Algebraic renormalisation of regularity structures

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Abstract

We give a systematic description of a canonical renormalisation procedure of stochastic PDEs containing nonlinearities involving generalised functions. This theory is based on the construction of a new class of regularity structures which comes with an explicit and elegant description of a subgroup of their group of automorphisms. This subgroup is sufficiently large to be able to implement a version of the BPHZ renormalisation prescription in this context. This is in stark contrast to previous works where one considered regularity structures with a much smaller group of automorphisms, which lead to a much more indirect and convoluted construction of a renormalisation group acting on the corresponding space of admissible models by continuous transformations.

Our construction is based on bialgebras of decorated coloured forests in cointeraction. More precisely, we have two Hopf algebras in cointeraction, coacting jointly on a vector space which represents the generalised functions of the theory. Two twisted antipodes play a fundamental role in the construction and provide a variant of the algebraic Birkhoff factorisation that arises naturally in perturbative quantum field theory.

Contents

1 Introduction

In a series of celebrated papers [Che54, Che57, Che58, Che71] Kuo-Tsai Chen discovered that, for any finite alphabet A, the family of iterated integrals of a smooth path $x : \mathbf{R}_+ \to \mathbf{R}^A$ has a number of interesting algebraic properties. Writing $\mathcal{T} = T(\mathbf{R}^A)$ for the tensor algebra on $\mathbf{R}^A$, which we identify with the space spanned by all finite words $\{(a_1 \cdots a_n)\}_{n \geq 0}$ with letters in A, we define the family of functionals $\mathbb{X}_{s,t}$ on $\mathcal{T}$ inductively by

$$\mathbb{X}_{s,t}() \stackrel{\text{def}}{=} 1, \qquad \mathbb{X}_{s,t}(a_1 \cdots a_n) \stackrel{\text{def}}{=} \int_s^t \mathbb{X}_{s,u}(a_1 \cdots a_{n-1})\, \dot{x}_{a_n}(u)\, du$$

where $0 \leq s \leq t$. Chen showed that this family yields for fixed s, t a character on $\mathcal{T}$ endowed with the *shuffle product* $\sqcup\!\sqcup$, namely

$$\mathbb{X}_{s,t}(v \sqcup\!\sqcup w) = \mathbb{X}_{s,t}(v)\, \mathbb{X}_{s,t}(w), \tag{1.1}$$

which furthermore satisfies the *flow relation*

$$(\mathbb{X}_{s,r} \otimes \mathbb{X}_{r,t})\Delta\tau = \mathbb{X}_{s,t}\tau, \qquad s \leq r \leq t,$$

where $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ is the *deconcatenation coproduct*

$$\Delta(a_1 \cdots a_n) = \sum_{k=0}^{n} (a_{k+1} \cdots a_n) \otimes (a_1 \cdots a_k).$$

In other words, we have a function $(s,t) \mapsto \mathbb{X}_{s,t} \in \mathcal{T}^*$ which takes values in the characters on the algebra $(\mathcal{T}, \sqcup\!\sqcup)$ and satisfies the *Chen relation*

$$\mathbb{X}_{s,r} \star \mathbb{X}_{r,t} = \mathbb{X}_{s,t}, \qquad s \le r \le t, \tag{1.2}$$

where $\star$ is the product dual to Δ. Note that $\mathcal{T}$, endowed with the shuffle product and the deconcatenation coproduct, is a Hopf algebra.

These two remarkable properties do not depend explicitly on the differentiability of the path $(x_t)_{t\ge 0}$. They can therefore serve as an important tool if one wants to consider non-smooth paths and still build a consistent calculus. This intuition was at the heart of Terry Lyons' definition [Lyo98] of a *geometric rough path* as a function $(s,t) \mapsto \mathbb{X}_{s,t} \in \mathcal{T}^*$ satisfying the two algebraic properties above and with a controlled modulus of continuity, for instance of Hölder type

$$|\mathbb{X}_{s,t}(a_1 \cdots a_n)| \le C|t-s|^{n\gamma}, \tag{1.3}$$

with some fixed $\gamma > 0$ (although the original definition involved rather a p-variation norm, which is natural in this context since it is invariant under reparametrisation). Lyons realised that this setting would allow to build a robust theory of integration and of associated differential equations. For instance, in the case of stochastic differential equations of Stratonovich type

$$dX_t = \sigma(X_t) \circ dW_t$$

with $W : \mathbf{R}_+ \to \mathbf{R}^d$ a d-dimensional Brownian motion and $\sigma : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ smooth, one can build rough paths $\mathbb{X}$ and $\mathbb{W}$ over X, respectively W, such that the map $\mathbb{W} \mapsto \mathbb{X}$ is *continuous*, while in general the map $W \mapsto X$ is simply measurable.

