



Recurrence relations for a family of generalized Cauchy numbers

J Chikhi

► To cite this version:

| J Chikhi. Recurrence relations for a family of generalized Cauchy numbers. 2016. hal-01390020

HAL Id: hal-01390020

<https://hal.science/hal-01390020>

Preprint submitted on 31 Oct 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RECURRENCE RELATIONS FOR A FAMILY OF GENERALIZED CAUCHY NUMBERS

J. CHIKHI

ABSTRACT. We obtain two sorts of recurrence relations for a class of generalized Cauchy numbers $c_{n,\alpha}^{(s)}$. The first recurrences are especially adapted to the so-called shifted poly-Cauchy numbers. The second relations are generalization, for the numbers $c_{n,\alpha}^{(s)}$, of the recurrences obtained by Agoh and Dilcher for the Bernoulli numbers of the second kind $b_n = c_{n,1}^{(1)}/n!$.

1. INTRODUCTION

The Bernoulli numbers of the second kind b_n can be defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n, \quad (|t| < 1).$$

These numbers satisfy $b_0 = 1$ and the recurrence formulae (see [3]),

$$(1.1) \quad \sum_{j=0}^n \frac{(-1)^j}{(n-j+1)} b_j = 0 \quad (n \geq 1).$$

In [1], Agoh and Dilcher obtained among others, this first generalization (theorem 4.1),

$$(1.2) \quad \sum_{j=0}^{n-k} \frac{s(n-j, k)}{(n-j)!} b_j = \frac{s(n-1, k-1)}{(n-1)!k} \quad (n \geq k \geq 1).$$

We recall that the Stirling numbers of the first kind $s(n, k)$ can be defined by

$$\log^k(1+t) = k! \sum_{n=k}^{\infty} s(n, k) \frac{t^n}{n!}, \quad (|t| < 1).$$

Their theorem 4.2 is a more general result. For $m \geq 0$, $k \geq 1$ and $n \geq k+m$,

$$(1.3) \quad \sum_{j=0}^{n-k-m} \binom{m+j}{m} \tilde{s}(n-j, k, m) b_{m+1+j} + (-1)^m \tilde{s}(n+m+1, k, m) \\ = \frac{1}{n!m!(k+m)} \sum_{j=0}^m (-1)^{m-j} s(m, m-j) s(n, k-1+j) \binom{k+m-1}{k+j-1}^{-1},$$

where the numbers $\tilde{s}(n, k, m)$ are given by

$$(1.4) \quad (1+t)^m \log^{k+m}(1+t) = (k+m)! \sum_{n=0}^{\infty} \tilde{s}(n, k, m) t^n, \quad (|t| < 1).$$

Remark 1.1. For $m = 0$, The relations (1.3) reduce to the relations (1.2). For $k = 1$, the relations (1.2) reduce to the relations (1.1).

Date: October 31, 2016.

2010 Mathematics Subject Classification : 11B83, 11B73,

Key words and phrases : generalized Cauchy numbers, shifted poly-Cauchy numbers, Stirling numbers, Bernoulli numbers, recurrences.

The aim of this note is to obtain some recurrence relations for the two parameters family of generalized Cauchy numbers defined by

$$F_{s,\alpha}(t) := \sum_{n=0}^{\infty} \frac{\log^n(1+t)}{n!(n+\alpha)^s} = \sum_{n=0}^{\infty} c_{n,\alpha}^{(s)} \frac{t^n}{n!}, \quad (|t| < 1),$$

where s and α are real parameters with, for simplicity, $s \geq 0$ and $\alpha > 0$. For s and α integers, they are called *shifted poly-Cauchy numbers* in [4].

Remark 1.2. For $s = 0$, $F_{0,\alpha}(t) = 1 + t$, then $c_{0,\alpha}^{(0)} = 1 = c_{1,\alpha}^{(0)}$ and $c_{n,\alpha}^{(0)} = 0$ for $n \geq 2$.

For $s = \alpha = 1$, $F_{1,1}(t) = t/\log(1+t)$, and the numbers $c_{n,1}^{(1)}$ are the ordinary Cauchy numbers $c_n = b_n/n!$.

We introduce some other numbers to be used in the sequel.

