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***Abstract*— This paper presents a novel discrete-time decentralized control law for the Voronoi-based self-deployment of a Multi-Agent dynamical system. The basic control objective is to let the agents deploy into a bounded convex polyhedral region and maximize the coverage quality by computing locally the control action for each agent. The Voronoi tessellation algorithm is employed to partition dynamically the deployed region and to allocate each agent to a corresponding bounded functioning zone at each time instant. The control synthesis is then locally computed based on an optimal formulation framework related to the Lloyd's algorithm but according to the discrete-time agent's dynamics equation. The performance of the discretized optimal solution will be demonstrated via an illustrative example.**

***Index Terms*— Multi-agent dynamical systems, Voronoi partition, discrete-time systems.**

I. Introduction

The attention for Multi-Agent Dynamical Systems (MAS) has been raised in recent years due to their various applications ranging from research operations (meteorological monitoring, geological exploration etc.) to domestic utilities (mini mobile robots, manufacturing etc.). Among the most well-known MAS applications the deployment task is widely considered leading to distributing a group of independent but cooperative mobile agents (vehicle, robots, etc.) over a predetermined region (e.g. [1], [2]). The ultimate goal of such applications is to maximize the coverage quality subject to selected constraints, e.g. energy-efficient consumption [3], [2], environmental disturbance [4], [5] or obstacle avoidance [6], [7]. This goal is translated in terms of driving the agents into a static partition considered as optimal coverage with respect to a given criterion. There are various works mentioned in the control literature employing dynamic Voronoi partition [8] as a conventional mathematical tool to approach a stationary configuration over a given bounded region. Many recent research works focus on driving the MAS into a Centroidal Voronoi Configuration (CVC) in which the position of each agent coincides with the centroid (center of mass) of its associated Voronoi cell[1] [9], [3], [10], [11]. The Lloyd's algorithm (see [12]) is widely used as a conventional tool to approximately compute a CVC by moving the agents toward their corresponding centers of mass.

In [9], the authors propose a generalized power-weighted Voronoi partition and modify the Lloyd's algorithm to solve the power constrained deployment. The results in [3] aim to reduce the step size in the Lloyd's algorithm. Other results [13] use Mixed-Integer Programming to minimize the energy consumption. In the notable recent work [10], the coverage task is formulated as an optimal control problem, which considers energy constraints in the control law computation. The authors show that the Lloyd's algorithm-based local control may waste too much energy and they propose an improved energy-efficient optimal control. The main results are developed for continuous-time systems.

The main contribution of the current paper is to extend the decentralized optimal control in [10] for discrete-time systems. The main novelty is to use the prediction of the dynamic Voronoi partition in order to enhance the coverage quality by driving the Multi-Agent system to a CVC. This is chosen as a conventional optimal coverage of the self-deployment problem.

This paper is structured as follows. Section II presents some useful preliminaries. The main results of [10] are briefly recalled in Section III. The proposed discrete-time solution is detailed in Section IV, followed by the stability proof for the entire MAS in Section V. Concluding remarks and perspectives are mentioned in the last section.

Notation: We use $||x||= \sqrt{x^\top x}$ to denote the Euclidean norm of the given vector x. We denote by $\mathcal{N}_i$ the set containing the neighbor's indices of the i^{th} agent.

II. Problem formulation

A. System description

Consider the Multi-Agent system (denoted by $\mathbf{\Sigma}$) composed of N mobile agents. The indices set is $\mathcal{N} = \{1,\dots,N\}$. Each agent has its own dynamics

$$\dot{x}_i = u_i, \text{with } i \in \mathcal{N} \tag{1}$$

which has the discrete-time zero-order-hold form

$$x_i(k+1) = x_i(k) + T_s u_i(k), \text{with } i \in \mathcal{N} \tag{2}$$

where $x_i \in \mathbb{R}^n$ is the state-space vector, $u_i \in \mathbb{R}^n$ is the input vector[2] and T_s is the sampling time. We use $x = [x_1^\top \dots x_N^\top]^\top$ and $u = [u_1^\top \dots u_N^\top]^\top$ to denote respectively the collective state and input of the system $\mathbf{\Sigma}$.

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[1]A Voronoi partition can be obtained in a distributed manner by considering that each agent construction relies only on its local state and the information of its closest neighbors, to construct its own Voronoi cell.

[2]The dimension choice of u_i follows the chosen dynamics (1). The results can be extended for $u_i \in \mathbb{R}^m$, with $m \neq n$.

