions on the geometry of the support of the solution.

Actually, in order to define such a bilinear form, we need to assume the existence of certain solutions to (1.1): the family of candidate solutions. In the case of \mathbb{S}^2 , we prove that such families of candidate solutions exist if f satisfies some hypothesis and then the BCN conjecture follows for these particular f .

The paper is organized as follows. In Section 2, we define what is a candidate family of solutions. Constructions of some of them on \mathbb{S}^2 are given in Section 3. In Section 4, we state and prove the main theorem concerning the existence of our traceless bilinear form. Then we apply it to the proof of the BCN conjecture.

Just after writing our paper, we learned that P. Mira [24] has also proved a result concerning to fully non-linear overdetermined elliptic problems in topological disks in \mathbb{R}^2 . Nevertheless, the main result (cf. [24, Theorem 2.4]) does not apply in our situation since the functional $F[u] = 0$ is not allow to depend on the base point $(x, y) \in \Omega$.

2 Family of candidate solutions

First, we need to define what is a smooth family of functions parametrized by some manifold N . So let N and M be two manifolds; N may have some boundary and M is endowed with some Riemannian metric g whose Levi-Civita connection is denoted by ∇ (we will use the notation $(g(\cdot, \cdot) = \langle \cdot, \cdot \rangle)$).

A domain in M is just a connected open subset of M with a C^2 boundary. For any $p \in N$, we choose a domain Ω_p in M in a continuous way. $(\Omega_p)_{p \in N}$ is continuous if, for any $p_0 \in N$, any closed subset F of Ω_{p_0} and any open subset U with $\overline{\Omega_{p_0}} \subset U$, we have $F \subset \Omega_p \subset U$ for any p close to p_0 . Let us then denote

$$A = \{(p, q) \in N \times M \mid q \in \overline{\Omega_p}\}$$

Definition 2.1. *With the above notations, a smooth family of C^3 functions on $(\Omega_p)_{p \in N}$ is a map $V : A \rightarrow \mathbb{R}$ such that*

- *the function $v_p : \overline{\Omega_p} \rightarrow \mathbb{R}; q \mapsto V(p, q)$ is C^3 ,*
- *there is an open neighborhood B of A in $N \times M$ such that V can be extended to B and the map*

$$\Phi : \begin{array}{ccc} B & \rightarrow & \mathbb{R} \times TM \oplus S^2M \\ (p, q) & \mapsto & (v_p(q), (q, \nabla v_p(q), \nabla^2 v_p(q))) \end{array}$$

is a C^1 map on A .

Here S^2M denotes the bundle of symmetric 2-forms on M and ∇^2 is the hessian operator.

Let us remark that in the following we will say that $(v_p)_{p \in N}$ is a smooth family of functions. Moreover if $(\Omega'_p)_{p \in N}$ is a continuous family of domains in M such that $\overline{\Omega}_p \subset \Omega'_p$ and N has no boundary, then $A' = \{(p, q) \in N \times M \mid q \in \overline{\Omega}'_p\}$ is a neighborhood of A .

Let us now fix a Lipschitz function f and a connected Riemannian surface (M, g) . Let us define $N = (TM \times \mathbb{R}_+) \setminus \{(q, 0, 0) \in TM \times \mathbb{R}; q \in M\}$.

Definition 2.2. *We say that the OEP*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \subset M, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle = \alpha & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

admits a family of candidate solutions, \mathcal{C} , if there is a smooth family of C^3 functions $(v_{q,w,a})_{(q,w,a) \in N}$ where $\Omega_{q,w,a} \subset M$ has C^2 boundary and $v_{q,w,a}$ solves

$$\begin{cases} \Delta v_{q,w,a} + f(v_{q,w,a}) = 0 & \text{in } \Omega_{q,w,a}, \\ v_{q,w,a} > 0 & \text{in } \Omega_{q,w,a}, \\ v_{q,w,a} = 0 & \text{on } \partial\Omega_{q,w,a}, \\ \langle \nabla v_{q,w,a}, \vec{\eta} \rangle = cte < 0 & \text{on } \partial\Omega_{q,w,a}, \end{cases}$$

such that

- (a) $q \in \overline{\Omega}_{q,w,a}$,
- (b) $v_{q,w,a}(q) = a$,
- (c) $\nabla v_{q,w,a}(q) = w$,
- (d) if $(q_0, w_0, a_0) \in N$ and denote $\Omega_0 = \Omega_{q_0, w_0, a_0}$ and $v_0 = v_{q_0, w_0, a_0}$, then, for any $q \in \overline{\Omega}_0$, we have

$$\Omega_{q, \nabla v_0(q), v_0(q)} = \Omega_0 \quad \text{and} \quad v_{q, \nabla v_0(q), v_0(q)} = v_0.$$

Let us remark that item (b) implies that $q \in \Omega_{q,w,a}$ if $a > 0$ and $q \in \partial\Omega_{q,w,a}$ if $a = 0$. Item (d) is a uniqueness property of the family of candidate solution. Actually, the same solution appears to be parametrized by several $(q, w, a) \in N$.

In the family of candidate solutions \mathcal{C} , are included all solutions with negative constant Neumann boundary values, not only those with fixed Neumann boundary values α . Such subset of \mathcal{C} , those with Neumann boundary values α , is denoted by $\mathcal{C}_\alpha \subset \mathcal{C}$. In the proof of our main theorem, we will see that only a part of \mathcal{C} will be used.

3 Two examples

A priori, constructing a family of candidate solutions is not easy to do. When we consider a sufficiently symmetric space, say for example \mathbb{R}^n , the Alexandrov moving plane method introduced by

Serrin tells us that if Ω is a bounded domain which admit a solution to the OEP (2.2) then Ω has to be a ball and u is rotationally symmetric. This suggest that at least in \mathbb{R}^2 , \mathbb{S}^2 and \mathbb{H}^2 , candidate solutions can be constructed by considering rotationally symmetric solutions to (2.2).

Actually, in this section, we construct two examples of families of candidate solutions in \mathbb{S}^2 . The work can also be done in \mathbb{R}^2 and \mathbb{H}^2 but only \mathbb{S}^2 is interesting with respect to our main theorem (see Corollary 4.2).

3.1 Example 1

Let $\lambda \in \mathbb{R}$, we consider $f(t) = \lambda t$ and the following OEP in $\Omega \subset \mathbb{S}^2$

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle = \alpha < 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{OEP}_\lambda)$$

First we remark that the existence of a solution implies $\lambda > 0$ by the maximum principle. So we focus on that case. The main result of this section is

Proposition 3.1. *For any $\lambda > 0$, (OEP_λ) admits a family of candidate solutions.*

The end of the Subsection 3.1 is devoted to the proof of the above proposition.

Let λ be positive, then there is $R_\lambda \in (0, \pi)$ such that the first eigenvalue of $-\Delta$ on a geodesic disk of radius R_λ in \mathbb{S}^2 is λ (see Theorem II.5.6 in [9]). So if $p \in \mathbb{S}^2$ and D_p is the geodesic disk in \mathbb{S}^2 of center p and radius R_λ , we have a solution u_p to

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } D_p, \\ u = 0 & \text{on } \partial D_p, \\ u(p) = 1. \end{cases}$$

Actually, u_p is invariant by rotation around p so $\langle \nabla u_p, \vec{\eta} \rangle = \alpha_\lambda$ on ∂D_p : u_p is a C^3 solution to (OEP_λ) . We notice that R_λ is decreasing in λ from π to 0.

If p is in \mathbb{S}^2 and q is a point at distance ρ from p (with $\rho < \pi$), the function u_p is just a function U of ρ . Moreover U solves the ODE

$$U'' + (\cot \rho)U' + \lambda U = 0. \quad (3.3)$$

We remark that $U''(0) = -\frac{\lambda}{2}$. Let us notice that if L is an isometry of \mathbb{S}^2 , $D_{L(p)} = L(D_p)$ and $u_{L(p)} = u_p \circ L^{-1}$.

The family $(tu_p)_{t \in \mathbb{R}_+, p \in \mathbb{S}^2}$ is then a smooth family of functions, each of them is a solution to (OEP_λ) . But it is not a family of candidate solutions as in the definition since we do not have the right parametrization. The rest of the section is devoted to prove we can reparametrize the family.

So let us consider $\varepsilon > 0$ such that $\bar{R} = R_\lambda + \varepsilon < \pi$. ε will be fixed by Lemma 3.2 below. First, the definition of u_p extends to the geodesic disk Δ_p of center p and radius \bar{R} by solving (3.3) up to \bar{R} . The choice of ε is given by the following lemma.

Lemma 3.2. *There is $\varepsilon > 0$ such that $U''U - U'^2$ does not vanish on $(0, \bar{R})$.*

The proof will be given later. The parameter ε is now fixed. Actually one has $U''U - U'^2 < 0$ in $(0, \bar{R})$ (this is the log-concavity of U on $(0, R_\lambda)$) and even in $(-\bar{R}, \bar{R})$ (by defining $U(-\rho) = U(\rho)$). The consequence of this is that $U' < 0$ on $(0, \bar{R})$. It also implies that the curve

$$\rho \in (-\bar{R}, \bar{R}) \mapsto (U(\rho), U'(\rho)) \in \mathbb{R}^2 \setminus \{(0, 0)\}$$

can be described as a polar curve $A(\rho)(\cos \beta(\rho), \sin \beta(\rho))$ with $\beta(-\bar{R}) = \bar{\beta} \in (\pi/2, \pi)$. The last claim follows since $U(R_\lambda) = 0$ and $U' < 0$ in $(0, \bar{R})$.

