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Sparse Spikes Super-resolution on Thin Grids II: the Continuous Basis Pursuit

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Abstract. This article analyzes the performance of the Continuous Basis Pursuit (C-BP) method for sparse super-resolution. The C-BP has been recently proposed by Ekanadham, Tranchina and Simoncelli as a refined discretization scheme for the recovery of spikes in inverse problems regularization. One of the most well known discretization scheme, the Basis Pursuit (BP, also known as LASSO) makes use of a finite dimensional ℓ^1 norm on a grid. In contrast, the C-BP rather uses a linear interpolation of the spikes positions to enable the recovery of spikes between grid's points. When the sought-after solution is constrained to be positive, a remarkable feature of this approach is that it retains the convexity of the initial ℓ^1 problem. For deconvolution problems, it is well known that ℓ^1 -type methods (including BP and C-BP) recover exactly the unknown sparse sum of positive Diracs when there is no noise in the measurements. Our main contribution identifies a non-degeneracy condition ensuring that the support of the solution enjoys some stability when noise is added. More precisely, we show that, in the small noise regime, when this non-degeneracy condition holds, the C-B method estimates a pair of Dirac masses around each spike of the input measure. We also derive the asymptotic of the recovery error when the grid size tends to zero. We show some numerical illustrations of this stability to noise for both the BP and C-BP methods, and evaluate numerically these non-degeneracy conditions.

1. Introduction

This article studies a specific convex optimization approach, the Continuous Basis-Pursuit (C-BP) for sparse super-resolution. A detailed review of the literature on super-resolution, and in particular variational regularization technics, can be found in the companion paper [17], which is dedicated to the Basis Pursuit (BP, or LASSO) method.

In the following, we focus for simplicity on the compact 1-D domain $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (i.e. an interval with periodic boundary conditions), but the algorithms considered (LASSO and C-BP) as well as our theoretical analysis can be extended to higher dimensional settings (see Section 1.6).

1.1. Super-resolution

We formalize the sparse super-resolution problem mathematically as the question of recovering an unknown Radon measure $m_0 \in \mathcal{M}(\mathbb{T})$ defined on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ from noisy linear observations in a separable Hilbert space \mathcal{H} , which we write as

$$y = \Phi(m_0) + w \in \mathcal{H} \tag{1}$$

where $w \in \mathcal{H}$ is some measurement noise. The terminology “super-resolution” refers here to the fact that $\Phi : \mathcal{M}(\mathbb{T}) \rightarrow \mathcal{H}$ is a bounded linear map defined through the integration against a *continuous* kernel function $\varphi : \mathbb{T} \rightarrow \mathcal{H}$

$$\forall x \in \mathbb{T}, \quad \Phi(m) = \int_{\mathbb{T}} \varphi(x) dm(x). \tag{2}$$

The continuity of φ is crucial, since it allows one to pose the problem over the space of measures, and thus enables the possible recovery of highly localized solutions (e.g. sums of Dirac masses). In order for our theoretical findings to hold, we furthermore require that $\varphi \in \mathcal{C}^3(\mathbb{T}, \mathcal{H})$.

A typical class of problems are the so-called deconvolution problems, when the operator Φ is translation invariant. This corresponds to using $\mathcal{H} = L^2(\mathbb{T})$, $\varphi(x) : x' \mapsto \tilde{\varphi}(x' - x)$ for some smooth kernel $\tilde{\varphi}$ defined on \mathbb{T} . This deconvolution setup is equivalent to computing the measurements over the Fourier domain. When the highest measured frequency f_c is finite, this corresponds to using $\mathcal{H} = \mathbb{C}^{2f_c+1}$ and $\varphi(x) = (\hat{\varphi}_k e^{2ikx\pi})_{k=-f_c}^{f_c}$ for some weights $\hat{\varphi}_k \in \mathbb{C}$ which are the Fourier coefficients of $\tilde{\varphi}$.

1.2. Sparse Regularization

Since Φ is usually ill-posed, a regularization scheme is needed to perform its inversion. Following several recent works (see for instance [5, 11, 7]), we study here sparse regularizations induced by the total variation of the measure (i.e. the total absolute mass)

$$|m|(\mathbb{T}) \stackrel{\text{def.}}{=} \sup \left\{ \int_{\mathbb{T}} \psi(x) dm(x) ; \psi \in \mathcal{C}(\mathbb{T}), \|\psi\|_{\infty} \leq 1 \right\}. \tag{3}$$

For discrete measures defined on a finite grid

$$\mathcal{G} \stackrel{\text{def.}}{=} \{z_i ; i \in \llbracket 0, G-1 \rrbracket\} \subset \mathbb{T} \tag{4}$$

of points, which take the form

$$m_{a,\mathcal{G}} \stackrel{\text{def.}}{=} \sum_{i=0}^{G-1} a_i \delta_{z_i} \in \mathcal{M}(\mathbb{T}),$$

the total variation is equal to the usual discrete ℓ^1 norm

$$|m_{a,\mathcal{G}}|(\mathbb{T}) = \|a\|_1 = \sum_i |a_i|,$$

which is at the heart of the celebrated Basis Pursuit [8] method (also known as the LASSO [30]). The total variation is thus the natural way to extend these intrinsically discrete and finite-dimensional methods to the infinite-dimensional class of Radon measures, to enable a grid-free regularization.

This total variation is used as a regularizer to define the BLASSO [11]

$$\min_{m \in \mathcal{M}(\mathbb{T})} \frac{1}{2} \|y - \Phi(m)\|_{\mathcal{H}}^2 + \lambda |m|(\mathbb{T}). \quad (5)$$

which is an infinite dimensional convex optimization problem. In the noiseless setting, $w = 0$, taking the limit $\lambda \rightarrow 0$ leads to the following constrained problem

$$\min_{m \in \mathcal{M}(\mathbb{T})} \{|m|(\mathbb{T}) ; \Phi(m) = \Phi(m_0)\}. \quad (6)$$

1.3. From BP to C-BP

The simplest way to compute approximate solutions to (3) is to restrict the solution to live in the finite dimensional sub-space of measures supported on the grid \mathcal{G} defined in (4). This is exactly equivalent to computing the weights a of the measure by solving the BP problem

$$\min_{a \in \mathbb{R}^G} \frac{1}{2} \|y - \Phi_{\mathcal{G}} a\|_{\mathcal{H}}^2 + \lambda \|a\|_1 \quad (7)$$

where we defined the restriction of Φ to the discretization grid as

$$\Phi_{\mathcal{G}} a \stackrel{\text{def.}}{=} \Phi(m_{a,\mathcal{G}}) = \sum_{i=0}^{G-1} a_i \varphi(z_i) \quad (8)$$

To obtain a better approximation of the infinite dimensional problem, [19] proposes to perform a first order approximation of the kernel. This method assumes that the unknown measure is positive, and we introduce the derivative operator restricted to the sampling grid

$$\Phi'_{\mathcal{G}} b \stackrel{\text{def.}}{=} \Phi'(m_{b,\mathcal{G}}) = \sum_{i=0}^{G-1} b_i \varphi'(z_i). \quad (9)$$

To ease the exposition, we consider a uniform grid $\mathcal{G} \stackrel{\text{def.}}{=} \{i/G ; i \in \llbracket 0, G-1 \rrbracket\}$ of G points, so that the grid size is $h \stackrel{\text{def.}}{=} 1/G$. The C-BP method of [19] solves

$$\min_{(a,b) \in \mathbb{R}^G \times \mathbb{R}^G} \frac{1}{2} \|y - \Phi_{\mathcal{G}} a - \Phi'_{\mathcal{G}} b\|_{\mathcal{H}}^2 + \lambda \|a\|_1 \quad \text{subject to} \quad |b| \leq \frac{h}{2} a, \quad (10)$$

where the inequality should be understood component-wise. Note also that the obtained a is always nonnegative, hence the C-BP method is tailored for the recovery of positive measures. This is a convex optimization problem, which can be solved using traditional conic optimization methods. As detailed in Section 2.2, this problem can also be re-cast as a LASSO in dimension $2G$ with positivity constraints (see Section 2.2). Hence it can

be solved using a large variety of first order proximal method, the most simple one being the Forward-Backward, see [2] and the references therein.

If (a^*, b^*) are solutions of (10), one recovers an output discrete measure defined by

$$m^* = \sum_{a_i^* \neq 0} a_i^* \delta_{x_i^*} \quad \text{where} \quad x_i^* \stackrel{\text{def.}}{=} ih + \frac{b_i^*}{a_i^*}, \quad (11)$$

where we set $\frac{b_i^*}{a_i^*} = 0$ whenever $a_i^* = 0$. The rationale behind (10) is to perform a first order Taylor approximation of the operator Φ , where the variable $\tau_i \stackrel{\text{def.}}{=} b_i/a_i \in [-h/2, h/2]$ encodes the horizontal shift of the Dirac location with respect to the grid sample ih . The landmark idea introduced in [19] is that, while the optimization is non-convex with respect to the pair (a, τ) , it is convex with respect to the pair (a, b) .

1.4. Previous Works

There is a large body of literature on the study of the recovery performance of the LASSO problem (7). We refer to the companion paper [17] for a detailed overview. Let us however insist on the fact that only a small part of these previous works tackles the super-resolution case where the operator Φ is highly “correlated”, which can be informally understood as cases associated to a continuous underlying kernel φ . As highlighted above, these cases are naturally paired with a continuous infinite dimensional problem (5). The theoretical analysis of this setting is recent, and we refer in particular to the works [7] which showed that some minimal separation between the spikes (the so-called “Rayleigh limit”) is necessary and sufficient to achieve exact recovery in the absence of noise, for deconvolution problems. Several related works have studied the impact of noise on this recovery, see for instance [5, 6, 20, 1]. In two recent papers, including the initial work [16] and the companion paper [17], we have shown how to transfer support stability properties of the infinite dimensional BLASSO (5) to the discretized LASSO problem (7). Most notably, these results show that solutions of the discrete LASSO problem estimate in general as much as twice the number of spikes as the input measure. It is important to note that in the special case where m_0 is a positive measure (which is the setup considered by the C-BP method), for deconvolution problems, then m_0 is always a solution to (6), as shown in [11] (see also [12] for a refined analysis of the stability to noise in this special case).

1.5. Contributions

The main purpose of this paper is to show how the results of the companion paper [17] generalize from the basic LASSO to the more involved C-BP method, at the expense of imposing positivity on the retrieved measure.

First of all, Section 2 shows that the C-BP problem can be recast as a LASSO under positivity constraints. It proposes an abstract analysis of support stability properties of such a class of problems, under a non-degeneracy condition (hypothesis (33)), see Theorem 1.

Working on a thin grid, we show the Γ -convergence of Problem (10) towards the Beurling LASSO (5) with positivity constraints. The main contributions of this paper then to provide a fine analysis of the support of the solutions as the grid stepsize tends to zero. More precisely, we first need to assume that m_0 is the solution of the BLASSO program (6) when $w = 0, \lambda = 0$. For positive measures, this is a rather weak hypothesis, and is for instance always true for deconvolution problems. We then identify a non-degeneracy condition (the ‘‘Twice Non-Degenerate Source Condition’’, see Definition 5), which is crucial to enable the stability of support under noise contaminating the measurements. Our main contribution, Theorem 2, shows that, under this non-degeneracy condition, the recovered spikes appear in pairs,

$$m_\lambda = \sum_{\nu=1}^N (\alpha_{\lambda,\nu} \delta_{x_{0,\nu}+t_\nu} + \beta_{\lambda,\nu} \delta_{x_{0,\nu}+\varepsilon_\nu h/2}) \quad \text{where} \quad \begin{cases} \varepsilon_\nu \in \{-1, +1\}, \\ -h/2 < t_\nu < h/2. \end{cases}$$

A closed form expression for ε is given, which depends on some corresponding *natural shift* intrinsic to the measure (which differs from the one of the LASSO). The corresponding low noise regime is characterized by $\frac{\|w\|_{\mathcal{H}}}{\lambda} = O(1)$ and $\lambda = O(h^3)$.

Eventually, we illustrate in Section 5 these theoretical results with numerical experiments comparing LASSO and C-BP. We first display the evolution of the solution path $\lambda \mapsto a_\lambda$ (a solution of (7)) and $\lambda \mapsto (a_\lambda, b_\lambda)$ (a solution of (10)). These paths are piecewise-affine, and our contributions (Theorem 2) precisely characterize the first affine segment of these paths, which perfectly matches the numerical observations.

1.6. Extensions

While we restrict here the exposition to 1-D problems, the C-BP formulation (10) can be extended to cope with measures in arbitrary dimension $d \geq 1$, i.e. to consider $m_0 \in \mathcal{M}(\mathbb{T}^d)$. This requires to define at each sampling grid point indexed by i a vector $b_i = (b_{i,k})_{k=1}^d \in \mathbb{R}^d$ together with the constraint $\|b_i\|_\infty \leq \frac{h}{2} a_i$, and also to use a matrix Φ'_G defined as

$$\Phi'_G b \stackrel{\text{def.}}{=} \sum_{i \in \mathcal{G}} \sum_{k=1}^d b_{i,k} \partial_k \varphi(x_i) \in \mathcal{H}$$

where ∂^k denote the differential operator with respect to the k^{th} direction in \mathbb{R}^d . Our analysis carries over to this setting without major difficulties.

The paper [19] also proposes other interpolation schemes than a first order Taylor expansion at the grid points. In particular, they develop a ‘‘polar’’ interpolation which makes use of two adjacent grid points. This method seems to outperform the linear interpolation in practice, and has been employed to perform spikes sorting in neuronal recordings [18].

Extending the results we propose in the present paper to these higher dimensional settings and alternative interpolation schemes is an interesting avenue for future work.

Let us also mention that an important problem is to extend the C-BP method (11) to measures with arbitrary signs and that can even be complex-valued. Unfortunately,

the corresponding constraint $|b| \leq |a|$ is then non-convex, which makes the mathematical analysis apparently much more involved. A non-convex and non-smooth optimization solver is proposed for this problem in [21], and shows promising practical performance for spectrum estimation.

1.7. Notations and preliminaries

The set of Radon measures (resp. positive Radon measures) is denoted by $\mathcal{M}(\mathbb{T})$ (resp. $\mathcal{M}^+(\mathbb{T})$). Endowed with the total variation norm (3), $\mathcal{M}(\mathbb{T})$ is a Banach space. Another useful topology on $\mathcal{M}(\mathbb{T})$ is the weak* topology: a sequence of measures $(m_n)_{n \in \mathbb{N}}$ weakly* converges towards $m \in \mathcal{M}(\mathbb{T})$ if and only if for all $\psi \in \mathcal{C}(\mathbb{T})$, $\lim_{n \rightarrow +\infty} \int_{\mathbb{T}} \psi dm_n = \int_{\mathbb{T}} \psi dm$. Any bounded subset of $\mathcal{M}(\mathbb{T})$ (for the total variation) is relatively sequentially compact for the weak* topology. Moreover the topology induced by the total variation is stronger than the weak* topology, and the total variation is sequentially lower semi-continuous for the weak* topology. Throughout the paper, given $\alpha \in \mathbb{R}^N$ and $x_0 \in \mathbb{T}^N$, the notation $m_{\alpha, x_0} \stackrel{\text{def.}}{=} \sum_{\nu=1}^N \alpha_{\nu} \delta_{x_{0,\nu}}$ hints that $\alpha_{\nu} \neq 0$ for all ν (contrary to the notation $m_{a, \mathcal{G}}$), and that the $x_{0,\nu}$'s are pairwise distinct.

The space \mathcal{H} denotes a separable Hilbert space. As explained in the companion paper [17, Lemma 1], the operator $\Phi : \mathcal{M}(\mathbb{T}) \rightarrow \mathcal{H}$ defined in (2) is weak-* to weak continuous, and so are the operators involving the derivatives, for $k \in \mathbb{N}$,

$$\forall m \in \mathcal{M}(\mathbb{T}), \quad \Phi^{(k)}(m) = \int_{\mathbb{T}} \varphi^{(k)}(x) dm(x).$$

Their respective adjoint operators, $\Phi^{(k),*} : \mathcal{H} \rightarrow \mathcal{C}(\mathbb{T})$, are given by $(\Phi^{(k),*} q)(t) = \langle q, \varphi^{(k)}(t) \rangle$ for all $q \in \mathcal{H}$, $t \in \mathbb{T}$. Moreover, $\Phi^{(k),*} q$ has the regularity of φ and $\frac{d^k}{dt^k}(\Phi^* q)(t) = (\Phi^{(k),*} q)(t)$. The ∞, \mathcal{H} -operator norm of $\Phi^* : \mathcal{H} \rightarrow \mathcal{C}(\mathbb{T})$ is defined as $\|\Phi^*\|_{\infty, \mathcal{H}} \stackrel{\text{def.}}{=} \sup \{ \|\Phi^* w\|_{\infty} ; w \in \mathcal{H}, \|w\|_{\mathcal{H}} \leq 1 \}$ (and the ∞, \mathcal{H} operator norm of a matrix is defined similarly). Given a vector $x_0 \in \mathbb{T}^N$, Φ_{x_0} refers to the linear operator $\mathbb{R}^N \rightarrow \mathcal{H}$, with

$$\forall \alpha \in \mathbb{R}^N, \quad \Phi_{x_0} \alpha \stackrel{\text{def.}}{=} \Phi(m_{\alpha, x_0}) = \sum_{\nu=1}^N \alpha_{\nu} \varphi(x_{0,\nu}).$$

It may also be seen as the restriction of Φ to measures supported on the set $\{x_{0,\nu} ; \nu \in [1, N]\}$. A similar notation is adopted for Φ'_{x_0} , resp. $\Phi_{x_0}^{(k)}$ (replacing $\varphi(x_{0,\nu})$ with $\varphi'(x_{0,\nu})$, resp. $\varphi^{(k)}(x_{0,\nu})$). The concatenation of Φ_{x_0} and Φ'_{x_0} is denoted by $\Gamma_{x_0} \stackrel{\text{def.}}{=} \begin{pmatrix} \Phi_{x_0} & \Phi'_{x_0} \end{pmatrix}$.

We shall rely on the notion of set convergence. Given a sequence $(C_n)_{n \in \mathbb{N}}$ of subsets of \mathbb{T} , we define

$$\limsup_{n \rightarrow +\infty} C_n = \left\{ x \in \mathbb{T} ; \liminf_{n \rightarrow +\infty} d(x, C_n) = 0 \right\} \tag{12}$$

$$\liminf_{n \rightarrow +\infty} C_n = \left\{ x \in \mathbb{T} ; \limsup_{n \rightarrow +\infty} d(x, C_n) = 0 \right\} \tag{13}$$

where d is defined by $d(x, C) = \inf_{x' \in C} |x' - x|$ and $|x - x'|$ refers to the distance between x and x' on the torus. If both sets are equal, let C be the corresponding set (then C is necessarily closed), we write

$$\lim_{n \rightarrow +\infty} C_n = C. \quad (14)$$

If the sequence $(C_n)_{n \in \mathbb{N}}$ is nondecreasing ($C_n \subset C_{n+1}$), then $\lim_{n \rightarrow \infty} C_n = \overline{\bigcup_{n \in \mathbb{N}} C_n}$, and if it is nonincreasing ($C_n \supset C_{n+1}$) then $\lim_{n \rightarrow \infty} C_n = \bigcap_{n \in \mathbb{N}} \overline{C_n}$ (where \overline{C} denotes the closure of C). We refer the reader to [26] for more detail about set convergence. We shall also use this notion in Hilbert spaces, with obvious adaptations.