B. Constraints on the agents' environment

Assume that the common working space $\mathcal{X} \subset \mathbb{R}^n$ is convex and bounded, represented by a polytope. A partition $\mathbb{V}(x_1, \ldots, x_N)$ of $\mathcal{X}$ is

$$\mathcal{X} = \bigcup_{i=1}^{N} \mathbb{V}_i, \quad \mathbb{V}_i \cap \mathbb{V}_j = \emptyset, \quad \forall i, j \in \mathcal{N} \tag{3}$$

A natural mathematical definition of such a decomposition is provided by the Voronoi partition, which characterizes the *neighborhood* $\mathbb{V}_i(x_i)$ as

$$\mathbb{V}_i = \{x \in \mathcal{X} | \ \|x_i - x\| \leq \|x_j - x\|, \ \forall j \neq i\} \tag{4}$$

It is worth to be mentioned that each set $\mathbb{V}_i$ is a polytope as a consequence of the boundedness of $\mathcal{X}$ and the structure of the constraints in (4). Using the available state measurement of the Multi-Agent system $\mathbf{\Sigma}$ at the time instant k, the geometric formulation (3) leads to a time-varying partition.

C. Coverage control

The ultimate goal of the deployment is to maximize the coverage. The density function[3] $\phi : \mathcal{X} \to \mathbb{R}_+$ denotes the priority of coverage at a point $q \in \mathcal{X}$. A candidate Lyapunov function $V(x)$ (see [11], [14]) is defined as

$$V(x) = \sum_{i=1}^{N} p_i \int_{\mathbb{V}_i} \|x_i - q\|^2 \phi(q) dq \tag{5}$$

with the positive scalar p_i denoting a weighting coefficient for $V(x)$. The optimal (maximized) coverage is achieved if $V(x)$ reaches its minimum.

Its local minimum points are obtained by solving $\frac{\partial V}{\partial x_i} = 0$ with the partial derivative of $V(x)$ with respect to x_i being

$$\begin{aligned} \frac{\partial V}{\partial x_i} &= 2p_i \int_{\mathbb{V}_i} (x_i - q)^\top \phi(q) dq \\ &= 2p_i \left(\int_{\mathbb{V}_i} \phi(q) dq \right) \left(x_i - \frac{\int_{\mathbb{V}_i} q\phi(q) dq}{\int_{\mathbb{V}_i} \phi(q) dq} \right)^\top \\ &= 2p_i M_{\mathbb{V}_i} (x_i - CM_{\mathbb{V}_i})^\top \end{aligned} \tag{6}$$

where the *mass* $M_{\mathbb{V}_i}$ and the *center of mass* $CM_{\mathbb{V}_i}$ of the Voronoi cell $\mathbb{V}_i$ are respectively defined as in [10]

$$M_{\mathbb{V}_i} = \int_{\mathbb{V}_i} \phi(q) dq \tag{7}$$

$$CM_{\mathbb{V}_i} = \frac{\int_{\mathbb{V}_i} q\phi(q) dq}{\int_{\mathbb{V}_i} \phi(q) dq} \tag{8}$$

Solving $\frac{\partial V}{\partial x_i} = 0, \forall i \in \mathcal{N}$ leads to an optimal configuration where $x_i = CM_{\mathbb{V}_i}$. Such optimal configuration is called *Centroidal Voronoi Configuration* (CVC). Using LaSalle's invariance principle (see [15]), it can be proved that the agent's local control $u_i = k_i(x_i - CM_{\mathbb{V}_i})$, with $k_i < 0$ can lead to the convergence of the entire MAS to a CVC.

The results obtained for the continuous-time MAS will be recalled in the next section.

[3]The function ϕ is continuously differentiable over $\mathcal{X}$.

III. Continuous-time decentralized optimal control

This section recalls the main results given in [10], applied for a MAS whose agents dynamics are characterized by the continuous-time equations (1). The decentralized control is obtained locally by solving the following optimization control problem

$$\inf_{u_i, i \in \mathcal{N}} \int_0^\infty L(x, u) d\tau \text{ s.t.: } \dot{x}_i = u_i \tag{9}$$

where $L(x, u) = \sum_{i=1}^{N} \left(s_i \left\| \int_{\mathbb{V}_i} (x_i - q)\phi(q) dq \right\|^2 + r_i \|u_i\|^2 \right)$ denotes the running cost. The scalars $s_i > 0, r_i > 0$ represent the weighting coefficients. The Hamilton-Jacobi-Bellman (HJB) equation of (9) with the dynamics (1) is

$$\inf_{u_i, i \in \mathcal{N}} H \left(L(x, u), \frac{\partial V}{\partial x} \right) = 0 \tag{10}$$

with the Hamiltonian

$$\begin{aligned} H &= L(x, u) + \frac{\partial V}{\partial x} \dot{x} \\ &= \sum_{i=1}^{N} \left(s_i \left\| \int_{\mathbb{V}_i} (x_i - q)\phi(q) dq \right\|^2 + r_i \|u_i\|^2 + \frac{\partial V}{\partial x_i} \dot{x}_i \right) \end{aligned}$$