The Itô stochastic integration was included in Lyons' theory although it can not be described in terms of geometric rough paths. A few years later Massimiliano Gubinelli [Gub10] introduced the concept of a *branched rough path* as a function $(s,t) \mapsto \mathbb{X}_{s,t} \in \mathcal{H}^*$ taking values in the characters of an algebra $(\mathcal{H}, \cdot)$ of rooted forests, satisfying the analogue of the Chen relation (1.2) with respect to the Grossman-Larsson $\star$-product, dual of the *Connes-Kreimer coproduct*, and with a regularity condition

$$|\mathbb{X}_{s,t}(\tau)| \le C|t-s|^{|\tau|\gamma} \tag{1.4}$$

where $|\tau|$ counts the number of nodes in the forest τ and $\gamma > 0$ is fixed. Again, this framework allows for a robust theory of integration and differential equations driven by branched rough paths. Moreover $\mathcal{H}$, endowed with the forest product and Connes-Kreimer coproduct, turns out to be a Hopf algebra.

The theory of *regularity structures* [Hai14], due to the second named author of this paper, arose from the desire to apply the above ideas to (stochastic) partial differential equations involving non-linearities of (random) space-time distributions. Prominent examples are the KPZ equation [Hai13, FH14, GP15], the Φ^4 stochastic

quantization equation [JLM85, AR91, DPD03, Hai14, CC13, Kup16], the continuous parabolic Anderson model [HL15b, HL15a, GIP15], and the stochastic Navier-Stokes equations [DPD02, ZZ14].

One major obstacle to the application of the rough paths framework to such SPDEs is that one would like to allow for the analogue of the map $s \mapsto \mathbb{X}_{s,t}\tau$ to be a space-time distribution for some $\tau \in \mathscr{H}$. However, the algebraic relations discussed above involve *products* of such quantities, which are in general ill-defined. One of the main ideas of [Hai14] was to replace the Hopf-algebra structure with a comodule structure: instead of a single space $\mathscr{H}$, we have two spaces $(\mathcal{T}, \mathcal{T}_+)$ and a coaction $\Delta^+ : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}_+$ such that $\mathcal{T}$ is a right comodule over the Hopf algebra $\mathcal{T}_+$. In this way, elements in the dual space $\mathcal{T}^*$ of $\mathcal{T}$ can code the distributional objects which are needed in the theory, while elements of the dual space $\mathcal{T}_+^*$ of $\mathcal{T}_+$ represent classical functions. Note that $\mathcal{T}$ is not assumed to be an algebra, which reflects the fact that we do not expect to be able to multiply arbitrary distributions with each other; $\mathcal{T}$ is not assumed to be a coalgebra either, so that $\mathcal{T}^*$ has no natural $\star$-product and the Chen relation (1.2) can not be given a meaning.

However the comodule structure allows to define the analogue of a rough path as a *pair*: a distribution-valued object indexed by $\mathcal{T}$ and a function-valued object indexed by $\mathcal{T}_+$. More precisely, we consider a distribution-valued function

$$\mathbf{R}^d \ni y \mapsto \Pi_y \tau \in \mathcal{T}^* \otimes \mathscr{D}'(\mathbf{R}^d) \,,$$

as well as a continuous function

$$\mathbf{R}^d \times \mathbf{R}^d \ni (x, y) \mapsto \gamma_{x,y} \in \mathcal{T}_+^* .$$

The analogue of the Chen relation (1.2) is then given by

$$\gamma_{xy} \star \gamma_{yz} = \gamma_{xz} \,, \qquad \Pi_y \star \gamma_{yz} = \Pi_z \,, \tag{1.5}$$

where the first $\star$-product is the convolution product on $\mathcal{T}_+^*$, while the second $\star$-product is given by the dual of the coaction Δ^+. This structure guarantees that all relevant expressions will be linear in the Π_y, so we never need to multiply distributions. Note that these algebraic conditions are not enough: analytic conditions analogous to (1.4) play an essential role in the analytical aspects of the theory. Once a *model* $\mathbb{X} = (\Pi, \gamma)$ has been constructed, it plays a role analogous to that of a rough path and allows to construct a robust solution theory for a class of rough (partial) differential equations.