2. Abstract analysis of the Lasso with cone constraint

This section studies a simple variant of the LASSO with cone constraint in an abstract setting. The results stated here shall be useful in Section 3, since this variant turns out to be the Continuous Basis-Pursuit when the degradation operator is the integration of an impulse response and its derivative.

2.1. Notations

Given a parameter $h > 0$, we consider the convex cone generated by the vectors $(1, \frac{h}{2})$ and $(1, -\frac{h}{2})$,

$$\mathcal{C}_h \stackrel{\text{def.}}{=} \left\{ (c, d) \in \mathbb{R} \times \mathbb{R} ; c \geq 0 \quad \text{and} \quad -c\frac{h}{2} + |d| \leq 0 \right\}. \quad (15)$$

We also define the cone \mathcal{C}_h^G as the set of vectors $(a, b) \in \mathbb{R}^G \times \mathbb{R}^G$ such that for all $k \in \llbracket 0, G-1 \rrbracket$ $(a_k, b_k) \in \mathcal{C}_h$.

Now, given a vector $(a_0, b_0) \in \mathcal{C}_h^G$ (i.e. $\forall k \in \llbracket 0, G-1 \rrbracket, a_{0,k} \geq \frac{2}{h}|b_{0,k}|$), we observe $y_0 \stackrel{\text{def.}}{=} Aa_0 + Bb_0$, where $A : \mathbb{R}^G \rightarrow \mathcal{H}$ and $B : \mathbb{R}^G \rightarrow \mathcal{H}$ are linear operators, or its noisy version $y = y_0 + w$ where $w \in \mathcal{H}$. To recover (a_0, b_0) from y or y_0 , we consider the following reconstruction problems:

$$\min_{(a,b) \in \mathcal{C}_h^G} \frac{1}{2} \|y - Aa - Bb\|_{\mathcal{H}}^2 + \lambda \|a\|_1, \quad (\mathcal{Q}_\lambda(y))$$

and for $\lambda = 0$,

$$\min_{(a,b) \in \mathcal{C}_h^G} \|a\|_1 \quad \text{such that } Aa + Bb = y_0. \quad (\mathcal{Q}_0(y_0))$$

Our main focus is on the support recovery properties of $(\mathcal{Q}_\lambda(y))$. Precisely, we split the ‘‘support’’ of $(a, b) \in \mathcal{C}_h^G$ into several parts:

$$I \stackrel{\text{def.}}{=} \text{supp}(a) \stackrel{\text{def.}}{=} \{i \in \llbracket 0, G-1 \rrbracket ; a_i > 0\} \quad (16)$$

$$= I^{(\mathbf{r})} \cup I^{(1)} \quad (17)$$

$$\text{where } I^{(\mathbf{r})} \stackrel{\text{def.}}{=} \left\{ i \in I ; a_i + \frac{2}{h}b_i > 0 \right\}, \quad I^{(1)} \stackrel{\text{def.}}{=} \left\{ i \in I ; a_i - \frac{2}{h}b_i > 0 \right\}. \quad (18)$$

In general $I^{(r)} \cap I^{(l)} \neq \emptyset$. If (a_λ, b_λ) is a solution of $(\mathcal{Q}_\lambda(y))$, we say that we have exact support recovery provided that $I^{(r)}(a_\lambda, b_\lambda) = I^{(r)}(a_0, b_0)$ and $I^{(l)}(a_\lambda, b_\lambda) = I^{(l)}(a_0, b_0)$.

Remark 1. The notation $I^{(r)}$, $I^{(l)}$ shall become clearer in the next section. When considering the Continuous Basis-Pursuit on a grid with stepsize $h > 0$, points i in $I^{(r)}$ correspond to Dirac masses which “tend to be on the right”, that is they do not coincide with the left half-grid point $ih - \frac{h}{2}$. Similarly, points in $I^{(l)}$ correspond to Dirac masses which “tend to be on the left”, as they do not coincide with the right half-grid point $ih + \frac{h}{2}$. In fact, if $i \in I^{(r)} \setminus I^{(l)}$, it corresponds to a Dirac mass at the right half-grid point: $\delta_{ih + \frac{h}{2}}$, and if $i \in I^{(l)} \setminus I^{(r)}$, it corresponds to a Dirac mass at the left half-grid point: $\delta_{ih - \frac{h}{2}}$. If $i \in I^{(r)} \cap I^{(l)}$, it corresponds to a Dirac mass which may belong “freely” to the interval $(ih - \frac{h}{2}, ih + \frac{h}{2})$ (see Figure 1).

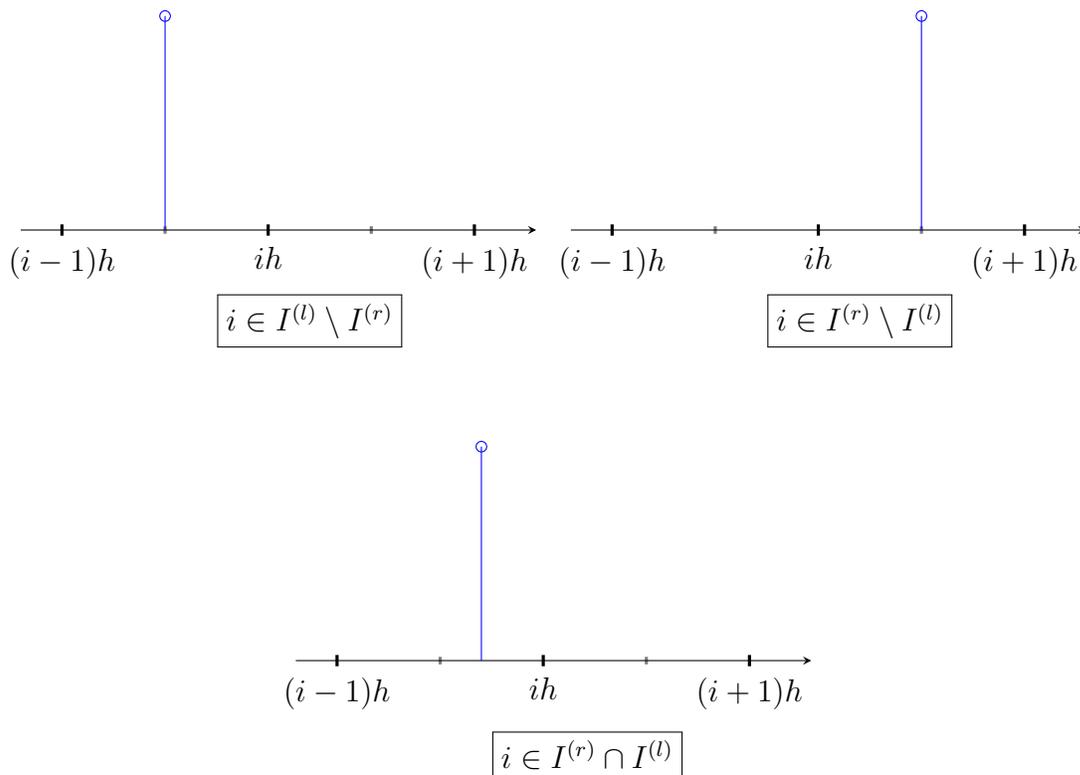


Figure 1: Depending on i being in $I^{(r)}$ or $I^{(l)}$, the spikes of the solutions to (10) are supported on half-grid points or in the interval $((i - 1/2)h, (i + 1/2)h)$.

2.2. Parametrization as a positive LASSO

To characterize the solutions of $(\mathcal{Q}_\lambda(y))$ and $(\mathcal{Q}_0(y_0))$, it is convenient to reparametrize the problem as a LASSO with positivity constraint, writing for all

$i \in \llbracket 0, G-1 \rrbracket$,

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \stackrel{\text{def.}}{=} \begin{pmatrix} 1 & 1 \\ \frac{h}{2} & -\frac{h}{2} \end{pmatrix} \begin{pmatrix} r_i \\ l_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} r_i \\ l_i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_i + \frac{2}{h}b_i \\ a_i - \frac{2}{h}b_i \end{pmatrix}. \quad (19)$$

In the following, we define the linear map

$$H_h : \begin{pmatrix} r \\ l \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix} \quad (20)$$

Observe that $(a_i, b_i) \in \mathcal{C}_h$ if and only if $r_i \geq 0$ and $l_i \geq 0$. Moreover, given $(a, b) \in \mathcal{C}_h^G$,

$$I^c = \{i \in \llbracket 0, G-1 \rrbracket ; (r_i, l_i) = (0, 0)\},$$

$$I^{(r)} = \{i \in \llbracket 0, G-1 \rrbracket ; r_i > 0\}, \quad \text{and} \quad I^{(l)} = \{i \in \llbracket 0, G-1 \rrbracket ; l_i > 0\}.$$

Therefore, Problems $(\mathcal{Q}_\lambda(y))$ and $(\mathcal{Q}_0(y_0))$ are respectively equivalent to the LASSO and Basis Pursuit with positivity constraint:

$$\min_{(r,l) \in (\mathbb{R}_+)^G \times (\mathbb{R}_+)^G} \lambda \left\| \begin{pmatrix} r \\ l \end{pmatrix} \right\|_1 + \frac{1}{2} \left\| y - \mathcal{A}_h \begin{pmatrix} r \\ l \end{pmatrix} \right\|_{\mathcal{H}}^2 \quad (\tilde{\mathcal{Q}}_\lambda(y))$$

$$\text{and} \quad \min_{(r,l) \in (\mathbb{R}_+)^G \times (\mathbb{R}_+)^G} \left\| \begin{pmatrix} r \\ l \end{pmatrix} \right\|_1 \quad \text{such that} \quad \mathcal{A}_h \begin{pmatrix} r \\ l \end{pmatrix} = y_0, \quad (\tilde{\mathcal{Q}}_0(y_0))$$

where $\mathcal{A}_h \stackrel{\text{def.}}{=} \begin{pmatrix} A + \frac{h}{2}B & A - \frac{h}{2}B \end{pmatrix} : \mathbb{R}^{2G} \rightarrow \mathcal{H}$.

The ‘‘support recovery’’ of (a_0, b_0) through $(\mathcal{Q}_\lambda(y))$ is equivalent to the support recovery of (r_0, l_0) through $(\tilde{\mathcal{Q}}_\lambda(y))$. As described below, the characterization of minimizers and the support recovery properties of the LASSO with positivity constraint $(\tilde{\mathcal{Q}}_\lambda(y))$ are quite similar to those of the classical LASSO described in the companion paper [17, Section 2].

Since the regularization term is of the form

$$F(r, l) \stackrel{\text{def.}}{=} \sum_{i=0}^{G-1} f(r_i) + \sum_{i=0}^{G-1} f(l_i), \quad \text{with} \quad f(x) \stackrel{\text{def.}}{=} \sup \{qx ; q \leq 1\} = \begin{cases} x & \text{if } x \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

its subdifferential is the product of the subdifferentials $\partial f(r_i)$ and $\partial f(l_i)$ for $1 \leq i \leq G-1$, where

$$\partial f(x) = \begin{cases} \{1\} & \text{if } x > 0, \\ (-\infty, 1] & \text{if } x = 0. \end{cases}$$

That is quite similar to the subdifferential of $|\cdot|$ at $x \in \mathbb{R}$ which is -1 , $[-1, 1]$ or 1 if $x < 0$, $x = 0$ or $x > 0$ respectively. Hence, one may adapt the standard results for the LASSO to the LASSO with positivity constraint, simply by replacing the conditions $\|\eta\|_\infty \leq 1$ with $\max \eta \leq 1$ (and similarly for strict inequalities) wherever they appear. We leave the detail to the reader, and in the following, we use those results freely to derive the properties of the LASSO with cone constraint $(\mathcal{Q}_\lambda(y))$.

2.3. Optimality conditions

The optimality conditions for $(\tilde{\mathcal{Q}}_\lambda(y))$ and $(\tilde{\mathcal{Q}}_0(y_0))$, written in terms of (a_λ, b_λ) , yield the following results.

Proposition 1. *Let $y \in \mathcal{H}$, $(a_\lambda, b_\lambda) \in \mathcal{C}_h^G$, and $I = I(a_\lambda, b_\lambda)$. Then (a_λ, b_λ) is a solution to $(\mathcal{Q}_\lambda(y))$ if and only if there exists $p_\lambda \in \mathcal{H}$ such that*

$$\max \left((A^* + \frac{h}{2}B^*)p_\lambda \right) \leq 1, \quad \text{and} \quad \max \left((A^* - \frac{h}{2}B^*)p_\lambda \right) \leq 1, \quad (21)$$

$$(A_{I^{(r)}}^* + \frac{h}{2}B_{I^{(r)}}^*)p_\lambda = \mathbf{1}_{I^{(r)}}, \quad \text{and} \quad (A_{I^{(l)}}^* - \frac{h}{2}B_{I^{(l)}}^*)p_\lambda = \mathbf{1}_{I^{(l)}}, \quad (22)$$

$$\lambda \begin{pmatrix} A^* \\ B^* \end{pmatrix} p_\lambda + \begin{pmatrix} A^* \\ B^* \end{pmatrix} (Aa_\lambda + Bb_\lambda - y) = 0. \quad (23)$$

Similarly, $(a_0, b_0) \in \mathcal{C}_h^G$ is a solution to $(\mathcal{Q}_0(y_0))$ if and only if $Aa_0 + Bb_0 = y_0$ and there exists $p \in \mathcal{H}$ such that

$$\max \left((A^* + \frac{h}{2}B^*)p \right) \leq 1, \quad \text{and} \quad \max \left((A^* - \frac{h}{2}B^*)p \right) \leq 1, \quad (24)$$

$$(A_{I^{(r)}}^* + \frac{h}{2}B_{I^{(r)}}^*)p = \mathbf{1}_{I^{(r)}}, \quad \text{and} \quad (A_{I^{(l)}}^* - \frac{h}{2}B_{I^{(l)}}^*)p = \mathbf{1}_{I^{(l)}}, \quad (25)$$

where $I = I(a_0, b_0)$.

If the inequalities outside the support are strict, it is possible to ensure the uniqueness of the solution.

Proposition 2. *Under the hypotheses of Proposition 1, if $\begin{pmatrix} (A + \frac{h}{2}B)_{I^{(r)}} & (A - \frac{h}{2}B)_{I^{(l)}} \end{pmatrix}$ has full rank and if p_λ (resp. p) satisfies*

$$\forall k \in I^c, \quad (A^*p_\lambda)_k + \frac{h}{2}|(B^*p_\lambda)_k| < 1, \quad (26)$$

$$\forall i \in I^{(l)} \setminus I^{(r)}, \quad ((A^* + \frac{h}{2}B^*)p_\lambda)_i < 1, \quad (27)$$

$$\forall i \in I^{(r)} \setminus I^{(l)}, \quad ((A^* - \frac{h}{2}B^*)p_\lambda)_i < 1, \quad (28)$$

then (a_λ, b_λ) (resp. (a_0, b_0)) is the unique solution to $(\mathcal{Q}_\lambda(y))$ (resp. $(\mathcal{Q}_0(y_0))$).

One may interpret the optimality conditions of Proposition 1 as the primal-dual relations between the solutions of $(\mathcal{Q}_\lambda(y))$ and $(\mathcal{Q}_0(y_0))$ with the solutions of their respective dual problems

$$\inf_{p \in D} \left\| \frac{y}{\lambda} - p \right\|_{\mathcal{H}}^2 \quad (\mathcal{E}_\lambda(y))$$

$$\sup_{p \in D} \langle y, p \rangle \quad (\mathcal{E}_0(y))$$

$$\text{where } D \stackrel{\text{def.}}{=} \left\{ p \in \mathcal{H} ; \max_{k \in [0, G-1]} (A^*p)_k + \frac{h}{2}|(B^*p)_k| \leq 1 \right\}. \quad (29)$$

Conversely, $p \in \mathcal{H}$ is a solution to $(\mathcal{E}_\lambda(y))$ (resp. $(\mathcal{E}_0(y))$) if and only if $p \in D$ and p satisfies the conditions of Proposition 1.

2.4. Low noise behavior of C-BP

The theorem of Fuchs [22] for the LASSO describes an almost necessary and sufficient condition for the support stability of the problem at low noise. Its adaptation to the positive LASSO is straightforward. However, the criterion that it provides is not satisfied in general, and the support at low noise is strictly larger than $(I^{(\mathbf{r})}(a_0, b_0), I^{(1)}(a_0, b_0))$. We provide below a finer description of that support by studying the minimal norm certificate.

Definition 1 (Minimal norm certificate). *Let $(a_0, b_0) \in \mathcal{C}_h^G$. Its minimal norm certificate is $\bar{\eta}_0 \stackrel{\text{def.}}{=} \begin{pmatrix} A^* + \frac{h}{2}B^* \\ A^* - \frac{h}{2}B^* \end{pmatrix} p_0$ where $p_0 \in \mathcal{H}$ is the solution to $(\mathcal{E}_0(y))$ with minimal norm.*

The extended support is $\text{ext}^{(\mathbf{r})}(a_0, b_0) = (\text{ext}^{(\mathbf{r})}(a_0, b_0), \text{ext}^{(1)}(a_0, b_0))$, where

$$\text{ext}^{(\mathbf{r})}(a_0, b_0) = \left\{ j \in \llbracket 0, G-1 \rrbracket ; ((A^* + \frac{h}{2}B^*)p_0)_j = 1 \right\}, \quad (30)$$

$$\text{ext}^{(1)}(a_0, b_0) = \left\{ j \in \llbracket 0, G-1 \rrbracket ; ((A^* - \frac{h}{2}B^*)p_0)_j = 1 \right\}. \quad (31)$$

From the optimality conditions, if (a_0, b_0) is a solution of $(\mathcal{Q}_0(y_0))$ then $I^{(\mathbf{r})} \subset \text{ext}^{(\mathbf{r})}(a_0, b_0)$ and $I^{(1)} \subset \text{ext}^{(1)}(a_0, b_0)$ (where $I = I(a_0, b_0)$), and p_0 can be characterized as

$$p_0 = \underset{q \in \mathcal{H}}{\text{argmin}} \left\{ \|q\| ; \begin{pmatrix} A^* + \frac{h}{2}B^* \\ A^* - \frac{h}{2}B^* \end{pmatrix} q \in \partial J(r_0, l_0) \right\}. \quad (32)$$

We are now in position to describe the behavior of $(\mathcal{Q}_\lambda(y))$ at low noise in the generic case. Some details about the proofs of the following Theorem can be found in [Appendix B](#).