Let S be the angular sector $\{(x, y) \in \mathbb{R}^2 \mid x > \sqrt{x^2 + y^2} \cos \bar{\beta}\}$ and consider the map

$$\begin{aligned} F : \mathbb{R}_+^* \times (-\bar{R}, \bar{R}) &\rightarrow S \\ (t, \rho) &\mapsto (tU(\rho), tU'(\rho)). \end{aligned}$$

The map F is C^1 and the above discussion about the curve $\rho \mapsto (U(\rho), U'(\rho))$ implies that F is a bijection. Let us denote $F^{-1}(x, y) = (T(x, y), R(x, y))$ and define

$$\tilde{N} = \{((q, w), a) \in T\mathbb{S}^2 \times \mathbb{R} \mid (a, |w|) \in S\}.$$

Since $\nabla u_{t,p}(q) = tU'(\rho)\partial_\rho$, the distance ρ can be computed as $\rho = R(u_{t,p}(q), -|\nabla u_{t,p}|(q))$ and the parameter t is $T(u_{t,p}(q), -|\nabla u_{t,p}|(q))$. Let us now define

$$G : (q, w, a) \in \tilde{N} \mapsto \exp_q\left(\frac{R(a, -|w|)}{|w|}w\right).$$

Then a right parametrization of the family of candidate solutions is given on \tilde{N} by

$$v_{q,w,a} = T(a, -|w|)u_{G(q,w,a)}.$$

So, if we prove that T and G are C^1 , we obtain a true family of candidate solutions. This smoothness is a consequence of the following lemma.

Lemma 3.3. *F is a C^1 diffeomorphism from $\mathbb{R}_+^* \times (-\bar{R}, \bar{R})$ onto S .*

Proof. We just have to check that this reciprocal map is smooth. Let us compute the differential of F . We have

$$\partial_t F(t, \rho) = (U(\rho), U'(\rho)) \quad \text{and} \quad \partial_\rho F(t, \rho) = (tU'(\rho), tU''(\rho)).$$

So the differential DF of F has not rank 2 if and only if $(U''U - U'^2)(\rho) = 0$ for some ρ . By Lemma 3.2, this never occurs on $(0, \bar{R})$ for our choice of ε .

We have then proved that DF is invertible on $\mathbb{R}_+^* \times (-\bar{R}, \bar{R})$. This implies that F^{-1} is smooth and finishes the proof. \square

The above lemma implies that the maps $t(q, w, a) = T(a, -|w|)$ and $p(q, w, a) = G(q, w, a)$ are C^1 when $w \neq 0$. Concerning the behaviour near $w = 0$, we use that, near $y = 0$, it holds

$$T(x, y) = x + o(y), \quad R(x, y) = -\frac{2}{\lambda x}y + o(y) \quad \text{and} \quad \frac{\partial R}{\partial y}(x, y) = -\frac{2}{\lambda x} + o(1).$$

Hence, near $w = 0$, $T(a, -|w|) = a + O(|w|^2)$ and $T(a, -|w|)$ is C^1 at $w = 0$. Clearly $\frac{R(a, -|w|)}{|w|}w = o(1)$ so it is continuous at $w = 0$. We also have

$$\begin{aligned} D_w\left(\frac{R(a, -|w|)}{|w|}w\right)(h) &= -\frac{\partial R}{\partial y}(a, -|w|)\left\langle\frac{w}{|w|^2}, h\right\rangle w + R(a, -|w|)\left(\left\langle-\frac{w}{|w|^3}, h\right\rangle w + \frac{h}{|w|}\right) \\ &= \frac{2}{\lambda a}\left\langle\frac{w}{|w|^2}, h\right\rangle w - \frac{2}{\lambda a}|w|\left\langle\frac{w}{|w|^3}, h\right\rangle w + \frac{2}{\lambda a}|w|\frac{h}{|w|} + |h|o(1) \\ &= \frac{2}{\lambda a}h + |h|o(1). \end{aligned}$$

So $(a, w) \mapsto \frac{R(a, -|w|)}{|w|}w$ is C^1 at $w = 0$ (the differential with respect to a is easier to deal with). Finally t and p are C^1 with respect to $(q, w, a) \in \tilde{N}$ and this finishes the proof of Proposition 3.1.

Let us now give the proof of Lemma 3.2.

Proof of Lemma 3.2. It is enough to prove that $U''U - U'^2 \neq 0$ in $(0, R_\lambda)$ since $(U''U - U'^2)(0) = -\frac{1}{2\lambda}$ and $(U''U - U'^2)(R_\lambda) = -\alpha_\lambda^2$. Let us consider $\bar{p} \in (0, R_\lambda)$.

Let $p = (0, 0, 1) \in \mathbb{S}^2 \subset \mathbb{R}^3$, $Y = (1, 0, 0) \in T_p\mathbb{S}^2$ and \tilde{Y} the Killing vectorfield

$$\tilde{Y}(q) = -\langle q, Y \rangle p + \langle p, q \rangle Y \in T_q\mathbb{S}^2.$$

We consider the polar parametrization of \mathbb{S}^2 given by $G(\rho, \theta) = (\sin \rho \cos \theta, \sin \rho \sin \theta, \cos \rho)$. So writing $q = G(\rho, \theta)$, we get

$$\tilde{Y}(q) = -\sin \rho \cos \theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \cos \rho \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Using $\frac{\partial}{\partial \rho} = \begin{pmatrix} \cos \rho \cos \theta \\ \cos \rho \sin \theta \\ -\sin \rho \end{pmatrix}$ and $\frac{\partial}{\partial \theta} = \begin{pmatrix} -\sin \rho \sin \theta \\ \sin \rho \cos \theta \\ 0 \end{pmatrix}$, we get

$$\tilde{Y} = \cos \theta \frac{\partial}{\partial \rho} - \cot \rho \sin \theta \frac{\partial}{\partial \theta} \text{ if } \rho \neq 0. \quad (3.4)$$

Let \bar{q} be the point with polar coordinates $\rho = \bar{\rho}$ and $\theta = 0$. Let v be the function defined by

$$v = \frac{\langle \tilde{Y}(\bar{q}), \nabla u_p(\bar{q}) \rangle}{u_p(\bar{q})} u_p - \langle \tilde{Y}, \nabla u_p \rangle \text{ in } D_p.$$

On the one hand, since $\tilde{Y}(\rho, -\theta) = \tilde{Y}(\rho, \theta)$ by (3.4), v is invariant by the map $\theta \rightarrow -\theta$. On the other hand, since \tilde{Y} is Killing, we have $\Delta v + \lambda v = 0$. Moreover, observe that $v(\bar{q}) = 0$.

At \bar{q} , $\langle \tilde{Y}(\bar{q}), \nabla u_p(\bar{q}) \rangle = U'(\bar{\rho})$. So, along the boundary of the disk D of center p and radius $\bar{\rho}$, we have $v(\bar{\rho}, \theta) = (1 - \cos \theta)U'(\bar{\rho})$ and v has the sign of $U'(\bar{\rho})$ on ∂D . If v change sign inside D , there is then a domain $\Omega \subset D$ such that v has constant sign in Ω and vanishes on $\partial\Omega$. Since $\Delta v + \lambda v = 0$, λ is then the first eigenvalue of $-\Delta$ in Ω . This is impossible since λ is the first eigenvalue of $-\Delta$ in $D_p \ni \Omega$.

Since $v(\bar{q}) = 0$, the boundary maximum principle implies $v \equiv 0$ in D or $\frac{\partial}{\partial \bar{\eta}} v(\bar{q}) \neq 0$. The first case would imply $0 = v(\rho) = \frac{U'(\bar{\rho})}{U(\bar{\rho})}$ so $U'(\bar{\rho}) = 0$ and then $0 \equiv v = -\cos \theta U'(\rho)$, *i.e.* $U'(\rho) = 0$, for any $\rho \in [0, \bar{\rho}]$ which is impossible. Therefore, we have $0 \neq \frac{\partial}{\partial \bar{\eta}} v(\bar{q}) = \frac{U'^2(\bar{\rho})}{U(\bar{\rho})} - U''(\bar{\rho})$, that is, $(U''U - U'^2)(\bar{\rho}) \neq 0$ and the proof is finished. \square

3.2 Example 2

In this subsection, we are going to generalize the preceding case. As above, we are looking for rotationally symmetric solutions to (2.2). So the problem reduces to the study of the ODE

$$U'' + (\cot \rho)U' + f(U) = 0. \quad (3.5)$$

We will assume that f has some particular property (see hypothesis (H) and Proposition 3.11 below) but let us begin by a general study.

3.2.1 Solutions to (3.5)

We are interested in understanding even solutions to the ODE (3.5) in $(-\pi, \pi)$ when the value $U(0)$ is prescribed. Let us notice that a function U is called a solution to (3.5) if U is C^2 and solves (3.5) on $(-\pi, \pi) \setminus \{0\}$.