(1) For a two indeterminate polynomial $P(X, Y)$ and a parameter β , we define the numbers $s(n, \beta, P)$ by

$$(1.5) \quad \left(\frac{\log(1+t)}{t} \right)^\beta P(1+t, \log(1+t)) = \sum_{n=0}^{\infty} s(n, \beta, P) t^n, \quad (|t| < 1).$$

A few formulas may be noted. If m is a nonnegative integer, $P = \frac{Y^m}{m!}$ and $\beta = 0$, then $s(n, 0, P) = s(n, m)/n!$.

Also, If m and k are nonnegative integers, $P = X^m \frac{Y^{k+m}}{(k+m)!}$ and $\beta = r$ is nonnegative integer, then

$$s(n, r, P) = r! \binom{k+m+r}{r} \tilde{s}(n+r, k, m+r).$$

(2) The weighted Stirling numbers of the second kind. For a nonnegative integer n and a parameter β , we define, see [2],

$$(1.6) \quad (x + \beta)^n = \sum_{k=0}^n R(n, k, \beta) (x)_k,$$

where $(x)_k$ is the falling factorial defined by $(x)_0 = 1$ and, for $k \geq 1$, $(x)_k = x(x-1)\dots(x-k+1)$. For $\beta = 0$, the coefficients in (1.6) are known to be the Stirling numbers of the second kind $S(n, k)$, so $R(n, k, 0) = S(n, k)$. Moreover we have (see [2]) $R(n, 0, \beta) = \beta^n$, $R(n, k, 1) = S(n+1, k+1)$ and the general expression $R(n, k, \beta) = \sum_{j=0}^{n-k} \binom{n}{j} \beta^j S(n-j, k)$.

2. THE FIRST RECURRENCE RELATIONS

Here is the first main theorem.

Theorem 2.1. For any positive integers n and N , we have

$$(2.1) \quad \sum_{k=1}^n \left(\alpha^N \frac{(-1)^{n-k}}{k!} + \sum_{j=1}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \\ + \sum_{k=n+1}^{n+N} \left(\sum_{j=k-n}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} = \sum_{k=1}^n (-1)^{n-k} \frac{c_{k,\alpha}^{(s)}}{k!}.$$

where $P_{N,j}$ is the polynomial $P_{N,j}(X, Y) = X^{j-1} \sum_{k=j}^N R(N, k, \alpha) S(k, j) Y^k$.

For $s = 0$, $c_{1,\alpha}^{(0)} = 1$ and $c_{n,\alpha}^{(0)} = 0$ for $n \geq 2$ then we obtain, in particular for the shifted poly-Cauchy numbers,

Corollary 2.2. For any positive integers n and N , we have

$$(2.2) \quad \sum_{k=1}^n \left(\alpha^N \frac{(-1)^{n-k}}{k!} + \sum_{j=1}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \\ + \sum_{k=n+1}^{n+N} \left(\sum_{j=k-n}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} = (-1)^{n-1}.$$

By more specialization, we obtain

Corollary 2.3. *If in addition of $s = 0$, $\alpha = N = 1$, then we recover the recurrence relations (1.1).*

2.1. Proof of the corollary (2.3). For $N = \alpha = 1$, the relations (2.2) write for positive integer n ,

$$\begin{aligned} (-1)^{n-1} &= \sum_{k=1}^n \left(\frac{(-1)^{n-k}}{k!} + \sum_{j=1}^{\min(k,1)} \frac{s(n+j-k, 0, P_{1,j})}{(k-j)!} \right) k! b_k \\ &\quad + \sum_{k=n+1}^{n+1} \left(\sum_{j=1}^{\min(k,1)} \frac{s(n+j-k, 0, P_{1,j})}{(k-j)!} \right) k! b_k \\ &= \sum_{k=1}^n \left(\frac{(-1)^{n-k}}{k!} + \frac{s(n+1-k, 0, P_{1,1})}{(k-1)!} \right) k! b_k + \frac{s(0, 0, P_{1,1})}{n!} (n+1)! b_{n+1}. \end{aligned}$$

