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# RECURRENCE RELATIONS FOR A FAMILY OF GENERALIZED CAUCHY NUMBERS 

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#### Abstract

We obtain two sorts of recurrence relations for a class of generalized Cauchy numbers $c_{n, \alpha}^{(s)}$. The first recurrences are especially adapted to the so-called shifted poly-Cauchy numbers. The second relations are generalization, for the numbers $c_{n, \alpha}^{(s)}$, of the recurrences obtained by Agoh and Dilcher for the Bernoulli numbers of the second kind $b_{n}=c_{n, 1}^{(1)} / n!$.


## 1. Introduction

The Bernoulli numbers of the second kind $b_{n}$ can be defined by the generating function

$$
\frac{t}{\log (1+t)}=\sum_{n=0}^{\infty} b_{n} t^{n}, \quad(|t|<1) .
$$

These numbers satisfy $b_{0}=1$ and the recurrence formulae (see [3]),

$$
\begin{equation*}
\sum_{j=0}^{n} \frac{(-1)^{j}}{(n-j+1)} b_{j}=0 \quad(n \geq 1) \tag{1.1}
\end{equation*}
$$

In [1], Agoh and Dilcher obtained among others, this first generalization (theorem 4.1),

$$
\begin{equation*}
\sum_{j=0}^{n-k} \frac{s(n-j, k)}{(n-j)!} b_{j}=\frac{s(n-1, k-1)}{(n-1)!k} \quad(n \geq k \geq 1) \tag{1.2}
\end{equation*}
$$

We recall that the Stirling numbers of the first kind $s(n, k)$ can be defined by

$$
\log ^{k}(1+t)=k!\sum_{n=k}^{\infty} s(n, k) \frac{t^{n}}{n!}, \quad(|t|<1)
$$

Their theorem 4.2 is a more general result. For $m \geq 0, k \geq 1$ and $n \geq k+m$,

$$
\begin{align*}
& \sum_{j=0}^{n-k-m}\binom{m+j}{m} \tilde{s}(n-j, k, m) b_{m+1+j}+(-1)^{m} \tilde{s}(n+m+1, k, m)  \tag{1.3}\\
= & \frac{1}{n!m!(k+m)} \sum_{j=0}^{m}(-1)^{m-j} s(m, m-j) s(n, k-1+j)\binom{k+m-1}{k+j-1}^{-1},
\end{align*}
$$

where the numbers $\tilde{s}(n, k, m)$ are given by

$$
\begin{equation*}
(1+t)^{m} \log ^{k+m}(1+t)=(k+m)!\sum_{n=0}^{\infty} \tilde{s}(n, k, m) t^{n}, \quad(|t|<1) . \tag{1.4}
\end{equation*}
$$

Remark 1.1. For $m=0$, The relations (1.3) reduce to the relations (1.2). For $k=1$, the relations (1.2) reduce to the relations (1.1).

[^0]The aim of this note is to obtain some recurrence relations for the two parameters family of generalized Cauchy numbers defined by

$$
F_{s, \alpha}(t):=\sum_{n=0}^{\infty} \frac{\log ^{n}(1+t)}{n!(n+\alpha)^{s}}=\sum_{n=0}^{\infty} c_{n, \alpha}^{(s)} \frac{t^{n}}{n!}, \quad(|t|<1)
$$

where $s$ and $\alpha$ are real parameters with, for simplicity, $s \geq 0$ and $\alpha>0$. For $s$ and $\alpha$ integers, they are called shifted poly-Cauchy numbers in [4].
Remark 1.2. For $s=0, F_{0, \alpha}(t)=1+t$, then $c_{0, \alpha}^{(0)}=1=c_{1, \alpha}^{(0)}$ and $c_{n, \alpha}^{(0)}=0$ for $n \geq 2$.
For $s=\alpha=1, F_{1,1}(t)=t / \log (1+t)$, and the numbers $c_{n, 1}^{(1)}$ are the ordinary Cauchy numbers $c_{n}=b_{n} / n!$.
We introduce some other numbers to be used in the sequel.
(1) For a two indeterminate polynomial $P(X, Y)$ and a parameter $\beta$, we define the numbers $s(n, \beta, P)$ by

$$
\begin{equation*}
\left.\left(\frac{\log (1+t)}{t}\right)^{\beta} P(1+t, \log (1+t))=\sum_{n=0}^{\infty} s(n, \beta, P)\right) t^{n}, \quad(|t|<1) \tag{1.5}
\end{equation*}
$$