associated with a value function $V = V(x)$

$$V(x) = \sum_{i=1}^{N} \sqrt{s_i r_i} \int_{\mathbb{V}_i} \|x_i - q\|^2 \phi(q) dq \tag{11}$$

with $s_i r_i = s_j r_j = p^2, \ \forall i, j \in \mathcal{N}$ and p denoting a positive scalar constant. By solving $\frac{\partial H}{\partial u_i} = 0$, we obtain the *continuous-time decentralized optimal control* (CDOC)

$$u_i = -\sqrt{\frac{s_i}{r_i}} \int_{\mathbb{V}_i} (x_i - q)\phi(q) dq, \quad i \in \mathcal{N} \tag{12}$$

Furthermore, the solution (12) satisfies LaSalle's invariance principle and thus ensures the Lyapunov convergence of the MAS into a CVC by means of $\dot{V} = -L \leq 0$.

IV. Discrete-time decentralized optimal control

This section proposes an optimal control for the discrete-time system's dynamics (2), based on the approach developed in [10]. For brevity, in the sequel we use $\mathbb{V}_i^+ = \mathbb{V}_i(x_i(k+1))$ to denote the Voronoi cell of $x_i(k+1)$, i.e. the i-th agent at time $k+1$. The mass and center of mass of the Voronoi cell $\mathbb{V}_i^+$ are respectively $M_{\mathbb{V}_i^+}$ and $CM_{\mathbb{V}_i^+}$.

Transposing the previous work [10] to the discrete-time dynamics (2) yields the following optimization problem

$$\begin{aligned} &\min_{u_i, i \in \mathcal{N}} \sum_{k=0}^{\infty} L(x(k), u(k)) \\ &\text{s.t: } x_i(k+1) = x_i(k) + T_s u_i(k) \end{aligned} \tag{13}$$

with $L(x(k), u(k)) = \sum_{i=1}^{N} \left(s_i \left\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q) dq \right\|^2 + r_i \|u_i(k)\|^2 \right)$ indicating the running cost.

Theorem 1. *A discrete-time decentralized stabilizing suboptimal control of the problem* (13) *is*

$$u_i(k) = -\frac{p_i}{r_i + p_i T_s M_{\mathbb{V}_i^+}} \int_{\mathbb{V}_i^+} (x_i(k) - q)\phi(q)dq \tag{14}$$

by considering the cost-to-go function

$$V(x(k)) = \sum_{i=1}^{N} p_i \int_{\mathbb{V}_i} \|x_i(k) - q\|^2 \phi(q)dq \tag{15}$$

with $p_i = \sqrt{s_i r_i}$ *and* $p_i = p_j = p > 0,\ \forall i, j \in \mathcal{N}$.

Proof. The discrete-time Hamiltonian (see [16]) of (13) is

$$H(k) = L(x(k), u(k)) + \frac{\Delta V(k)}{T_s} \tag{16}$$

with $\Delta V(k) = V(x(k+1)) - V(x(k))$. By using (15), we can express $\Delta V(k)$ as $\Delta V(k) = \sum_{i=1}^{N} p \int_{\mathbb{V}_i^+} \|x_i(k+1) - q\|^2 \phi(q)dq - \sum_{i=1}^{N} p \int_{\mathbb{V}_i} \|x_i(k) - q\|^2 \phi(q)dq$. Using the expressions (6)-(8), this can be rewritten in terms of the mass and the center of mass such as $\Delta V(k) = \sum_{i=1}^{N} p\Big(||x_i(k+1)||^2 M_{\mathbb{V}_i^+} + \int_{\mathbb{V}_i^+} ||q||^2 \phi(q)dq - 2x_i^\top(k+1) M_{\mathbb{V}_i^+} CM_{\mathbb{V}_i^+}\Big) - \sum_{i=1}^{N} p\Big(||x_i(k)||^2 M_{\mathbb{V}_i} - 2x_i^\top(k) M_{\mathbb{V}_i} CM_{\mathbb{V}_i} + \int_{\mathbb{V}_i} ||q||^2 \phi(q)dq\Big)$. By following the definition (3)-(4), we have $\sum_{i=1}^{N} \int_{\mathbb{V}_i} ||q||^2 \phi(q)dq = \int_{\bigcup_{i=1}^{N} \mathbb{V}_i} ||q||^2 \phi(q)dq = \int_{\mathcal{X}} ||q||^2 \phi(q)dq$ and this leads to $\sum_{i=1}^{N} \int_{\mathbb{V}_i^+} ||q||^2 \phi(q)dq - \sum_{i=1}^{N} \int_{\mathbb{V}_i} ||q||^2 \phi(q)dq = 0$. This allows us to rewrite $\Delta V(k)$ as follows