We have $P_{1,1} = R(1, 1, 1)S(1, 1)Y = Y$ then $s(0, 0, P_{1,1}) = 0$ and $s(l, 0, P_{1,1}) = (-1)^{l-1}/l$ for $l \geq 1$. Hence

$$\begin{aligned} (-1)^{n-1} &= \sum_{k=1}^n \left(\frac{(-1)^{n-k}}{k!} + \frac{(-1)^{n-k}}{(k-1)!(n+1-k)} \right) k! b_k \\ &= \sum_{k=1}^n (-1)^{n-k} \left(1 + \frac{k}{n+1-k} \right) b_k = (n+1) \sum_{k=1}^n \frac{(-1)^{n-k}}{n+1-k} b_k, \end{aligned}$$

but $b_0 = 1$, then

$$\frac{(-1)^n}{n+1} b_0 + \sum_{k=1}^n \frac{(-1)^{n-k}}{n+1-k} b_k = 0 = (-1)^n \sum_{k=0}^n \frac{(-1)^k}{n+1-k} b_k,$$

and we arrive to desired recurrence (1.1).

2.2. Proof of theorem (2.1). Let us consider the function $E_{s,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\alpha)^s}$ so that $E_{s,\alpha}(\log(1+t)) = F_{s,\alpha}(t)$, the generating function of our generalized Cauchy numbers. Then,

Lemma 2.1. *For any positive integer N , any real number x , we have*

$$(2.3) \quad E_{s,\alpha}(x) = \sum_{k=0}^{\infty} R(N, k, \alpha) x^k \frac{d^k}{dx^k} E_{s+N,\alpha}(x).$$

Indeed, we write

$$E_{s,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\alpha)^s} = \sum_{n=0}^{\infty} (n+\alpha)^N \frac{x^n}{n!(n+\alpha)^{s+N}},$$

then by (1.6),

$$\begin{aligned} E_{s,\alpha}(x) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^N R(N, k, \alpha) (n)_k \right) \frac{x^n}{n!(n+\alpha)^{s+N}} \\ &= \sum_{k=0}^N R(N, k, \alpha) x^k \sum_{n=0}^{\infty} (n)_k \frac{x^{n-k}}{n!(n+\alpha)^{s+N}} \\ &= \sum_{k=0}^N R(N, k, \alpha) x^k \frac{d^k}{dx^k} E_{s+N,\alpha}(x). \end{aligned}$$

To translate this result to $F_{s,\alpha}(t)$, we shall use the following lemma.

Lemma 2.2. *If $F(x)$ and $F(\log(1+t))$ admit m derivatives, then it holds*

$$(2.4) \quad \frac{d^m}{dt^m} F(\log(1+t)) = \frac{1}{(1+t)^m} \sum_{j=0}^m s(m, j) F^{(j)}(\log(1+t))$$

$$(2.5) \quad \text{and } F^{(m)}(\log(1+t)) = \sum_{j=0}^m S(m, j) (1+t)^j \frac{d^j}{dt^j} F(\log(1+t)) .$$

Indeed, the first part is classical and given in [3] with t in place of $1+t$. It is also used in [1], lemma 4.1, with the function $F(x) = 1/x$. The proof is by recurrence and uses the fundamental relations on the Stirling numbers of the first kind, see [3] :

$$\begin{aligned} s(n, k) &= s(n-1, k-1) - (n-1)s(n-1, k) , \quad n \geq k \geq 1 ; \\ s(n, 0) &= s(0, k) = 0, \quad \text{except } s(0, 0) = 1 . \end{aligned}$$

While the second part is the usual inversion formula based on the orthogonality of the Stirling numbers of the two kinds, see [3]. By combining the lemma (2.3) and the lemma (2.4), we obtain for $|t| < 1$,

$$\begin{aligned} F_{s, \alpha}(t) &= \sum_{k=0}^N R(N, k, \alpha) \log^k(1+t) E_{s+N, \alpha}^{(k)}(\log(1+t)) \\ &= \sum_{k=0}^N R(N, k, \alpha) \log^k(1+t) \sum_{j=0}^k S(k, j) (1+t)^j \frac{d^j}{dt^j} E_{s+N, \alpha}(\log(1+t)) \\ &= \sum_{k=0}^N \sum_{j=0}^k R(N, k, \alpha) S(k, j) (1+t)^j \log^k(1+t) F_{s+N, \alpha}^{(j)}(t) \\ &= R(N, 0, \alpha) S(0, 0) F_{s+N, \alpha}(t) + \sum_{k=1}^N \sum_{j=1}^k R(N, k, \alpha) S(k, j) (1+t)^j \log^k(1+t) F_{s+N, \alpha}^{(j)}(t) \\ &= \alpha^N F_{s+N, \alpha}(t) + (1+t) \sum_{j=1}^N (1+t)^{j-1} \left(\sum_{k=j}^N R(N, k, \alpha) S(k, j) \log^k(1+t) \right) F_{s+N, \alpha}^{(j)}(t) , \end{aligned}$$