Theorem 1. *Let $(a_0, b_0) \in \mathcal{C}_h^G \setminus \{0\}$ be an identifiable signal, $(J^{(\mathbf{r})}, J^{(1)}) \stackrel{\text{def.}}{=} \text{ext}^{(\mathbf{r})}(a_0, b_0)$ such that $\hat{\mathcal{A}}_h \stackrel{\text{def.}}{=} \begin{pmatrix} (A + \frac{h}{2}B)_{J^{(\mathbf{r})}} & (A - \frac{h}{2}B)_{J^{(1)}} \end{pmatrix}$ has full rank. Let*

$$\begin{pmatrix} u_{J^{(\mathbf{r})}} \\ v_{J^{(1)}} \end{pmatrix} \stackrel{\text{def.}}{=} -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} s \quad \text{where } s \stackrel{\text{def.}}{=} (1, \dots, 1)^* \in \mathbb{R}^{|J^{(\mathbf{r})}| + |J^{(1)}|},$$

and assume that the following non-degeneracy condition holds

$$\forall j \in J^{(\mathbf{r})} \setminus I^{(\mathbf{r})}, u_j > 0, \quad \text{and} \quad \forall j \in J^{(1)} \setminus I^{(1)}, v_j > 0. \quad (33)$$

Then, there exists constants $C^{(1)} > 0$, $C^{(2)} > 0$ such that for

$$\lambda \leq C^{(1)} \min \left(\left\{ b_{0,i} + \frac{2}{h}a_{0,i} ; i \in I^{(\mathbf{r})}(a_0, b_0) \right\} \cup \left\{ b_{0,i} - \frac{2}{h}a_{0,i} ; i \in I^{(1)}(a_0, b_0) \right\} \right) \quad (34)$$

and $\|w\|_{\mathcal{H}} \leq C^{(2)}\lambda$, the solution (a_λ, b_λ) to $(\mathcal{Q}_\lambda(y))$ is unique, $I^{(\mathbf{r})}(a_\lambda, b_\lambda) = J^{(\mathbf{r})}$, $I^{(1)}(a_\lambda, b_\lambda) = J^{(1)}$, and it reads

$$\begin{pmatrix} a_\lambda \\ b_\lambda \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + H_h \hat{\mathcal{A}}_h^\dagger w - \lambda H_h (\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} s,$$

where H_h is defined in (20).

3. Continuous-Basis Pursuit on thin grids

Now, we turn to the “continuous” inverse problem described in the introduction, and we assume that each α_ν ($1 \leq \nu \leq N$) is positive. We aim at recovering m_0 using the Continuous Basis-Pursuit (C-BP) proposed in [19]. Given a grid \mathcal{G}_n , the goal is to reconstruct a measure $m = \sum_{i=0}^{G_n-1} a_i \delta_{ih_n+t_i}$ where $t_i \in [-\frac{h_n}{2}, \frac{h_n}{2}]$ which estimates m_0 . Applying a Taylor expansion to φ and setting $b_i = t_i a_i$, the authors of [19] are led to solve

$$\min_{(a,b) \in \mathcal{C}_{h_n}^{G_n}} \frac{1}{2} \|y - \Phi_{\mathcal{G}_n} a - \Phi'_{\mathcal{G}_n} b\|_{\mathcal{H}}^2 + \lambda \|a\|_1 \quad (\mathcal{Q}_\lambda^n(y))$$

$$\min_{(a,b) \in \mathcal{C}_{h_n}^{G_n}} \|a\|_1 \text{ such that } \Phi_{\mathcal{G}_n} a + \Phi'_{\mathcal{G}_n} b = y_0. \quad (\mathcal{Q}_0^n(y_0))$$

which are particular instances of $(\mathcal{Q}_\lambda(y))$ and $(\mathcal{Q}_0(y_0))$, with the choice $(A, B) = (\Phi_{\mathcal{G}_n}, \Phi'_{\mathcal{G}_n})$. The dual problems are respectively:

$$\inf_{q \in D^n} \left\| \frac{y}{\lambda} - q \right\|_{\mathcal{H}}^2 \quad (\mathcal{E}_\lambda^n(y))$$

$$\sup_{q \in D^n} \langle y_0, q \rangle \quad (\mathcal{E}_0^n(y_0))$$

$$\text{where } D^n \stackrel{\text{def.}}{=} \left\{ q \in \mathcal{H} ; \max_{k \in [0, G_n-1]} (\Phi^* q)(kh_n) + \frac{h_n}{2} |(\Phi^* q)'(kh_n)| \leq 1 \right\}, \quad (35)$$

To study the behavior of the solutions to these problems as n increases, we shall apply the results of the previous section, and in particular Lemma 3.

3.1. The Positive Beurling LASSO

As we explain below, a natural limit model of C-BP on thin grids is the *positive Beurling LASSO*,

$$\min_{m \in \mathcal{M}^+(\mathbb{T})} \frac{1}{2} \|y - \Phi m\|_{\mathcal{H}}^2 + \lambda m(\mathbb{T}), \quad (\mathcal{Q}_\lambda^\infty(y))$$

$$\text{and } \min_{m \in \mathcal{M}^+(\mathbb{T})} m(\mathbb{T}) \text{ such that } \Phi m = y_0, \quad (\mathcal{Q}_0^\infty(y_0))$$

where $\mathcal{M}^+(\mathbb{T})$ refers to the space of positive Radon measures. The indicator function of positive measures plus the total mass may be encoded in the quantity:

$$m(\mathbb{T}) + \iota_{\mathcal{M}^+(\mathbb{T})}(m) = \sup \left\{ \int_{\mathbb{T}} \psi(t) dm(t) ; \psi \in \mathcal{C}(\mathbb{T}) \text{ and } \sup_{t \in \mathbb{T}} \psi(t) \leq 1 \right\}. \quad (36)$$

The characterization of optimality, the notions of minimal norm certificates and extended support are straightforward adaptations of those of the Beurling LASSO exposed in the companion paper [17, Section 3]. Again, it essentially amounts to replacing

condition $\|\eta\|_\infty \leq 1$ with $\sup_{t \in \mathbb{T}} \eta(t) \leq 1$ where $\eta = \Phi^* p$ for $p \in \mathcal{H}$. For instance, up to the addition of a constant, the dual problems to $(\mathcal{Q}_\lambda^\infty(y))$ and $(\mathcal{Q}_0^\infty(y_0))$ are respectively:

$$\begin{aligned} \inf_{p \in D^\infty} \left\| \frac{y}{\lambda} - p \right\|_{\mathcal{H}}^2 & \quad (\mathcal{E}_\lambda^\infty(y)) \\ \sup_{p \in D^\infty} \langle y_0, p \rangle & \quad (\mathcal{E}_0^\infty(y_0)) \end{aligned}$$

$$\text{where } D^\infty \stackrel{\text{def.}}{=} \left\{ p \in \mathcal{H} ; \sup_{t \in \mathbb{T}} (\Phi^* p)(t) \leq 1 \right\}. \quad (37)$$

As for the Beurling LASSO, the low noise behavior of $(\mathcal{Q}_\lambda^\infty(y))$ is governed by the minimal norm solution of $(\mathcal{E}_0^\infty(y_0))$ (see [17, Section 3]). That solution being difficult to compute in general, one is led to study a “good candidate” for it, the vanishing derivatives precertificate, which can easily be computed by solving a linear system in the least square sense. In this paper, we are not directly interested in the low noise behavior of $(\mathcal{Q}_\lambda^\infty(y))$ but we shall use this precertificate as an auxiliary quantity, hence we may adopt the following definition.

Definition 2. Let $m_0 = \sum_{\nu=1}^N \alpha_{0,\nu} \delta_{x_{0,\nu}}$ an identifiable measure for the problem $(\mathcal{Q}_0^\infty(y_0))$ such that $\Gamma_{x_0} \stackrel{\text{def.}}{=} \begin{pmatrix} \Phi_{x_0} & \Phi'_{x_0} \end{pmatrix}$ has full rank.

The vanishing derivatives precertificate is defined as

$$\eta_{V,\infty} \stackrel{\text{def.}}{=} \Phi^* p_{V,\infty} \quad \text{where} \quad p_{V,\infty} \stackrel{\text{def.}}{=} \Gamma_{x_0}^{+,*} \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad (38)$$

where $s = (1, \dots, 1)^T \in \mathbb{R}^N$ and $\Gamma_{x_0}^{+,*} = \Gamma_{x_0} (\Gamma_{x_0}^* \Gamma_{x_0})^{-1}$.

3.2. The Limit Problem for Thin Grids

Let us recall that we obtain a measure from the vector $(a, b) \in \mathcal{C}_{h_n}^{G_n}$ by setting

$$m = \sum_{i=0}^{G_n-1} a_i \delta_{ih_n + b_i/a_i} \quad (39)$$

with the convention that $b_i/a_i = 0$ if $a_i = 0$. It should be noticed that $b_i/a_i \in [-\frac{h_n}{2}, \frac{h_n}{2}]$.

In this section we prove the Γ -convergence of $(\mathcal{Q}_\lambda^n(y))$ towards $(\mathcal{Q}_\lambda^\infty(y))$. Since all the solutions to $(\mathcal{Q}_\lambda^n(y))$ belong to $X_+ \stackrel{\text{def.}}{=} \{m \in \mathcal{M}^+(\mathbb{T}) ; \lambda |m|(\mathbb{T}) \leq \frac{1}{2} \|y\|_{\mathcal{H}}^2\}$, we may restrict the problems to X_+ , which is metrizable for the weak* topology. Hence, we may use the following formulation of Γ -convergence valid in metric spaces (see [10]).

Definition 3. We say that the Problem $(\mathcal{Q}_\lambda^n(y))$ Γ -converges towards Problem $(\mathcal{Q}_\lambda^\infty(y))$ if, for all $m \in X_+$, the following conditions hold

- (Liminf inequality) for any sequence of measures $(m_n)_{n \in \mathbb{N}} \in X_+^{\mathbb{N}}$ of the form (39) with $(a^{(n)}, b^{(n)}) \in \mathcal{C}_{h_n}^{G_n}$ such that m_n weakly* converges towards m ,

$$\liminf_{n \rightarrow +\infty} \left(\lambda \|a^{(n)}\|_1 + \frac{1}{2} \|\Phi_{\mathcal{G}_n} a^{(n)} + \Phi'_{\mathcal{G}_n} b^{(n)} - y\|_{\mathcal{H}}^2 \right) \geq \lambda m(\mathbb{T}) + \frac{1}{2} \|\Phi m - y\|_{\mathcal{H}}^2.$$

- (Limsup inequality) there exists a sequence of measures $(m_n)_{n \in \mathbb{N}} \in X_+^{\mathbb{N}}$ of the form (39) with $(a^{(n)}, b^{(n)}) \in \mathcal{C}_{h_n}^{G_n}$ such that m_n weakly* converges towards m and

$$\limsup_{n \rightarrow +\infty} \left(\lambda \|a^{(n)}\|_1 + \frac{1}{2} \|\Phi_{\mathcal{G}_n} a^{(n)} + \Phi'_{\mathcal{G}_n} b^{(n)} - y\|_{\mathcal{H}}^2 \right) \leq \lambda m(\mathbb{T}) + \frac{1}{2} \|\Phi m - y\|_{\mathcal{H}}^2.$$

The following proposition, which is proved in [Appendix C.1](#), states the Γ -convergence of the model and its consequences.

Proposition 3. *The Problem $(\mathcal{Q}_\lambda^n(y))$ Γ -converges towards $(\mathcal{Q}_\lambda^\infty(y))$, and*

$$\lim_{n \rightarrow +\infty} (\inf (\mathcal{Q}_\lambda^n(y))) = \inf (\mathcal{Q}_\lambda^\infty(y)). \quad (40)$$

Each sequence $(m_{\lambda,n})_{n \in \mathbb{N}}$ such that $m_{\lambda,n}$ is a minimizer of $(\mathcal{Q}_\lambda^n(y))$ has accumulation points (for the weak* topology, and each of these accumulation points is a minimizer of $(\mathcal{Q}_\lambda^\infty(y))$).

In particular, if the solution $m_{\lambda,\infty}$ to $(\mathcal{Q}_\lambda^\infty(y))$ is unique, the whole sequence $(m_{\lambda,n})_{n \in \mathbb{N}}$ converges towards $m_{\lambda,\infty}$.

4. Convergence of the support

Though Proposition 3 states the convergence of the solutions of $(\mathcal{Q}_\lambda^n(y))$ towards those of $(\mathcal{Q}_\lambda^\infty(y))$, it does not describe the supports of the solutions. We now study the convergence of those supports using dual certificates and the optimality conditions (Proposition 1). In this continuous context, a dual certificate is determined by a function $\eta = \Phi^* p \in \mathcal{C}(\mathbb{T})$ where $q \in \mathcal{H}$, and if (a, b) is a solution to $(\mathcal{Q}_\lambda^n(y))$,

$$I^{(r)} = \left\{ i \in \llbracket 0, G_n - 1 \rrbracket ; b_i > -\frac{h_n}{2} a_i \right\} \subset \left\{ i \in \llbracket 0, G_n - 1 \rrbracket ; \left(\eta + \frac{h_n}{2} \eta' \right) (i h_n) = 1 \right\},$$

$$I^{(l)} = \left\{ i \in \llbracket 0, G_n - 1 \rrbracket ; b_i < \frac{h_n}{2} a_i \right\} \subset \left\{ i \in \llbracket 0, G_n - 1 \rrbracket ; \left(\eta - \frac{h_n}{2} \eta' \right) (i h_n) = 1 \right\}.$$

To sum up, we shall exploit the following observations

- if $(\eta + \frac{h_n}{2} \eta') (i h_n) = 1$ but $(\eta - \frac{h_n}{2} \eta') (i h_n) < 1$, a spike may appear at $i h_n + \frac{h_n}{2}$,
- if $(\eta - \frac{h_n}{2} \eta') (i h_n) = 1$ but $(\eta + \frac{h_n}{2} \eta') (i h_n) < 1$, a spike may appear at $i h_n - \frac{h_n}{2}$,
- if $(\eta + \frac{h_n}{2} \eta') (i h_n) = 1$ and $(\eta - \frac{h_n}{2} \eta') (i h_n) = 1$, a spike may appear anywhere in the interval $[i h_n - \frac{h_n}{2}, i h_n + \frac{h_n}{2}]$.

In the next two paragraphs, we describe the behavior of the support. Those results rely on auxiliary Lemmas in [Appendix D](#).

4.1. Asymptotics of the Support for Fixed $\lambda > 0$

The following proposition relies on the convergence of the dual certificates (see Lemma 6 in Appendix). It states that in the generic case, one may observe up to two pairs of spikes for each spike of the solution of the positive Beurling-lasso. As before, r is chosen such that $0 < r < \frac{1}{2} \min_{\nu \neq \nu'} |x_\nu - x_{\nu'}|$.

Proposition 4. *Let $\lambda > 0$, and assume that there exists a solution $m_{\lambda,\infty}$ to $(\mathcal{Q}_\lambda^\infty(y))$ which is a sum of a finite number of Dirac masses, $m_{\lambda,\infty} = \sum_{\nu=1}^N \alpha_\nu \delta_{x_\nu}$ where $\alpha_\nu > 0$. Assume that $\eta_{\lambda,\infty}$ satisfies $|\eta_{\lambda,\infty}(t)| < 1$ for all $t \in \mathbb{T} \setminus \{x_1, \dots, x_N\}$.*

Then any sequence of solution $m_{\lambda,n} = \sum_{i=0}^{G_n-1} a_{\lambda,i} \delta_{ih_n + b_{\lambda,i}/a_{\lambda,i}}$ to $(\mathcal{Q}_\lambda^n(y))$ satisfies

$$\limsup_{n \rightarrow +\infty} (\text{supp } m_{\lambda,n}) \subset \{x_1, \dots, x_N\}.$$

If, moreover, $m_{\lambda,\infty}$ is the unique solution to $(\mathcal{Q}_\lambda^\infty(y))$,

$$\lim_{n \rightarrow +\infty} (\text{supp}(m_{\lambda,n})) = \{x_1, \dots, x_N\}. \tag{41}$$

If, additionally, $(\eta_{\lambda,\infty})''(x_\nu) \neq 0$ for some $\nu \in \{1, \dots, N\}$, then for all n large enough, the restriction of $m_{\lambda,n}$ to $(x_\nu - r, x_\nu + r)$ is a sum of Dirac masses whose configuration is given in Table 1, and if $(\eta_{\lambda,\infty})^{(3)}(x_\nu) \neq 0$, then only the cases indicated with () may appear.*

Remark 2. Proposition 4 states that the support of the C-BP on thin grids actually depends on the properties of the dual certificate $\eta_{\lambda,\infty}$ of the (positive) Beurling LASSO. The condition $\eta_{\lambda,\infty}''(x_\nu) \neq 0$ seems to be overwhelming, or generic, and it is ensured for instance if λ is small and the Non-Degenerate Source Condition holds (see [16]). As for the condition $\eta_{\lambda,\infty}^{(3)}(x_\nu) \neq 0$, it also seems to be generic, as there is nothing to impose $\eta_{\lambda,\infty}^{(3)}(x_\nu) = 0$ in the positive Beurling LASSO. As a result, in practice, *one does not observe all the configurations given in Table 1, and only the cases indicated with (*) appear*, the case of two spikes being again overwhelming.

This means that when approximating the positive Beurling LASSO with the Continuous Basis-Pursuit, one generally sees two spikes instead of one, and those spikes are at successive half-grid points: $(ih + \frac{h}{2}, (i+1)h + \frac{h}{2})$ or $(ih - \frac{h}{2}, (i+1)h - \frac{h}{2})$.

4.2. Asymptotic of the Low Noise Support

Now, we focus on the low noise behavior of the model. This analysis is more difficult in whole generality, since it involves the minimal norm solutions of linear problems, for which it is non-trivial to pass to the limit. Therefore, we are led to assume that $\{x_{0,1}, \dots, x_{0,N}\} \subset \mathcal{G}_n$ for n large enough, and the measure now reads $m_0 = \sum_{i=0}^{G_n-1} a_{0,i} \delta_{ih_n}$.

Even so, the minimal norm solutions of $(\mathcal{E}_0^n(y_0))$ do not converge towards the minimal norm solution of $(\mathcal{E}_0^\infty(y_0))$, and we introduce a new variational problem to carry the study further.

Number of Dirac masses	Saturations of the certificates ($S_{n,\nu}^{(r)}/S_{n,\nu}^{(1)}$)	Possible Dirac Locations
One	$\{ih_n\}/\emptyset$ or $\emptyset/\{ih_n\}$ $\{ih_n\}/\{ih_n\}$	$ih_n + \varepsilon_n \frac{h_n}{2}$, with $\varepsilon_n \in \{-1, 1\}$ (*) $ih_n + t_i$, with $-\frac{h_n}{2} \leq t_i \leq \frac{h_n}{2}$
Two	$\{(i-1)h_n, ih_n\}/\emptyset$ or $\emptyset/\{ih_n, (i+1)h_n\}$ $\{ih_n\}/\{jh_n\}$ $\{ih_n\}/\{jh_n, (i+1)h_n\}$ or $\{(i-1)h_n, ih_n\}/\{ih_n\}$	$\left((i-\varepsilon_n)h_n + \varepsilon_n \frac{h_n}{2}, ih_n + \varepsilon_n \frac{h_n}{2}\right)$, with $\varepsilon_n \in \{-1, 1\}$ (*) $\left(ih_n + \frac{h_n}{2}, jh_n - \frac{h_n}{2}\right)$, $i < j$ $\left((i-\varepsilon_n)h_n + \varepsilon_n \frac{h_n}{2}, ih_n + t_i\right)$, $\varepsilon_n \in \{-1, 1\}$, $-\frac{h_n}{2} \leq t_i \leq \frac{h_n}{2}$
Three	$\{ih_n\}/\{jh_n, (j+1)h_n\}$ $\{(i-1)h_n, ih_n\}/\{jh_n\}$ $\{(i-1)h_n, ih_n\}/\{ih_n, (i+1)h_n\}$	$\left(ih_n + \frac{h_n}{2}, jh_n - \frac{h_n}{2}, (j+1)h_n - \frac{h_n}{2}\right)$, with $i < j$ $\left((i-1)h_n + \frac{h_n}{2}, ih_n + \frac{h_n}{2}, jh_n - \frac{h_n}{2}\right)$, with $i < j$ $\left((i-1)h_n + \frac{h_n}{2}, ih_n + t_i, (i+1)h_n - \frac{h_n}{2}\right)$, $-\frac{h_n}{2} \leq t_i \leq \frac{h_n}{2}$
Four	$\{(i-1)h_n, ih_n\}/\{jh_n, (j+1)h_n\}$	$\left((i-1)h_n + \frac{h_n}{2}, ih_n + \frac{h_n}{2}, jh_n - \frac{h_n}{2}, (j+1)h_n - \frac{h_n}{2}\right)$, $i < j$

Table 1: Number of Dirac masses that may appear if $\eta''_{\lambda,\infty}(x_\nu) \neq 0$. For the sake of the simplicity of the table, and since we focus on the saturations of dual certificates, we regard sums like $\delta_{ih_n+h_n/2} + \delta_{(i+1)h_n-h_n/2}$ as “two” Dirac masses.