$$\begin{aligned}\Delta V(k) = &\sum_{i=1}^{N} p\Big(||x_i(k+1)||^2 - 2x_i^\top(k+1) CM_{\mathbb{V}_i^+}\Big) M_{\mathbb{V}_i^+} \\ &- \sum_{i=1}^{N} p\Big(||x_i(k)||^2 M_{\mathbb{V}_i} - 2x_i^\top(k) M_{\mathbb{V}_i} CM_{\mathbb{V}_i}\Big)\end{aligned} \tag{17}$$

Differentiating $H(k)$ with respect to $u_i(k)$, we obtain

$$\frac{\partial H(k)}{\partial u_i(k)} = 2 r_i u_i^\top(k) + \frac{1}{T_s} \frac{\partial \Delta V(k)}{\partial u_i(k)} \tag{18}$$

where the partial derivative of $\Delta V(k)$ with respect to $u_i(k)$ can be obtained by differentiating (17) by $u_i(k)$ (using the dynamics (2)), i.e.

$$\frac{\partial \Delta V(k)}{\partial u_i(k)} = 2pT_s M_{\mathbb{V}_i^+} \Big(x_i(k+1) - CM_{\mathbb{V}_i^+}\Big)^\top + T_s \theta_i \tag{19}$$

where $\theta_i = \frac{p}{T_s}\Big(||x_i(k+1)||^2 \frac{\partial M_{\mathbb{V}_i^+}}{\partial u_i(k)} - 2x_i^\top(k+1) \frac{\partial (M_{\mathbb{V}_i^+} CM_{\mathbb{V}_i^+})}{\partial u_i(k)}\Big) + \frac{1}{T_s} \sum_{j \in \mathcal{N}_i} p\Big(||x_j(k+1)||^2 \frac{\partial M_{\mathbb{V}_j^+}}{\partial u_i(k)} - 2x_j^\top(k+1) \frac{\partial (M_{\mathbb{V}_j^+} CM_{\mathbb{V}_j^+})}{\partial u_i(k)}\Big)$ collects all partial derivatives of the mass and center of mass with respect to $u_i(k)$.

Replacing (19) in (18), we get $\frac{\partial H(k)}{\partial u_i(k)} = 2 r_i u_i^\top(k) + 2p x_i^\top(k+1) M_{\mathbb{V}_i^+} - 2p M_{\mathbb{V}_i^+} CM_{\mathbb{V}_i^+}^\top + \theta_i$. Using the discrete-time dynamics (2) and regrouping the terms related to the control $u_i(k)$, the following expression is obtained

$$\begin{aligned}\frac{\partial H(k)}{\partial u_i(k)} = &\,2(r_i + pT_s M_{\mathbb{V}_i^+}) u_i^\top(k) \\ &+ 2p M_{\mathbb{V}_i^+} (x_i(k) - CM_{\mathbb{V}_i^+})^\top + \theta_i\end{aligned} \tag{20}$$

A suboptimal solution $u_i(k)$ can be obtained by solving $(r_i + pT_s M_{\mathbb{V}_i^+}) u_i^\top(k) + p M_{\mathbb{V}_i^+} (x_i(k) - CM_{\mathbb{V}_i^+})^\top = 0$, i.e.

$$u_i(k) = -\frac{p M_{\mathbb{V}_i^+}}{r_i + pT_s M_{\mathbb{V}_i^+}} (x_i(k) - CM_{\mathbb{V}_i^+}) \tag{21}$$

which is also the control solution (14) and the proof of the main claim is completed.

With respect to the local stability of the i^{th} agent in case of static Voronoi partition, by replacing (14) in (2), the closed-loop dynamics becomes $x_i(k+1) = \frac{r_i}{r_i + p_i T_s M_{\mathbb{V}_i^+}} x_i(k) + \frac{p_i T_s M_{\mathbb{V}_i^+}}{r_i + p_i T_s M_{\mathbb{V}_i^+}} CM_{\mathbb{V}_i^+}$. Obviously,

$$\frac{r_i}{r_i + p_i T_s M_{\mathbb{V}_i^+}} \leq 1$$

because $r_i, p_i, T_s, M_{\mathbb{V}_i^+} > 0$. □

Remark 1. *In Theorem 1, the general case* $p_i = p_j = p$ *was considered. Notice that we can choose* $p = 1$ *in order to keep the consistency with the results proposed in [10].*