then

$$(2.6) \quad \frac{F_{s, \alpha}(t)}{1+t} = \alpha^N \frac{F_{s+N, \alpha}(t)}{1+t} + \sum_{j=1}^N P_{N, j} (1+t, \log(1+t)) F_{s+N, \alpha}^{(j)}(t) .$$

To conclude, we consider the corresponding expansions. On one hand, we have

$$(2.7) \quad \frac{F_{s, \alpha}(t)}{1+t} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k, \alpha}^{(s)}}{k!} \right) t^n , \quad \frac{F_{s+N, \alpha}(t)}{1+t} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k, \alpha}^{(s+N)}}{k!} \right) t^n .$$

On an other hand, we have

$$\begin{aligned} P_{N, j} (1+t, \log(1+t)) F_{s+N, \alpha}^{(j)}(t) &= \left(\sum_{n=0}^{\infty} s(n, 0, P_{N, j}) t^n \right) \left(\sum_{n=0}^{\infty} c_{n+j, \alpha}^{(s+N)} \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n s(n-k, 0, P_{N, j}) \frac{c_{k+j, \alpha}^{(s+N)}}{k!} \right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=j}^{n+j} s(n+j-k, 0, P_{N, j}) \frac{c_{k, \alpha}^{(s+N)}}{(k-j)!} \right) t^n . \end{aligned}$$

Hence, we obtain

$$(2.8) \quad \sum_{j=1}^N P_{N,j}(1+t, \log(1+t)) F_{s+N,\alpha}^{(j)}(t) \\ = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+N} \left(\sum_{j=\max(1,k-n)}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \right) t^n.$$

The identities (2.6), (2.7) and (2.8) lead to the equality

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k,\alpha}^{(s)}}{k!} \right) t^n = \alpha^N \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k,\alpha}^{(s+N)}}{k!} \right) t^n \\ + \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+N} \left(\sum_{j=\max(1,k-n)}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \right) t^n.$$

Equating the coefficients of t^n in the expansions above, achieves the proof of the theorem (2.1).

3. THE SECOND RECURRENCE RELATIONS

3.1. Preliminaries. The starting point is the following lemma.

Lemma 3.1. *For any real number t , with $0 < |t| < 1$, we have*

$$(3.1) \quad F_{s+1,\alpha}(t) = \frac{1}{\log^\alpha(1+t)} \int_0^t \log^{\alpha-1}(1+u) \frac{F_{s,\alpha}(u)}{1+u} du.$$

Proof. We just compute, for $|t| < 1$, the derivative

$$\frac{d}{dt} (\log^\alpha(1+t) F_{s+1,\alpha}(t)) = \frac{d}{dt} \left(\sum_{n=0}^{\infty} \frac{\log^{n+\alpha}(1+t)}{n!(n+\alpha)^{s+1}} \right) \\ = \frac{1}{1+t} \sum_{n=0}^{\infty} \frac{\log^{n+\alpha-1}(1+t)}{n!(n+\alpha)^s} = \log^{\alpha-1}(1+t) \frac{F_{s,\alpha}(t)}{1+t},$$

and observe that the function $\log^\alpha(1+t) F_{s+1,\alpha}(t)$ vanishes for $t = 0$. □

We define the function

$$(3.2) \quad G_{s,\alpha}(t) := \frac{1}{t^\alpha} \int_0^t \log^{\alpha-1}(1+u) \frac{F_{s,\alpha}(u)}{1+u} du,$$

and look for its expansion in powers of t .