A few formulas may be noted. If $m$ is a nonnegative integer, $P=\frac{Y^{m}}{m!}$ and $\beta=0$, then $s(n, 0, P)=s(n, m) / n!$. Also, If $m$ and $k$ are nonnegative integers, $P=X^{m} \frac{Y^{k+m}}{(k+m)!}$ and $\beta=r$ is nonnegative integer, then

$$
s(n, r, P)=r!\binom{k+m+r}{r} \tilde{s}(n+r, k, m+r) .
$$

(2) The weighted Stirling numbers of the second kind. For a nonnegative integer $n$ and a parameter $\beta$, we define, see [2],

$$
\begin{equation*}
(x+\beta)^{n}=\sum_{k=0}^{n} R(n, k, \beta)(x)_{k} \tag{1.6}
\end{equation*}
$$

where $(x)_{k}$ is the falling factorial defined by $(x)_{0}=1$ and, for $k \geq 1,(x)_{k}=x(x-1) . .(x-k+1)$. For $\beta=0$, the coefficients in (1.6) are known to be the Stirling numbers of the second kind $S(n, k)$, so $R(n, k, 0)=S(n, k)$. Moreover we have (see [2]) $R(n, 0, \beta)=\beta^{n}, R(n, k, 1)=S(n+1, k+1)$ and the general expression $R(n, k, \beta)=$ $\sum_{j=0}^{n-k}\binom{n}{j} \beta^{j} S(n-j, k)$.

## 2. The first recurrence relations

Here is the first main theorem.
Theorem 2.1. For any positive integers $n$ and $N$, we have

$$
\begin{align*}
& \sum_{k=1}^{n}\left(\alpha^{N} \frac{(-1)^{n-k}}{k!}+\sum_{j=1}^{\min (k, N)} \frac{s\left(n+j-k, 0, P_{N, j}\right)}{(k-j)!}\right) c_{k, \alpha}^{(s+N)}  \tag{2.1}\\
+ & \sum_{k=n+1}^{n+N}\left(\sum_{j=k-n}^{\min (k, N)} \frac{s\left(n+j-k, 0, P_{N, j}\right)}{(k-j)!}\right) c_{k, \alpha}^{(s+N)}=\sum_{k=1}^{n}(-1)^{n-k} \frac{c_{k, \alpha}^{(s)}}{k!} .
\end{align*}
$$

where $P_{N, j}$ is the polynomial $P_{N, j}(X, Y)=X^{j-1} \sum_{k=j}^{N} R(N, k, \alpha) S(k, j) Y^{k}$.
For $s=0, c_{1, \alpha}^{(0)}=1$ and $c_{n, \alpha}^{(0)}=0$ for $n \geq 2$ then we obtain, in particular for the shifted poly-Cauchy numbers,
Corollary 2.2. For any positive integers $n$ and $N$, we have

$$
\begin{align*}
& \sum_{k=1}^{n}\left(\alpha^{N} \frac{(-1)^{n-k}}{k!}+\sum_{j=1}^{\min (k, N)} \frac{s\left(n+j-k, 0, P_{N, j}\right)}{(k-j)!}\right) c_{k, \alpha}^{(s+N)}  \tag{2.2}\\
+ & \sum_{k=n+1}^{n+N}\left(\sum_{j=k-n}^{\min (k, N)} \frac{s\left(n+j-k, 0, P_{N, j}\right)}{(k-j)!}\right) c_{k, \alpha}^{(s+N)}=(-1)^{n-1} .
\end{align*}
$$