The theory yields a *canonical lift* of any smoothened realisation of the driving noise for the stochastic PDE under consideration to a model $\mathbb{X}^\varepsilon$. Another major difference with what one sees in the rough paths setting is the following phenomenon: if we remove the regularisation as $\varepsilon \to 0$, neither the canonical model $\mathbb{X}^\varepsilon$ nor the solution to the regularised equation converge in general to a limit. This is a structural problem which reflects again the fact that some products are intrinsically ill-defined.

This is where *renormalisation* enters the game. It was already recognised in [Hai14] that one should find a group $\mathfrak{R}$ of transformations on the space of models

and elements M_ε in $\mathfrak{R}$ in such a way that, when applying M_ε to the canonical lift $\mathbb{X}^\varepsilon$, the resulting sequence of models converges to a limit. Then the theory essentially provides a *black box*, allowing to build maximal solutions for the stochastic PDE in question.

One aspect of the theory developed in [Hai14] that is far from satisfactory is that while one has in principle a characterisation of $\mathfrak{R}$, this characterisation is very indirect. The methodology pursued so far has been to first make an educated guess for a sufficiently large family of renormalisation maps, then verify *by hand* that these do indeed belong to $\mathfrak{R}$ and finally show, again *by hand*, that the renormalised models converge to a limit. Since these steps did not rely on any general theory, they had to be performed separately for each new class of stochastic PDEs.

The main aim of the present article is to define an algebraic framework allowing to build regularity structures which, on the one hand, extend the ones built in [Hai14] and, on the other hand, admit sufficiently many automorphisms (in the sense of [Hai14, Def. 2.28]) to cover the renormalisation procedures of all subcritical stochastic PDEs that have been studied to date.

Moreover our construction is not restricted to the Gaussian setting and applies to *any* choice of the driving noise with minimal integrability conditions. In particular this allows to recover all the renormalisation procedures used so far in applications of the theory [Hai14, HP15, HQ15, HS15, Hos16, SX16]. It reaches however far beyond this and shows that the BPHZ renormalisation procedure belongs to the renormalisation group of the regularity structure associated to *any* class of subcritical semilinear stochastic PDEs. In particular, this is the case for the generalised KPZ equation which is the most natural stochastic evolution on loop space and is (formally!) given in local coordinates by

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\,\xi_i\,, \tag{1.6}$$

where the ξ_i are independent space-time white noises, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the underlying manifold, and the σ_i are a collection of vector field with the property that $\sum_i L^2_{\sigma_i} = \Delta$, where L_σ is the Lie derivative in the direction of σ and Δ is the Laplace-Beltrami operator. Another example is given by the stochastic sine-Gordon equation [HS14] close to the Kosterlitz-Thouless transition. In both of these examples, the relevant group describing the renormalisation procedures is of very large dimension (about 100 in the first example and arbitrarily large in the second one), so that the verification "by hand" that it does indeed belong to the "renormalisation group" as done for example in [Hai14, HP15], would be impractical.

The renormalisation procedure of distributions coded by $\mathcal{T}$ is described in this paper by a new Hopf algebra $\mathcal{T}_-$ and a coaction $\Delta^- : \mathcal{T} \to \mathcal{T}_- \otimes \mathcal{T}$ turning $\mathcal{T}$ into a left comodule over $\mathcal{T}_-$. This construction is, crucially, compatible with the comodule structure of $\mathcal{T}$ over $\mathcal{T}_+$. In particular, $\mathcal{T}_-$ and $\mathcal{T}_+$ are in *cointeraction* (in the terminology of [Foi16]), namely $\mathcal{T}_+$ is a left-comodule over $\mathcal{T}_-$ and this coaction allows to define a skew-product Hopf algebra $\mathcal{T}_- \ltimes \mathcal{T}_+$; moreover this is

compatible with the two comodule structures of $\mathcal{T}$ over $\mathcal{T}_-$ and $\mathcal{T}_+$ respectively, making $\mathcal{T}$ a comodule over $\mathcal{T}_- \ltimes \mathcal{T}_+$.

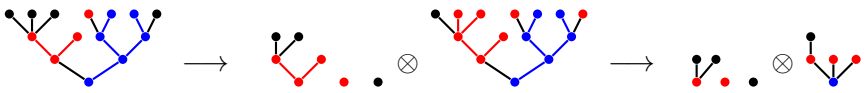
Once this structure is obtained, we can define *renormalised models* as follows: for a suitable choice of a functional $g : \mathcal{T}_- \to \mathbf{R}$,

$$\gamma^g_{z\bar{z}} = (g \otimes \gamma_{z\bar{z}})\Delta^- \,, \qquad \Pi^g_z = (g \otimes \Pi_z)\Delta^- \,.$$

With this definition one can construct a model $\mathbb{X}^g$ satisfying again the generalised Chen relation (1.5) and, crucially, the associated analytical conditions.