For $0 < \varepsilon < \pi$, we denote $C_e^k(\varepsilon)$ the space of even C^k functions on $[-\varepsilon, \varepsilon]$ endowed with the C^k norm $\|\cdot\|_{k,\varepsilon}$. We denote $C_e^0(\varepsilon) = C_e(\varepsilon)$ and $\|\cdot\|_{0,\varepsilon} = \|\cdot\|_\varepsilon$. For $g \in C_e(\varepsilon)$, we consider first the ODE:

$$U'' + (\cot \rho)U' + g = 0 \quad (3.6)$$

with $U(0) = 0$. This equation is equivalent to $((\sin \rho)U')' + (\sin \rho)g = 0$ so the solution must be $A(g)$ defined by

$$A(g)(\rho) = \begin{cases} -\int_0^\rho \frac{1}{\sin s} \int_0^s (\sin x)g(x)dx ds & \text{if } \rho \neq 0 \\ 0 & \text{if } \rho = 0 \end{cases}$$

Lemma 3.4. *The map A has the following properties.*

- (i) $A : C_e(\varepsilon) \rightarrow C_e^2(\varepsilon)$ is linear and continuous.
- (ii) $A(g)$ solves $U'' + (\cot \rho)U' + g = 0$ on $[-\varepsilon, \varepsilon]$.
- (iii) If $g \in C_e^1(\varepsilon)$, then $A(g) \in C_e^3(\varepsilon)$.

Proof. Let us notice that $A(g)$ is clearly C^2 on $[-\varepsilon, \varepsilon] \setminus \{0\}$. Moreover we have

$$\begin{aligned} |A(g)(\rho)| &\leq \int_0^{|\rho|} \frac{1}{\sin s} \int_0^s (\sin x) \|g\|_{|\rho|} dx ds \\ &\leq \|g\|_{|\rho|} \int_0^{|\rho|} \frac{\sin \frac{s}{2}}{\cos \frac{s}{2}} ds \leq 2 \ln \cos \frac{|\rho|}{2} \|g\|_{|\rho|}. \end{aligned} \quad (3.7)$$

This proves that $A(g)$ is continuous at 0 and A is a continuous linear map with image in $C_e(\varepsilon)$.

We have

$$A(g)'(\rho) = \frac{-1}{\sin \rho} \int_0^\rho (\sin x)g(x)dx$$

and hence

$$|A(g)'(\rho)| \leq \frac{1}{\sin |\rho|} \int_0^{|\rho|} (\sin x)\|g\|_{|\rho|}dx \leq \tan \frac{|\rho|}{2} \|g\|_{|\rho|},$$

so $A(g)$ is C^1 with $A(g)'(0) = 0$ and A is continuous with image in $C_\varepsilon^1(\varepsilon)$.

We also have

$$A(g)''(\rho) = \frac{\cos \rho}{\sin^2 \rho} \int_0^\rho (\sin x)g(x)dx - g(\rho),$$

therefore $A(g)$ solves (3.6). Besides it holds

$$\begin{aligned} |A(g)''(\rho) + (1 - \frac{\cos \rho}{2 \cos^2 \frac{\rho}{2}})g(0)| &\leq \frac{\cos \rho}{\sin^2 \rho} \int_0^{|\rho|} (\sin x)\|g - g(0)\|_{|\rho|}dx + \|g - g(0)\|_{|\rho|} \\ &\leq (1 + \frac{\cos \rho}{2 \cos^2 \frac{\rho}{2}})\|g - g(0)\|_{|\rho|}, \end{aligned}$$

thus $A(g)$ is C^2 with $A(g)''(0) = -\frac{1}{2}g(0)$ and A is continuous with image in $C_\varepsilon^2(\varepsilon)$.

If g is C^1 we have

$$A(g)'''(\rho) = \frac{-1}{\sin \rho} \int_0^\rho (\sin x)g(x)dx - \frac{2 \cot^2 \rho}{\sin \rho} \int_0^\rho (\sin x)g(x)dx + \cot \rho g(\rho) - g'(\rho),$$

so, integrating by parts and using $g'(0) = 0$, we obtain

$$\begin{aligned} \left| \cot \rho g(\rho) - \frac{2 \cot^2 \rho}{\sin \rho} \int_0^\rho (\sin x)g(x)dx \right| &\leq \left| \cot \rho \left(1 - \frac{2 \cos \rho (1 - \cot \rho)}{\sin \rho}\right) g(0) \right| \\ &\quad + \left| \cot \rho \left(1 - \frac{2 \cos \rho (1 - \cot \rho)}{\sin \rho}\right) \int_0^\rho g'(x)dx \right| \\ &\quad + \left| \frac{2 \cot^2 \rho}{\sin \rho} \int_0^\rho (1 - \cos x)g'(x)dx \right| \\ &\leq \frac{\cos \rho \sin \frac{|\rho|}{2}}{2 \cos^3 \frac{\rho}{2}} (|g(0)| + \rho \|g'\|_{|\rho|}) \\ &\quad + \left| \frac{2 \cot^2 \rho (\rho - \sin \rho)}{\sin \rho} \right| \|g'\|_{|\rho|}. \end{aligned}$$

This implies

$$|A(g)'''(\rho)| \leq \left(\tan \frac{|\rho|}{2} + \frac{\cos \rho \sin \frac{|\rho|}{2}}{2 \cos^3 \frac{\rho}{2}} \right) \|g\|_{|\rho|} + \left(1 + \rho \frac{\cos \rho \sin \frac{|\rho|}{2}}{2 \cos^3 \frac{\rho}{2}} + \left| \frac{2 \cot^2 \rho (\rho - \sin \rho)}{\sin \rho} \right| \right) \|g'\|_{|\rho|},$$

then $A(g)$ is C^3 and A is continuous as a map from $C_\varepsilon^1(\varepsilon)$ to $C_\varepsilon^3(\varepsilon)$. Thus (i), (ii) and (iii) are proved. \square

Let us now construct a solution to (3.5).

Lemma 3.5. *Let us assume that f is C^1 . For any $t \in \mathbb{R}$, there is a unique even solution U_t to (3.5) on $(-\pi, \pi)$ such that $U_t(0) = t$. Moreover the solution is C^1 in t .*

Proof. Let us notice that the only problem is near 0. So if we solve the problem on $(-\varepsilon, \varepsilon)$, then the solution extends to $(-\pi, \pi)$ without any problem.

If U is an even solution to (3.5), then $((\sin \rho)U')' + (\sin \rho)f(U) = 0$ and then

$$U(\rho) - U(0) = - \int_0^\rho \frac{1}{\sin s} \int_0^s (\sin x) f(U(x)) dx ds = A(f \circ U)(\rho).$$

Let us then define, for $(v, t) \in C_e(\varepsilon) \times \mathbb{R}$, the function $\mathcal{A}(v, t)$ in $[-\varepsilon, \varepsilon]$ by $\mathcal{A}(v, t) = A(f(t + v))$. Then U is a solution of (3.5) if $U - U(0) = \mathcal{A}(U - U(0), U(0))$. Observe that $\mathcal{A}(v, t)$ is continuous in $[-\varepsilon, \varepsilon]$ and, by (3.7),

$$\|\mathcal{A}(v, t)\|_\varepsilon \leq 2 \left| \ln \left(\cos \frac{\varepsilon}{2} \right) \right| \sup_{[t - \|v\|_\varepsilon, t + \|v\|_\varepsilon]} |f|.$$

So \mathcal{A} is a map from $C_e(\varepsilon) \times \mathbb{R}$ to $C_e(\varepsilon)$. Moreover for any $t_0 \in \mathbb{R}$, there is $\varepsilon > 0$ such that the ball $B_1 \subset C_e(\varepsilon)$ of radius 1 and center 0 is stable by $\mathcal{A}(\cdot, t)$ for any t close to t_0 . By the above description, solving (3.5) with initial data t consists then in finding fixed points of $\mathcal{A}(\cdot, t)$.

Let us notice that the map \mathcal{A} is C^1 and

$$D_{\mathcal{A}|_{(v,t)}}(y, h)(\rho) = - \int_0^\rho \frac{1}{\sin s} \int_0^s (\sin x) f'(t + v(x))(y + h(x)) dx ds,$$

this implies that

$$\|D_{\mathcal{A}|_{(v,t)}}\| \leq 2 \left| \ln \left(\cos \frac{\varepsilon}{2} \right) \right| \sup_{[t - \|v\|_\varepsilon, t + \|v\|_\varepsilon]} |f'|.$$

So, there is ε such that B_1 is stable by $\mathcal{A}(\cdot, t)$ and $\mathcal{A}(\cdot, t)$ is a contraction on B_1 . So, $\mathcal{A}(\cdot, t)$ has a unique fixed point v_t in B_1 . *A priori*, $t + v_t$ is just a weak solution to (3.5) ($t + v_t$ is just continuous). But Lemma 3.4 tells that $t + v_t$ is C^2 and a true solution of (3.5). Thus $t + v_t$ extends to $(-\pi, \pi)$ as a solution of (3.5) with initial value t . Moreover, since any continuous function lies in B_1 if ε is sufficiently small, the function $t + v_t$ is the unique solution to (3.5) in $(-\pi, \pi)$ with initial value t .