By more specialization, we obtain
Corollary 2.3. If in addition of $s=0, \alpha=N=1$, then we recover the recurrence relations (1.1).
2.1. Proof of the corollary 2.3 . For $N=\alpha=1$, the relations 2.2 write for positive integer $n$,

$$
\begin{array}{r}
(-1)^{n-1}=\sum_{k=1}^{n}\left(\frac{(-1)^{n-k}}{k!}+\sum_{j=1}^{\min (k, 1)} \frac{s\left(n+j-k, 0, P_{1, j}\right)}{(k-j)!}\right) k!b_{k} \\
+\sum_{k=n+1}^{n+1}\left(\sum_{j=1}^{\min (k, 1)} \frac{s\left(n+j-k, 0, P_{1, j}\right)}{(k-j)!}\right) k!b_{k} \\
=\sum_{k=1}^{n}\left(\frac{(-1)^{n-k}}{k!}+\frac{s\left(n+1-k, 0, P_{1,1}\right)}{(k-1)!}\right) k!b_{k}+\frac{s\left(0,0, P_{1,1}\right)}{n!}(n+1)!b_{n+1} .
\end{array}
$$

We have $P_{1,1}=R(1,1,1) S(1,1) Y=Y$ then $s\left(0,0, P_{1,1}\right)=0$ and $s\left(l, 0, P_{1,1}\right)=(-1)^{l-1} / l$ for $l \geq 1$. Hence

$$
\begin{aligned}
& (-1)^{n-1}=\sum_{k=1}^{n}\left(\frac{(-1)^{n-k}}{k!}+\frac{(-1)^{n-k}}{(k-1)!(n+1-k)}\right) k!b_{k} \\
= & \sum_{k=1}^{n}(-1)^{n-k}\left(1+\frac{k}{n+1-k}\right) b_{k}=(n+1) \sum_{k=1}^{n} \frac{(-1)^{n-k}}{n+1-k} b_{k},
\end{aligned}
$$

but $b_{0}=1$, then

$$
\frac{(-1)^{n}}{n+1} b_{0}+\sum_{k=1}^{n} \frac{(-1)^{n-k}}{n+1-k} b_{k}=0=(-1)^{n} \sum_{k=0}^{n} \frac{(-1)^{k}}{n+1-k} b_{k}
$$

and we arrive to desired recurrence (1.1).
2.2. Proof of theorem 2.1). Let us consider the function $E_{s, \alpha}(x)=\sum_{n=0}^{\infty} \frac{x^{n}}{n!(n+\alpha)^{s}}$ so that $E_{s, \alpha}(\log (1+t))=F_{s, \alpha}(t)$, the generating function of our generalized Cauchy numbers. Then,

Lemma 2.1. For any positive integer $N$, any real number $x$, we have

$$
\begin{equation*}
E_{s, \alpha}(x)=\sum_{k=0}^{\infty} R(N, k, \alpha) x^{k} \frac{d^{k}}{d x^{k}} E_{s+N, \alpha}(x) . \tag{2.3}
\end{equation*}
$$

Indeed, we write

$$
E_{s, \alpha}(x)=\sum_{n=0}^{\infty} \frac{x^{n}}{n!(n+\alpha)^{s}}=\sum_{n=0}^{\infty}(n+\alpha)^{N} \frac{x^{n}}{n!(n+\alpha)^{s+N}}
$$

then by 1.6,

$$
\begin{array}{r}
E_{s, \alpha}(x)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{N} R(N, k, \alpha)(n)_{k}\right) \frac{x^{n}}{n!(n+\alpha)^{s+N}} \\
=\sum_{k=0}^{N} R(N, k, \alpha) x^{k} \sum_{n=0}^{\infty}(n)_{k} \frac{x^{n-k}}{n!(n+\alpha)^{s+N}} \\
=\sum_{k=0}^{N} R(N, k, \alpha) x^{k} \frac{d^{k}}{d x^{k}} E_{s+N, \alpha}(x) .
\end{array}
$$

To translate this result to $F_{s, \alpha}(t)$, we shall use the following lemma.

Lemma 2.2. If $F(x)$ and $F(\log (1+t))$ admit $m$ derivatives, then it holds

$$
\begin{align*}
& \frac{d^{m}}{d t^{m}} F(\log (1+t))=\frac{1}{(1+t)^{m}} \sum_{j=0}^{m} s(m, j) F^{(j)}(\log (1+t))  \tag{2.4}\\
\text { and } & F^{(m)}(\log (1+t))=\sum_{j=0}^{m} S(m, j)(1+t)^{j} \frac{d^{j}}{d t^{j}} F(\log (1+t)) . \tag{2.5}
\end{align*}
$$