Proposition 3.1. *For any $|t| < 1$, we have $G_{s,\alpha}(t) = \sum_{n=0}^{\infty} g_{n,\alpha}^{(s)} t^n$ with*

$$(3.3) \quad g_{n,\alpha}^{(s)} = \frac{1}{\alpha(n+\alpha)} \sum_{k=0}^n (n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k,\alpha}^{(s)}}{k!}.$$

Indeed, we first expand, for $|u| < 1$,

$$\frac{\log^{\alpha-1}(1+u)}{1+u} = \frac{1}{\alpha} (\log^\alpha(1+u))' \\ = \frac{1}{\alpha} \left(\sum_{n=0}^{\infty} s(n, \alpha, 1) u^{n-\alpha} \right)' = \frac{1}{\alpha} \sum_{n=0}^{\infty} (n-\alpha) s(n, \alpha, 1) u^{n-\alpha-1}.$$

Next we put it in (3.2),

$$\begin{aligned} G_{s,\alpha}(t) &= \frac{1}{\alpha t^\alpha} \int_0^t \left(\sum_{n=0}^{\infty} (n+\alpha) s(n, \alpha, 1) u^{n+\alpha-1} \right) F_{s,\alpha}(u) du \\ &= \frac{1}{\alpha t^\alpha} \int_0^t \left(\sum_{n=0}^{\infty} (n+\alpha) s(n, \alpha, 1) u^{n+\alpha-1} \right) \left(\sum_{n=0}^{\infty} c_{n,\alpha}^{(s)} \frac{u^n}{n!} \right) du \\ &= \frac{1}{\alpha t^\alpha} \int_0^t u^{\alpha-1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k,\alpha}^{(s)}}{k!} \right) u^n du. \end{aligned}$$

After integration, it holds

$$G_{s,\alpha}(t) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k,\alpha}^{(s)}}{k!} \right) \frac{t^n}{n+\alpha},$$

and the expression of $g_{n,\alpha}^{(s)}$ appears clearly.

3.2. The second theorem. We are now able to state our second main result.

Theorem 3.1. *Let $N > m \geq 0$ be given integers. Then for any integer $n \geq N$, we have*

$$\begin{aligned} (3.4) \quad & \sum_{j=0}^{n-N} s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m}{j!} c_{j,\alpha}^{(s+1)} \\ &= \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j, l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j, N-l-1}) \right), \end{aligned}$$

where the polynomials $Q_{N,m}$ are $Q_{N,m}(X, Y) = X^m Y^N$.

Proof. We shall apply the operator $(1+t)^m \log^{N+\alpha-1} (1+t) d^m/dt^m$ on both sides of the equality $\frac{F_{s+1,\alpha}(t)}{t^\alpha} = \frac{G_{s,\alpha}(t)}{\log^\alpha(1+t)}$ obtained from (3.1). For the left side, we have for the derivatives

$$\frac{d^m}{dt^m} \frac{F_{s+1,\alpha}(t)}{t^\alpha} = \frac{d^m}{dt^m} \sum_{n=0}^{\infty} c_{n,\alpha}^{(s+1)} \frac{t^{n-\alpha}}{n!} = \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^{n-\alpha-m}}{n!},$$

then

$$\begin{aligned} (1+t)^m \log^{N+\alpha-1} (1+t) \frac{d^m}{dt^m} \frac{F_{s+1,\alpha}(t)}{t^\alpha} &= (1+t)^m \log^{N+\alpha-1} (1+t) \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^{n-\alpha-m}}{n!} \\ &= t^{-m-1} \left(\frac{\log(1+t)}{t} \right)^{\alpha-1} (1+t)^m \log^N (1+t) \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^n}{n!}, \end{aligned}$$

and by (1.5), we expand

$$\begin{aligned} & \left(\frac{\log(1+t)}{t} \right)^{\alpha-1} (1+t)^m \log^N (1+t) \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^n}{n!} \\ &= t^{-m-1} \left(\sum_{n=0}^{\infty} s(n, \alpha-1, Q_{m,N}) t^n \right) \left(\sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^n}{n!} \right) \\ &= t^{-m-1} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m}{j!} c_{j,\alpha}^{(s+1)} \right) t^n. \end{aligned}$$

Thus we have obtained that,

$$(3.5) \quad \begin{aligned} & (1+t)^m \log^{N+\alpha-1}(1+t) \log^{N+\alpha-1}(1+t) \frac{d^m}{dt^m} \frac{F_{s+1,\alpha}(t)}{t^\alpha} \\ &= t^{-m-1} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m}{j!} c_{j,\alpha}^{(s+1)} \right) t^n. \end{aligned}$$