All the coproducts and coactions mentioned above are a priori different operators, but we describe them in a unified framework as special cases of a contraction / extraction operation of subforests, as arising in the BPHZ renormalisation procedure / forest formula [BP57, Hep69, Zim69, FMRS85]. It is interesting to remark that the structure described in this article is an extension of that previously described in [CHV05, CHV10, CEFM11] in the context of the analysis of B-series for numerical ODE solvers, which is itself an extension of the Connes-Kreimer Hopf algebra of rooted trees [CK98, CK00] arising in the abovementioned forest formula in perturbative QFT. It is also closely related to incidence Hopf algebras associated to families of posets [Sch87, Sch94].

There are however a number of substantial differences with respect to the existing literature. First we propose a new approach based on coloured forests; for instance we shall consider operations like



of colouring, extraction and contraction of subforests. Further, the abovementioned articles deal with *two* spaces in cointeraction, analogous to our Hopf algebras $\mathcal{T}_-$ and $\mathcal{T}_+$, while our third space $\mathcal{T}$ is the crucial ingredient which allows for distributions in the analytical part of the theory. Indeed, one of the main novelties of regularity structures is that they allow to study random distributional objects in a pathwise sense rather than through Feynman integrals / correlation functions and the space $\mathcal{T}$ encodes the fundamental bricks of this construction. Another important difference is that the structure described here does not consist of simple trees / forests, but they are decorated with multiindices on both their edges and their vertices. These decorations are *not inert* but transform in a non-trivial way under our coproducts, interacting with other operations like the contraction of sub-forests and the computation of suitable gradings.

In this article, Taylor sums play a very important role, just as in the BPHZ renormalisation procedure, and they appear in the actions of both $\mathcal{T}_-$ (the *renormalisation*) and $\mathcal{T}_+$ (the *recentering*). In both operations, the group elements used to perform such operations are constructed with the help of a *twisted antipode*, providing a variant of the algebraic Birkhoff factorisation that was previously shown to arise naturally in the context of perturbative quantum field theory, see for example [Kre98, CK98, CK00, CK01, EFGK04, Guo10].

In general, the context for a twisted antipode / Birkhoff factorisation is that of a group G acting on some vector space A which comes with a valuation. Given an element of A, one then wants to renormalise it by acting on it with a suitable element such that its valuation vanishes. In the context of dimensional regularisation, elements of A assign to each Feynman diagram a Laurent series in some parameter ε and the valuation extracts the pole part of this series. In our case, the space A consists of stationary linear maps $\mathbf{\Pi}\colon \mathcal{T} \to \mathcal{C}^\infty$ and we have *two* actions on it, by the group of characters $\mathcal{G}_\pm$ of $\mathcal{T}_\pm$, corresponding to two different valuations. The *renormalisation group* $\mathcal{G}_-$ is associated to the valuation that extracts the value of $\mathbf{E}(\mathbf{\Pi}\tau)(0)$ for every homogeneous element $\tau \in \mathcal{T}$ of negative degree. The *structure group* $\mathcal{G}_+$ on the other hand is associated to the valuations that extract the value $(\mathbf{\Pi}\tau)(x)$ for homogeneous elements $\tau \in \mathcal{T}$ of positive degree.

We show in particular that the twisted antipode related to the action of $\mathcal{G}_+$ is intimately related to the algebraic properties of Taylor *remainders*. Also in this respect, regularity structures provide a far-reaching generalisation of rough paths, expanding Massimiliano Gubinelli's investigation of the algebraic and analytic properties of *increments* of functions of a real variable achieved in the theory of *controlled rough paths* [Gub04].

Let us give a short survey of the content of this article. First, in Section 2, we introduce some of the tools and definitions required for our construction. In particular, we introduce a notion of bigraded space which is essential to formalise the coproducts we introduce later on. This is because our coproducts are described by infinite formal series instead of finite sums.

In Section 3, we then use these tools to give a general construction of a sequence of Hopf algebras and obtain a cointeraction property between bialgebras of forests.

In Section 4, we specialise this construction to the case relevant for the renormalisation of stochastic PDEs. This leads to the construction of two Hopf algebras $\mathcal{H}_1$ and $\mathcal{H}_2$ as well as a vector space $\mathcal{H}_\circ$ which is a left comodule for $\mathcal{H}_1$ and a right comodule for $\mathcal{H}_2$. Furthermore, $\mathcal{H}_2$ is itself a left comodule over $\mathcal{H}_1$, which allows to construct a skew-product Hopf algebra $\mathcal{H}_{12} = \mathcal{H}_1 \ltimes \mathcal{H}_2$. Finally, $\mathcal{H}_\circ$ is a left-comodule over $\mathcal{H}_{12}$. This construction is universal and does not depend on the specific class of stochastic PDEs one considers. In particular, the space $\mathcal{H}_\circ$ is actually an algebra and the coaction of $\mathcal{H}_2$ is compatible with that algebra structure.