Remark 2. *The computation of the control* (14) *requires the predicted mass* $M_{\mathbb{V}_i^+}$ *and the predicted center of mass* $CM_{\mathbb{V}_i^+}$ *of the Voronoi cell* $\mathbb{V}_i^+$. *This is considered as the main difference by comparison with the continuous-time case.*

Lemma 1. *The suboptimal solution* (21) *can be considered as a discrete-time approximation of the continuous-time decentralized optimal control of the problem* (13) *because the terms* θ_i *satisfy*

$$\sum_{i=1}^{N} \int \theta_i du_i(k) = 0 \tag{22}$$

Proof. Using the control solution (21) translates the expression (20) into $\frac{\partial H(k)}{\partial u_i(k)} = \theta_i$. According to the definition of total derivative of multi-variable function, we can integrate $\dfrac{\partial H(k)}{\partial u_i(k)}$ with respect to all $u_i(k)$ and thus we obtain

$$H(k)=\sum_{i=1}^{N}\int \theta_i du_i(k) \tag{23}$$

Consider the mass conservation law (see [14]), i.e.

$$\frac{\partial}{\partial x_i}\int_{\mathbb{V}_i}\|x_i-q\|^2\phi(q)dq=\int_{\mathbb{V}_i}\frac{\partial}{\partial x_i}\|x_i-q\|^2\phi(q)dq + \int_{\partial\mathbb{V}_i}\|x_i-\gamma\|^2\phi(\gamma)n^\top(\gamma)\frac{\partial\gamma}{\partial x_i}d\gamma \tag{24}$$

where $\partial\mathbb{V}_i$ is the boundary of the set $\mathbb{V}_i$, i.e. $\partial\mathbb{V}_i=\bigcup\limits_{\forall j\in\mathcal{N}_i}(\mathbb{V}_i\cap\mathbb{V}_j)$. Here, $n(\gamma)$ is the unit outward normal to $\partial\mathbb{V}_i$ which is parameterized by the scalar γ. Similar to (24), for the function (15), we obtain $\dfrac{\partial V(x)}{\partial x_i}=p\dfrac{\partial}{\partial x_i}\displaystyle\int_{\mathbb{V}_i}\|x_i-q\|^2\phi(q)dq$ and derive $\dfrac{\partial V(x)}{\partial x_i}=2p\displaystyle\int_{\mathbb{V}_i}(x_i-q)^\top\phi(q)dq+p\int_{\partial\mathbb{V}_i}\|x_i-\gamma\|^2\phi(\gamma)n^\top(\gamma)\frac{\partial\gamma}{\partial x_i}d\gamma$.

Based on (6), the discrete-time form at the time instant $k+1$ is further obtained

$$\frac{\partial V(x(k+1))}{\partial x_i(k+1)}=2p\Big(M_{\mathbb{V}_i^+}x_i(k+1)-M_{\mathbb{V}_i^+}CM_{\mathbb{V}_i^+}\Big)^\top + p\int_{\partial\mathbb{V}_i^+}\|x_i(k+1)-\gamma\|^2\phi(\gamma)n^\top(\gamma)\frac{\partial\gamma}{\partial x_i}d\gamma \tag{25}$$

Using (2), we have additionally

$$\frac{\partial\Delta V}{\partial u_i(k)}=\frac{\partial V(x(k+1))}{\partial x_i(k+1)}\frac{\partial x_i(k+1)}{\partial u_i(k)}=\frac{\partial V(x(k+1))}{\partial x_i(k+1)}T_s \tag{26}$$

By substituting (25) in (26), we obtain

$$\frac{\partial\Delta V}{\partial u_i(k)}=2pT_s\Big(M_{\mathbb{V}_i^+}x_i(k+1)-M_{\mathbb{V}_i^+}CM_{\mathbb{V}_i^+}\Big)^\top + T_sp\int_{\partial\mathbb{V}_i^+}\|x_i(k+1)-\gamma\|^2\phi(\gamma)n^\top(\gamma)\frac{\partial\gamma}{\partial x_i}d\gamma \tag{27}$$

Subsequently, from (19) and (27), we get another equation characterizing θ_i

$$\theta_i=p\int_{\partial\mathbb{V}_i^+}\|x_i(k+1)-\gamma\|^2\phi(\gamma)n^\top(\gamma)\frac{\partial\gamma}{\partial x_i}d\gamma \tag{28}$$