In order to prove that v_t depends in a C^1 way of t , we apply the Implicit Function Theorem to the equation $\mathcal{A}(v_t, t) - v_t = 0$. Let $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $D_{v, \mathcal{A}|_{(v_{t_0}, t_0)}}$ is a contraction on $C_e(\varepsilon)$. Then $D_{v, \mathcal{A}|_{(v_{t_0}, t_0)}} - \text{id}$ is invertible from $C_e(\varepsilon)$ to $C_e(\varepsilon)$ and the Implicit Function Theorem applies. \square

Let us remark that if f is C^1 then item (iii) in Lemma 3.4 implies that the solution constructed by the above lemma is C^3 .

3.2.2 The study of the solutions U_t

We recall that we will use the solutions U_t to produce rotationally symmetric solutions to (2.2). If u is a solution to (2.2), then, at the maximum point \bar{p} , the maximum principle implies $f(u(\bar{p})) > 0$. So, some positiveness assumption could be interesting on f . The second thing is that we want to be sure that a solution U_t vanishes somewhere in $(0, \pi)$ if $t > 0$. We then have the following lemma.

Lemma 3.6. *Assume that f is C^1 and positive on \mathbb{R}_+^* , then the solution U_t given by Lemma 3.5 with positive initial value t vanishes at some $r_t \in (0, \pi)$, we denote by r_t the smallest positive zero of U_t . Moreover, if $r_t > \pi/2$, U_t is concave on $[\pi/2, r_t]$.*

Proof. If U_t never vanishes in $(0, \pi)$, then

$$U_t'(\rho) = -\frac{1}{\sin \rho} \int_0^\rho \sin(x) f(U_t(x)) dx < 0,$$

which implies that U_t is decreasing and positive, and hence, it has a limit. But, for ρ close to π , it holds

$$U_t'(\rho) \sim \pi - \frac{1}{\pi - \rho} \int_0^\pi \sin(x) f(U_t(x)) dx,$$

which is not integrable, a contradiction. Therefore, r_t is well defined.

To finish the proof, observe that if $\rho \in [\pi/2, r_t]$ then $U_t''(\rho) = -\cot \rho U_t'(\rho) - f(U_t(\rho)) \leq 0$, so U_t is concave, as desired. \square

Remark 3.7. *Let us point out that since $U_t'(r_t) < 0$, r_t depends in a C^1 way of t .*

So let us assume that $f > 0$ on \mathbb{R}_+^* . For any $p \in \mathbb{S}^2$ and $t > 0$, we denote by $D_{t,p}$ the disk of center p and radius r_t . In $D_{t,p}$, the function U_t define a radial function $u_{t,p}$ which is a solution to the OEP (2.2) in $D_{t,p}$. Our aim is to prove that using some extra hypotheses on f we can construct a family of candidate solution from $(u_{t,p})_{(t,p) \in \mathbb{R}_+^* \times \mathbb{S}^2}$.

As in the case of (OEP $_\lambda$), we need to go further in the description of the solutions of (3.5). So for the rest of this section, we fix some extra assumptions on f , the assumption is

$$f \text{ is a } C^1 \text{ function so that } f(x) > 0 \text{ and } f(x) \geq x f'(x) \text{ for any } x \in \mathbb{R}_+^*. \quad (H)$$

Since the family of functions U_t is C^1 in t , we define $H_t = \frac{\partial}{\partial t} U_t$. Observe that H_t is a solution to

$$H_t'' + (\cot \rho) H_t' + f'(U_t) H_t = 0, \quad (3.8)$$

with $H_t(0) = 1$. We use the notation $h_{t,p}$ to denote the rotationally symmetric function in $D_{t,p}$ associated to H_t .

Lemma 3.8. *Let f be a function satisfying (H), then for any t , H_t is positive in $(-r_t, r_t)$.*

Proof. Since $H_t(0) = 1$, if the lemma were not true, there is $\bar{\rho} \in (0, r_t)$ such that $H_t(\bar{\rho}) = 0$ and $H_t(\rho) > 0$ for $\rho \in (-\bar{\rho}, \bar{\rho})$. Given $p \in \mathbb{S}^2$, let $D_{\bar{\rho}}$ be the geodesic disk of center p and radius $\bar{\rho}$. The function $h_{t,p}$ is positive in $D_{\bar{\rho}}$ and vanishes on $\partial D_{\bar{\rho}}$. $u_{p,t}$ is positive in $\bar{D}_{\bar{\rho}}$ so $0 < \delta = \min_{D_{\bar{\rho}}} \frac{u_{t,p}}{h_{t,p}}$ is well defined. Then the function $w = u_{t,p} - \delta h_{t,p}$ is non negative in $D_{t,p}$ and vanishes at some point inside it. By Hypothesis (H), we have

$$\Delta w + f'(u_{t,p}) w = f'(u_{t,p}) u_{t,p} - f(u_{t,p}) \leq 0.$$

By the maximum principle, $w \equiv 0$ which is impossible since $w > 0$ on $\partial D_{\bar{\rho}}$. \square

From this lemma we see that U_t is increasing in t . We are now interested in the behavior when t goes to $+\infty$.

Lemma 3.9. *Let f be a function satisfying (H). Let t_n be a sequence going to $+\infty$ and $\rho_n \in [0, r_{t_n}]$ then $\|(U_{t_n}(\rho_n), U'_{t_n}(\rho_n))\| \rightarrow +\infty$. Actually, either $U_{t_n}(\rho_n) \rightarrow +\infty$ or $U'_{t_n}(\rho_n) \rightarrow -\infty$.*

Proof. Let us assume the lemma is false. For $t > 0$, we denote $\mathcal{U}_t(\rho) = U_t^2(\rho) + U_t'^2(\rho)$. So we can find $t_n \rightarrow +\infty$ and $\rho_n \in [0, r_{t_n}]$ such that $(\mathcal{U}_{t_n}(\rho_n))_n$ is bounded. Since r_t is non decreasing, we can define $r_\infty = \lim_{t \rightarrow \infty} r_t$.

Since f satisfies (H), $f(x) \leq f(1)x = \lambda x$ for $x \in [1, +\infty)$ and there is $a > 0$ such that $f(x) \leq a + \lambda x$ on \mathbb{R}_+ . If $\rho_n > \pi/2$, the concavity of U_{t_n} in $[\pi/2, \rho_n]$ implies that

$$0 \leq U_{t_n}(\pi/2) \leq U_{t_n}(\rho_n) + U'_{t_n}(\rho_n)(\pi/2 - \rho_n) \text{ and } U'_{t_n}(\rho_n) \leq U'_{t_n}(\pi/2) \leq 0.$$

Since $\mathcal{U}_{t_n}(\rho_n)$ is bounded, this implies $\mathcal{U}_{t_n}(\pi/2)$ is bounded and we can always assume that $\rho_n \leq \pi/2$. Using $0 \leq f(x) \leq a + \lambda x$ on \mathbb{R}_+ , for $0 < \rho < \min(r_t, \pi/2)$, we have

$$-(1 + 2 \cot \rho) \mathcal{U}_t \leq \mathcal{U}_t' = 2U_t U_t' - 2((\cot \rho) U_t' + f(U_t)) U_t' \leq 2a \sqrt{\mathcal{U}_t} + \lambda \mathcal{U}_t,$$

thus, since $\mathcal{U}_{t_n}(\rho_n)$ stays bounded, $(\mathcal{U}_{t_n}(\rho))_n$ is bounded for any $\rho \in (0, \min(r_\infty, \pi/2))$.

Let us now prove that $U_{t_n}(\rho)$ goes to ∞ for ρ close to 0 which is a contradiction with the boundedness of \mathcal{U} .

There is $d_0 \in (0, r_\infty)$ such that $U_2(\rho) \geq 1$ for $|\rho| \leq d_0$. From the monotonicity $t \mapsto U_t$, we thus have $U_t(\rho) \geq 1$ for $|\rho| \leq d_0$ and $t \geq 2$. Let $\mu > 0$ be the first eigenvalue of $-\Delta$ in a geodesic disk of \mathbb{S}^2 of radius d_0 . Let $\tilde{\mu} \geq \mu$ such that $a + \lambda x < \tilde{\mu} x$ if $x \geq 1$. Let $d_1 \leq d_0$ be the radius of a geodesic disk in \mathbb{S}^2 whose first eigenvalue is $\tilde{\mu}$.

