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► **To cite this version:**

| J Chikhi. Recurrence relations for a family of generalized Cauchy numbers. 2016. hal-01390020

HAL Id: hal-01390020

<https://hal.science/hal-01390020v1>

Preprint submitted on 31 Oct 2016

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RECURRENCE RELATIONS FOR A FAMILY OF GENERALIZED CAUCHY NUMBERS

J. CHIKHI

ABSTRACT. We obtain two sorts of recurrence relations for a class of generalized Cauchy numbers $c_{n,\alpha}^{(s)}$. The first recurrences are especially adapted to the so-called shifted poly-Cauchy numbers. The second relations are generalization, for the numbers $c_{n,\alpha}^{(s)}$, of the recurrences obtained by Agoh and Dilcher for the Bernoulli numbers of the second kind $b_n = c_{n,1}^{(1)}/n!$.

1. INTRODUCTION

The Bernoulli numbers of the second kind b_n can be defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n, \quad (|t| < 1).$$

These numbers satisfy $b_0 = 1$ and the recurrence formulae (see [3]),

$$(1.1) \quad \sum_{j=0}^n \frac{(-1)^j}{(n-j+1)} b_j = 0 \quad (n \geq 1).$$

In [1], Agoh and Dilcher obtained among others, this first generalization (theorem 4.1),

$$(1.2) \quad \sum_{j=0}^{n-k} \frac{s(n-j, k)}{(n-j)!} b_j = \frac{s(n-1, k-1)}{(n-1)!k} \quad (n \geq k \geq 1).$$

We recall that the Stirling numbers of the first kind $s(n, k)$ can be defined by

$$\log^k(1+t) = k! \sum_{n=k}^{\infty} s(n, k) \frac{t^n}{n!}, \quad (|t| < 1).$$

Their theorem 4.2 is a more general result. For $m \geq 0$, $k \geq 1$ and $n \geq k+m$,

$$(1.3) \quad \begin{aligned} & \sum_{j=0}^{n-k-m} \binom{m+j}{m} \tilde{s}(n-j, k, m) b_{m+1+j} + (-1)^m \tilde{s}(n+m+1, k, m) \\ &= \frac{1}{n!m!(k+m)} \sum_{j=0}^m (-1)^{m-j} s(m, m-j) s(n, k-1+j) \binom{k+m-1}{k+j-1}^{-1}, \end{aligned}$$

where the numbers $\tilde{s}(n, k, m)$ are given by

$$(1.4) \quad (1+t)^m \log^{k+m}(1+t) = (k+m)! \sum_{n=0}^{\infty} \tilde{s}(n, k, m) t^n, \quad (|t| < 1).$$

Remark 1.1. For $m = 0$, The relations (1.3) reduce to the relations (1.2). For $k = 1$, the relations (1.2) reduce to the relations (1.1).

Date: October 31, 2016.

2010 Mathematics Subject Classification : 11B83, 11B73,

Key words and phrases : generalized Cauchy numbers, shifted poly-Cauchy numbers, Stirling numbers, Bernoulli numbers, recurrences.

The aim of this note is to obtain some recurrence relations for the two parameters family of generalized Cauchy numbers defined by

$$F_{s,\alpha}(t) := \sum_{n=0}^{\infty} \frac{\log^n(1+t)}{n!(n+\alpha)^s} = \sum_{n=0}^{\infty} c_{n,\alpha}^{(s)} \frac{t^n}{n!}, \quad (|t| < 1),$$

where s and α are real parameters with, for simplicity, $s \geq 0$ and $\alpha > 0$. For s and α integers, they are called *shifted poly-Cauchy numbers* in [4].

Remark 1.2. For $s = 0$, $F_{0,\alpha}(t) = 1 + t$, then $c_{0,\alpha}^{(0)} = 1 = c_{1,\alpha}^{(0)}$ and $c_{n,\alpha}^{(0)} = 0$ for $n \geq 2$.

For $s = \alpha = 1$, $F_{1,1}(t) = t/\log(1+t)$, and the numbers $c_{n,1}^{(1)}$ are the ordinary Cauchy numbers $c_n = b_n/n!$.

We introduce some other numbers to be used in the sequel.