Furthermore, the authors of [10] proved that

$$\int_{\partial\mathbb{V}_i}\|x_i-\gamma\|^2\phi(\gamma)n^\top(\gamma)\frac{\partial\gamma}{\partial x_i}d\gamma = \sum_{j\in\mathcal{N}_i}\int_{\mathbb{V}_i\cap\mathbb{V}_j}\|x_i-\gamma_{ij}\|^2\phi(\gamma_{ij})n^\top(\gamma_{ij})\frac{\partial\gamma_{ij}}{\partial x_i}d\gamma_{ij} \tag{29}$$

which represents the mass variation through the boundary of $\mathbb{V}_i$ as a collection of mass flow through each facet $\mathbb{V}_i\cap\mathbb{V}_j$ defining $\partial\mathbb{V}_i$. Therefore, we obtain

$$\theta_i=p\sum_{j\in\mathcal{N}_i}\int_{\mathbb{V}_i^+\cap\mathbb{V}_j^+}\|x_i(k{+}1)-\gamma_{ij}\|^2\phi(\gamma_{ij})n^\top(\gamma_{ij})\frac{\partial\gamma_{ij}}{\partial x_i}d\gamma_{ij} \tag{30}$$

Additionally $\displaystyle\sum_{i=1}^{N}\int\theta_i du_i(k)=\sum_{i=1}^{N}\int\frac{\theta_i}{T_s}dx_i(k+1)$ with

$$\sum_{i=1}^{N}\int\frac{\theta_i}{T_s}dx_i(k+1)=\sum_{i=1}^{N}\frac{p_i}{T_s}\int\Big(\sum_{j\in\mathcal{N}_i}\int_{\mathbb{V}_i^+\cap\mathbb{V}_j^+}\|x_i(k+1)-\gamma_{ij}\|^2\phi(\gamma_{ij})n^\top(\gamma_{ij})\frac{\partial\gamma_{ij}}{\partial x_i}d\gamma_{ij}\Big)dx_i(k+1)$$

Since

$$\|x_i-\gamma_{ij}\|=\|x_j-\gamma_{ji}\|,\frac{\partial\gamma_{ij}}{\partial x_i}=\frac{\partial\gamma_{ji}}{\partial x_j},n(\gamma_{ij})=-n(\gamma_{ji})$$

it is possible to write[4]

$$\int_{\mathbb{V}_i\cap\mathbb{V}_j}\|x_i-\gamma_{ij}\|^2\phi(\gamma_{ij})n^\top(\gamma_{ij})\frac{\partial\gamma_{ij}}{\partial x_i}d\gamma_{ij} = -\int_{\mathbb{V}_j\cap\mathbb{V}_i}\|x_j-\gamma_{ji}\|^2\phi(\gamma_{ji})n^\top(\gamma_{ji})\frac{\partial\gamma_{ji}}{\partial x_j}d\gamma_{ji} \tag{31}$$

and thus the integral

$$\sum_{i=1}^{N}p\int\Big(\sum_{j\in\mathcal{N}_i}\int_{\mathbb{V}_i\cap\mathbb{V}_j}\|x_i-\gamma_{ij}\|^2\phi(\gamma_{ij})n^\top(\gamma_{ij})\frac{\partial\gamma_{ij}}{\partial x_i}d\gamma_{ij}\Big)dx_i$$

vanishes. This result leads to $\displaystyle\sum_{i=1}^{N}\int\theta_i du_i(k)=0$ and thus, the equation (23) becomes $H(k)=0$, proving that the control (21) can be considered as a discrete-time approximation of the optimal control solution of the problem (13). □

V. Equivalence between discrete-time approximation of CDOC and CDOC-Stability proof

The previous section presents the proposed discrete-time approximation of CDOC (14). It will be proved to converge to the CDOC of [10].

Theorem 2. *If the sampling time T_s goes to zero, then the next three statements are true:*
i. The discretized equation (2) *approaches the continuous-time dynamics* (1)*;*
ii. The discrete-time approximation of CDOC (14) *approaches CDOC* (12)*;*
iii. The HJB equation $\min\limits_u H=0$ is ensured.

Proof. i. The first statement is obvious.
ii. Consider the discrete-time control solution (14). If $T_s\to 0$, this solution (14) approaches the limit value $u_i(t)=-\dfrac{p}{r_i}\displaystyle\int_{\mathbb{V}_i}(x_i(t)-q)\phi(q)dq=-\sqrt{\frac{s_i}{r_i}}\int_{\mathbb{V}_i}(x_i(t)-q)\phi(q)dq$ which is exactly the CDOC solution, for $p^2=s_ir_i$ (see (6) and (11)).