Indeed, the first part is classical and given in [3] with $t$ in place of $1+t$. It is also used in [1], lemma 4.1, with the function $F(x)=1 / x$. The proof is by recurrence and uses the fundamental relations on the Stirling numbers of the first kind, see [3] :

$$
\begin{array}{r}
s(n, k)=s(n-1, k-1)-(n-1) s(n-1, k), \quad n \geq k \geq 1 \\
s(n, 0)=s(0, k)=0, \quad \text { except } s(0,0)=1
\end{array}
$$

While the second part is the usual inversion formula based on the orthogonality of the Stirling numbers of the two kinds, see [3]. By combining the lemma (2.3) and the lemma 2.4], we obtain for $|t|<1$,

$$
\begin{array}{r}
F_{s, \alpha}(t)=\sum_{k=0}^{N} R(N, k, \alpha) \log ^{k}(1+t) E_{s+N, \alpha}^{(k)}(\log (1+t)) \\
=\sum_{k=0}^{N} R(N, k, \alpha) \log ^{k}(1+t) \sum_{j=0}^{k} S(k, j)(1+t)^{j} \frac{d^{j}}{d t^{j}} E_{s+N, \alpha}(\log (1+t)) \\
=\sum_{k=0}^{N} \sum_{j=0}^{k} R(N, k, \alpha) S(k, j)(1+t)^{j} \log ^{k}(1+t) F_{s+N, \alpha}^{(j)}(t) \\
=R(N, 0, \alpha) S(0,0) F_{s+N, \alpha}(t)+\sum_{k=1}^{N} \sum_{j=1}^{k} R(N, k, \alpha) S(k, j)(1+t)^{j} \log ^{k}(1+t) F_{s+N, \alpha}^{(j)}(t) \\
=\alpha^{N} F_{s+N, \alpha}(t)+(1+t) \sum_{j=1}^{N}(1+t)^{j-1}\left(\sum_{k=j}^{N} R(N, k, \alpha) S(k, j) \log ^{k}(1+t)\right) F_{s+N, \alpha}^{(j)}(t),
\end{array}
$$

then

$$
\begin{equation*}
\frac{F_{s, \alpha}(t)}{1+t}=\alpha^{N} \frac{F_{s+N, \alpha}(t)}{1+t}+\sum_{j=1}^{N} P_{N, j}\left(1+t, \log (1+t) F_{s+N, \alpha}^{(j)}(t) .\right. \tag{2.6}
\end{equation*}
$$

To conclude, we consider the corresponding expansions. On one hand, we have

$$
\begin{equation*}
\frac{F_{s, \alpha}(t)}{1+t}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k} \frac{c_{k, \alpha}^{(s)}}{k!}\right) t^{n}, \frac{F_{s+N, \alpha}(t)}{1+t}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k} \frac{c_{k, \alpha}^{(s+N)}}{k!}\right) t^{n} \tag{2.7}
\end{equation*}
$$

On an other hand, we have

$$
\begin{aligned}
P_{N, j}\left(1+t, \log (1+t) F_{s+N, \alpha}^{(j)}(t)=\right. & \left(\sum_{n=0}^{\infty} s\left(n, 0, P_{N, j}\right) t^{n}\right)\left(\sum_{n=0}^{\infty} c_{n+j, \alpha}^{(s+N)} \frac{t^{n}}{n!}\right) \\
& =\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} s\left(n-k, 0, P_{N, j}\right) \frac{c_{k+j, \alpha}^{(s+N)}}{k!}\right) t^{n} \\
= & \sum_{n=0}^{\infty}\left(\sum_{k=j}^{n+j} s\left(n+j-k, 0, P_{N, j}\right) \frac{c_{k, \alpha}^{(s+N)}}{(k-j)!}\right) t^{n}
\end{aligned}
$$

Hence, we obtain

$$
\begin{array}{r}
\sum_{j=1}^{N} P_{N, j}\left(1+t, \log (1+t) F_{s+N, \alpha}^{(j)}(t)\right.  \tag{2.8}\\
=\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n+N}\left(\sum_{j=\max (1, k-n)}^{\min (k, N)} \frac{s\left(n+j-k, 0, P_{N, j}\right)}{(k-j)!}\right) c_{k, \alpha}^{(s+N)}\right) t^{n} .
\end{array}
$$

The identities 2.6, 2.7) and 2.8 lead to the equality

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k} \frac{c_{k, \alpha}^{(s)}}{k!}\right) t^{n}=\alpha^{N} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k} \frac{c_{k, \alpha}^{(s+N)}}{k!}\right) t^{n} \\
& \quad+\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n+N}\left(\sum_{j=\max (1, k-n)}^{\min (k, N)} \frac{s\left(n+j-k, 0, P_{N, j}\right)}{(k-j)!}\right) c_{k, \alpha}^{(s+N)}\right) t^{n} .
\end{aligned}
$$

Equating the coefficients of $t^{n}$ in the expansions above, achieves the proof of the theorem 2.1.