(1) For a two indeterminate polynomial $P(X, Y)$ and a parameter β , we define the numbers $s(n, \beta, P)$ by

$$(1.5) \quad \left(\frac{\log(1+t)}{t} \right)^\beta P(1+t, \log(1+t)) = \sum_{n=0}^{\infty} s(n, \beta, P) t^n, \quad (|t| < 1).$$

A few formulas may be noted. If m is a nonnegative integer, $P = \frac{Y^m}{m!}$ and $\beta = 0$, then $s(n, 0, P) = s(n, m)/n!$. Also, if m and k are nonnegative integers, $P = X^m \frac{Y^{k+m}}{(k+m)!}$ and $\beta = r$ is nonnegative integer, then

$$s(n, r, P) = r! \binom{k+m+r}{r} \tilde{s}(n+r, k, m+r).$$

(2) The weighted Stirling numbers of the second kind. For a nonnegative integer n and a parameter β , we define, see [2],

$$(1.6) \quad (x + \beta)^n = \sum_{k=0}^n R(n, k, \beta) (x)_k,$$

where $(x)_k$ is the falling factorial defined by $(x)_0 = 1$ and, for $k \geq 1$, $(x)_k = x(x-1)\dots(x-k+1)$. For $\beta = 0$, the coefficients in (1.6) are known to be the Stirling numbers of the second kind $S(n, k)$, so $R(n, k, 0) = S(n, k)$. Moreover we have (see [2]) $R(n, 0, \beta) = \beta^n$, $R(n, k, 1) = S(n+1, k+1)$ and the general expression $R(n, k, \beta) = \sum_{j=0}^{n-k} \binom{n}{j} \beta^j S(n-j, k)$.

2. THE FIRST RECURRENCE RELATIONS

Here is the first main theorem.

Theorem 2.1. For any positive integers n and N , we have

$$(2.1) \quad \sum_{k=1}^n \left(\alpha^N \frac{(-1)^{n-k}}{k!} + \sum_{j=1}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \\ + \sum_{k=n+1}^{n+N} \left(\sum_{j=k-n}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} = \sum_{k=1}^n (-1)^{n-k} \frac{c_{k,\alpha}^{(s)}}{k!}.$$

where $P_{N,j}$ is the polynomial $P_{N,j}(X, Y) = X^{j-1} \sum_{k=j}^N R(N, k, \alpha) S(k, j) Y^k$.

For $s = 0$, $c_{1,\alpha}^{(0)} = 1$ and $c_{n,\alpha}^{(0)} = 0$ for $n \geq 2$ then we obtain, in particular for the shifted poly-Cauchy numbers,

Corollary 2.2. For any positive integers n and N , we have

$$(2.2) \quad \sum_{k=1}^n \left(\alpha^N \frac{(-1)^{n-k}}{k!} + \sum_{j=1}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \\ + \sum_{k=n+1}^{n+N} \left(\sum_{j=k-n}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} = (-1)^{n-1}.$$

By more specialization, we obtain

Corollary 2.3. *If in addition of $s = 0$, $\alpha = N = 1$, then we recover the recurrence relations (1.1).*

2.1. **Proof of the corollary (2.3).** For $N = \alpha = 1$, the relations (2.2) write for positive integer n ,

$$\begin{aligned} (-1)^{n-1} &= \sum_{k=1}^n \left(\frac{(-1)^{n-k}}{k!} + \sum_{j=1}^{\min(k,1)} \frac{s(n+j-k, 0, P_{1,j})}{(k-j)!} \right) k! b_k \\ &\quad + \sum_{k=n+1}^{n+1} \left(\sum_{j=1}^{\min(k,1)} \frac{s(n+j-k, 0, P_{1,j})}{(k-j)!} \right) k! b_k \\ &= \sum_{k=1}^n \left(\frac{(-1)^{n-k}}{k!} + \frac{s(n+1-k, 0, P_{1,1})}{(k-1)!} \right) k! b_k + \frac{s(0, 0, P_{1,1})}{n!} (n+1)! b_{n+1}. \end{aligned}$$