[4]From the point of view of mass conservation, we consider the case when $\mathcal{X}=\mathbb{V}_i\cup\mathbb{V}_j$. If one part of the mass belonging to the $\mathbb{V}_i$ cell passes through the facet $\mathbb{V}_i\cap\mathbb{V}_j$ into the $\mathbb{V}_j$ cell, then $\mathbb{V}_i$ loses it but the cell $\mathbb{V}_j$ gets it. In general, the mass over $\mathcal{X}$ is conserved.

iii. According to the definition of the total derivative of a multivariable function, it is inferred that

$$\Delta V(k) = \sum_{i=1}^{N} \int \frac{\partial \Delta V(k)}{\partial u_i(k)} du_i(k) \tag{32}$$

Replacing $\Delta V(k)$ in the Hamiltonian (16) leads to $H(k) = \sum_{i=1}^{N} \Big(s_i \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 + r_i \|u_i(k)\|^2 \Big) + \frac{1}{T_s} \sum_{i=1}^{N} \int \frac{\partial \Delta V(k)}{\partial u_i(k)} du_i(k)$. Substitute $\frac{\partial \Delta V(k)}{\partial u_i(k)}$ with (19) and use (22) to obtain

$$\begin{aligned} H(k) &= \sum_{i=1}^{N} \Big(s_i \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 + r_i \|u_i(k)\|^2 \Big) \\ &+ \sum_{i=1}^{N} 2p \int \Big(M_{\mathbb{V}_i^+} x_i(k+1) - M_{\mathbb{V}_i^+} CM_{\mathbb{V}_i^+} \Big)^\top du_i(k) \end{aligned} \tag{33}$$

and thus

$$\begin{aligned} H(k) &= \sum_{i=1}^{N} \Big(s_i \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 + r_i \|u_i(k)\|^2 \Big) \\ &+ \sum_{i=1}^{N} 2p M_{\mathbb{V}_i^+} \Big(\int x_i^\top (k+1) du_i(k) - CM_{\mathbb{V}_i^+}^\top \int du_i(k) \Big) \end{aligned} \tag{34}$$

Using the dynamics (2), this can be rewritten as

$$\begin{aligned} H(k) &= \sum_{i=1}^{N} \Big(s_i \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 + r_i \|u_i(k)\|^2 \Big) \\ &+ \sum_{i=1}^{N} 2p M_{\mathbb{V}_i^+} \Big(T_s \frac{||u_i(k)||^2}{2} + (x_i(k) - CM_{\mathbb{V}_i^+})^\top u_i(k) \Big) \end{aligned} \tag{35}$$

Regrouping the terms in $||u_i(k)||^2$, the expression (35) becomes $H(k) = \sum_{i=1}^{N} \Big(s_i \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 + (r_i + pT_s M_{\mathbb{V}_i^+}) \|u_i(k)\|^2 + 2pM_{\mathbb{V}_i^+} (x_i(k) - CM_{\mathbb{V}_i^+})^\top u_i(k) \Big)$. Replacing $u_i(k)$ by the solution (14), this is equivalent to $H(k) = \sum_{i=1}^{N} \Big(s_i \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 - \frac{p^2}{r_i + pT_s M_{\mathbb{V}_i^+}} \Big\| \int_{\mathbb{V}_i^+} (x_i(k) - q)\phi(q)dq \Big\|^2 \Big)$.

When $T_s \to 0$, using $CM_{\mathbb{V}_i^+} \to CM_{\mathbb{V}_i}$ and $M_{\mathbb{V}_i^+} \to M_{\mathbb{V}_i}$, the control law (14) becomes

$$u_i(t) = -\frac{p}{r_i} \int_{\mathbb{V}_i} (x_i(t) - q)\phi(q)dq \tag{36}$$

which is further used to find a simplified form of $\min_u H$

$$\min_u H = \sum_{i=1}^{N} \Big(s_i - \frac{p^2}{r_i} \Big) \Big\| \int_{\mathbb{V}_i} (x_i(k) - q)\phi(q)dq \Big\|^2 \tag{37}$$

Using $p^2 = s_i r_i$, we conclude that if $T_s \to 0$ then $\min_u H \to 0$. The proof is thus completed. $\square$

VI. Numerical example

We consider a Multi-Agent system $\mathbf{\Sigma}$ composed of $N = 7$ homogeneous agents having the common dynamics (2), with the sampling time $T_s = 0.01$s, $x_i \in \mathbb{R}^2$ and $u_i \in \mathbb{R}^2$ denoting respectively the agents position and speed.

The agents are deployed within a bounded region $\mathcal{X} = \text{conv}\{(0,0),(0,6),(6,6),(6,0)\}$, see Fig. 1. The density function over $\mathcal{X}$ is uniform, i.e. $\phi(q) = 1, \forall q \in \mathcal{X}$.

In Figs. 1 and 3, the blue lines represent the motion of the agents in the considered discrete-time case. The agents initial positions $x_i(0)$ are marked by the red points (see the zoom shown in Figs. 1 and 3). The last configuration of the entire MAS is shown in each figure, with the green dots denoting the last positions of the agents. The evolution of the coverage criterion $M_{\mathbb{V}_i}\|x_i - CM_{\mathbb{V}_i}\|$ for each agent is shown in Figs. 2 and 4.