For the second side, the derivatives are

$$\frac{d^m}{dt^m} \frac{G_{s,\alpha}(t)}{\log^\alpha(1+t)} = \sum_{j=0}^m \binom{m}{j} G_{s,\alpha}^{(m-j)}(t) \frac{d^j}{dt^j} \log^{-\alpha}(1+t),$$

and by lemma (2.4), we have

$$\frac{d^j}{dt^j} \log^{-\alpha}(1+t) = \frac{1}{(1+t)^j} \sum_{l=0}^j \frac{(-\alpha)_l s(j,l)}{\log^{\alpha+l}(1+t)},$$

then we obtain

$$\begin{aligned} & \frac{d^m}{dt^m} \frac{G_{s,\alpha}(t)}{\log^\alpha(1+t)} \\ &= \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} G_{s,\alpha}^{(m-j)}(t) \frac{(-\alpha)_l s(j,l)}{(1+t)^j \log^{\alpha+l}(1+t)}, \end{aligned}$$

then

$$(3.6) \quad \begin{aligned} & (1+t)^m \log^{N+\alpha-1}(1+t) \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} G_{s,\alpha}^{(m-j)}(t) \frac{(-\alpha)_l s(j,l)}{(1+t)^j \log^{\alpha+l}(1+t)} \\ &= \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t). \end{aligned}$$

According to (3.2) and (1.5), we have

$$\begin{aligned} G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t) &= t^{j-m} \left(\sum_{n=0}^{\infty} (n)_{m-j} g_{n,\alpha}^{(s)} t^n \right) \left(\sum_{n=0}^{\infty} s(n, 0, Q_{m-j, N-l-1}) t^n \right) \\ &= t^{j-m} \sum_{n=0}^{\infty} \left(\sum_{r=0}^n (r)_{m-j} g_{r,\alpha}^{(s)} s(n-r, 0, Q_{m-j, N-l-1}) \right) t^n \\ &= t^{-m-1} \sum_{n=0}^{\infty} \left(\sum_{r=0}^n (r)_{m-j} g_{r,\alpha}^{(s)} s(n-r, 0, Q_{m-j, N-l-1}) \right) t^{n+j+1}. \end{aligned}$$

By changing $n+j+1$ to n in the summation, we obtain

$$G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t) = t^{-m-1} \sum_{n=j+1}^{\infty} \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j, N-l-1}) \right) t^n,$$

which leads to the expression of the right hand of the equality (3.6),

$$\begin{aligned} & \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t) \\ &= t^{-m-1} \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \sum_{n=j+1}^{\infty} \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j, N-l-1}) \right) t^n \\ &= t^{-m-1} \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\min(m, n-1)} \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j, N-l-1}) \right) \right) t^n. \end{aligned}$$

We obtain here that

$$(3.7) \quad (1+t)^m \log^{N+\alpha-1}(1+t) \log^{N+\alpha-1}(1+t) \frac{d^m}{dt^m} \frac{G_{s,\alpha}(t)}{\log^\alpha(1+t)} \\ = t^{-m-1} \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\min(m,n-1)} \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j,N-l-1}) \right) \right) t^n.$$

Finally, (3.5) and (3.7) together give

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m}{j!} c_{j,\alpha}^{(s+1)} \right) t^n \\ = \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\min(m,n-1)} \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j,N-l-1}) \right) \right) t^n.$$

To conclude, we equate the coefficients of t^n , for $n \geq N$, in the summations after observing that for $n-j > N$, $s(n-j, \alpha-1, Q_{m,N}) = 0$ and $\min(m, n-1) = m$ for $n > m$. $\square$

3.3. The special case $s = 0$ and $\alpha = 1$. We show here that we recover the recurrences of Agoh and Dilcher for the Bernoulli numbers of the second kind. Let n be an integer $\geq N$ and denote by L the left hand of the recurrence (3.4), with $s = 0$, $\alpha = 1$, and R its right one.