## 3. The second recurrence relations

3.1. Preliminaries. The starting point is the following lemma.

Lemma 3.1. For any real number $t$, with $0<|t|<1$, we have

$$
\begin{equation*}
F_{s+1, \alpha}(t)=\frac{1}{\log ^{\alpha}(1+t)} \int_{0}^{t} \log ^{\alpha-1}(1+u) \frac{F_{s, \alpha}(u)}{1+u} d u \tag{3.1}
\end{equation*}
$$

Proof. We just compute, for $|t|<1$, the derivative

$$
\begin{aligned}
& \frac{d}{d t}\left(\log ^{\alpha}(1+t) F_{s+1, \alpha}(t)\right)=\frac{d}{d t}\left(\sum_{n=0}^{\infty} \frac{\log ^{n+\alpha}(1+t)}{n!(n+\alpha)^{s+1}}\right) \\
& =\frac{1}{1+t} \sum_{n=0}^{\infty} \frac{\log ^{n+\alpha-1}(1+t)}{n!(n+\alpha)^{s}}=\log ^{\alpha-1}(1+t) \frac{F_{s, \alpha}(t)}{1+t}
\end{aligned}
$$

and observe that the function $\log ^{\alpha}(1+t) F_{s+1, \alpha}(t)$ vanishes for $t=0$.
We define the function

$$
\begin{equation*}
G_{s, \alpha}(t):=\frac{1}{t^{\alpha}} \int_{0}^{t} \log ^{\alpha-1}(1+u) \frac{F_{s, \alpha}(u)}{1+u} d u \tag{3.2}
\end{equation*}
$$

and look for its expansion in powers of $t$.
Proposition 3.1. For any $|t|<1$, we have $G_{s, \alpha}(t)=\sum_{n=0}^{\infty} g_{n, \alpha}^{(s)} t^{n}$ with

$$
\begin{equation*}
g_{n, \alpha}^{(s)}=\frac{1}{\alpha(n+\alpha)} \sum_{k=0}^{n}(n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k, \alpha}^{(s)}}{k!} \tag{3.3}
\end{equation*}
$$

Indeed, we first expand, for $|u|<1$,

$$
\begin{array}{r}
\frac{\log ^{\alpha-1}(1+u)}{1+u}=\frac{1}{\alpha}\left(\log ^{\alpha}(1+u)\right)^{\prime} \\
=\frac{1}{\alpha}\left(\sum_{n=0}^{\infty} s(n, \alpha, 1) u^{n-\alpha}\right)^{\prime}=\frac{1}{\alpha} \sum_{n=0}^{\infty}(n-\alpha) s(n, \alpha, 1) u^{n-\alpha-1}
\end{array}
$$

Next we put it in 3.2,

$$
\begin{array}{r}
G_{s, \alpha}(t)=\frac{1}{\alpha t^{\alpha}} \int_{0}^{t}\left(\sum_{n=0}^{\infty}(n+\alpha) s(n, \alpha, 1) u^{n+\alpha-1}\right) F_{s, \alpha}(u) d u \\
=\frac{1}{\alpha t^{\alpha}} \int_{0}^{t}\left(\sum_{n=0}^{\infty}(n+\alpha) s(n, \alpha, 1) u^{n+\alpha-1}\right)\left(\sum_{n=0}^{\infty} c_{n, \alpha}^{(s)} \frac{u^{n}}{n!}\right) d u \\
=\frac{1}{\alpha t^{\alpha}} \int_{0}^{t} u^{\alpha-1} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k, \alpha}(s)}{k!}\right) u^{n} d u
\end{array}
$$

After integration, it holds

$$
G_{s, \alpha}(t)=\frac{1}{\alpha} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k, \alpha}^{(s)}}{k!}\right) \frac{t^{n}}{n+\alpha}
$$

and the expression of $g_{n, \alpha}^{(s)}$ appears clearly.
3.2. The second theorem. We are now able to state our second main result.