Definition 4 (Third derivative precertificate). *Given $m_0 \in \mathcal{M}(\mathbb{T})$, we define the third derivative precertificate as $\eta_T \stackrel{\text{def.}}{=} \Phi^* p_T$ where*

$$p_T \stackrel{\text{def.}}{=} \underset{p \in \mathcal{H}}{\operatorname{argmin}} \left\{ \|p\|_{\mathcal{H}}; \forall i \in \{1, \dots, N\}, (\Phi^* p)(x_{0,i}) = 1, \right. \\ \left. (\Phi^* p)'(x_{0,i}) = 0 \text{ and } (\Phi^* p)^{(3)}(x_{0,i}) = 0 \right\}, \quad (42)$$

whenever the above set is not empty.

It is clear that the set defined in (42) is a closed convex set. It is nonempty for instance if the conditions of Lemma 7 hold. Note that p_T corresponds to a quadratic minimization under linear constraint, and can hence be computed by solving a linear system,

$$p_T = \begin{pmatrix} \Gamma_{x_0}^* \\ \Phi_{x_0}^{(3)*} \end{pmatrix}^+ \begin{pmatrix} \mathbf{1}_N \\ 0 \\ 0 \end{pmatrix} = \Gamma_{x_0}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} - \tilde{\Pi} \Phi_{x_0}^{(3)*} (\Phi_{x_0}^{(3)*} \tilde{\Pi} \Phi_{x_0}^{(3)*})^{-1} \Phi_{x_0}^{(3)*} \Gamma_{x_0}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} \quad (43)$$

$$= p_{V,\infty} - \tilde{\Pi} \Phi_{x_0}^{(3)*} (\Phi_{x_0}^{(3)*} \tilde{\Pi} \Phi_{x_0}^{(3)*})^{-1} \Phi_{x_0}^{(3)*} p_{V,\infty}, \quad (44)$$

where $\tilde{\Pi}$ is the orthogonal projector onto $(\text{Im } \Gamma_{x_0})^\perp$, and $\Gamma_{x_0} = \begin{pmatrix} \Phi_{x_0} & \Phi'_{x_0} \end{pmatrix}$.

Definition 5 (Twice Non-Degenerate Source Condition). *We say that m_0 satisfies the Twice Non-Degenerate Source Condition (TNDSC) if p_T in (42) is well defined and if it satisfies, for $\eta_T = \Phi^* p_T$,*

$$\begin{aligned} \forall t \in \mathbb{T} \setminus \{x_{0,1}, \dots, x_{0,N}\}, \quad \eta_T(t) < 1, \\ \forall \nu \in \{1, \dots, N\}, \quad \eta_T''(x_{0,\nu}) < 0 \quad \text{and} \quad \eta_T^{(4)}(x_{0,\nu}) > 0. \end{aligned}$$

We are now in position to provide a sufficient condition for the spikes to appear in pair, with a prediction on the location of the neighbor. The proof of this Theorem can be found in [Appendix D.3](#).

Theorem 2. *Assume that the operator $\begin{pmatrix} \Phi_{x_0} & \Phi'_{x_0} & \Phi_{x_0}^{(3)} \end{pmatrix}$ has full rank and that the Twice Non Degenerate Source condition (Definiton 5) holds. Moreover, assume that all the components of the natural shift*

$$\rho \stackrel{\text{def.}}{=} (\Phi_{x_0}^{(3)*} \tilde{\Pi} \Phi_{x_0}^{(3)})^{-1} \Phi_{x_0}^{(3)*} \Gamma_{x_0}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} \quad (45)$$

are nonzero. Then, for n large enough the extended support of m_0 on the grid \mathcal{G}_n has the form

$$\begin{aligned} \text{ext}_n^{(\mathbf{r})}(m_0) &= \{x_{0,1}, \dots, x_{0,N}\} \cup \{x_{0,\nu} - h_n; \nu \in \llbracket 1, N \rrbracket \quad \text{and} \quad \rho_\nu > 0\} \\ \text{ext}_n^{(\mathbf{l})}(m_0) &= \{x_{0,1}, \dots, x_{0,N}\} \cup \{x_{0,\nu} + h_n; \nu \in \llbracket 1, N \rrbracket \quad \text{and} \quad \rho_\nu < 0\}. \end{aligned}$$

Combining the above Theorem with Theorem 1, we get

Corollary 1. *Under the hypotheses of Theorem 2, for n large enough, there exists constants $C_n^{(1)} > 0$, $C_n^{(2)} > 0$ such that for $\lambda \leq C_n^{(1)} \min_{1 \leq \nu \leq N} |\alpha_{0,\nu}|$, and for all $w \in \mathcal{H}$ such that $\|w\|_{\mathcal{H}} \leq C_n^{(2)} \lambda$, the solution to $(\mathcal{Q}_\lambda^n(y))$ is unique, and reads $m_\lambda = \sum_{\nu=1}^N (\alpha_{\lambda,\nu} \delta_{x_{0,\nu}+t_\nu} + \beta_{\lambda,\nu} \delta_{x_{0,\nu}+\varepsilon_\nu h_n})$, where*

$$-h_n/2 < t_\nu < h_n/2 \quad \text{and} \quad \varepsilon \stackrel{\text{def.}}{=} -\text{sign}(\rho).$$

Moreover, the constants $C_n^{(1)}, C_n^{(2)}$ can be chosen as $C_n^{(1)} = O(h_n^3)$ and $C_n^{(2)} = O(1)$, and one has

$$\left\| \begin{pmatrix} \alpha_\lambda \\ \beta_\lambda \end{pmatrix} - \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix} \right\|_\infty = O\left(\frac{w}{h_n^3}, \frac{\lambda}{h_n^3}\right). \quad (46)$$

The proof is given in [Appendix D.4](#).

5. Numerical illustrations

In this section, we illustrate the relevance of our analysis to gain a precise understanding of the recovery performance of ℓ^1 -type methods (LASSO and C-BP) for deconvolution. The code to reproduce these numerical experiments is available online[‡].

[‡] <https://github.com/gpeyre/2015-IP-lasso-cbp/>

5.1. Convergence of pre-certificates

In this section and in Section 5.2, we consider the deconvolution problem, that is $\varphi(x) = \tilde{\varphi}(\cdot - x)$, in the case where $\tilde{\varphi}$ is an ideal filter, i.e. whose Fourier coefficients

$$\forall k \in \mathbb{Z}, \quad \hat{\tilde{\varphi}}(k) \stackrel{\text{def.}}{=} \int_{\mathbb{T}} \tilde{\varphi}(t) e^{-2i\pi kt} dt$$

satisfy $\hat{\tilde{\varphi}}(k) = 1$ if $k \in \{-f_c, \dots, f_c\}$ and $\hat{\tilde{\varphi}}(k) = 0$ otherwise. This allows us to implement exactly the Φ operator appearing in the LASSO and C-BP problem since $\text{Im}(\Phi)$ is a finite dimensional space of dimension $Q = 2f_c + 1$, i.e. it can be represented using a matrix of size (Q, P) when evaluated on a grid of P points. In Figures 2 and 3 we used $f_c = 10$.

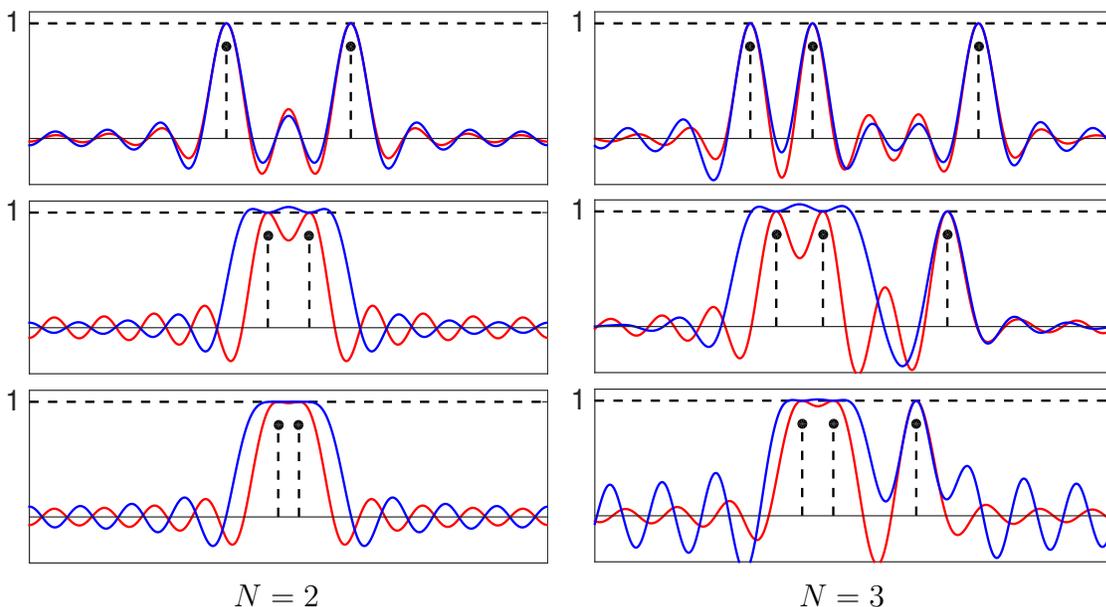


Figure 2: Display of $\eta_{V,\infty}$ (red) and η_T (blue) pre-certificate for different input positive measures m_0 (showed as black dots to symbolize the position of the Diracs).

Figure 2 illustrates for the case of two ($N = 2$) and three ($N = 3$) spikes the behavior of the vanishing pre-certificate $\eta_{V,\infty}$ (see Definition 2) useful to analyze LASSO/BLASSO problems and of the pre-certificate η_T (see Definition 4) useful to analyze C-BP problems.

We first notice that for all the (positive) input measures (i.e. whatever the spacing between the Diracs), $\eta_{V,\infty}$ is always a non-degenerate certificate (in the sense of Proposition 2), meaning that one actually has $\eta_{V,\infty} = \eta_{0,\infty}$ (where the minimal norm certificate $\eta_{0,\infty}$ is the minimal norm solution of $(\mathcal{D}_0^\infty(y_0))$ in [17]). This empirical finding is the subject of another recent work on the asymptotic of sparse recovery of positive measures when the spacing between the Diracs tends to zero [12]. Since $\eta_{0,\infty}$ is non-degenerate, one can thus apply [17, Theorem 2] to analyze the extended support of the LASSO on a thin grid (see Section 5.2 below for a numerical illustration).

For the C-BP problem, the situation is however more contrasted. We observe that when the Dirac masses are separated enough (first row) then the pre-certificate μ_T is a valid certificate, meaning the the Twice Non-Degenerate Source Condition (see Definition 5) holds. This means that Theorem 2 can be applied to analyze the extended support of C-BP on a thin grid (see Section 5.2 below for a numerical illustration). But when the Dirac masses are too close (second and third rows), one has $\|\eta_T\|_\infty > 1$, so that one cannot ensure the support stability of the C-BP solution with our result.

5.2. Extended support for deconvolution on a thin grid

We still consider the case of an ideal low pass filter. Figure 3 displays the evolution, as a function of λ (in abscissa) of the solution a_λ of the LASSO (Eq. (7)) and of the solution (a_λ, b_λ) of the C-BP ($\mathcal{Q}_\lambda^n(y)$), on a thin grid. We consider here the case of an input measure with two nearby Diracs (displayed as red/blue dots in the upper-left part of the Figure) and when there is no noise, i.e. $w = 0$. Each 1-D curve (either plain or dashed) represents the evolution of a single coefficient, e.g. $(a_\lambda)_i$, for some index i (only non-zero coefficients are displayed).

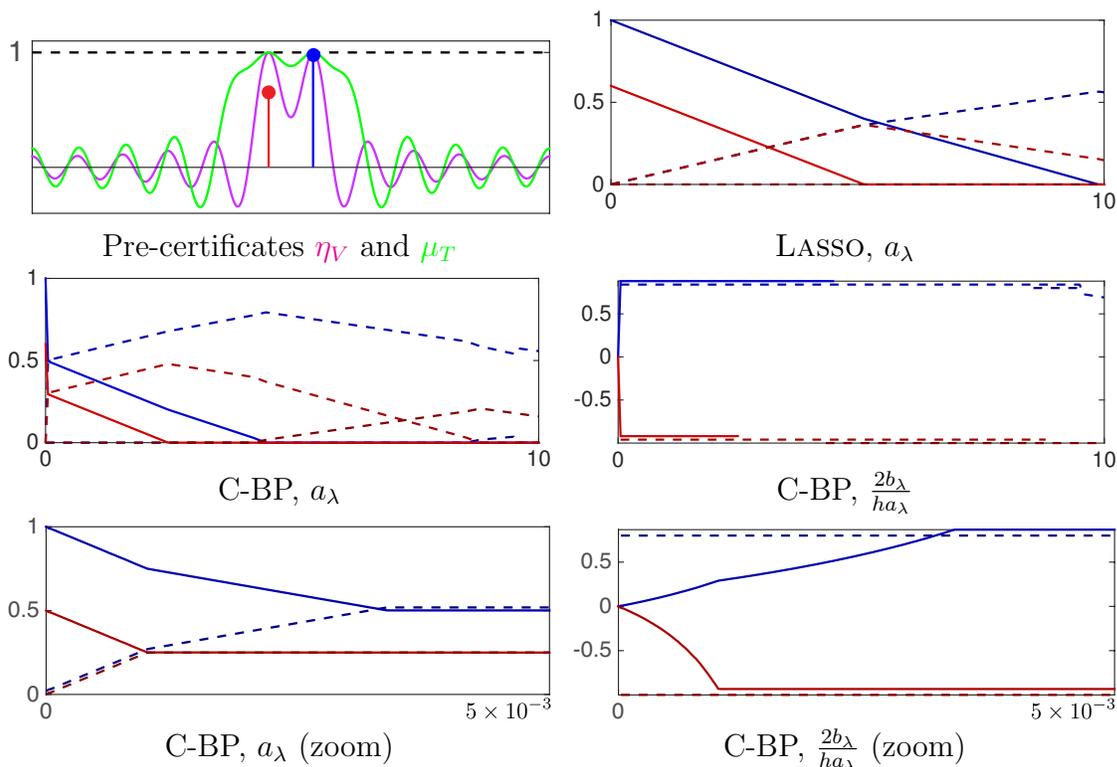


Figure 3: Display of the evolution as a function of λ of the solutions of the LASSO and C-BP problems. Note that dashed curved have been (artificially) slightly shifted to avoid that they overlap with the plain curve.

The solutions path $\lambda \mapsto a_\lambda$ (for LASSO) and $\lambda \mapsto (a_\lambda, b_\lambda)$ (for C-BP) are continuous and piecewise affine, which is to be expected since the regularizations (ℓ^1 and ℓ^1 under

conic constraints) are polyhedral. The upper-left plot in the figure displays the precertificate $\eta_{V,\infty}$ (in magenta, see Definition 2) and η_T (in green, see Definition 4). This shows graphically that these two precertificates are non-degenerate (in the sense of [17, Proposition 3] and Definition 5) so that the conclusions of [17, Theorem 2] and Theorem 2 hold, hence precisely describing the evolution of the solution on the extended support when λ is small. On these graphs, this corresponds to the first segment of the corresponding piecewise affine paths.

The behavior for BP agrees with our analysis. As predicted by [17, Corollary 1], there exists a range of values $0 < \lambda < \lambda_0$ on which the solution is exactly supported on the extended support J , which is composed of four spikes (the plain curve corresponds to the support I and the dashed curve corresponds to $J \setminus I$). Also, as predicted by [17, Proposition 7] in the case $w = 0$, we verify that $\lambda_0 = O(h_n)$ and that the Lipschitz constant of $\lambda \mapsto a_\lambda$ is of order $O(1/h_n)$.

In sharp contrast, the behavior for C-BP is less regular, since the range $0 < \lambda < \lambda_0$ on which the solution is supported on the extended support is shorter, as it can be clearly seen on the zoom for very small values of λ . This is in agreement with Proposition ?? which shows that λ_0 is of the order of $O(h_n^3)$ and that the Lipschitz constant of $\lambda \mapsto (a_\lambda, b_\lambda)$ is of order $O(1/h_n^3)$. On this range of small λ , as predicted by Theorem 2, the support of the solutions (which correspond to the extended support J described in Theorem 2) is composed of one pair of neighboring spikes for each original spike. For indices on the support $i \in I$, one has $|(b_\lambda)_i|/(a_\lambda)_i < h/2$ (the constraint is non-saturating, and the spike moves “freely” inside $(ih - \frac{h}{2}, ih + \frac{h}{2})$) while for indices on the extended part $i \in J \setminus I$, one has $|(b_\lambda)_i|/(a_\lambda)_i = h/2$ (the constraint is saturating, the spikes are fixed at half-grid points). Another part of the path is interesting, for λ not so small (say $\lambda > \lambda_1$), which is in fact the prominent regime in the non-zoomed figure. For this range of λ , there is still a pair of spikes for each original spike, but this time both spikes saturate, on same side. This observation should be related to Proposition 4 and Remark 2 which predict that, in the case where $\eta_{\lambda,\infty}^{(3)}(x_{\lambda,\nu}) \neq 0$, the C-BP yields either one spike or a pair of spikes with the same shift (the latter case is in fact overwhelming).

Conclusion

In this paper, we provided a detailed analysis of the support recovery property of the C-BP method. We showed that in general the solution path (for small noise w and small regularization parameter λ) is delocalized on pairs of spikes surrounding the sought-after spikes. This in turn allow to study the asymptotic behavior of the constant involved.

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Appendix A. Asymptotic expansion of the inverse of a Gram matrix

In this Appendix, we gather some useful lemmas on the asymptotic behavior of inverse Gram matrices. The proof of the following lemma can be found in [17].