The first eigenfunction in a disk of radius d_1 is a rotationally symmetric function generated by a profile curve W solving in $[-d_1, d_1]$ the equation

$$W'' + (\cot \rho) W' + \tilde{\mu} W = 0$$

with $W(0) = 1$. Let us prove that for $t \geq 2$, $U_t \geq tW$ on $[-d_1, d_1]$. Notice that $U_t \geq 1$ on $[-d_1, d_1]$, so $U_t > tW$ close to $-d_1$ and d_1 . Moreover, at $\rho = 0$, it holds

$$U_t(0) = t = tW(0) \text{ and } U_t''(0) = -\frac{1}{2}f(t) > -\frac{\tilde{\mu}}{2}t = tW''(0).$$

So $U_t - tW$ vanishes at 0 and is positive near 0. If $U_t - tW \geq 0$ were not true, there is $d_2 \in (0, d_1)$ such that $U_t(d_2) - tW(d_2) = 0$. Let $\delta = \sup_{[-d_2, d_2]} U_t/W$. Since $U_t - tW$ is positive near 0, $\delta > t$. We have $U_t - \delta W \leq 0$ on $[-d_2, d_2]$, $U_t(d_2) - \delta W(d_2) < 0$ and there is some d_3 such that $U_t(d_3) - \delta W(d_3) = 0$. At that value d_3 we obtain

$$\begin{aligned} 0 &\geq (U_t - \delta W)''(d_3) = -(\cot d_3) U_t'(d_3) - f(U_t)(d_3) + (\cot d_3) \delta W'(d_3) + \tilde{\mu} \delta W(d_3) \\ &> -(\cot d_3) (U_t - \delta W)'(d_3) - \tilde{\mu} (U_t - \delta W)(d_3) > 0, \end{aligned}$$

which gives us a contradiction and $U_t \geq tW$ on $[-d_1, d_1]$ as claimed.

Using this last inequality, we obtain $\lim U_{t_n}(\rho) = +\infty$ for any $\rho \in (-d_1, d_1)$. This contradicts $\mathcal{U}_{t_n}(\rho)$ stays bounded as $n \rightarrow \infty$. The proof is finished. \square

We need one more lemma concerning U_t and H_t which is similar to Lemma 3.2.

Lemma 3.10. *Let f be a function satisfying (H). If $t > 0$ then $H_t U_t'' - U_t' H_t' \neq 0$ on $[-r_t, r_t]$.*

Proof. First we see that $(H_t U_t'' - U_t' H_t')(0) = U_t''(0) = -\frac{1}{2}f(t) < 0$. Consider $\bar{\rho} \in (0, r_t]$. If $H_t(\bar{\rho}) = 0$, then $H_t'(\bar{\rho}) \neq 0$ since H_t is a non vanishing solution of (3.8) and $(H_t U_t'' - U_t' H_t')(\bar{\rho}) = -(U_t' H_t')(\bar{\rho}) \neq 0$. So we can assume $H_t(\bar{\rho}) \neq 0$.

As in the proof of Lemma 3.2, let $p = (0, 0, 1) \in \mathbb{S}^2$, $Y = (1, 0, 0) \in T_p \mathbb{S}^2$ and \tilde{Y} the Killing vectorfield $\tilde{Y} = -\langle q, Y \rangle p - \langle p, q \rangle Y$. We also consider the polar coordinates (ρ, θ) around p and \bar{q} the point with polar coordinates $\rho = \bar{\rho}$ and $\theta = 0$. Let v be the function defined by

$$v = \frac{\langle \tilde{Y}(\bar{q}), \nabla u_{t,p}(\bar{q}) \rangle}{h_{t,p}(\bar{q})} h_{t,p} - \langle \tilde{Y}, \nabla u_{t,p} \rangle \text{ in } D_p,$$

such that $v(\bar{q}) = 0$ (recall that $h_{t,p}(\bar{q}) \neq 0$ since $H_t(\bar{\rho}) \neq 0$). Since \tilde{Y} is Killing, we have $\Delta v + f'(u_{t,p})v = 0$.

Using polar coordinates, we have $v(\rho, \theta) = \frac{U_t'(\bar{\rho})}{H_t(\bar{\rho})} H_t(\rho) - \cos \theta U_t'(\rho)$. Let $D_{\bar{\rho}}$ denote the disk of radius $\bar{\rho}$ and center p . Since $U_t'(\bar{\rho}) < 0$, the maximum of v on $\partial D_{\bar{\rho}}$ is reached at $\theta = 0$ and $v \leq 0$ on $\partial D_{\bar{\rho}}$.

Let us prove $v \leq 0$ in $D_{\bar{\rho}}$. If this were not the case, we consider $\Omega = \{q \in D_{\bar{\rho}} \mid v(q) > 0\}$. Since $h_{t,p} > 0$ in $D_{\bar{\rho}}$, $0 < \delta = \min_{\Omega} \frac{h_{t,p}}{v}$ is well defined and satisfies $h_{t,p} - \delta v \geq 0$ in Ω and $h_{t,p} - \delta v$ vanishes at some point in Ω . Since $h_{t,p} - \delta v$ solves $\Delta u + f'(u_t)u = 0$, the maximum principle gives $h_{t,p} - \delta v \equiv 0$ in Ω but $h_{t,p} - \delta v > 0$ on $\partial \Omega$. We get our contradiction and $v \leq 0$ in $D_{\bar{\rho}}$.

By the boundary maximum principle this implies either $v \equiv 0$ in $D_{\bar{\rho}}$ or $\frac{\partial}{\partial \bar{\eta}} v(\bar{q}) > 0$. The first case is impossible since $v(p) = \frac{U_t'(\bar{\rho})}{H_t(\bar{\rho})} \neq 0$. The second case implies $0 < \frac{\partial}{\partial \bar{\eta}} v(\bar{q}) = \frac{U_t'(\bar{\rho})}{H_t(\bar{\rho})} H_t'(\bar{\rho}) - U_t''(\bar{\rho})$ and the lemma is proved. \square

3.2.3 The family of candidate solutions

Now we are interested in finding a good reparametrization of $(u_{t,p})_{(t,p) \in \mathbb{R}_+^* \times \mathbb{S}^2}$. More precisely we are going to prove the main result of this section.

Proposition 3.11. *Let us assume that f satisfies (H). Then (2.2) admits a family of candidate solutions.*

Let us consider

$$F : \begin{array}{ccc} \mathbb{R} \times (-\pi, \pi) & \rightarrow & \mathbb{R}^2 \\ (t, \rho) & \mapsto & (U_t(\rho), U_t'(\rho)). \end{array}$$

Let us also define $\Delta = \{(t, \rho) \in \mathbb{R}_+^* \times (-\pi, \pi) \mid -r_t < \rho < r_t\}$ and $\bar{\Delta}$ be the closure of Δ in $\mathbb{R}_+^* \times (\pi, \pi)$.

Lemma 3.12. *The map F is a diffeomorphism from Δ to $\mathbb{R}_+^* \times \mathbb{R}$.*

Proof. First we notice that $F(\Delta) \subset \mathbb{R}_+^* \times \mathbb{R}$. We have

$$\partial_t F(t, \rho) = (H_t(\rho), H_t'(\rho)) \quad \text{and} \quad \partial_\rho F(t, \rho) = (U_t'(\rho), U_t''(\rho)).$$

So the differential DF of F has not rank 2 if and only if $(H_t U_t'' - H_t' U_t')(\rho) = 0$ for some $\rho \in (-r_t, r_t)$. By Lemma 3.10, this never occurs so F is a local diffeomorphism.

Let us prove that F is proper. Let $(t_n, \rho_n)_n$ be a proper sequence in Δ . If $t_n \rightarrow +\infty$, Lemma 3.9 tells that $\|F(t_n, \rho_n)\| \rightarrow +\infty$. If $t_n \rightarrow 0$, by monotonicity, $U_{t_n}(\rho_n) \leq U_{t_n}(0) = t_n \rightarrow 0$ and $F(t_n, \rho_n)$ goes to $\partial(\mathbb{R}_+^* \times \mathbb{R})$. If $t_n \rightarrow a > 0$ then $\rho_n \rightarrow r_a$ and $U_{t_n}(\rho_n) \rightarrow U_a(r_a) = 0$. Therefore, in any case, $F(t_n, \rho_n)$ goes to $\partial(\mathbb{R}_+^* \times \mathbb{R})$. Thus F is proper which implies that $F : \Delta \rightarrow \mathbb{R}_+^* \times \mathbb{R}$ is a covering map. Since $\mathbb{R}_+^* \times \mathbb{R}$ is simply connected F is a diffeomorphism. \square

We have $F(t, r_t) = (0, U'_t(r_t))$ and $\partial_t U'_t(r_t) = H'_t(r_t) + U''_t(r_t) \partial_t r_t = H'_t(r_t) - \frac{H_t(r_t)}{U'_t(r_t)} U''_t(r_t) < 0$ by Lemma 3.10. This implies that F extend to a bijective map on $\bar{\Delta}$, notice that $F(t, -r_t) = (0, -U'_t(r_t))$. Actually we have much more.

Lemma 3.13. *There is an open neighborhood $\tilde{\Delta}$ of $\bar{\Delta}$ in $\mathbb{R}_+^* \times (-\pi, \pi)$ such that F is a diffeomorphism from $\tilde{\Delta}$ onto its image.*

Proof. Let P be the projection map

$$P : \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \{0\} \times \mathbb{R} \\ (x, y) & \mapsto & (0, y). \end{array}$$

Let $p \in \partial\Delta$, since $DF(p)$ has rank 2 there is a neighborhood \mathcal{O}_p of p such that F is a diffeomorphism from \mathcal{O}_p onto its image. Moreover by reducing \mathcal{O}_p if necessary, we can assume $F(\mathcal{O}_p \setminus \bar{\Delta}) \subset \mathbb{R}_-^* \times \mathbb{R}$.