Theorem 3.1. Let $N>m \geq 0$ be given integers. Then for any integer $n \geq N$, we have

$$
\begin{array}{r}
\sum_{j=0}^{n-N} s\left(n-j, \alpha-1, Q_{m, N}\right) \frac{(j-\alpha)_{m}}{j!} c_{j, \alpha}^{(s+1)}  \tag{3.4}\\
=\sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l)\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right),
\end{array}
$$

where the polynomials $Q_{N, m}$ are $Q_{N, m}(X, Y)=X^{m} Y^{N}$.
Proof. We shall apply the operator $(1+t)^{m} \log ^{N+\alpha-1}(1+t) d^{m} / d t^{m}$ on both sides of the equality $\frac{F_{s+1, \alpha}(t)}{t^{\alpha}}=\frac{G_{s, \alpha}(t)}{\log ^{\alpha}(1+t)}$ obtained from 3.1. For the left side, we have for the derivatives

$$
\frac{d^{m}}{d t^{m}} \frac{F_{s+1, \alpha}(t)}{t^{\alpha}}=\frac{d^{m}}{d t^{m}} \sum_{n=0}^{\infty} c_{n, \alpha}^{(s+1)} \frac{t^{n-\alpha}}{n!}=\sum_{n=0}^{\infty}(n-\alpha)_{m} c_{n, \alpha}^{(s+1)} \frac{t^{n-\alpha-m}}{n!}
$$

then

$$
\begin{array}{r}
(1+t)^{m} \log ^{N+\alpha-1}(1+t) \frac{d^{m}}{d t^{m}} \frac{F_{s+1, \alpha}(t)}{t^{\alpha}}=(1+t)^{m} \log ^{N+\alpha-1}(1+t) \sum_{n=0}^{\infty}(n-\alpha)_{m} c_{n, \alpha}^{(s+1)} \frac{t^{n-\alpha-m}}{n!} \\
=t^{-m-1}\left(\frac{\log (1+t)}{t}\right)^{\alpha-1}(1+t)^{m} \log ^{N}(1+t) \sum_{n=0}^{\infty}(n-\alpha)_{m} c_{n, \alpha}^{(s+1)} \frac{t^{n}}{n!}
\end{array}
$$

and by 1.5 , we expand

$$
\begin{aligned}
& \left(\frac{\log (1+t)}{t}\right)^{\alpha-1}(1+t)^{m} \log ^{N}(1+t) \sum_{n=0}^{\infty}(n-\alpha)_{m} c_{n, \alpha}^{(s+1)} \frac{t^{n}}{n!} \\
= & t^{-m-1}\left(\sum_{n=0}^{\infty} s\left(n, \alpha-1, Q_{m, N}\right) t^{n}\right)\left(\sum_{n=0}^{\infty}(n-\alpha)_{m} c_{n, \alpha}^{(s+1)} \frac{t^{n}}{n!}\right) \\
= & t^{-m-1} \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} s\left(n-j, \alpha-1, Q_{m, N}\right) \frac{(j-\alpha)_{m}}{j!} c_{j, \alpha}^{(s+1)}\right) t^{n} .
\end{aligned}
$$

Thus we have obtained that,

$$
\begin{align*}
& (1+t)^{m} \log ^{N+\alpha-1}(1+t) \log ^{N+\alpha-1}(1+t) \frac{d^{m}}{d t^{m}} \frac{F_{s+1, \alpha}(t)}{t^{\alpha}}  \tag{3.5}\\
= & t^{-m-1} \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} s\left(n-j, \alpha-1, Q_{m, N}\right) \frac{(j-\alpha)_{m}}{j!} c_{j, \alpha}^{(s+1)}\right) t^{n} .
\end{align*}
$$