We have $P_{1,1} = R(1, 1, 1)S(1, 1)Y = Y$ then $s(0, 0, P_{1,1}) = 0$ and $s(l, 0, P_{1,1}) = (-1)^{l-1}/l$ for $l \geq 1$. Hence

$$\begin{aligned} (-1)^{n-1} &= \sum_{k=1}^n \left(\frac{(-1)^{n-k}}{k!} + \frac{(-1)^{n-k}}{(k-1)!(n+1-k)} \right) k! b_k \\ &= \sum_{k=1}^n (-1)^{n-k} \left(1 + \frac{k}{n+1-k} \right) b_k = (n+1) \sum_{k=1}^n \frac{(-1)^{n-k}}{n+1-k} b_k, \end{aligned}$$

but $b_0 = 1$, then

$$\frac{(-1)^n}{n+1} b_0 + \sum_{k=1}^n \frac{(-1)^{n-k}}{n+1-k} b_k = 0 = (-1)^n \sum_{k=0}^n \frac{(-1)^k}{n+1-k} b_k,$$

and we arrive to desired recurrence (1.1).

2.2. **Proof of theorem (2.1).** Let us consider the function $E_{s,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\alpha)^s}$ so that $E_{s,\alpha}(\log(1+t)) = F_{s,\alpha}(t)$, the generating function of our generalized Cauchy numbers. Then,

Lemma 2.1. *For any positive integer N , any real number x , we have*

$$(2.3) \quad E_{s,\alpha}(x) = \sum_{k=0}^{\infty} R(N, k, \alpha) x^k \frac{d^k}{dx^k} E_{s+N,\alpha}(x).$$

Indeed, we write

$$E_{s,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\alpha)^s} = \sum_{n=0}^{\infty} (n+\alpha)^N \frac{x^n}{n!(n+\alpha)^{s+N}},$$

then by (1.6),

$$\begin{aligned} E_{s,\alpha}(x) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^N R(N, k, \alpha) (n)_k \right) \frac{x^n}{n!(n+\alpha)^{s+N}} \\ &= \sum_{k=0}^N R(N, k, \alpha) x^k \sum_{n=0}^{\infty} (n)_k \frac{x^{n-k}}{n!(n+\alpha)^{s+N}} \\ &= \sum_{k=0}^N R(N, k, \alpha) x^k \frac{d^k}{dx^k} E_{s+N,\alpha}(x). \end{aligned}$$

To translate this result to $F_{s,\alpha}(t)$, we shall use the following lemma.

Lemma 2.2. *If $F(x)$ and $F(\log(1+t))$ admit m derivatives, then it holds*

$$(2.4) \quad \frac{d^m}{dt^m} F(\log(1+t)) = \frac{1}{(1+t)^m} \sum_{j=0}^m s(m, j) F^{(j)}(\log(1+t))$$

$$(2.5) \quad \text{and } F^{(m)}(\log(1+t)) = \sum_{j=0}^m S(m, j) (1+t)^j \frac{d^j}{dt^j} F(\log(1+t)).$$

Indeed, the first part is classical and given in [3] with t in place of $1+t$. It is also used in [1], lemma 4.1, with the function $F(x) = 1/x$. The proof is by recurrence and uses the fundamental relations on the Stirling numbers of the first kind, see [3] :

$$\begin{aligned} s(n, k) &= s(n-1, k-1) - (n-1)s(n-1, k), \quad n \geq k \geq 1; \\ s(n, 0) &= s(0, k) = 0, \quad \text{except } s(0, 0) = 1. \end{aligned}$$

While the second part is the usual inversion formula based on the orthogonality of the Stirling numbers of the two kinds, see [3]. By combining the lemma (2.3) and the lemma (2.4), we obtain for $|t| < 1$,

$$\begin{aligned} F_{s, \alpha}(t) &= \sum_{k=0}^N R(N, k, \alpha) \log^k(1+t) E_{s+N, \alpha}^{(k)}(\log(1+t)) \\ &= \sum_{k=0}^N R(N, k, \alpha) \log^k(1+t) \sum_{j=0}^k S(k, j) (1+t)^j \frac{d^j}{dt^j} E_{s+N, \alpha}(\log(1+t)) \\ &= \sum_{k=0}^N \sum_{j=0}^k R(N, k, \alpha) S(k, j) (1+t)^j \log^k(1+t) F_{s+N, \alpha}^{(j)}(t) \\ &= R(N, 0, \alpha) S(0, 0) F_{s+N, \alpha}(t) + \sum_{k=1}^N \sum_{j=1}^k R(N, k, \alpha) S(k, j) (1+t)^j \log^k(1+t) F_{s+N, \alpha}^{(j)}(t) \\ &= \alpha^N F_{s+N, \alpha}(t) + (1+t) \sum_{j=1}^N (1+t)^{j-1} \left(\sum_{k=j}^N R(N, k, \alpha) S(k, j) \log^k(1+t) \right) F_{s+N, \alpha}^{(j)}(t), \end{aligned}$$