In Section 5, we show how to further specialise this construction to specific classes of stochastic PDEs. For this, we introduce the new notion of a *rule* which formalises the notion of a *class of stochastic PDEs*. Rules come with a notion of subcriticality, which makes more precise the corresponding notion used in [Hai14] in a more informal way, as well as a notion of completeness which guarantees that the class of stochastic PDEs in question is invariant under our renormalisation procedure. Each complete subcritical rule then gives rise to subspaces $\mathcal{T}_-^{\mathrm{ex}}$ / $\mathcal{T}^{\mathrm{ex}}$ / $\mathcal{T}_+^{\mathrm{ex}}$ of $\mathcal{H}_1$ / $\mathcal{H}_\circ$ / $\mathcal{H}_2$ which still carry all of the structure described above (except that $\mathcal{T}^{\mathrm{ex}}$ is no longer an algebra), but the space $\mathcal{T}^{\mathrm{ex}}$ furthermore comes with a discrete $\mathbf{R}$-grading which turns $\mathscr{T}^{\mathrm{ex}} = (\mathcal{T}^{\mathrm{ex}}, \mathcal{G}_+^{\mathrm{ex}})$ into a regularity structure, where

$\mathcal{G}_\pm^{\mathrm{ex}}$ denotes the character group of $\mathcal{T}_\pm^{\mathrm{ex}}$. The group $\mathcal{G}_-^{\mathrm{ex}}$ then acts on this regularity structure by automorphisms in the sense of [Hai14, Def. 2.28].

Finally, in Section 6 we give a general and systematic construction of renormalised models for a subcritical SPDE with a stationary noise. In this construction the two twisted antipodes mentioned above play a critical role. We conclude by showing that although the regularity structure $\mathscr{T}^{\mathrm{ex}}$ is different from the structure $\mathscr{T}$ constructed in [Hai14] (it extends it, which is the reason for the superscript 'ex'), the action of a suitable subgroup $\mathcal{G}_-$ of $\mathcal{G}_-^{\mathrm{ex}}$ onto the space of those models for $\mathscr{T}^{\mathrm{ex}}$ that are "compatible" with $\mathscr{T}$ in a suitable sense also yields a continuous action of $\mathcal{G}_-$ onto the space of (admissible) models for $\mathscr{T}$.

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2 Rooted forests and bigraded spaces

Given a finite set S and a map $\ell\colon S \to \mathbf{N}$, we write

$$\ell! \stackrel{\text{def}}{=} \prod_{x\in S} \ell(x)!\;,$$

and we define the corresponding binomial coefficients accordingly. Note that if ℓ_1 and ℓ_2 have disjoint supports, then $(\ell_1+\ell_2)! = \ell_1!\ell_2!$. Given a map $\pi\colon S \to \bar{S}$, we also define $\pi^\star\ell\colon \bar{S} \to \mathbf{N}$ by $\pi^\star\ell(x) = \sum_{y\in\pi^{-1}(x)} \ell(y)$.

For $k, \ell : S \to \mathbf{N}$ we define

$$\binom{k}{\ell} \stackrel{\text{def}}{=} \prod_{x\in S} \binom{k(x)}{\ell(x)}\;,$$

with the convention $\binom{k}{\ell} = 0$ unless $0 \le \ell \le k$, which will be used throughout the paper. With these definitions at hand, one has the following slight reformulation of the classical Chu-Vandermonde identity.

Lemma 2.1 (Chu-Vandermonde) *For every $k\colon S \to \mathbf{N}$, one has the identity*

$$\sum_{\ell\,:\,\pi^\star\ell} \binom{k}{\ell} = \binom{\pi^\star k}{\pi^\star \ell}\;,$$

where the sum runs over all possible choices of ℓ such that $\pi^\star\ell$ is fixed. □

Remark 2.2 These notations are also consistent with the case where the maps k and ℓ are multi-index valued under the natural identification of a map $S \to \mathbf{N}^d$ with a map $S \times \{1,\dots,\infty\} \to \mathbf{N}$ given by $\ell(x)_i \leftrightarrow \ell(x,i)$.