Lemma 1. *Let $A: \mathbb{R}^N \rightarrow \mathcal{H}$, $B: \mathbb{R}^N \rightarrow \mathbb{R}^N$ be linear operators such that A has full rank and B is invertible. Then $(AB)^+ = B^{-1}A^+$.*

Lemma 2. *Let $A, B, C, C_h: \mathbb{R}^N \rightarrow \mathcal{H}$ be linear operators such that $C_h = C + o(1)$ for $h > 0$, and that $\begin{pmatrix} A & B & C \end{pmatrix}$ has full rank. Let $\tilde{\Pi}$ be the orthogonal projector onto $(\text{Im} \begin{pmatrix} A & B \end{pmatrix})^\perp$, and let*

$$G_h \stackrel{\text{def.}}{=} \begin{pmatrix} (A + \frac{h}{2}B)^* \\ (A - \frac{h}{2}B)^* \\ (A + \frac{h}{2}B + h^3C_h)^* \end{pmatrix} \begin{pmatrix} A + \frac{h}{2}B & A - \frac{h}{2}B & A + \frac{h}{2}B + h^3C_h \end{pmatrix}.$$

Then for $h > 0$ small enough, G_h and $C^*\tilde{\Pi}C$ are invertible, and

$$G_h^{-1} \begin{pmatrix} \mathbf{1}_N \\ \mathbf{1}_N \\ \mathbf{1}_N \end{pmatrix} = -\frac{1}{h^3} \begin{pmatrix} -I_N \\ 0 \\ I_N \end{pmatrix} (C^*\tilde{\Pi}C)^{-1}C^* \begin{pmatrix} A & B \end{pmatrix}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} + o\left(\frac{1}{h^3}\right)$$

$$\begin{pmatrix} A + \frac{h}{2}B & A - \frac{h}{2}B & A + \frac{h}{2}B + h^3C_h \end{pmatrix}^+ = \frac{1}{h^3} \begin{pmatrix} -(C^*\tilde{\Pi}C)^{-1}C^*\tilde{\Pi} \\ 0 \\ (C^*\tilde{\Pi}C)^{-1}C^*\tilde{\Pi} \end{pmatrix} + o\left(\frac{1}{h^3}\right),$$

but

$$\begin{pmatrix} A^* + \frac{h}{2}B^* \\ A^* - \frac{h}{2}B^* \\ A^* + \frac{h}{2}B^* + h^3C_h^* \end{pmatrix}^+ \begin{pmatrix} \mathbf{1}_N \\ \mathbf{1}_N \\ \mathbf{1}_N \end{pmatrix} = \begin{pmatrix} A^* \\ B^* \end{pmatrix}^+ \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} - \tilde{\Pi}C(C^*\tilde{\Pi}C)^{-1}C^* \begin{pmatrix} A^* \\ B^* \end{pmatrix}^+ \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} + o(1).$$

Proof. Observe that

$$\begin{pmatrix} A + \frac{h}{2}B & A - \frac{h}{2}B & A + \frac{h}{2}B + h^3C_h \end{pmatrix} = \begin{pmatrix} A & B & C_h \end{pmatrix} \text{diag} \left(1, \frac{h}{2}, h^3 \right) \begin{pmatrix} I_N & I_N & I_N \\ I_N & -I_N & I_N \\ 0 & 0 & I_N \end{pmatrix}$$

As a result, for $h > 0$ small enough G_h is invertible and

$$G_h^{-1} = \begin{pmatrix} \frac{1}{2}I_N & \frac{1}{2}I_N & -I_N \\ \frac{1}{2}I_N & -\frac{1}{2}I_N & 0 \\ 0 & 0 & I_N \end{pmatrix} \text{diag} \left(1, \frac{2}{h}, \frac{1}{h^3} \right) \begin{pmatrix} A^*A & A^*B & A^*C_h \\ B^*A & B^*B & B^*C_h \\ C_h^*A & C_h^*B & C_h^*C_h \end{pmatrix}^{-1} \\ \times \text{diag} \left(1, \frac{2}{h}, \frac{1}{h^3} \right) \begin{pmatrix} \frac{1}{2}I_N & \frac{1}{2}I_N & 0 \\ \frac{1}{2}I_N & -\frac{1}{2}I_N & 0 \\ -I_N & 0 & I_N \end{pmatrix}$$

the middle matrix being invertible from the full rank assumption on $(A \ B \ C)$.

Moreover, writing $\tilde{\Gamma} \stackrel{\text{def.}}{=} (A \ B)$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\text{def.}}{=} \begin{pmatrix} \tilde{\Gamma}^*\tilde{\Gamma} & \tilde{\Gamma}^*C_h \\ C_h^*\tilde{\Gamma} & C_h^*C_h \end{pmatrix}$, we obtain

$$\begin{pmatrix} A^*A & A^*B & A^*C_h \\ B^*A & B^*B & B^*C_h \\ C_h^*A & C_h^*B & C_h^*C_h \end{pmatrix}^{-1} = \begin{pmatrix} u & -a^{-1}bS^{-1} \\ -S^{-1}ca^{-1} & S^{-1} \end{pmatrix}$$

where $u \stackrel{\text{def.}}{=} a^{-1} + a^{-1}bS^{-1}ca^{-1}$, $S \stackrel{\text{def.}}{=} d - ca^{-1}b = C_h^*\tilde{\Pi}C_h$, $a^{-1}bS^{-1} = (\tilde{\Gamma}^*\tilde{\Gamma})^{-1}\tilde{\Gamma}^*C_h(C_h^*\tilde{\Pi}C_h)^{-1}$, $S^{-1}ca^{-1} = (C_h^*\tilde{\Pi}C_h)^{-1}C_h^*\tilde{\Gamma}(\tilde{\Gamma}^*\tilde{\Gamma})^{-1}$, and $\tilde{\Pi}$ is the orthogonal projector onto $(\text{Im } \tilde{\Gamma})^\perp$. Thus

$$G_h^{-1} \begin{pmatrix} \mathbf{1}_N \\ \mathbf{1}_N \\ \mathbf{1}_N \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I_N & \frac{1}{2}I_N & -I_N \\ \frac{1}{2}I_N & -\frac{1}{2}I_N & 0 \\ 0 & 0 & I_N \end{pmatrix} \text{diag} \left(1, \frac{2}{h}, \frac{1}{h^3} \right) \begin{pmatrix} u & -a^{-1}bS^{-1} \\ -S^{-1}ca^{-1} & S^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_N \\ 0 \\ 0 \end{pmatrix} \\ = \frac{1}{h^3} \begin{pmatrix} 0 & 0 & -I_N \\ 0 & 0 & 0 \\ 0 & 0 & I_N \end{pmatrix} \begin{pmatrix} u \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} \\ -S^{-1}ca^{-1} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} \end{pmatrix} + o\left(\frac{1}{h^3}\right) \\ = -\frac{1}{h^3} \begin{pmatrix} -I_N \\ 0 \\ I_N \end{pmatrix} (C^*\tilde{\Pi}C)^{-1}C^*\tilde{\Gamma}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} + o\left(\frac{1}{h^3}\right).$$

Eventually, one has

$$\begin{aligned}
 & \left(A + \frac{h}{2}B \quad A - \frac{h}{2}B \quad A + \frac{h}{2}B + h^3C_h \right)^+ \\
 &= \begin{pmatrix} \frac{1}{2}I_N & \frac{1}{2}I_N & -I_N \\ \frac{1}{2}I_N & -\frac{1}{2}I_N & 0 \\ 0 & 0 & I_N \end{pmatrix} \text{diag}\left(1, \frac{2}{h}, \frac{1}{h^3}\right) \left(\tilde{\Gamma} \quad C_h \right)^+ \\
 &= \frac{1}{h^3} \begin{pmatrix} 0 & 0 & -I_N \\ 0 & 0 & 0 \\ 0 & 0 & I_N \end{pmatrix} \begin{pmatrix} u & -a^{-1}bS^{-1} \\ -S^{-1}ca^{-1} & S^{-1} \end{pmatrix} \left(\tilde{\Gamma}^* \quad C_h^* \right) + o\left(\frac{1}{h^3}\right) \\
 &= \frac{1}{h^3} \begin{pmatrix} -(C^*\tilde{\Pi}C)^{-1}C^*\tilde{\Pi} \\ 0 \\ (C^*\tilde{\Pi}C)^{-1}C^*\tilde{\Pi} \end{pmatrix} + o\left(\frac{1}{h^3}\right),
 \end{aligned}$$

and

$$\begin{aligned}
 & \left(A + \frac{h}{2}B \quad A - \frac{h}{2}B \quad A + \frac{h}{2}B + h^3C_h \right)^{+,*} \begin{pmatrix} \mathbf{1}_N \\ \mathbf{1}_N \\ \mathbf{1}_N \end{pmatrix} \\
 &= \begin{pmatrix} \tilde{\Gamma}^* \\ C_h^* \end{pmatrix}^+ \text{diag}\left(1, \frac{2}{h}, \frac{1}{h^3}\right) \begin{pmatrix} \frac{1}{2}I_N & \frac{1}{2}I_N & 0 \\ \frac{1}{2}I_N & -\frac{1}{2}I_N & -I_N \\ 0 & 0 & I_N \end{pmatrix} \begin{pmatrix} \mathbf{1}_N \\ \mathbf{1}_N \\ \mathbf{1}_N \end{pmatrix} \\
 &= \left[\tilde{\Gamma}^{+,*} - \tilde{\Pi}C_h(C_h^*\tilde{\Pi}C_h)^{-1}C_h^*\tilde{\Gamma}^{+,*} \right] \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} \\
 &= \tilde{\Gamma}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} - \tilde{\Pi}C(C^*\tilde{\Pi}C)^{-1}C^*\tilde{\Gamma}^{+,*} \begin{pmatrix} \mathbf{1}_N \\ 0 \end{pmatrix} + o(1)
 \end{aligned}$$

□

Appendix B. Proofs for Section 2

The proof of Theorem 1 is similar to the proofs of Theorem 2 and Corollary 1 which are detailed below, therefore we omit it. Let us mention however that it relies on the following fundamental Lemma (which is a simplified version of [17, Lemma 4]), which characterizes $\bar{\eta}_0$.

Lemma 3. *Let $J^{(r)}, J^{(l)} \subset \llbracket 0, G-1 \rrbracket$, and $(a_0, b_0) \in \mathcal{C}_h^G$. Assume that $(I^{(r)}, I^{(l)}) \stackrel{\text{def.}}{=} (I^{(r)}(a_0, b_0), I^{(l)}(a_0, b_0))$ is such that*

$$I^{(r)} \subset J^{(r)}, \quad I^{(l)} \subset J^{(l)} \quad \text{and} \quad \hat{\mathcal{A}}_h \stackrel{\text{def.}}{=} \left((A + \frac{h}{2}B)_{J^{(r)}} \quad (A - \frac{h}{2}B)_{J^{(l)}} \right)$$

has full rank. Define $\begin{pmatrix} u_{J^{(\mathbf{r})}} \\ v_{J^{(\mathbf{l})}} \end{pmatrix} \stackrel{\text{def.}}{=} -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} s$ where $s \stackrel{\text{def.}}{=} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{|J^{(\mathbf{r})}|+|J^{(\mathbf{l})}|}$. Then

$(J^{(\mathbf{r})}, J^{(\mathbf{l})})$ is the extended support of (a_0, b_0) if and only if the following two conditions hold:

- for all $j \in J^{(\mathbf{r})} \setminus I^{(\mathbf{r})}$, $u_j \geq 0$, and for all $j \in J^{(\mathbf{l})} \setminus I^{(\mathbf{l})}$, $v_j \geq 0$.
- $\max \left[\begin{pmatrix} (A + \frac{h}{2}B)_{J^{(\mathbf{r})}^c}^* \\ (A - \frac{h}{2}B)_{J^{(\mathbf{l})}^c}^* \end{pmatrix} \cdot \hat{\mathcal{A}}_h \begin{pmatrix} u_{J^{(\mathbf{r})}} \\ v_{J^{(\mathbf{l})}} \end{pmatrix} \right] < 1$.

Moreover, in that case, the minimal norm certificate is given by

$$\bar{\eta}_0 = -\mathcal{A}_h^* (\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} s.$$

Appendix C. Proofs for Section 3

Appendix C.1. Proof of Proposition 3

We observe that our embedding of the discrete problem into the continuous one, that is $m_n = \sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n + b_i/a_i}$, yields $\|a^{(n)}\|_1 = |m_n|(\mathbb{T}) = m_n(\mathbb{T})$. For the liminf inequality, let $(m_n)_{n \in \mathbb{N}}$ be of the form (39) which weakly* converges towards m . We notice that $\Phi'_{\mathcal{G}_n} b^{(n)} = \Phi'(\sum_{i=0}^{G_n-1} b_i^{(n)} \delta_{ih_n})$, where $\Phi' : m \mapsto \int_{\mathbb{T}} \varphi'(x) dm(x)$ is continuous from $\mathcal{M}(\mathbb{T})$ to \mathcal{H} (in the strong topologies). Moreover,

$$\left| \sum_{i=0}^{G_n-1} b_i^{(n)} \delta_{ih_n} \right|(\mathbb{T}) \leq \frac{h_n}{2} \left(\sum_{i=0}^{G_n-1} a_i^{(n)} \right) \leq \frac{h_n}{2\lambda} \left(\frac{1}{2} \|y\|_{\mathcal{H}}^2 + 1 \right) \rightarrow 0,$$

so that $\Phi'_{\mathcal{G}_n} b^{(n)}$ converges strongly towards 0 in \mathcal{H} . Additionally, $\Phi_{\mathcal{G}_n} a^{(n)} = \Phi(\sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n})$ and for all $\psi \in \mathcal{C}(\mathbb{T})$,

$$\begin{aligned} \left| \left\langle \sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n + b_i/a_i} - \sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n}, \psi \right\rangle \right| &= \left| \sum_{i=0}^{G_n-1} a_i^{(n)} (\psi(ih_n + b_i/a_i) - \psi(ih_n)) \right| \\ &\leq \sum_{i=0}^{G_n-1} a_i^{(n)} \omega_\psi \left(\frac{h_n}{2} \right) \rightarrow 0 \end{aligned}$$

where $\omega_\psi : t \mapsto \sup_{|x'-x| \leq t} |\psi(x) - \psi(x')|$ is the modulus of continuity of ψ . As a result, $\sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n} - m_n \xrightarrow{*} 0$ so that $\sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n}$ weakly* converges to m . Hence, $\Phi_{\mathcal{G}_n} a^{(n)}$ weakly converges towards Φm in \mathcal{H} . To sum up, $\Phi_{\mathcal{G}_n} a^{(n)} + \Phi'_{\mathcal{G}_n} b^{(n)} - y$ weakly converges towards $\Phi m - y$ and we conclude by invoking the lower semi-continuity of both terms:

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \left(\lambda \|a^{(n)}\|_1 + \frac{1}{2} \|\Phi_{\mathcal{G}_n} a^{(n)} + \Phi'_{\mathcal{G}_n} b^{(n)} - y\|_{\mathcal{H}}^2 \right) &= \liminf_{n \rightarrow +\infty} \left(\lambda |m_n|(\mathbb{T}) + \frac{1}{2} \|\Phi_{\mathcal{G}_n} a^{(n)} + \Phi'_{\mathcal{G}_n} b^{(n)} - y\|_{\mathcal{H}}^2 \right) \\ &\geq \lambda m(\mathbb{T}) + \frac{1}{2} \|\Phi m - y\|_{\mathcal{H}}^2. \end{aligned}$$

For the limsup inequality, we build a recovery sequence m_n by choosing $a_k^{(n)} = m(\llbracket kh_n, (k+1)h_n \rrbracket)$ and $b_k^{(n)} = 0$ for all $k \in \llbracket 0, G_n - 1 \rrbracket$. Then, for all n , $\|a^{(n)}\|_1 = m(\mathbb{T})$ and

$$\begin{aligned} \left\| \Phi \left(\sum_{i=0}^{G_n-1} a_i^{(n)} \delta_{ih_n} \right) - \Phi m \right\|_{\mathcal{H}} &\leq \sum_{i=0}^{G_n-1} \left\| \int_{[ih_n, (i+1)h_n)} (\varphi(t) - \varphi(ih_n)) dm(t) \right\|_{\mathcal{H}} \\ &\leq m(\mathbb{T}) \omega_\varphi(h_n) \rightarrow 0 \end{aligned}$$

since φ is uniformly continuous on \mathbb{T} . As a result, $\Phi m_n - y$ strongly converges towards $\Phi m - y$ and the limsup inequality is proved.

Appendix D. Proofs of Section 4

The following lemma is central in our analysis. It studies sequences of dual certificates $(\eta_n)_{n \in \mathbb{N}}$. For $0 < r < \frac{1}{2} \min_{\nu \neq \nu'} |x_\nu - x_{\nu'}|$, $\nu \in \{1, \dots, N\}$, it will be useful to consider the sets:

$$\begin{aligned} S_{n,\nu}^{(\mathbf{r})}(r) &\stackrel{\text{def.}}{=} \left\{ t \in \mathcal{G}_n \cap (x_\nu - r, x_\nu + r) ; \left(\eta_n + \frac{h_n}{2} \eta_n' \right) (t) = 1 \right\}, \\ S_{n,\nu}^{(\mathbf{l})}(r) &\stackrel{\text{def.}}{=} \left\{ t \in \mathcal{G}_n \cap (x_\nu - r, x_\nu + r) ; \left(\eta_n - \frac{h_n}{2} \eta_n' \right) (t) = 1 \right\}. \end{aligned}$$

Lemma 4. *Let $(x_1, \dots, x_N) \in \mathbb{T}^N$ pairwise distinct, and let $\{\eta_n\}_{n \in \mathbb{N}} \in (\mathcal{C}^3(\mathbb{T}))^{\mathbb{N}}$ be a sequence of functions which converges uniformly towards some η_∞ (and similarly for the derivatives) such that for all $\nu \in \{1, \dots, N\}$, $\eta_\infty(x_\nu) = 1$ and for all $t \in \mathbb{T} \setminus \{x_1, \dots, x_N\}$, $\eta_\infty(t) < 1$.*

(i) *Then*

$$\limsup_{n \rightarrow +\infty} \left\{ t \in \mathcal{G}_n ; \eta_n(t) + \frac{h_n}{2} |\eta_n'(t)| = 1 \right\} \subset \{x_1, \dots, x_N\}. \quad (\text{D.1})$$

In particular for $r > 0$ small enough, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\left\{ t \in \mathcal{G}_n ; \eta_n(t) + \frac{h_n}{2} |\eta_n'(t)| = 1 \right\} = \bigcup_{\nu=1}^N (S_{n,\nu}^{(\mathbf{r})}(r) \cup S_{n,\nu}^{(\mathbf{l})}(r)) \subset \bigcup_{\nu=1}^N (x_\nu - r, x_\nu + r).$$

Assume moreover that for all $n \in \mathbb{N}$ and all $t \in \mathcal{G}_n$, $\eta_n(t) + \frac{h_n}{2} |\eta_n'(t)| \leq 1$. For each $\nu \in \{1, \dots, N\}$:

(ii) *If $\eta_\infty''(x_\nu) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, each set $S_{n,\nu}^{(\mathbf{r})}(r)$ and $S_{n,\nu}^{(\mathbf{l})}(r)$ is of the form \emptyset , $\{ih_n\}$, or $\{ih_n, (i+1)h_n\}$, and if both sets are nonempty:*

$$\max S_{n,\nu}^{(\mathbf{r})}(r) \leq \min S_{n,\nu}^{(\mathbf{l})}(r).$$

(iii) *If $\eta_\infty^{(3)}(x_\nu) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $S_{n,\nu}^{(\mathbf{r})}(r) = \emptyset$ or $S_{n,\nu}^{(\mathbf{l})}(r) = \emptyset$.*

(iv) If $\eta_\infty^{(4)}(x_\nu) \neq 0$, the set of $n \in \mathbb{N}$ such that $S_{n,\nu}^{(r)} = \{(i_n - 1)h_n, i_n h_n\}$ and $S_{n,\nu}^{(1)} = \{i_n h_n, (i_n + 1)h_n\}$ (with the same $i_n \in \llbracket 0, G_n - 1 \rrbracket$) is finite.