then

$$(2.6) \quad \frac{F_{s, \alpha}(t)}{1+t} = \alpha^N \frac{F_{s+N, \alpha}(t)}{1+t} + \sum_{j=1}^N P_{N, j} (1+t, \log(1+t)) F_{s+N, \alpha}^{(j)}(t).$$

To conclude, we consider the corresponding expansions. On one hand, we have

$$(2.7) \quad \frac{F_{s, \alpha}(t)}{1+t} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k, \alpha}^{(s)}}{k!} \right) t^n, \quad \frac{F_{s+N, \alpha}(t)}{1+t} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k, \alpha}^{(s+N)}}{k!} \right) t^n.$$

On an other hand, we have

$$\begin{aligned} P_{N, j} (1+t, \log(1+t)) F_{s+N, \alpha}^{(j)}(t) &= \left(\sum_{n=0}^{\infty} s(n, 0, P_{N, j}) t^n \right) \left(\sum_{n=0}^{\infty} c_{n+j, \alpha}^{(s+N)} \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n s(n-k, 0, P_{N, j}) \frac{c_{k+j, \alpha}^{(s+N)}}{k!} \right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=j}^{n+j} s(n+j-k, 0, P_{N, j}) \frac{c_{k, \alpha}^{(s+N)}}{(k-j)!} \right) t^n. \end{aligned}$$

Hence, we obtain

$$(2.8) \quad \sum_{j=1}^N P_{N,j}(1+t, \log(1+t)) F_{s+N,\alpha}^{(j)}(t) \\ = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+N} \left(\sum_{j=\max(1,k-n)}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \right) t^n.$$

The identities (2.6), (2.7) and (2.8) lead to the equality

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k,\alpha}^{(s)}}{k!} \right) t^n = \alpha^N \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} \frac{c_{k,\alpha}^{(s+N)}}{k!} \right) t^n \\ + \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+N} \left(\sum_{j=\max(1,k-n)}^{\min(k,N)} \frac{s(n+j-k, 0, P_{N,j})}{(k-j)!} \right) c_{k,\alpha}^{(s+N)} \right) t^n.$$

Equating the coefficients of t^n in the expansions above, achieves the proof of the theorem (2.1).

3. THE SECOND RECURRENCE RELATIONS

3.1. Preliminaries. The starting point is the following lemma.

Lemma 3.1. *For any real number t , with $0 < |t| < 1$, we have*

$$(3.1) \quad F_{s+1,\alpha}(t) = \frac{1}{\log^\alpha(1+t)} \int_0^t \log^{\alpha-1}(1+u) \frac{F_{s,\alpha}(u)}{1+u} du.$$

Proof. We just compute, for $|t| < 1$, the derivative

$$\frac{d}{dt} (\log^\alpha(1+t) F_{s+1,\alpha}(t)) = \frac{d}{dt} \left(\sum_{n=0}^{\infty} \frac{\log^{n+\alpha}(1+t)}{n!(n+\alpha)^{s+1}} \right) \\ = \frac{1}{1+t} \sum_{n=0}^{\infty} \frac{\log^{n+\alpha-1}(1+t)}{n!(n+\alpha)^s} = \log^{\alpha-1}(1+t) \frac{F_{s,\alpha}(t)}{1+t},$$

and observe that the function $\log^\alpha(1+t) F_{s+1,\alpha}(t)$ vanishes for $t = 0$. □

We define the function

$$(3.2) \quad G_{s,\alpha}(t) := \frac{1}{t^\alpha} \int_0^t \log^{\alpha-1}(1+u) \frac{F_{s,\alpha}(u)}{1+u} du,$$

and look for its expansion in powers of t .