2.1 Rooted trees and forests

Recall that a rooted tree T is a finite tree (a finite connected simple graph without cycles) with a distinguished vertex, $\varrho = \varrho_T$, called *the root*. Vertices of T, also called nodes, are denoted by $N = N_T$ and edges by $E = E_T \subset N^2$. Since we want our trees to be rooted, they need to have at least one node, so that we do not allow for trees with $N_T = \emptyset$. We do however allow for the *trivial* tree consisting of an empty edge set and a vertex set with only one element. This tree will play a special role in the sequel and will be denoted by $\bullet$. We will always assume that our trees are combinatorial meaning that there is no particular order imposed on edges leaving any given vertex.

Given a rooted tree T, we also endow N_T with the partial order $\leq$ where $w \leq v$ if and only if w is on the unique path connecting v to the root, and we orient edges in E_T so that if $(x, y) = (x \to y) \in E_T$, then $x \leq y$. In this way, we can always view a tree as a directed graph.

Two rooted trees T and T' are *isomorphic* if there exists a bijection $\iota\colon E_T \to E_{T'}$ which is coherent in the sense that there exists a bijection $\iota_N\colon N_T \to N_{T'}$ such that $\iota(x, y) = (\iota_N(x), \iota_N(y))$ for any edge $(x, y) \in e$ and such that the roots are mapped onto each other.

We say that a rooted tree is *typed* if it is furthermore endowed with a function $\mathfrak{t}\colon E_T \to \mathfrak{L}$, where $\mathfrak{L}$ is some finite set of types. We think of $\mathfrak{L}$ as being fixed once and for all and will sometimes omit to mention it in the sequel. In particular, we will never make explicit the dependence on the choice of $\mathfrak{L}$ in our notations. Two typed trees $(T, \mathfrak{t})$ and $(T', \mathfrak{t}')$ are isomorphic if T and T' are isomorphic and $\mathfrak{t}$ is pushed onto $\mathfrak{t}'$ by the corresponding isomorphism ι in the sense that $\mathfrak{t}' \circ \iota = \mathfrak{t}$.

Similarly to a tree, a *forest* F is a finite simple graph (again with nodes N_F and edges $E_F \subset N_F^2$) without cycles. A forest F is *rooted* if every connected component T of F is a rooted tree with root ϱ_T. As above, we will consider forests that are typed in the sense that they are endowed with a map $\mathfrak{t}\colon E_F \to \mathfrak{L}$, and we consider the same notion of isomorphism between typed forests as for typed trees. Note that while a tree is non-empty by definition, a forest can be empty. We denote the empty forest by either $\mathbf{1}$ or $\emptyset$.

Given a typed forest F, a subforest $A \subset F$ consists of subsets $E_A \subset E_F$ and $N_A \subset N_F$ such that if $(x, y) \in E_A$ then $\{x, y\} \subset N_A$. Types in A are inherited from F. A connected component of A is a tree whose root is defined to be the minimal node in the partial order inherited from F. We say that subforests A and B are disjoint, and write $A \cap B = \emptyset$, if one has $N_A \cap N_B = \emptyset$ (which also implies that $E_A \cap E_B = \emptyset$). Given two typed forests F, G, we write $F \sqcup G$ for the typed forest obtained by taking the disjoint union (as graphs) of the two forests F and G and adjoining to it the natural typing inherited from F and G. If furthermore $A \subset F$ and $B \subset G$ are subforests, then we write $A \sqcup B$ for the corresponding subforest of $F \sqcup G$.

We fix once and for all an integer $d \geq 1$, dimension of the parameter-space $\mathbf{R}^d$. We also denote by $\mathbf{Z}(\mathfrak{L})$ the free abelian group generated by $\mathfrak{L}$.

2.2 Coloured and decorated forests

Given a typed forest F, we want now to consider families of *disjoint subforests* of F, denoted by $(\hat{F}_i, i > 0)$. It is convenient for us to code this family with a single function $\hat{F} : E_F \sqcup N_F \to \mathbf{N}$ as given by the next definition.

Definition 2.3 A *coloured forest* is a pair $(F, \hat{F})$ such that
1. $F = (E_F, N_F, \mathfrak{t})$ is a typed rooted forest
2. $\hat{F}\colon E_F \sqcup N_F \to \mathbf{N}$ is such that if $\hat{F}(e) \neq 0$ for $e = (x, y) \in E_F$ then $\hat{F}(x) = \hat{F}(y) = \hat{F}(e)$.