For the second side, the derivatives are

$$
\frac{d^{m}}{d t^{m}} \frac{G_{s, \alpha}(t)}{\log ^{\alpha}(1+t)}=\sum_{j=0}^{m}\binom{m}{j} G_{s, \alpha}^{(m-j)}(t) \frac{d^{j}}{d t^{j}} \log ^{-\alpha}(1+t)
$$

and by lemma 2.4, we have

$$
\frac{d^{j}}{d t^{j}} \log ^{-\alpha}(1+t)=\frac{1}{(1+t)^{j}} \sum_{l=0}^{j} \frac{(-\alpha)_{l} s(j, l)}{\log ^{\alpha+l}(1+t)}
$$

then we obtain

$$
\begin{array}{r}
\frac{d^{m}}{d t^{m}} \frac{G_{s, \alpha}(t)}{\log ^{\alpha}(1+t)} \\
=\sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j} G_{s, \alpha}^{(m-j)}(t) \frac{(-\alpha)_{l} s(j, l)}{(1+t)^{j} \log ^{\alpha+l}(1+t)}
\end{array}
$$

then

$$
\begin{align*}
& (1+t)^{m} \log ^{N+\alpha-1}(1+t) \sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j} G_{s, \alpha}^{(m-j)}(t) \frac{(-\alpha)_{l} s(j, l)}{(1+t)^{j} \log ^{\alpha+l}(1+t)}  \tag{3.6}\\
& \quad=\sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l) G_{s, \alpha}^{(m-j)}(t)(1+t)^{m-j} \log ^{N-l-1}(1+t)
\end{align*}
$$

According to (3.2) and (1.5), we have

$$
\begin{array}{r}
G_{s, \alpha}^{(m-j)}(t)(1+t)^{m-j} \log ^{N-l-1}(1+t)=t^{j-m}\left(\sum_{n=0}^{\infty}(n)_{m-j} g_{n, \alpha}^{(s)} t^{n}\right)\left(\sum_{n=0}^{\infty} s\left(n, 0, Q_{m-j, N-l-1}\right) t^{n}\right) \\
=t^{j-m} \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-r, 0, Q_{m-j, N-l-1}\right)\right) t^{n} \\
=t^{-m-1} \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-r, 0, Q_{m-j, N-l-1}\right)\right) t^{n+j+1}
\end{array}
$$

By changing $n+j+1$ to $n$ in the summation, we obtain

$$
G_{s, \alpha}^{(m-j)}(t)(1+t)^{m-j} \log ^{N-l-1}(1+t)=t^{-m-1} \sum_{n=j+1}^{\infty}\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right) t^{n}
$$

witch leads to the expression of the right hand of the equality (3.6),

$$
\begin{array}{r}
\sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l) G_{s, \alpha}^{(m-j)}(t)(1+t)^{m-j} \log ^{N-l-1}(1+t) \\
=t^{-m-1} \sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l) \sum_{n=j+1}^{\infty}\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right) t^{n} \\
=t^{-m-1} \sum_{n=1}^{\infty}\left(\sum_{j=0}^{\min (m, n-1)} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l)\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right)\right) t^{n} .
\end{array}
$$

We obtain here that

$$
\begin{array}{r}
(1+t)^{m} \log ^{N+\alpha-1}(1+t) \log ^{N+\alpha-1}(1+t) \frac{d^{m}}{d t^{m}} \frac{G_{s, \alpha}(t)}{\log ^{\alpha}(1+t)}  \tag{3.7}\\
=t^{-m-1} \sum_{n=1}^{\infty}\left(\sum_{j=0}^{\min (m, n-1)} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l)\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right)\right) t^{n} .
\end{array}
$$

Finally, (3.5) and (3.7) together give

$$
\begin{array}{r}
\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} s\left(n-j, \alpha-1, Q_{m, N}\right) \frac{(j-\alpha)_{m}}{j!} c_{j, \alpha}^{(s+1)}\right) t^{n} \\
=\sum_{n=1}^{\infty}\left(\sum_{j=0}^{\min (m, n-1)} \sum_{l=0}^{j}\binom{m}{j}(-\alpha)_{l} s(j, l)\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, \alpha}^{(s)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right)\right) t^{n} .
\end{array}
$$

To conclude, we equate the coefficients of $t^{n}$, for $n \geq N$, in the summations after observing that for $n-j>N$, $s\left(n-j, \alpha-1, Q_{m, N}\right)=0$ and $\min (m, n-1)=m$ for $n>m$.
3.3. The special case $s=0$ and $\alpha=1$. We show here that we recover the recurrences of Agoh and Dilcher for the Bernoulli numbers of the second kind. Let $n$ be an integer $\geq N$ and denote by $L$ the left hand of the recurrence (3.4), with $s=0, \alpha=1$, and $R$ its right one.