Proof. (i) For all $\tilde{r} \in (0, r)$, by compactness, $\sup \{\eta_\infty(t) ; t \in \mathbb{T} \setminus \bigcup (x_\nu - \tilde{r}, x_\nu + \tilde{r})\} < 1$. Thus by uniform convergence there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $(\eta_n \pm \frac{h_n}{2}\eta_n') < 1$ on $\mathbb{T} \setminus \bigcup_{\nu=1}^N (x_\nu - \tilde{r}, x_\nu + \tilde{r})$, and the first claim is proved.

(ii) If moreover $\eta_\infty''(x_\nu) \neq 0$, it is in fact negative. Choosing $\tilde{r} \in (0, r)$ small enough and then n large enough, we may assume that $\eta_n'' < -k_0$ in $(x_\nu - \tilde{r}, x_\nu + \tilde{r})$, for some $k_0 > 0$, and by (D.1) that $S_{n,\nu}^{(r)} \cup S_{n,\nu}^{(1)} \subset (x_\nu - \tilde{r}, x_\nu + \tilde{r})$. By uniform convergence, $\eta_n'' + \frac{h_n}{2}|\eta_n^{(3)}| < -\frac{k_0}{2}$ for n large enough, so that both functions $\eta_n + \frac{h_n}{2}\eta_n'$ and $\eta_n - \frac{h_n}{2}\eta_n'$ are strictly concave in $(x_\nu - \tilde{r}, x_\nu + \tilde{r})$. This implies that $S_{n,\nu}^{(r)}$ (resp. $S_{n,\nu}^{(1)}$) is of the form \emptyset , $\{ih_n\}$, or $\{ih_n, (i+1)h_n\}$. Observe also that $\eta_n + \frac{h_n}{2}\eta_n' - (\eta_n - \frac{h_n}{2}\eta_n') = h_n\eta_n'$. Since the function η_n' is strictly decreasing in $(x_\nu - \tilde{r}, x_\nu + \tilde{r})$, it vanishes at most once. If $S_{n,\nu}^{(r)} \neq \emptyset$ and $S_{n,\nu}^{(1)} \neq \emptyset$, it must change sign in $(x_\nu - \tilde{r}, x_\nu + \tilde{r})$ and thus it vanishes exactly once, at some $\xi \in (x_\nu - \tilde{r}, x_\nu + \tilde{r})$. Then for $t \in (x_\nu - \tilde{r}, \xi)$,

$$(\eta_n - \frac{h_n}{2}\eta_n')(t) = (\eta_n + \frac{h_n}{2}\eta_n')(t) - h_n\eta_n'(t) \leq 1 - h_n\eta_n'(t) < 1$$

so that $\min S_{n,\nu}^{(1)} \geq \xi$. Similarly $\max S_{n,\nu}^{(r)} \leq \xi$.

(iii) By contradiction, assume that the set of $n' \in \mathbb{N}$ such that $S_{n',\nu}^{(r)} \neq \emptyset$ and $S_{n',\nu}^{(1)} \neq \emptyset$ is infinite. We may extract a subsequence $n = n'(m)$ such that there exists $i_n, j_n \in \llbracket 0, G_n - 1 \rrbracket$ (denoted hereafter i, j) with $ih_n \in S_{n,\nu}^{(r)}$, $jh_n \in S_{n,\nu}^{(1)}$. Combining the Taylor expansions of η_n and $(\eta_n)'$ around ih_n (resp. jh_n), we get

$$\begin{aligned} 1 &\geq \eta_n((i+1)h_n) - \frac{h_n}{2}\eta_n'((i+1)h_n) \\ &= \underbrace{\eta_n(ih_n) + h_n\eta_n'(ih_n)(1 - \frac{1}{2})}_{=1} + \underbrace{h_n^2\eta_n''(ih_n)\left(\frac{1}{2!} - \frac{1}{2}\right)}_{=0} + h_n^3\eta_n^{(3)}(ih_n)\alpha_3 \\ &\quad + h_n^4 \int_0^1 \eta_n^{(4)}(ih_n + th_n) \left(\frac{(1-t)^3}{3!} - \frac{(1-t)^2}{2! \times 2} \right) dt, \text{ and} \\ 1 &\geq \eta_n((j-1)h_n) + \frac{h_n}{2}\eta_n'((j-1)h_n) \\ &= \underbrace{\eta_n(jh_n) - h_n\eta_n'(jh_n)(1 - \frac{1}{2})}_{=1} + \underbrace{h_n^2\eta_n''(jh_n)\left(\frac{1}{2!} - \frac{1}{2}\right)}_{=0} - h_n^3\eta_n^{(3)}(jh_n)\alpha_3 \\ &\quad + h_n^4 \int_0^1 \eta_n^{(4)}(jh_n - th_n) \left(\frac{(1-t)^3}{3!} - \frac{(1-t)^2}{2! \times 2} \right) dt \end{aligned}$$

where α_k is defined in (D.11). Now, let $n \rightarrow +\infty$. By (D.1), $ih_n \rightarrow x_\nu$ and $jh_n \rightarrow x_\nu$, and using the uniform convergence of $\eta_n^{(k)}$ towards $\eta_\infty^{(k)}$, dividing by h_n^3 , we obtain respectively $0 \geq -\eta_\infty^{(3)}(x_\nu) \times \frac{1}{12}$ and $0 \geq \eta_\infty^{(3)}(x_\nu) \times \frac{1}{12}$, thus $\eta_\infty^{(3)}(x_\nu) = 0$.

(iv) Assume, by contradiction, that the mentioned set is infinite. For such n , a Taylor expansion at ih_n yields (we write i for i_n):

$$\begin{aligned}
 1 &= \eta_n((i+1)h_n) - \frac{h_n}{2}\eta_n'((i+1)h_n) \\
 &= \underbrace{\eta_n(ih_n) + \frac{h_n}{2}\eta_n'(ih_n)}_{=1} + \gamma_3 h_n^3 \eta_n^{(3)}(ih_n) + \gamma_4 h_n^4 \eta_n^{(4)}(ih_n) \\
 &\quad + h_n^5 \int_0^1 \eta_n^{(5)}(ih_n + th_n) \left(\frac{(1-t)^4}{4!} - \frac{(1-t)^3}{3! \times 2} \right) dt, \text{ and} \\
 1 &= \eta_n((i-1)h_n) + \frac{h_n}{2}\eta_n'((i-1)h_n) \\
 &= \underbrace{\eta_n(ih_n) - \frac{h_n}{2}\eta_n'(ih_n)}_{=1} - \gamma_3 h_n^3 \eta_n^{(3)}(ih_n) + \gamma_4 h_n^4 \eta_n^{(4)}(ih_n) \\
 &\quad + h_n^5 \int_0^1 \eta_n^{(5)}(ih_n + th_n) \left(\frac{(1-t)^4}{4!} - \frac{(1-t)^3}{3! \times 2} \right) dt,
 \end{aligned}$$

with $\gamma_k = \frac{1}{k!} - \frac{1}{(k-1)! \times 2}$. Summing both equalities, dividing by h_n^4 and taking the limit $n \rightarrow +\infty$ yields $(\eta_\infty)^{(4)}(x_\nu) = 0$, a contradiction. \square

This other lemma focusses on the limit of the sets D^n defined in (35).

Lemma 5. *As $n \rightarrow +\infty$, the sets D^n converge towards D^∞ defined in (37) (in the sense of set convergence).*

Proof. We observe that $E^n \subset D^n \subset F^n$, where

$$\begin{aligned}
 E^n &\stackrel{\text{def.}}{=} \left\{ p \in \mathcal{H} ; \max_{t \in \mathbb{T}} (\Phi^* p)(t) + \frac{h_n}{2} |(\Phi^* p)'(t)| \leq 1 \right\}, \\
 F^n &\stackrel{\text{def.}}{=} \left\{ p \in \mathcal{H} ; \max_{k \in \llbracket 0, G_n - 1 \rrbracket} \Phi^* p(kh_n) \leq 1 \right\}
 \end{aligned}$$

so that it suffices to prove that E^n and F^n converge towards D^∞ . On the one hand, it is clear that $D^\infty = \bigcap_{n \in \mathbb{N}} F^n$, and the sequence F^n is non-increasing. On the other hand, it is possible to check that $D^\infty = \overline{\bigcup_{n \in \mathbb{N}} E^n}$, and the sequence E^n is non-decreasing. As a consequence, the claimed set convergences hold (see [26, Ex. 4.3]). \square

Let us recall that the dual problem to $(\mathcal{Q}_\lambda^n(y))$ is the projection onto the closed convex set

$$D^n \stackrel{\text{def.}}{=} \left\{ p \in \mathcal{H} ; (\Phi^* p)(ih_n) + \frac{h_n}{2} |(\Phi^* p)'(ih_n)| \leq 1 \right\}.$$

Since the set convergence of D^n (see Lemma 5 above) implies the convergence of the projections onto D^n (see [26], or [16] for a direct proof in a similar context), we obtain:

Lemma 6. Let $p_{\lambda,n}$ (resp. $p_{\lambda,\infty}$) be a solution of $(\mathcal{E}_\lambda^n(y))$ (resp. $(\mathcal{E}_\lambda^\infty(y))$), and $\eta_{\lambda,n} = \Phi^* p_{\lambda,n}$ (resp. $\eta_{\lambda,\infty} = \Phi^* p_{\lambda,\infty}$). Then

$$\lim_{n \rightarrow +\infty} p_{\lambda,n} = p_{\lambda,\infty} \text{ strongly in } \mathcal{H},$$

$$\lim_{n \rightarrow +\infty} \eta_{\lambda,n}^{(k)} = \eta_{\lambda,\infty}^{(k)} \text{ in the sense of the uniform convergence, for all } k \in \mathbb{N} \text{ up to the regularity of } \varphi.$$

We are now in position to prove Proposition 4.

Proof of Proposition 4. By Lemma 6, we know that the dual certificates $\eta_{\lambda,n}$ converge towards $\eta_{\lambda,\infty}$. By Lemma 4 (i) and the optimality conditions, we have thus $\limsup_{n \rightarrow +\infty} (\text{supp}(m_{\lambda,n})) \subset \{x_1, \dots, x_N\}$.

If $m_{\lambda,\infty}$ is the unique solution, assume by contradiction that $\liminf(\text{supp}(m_{\lambda,n})) \subsetneq \{x_1, \dots, x_N\}$. Then there is some ν , some $\varepsilon > 0$ such that (up to a subsequence) $(\text{supp}(m_{\lambda,n}) \cap (x_\nu - \varepsilon, x_\nu + \varepsilon)) = \emptyset$. This contradicts the Γ -convergence result (Prop. 3) which ensures that $m_{\lambda,n}$ converges towards $m_{\lambda,\infty}$ in the weak* topology. As a result $\lim_{n \rightarrow +\infty} (\text{supp}(m_{\lambda,n})) = \{x_1, \dots, x_N\}$.

If $\mu''_{\lambda,\infty}(x_\nu) \neq 0$, Lemma 4 ensures that the sets $S_{n,\nu}^{(\mathbf{r})}(r)$ and $S_{n,\nu}^{(1)}(r)$ are of the form \emptyset , $\{ih_n\}$, or $\{ih_n, (i+1)h_n\}$. Moreover, since $\lim_{n \rightarrow +\infty} (\text{supp } m_{\lambda,n}) = \{x_1, \dots, x_N\}$ we must have $S_{n,\nu}^{(\mathbf{r})}(r) \neq \emptyset$ or $S_{n,\nu}^{(1)}(r) \neq \emptyset$. Using the fact that $\max S_{n,\nu}^{(\mathbf{r})}(r) \leq \min S_{n,\nu}^{(1)}(r)$, one may check that the only possible saturation points of $\eta_{\lambda,n} + \frac{h_n}{2} \eta'_{\lambda,n}$ and $\eta_{\lambda,n} - \frac{h_n}{2} \eta'_{\lambda,n}$ are given in Table 1. The optimality conditions of Proposition 1 imply that $m_{\lambda,n}$ is at most a sum of Dirac masses at those locations.

If $\eta_{\lambda,\infty}^{(3)}(x_\nu) \neq 0$, Lemma 4 (iii) implies that for n large enough, $S_{n,\nu}^{(\mathbf{r})}(r) = \emptyset$ or $S_{n,\nu}^{(1)}(r) = \emptyset$ (but not both). Hence there are at most two (successive) saturations, produced either by $\eta_{\lambda,n} + \frac{h_n}{2} \eta'_{\lambda,n}$ or by $\eta_{\lambda,n} - \frac{h_n}{2} \eta'_{\lambda,n}$. \square

Appendix D.1. Sufficient Condition for Asymptotic Solutions

The following property ensures that m_0 is a solution to $(\mathcal{Q}_0^n(y_0))$ for each n large enough.

Lemma 7. Assume that there exists a function $\eta \in \text{Im } \Phi^*$, such that for all $t \in \mathbb{T} \setminus \{x_{0,1}, \dots, x_{0,N}\}$, $\eta(t) < 1$ and

$$\forall \nu \in \{1, \dots, N\}, \eta(x_{0,\nu}) = 1, \eta''(x_{0,\nu}) \neq 0, \eta^{(3)}(x_{0,\nu}) = 0, \eta^{(4)}(x_{0,\nu}) > 0.$$

Then, for all n large enough, η is a dual certificate for $m_0 = \sum_{\nu=1}^N \alpha_{0,\nu} \delta_{x_{0,\nu}}$ for $(\mathcal{Q}_0^n(y_0))$, and m_0 is a solution to $(\mathcal{Q}_0^n(y_0))$. Moreover, if Γ_{x_0} has full rank, this solution is unique.

Remark 3. The condition $\eta^{(3)}(x_{0,\nu}) = 0$ is natural since our aim is to build a certificate which is valid for all n , hence Lemma 4 applies with $S_{n,\nu}^{(\mathbf{r})}(r) \neq \emptyset$ and $S_{n,\nu}^{(1)}(r) \neq \emptyset$.

Proof. Let $\nu \in \{1, \dots, N\}$ and $r_\nu \in (0, r)$ such that $\eta''(t) < 0$, and $\eta^{(4)}(t) > 0$ in $(x_{0,\nu} - r_\nu, x_{0,\nu} + r_\nu)$. We shall prove that $\eta(kh_n) + \frac{h_n}{2} |\eta'(kh_n)| < 1$ for all k such that

t	$-\tilde{r}$	0	\tilde{r}
$\eta^{(4)}$		+	
$\eta^{(3)}$		0	
η''	$\eta''(-\tilde{r}) < 0$	$\eta''(0)$	$\eta''(\tilde{r}) < 0$
η'		0	
η		1	

 Table D1: Variations of η and its derivatives.

$kh_n \in (x_{0,\nu} - r_\nu, x_{0,\nu} + r_\nu) \setminus \{x_{0,\nu}\}$. To simplify the notation, we assume without loss of generality that $x_{0,\nu} = 0$ and we write $\tilde{r} = r_\nu$. The variations of η and its derivatives are given in Table D1.

Let us observe that the function $\theta : t \mapsto \eta(t) - \frac{t}{2}\eta'(t)$ is (strictly) decreasing in $[0, \tilde{r})$, since

$$\forall t \in (0, \tilde{r}), \quad \theta'(t) = \frac{1}{2}(\eta'(t) - t\eta''(t)) = \frac{1}{2} \int_0^t \underbrace{(\eta''(u) - \eta''(t))}_{<0} du < 0. \quad (\text{D.2})$$

Hence, for all k such that $kh_n \in (0, \tilde{r})$,

$$\eta(kh_n) - \frac{h_n}{2}\eta'(kh_n) = \underbrace{\eta(kh_n) - \frac{kh_n}{2}\eta'(kh_n)}_{=\theta(kh_n) < \theta(0)=1} + \underbrace{\frac{(k-1)h_n}{2}\eta'(kh_n)}_{<0} < 1. \quad (\text{D.3})$$

On the other hand, θ is (strictly) increasing on $(-\tilde{r}, 0]$ since

$$\forall t \in (-\tilde{r}, 0), \quad \theta'(t) = \frac{1}{2}(\eta'(t) - t\eta''(t)) = \frac{1}{2} \int_0^t \underbrace{(\eta''(u) - \eta''(t))}_{<0} du > 0. \quad (\text{D.4})$$

As a consequence, for all k such that $kh_n \in (-\tilde{r}, 0)$,

$$\eta(kh_n) + \frac{h_n}{2}\eta'(kh_n) = \underbrace{\eta(kh_n) - \frac{kh_n}{2}\eta'(kh_n)}_{=\theta(kh_n) < \theta(0)=1} + \underbrace{\frac{(k+1)h_n}{2}\eta'(kh_n)}_{\leq 0} < 1. \quad (\text{D.5})$$

Thus we see that $\eta(kh_n) + \frac{h_n}{2}|\eta'(kh_n)| < 1$ for all $kh_n \in (-\tilde{r}, \tilde{r}) \setminus \{0\}$, and we proceed similarly on all the intervals of the form $(x_{0,\nu} - r_\nu, x_{0,\nu} + r_\nu)$. By a compactness argument, there exists a constant $\beta < 1$ such that $\eta(t) \leq \beta$ for all $t \in \mathbb{T} \setminus \bigcup_{\nu=1}^N (x_{0,\nu} - r_\nu, x_{0,\nu} + r_\nu)$.

For n large enough, the inequality $\frac{h_n}{2} (\sup_{t \in \mathbb{T}} |\eta'(t)|) < 1 - \beta$ holds, and we see that $\eta(kh_n) + \frac{h_n}{2} |\eta'(kh_n)| < 1$ for all $t \in \mathbb{T} \setminus \bigcup_{\nu=1}^N (x_{0,\nu} - r_\nu, x_{0,\nu} + r_\nu)$.

As a conclusion, we see that η is a valid certificate for $(a_0, 0)$ (see the optimality conditions of Proposition 1), thus $(a_0, 0)$ is a solution of $(\mathcal{Q}_0^n(y_0))$. \square

Appendix D.2. Asymptotics of the Minimal Norm Certificates

Observe that if the Twice Non-Degenerate Source Condition (Definition 5) holds, the hypotheses of Lemma 7 are satisfied and m_0 is a solution to $(\mathcal{Q}_0^n(y_0))$ for n large enough. In fact the associated minimal norm certificates (which thus exist) converge towards η_T .

Proposition 5. *Let $m_0 \in \mathcal{M}(\mathbb{T})$ satisfy the Twice Non-Degenerate Source Condition (and η_T the corresponding Third derivative (pre)certificate). Let $p_{0,n}$ be the minimal norm solution of $(\mathcal{E}_0^n(y_0))$, and $\eta_{0,n} = \Phi^* p_{0,n}$. Then,*

$$\lim_{n \rightarrow +\infty} p_{0,n} = p_T \text{ for the } \mathcal{H} \text{ strong topology,} \quad (\text{D.6})$$

$$\lim_{n \rightarrow +\infty} \eta_{0,n}^{(k)} = \eta_T^{(k)} \text{ in the sense of the uniform convergence, for all } k \in \mathbb{N} \text{ up to the regularity of } \varphi. \quad (\text{D.7})$$

Proof. As mentioned above, the Twice Non-Degenerate Source Condition implies that Lemma 7 applies to the function η_T , hence η_T is a certificate for $(\mathcal{Q}_0^n(y_0))$. As a result, $\|p_{0,n}\| \leq \|p_T\|$ and the sequence $(p_{0,n})_{n \in \mathbb{N}}$ is bounded in \mathcal{H} . We may extract a subsequence $p_{0,n'}$ which weakly converges towards some $\tilde{p} \in \mathcal{H}$, and then $\|\tilde{p}\| \leq \liminf_{n' \rightarrow +\infty} \|p_{0,n'}\| \leq \|p_T\|$. Since Φ^* and $\Phi^{(k),*}$ are compact (see [17, Lemma 1]), we obtain that $\eta_{0,n'}^{(k)} = (\Phi^* p_{0,n'})^{(k)}$ converges toward $\tilde{\eta}^{(k)} \stackrel{\text{def.}}{=} (\Phi^* \tilde{p})^{(k)}$ in the (strong) topology of uniform convergence. We immediately obtain that $\tilde{\eta}(t) \leq 1$ for all $t \in \mathbb{T}$, and $\tilde{\eta}(x_{0,\nu}) = 1$, $\tilde{\eta}(x_{0,\nu}) = 0$ for all $\nu \in \{1, \dots, N\}$.