Proposition 3.1. *For any $|t| < 1$, we have $G_{s,\alpha}(t) = \sum_{n=0}^{\infty} g_{n,\alpha}^{(s)} t^n$ with*

$$(3.3) \quad g_{n,\alpha}^{(s)} = \frac{1}{\alpha(n+\alpha)} \sum_{k=0}^n (n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k,\alpha}^{(s)}}{k!}.$$

Indeed, we first expand, for $|u| < 1$,

$$\frac{\log^{\alpha-1}(1+u)}{1+u} = \frac{1}{\alpha} (\log^\alpha(1+u))' \\ = \frac{1}{\alpha} \left(\sum_{n=0}^{\infty} s(n, \alpha, 1) u^{n-\alpha} \right)' = \frac{1}{\alpha} \sum_{n=0}^{\infty} (n-\alpha) s(n, \alpha, 1) u^{n-\alpha-1}.$$

Next we put it in (3.2),

$$\begin{aligned} G_{s,\alpha}(t) &= \frac{1}{\alpha t^\alpha} \int_0^t \left(\sum_{n=0}^{\infty} (n+\alpha) s(n, \alpha, 1) u^{n+\alpha-1} \right) F_{s,\alpha}(u) du \\ &= \frac{1}{\alpha t^\alpha} \int_0^t \left(\sum_{n=0}^{\infty} (n+\alpha) s(n, \alpha, 1) u^{n+\alpha-1} \right) \left(\sum_{n=0}^{\infty} c_{n,\alpha}^{(s)} \frac{u^n}{n!} \right) du \\ &= \frac{1}{\alpha t^\alpha} \int_0^t u^{\alpha-1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k,\alpha}^{(s)}}{k!} \right) u^n du. \end{aligned}$$

After integration, it holds

$$G_{s,\alpha}(t) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (n-k+\alpha) s(n-k, \alpha, 1) \frac{c_{k,\alpha}^{(s)}}{k!} \right) \frac{t^n}{n+\alpha},$$

and the expression of $g_{n,\alpha}^{(s)}$ appears clearly.

3.2. The second theorem. We are now able to state our second main result.

Theorem 3.1. *Let $N > m \geq 0$ be given integers. Then for any integer $n \geq N$, we have*

$$(3.4) \quad \sum_{j=0}^{n-N} s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m c_{j,\alpha}^{(s+1)}}{j!} \\ = \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j, l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, Q_{m-j, N-l-1}) \right),$$

where the polynomials $Q_{N,m}$ are $Q_{N,m}(X, Y) = X^m Y^N$.

Proof. We shall apply the operator $(1+t)^m \log^{N+\alpha-1} (1+t) d^m / dt^m$ on both sides of the equality $\frac{F_{s+1,\alpha}(t)}{t^\alpha} = \frac{G_{s,\alpha}(t)}{\log^\alpha(1+t)}$ obtained from (3.1). For the left side, we have for the derivatives

$$\frac{d^m}{dt^m} \frac{F_{s+1,\alpha}(t)}{t^\alpha} = \frac{d^m}{dt^m} \sum_{n=0}^{\infty} c_{n,\alpha}^{(s+1)} \frac{t^{n-\alpha}}{n!} = \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^{n-\alpha-m}}{n!},$$

then

$$\begin{aligned} (1+t)^m \log^{N+\alpha-1} (1+t) \frac{d^m}{dt^m} \frac{F_{s+1,\alpha}(t)}{t^\alpha} &= (1+t)^m \log^{N+\alpha-1} (1+t) \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^{n-\alpha-m}}{n!} \\ &= t^{-m-1} \left(\frac{\log(1+t)}{t} \right)^{\alpha-1} (1+t)^m \log^N (1+t) \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^n}{n!}, \end{aligned}$$

and by (1.5), we expand

$$\begin{aligned} &\left(\frac{\log(1+t)}{t} \right)^{\alpha-1} (1+t)^m \log^N (1+t) \sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^n}{n!} \\ &= t^{-m-1} \left(\sum_{n=0}^{\infty} s(n, \alpha-1, Q_{m,N}) t^n \right) \left(\sum_{n=0}^{\infty} (n-\alpha)_m c_{n,\alpha}^{(s+1)} \frac{t^n}{n!} \right) \\ &= t^{-m-1} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m c_{j,\alpha}^{(s+1)}}{j!} \right) t^n. \end{aligned}$$