$$
L=\sum_{j=0}^{n-N} s\left(n-j, 0, Q_{m, N}\right)(j-1)_{m} b_{j}
$$

and

$$
R=\sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j}(-1)_{l} s(j, l)\left(\sum_{r=0}^{n-j-1}(r)_{m-j} g_{r, 1}^{(0)} s\left(n-j-1-r, 0, Q_{m-j, N-l-1}\right)\right)
$$

As $(j-1)_{m}=(j-1)(j-2) . .(j-m)$ vanishes for $j=1,2, . ., m$ and equals $(-1)^{m} m!$ for $j=0$, we have

$$
\begin{array}{r}
L=(-1)^{m} m!s\left(n, 0, Q_{m, N}\right)+\sum_{j=m+1}^{n-N} s\left(n-j, 0, Q_{m, N}\right)(j-1)_{m} b_{j} \\
=(-1)^{m} m!N!\tilde{s}(n, N-m, m)+N!\sum_{j=m+1}^{n-N} \tilde{s}(n-j, N-m, m)(j-1)_{m} b_{j} \\
=(-1)^{m} m!N!\tilde{s}(n, N-m, m)+m!N!\sum_{j=m+1}^{n-N} \tilde{s}(n-j, N-m, m)\binom{j-1}{m} b_{j},
\end{array}
$$

and by a little shift

$$
\begin{equation*}
L=(-1)^{m} m!N!\tilde{s}(n, N-m, m)+m!N!\sum_{j=0}^{n-N-m-1} \tilde{s}(n-j-m-1, N-m, m)\binom{j+m}{m} b_{j+m+1} \tag{3.8}
\end{equation*}
$$

For the right hand $R$, as $G_{0,1}(t)=\frac{1}{t} \int_{0}^{t} \frac{1+u}{1+u} d u=1$, then $g_{0,1}^{(0)}=1$ and $g_{r, 1}^{(0)}=0$ for $r \geq 1$, so

$$
R=\sum_{j=0}^{m} \sum_{l=0}^{j}\binom{m}{j}(-1)_{l} s(j, l)(0)_{m-j} s\left(n-j-1,0, Q_{m-j, N-l-1}\right) .
$$

But $(0)_{m-j}$ vanishes for $m>j$ and equals 1 for $m=j$, then

$$
\begin{array}{r}
R=\sum_{l=0}^{m}(-1)^{l} l!s(m, l) s\left(n-m-1,0, Q_{0, N-l-1}\right) \\
=\sum_{l=0}^{m}(-1)^{l} l!s(m, l)(N-l-1)!\tilde{s}(n-m-1, N-l-1,0) .
\end{array}
$$

We use $\tilde{s}(n-m-1, N-l-1,0)=s(n-m-1, N-l-1)$ and introduce a binomial coefficient,

$$
R=\frac{(N-1)!}{(n-m-1)!} \sum_{l=0}^{m}(-1)^{l} s(m, l) s(n-m-1, N-l-1)\binom{N-1}{N-1-l}^{-1}
$$

Now, put $l=m-j$ in the summation above and obtain

$$
\begin{equation*}
R=\frac{(N-1)!}{(n-m-1)!} \sum_{j=0}^{m}(-1)^{m-j} s(m, m-j) s(n-m-1, N-m+j-1)\binom{N-1}{N-m+j-1}^{-1} \tag{3.9}
\end{equation*}
$$

Finally, we put $N=m+k$ in both 3.8 and 3.9 with $n+m+1$ instead of $n$ and obtain

$$
\begin{gathered}
L=(-1)^{m} m!(m+k)!\tilde{s}(n+m+1, k, m)+m!(m+k)!\sum_{j=0}^{n-m-k} \tilde{s}(n-j, k, m)\binom{j+m}{m} b_{j+m+1} \\
R=\frac{(m+k-1)!}{n!} \sum_{j=0}^{m}(-1)^{m-j} s(m, m-j) s(n, k+j-1)\binom{m+k-1}{k+j-1}^{-1}
\end{gathered}
$$

and $\frac{L}{m!(m+k)!}=\frac{R}{m!(m+k)!}$ is exactly the desired identity 1.3 .

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