Moreover, applying Lemma 4 to $\Phi^* p_{0,n}$ (observing that $x_\nu \in S_{n,\nu}^{(r)}(r) \cap S_{n,\nu}^{(l)}(r)$), we get $\tilde{\eta}^{(3)}(x_{0,\nu}) = 0$. As a result, \tilde{p} is admissible for (42), hence $\|p_T\| \leq \|\tilde{p}\|$. Thus in fact $\|p_T\| = \|\tilde{p}\|$ and $p_T = \tilde{p}$. Since the limit of the extracted subsequence does not depend on the choice of the subsequence, in fact the whole sequence converges. Moreover, the convergence is strong in \mathcal{H} since $\lim_{n \rightarrow +\infty} \|p_{0,n}\| = \|p_T\|$. \square

As a consequence of the above convergence result, the third derivative precertificate controls the extended support on thin grids.

Proposition 6. *Let $m_0 \in \mathcal{M}(\mathbb{T})$ (with $\{x_{0,1}, \dots, x_{0,N}\} \subset \mathcal{G}_n$) such that the Twice Non Degenerate Source Condition holds. Then, for n large enough, m_0 is a solution to $(\mathcal{Q}_0^n(y_0))$ and its extended support is given by:*

$$\text{ext}_n^{(r)}(m_0) = \bigcup_{\nu=1}^N S_{n,\nu}^{(r)}(r), \quad \text{and} \quad \text{ext}_n^{(l)}(m_0) = \bigcup_{\nu=1}^N S_{n,\nu}^{(l)}(r), \quad (\text{D.8})$$

where

- $S_{n,\nu}^{(\mathbf{r})}(r)$ is equal to $\{x_{0,\nu}\}$ or $\{x_{0,\nu} - h_n, x_{0,\nu}\}$,
- $S_{n,\nu}^{(\mathbf{l})}(r)$ is equal to $\{x_{0,\nu}\}$ or $\{x_{0,\nu}, x_{0,\nu} + h_n\}$.

Moreover, one cannot have simultaneously $S_{n,\nu}^{(\mathbf{r})}(r) = \{x_{0,\nu} - h_n, x_{0,\nu}\}$ and $S_{n,\nu}^{(\mathbf{l})}(r) = \{x_{0,\nu}, x_{0,\nu} + h_n\}$.

Proof. By Lemma 7, m_0 is a solution to $(\mathcal{Q}_0^n(y_0))$ and η_T is a solution to $(\mathcal{E}_0^n(y_0))$. Applying Lemma 4 to $\eta_{0,n}$, η_T , we see that $S_{n,\nu}^{(\mathbf{r})}(r)$ is of the form \emptyset , $\{ih_n\}$ or $\{(i-1)h_n, ih_n\}$, and that $S_{n,\nu}^{(\mathbf{l})}(r)$ is of the form \emptyset , $\{jh_n\}$ or $\{jh_n, (j+1)h_n\}$, with $i \leq j$. On the other hand, by the extremality relations between $\eta_{0,n}$ (solution of $(\mathcal{E}_0^n(y_0))$) and $m_{0,n}$ (solution of $(\mathcal{Q}_0^n(y_0))$), $x_{0,\nu} \in S_{n,\nu}^{(\mathbf{r})}(r)$ and $x_{0,\nu} \in S_{n,\nu}^{(\mathbf{l})}(r)$. As a consequence $S_{n,\nu}^{(\mathbf{r})}(r)$ is equal to $\{x_{0,\nu}\}$ or $\{x_{0,\nu} - h_n, x_{0,\nu}\}$, and $S_{n,\nu}^{(\mathbf{l})}(r)$ is equal to $\{x_{0,\nu}\}$ or $\{x_{0,\nu}, x_{0,\nu} + h_n\}$.

Now, since $\eta_T^4(0) \neq 0$, the fourth point of Lemma 4 ensures that for n large enough, one cannot have simultaneously $S_{n,\nu}^{(\mathbf{r})}(r) = \{x_{0,\nu} - h_n, x_{0,\nu}\}$ and $S_{n,\nu}^{(\mathbf{l})}(r) = \{x_{0,\nu}, x_{0,\nu} + h_n\}$. \square

Remark 4. As Proposition 6 shows, for each original spike, at most one pair of spikes appears at low noise: the original spike slightly shifted and either the immediate left neighbor shifted by $+h_n/2$ or the immediate right neighbor shifted by $-h_n/2$.

Appendix D.3. Proof of Theorem 2

We proceed by building a good candidate for $\eta_{0,n}$, making the ansatz that for all $\nu \in \{1, \dots, N\}$, its saturation points satisfy

$$\text{if } \rho_\nu > 0, \text{ then } S_{n,\nu}^{(\mathbf{r})}(r) = \{x_{0,\nu} - h_n, x_{0,\nu}\}, \quad \text{and} \quad S_{n,\nu}^{(\mathbf{l})}(r) = \{x_{0,\nu}\}, \quad (\text{D.9})$$

$$\text{if } \rho_\nu < 0, \text{ then } S_{n,\nu}^{(\mathbf{r})}(r) = \{x_{0,\nu}\}, \quad \text{and} \quad S_{n,\nu}^{(\mathbf{l})}(r) = \{x_{0,\nu}, x_{0,\nu} + h_n\}, \quad (\text{D.10})$$

and then, using Lemma 3, we prove that this candidate is indeed the minimal norm certificate.

To comply with the notations of Lemma 3, let us write $\sum_{\nu=1}^N \alpha_{0,i} \delta_{x_{0,\nu}} = \sum_{k=0}^{G_n-1} a_{0,k} \delta_{kh_n}$, and $I^{(\mathbf{r})} \stackrel{\text{def.}}{=} I^{(\mathbf{l})} \stackrel{\text{def.}}{=} I \stackrel{\text{def.}}{=} \{i \in \llbracket 0, G_n - 1 \rrbracket ; a_{0,i} \neq 0\}$.

For any choice of shift $(\varepsilon_i)_{i \in I} \in \{-1, +1\}^N$, we set $J^{(\mathbf{r})} \stackrel{\text{def.}}{=} I^{(\mathbf{r})} \cup \{i + \varepsilon_i ; i \in I \text{ and } \varepsilon_i = -1\}$ and $J^{(\mathbf{l})} \stackrel{\text{def.}}{=} I^{(\mathbf{l})} \cup \{i + \varepsilon_i ; i \in I \text{ and } \varepsilon_i = +1\}$. Since $|x_{0,\nu} - x_{0,\nu'}| > 2h_n$ for $\nu' \neq \nu$ and n large enough, we have $\text{Card } J^{(\mathbf{r})} + \text{Card } J^{(\mathbf{l})} = 3 \times \text{Card } I = 3N$. The idea is to find a choice of ε such that $u_j > 0$ for all $j \in J^{(\mathbf{r})} \setminus I$, and $v_j > 0$ for all $j \in J^{(\mathbf{l})} \setminus I$, where

$$\begin{pmatrix} u_{J^{(\mathbf{r})}} \\ v_{J^{(\mathbf{l})}} \end{pmatrix} \stackrel{\text{def.}}{=} -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \begin{pmatrix} \mathbf{1}_{J^{(\mathbf{r})}} \\ \mathbf{1}_{J^{(\mathbf{l})}} \end{pmatrix}, \quad \hat{\mathcal{A}}_h \stackrel{\text{def.}}{=} \begin{pmatrix} (A + \frac{h}{2}B)_{J^{(\mathbf{r})}} & (A - \frac{h}{2}B)_{J^{(\mathbf{l})}} \end{pmatrix}$$

$$A \stackrel{\text{def.}}{=} \Phi_{\mathcal{G}_n} \quad \text{and} \quad B \stackrel{\text{def.}}{=} \Phi'_{\mathcal{G}_n}.$$

In this particular case where $I^{(\mathbf{r})} = I^{(\mathbf{l})} = I$, all j in $(J^{(\mathbf{r})} \setminus I) \cup (J^{(\mathbf{l})} \setminus I)$ may be uniquely written as $j = i + \varepsilon_i$ for some $i \in I$, where $\varepsilon_i \in \{-1, +1\}$. We may swap the

columns of $\hat{\mathcal{A}}_h$ so as to reformulate the condition $\begin{pmatrix} u_{J^{(r)}} \\ v_{J^{(l)}} \end{pmatrix} = -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N}$ into

$$\begin{pmatrix} \tilde{u}_I \\ \tilde{v}_I \\ \tilde{t}_I \end{pmatrix} = -(\bar{\mathcal{A}}_h^* \bar{\mathcal{A}}_h)^{-1} \begin{pmatrix} \mathbf{1}_N \\ \mathbf{1}_N \\ \mathbf{1}_N \end{pmatrix},$$

where $\bar{\mathcal{A}}_h \stackrel{\text{def.}}{=} \begin{pmatrix} A_I + \frac{h_n}{2} B_I \text{diag}(\varepsilon) & A_I - \frac{h_n}{2} B_I \text{diag}(\varepsilon) & A_{I+\varepsilon} - \frac{h_n}{2} B_{I+\varepsilon} \text{diag}(\varepsilon) \end{pmatrix}$ and $\tilde{t}_i > 0$ for all $i \in I$. But a Taylor expansion yields

$$A_{I+\varepsilon} - \frac{h_n}{2} B_{I+\varepsilon} \text{diag}(\varepsilon) = \underbrace{\Phi_{x_0}}_{=A_I} + \frac{h_n}{2} \underbrace{\Phi'_{x_0} \text{diag}(\varepsilon)}_{=B_I \text{diag}(\varepsilon)} + (h_n)^3 \gamma_3 \Phi_{x_0}^{(3)} \text{diag}(\varepsilon) + o(h_n^3),$$

where we defined

$$\gamma_k \stackrel{\text{def.}}{=} \frac{1}{k!} - \frac{1}{(k-1)! \times 2}. \quad (\text{D.11})$$

Hence, we may apply Lemma 2 to Φ_{x_0} , $\Phi'_{x_0} \text{diag}(\varepsilon)$ and $\gamma_3 \Phi_{x_0}^{(3)} \text{diag}(\varepsilon)$ so as to obtain

$$\tilde{t}_I = -\frac{1}{\gamma_3 h_n^3} \text{diag}(\varepsilon) \rho + o\left(\frac{1}{h_n^3}\right).$$

Therefore it is sufficient to choose $\varepsilon = -\text{sign}(\rho)$ to make all the components of \tilde{t}_I nonnegative.

With that choice of ε , it remains to prove that

$$\max \left[\begin{pmatrix} (A + \frac{h}{2} B)_{J^{(r)c}}^* \\ (A - \frac{h}{2} B)_{J^{(l)c}}^* \end{pmatrix} \hat{\mathcal{A}}_h \begin{pmatrix} u_{J^{(r)}} \\ v_{J^{(l)}} \end{pmatrix} \right] < 1.$$

Let us write $\tilde{p}_n \stackrel{\text{def.}}{=} \hat{\mathcal{A}}_h \begin{pmatrix} u_{J^{(r)}} \\ v_{J^{(l)}} \end{pmatrix}$. Since $\begin{pmatrix} u_{J^{(r)}} \\ v_{J^{(l)}} \end{pmatrix} = -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N}$, we get $\tilde{p}_n = \hat{\mathcal{A}}_h^{+,*} \mathbf{1}_{3N}$, and applying Lemma 2 to Φ_{x_0} , $\Phi'_{x_0} \text{diag}(\varepsilon)$ and $\gamma_3 \Phi_{x_0}^{(3)} \text{diag}(\varepsilon)$, we see that \tilde{p}_n converges towards p_T (using (43)).

By construction of \tilde{p}_n ,

$$\begin{aligned} \forall j \in J^{(r)} \setminus I, \quad (\Phi^* \tilde{p}_n + \frac{h_n}{2} (\Phi^* \tilde{p}_n)')(jh_n) &= 1, \\ \text{and } \forall j \in J^{(l)} \setminus I, \quad (\Phi^* \tilde{p}_n - \frac{h_n}{2} (\Phi^* \tilde{p}_n)')(jh_n) &= 1, \end{aligned} \quad (\text{D.12})$$

which may be summarized as

$$\forall i \in I, \quad (\Phi^* \tilde{p}_n - \varepsilon_i \frac{h_n}{2} (\Phi^* \tilde{p}_n)')((i + \varepsilon_i)h_n) = 1.$$

Arguing as in the proof of point (iv) in Lemma 4 (replacing “1 = ...” with “1 \geq ...” and using that $\eta_T^{(4)}(x_{0,\nu}) > 0$), we may prove that for n large enough, $(\Phi^* \tilde{p}_n + \varepsilon_i \frac{h_n}{2} (\Phi^* \tilde{p}_n)')((i - \varepsilon_i)h_n) < 1$.

Then, by the same argument of compactness and local concavity as in point (ii) of Lemma 4, we observe that

$$\begin{aligned} & \left\{ k \in \llbracket 0, G_n - 1 \rrbracket ; (\Phi^* \tilde{p}_n + \frac{h_n}{2} (\Phi^* \tilde{p}_n)')(kh_n) \geq 1 \right\} \subset J^{(\mathbf{r})}, \\ & \left\{ k \in \llbracket 0, G_n - 1 \rrbracket ; (\Phi^* \tilde{p}_n - \frac{h_n}{2} (\Phi^* \tilde{p}_n)')(kh_n) \geq 1 \right\} \subset J^{(\mathbf{l})}, \end{aligned}$$

and those inclusions are in fact equalities. That precisely means that

$$\max \left[\begin{pmatrix} (A + \frac{h}{2} B)_{J^{(\mathbf{r})}^c}^* \\ (A - \frac{h}{2} B)_{J^{(\mathbf{l})}^c}^* \end{pmatrix} \tilde{p}_n \right] < 1.$$

Hence, by Lemma 3, $\Phi^* \tilde{p}_n$ is the minimal norm certificate $\eta_{0,n}$ and $(J^{(\mathbf{r})} h_n, J^{(\mathbf{l})} h_n)$ is the extended support. This concludes the proof.

Appendix D.4. Proof of Corollary 1

We build a pair of candidate solutions for the primal and dual problems, and we prove that they satisfy the optimality conditions. Again, we change variables so as to deal with the positive LASSO, working with (r_λ, l_λ) rather than (a_λ, b_λ) (see (19)).

Recall the notation $A \stackrel{\text{def.}}{=} \Phi_{\mathcal{G}_n}$, $B \stackrel{\text{def.}}{=} \Phi'_{\mathcal{G}_n}$ and $\mathcal{A}_h \stackrel{\text{def.}}{=} \begin{pmatrix} A + \frac{h}{2} B & A - \frac{h}{2} B \end{pmatrix}$. Defining $(J^{(\mathbf{r})}, J^{(\mathbf{l})})$ as in the proof of Theorem 2, we let $\hat{\mathcal{A}}_h \stackrel{\text{def.}}{=} \begin{pmatrix} (A + \frac{h}{2} B)_{J^{(\mathbf{r})}} & (A - \frac{h}{2} B)_{J^{(\mathbf{l})}} \end{pmatrix}$ (again, $\hat{\mathcal{A}}_h$ has full rank for h small enough since $\begin{pmatrix} \Phi_{x_0} & \Phi'_{x_0} & \Phi_{x_0}^{(3)} \end{pmatrix}$ has full rank).

Now, we let

$$\left\{ \begin{aligned} & \begin{pmatrix} (r_\lambda)_{J^{(\mathbf{r})}} \\ (l_\lambda)_{J^{(\mathbf{l})}} \end{pmatrix} \stackrel{\text{def.}}{=} \hat{\mathcal{A}}_h^+ y - \lambda (\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N} = \begin{pmatrix} (r_0)_{J^{(\mathbf{r})}} \\ (l_0)_{J^{(\mathbf{l})}} \end{pmatrix} + \hat{\mathcal{A}}_h^+ w + \lambda \begin{pmatrix} u_{J^{(\mathbf{r})}} \\ v_{J^{(\mathbf{l})}} \end{pmatrix}, \\ & p_\lambda \stackrel{\text{def.}}{=} \frac{1}{\lambda} \left(y - \hat{\mathcal{A}}_h \begin{pmatrix} (r_\lambda)_{J^{(\mathbf{r})}} \\ (l_\lambda)_{J^{(\mathbf{l})}} \end{pmatrix} \right) \end{aligned} \right. \quad (\text{D.13})$$

where u, v are defined as in the proof of Theorem 2 by $\begin{pmatrix} u_{J^{(\mathbf{r})}} \\ v_{J^{(\mathbf{l})}} \end{pmatrix} \stackrel{\text{def.}}{=} -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \begin{pmatrix} \mathbf{1}_{J^{(\mathbf{r})}} \\ \mathbf{1}_{J^{(\mathbf{l})}} \end{pmatrix} = -(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N}$.