Thus we have obtained that,

$$(3.5) \quad \begin{aligned} & (1+t)^m \log^{N+\alpha-1}(1+t) \log^{N+\alpha-1}(1+t) \frac{d^m F_{s+1,\alpha}(t)}{dt^m t^\alpha} \\ &= t^{-m-1} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n s(n-j, \alpha-1, Q_{m,N}) \frac{(j-\alpha)_m {}^{(s+1)}c_{j,\alpha}}{j!} \right) t^n. \end{aligned}$$

For the second side, the derivatives are

$$\frac{d^m G_{s,\alpha}(t)}{dt^m \log^\alpha(1+t)} = \sum_{j=0}^m \binom{m}{j} G_{s,\alpha}^{(m-j)}(t) \frac{d^j \log^{-\alpha}(1+t)}{dt^j},$$

and by lemma (2.4), we have

$$\frac{d^j \log^{-\alpha}(1+t)}{dt^j} = \frac{1}{(1+t)^j} \sum_{l=0}^j \frac{(-\alpha)_l s(j,l)}{\log^{\alpha+l}(1+t)},$$

then we obtain

$$\begin{aligned} & \frac{d^m G_{s,\alpha}(t)}{dt^m \log^\alpha(1+t)} \\ &= \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} G_{s,\alpha}^{(m-j)}(t) \frac{(-\alpha)_l s(j,l)}{(1+t)^j \log^{\alpha+l}(1+t)}, \end{aligned}$$

then

$$(3.6) \quad \begin{aligned} & (1+t)^m \log^{N+\alpha-1}(1+t) \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} G_{s,\alpha}^{(m-j)}(t) \frac{(-\alpha)_l s(j,l)}{(1+t)^j \log^{\alpha+l}(1+t)} \\ &= \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t). \end{aligned}$$

According to (3.2) and (1.5), we have

$$\begin{aligned} G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t) &= t^{j-m} \left(\sum_{n=0}^{\infty} (n)_{m-j} g_{n,\alpha}^{(s)} t^n \right) \left(\sum_{n=0}^{\infty} s(n,0, Q_{m-j,N-l-1}) t^n \right) \\ &= t^{j-m} \sum_{n=0}^{\infty} \left(\sum_{r=0}^n (r)_{m-j} g_{r,\alpha}^{(s)} s(n-r,0, Q_{m-j,N-l-1}) \right) t^n \\ &= t^{-m-1} \sum_{n=0}^{\infty} \left(\sum_{r=0}^n (r)_{m-j} g_{r,\alpha}^{(s)} s(n-r,0, Q_{m-j,N-l-1}) \right) t^{n+j+1}. \end{aligned}$$

By changing $n+j+1$ to n in the summation, we obtain

$$G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t) = t^{-m-1} \sum_{n=j+1}^{\infty} \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r,0, Q_{m-j,N-l-1}) \right) t^n,$$

which leads to the expression of the right hand of the equality (3.6),

$$\begin{aligned} & \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) G_{s,\alpha}^{(m-j)}(t) (1+t)^{m-j} \log^{N-l-1}(1+t) \\ &= t^{-m-1} \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \sum_{n=j+1}^{\infty} \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r,0, Q_{m-j,N-l-1}) \right) t^n \\ &= t^{-m-1} \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\min(m,n-1)} \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r,0, Q_{m-j,N-l-1}) \right) \right) t^n. \end{aligned}$$

We obtain here that

$$(3.7) \quad (1+t)^m \log^{N+\alpha-1}(1+t) \log^{N+\alpha-1}(1+t) \frac{d^m}{dt^m} \frac{G_{s,\alpha}(t)}{\log^\alpha(1+t)} \\ = t^{-m-1} \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\min(m,n-1)} \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, \mathcal{Q}_{m-j,N-l-1}) \right) \right) t^n.$$

Finally, (3.5) and (3.7) together give

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n s(n-j, \alpha-1, \mathcal{Q}_{m,N}) \frac{(j-\alpha)_m}{j!} c_{j,\alpha}^{(s+1)} \right) t^n \\ = \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\min(m,n-1)} \sum_{l=0}^j \binom{m}{j} (-\alpha)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,\alpha}^{(s)} s(n-j-1-r, 0, \mathcal{Q}_{m-j,N-l-1}) \right) \right) t^n.$$