Actually, by reducing \mathcal{O}_p once more, we can assume that for any $q \in \mathcal{O}_p$, the segment $[F(q), P(F(q))]$ is contained in $F(\mathcal{O}_p)$. We claim that $\tilde{\Delta} = \Delta \cup (\bigcup_{p \in \partial\Delta} \mathcal{O}_p)$ satisfies the expected property.

Since F is a local diffeomorphism on $\tilde{\Delta}$, we only have to check the injectivity. Since $F(\mathcal{O}_p \setminus \bar{\Delta}) \subset \mathbb{R}_-^* \times \mathbb{R}$ and F is injective in $\bar{\Delta}$, if F were not injective, there is $q_i \in \mathcal{O}_{p_i}$ ($i = 1, 2$) such that $F(q_1) = F(q_2)$. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be the segment $[F(q_i), P(F(q_i))]$, with $\gamma(0) = F(q_i)$ and $\gamma(1) = P(F(q_i))$, and $\gamma_i = F^{-1}(\gamma)$ in \mathcal{O}_{p_i} . We have $F(\gamma_i(1)) = P(F(q_i))$. So since F is injective on $\partial\Delta$, $\gamma_1(1) = \gamma_2(1)$. Since $F(\gamma_1(s)) = F(\gamma_2(s))$ and F is a local diffeomorphism, $\gamma_1 = \gamma_2$ and $q_1 = \gamma_1(0) = \gamma_2(0) = q_2$: F is injective. \square

Now the end of the proof of Proposition 3.11 is similar to the one of the (OEP $_\lambda$) case. More precisely, let us define $S = F(\tilde{\Delta})$, $\tilde{N} = \{((q, w), a) \in T\mathbb{S}^2 \times \mathbb{R} \mid (a, |w|) \in S\}$ and

$$F^{-1}(x, y) = (T(x, y), R(x, y)) \text{ for } (x, y) \in S.$$

Then the computations are the same except that the estimates of R become

$$R(x, y) = -\frac{2}{f(x)}y + o(y) \quad \text{and} \quad \frac{\partial R}{\partial y}(x, y) = -\frac{2}{f(x)} + o(1).$$

4 Main Theorem and applications

Before stating our main theorem, let us give a last definition. Let (M, g) be a Riemannian surface and Q be a quadratic form on M . A curve γ on M is a *line of curvature* of Q if $\gamma'(t)$ is an eigenvector of $Q_{\gamma(t)}$ for any t .

Now, with the definition of a family of candidate solutions associated to an OEP, we can announce:

Theorem A. *Let (M, g) be a complete connected Riemannian surface and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. Let $\Omega \subset M$ be a bounded connected domain with a C^2 boundary $\partial\Omega$ and $u \in C^3(\overline{\Omega})$ be a solution to the OEP*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle = \alpha & \text{on } \partial\Omega. \end{cases}$$

Assume that the above OEP admits a family of candidate solutions \mathcal{C} . Then, there exists a C^1 traceless symmetric bilinear form Q on $\overline{\Omega}$ such that

1. *Q vanishes at some $x \in \overline{\Omega}$ if, and only if, u has a contact of order $k \geq 2$ at x with some $\bar{v} \in \mathcal{C}$.*
2. *Q vanishes identically on $\overline{\Omega}$ if, and only if, $u \in \mathcal{C}_\alpha$.*
3. *If Q does not vanish identically on $\overline{\Omega}$, then Q has only isolated (interior and boundary) zeroes. Moreover, the null directions of Q determine on $\overline{\Omega}$ two C^1 line fields with isolated singularities of negative index.*
4. *The boundary $\partial\Omega$ is a line of curvature of Q .*

4.1 Applications

As a consequence of Theorem A, we can prove some uniqueness property for solutions of some OEP.

Theorem 4.1. *Let f be a Lipschitz function and $\Omega \subset M$ be a compact simply-connected domain with a C^2 boundary $\partial\Omega$ where the OEP (1.1) admits a solution $u \in C^3(\Omega)$. If the OEP admits a family of candidate solutions \mathcal{C} , then $u \in \mathcal{C}_\alpha$.*

Combining the above theorem with the candidate families constructed in Section 3, we obtain:

Corollary 4.2 (BCN conjecture in \mathbb{S}^2). *Let us assume that f satisfies (H). Let u be a solution of (1.1) in some topological disk $\Omega \subset \mathbb{S}^2$ with C^2 boundary. Then, Ω is a geodesic disk (centered at some point $p \in \mathbb{S}^2$) and u is rotationally symmetric (with respect to the center p).*

Proof of Theorem 4.1. Let Q be the traceless symmetric bilinear form given by Theorem A. If Q vanishes identically then $u \in \mathcal{C}_\alpha$ and we are done. So, we assume that Q does not vanish identically and we will get a contradiction. We follow ideas already appearing in the work of Choe [11].

Let Φ_+ be a diffeomorphism from the north hemisphere $\mathbb{S}_+ = \{(x, y, z) \in \mathbb{S}^2 \mid z \geq 0\}$ to $\overline{\Omega}$. Let us define $\Phi_- = \Phi_+ \circ S$ on the south hemisphere \mathbb{S}_- where S is the symmetry with respect to the equator E . We define $\tilde{Q} = \Phi_+^* Q$ on \mathbb{S}_+ and $\tilde{Q} = \Phi_-^* Q$ on \mathbb{S}_- . Since $\partial\Omega$ is a line of curvature of Q these two definitions coincide on E . By Theorem A, \tilde{Q} is a symmetric bilinear form on \mathbb{S}^2 which has isolated zeroes and is a Lorentzian metric outside its zeroes. Moreover the null directions of \tilde{Q} determine on \mathbb{S}^2 two line fields with isolated singularities of negative index. This gives a contradiction by the Poincaré-Hopf Index Theorem. \square

Another consequence is the classification of simply-connected harmonics domains in \mathbb{S}^2 . A domain $\Omega \subset \mathbb{S}^2$ with regular boundary is harmonic if the mean value of any harmonic function on Ω equals its mean value on $\partial\Omega$ and they are characterized by supporting a solution to the *Serrin Problem*

$$\begin{cases} \Delta u + 1 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle_g = \alpha & \text{on } \partial\Omega. \end{cases}$$

Note that in the above case, u must be positive on the interior of Ω by the maximum principle. Thus, Theorem 4.1 and the candidate families constructed in Section 3 give

Corollary 4.3 (Serrin Problem in \mathbb{S}^2). *Any simply connected harmonic domain in \mathbb{S}^2 is a geodesic ball.*

The Schiffer conjecture D can be stated as (cf. [33, 35]): Given a bounded connected open domain $\Omega \subset \mathbb{S}^2$ with a regular boundary and such that the complement of its closure is connected, the existence of a solution to

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle_g = \alpha & \text{on } \partial\Omega, \end{cases}$$

implies that Ω is a geodesic ball. When λ is the first eigenvalue of the Laplacian then $u > 0$ on Ω . Thus, the Schiffer conjecture D for the first eigenvalue can be stated as:

First Schiffer conjecture D in \mathbb{S}^2 : *If there exists a solution to*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \vec{\eta} \rangle_g = \alpha & \text{on } \partial\Omega, \end{cases}$$

then Ω is a geodesic ball.

Therefore, the previous considerations lead us to

Corollary 4.4 (First Schiffer Conjecture D in \mathbb{S}^2). *The first Schiffer conjecture is true in \mathbb{S}^2 .*

A last application concerns the case where $f(x) = x - x^3$ which is the usual Allen-Cahn non-linearity. Actually such f does not satisfy (H) since f is negative on $(1, \infty)$ but $f(x) \geq xf'(x)$ for $x \geq 0$. So if u is solution of (1.1) on $\Omega \subset \mathbb{S}^2$ and $M = \max u$, the maximum principle implies that $f(M) > 0$ and $M < 1$. We can then consider a function g which coincides with f on $[0, M]$ and satisfying (H). As a consequence Ω is a g -extremal domain. We thus obtain

Corollary 4.5. *If $f(x) = x - x^3$, a f -extremal disk in \mathbb{S}^2 with C^2 boundary is a geodesic disk.*

4.2 Proof of Theorem A

Once the quadratic form Q will be defined, we will consider its $(2,0)$ part \mathcal{P} : if z is a local conformal parameter, $\mathcal{P} = P(z)dz^2$ where $P(z) = Q(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$. Properties of Q can then be deduced from properties of \mathcal{P} since $Q = \mathcal{P} + \overline{\mathcal{P}}$ (cf. [21]).

Let $\Omega \subset (M, g)$ be a bounded domain with C^2 boundary $\partial\Omega$. Let u be a solution to the problem (2.2).

Let $(v_{q,w,a})_{(q,w,a) \in N}$ be a candidate family associated to the OEP (1.1), we recall that $N = TM \times \mathbb{R}_+ \setminus \{(q, 0, 0) \in TM \times \mathbb{R}; q \in M\}$. For any $x \in \Omega$, let us define the symmetric bilinear form $Q_x : T_x M \times T_x M \rightarrow \mathbb{R}$ given by

$$Q_x := \nabla^2 u(x) - \nabla^2 v_{x, \nabla u(x), u(x)}(x)$$

where ∇^2 is the Hessian operator. Observe that Q is well-defined in $\overline{\Omega}$ and is C^1 since u is C^3 and $(v_{q,w,a})_{(q,w,a) \in N}$ is a smooth family. The definition of Q proves item 1 in Theorem A.