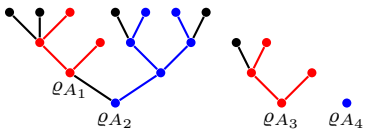
We say that $\hat{F}$ is a *colouring* of F. For $i > 0$, we define the subforest of F

$$\hat{F}_i = (\hat{E}_i, \hat{N}_i), \qquad \hat{E}_i = \hat{F}^{-1}(i) \cap E_F, \quad \hat{N}_i = \hat{F}^{-1}(i) \cap N_F,$$

as well as $\hat{E} = \bigcup_{i>0} \hat{E}_i$.

The condition on $\hat{F}$ guarantees that every $\hat{F}_i$ is indeed a subforest for $i > 0$ and that they are all disjoint. On the other hand, $\hat{F}^{-1}(0)$ is not supposed to have any particular structure and 0 is not counted as a colour.

Example 2.4 This is an example of a forest with two colours: red for 1 and blue for 2 (and black for 0)

$(F, \hat{F}) =$ 

We then have $\hat{F}_1 = \hat{F}^{-1}(1) = A_1 \sqcup A_3$ and $\hat{F}_2 = \hat{F}^{-1}(2) = A_2 \sqcup A_4$.

We add now *decorations* on the nodes and edges of a coloured forest. For this, we fix throughout this article an arbitrary "dimension" $d \in \mathbf{N}$ and we give the following definition.

Definition 2.5 We denote by $\mathfrak{F}$ the set of all 5-tuples $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ such that
1. $(F, \hat{F})$ is a coloured forest in the sense of Definition 2.3.
2. One has $\mathfrak{n}\colon N_F \to \mathbf{N}^d$
3. One has $\mathfrak{o}\colon N_F \to \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$ with $\operatorname{supp} \mathfrak{o} \subset \operatorname{supp} \hat{F}$.
4. One has $\mathfrak{e}\colon E_F \to \mathbf{N}^d$ with $\operatorname{supp} \mathfrak{e} \subset \{e \in E_F : \hat{F}(e) = 0\}$.

We identify $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ and $(F', \hat{F}', \mathfrak{n}', \mathfrak{o}', \mathfrak{e}')$ whenever F is isomorphic to F', the corresponding isomorphism maps $\hat{F}$ to $\hat{F}'$ and pushes the three decoration functions onto their counterparts. We call elements of $\mathfrak{F}$ *decorated forests*. We will also sometimes use the notation $(F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$ instead of $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$.

The set $\mathfrak{F}$ is a commutative monoid under the *forest product*

$$(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \cdot (G, \hat{G}, \mathfrak{n}', \mathfrak{o}', \mathfrak{e}') = (F \sqcup G, \hat{F} + \hat{G}, \mathfrak{n} + \mathfrak{n}', \mathfrak{o} + \mathfrak{o}', \mathfrak{e} + \mathfrak{e}')\,, \quad (2.1)$$

where decorations defined on one of the forests are extended to the disjoint union by setting them to vanish on the other forest. The identity element of $\mathfrak{F}$ is the empty forest $\mathbf{1}$.

Definition 2.6 For any coloured forest $(F, \hat{F})$, we define an equivalence relation $\sim$ on the node set N_F by saying that $x \sim y$ if x and y are connected in $\hat{E}$; this is the smallest equivalence relation for which $x \sim y$ whenever $(x, y) \in \hat{E}$.

Remark 2.7 We want to show the intuition behind decorated forests. We think of each $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ as defining a function on $(\mathbf{R}^d)^{N_F}$ in the following way. We associate to each type $\mathfrak{t} \in \mathfrak{L}$ a kernel $\varphi_{\mathfrak{t}} : \mathbf{R}^d \to \mathbf{R}$ and we define the domain

$$U_F \stackrel{\text{def}}{=} \{x \in (\mathbf{R}^d)^{N_F} : x_v = x_w \quad \text{if} \quad v \sim w\} ,$$

where $\sim$ is the equivalence relation of Definition 2.6. Then we set $H_\tau \in \mathscr{C}^\infty(U_F)$,

$$H_\tau(x_v, v \in N_F) \stackrel{\text{def}}{=} \prod_{v \in N_F} (x_v)^{\mathfrak{n}(v)} \prod_{e=(u,v) \in E_F \setminus \hat{E}} \partial^{\mathfrak{e}(e)} \varphi_{\mathfrak{t}(e)}(x_u - x_v), \tag{2.2}$$

where, for $x = (x^1, \dots, x^d) \in \mathbf{R}^d$, $n = (n^1, \dots, n^d) \in \mathbf{N}^d$ and $\varphi \in \mathscr{C}^\infty(\mathbf{R}^d)$

$$(x)^n \stackrel{\text{def}}{=} \prod_{j=1}^d (x^j)^{n^j}, \qquad \partial^n \varphi = \partial_{x^1}^{n^1} \cdots \partial_{x^d}^{n^d} \varphi \in \mathscr{C}^\infty(\mathbf{R}^d) .$$