$$L = \sum_{j=0}^{n-N} s(n-j, 0, Q_{m,N}) (j-1)_m b_j$$

and

$$R = \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-1)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,1}^{(0)} s(n-j-1-r, 0, Q_{m-j,N-l-1}) \right).$$

As $(j-1)_m = (j-1)(j-2)\dots(j-m)$ vanishes for $j = 1, 2, \dots, m$ and equals $(-1)^m m!$ for $j = 0$, we have

$$L = (-1)^m m! s(n, 0, Q_{m,N}) + \sum_{j=m+1}^{n-N} s(n-j, 0, Q_{m,N}) (j-1)_m b_j \\ = (-1)^m m! N! \tilde{s}(n, N-m, m) + N! \sum_{j=m+1}^{n-N} \tilde{s}(n-j, N-m, m) (j-1)_m b_j \\ = (-1)^m m! N! \tilde{s}(n, N-m, m) + m! N! \sum_{j=m+1}^{n-N} \tilde{s}(n-j, N-m, m) \binom{j-1}{m} b_j,$$

and by a little shift

$$(3.8) \quad L = (-1)^m m! N! \tilde{s}(n, N-m, m) + m! N! \sum_{j=0}^{n-N-m-1} \tilde{s}(n-j-m-1, N-m, m) \binom{j+m}{m} b_{j+m+1}.$$

For the right hand R , as $G_{0,1}(t) = \frac{1}{t} \int_0^t \frac{1+u}{1+u} du = 1$, then $g_{0,1}^{(0)} = 1$ and $g_{r,1}^{(0)} = 0$ for $r \geq 1$, so

$$R = \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-1)_l s(j,l) (0)_{m-j} s(n-j-1, 0, Q_{m-j,N-l-1}).$$

But $(0)_{m-j}$ vanishes for $m > j$ and equals 1 for $m = j$, then

$$R = \sum_{l=0}^m (-1)^l l! s(m,l) s(n-m-1, 0, Q_{0,N-l-1}) \\ = \sum_{l=0}^m (-1)^l l! s(m,l) (N-l-1)! \tilde{s}(n-m-1, N-l-1, 0).$$

We use $\tilde{s}(n-m-1, N-l-1, 0) = s(n-m-1, N-l-1)$ and introduce a binomial coefficient,

$$R = \frac{(N-1)!}{(n-m-1)!} \sum_{l=0}^m (-1)^l s(m, l) s(n-m-1, N-l-1) \binom{N-1}{N-l-1}^{-1}.$$

Now, put $l = m-j$ in the summation above and obtain

$$(3.9) \quad R = \frac{(N-1)!}{(n-m-1)!} \sum_{j=0}^m (-1)^{m-j} s(m, m-j) s(n-m-1, N-m+j-1) \binom{N-1}{N-m+j-1}^{-1}.$$

Finally, we put $N = m+k$ in both (3.8) and (3.9) with $n+m+1$ instead of n and obtain

$$L = (-1)^m m! (m+k)! \tilde{s}(n+m+1, k, m) + m! (m+k)! \sum_{j=0}^{n-m-k} \tilde{s}(n-j, k, m) \binom{j+m}{m} b_{j+m+1},$$

$$R = \frac{(m+k-1)!}{n!} \sum_{j=0}^m (-1)^{m-j} s(m, m-j) s(n, k+j-1) \binom{m+k-1}{k+j-1}^{-1},$$

and $\frac{L}{m!(m+k)!} = \frac{R}{m!(m+k)!}$ is exactly the desired identity (1.3).

REFERENCES

1. T. Agoh, K. Dilcher, *Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind*, Fibonacci Quart., 48 (2010), no.1, 4-12.
2. L. Carlitz, *Weighted Stirling numbers of the first and second kind-I, II*, Fibonacci Quart., 1980, 18: 147-162, 242-257.
3. L. Comtet, *Advanced combinatorics*, Reidel (1974).
4. T. Komatsu, V. Laohakosol, K. Liptai, *A generalization of poly-Cauchy numbers and their proprieties*, Abstract and Applied Analysis, Volume 2013, Article ID 179841, 8 pages.

JAMEL CHIKHI, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'EVRY VAL D'ESSONNE, 23 BD. DE FRANCE, 91037 EVRY CEDEX, FRANCE,

E-mail address: jamel.chikhi@univ-evry.fr