Claim A. Q_x is traceless so either $Q_x = 0$ or Q_x is a Lorentzian metric.

Proof of Claim A. Let us compute the trace of Q_x (w.r.t. g) at each point $x \in \overline{\Omega}$:

$$\text{Tr}_g Q_x = \Delta u(x) - \Delta v_{x, \nabla u(x), u(x)}(x) = -f(u(x)) + f(v_{x, \nabla u(x), u(x)}(x)) = -f(u(x)) + f(u(x)) = 0,$$

where we have used item (b) of the family of candidate solutions. This finishes the proof of Claim A. \square

A second point is easy to verify, it is statement 4 in Theorem A.

Claim B. The boundary $\partial\Omega$ is a line of curvature of Q .

Proof of Claim B. Let $\vec{\tau}$ the unit tangent vector field along $\partial\Omega$. Since $\langle \nabla u, \vec{\eta} \rangle = \alpha$ is constant along $\partial\Omega$, if we differentiate with respect to $\vec{\tau}$ we get

$$0 = \langle \nabla_{\vec{\tau}} \nabla u, \vec{\eta} \rangle + \langle \nabla u, \nabla_{\vec{\tau}} \vec{\eta} \rangle = \nabla^2 u(\vec{\tau}, \vec{\eta}) + \alpha \langle \vec{\eta}, \nabla_{\vec{\tau}} \vec{\eta} \rangle = \nabla^2 u(\vec{\tau}, \vec{\eta}),$$

where we have used the full overdetermined condition and $\langle \vec{\eta}, \nabla_{\vec{\tau}} \vec{\eta} \rangle = 0$ (observe we use $\partial\Omega \in C^2$). Note that the above holds for any solution satisfying both boundary conditions, as it does the candidate family, hence

$$Q_x(\vec{\tau}(x), \vec{\eta}(x)) = \nabla^2 u(x)(\vec{\tau}(x), \vec{\eta}(x)) - \nabla^2 v_{x, \nabla u(x), u(x)}(x)(\vec{\tau}(x), \vec{\eta}(x)) = 0,$$

that is, $\partial\Omega$ is a line of curvature of Q since Q is trace free. \square

Now, we study the behavior of Q near a point where $Q_x = 0$. To do this we introduce $z \in \mathcal{U} \subset \Omega$ a local conformal parameter in \mathcal{U} , i.e., there exists a positive function $\lambda \in C^2(\mathcal{U})$ such that $g = \lambda(z)|dz|^2$. We have a first observation.

Lemma 4.6. Given $v \in C^2(\mathcal{U})$ a solution to $\Delta v + f(v) = 0$, consider the quadratic differential

$$\mathcal{H}_v = H_v(z) dz^2 = (\nabla^2 v) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) dz^2. \quad (4.9)$$

Then, it holds

$$(H_v)_{\bar{z}} = -\frac{\lambda}{4} (f'(u) + 2K_g) \langle \nabla v, \frac{\partial}{\partial z} \rangle, \quad (4.10)$$

where K_g denote the Gaussian curvature of g .

Proof. First, note that $\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial \bar{z}} = \nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial z} = 0$ since z is a local conformal parameter for g . Moreover, a straightforward computation shows

$$\langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \frac{\partial}{\partial z} \rangle = \frac{\lambda}{4} \Delta v.$$

Then,

$$\begin{aligned} (H_v)_{\bar{z}} &= \frac{\partial}{\partial \bar{z}} \left((\nabla^2 v) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right) = \frac{\partial}{\partial \bar{z}} \left(\langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \frac{\partial}{\partial z} \rangle \right) = \langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \frac{\partial}{\partial z} \rangle \\ &= \langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \frac{\partial}{\partial z} \rangle + \langle R \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z} \right) \nabla v, \frac{\partial}{\partial z} \rangle \\ &= \frac{\partial}{\partial z} \left(\langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \frac{\partial}{\partial z} \rangle \right) - \langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial z} \rangle + \langle R \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z} \right) \nabla v, \frac{\partial}{\partial z} \rangle \\ &= \frac{\partial}{\partial z} \left(\frac{\lambda}{4} \Delta v \right) - \frac{\lambda_z}{\lambda} \langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla v, \frac{\partial}{\partial z} \rangle - \frac{\lambda K_g}{2} \langle \nabla v, \frac{\partial}{\partial z} \rangle \\ &= \frac{\lambda}{4} \frac{\partial}{\partial z} (\Delta v) - \frac{\lambda K_g}{2} \langle \nabla v, \frac{\partial}{\partial z} \rangle = \frac{\lambda}{4} \left(\langle \nabla \Delta v, \frac{\partial}{\partial z} \rangle - 2K_g \langle \nabla v, \frac{\partial}{\partial z} \rangle \right), \end{aligned}$$

now, using that $\Delta v = -f(v)$ we get $\nabla \Delta v = -f'(v) \nabla v$, and (4.10) holds. \square

Now we introduce \mathcal{P} the $(2,0)$ part of Q : i.e. $\mathcal{P} = Q_z \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) dz^2 = P(z) dz^2$ where z is a local conformal parameter. Since Q is trace free, we have

$$Q = \mathcal{P} + \overline{\mathcal{P}}.$$

Claim C. *The function P satisfies $P_{\bar{z}}(z) = \beta(z) \overline{P}(z)$ where β is some continuous complex function.*

Proof of Claim C. Let us fix some z_0 where we are going to compute $P_{\bar{z}}$, we denote by $u_0 = u(z_0)$ and $w_0 = \nabla u(z_0)$. We finally define \bar{v} to be v_{z_0, w_0, u_0} . u and \bar{v} are then two solutions to (1.1) which satisfy $u(z_0) = \bar{v}(z_0)$ and $\nabla u(z_0) = \nabla \bar{v}(z_0)$. In the following computations, we use the notation $v_{q,w,a}(z) = v(z, q, w, a)$ since we will compute derivatives with respect to all these parameters (this computation can be done because of the smoothness of a family of candidate solutions). Hence

from (4.10), we get

$$\begin{aligned}
 P_{\bar{z}}(z_0) &= (H_u)_{\bar{z}}(z_0) - (H_{\bar{v}})_{\bar{z}}(z_0) - D_q \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial \bar{z}} \right) \\
 &\quad - D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) (\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u) - D_a \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial \bar{z}} u \right) \\
 &= -\frac{\lambda}{4} (f'(u_0) + 2K_g) \langle w_0, \frac{\partial}{\partial z} \rangle + \frac{\lambda}{4} (f'(u_0) + 2K_g) \langle w_0, \frac{\partial}{\partial z} \rangle \\
 &\quad - D_q \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial \bar{z}} \right) - D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) (\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u) \\
 &\quad - D_a \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \langle w_0, \frac{\partial}{\partial \bar{z}} \rangle \\
 &= -D_q \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial \bar{z}} \right) - D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) (\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u) \\
 &\quad - D_a \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \langle w_0, \frac{\partial}{\partial \bar{z}} \rangle.
 \end{aligned}$$

We can do the same computation by replacing u by \bar{v} but in this case, since \bar{v} is a candidate solution $\bar{v} = v_{x, \nabla \bar{v}(x), \bar{v}(x)}$ for any x , the associated $\mathcal{P}^{\bar{v}}$ vanishes. Let us explain this. Consider the symmetric quadratic form

$$Q_x^{\bar{v}} := \nabla^2 \bar{v}(x) - \nabla^2 v_{x, \nabla \bar{v}(x), \bar{v}(x)}(x),$$

and hence we can write

$$Q^{\bar{v}} = \mathcal{P}^{\bar{v}} + \overline{\mathcal{P}^{\bar{v}}},$$

but, since \bar{v} is a candidate solution, we have $\mathcal{P}^{\bar{v}} \equiv 0$. Hence, we obtain

$$\begin{aligned}
 0 &= -D_q \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial \bar{z}} \right) - D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) (\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla \bar{v}) \\
 &\quad - D_a \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \langle w_0, \frac{\partial}{\partial \bar{z}} \rangle.
 \end{aligned}$$

So gathering the above two equations gives

$$P_{\bar{z}}(z_0) = -D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) (\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u - \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla \bar{v}).$$

We also have

$$\begin{aligned}
 (\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u - \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla \bar{v})(z_0) &= \frac{2}{\lambda} \left(\langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u - \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla \bar{v}, \frac{\partial}{\partial \bar{z}} \rangle \frac{\partial}{\partial z} + \langle \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla u - \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla \bar{v}, \frac{\partial}{\partial z} \rangle \frac{\partial}{\partial \bar{z}} \right) \\
 &= \frac{2}{\lambda} \left((\nabla^2 u \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}} \right) - \nabla^2 \bar{v} \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}} \right)) \frac{\partial}{\partial z} + (\nabla^2 u \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z} \right) - \nabla^2 \bar{v} \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z} \right)) \frac{\partial}{\partial \bar{z}} \right) \\
 &= \frac{2}{\lambda} \left(\bar{P}(z_0) \frac{\partial}{\partial z} + \frac{\lambda}{4} (\Delta u(z_0) - \Delta \bar{v}(z_0)) \frac{\partial}{\partial \bar{z}} \right) \\
 &= \frac{2}{\lambda} \left(\bar{P}(z_0) \frac{\partial}{\partial z} + \frac{\lambda}{4} (-f(u_0) + f(u_0)) \frac{\partial}{\partial \bar{z}} \right) = \frac{2}{\lambda} \bar{P}(z_0) \frac{\partial}{\partial z}.
 \end{aligned}$$

Thus we obtain

$$P_{\bar{z}}(z_0) = -D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \left(\frac{2}{\lambda} \bar{P}(z_0) \frac{\partial}{\partial z} \right) = -\frac{2}{\lambda} D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \left(\frac{\partial}{\partial z} \right) \bar{P}(z_0)$$

which is exactly $P_{\bar{z}}(z_0) = \beta(z_0) \bar{P}(z_0)$ for $\beta(z_0) = -\frac{2}{\lambda} D_w \nabla^2 v(z_0, z_0, w_0, u_0) \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \left(\frac{\partial}{\partial z} \right)$. \square

In order to study the behaviour of Q near a vanishing point we will use two lemmas.