To conclude, we equate the coefficients of t^n , for $n \geq N$, in the summations after observing that for $n-j > N$, $s(n-j, \alpha-1, \mathcal{Q}_{m,N}) = 0$ and $\min(m, n-1) = m$ for $n > m$. \square

3.3. The special case $s = 0$ and $\alpha = 1$. We show here that we recover the recurrences of Agoh and Dilcher for the Bernoulli numbers of the second kind. Let n be an integer $\geq N$ and denote by L the left hand of the recurrence (3.4), with $s = 0$, $\alpha = 1$, and R its right one.

$$L = \sum_{j=0}^{n-N} s(n-j, 0, \mathcal{Q}_{m,N}) (j-1)_m b_j$$

and

$$R = \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-1)_l s(j,l) \left(\sum_{r=0}^{n-j-1} (r)_{m-j} g_{r,1}^{(0)} s(n-j-1-r, 0, \mathcal{Q}_{m-j,N-l-1}) \right).$$

As $(j-1)_m = (j-1)(j-2)\dots(j-m)$ vanishes for $j = 1, 2, \dots, m$ and equals $(-1)^m m!$ for $j = 0$, we have

$$L = (-1)^m m! s(n, 0, \mathcal{Q}_{m,N}) + \sum_{j=m+1}^{n-N} s(n-j, 0, \mathcal{Q}_{m,N}) (j-1)_m b_j \\ = (-1)^m m! N! \bar{s}(n, N-m, m) + N! \sum_{j=m+1}^{n-N} \bar{s}(n-j, N-m, m) (j-1)_m b_j \\ = (-1)^m m! N! \bar{s}(n, N-m, m) + m! N! \sum_{j=m+1}^{n-N} \bar{s}(n-j, N-m, m) \binom{j-1}{m} b_j,$$

and by a little shift

$$(3.8) \quad L = (-1)^m m! N! \bar{s}(n, N-m, m) + m! N! \sum_{j=0}^{n-N-m-1} \bar{s}(n-j-m-1, N-m, m) \binom{j+m}{m} b_{j+m+1}.$$

For the right hand R , as $G_{0,1}(t) = \frac{1}{t} \int_0^t \frac{1+u}{1+u} du = 1$, then $g_{0,1}^{(0)} = 1$ and $g_{r,1}^{(0)} = 0$ for $r \geq 1$, so

$$R = \sum_{j=0}^m \sum_{l=0}^j \binom{m}{j} (-1)_l s(j,l) (0)_{m-j} s(n-j-1, 0, \mathcal{Q}_{m-j,N-l-1}).$$

But $(0)_{m-j}$ vanishes for $m > j$ and equals 1 for $m = j$, then

$$R = \sum_{l=0}^m (-1)^l l! s(m,l) s(n-m-1, 0, \mathcal{Q}_{0,N-l-1}) \\ = \sum_{l=0}^m (-1)^l l! s(m,l) (N-l-1)! \bar{s}(n-m-1, N-l-1, 0).$$

We use $\tilde{s}(n-m-1, N-l-1, 0) = s(n-m-1, N-l-1)$ and introduce a binomial coefficient,

$$R = \frac{(N-1)!}{(n-m-1)!} \sum_{l=0}^m (-1)^l s(m, l) s(n-m-1, N-l-1) \binom{N-1}{N-1-l}^{-1}.$$

Now, put $l = m - j$ in the summation above and obtain

$$(3.9) \quad R = \frac{(N-1)!}{(n-m-1)!} \sum_{j=0}^m (-1)^{m-j} s(m, m-j) s(n-m-1, N-m+j-1) \binom{N-1}{N-m+j-1}^{-1}.$$

Finally, we put $N = m + k$ in both (3.8) and (3.9) with $n + m + 1$ instead of n and obtain

$$L = (-1)^m m!(m+k)! \tilde{s}(n+m+1, k, m) + m!(m+k)! \sum_{j=0}^{n-m-k} \tilde{s}(n-j, k, m) \binom{j+m}{m} b_{j+m+1},$$

$$R = \frac{(m+k-1)!}{n!} \sum_{j=0}^m (-1)^{m-j} s(m, m-j) s(n, k+j-1) \binom{m+k-1}{k+j-1}^{-1},$$

and $\frac{L}{m!(m+k)!} = \frac{R}{m!(m+k)!}$ is exactly the desired identity (1.3).

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