Observe that by construction of p_λ , $\begin{pmatrix} A + \frac{h}{2} B & A - \frac{h}{2} B \end{pmatrix}^* p_\lambda = \hat{\mathcal{A}}_h^* p_\lambda = \mathbf{1}_{3N}$. It remains to prove that

- (i) For all $i \in I$, $r_{\lambda,i} \geq 0$ and $l_{\lambda,i} \geq 0$.
- (ii) For all $j \in J^{(\mathbf{r})} \setminus I$ (resp. $j \in J^{(\mathbf{l})} \setminus I$), $r_{\lambda,j} \geq 0$ (resp. $l_{\lambda,j} \geq 0$).
- (iii) For $k \notin J^{(\mathbf{r})}$ (resp. $k \notin J^{(\mathbf{l})}$) $((A^* + \frac{h_n}{2} B^*) p_\lambda)_k < 1$ (resp. $((A^* - \frac{h_n}{2} B^*) p_\lambda)_k < 1$)

Regarding the first point, the following componentwise inequality holds,

$$\begin{aligned} \begin{pmatrix} (r_\lambda)_I \\ (l_\lambda)_I \end{pmatrix} &= \begin{pmatrix} (r_0)_I \\ (l_0)_I \end{pmatrix} + (\hat{\mathcal{A}}_h^+ w)_{I,I} - \lambda((\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N})_{I,I} \\ &\geq \begin{pmatrix} (r_0)_I \\ (l_0)_I \end{pmatrix} - \|\hat{\mathcal{A}}_h^+\|_{\infty, \mathcal{H}} \|w\|_{\mathcal{H}} - \lambda \left\| \begin{pmatrix} u_I \\ v_I \end{pmatrix} \right\|_{\infty} \end{aligned}$$

From Lemma 2 we get

$$\|\hat{\mathcal{A}}_h^+\|_{\infty, \mathcal{H}} \sim \frac{1}{(h_n)^3} \left\| \begin{pmatrix} (\Phi_{x_0}^{(3),*} \tilde{\Pi} \Phi_{x_0}^{(3)})^{-1} \Phi_{x_0}^{(3),*} \tilde{\Pi} \\ 0 \\ -(\Phi_{x_0}^{(3),*} \tilde{\Pi} \Phi_{x_0}^{(3)})^{-1} \Phi_{x_0}^{(3),*} \tilde{\Pi} \end{pmatrix} \right\|_{\infty, \mathcal{H}} \quad \text{and} \quad \left\| \begin{pmatrix} u_I \\ v_I \end{pmatrix} \right\|_{\infty} \sim \frac{1}{(h_n)^3} \left\| \begin{pmatrix} \rho \\ 0 \end{pmatrix} \right\|_{\infty}$$

so that $(r_\lambda)_I$ and $(l_\lambda)_I$ have positive components provided $c_1 \|w\|_{\mathcal{H}} + c_2 \lambda \leq \frac{(h_n)^3}{2} \min \bigcup_{i \in I} \{r_i, l_i\}$, with $c_1 \stackrel{\text{def.}}{=} \left\| \begin{pmatrix} (\Phi_{x_0}^{(3),*} \tilde{\Pi} \Phi_{x_0}^{(3)})^{-1} \Phi_{x_0}^{(3),*} \tilde{\Pi} \\ 0 \\ -(\Phi_{x_0}^{(3),*} \tilde{\Pi} \Phi_{x_0}^{(3)})^{-1} \Phi_{x_0}^{(3),*} \tilde{\Pi} \end{pmatrix} \right\|_{\infty, \mathcal{H}}$ and $c_2 \stackrel{\text{def.}}{=} \left\| \begin{pmatrix} \rho \\ 0 \end{pmatrix} \right\|_{\infty}$.

As for the second point, we have

$$\begin{pmatrix} (r_\lambda)_{J^{(r)} \setminus I} \\ (l_\lambda)_{J^{(l)} \setminus I} \end{pmatrix} = (\hat{\mathcal{A}}_h^+ w)_{J^{(r)} \setminus I, J^{(l)} \setminus I} + \lambda \begin{pmatrix} u_{J^{(r)} \setminus I} \\ v_{J^{(l)} \setminus I} \end{pmatrix}$$

and from the proof of Theorem 2, we know that

$$\begin{pmatrix} u_{J^{(r)} \setminus I} \\ v_{J^{(l)} \setminus I} \end{pmatrix} = \frac{1}{\gamma_3 h_n^3} |\rho| + o\left(\frac{1}{h_n^3}\right),$$

where γ_k is defined in (D.11). Therefore, the components of $(r_\lambda)_{J^{(r)} \setminus I}$ and $(l_\lambda)_{J^{(l)} \setminus I}$ are positive provided $\left\| \hat{\mathcal{A}}_h^+ \right\|_{\infty, \mathcal{H}} \|w\|_{\mathcal{H}} < \lambda \min \{u_j, v_{j'}; j \in J^{(r)} \setminus I, j' \in J^{(l)} \setminus I\} > 0$, and this condition holds for large n provided $\|w\|_{\mathcal{H}} / \lambda \leq c_3$, where

$$c_3 = \frac{\min_{i \in I} |\rho_i|}{2|\gamma_3| \left\| (\Phi_{x_0}^{(3),*} \tilde{\Pi} \Phi_{x_0}^{(3)})^{-1} \Phi_{x_0}^{(3),*} \tilde{\Pi} \right\|_{\infty, \mathcal{H}}}. \quad (\text{D.14})$$

For the last point, we have to ensure that

$$\max \left[\begin{pmatrix} (A^* + \frac{h}{2} B^*)_{(J^{(r)})^c} \\ (A^* - \frac{h}{2} B^*)_{(J^{(l)})^c} \end{pmatrix} p_\lambda \right] < 1$$

Since by construction

$$\begin{aligned} p_\lambda &= \frac{1}{\lambda} \left(w - \hat{\mathcal{A}}_h (\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \hat{\mathcal{A}}_h^* w + \lambda \hat{\mathcal{A}}_h (\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N} \right) \\ &= \frac{1}{\lambda} \left(\tilde{\Pi} w + \lambda \hat{\mathcal{A}}_h^{+,*} \mathbf{1}_{3N} \right) = \frac{1}{\lambda} \left(\tilde{\Pi} w \right) + p_{0,n}, \end{aligned}$$

we are led to check that

$$(\omega + \lambda\eta_{0,n})(jh) + \frac{h}{2}(\omega + \lambda\eta_{0,n})'(jh) < \lambda \quad \text{for all } j \in (J^{(\mathbf{r})})^C, \quad (\text{D.15})$$

$$(\omega + \lambda\eta_{0,n})(jh) - \frac{h}{2}(\omega - \lambda\eta_{0,n})'(jh) < \lambda \quad \text{for all } j \in (J^{(\mathbf{l})})^C, \quad (\text{D.16})$$

where $\omega \stackrel{\text{def.}}{=} \Phi^* \tilde{\Pi} w$ and $\eta_{0,n} \stackrel{\text{def.}}{=} \Phi^*(\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N} = \Phi^* p_{0,n}$ yields the minimal norm certificate

$$\bar{\eta}_{0,n} = \begin{pmatrix} (\eta_{0,n} + \frac{h}{2}\eta'_{0,n})(\mathcal{G}_n) \\ (\eta_{0,n} - \frac{h}{2}\eta'_{0,n})(\mathcal{G}_n) \end{pmatrix} = \begin{pmatrix} (A + \frac{h}{2}B)^* \\ (A - \frac{h}{2}B)^* \end{pmatrix} (\hat{\mathcal{A}}_h^* \hat{\mathcal{A}}_h)^{-1} \mathbf{1}_{3N}.$$

Given $0 < r < \frac{1}{2} \min_{\nu \neq \nu'} |x_{0,\nu} - x_{0,\nu'}|$, let $N(r) \stackrel{\text{def.}}{=} \bigcup_{\nu} (x_{0,\nu} - r, x_{0,\nu} + r)$ be a neighborhood of the $x_{0,\nu}$'s. By the Twice Non-Degenerate Source condition, we may choose $r > 0$, such that

$$-\tilde{k}_1 \stackrel{\text{def.}}{=} \sup_{t \in N(r)} \eta_T''(t) < 0, \quad \text{and} \quad \tilde{k}_2 \stackrel{\text{def.}}{=} \inf_{t \in N(r)} \eta_T^{(4)}(t) > 0.$$

By compactness, $\tilde{k}_3 \stackrel{\text{def.}}{=} \sup_{t \in \mathbb{T} \setminus N(r)} \eta_T(t) < 1$.

Let us recall that $\eta_{0,n} \rightarrow \eta_T$ in the sense of the uniform convergence (and similarly for the derivatives). As a result, for $n \in \mathbb{N}$ large enough,

$$\sup_{t \in N(r)} (\eta_{0,n})''(t) < -\frac{\tilde{k}_1}{2} < 0, \quad \inf_{t \in N(r)} (\eta_{0,n})^{(4)}(t) > \frac{\tilde{k}_2}{2} > 0, \quad \sup_{t \in \mathbb{T} \setminus N(r)} \eta_{0,n}(t) < \frac{1 + \tilde{k}_3}{2} < 1, \quad (\text{D.17})$$

$$\frac{h}{2} \|(\bar{\eta}_{0,n})^{(3)}\|_{\infty} \leq \frac{\tilde{k}_1}{8}, \quad \text{and} \quad \frac{h}{2} \|(\eta_{0,n})'\|_{\infty} \leq \frac{1 - \tilde{k}_3}{6}.$$

Now, we assume that $\frac{\|w\|_{\mathcal{H}}}{\lambda}$ is small enough, so that

$$\|(\Phi^{(k)})^*\|_{\infty, \mathcal{H}} \frac{\|w\|_{\mathcal{H}}}{\lambda} < \frac{\tilde{k}_1}{8}, \quad \text{for } k \in \{2, 3\}, \quad \|(\Phi^{(4)})^*\|_{\infty, \mathcal{H}} \frac{\|w\|_{\mathcal{H}}}{\lambda} < \frac{\tilde{k}_2}{4},$$

$$\text{and} \quad \|(\Phi^{(k)})^*\|_{\infty, \mathcal{H}} \frac{\|w\|_{\mathcal{H}}}{\lambda} < \frac{1 - \tilde{k}_3}{6}, \quad \text{for } k \in \{0, 1\}, \quad (\text{D.18})$$

Then, using the fact that $|\omega^{(k)}|(t) \leq \|(\Phi^{(k)})^*\|_{\infty, \mathcal{H}} \|w\|_{\mathcal{H}}$ and $h \leq 1$, we obtain

$$\sup_{t \in \mathbb{T} \setminus N(r)} \left(\frac{\omega}{\lambda} + \eta_{0,n} + \frac{h}{2} \left| \left(\frac{\omega}{\lambda} + \eta_{0,n} \right)' \right| \right) (t) < 1.$$

Thus it remains to prove that for each $\nu \in \{1, \dots, N\}$,

$$\left(\frac{\omega}{\lambda} + \eta_{0,n} + \frac{h}{2} \left(\frac{\omega}{\lambda} + \eta_{0,n} \right)' \right) (t) < 1 \quad \text{for } t \in (x_{0,\nu} - r, x_{0,\nu} + r) \setminus S_{n,\nu}^{(\mathbf{r})}(r), \quad (\text{D.19})$$

$$\text{and} \quad \left(\frac{\omega}{\lambda} + \eta_{0,n} - \frac{h}{2} \left(\frac{\omega}{\lambda} + \eta_{0,n} \right)' \right) (t) < 1 \quad \text{for } t \in (x_{0,\nu} - r, x_{0,\nu} + r) \setminus S_{n,\nu}^{(\mathbf{l})}(r). \quad (\text{D.20})$$

We only deal with the case $S_{n,\nu}^{(r)} = \{x_{0,\nu}\}$, $S_{n,\nu}^{(l)} = \{x_{0,\nu}, x_{0,\nu} + h\}$, the symmetric case being similar. Let $f \stackrel{\text{def.}}{=} \frac{1}{\lambda}\omega(\cdot - x_{0,\nu}) + \eta_{0,n}(\cdot - x_{0,\nu})$. By definition of $\tilde{\Pi}$, $\omega(x_{0,\nu}) = \omega'(x_{0,\nu}) = \omega(x_{0,\nu} + h) - \frac{h}{2}\omega(x_{0,\nu} + h) = 0$, so that

$$f(0) = 1, \quad f'(0) = 1, \quad \text{and} \quad f(h) - \frac{h}{2}f'(h) = 1. \quad (\text{D.21})$$

Moreover, from Eq. (D.17) to (D.18), and letting $k_1 = \frac{\tilde{k}_1}{8}$, $k_2 = \frac{\tilde{k}_2}{4}$, we deduce that

$$\forall t \in (-r, r), \quad f''(t) + \frac{h}{2}|f^{(3)}(t)| < -k_1 < 0, \quad \text{and} \quad f^{(4)}(t) > k_2 > 0, \quad (\text{D.22})$$

so that the strict concavity of $f - \frac{h}{2}f'$ implies that $(f - \frac{h}{2}f')(t) < 1$ for $t \in (-r, -h) \cup (0, r)$.

It remains to prove that $(f + \frac{h}{2}f')(t) < 1$ for $t \in (-r, r) \setminus (-h, 0]$. A Taylor expansion of f and f' yields (writing as usual $\gamma_k = \frac{1}{k!} - \frac{1}{(k-1)! \times 2}$)

$$\begin{aligned} \underbrace{1 - (f(h) - \frac{h}{2}f'(h))}_{=0} &= \underbrace{1 - f(0) - \frac{h}{2}f'(0)}_{=0} - h^3\gamma_3f^{(3)}(0) - h^4\gamma_4f^{(4)}(0) + R_1(h), \\ 1 - (f(-h) + \frac{h}{2}f'(-h)) &= 1 - f(0) + \underbrace{\frac{h}{2}f'(0)}_{=0} + h^3\gamma_3f^{(3)}(0) - h^4\gamma_4f^{(4)}(0) + R_2(h). \end{aligned}$$

Adding both equations we get

$$1 - (f(-h) + \frac{h}{2}f'(-h)) = -2h^4\gamma_4f^{(4)}(0) + (R_1 + R_2)(h) \quad (\text{D.23})$$

$$\geq 2(-\gamma_4)k_2h^4 + (R_1 + R_2)(h), \quad (\text{D.24})$$

where

$$R_1 + R_2(h) = h^5 \int_0^1 (-f^{(5)}(sh) + f^{(5)}(-sh)) \left(\frac{(1-s)^4}{4!} - \frac{(1-s)^3}{2 \times 3!} \right) ds, \quad (\text{D.25})$$

$$\text{with} \quad \|f^{(5)}\|_\infty \leq \left\| \frac{1}{\lambda}\omega^{(5)} \right\|_\infty + \|(\eta_{0,n})^{(5)}\|_\infty = O(1). \quad (\text{D.26})$$

Hence,

$$1 - (f(-h) + \frac{h}{2}f'(-h)) \geq \underbrace{2(-\gamma_4)k_2}_{>0} h^4 + O(h^5) > 0. \quad (\text{D.27})$$

Moreover, by the strict concavity of $f + \frac{h}{2}f'$, we also deduce that $(f + \frac{h}{2}f')(t) < 1$ for $t \in (-r, -h] \cup (0, r)$, thus we get the local inequalities (D.19) and (D.20), hence the global inequalities (D.15) and (D.16).

To conclude, the (strict) extremality relations hold for (r_λ, l_λ) and p_λ and they are unique solutions of the primal and dual problems respectively. The condition on $\frac{\|w\|_{\mathcal{H}}}{\lambda}$ are $O(1)$, and we eventually obtain $C_n^{(1)} = O(h_n^3)$, $C_n^{(2)} = O(1)$. The proof of (46) follows from the expression of (r_λ, l_λ) .

References

- [1] J-M. Azais, Y. De Castro, and F. Gamboa. *Spike detection from inaccurate samplings*. Tech. rep. Preprint hal-00780808, Jan. 2013.
- [2] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. New York: Springer-Verlag, 2011.
- [3] B.N. Bhaskar and B. Recht. “Atomic norm denoising with applications to line spectral estimation”. In: *2011 49th Annual Allerton Conference on Communication, Control, and Computing*. 2011, pp. 261–268.
- [4] T. Blu et al. “Sparse Sampling of Signal Innovations”. In: *IEEE Signal Processing Magazine* 25.2 (2008), pp. 31–40.
- [5] K. Bredies and H.K. Pikkarainen. “Inverse problems in spaces of measures”. In: *ESAIM: Control, Optimisation and Calculus of Variations* 19.1 (2013), pp. 190–218.
- [6] E. J. Candès and C. Fernandez-Granda. “Super-Resolution from Noisy Data”. In: *Journal of Fourier Analysis and Applications* 19.6 (2013), pp. 1229–1254.
- [7] E. J. Candès and C. Fernandez-Granda. “Towards a Mathematical Theory of Super-Resolution”. In: *Communications on Pure and Applied Mathematics* 67.6 (2013), pp. 906–956.
- [8] S.S. Chen, D.L. Donoho, and M.A. Saunders. “Atomic decomposition by basis pursuit”. In: *SIAM journal on scientific computing* 20.1 (1999), pp. 33–61.
- [9] J. F. Claerbout and F. Muir. “Robust modeling with erratic data”. In: *Geophysics* 38.5 (1973), pp. 826–844.
- [10] G. Dal Maso. *An Introduction to Γ -convergence*. Vol. 8. Progress in nonlinear differential equations and their applications. Boston, MA: Birkhauser, 1993.
- [11] Y. de Castro and F. Gamboa. “Exact reconstruction using Beurling minimal extrapolation”. In: *Journal of Mathematical Analysis and Applications* 395.1 (2012), pp. 336–354.
- [12] Q. Denoyelle, V. Duval, and G. Peyré. “Support Recovery for Sparse Deconvolution of Positive Measures”. In: 1506.08264 (2015). URL: <http://arxiv.org/abs/1506.08264>.
- [13] D. L. Donoho. “Superresolution via sparsity constraints”. In: *SIAM J. Math. Anal.* 23.5 (1992), pp. 1309–1331.
- [14] C. Dossal. “A necessary and sufficient condition for exact recovery by ℓ^1 minimization”. In: *Comptes Rendus Mathématique* 350.1 (2011), pp. 117–120.
- [15] C. Dossal and S. Mallat. “Sparse spike deconvolution with minimum scale”. In: *Proceedings of SPARS*. Nov. 2005, pp. 123–126.
- [16] V. Duval and G. Peyré. “Exact Support Recovery for Sparse Spikes Deconvolution”. In: *to appear in Foundation of Computational Mathematics* (2015).

- [17] V. Duval and G. Peyré. *Sparse Spikes Retrieval on Thin Grids I: the LASSO*. Tech. rep. 2016.
- [18] C. Ekanadham, D. Tranchina, and E.P. Simoncelli. “A unified framework and method for automatic neural spike identification”. In: *Journal of Neuroscience Methods* 222 (2014), pp. 47–55.
- [19] C. Ekanadham, D. Tranchina, and E.P. Simoncelli. “Recovery of Sparse Translation-Invariant Signals With Continuous Basis Pursuit”. In: *Signal Processing, IEEE Transactions on* 59.10 (2011), pp. 4735–4744.
- [20] C. Fernandez-Granda. “Support detection in super-resolution”. In: *Proc. Proceedings of the 10th International Conference on Sampling Theory and Applications* (2013), pp. 145–148.
- [21] A. Florescu et al. *A Constrained Optimization Approach For Complex Sparse Perturbed Models*. Tech. rep. Preprint hal-00783298, 2013.
- [22] J.J. Fuchs. “On sparse representations in arbitrary redundant bases”. In: *IEEE Transactions on Information Theory* 50.6 (2004), pp. 1341–1344.
- [23] P. Heins. In: *PhD thesis* (2014).
- [24] J. Min et al. “FALCON: Fast and Unbiased Reconstruction of High-Density Super-Resolution Microscopy Data”. In: *Scientific Reports* 4577 (2014), pp. 1–9.
- [25] J. W. Odendaal, E. Barnard, and C. W. I. Pistorius. “Two-dimensional superresolution radar imaging using the MUSIC algorithm”. In: *IEEE Transactions on Antennas and Propagation* 42 (Oct. 1994), pp. 1386–1391.
- [26] R. T. Rockafellar, R. J-B. Wets, and M. Wets. *Variational analysis*. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg, New York: Springer, 1998.
- [27] G. Still. “Discretization in semi-infinite programming: the rate of convergence”. In: *Mathematical Programming* 91.1 (2001), pp. 53–69.
- [28] G. Tang, B. Narayan Bhaskar, and B. Recht. “Near Minimax Line Spectral Estimation”. In: *CoRR* abs/1303.4348 (2013).
- [29] G. Tang, B. Narayan Bhaskar, and B. Recht. “Sparse recovery over continuous dictionaries-just discretize”. In: *Proc. ACSSC. IEEE*, 2013, pp. 1043–1047.
- [30] R. Tibshirani. “Regression shrinkage and selection via the Lasso”. In: *Journal of the Royal Statistical Society. Series B. Methodological* 58.1 (1996), pp. 267–288.