Lemma 4.7. *Let $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a complex function defined in an open subset U . Assume that*

$$|f_{\bar{z}}| \leq h|f|$$

where h is a continuous non-negative function. Assume further that $z = z_0 \in U$ is a zero of f . Then either $f \equiv 0$ in a neighborhood $V \subset U$ of z_0 or there is $k \in \mathbb{N}$ such that

$$f(z) = (z - z_0)^k \tilde{f}(z), \quad z \in V,$$

where \tilde{f} is a continuous function with $\tilde{f}(z_0) \neq 0$.

This is [22, Lemma 2.7.1, pp 75]. Actually we will also use a version of this lemma on the boundary. So let D denote the unit disk in \mathbb{C} and $D_{\pm} = \{z \in D \mid \pm \Im z > 0\}$. We then have the following variant of the above lemma.

Lemma 4.8. *Let $f : \bar{D}_+ \rightarrow \mathbb{C}$ be a continuous complex function which is C^1 in D_+ and satisfies $f(z) \in \mathbb{R}$ if $z \in \mathbb{R}$. Assume that in D_+*

$$|f_{\bar{z}}| \leq h|f|$$

where h is a continuous non-negative function on \bar{D} . Assume further that $f(0) = 0$. Then either $f \equiv 0$ in a neighborhood V of 0 or there is $k \in \mathbb{N}$ such that

$$f(z) = z^k \tilde{f}(z), \quad z \in V,$$

where \tilde{f} is a continuous function with $\tilde{f}(0) \neq 0$.

Proof. For $c > 0$ and $\zeta \in \mathbb{C}$, we denote by $D_c(\zeta)$ the disk in \mathbb{C} of center ζ and radius c . Let $w \in D_R(0) \setminus \{0\}$, $R < 1$, and define $W = D_R(0) \setminus (D_a(0) \cup D_a(w))$, where $a < \min(|w|/2, R - |w|)$.

Let us extend the definition of f to D_- by $f(z) = f(\bar{z})$. Since f is real on $D \cap \mathbb{R}$, this gives us a continuous complex function in \bar{D} . If we extend h by $h(z) = h(\bar{z})$ we get in D_- , $|f_{\bar{z}}| \leq h|f|$.

If $r \in \mathbb{N}$, we can then define a continuous complex 1-form by

$$\phi = \frac{f(z)}{z^r(z-w)} dz$$

which is C^1 in $D_+ \cup D_-$. Moreover $d\phi = -\frac{f_{\bar{z}}}{z^r(z-w)} dz \wedge d\bar{z}$. We then have the following computation

$$\begin{aligned} \int_W d\phi &= \int_{W \cap D_+} d\phi + \int_{W \cap D_-} d\phi \\ &= \int_{\partial D_R(0) \cap D_+} \phi - \int_{\partial D_a(0) \cap D_+} \phi - \int_{\partial D_a(w) \cap D_+} \phi + \int_{[-1,1] \cap W} \phi \\ &\quad + \int_{\partial D_R(0) \cap D_-} \phi - \int_{\partial D_a(0) \cap D_-} \phi - \int_{\partial D_a(w) \cap D_-} \phi - \int_{[-1,1] \cap W} \phi \\ &= \int_{\partial D_R(0)} \phi - \int_{\partial D_a(0)} \phi - \int_{\partial D_a(w)} \phi. \end{aligned}$$

Now, the end of the proof is similar to the one of [22, Lemma 2.7.1, pp 75] which only uses the continuity of f and a uniform bound on h . \square

We can now obtain statement 3 in Theorem A.

Claim D. *Either $Q \equiv 0$ on $\overline{\Omega}$ or Q has isolated zeroes. Moreover, in the second case, the null directions of Q determine on $\overline{\Omega}$ two C^1 line fields with isolated singularities of negative index.*

Proof of Claim D. Let x_0 be a zero of Q . Assume that $x_0 \in \Omega$ and choose some complex conformal coordinate near x_0 such that $z = 0$ is x_0 . The $(2,0)$ -part $\mathcal{P} = P(z)dz^2$ vanishes then at 0. By Claim C and Lemma 4.7, either P vanishes in a neighborhood of 0 or $P(z) = z^n \tilde{f}(z)$ for some $n \in \mathbb{N}$ and $\tilde{f}(0) \neq 0$. In the second case, 0 is an isolated zero of P so is x_0 for Q . The writing $z^n \tilde{f}(z)$ implies that the null directions of Q have negative index around x_0 . We have then proved that either $Q \equiv 0$ or it has isolated zeroes in Ω .

If $x_0 \in \partial\Omega$, we choose some complex conformal coordinate $z \in \overline{D}_+$ near x_0 such that $z = 0$ is x_0 and $z \in \mathbb{R}$ correspond to $\partial\Omega$. We then have $\frac{\partial}{\partial z} = \frac{\sqrt{\lambda}}{2}(\vec{\tau} + i\vec{\eta})$ along $\overline{D}_+ \cap \mathbb{R}$. By Claim B, this implies

$$P = Q\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \frac{\lambda}{4}(Q(\vec{\tau}, \vec{\tau}) - Q(\vec{\eta}, \vec{\eta}) + 2iQ(\vec{\tau}, \vec{\eta})) = \frac{\lambda}{4}(Q(\vec{\tau}, \vec{\tau}) - Q(\vec{\eta}, \vec{\eta})) \in \mathbb{R}.$$

As above, the conclusion follows using Lemma 4.8. \square

Next, we verify statement 2 in Theorem A.

Claim E. *If $Q \equiv 0$ on Ω then $u \in \mathcal{C}_\alpha$.*

Proof of Claim E. Let $x_0 \in \Omega$ and write $\bar{v} = v_{x_0, \nabla u(x_0), u(x_0)}$. Let X be normal coordinates at x_0 . At x_0 , u and \bar{v} have a contact of order at least 2. Moreover

$$\Delta(u - \bar{v}) = f(u) - f(\bar{v}) = g \times (u - \bar{v})$$

where $g = \int_0^1 f'(tu + (1-t)\bar{v})dt$. So by Bers Theorem [8], either $u = \bar{v}$ near 0 or $(u - \bar{v})(X) = q(X) + o(\|X\|^k)$ where $q \neq 0$ is a homogeneous harmonic polynomial of degree $k \geq 2$. Assume we are in this second case. So we can write

$$\begin{aligned} Q_x &= \nabla^2 u(x) - \nabla^2 v_{x, \nabla u(x), u(x)}(x) \\ &= (\nabla^2 u(x) - \nabla^2 \bar{v}(x)) + (\nabla^2 \bar{v}(x) - \nabla^2 v_{x, \nabla u(x), u(x)}(x)). \end{aligned}$$

In coordinates the hessian operator ∇^2 can be written as an operator which is the Euclidean hessian H up to a $O(\|X\|)$ error term. So the first term in the above computation is just $Hq(X) + o(\|X\|^{k-2})$. For the second term, we notice that $\bar{v} = v_{x, \bar{v}(x), \nabla \bar{v}(x)}$ and the map $(w, a) \in T_x M \times \mathbb{R} \mapsto \nabla^2 v_{x, w, a}(x)$ is smooth. So the second term can be evaluated by

$$\nabla^2 \bar{v}(x) - \nabla^2 v_{x, u(x), \nabla u(x)}(x) = O(|u - \bar{v}|(x) + \|\nabla u - \nabla \bar{v}\|(x)) = O(\|X\|^{k-1})$$

So $Q_x = Hq(X) + o(\|X\|^{k-2})$. As Q vanishes around x_0 this implies that $q = 0$. So we get a contradiction and $u = \bar{v}$ near x_0 and by connectedness, $u = \bar{v}$ everywhere and $\Omega = \Omega_{x_0, \nabla u(x_0), u(x_0)}$. \square

Remark 4.9. *Let us notice that the argument used in Claim E can be applied to prove that Q has isolated zeroes of negative index in Ω but it seems difficult to do the same at $x_0 \in \partial\Omega$ since we do not have a result similar to Bers Theorem at a boundary point.*

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