Two scenarios are considered. In the first scenario, we apply the decentralized control (14) with the weighting coefficients $s_i = 10$ and $r_i = 1$. The deployment result is shown in Fig. 1 with the evolution of the agents and also the Voronoi partition obtained after 100 sampling periods. The coverage criterion curves in Fig. 2 drop asymptotically to zero, proving that the entire MAS $\mathbf{\Sigma}$ is close to a CVC.

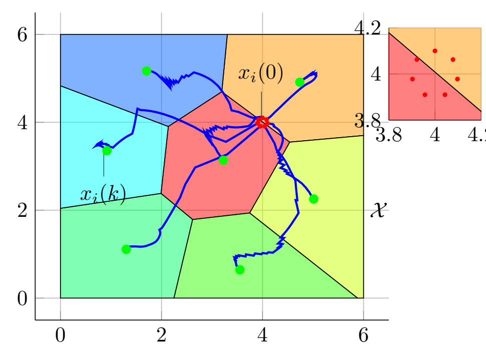

Fig. 1. Coverage of $\mathcal{X}$ with $s_i = 10$ and $r_i = 1$.

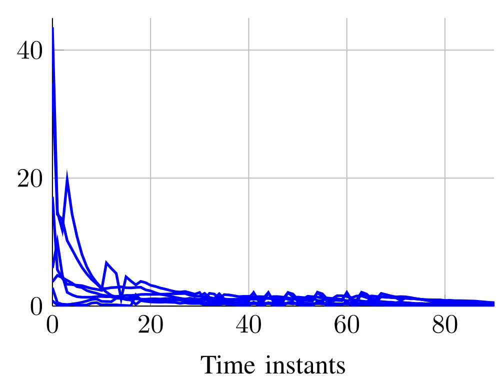

Fig. 2. Coverage criterion of N agents with $s_i = 10$ and $r_i = 1$.

In the second scenario, the weighting coefficients are $s_i = 100$ and $r_i = 1$. The MAS converges into a CVC different from the previous scenario (notice that a CVC is not not unique [17]) with a faster convergence rate. The drop to zero of the coverage criterion proves that $\mathbf{\Sigma}$ approaches the CVC (see Fig 2), although there are some discontinuities due to the abrupt change of the shape of the dynamic Voronoi cells.

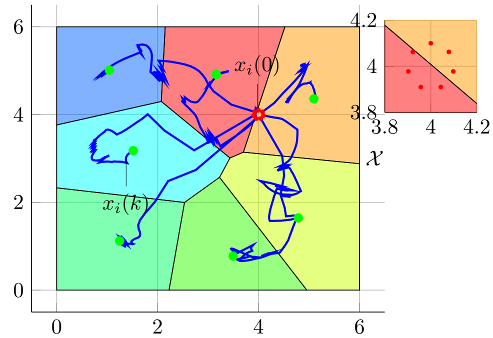

Fig. 3. Coverage of $\mathcal{X}$ with $s_i = 100$ and $r_i = 1$.

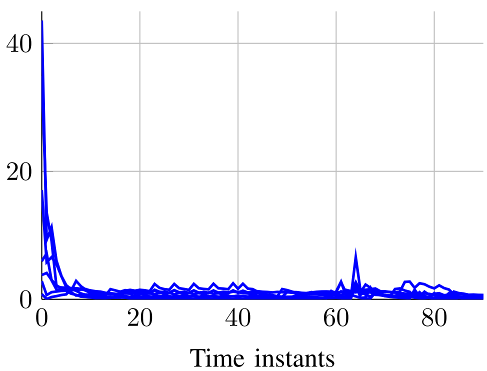

Fig. 4. Coverage criterion of N agents with $s_i = 100$ and $r_i = 1$.

VII. Conclusion

This paper provides a novel discrete-time decentralized control for the Multi-Agent Voronoi-based coverage/deployment problem, by following the optimal control framework. Similar to the continuous-time case, the discrete-time solution is also spatially distributed over Delaunay graphs. One of the contributions consists of taking into account the prediction of the Voronoi partition. However, finding an explicit equation to predict this partition appears to be complicated and it is part of our current research work. Some simulations showed small oscillations/discontinuities in the evolution of the centroids which will be analyzed in the future. Another interesting topic to be addressed in future work relates to the choice of the weighting coefficients for the energy-efficiency problem. Extending the control solutions for the case of finite horizon is part of current work. Future work will focus on the application of the proposed results on Smart Grid, related to distributed/decentralized energy production and distribution.

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