In this way, a decorated forest encodes a function: every node in $N_F / \sim$ represents a variable in $\mathbf{R}^d$, every uncoloured edge of a certain type $\mathfrak{t}$ a function $\varphi_{\mathfrak{t}(e)}$ of the difference of the two variables sitting at each one of its nodes; the decoration $\mathfrak{n}(v)$ gives a power of x_v and $\mathfrak{e}(e)$ a derivative of the kernel $\varphi_{\mathfrak{t}(e)}$.

In this example the decoration $\mathfrak{o}$ plays no role. We shall see below that $\mathfrak{o}$ allows us to encode some additional information relevant for the various algebraic manipulations we wish to subject these functions to.

Remark 2.8 Every forest $F = (N_F, E_F)$ has a unique decomposition into non-empty connected components. This property naturally extends to decorated forests $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$, by considering the connected components of the underlying forest F and restricting the colouring $\hat{F}$ and the decorations $\mathfrak{n}, \mathfrak{o}, \mathfrak{e}$.

2.3 Bigraded spaces and triangular maps

It will be convenient in the sequel to consider a particular category of *bigraded spaces* as follows.

Definition 2.9 For a collection of vector spaces $\{V_k : k \in \mathbf{N}^2\}$, we define the vector space

$$V = \boxplus_{n \in \mathbf{N}^2} V_n ,$$

as the space of all formal sums $\sum_{n\in\mathbf{N}^2} v_n$ with $v_n \in V_n$ and such that there exists $k \in \mathbf{N}$ such that $v_n = 0$ as soon as $n_2 > k$. Given two bigraded spaces V and W, we write $V \hat{\otimes} W$ for the bigraded space

$$V \hat{\otimes} W \stackrel{\text{def}}{=} \bigboxplus_{n\in\mathbf{N}^2} \left[\bigoplus_{k+\ell=n} (V_k \otimes W_\ell) \right]. \tag{2.3}$$

One has a canonical inclusion $V \otimes W \subset V \hat{\otimes} W$, but in general the latter is strictly larger.

Definition 2.10 We introduce a partial order on $\mathbf{N}^2$ by

$$(m_1, m_2) \geq (n_1, n_2) \qquad \Leftrightarrow \qquad m_1 \geq n_1 \;\&\; m_2 \leq n_2 \;.$$

Given two such bigraded spaces V and $\bar{V}$, a family $\{A_{mn}\}_{m,n\in\mathbf{N}^2}$ of linear maps $A_{mn} : V_n \to \bar{V}_m$ is called *triangular* if $A_{mn} = 0$ unless $m \geq n$.

Lemma 2.11 *Let V and $\bar{V}$ be two bigraded spaces and $\{A_{mn}\}_{m,n\in\mathbf{N}^2}$ a triangular family of linear maps $A_{mn} : V_n \to \bar{V}_m$. Then the map*

$$Av \stackrel{\text{def}}{=} \sum_m \Big(\sum_n A_{mn} v_n \Big) \in \bigboxplus_{m\in\mathbf{N}^2} \bar{V}_m, \qquad v = \sum_n v_n \in \bigboxplus_{n\in\mathbf{N}^2} V_n$$

is well defined from V to $\bar{V}$ and linear. We call $A : V \to V$ a triangular map.

Proof. Let $v = \sum_n v_n \in V$ and $k \in \mathbf{N}$ such that $v_n = 0$ whenever $n_2 > k$.

First we note that, for fixed $m \in \mathbf{N}^2$, the family $(A_{mn}v_n)_{n\in\mathbf{N}^2}$ is zero unless $n \in [0, m_1]\times[0, k]$; indeed if $n_2 > k$ then $v_n = 0$, while if $n_1 > m_1$ then $A_{nm} = 0$. Therefore the sum $\sum_n A_{mn}v_n$ is well defined and equal to some $\bar{v}_m \in \bar{V}_m$.

We now prove that $\bar{v}_m = 0$ whenever $m_2 > k$, so that indeed $\sum_m \bar{v}_m \in \bigboxplus_{m\in\mathbf{N}^2} \bar{V}_m$. Let $m_2 > k$; for $n_2 > k$, v_n is 0, while for $n_2 \leq k$ we have $n_2 < m_2$ and therefore $A_{nm} = 0$ and this proves the claim. □

A linear function $A : V \to \bar{V}$ which can be obtained as in Lemma 2.11 is called *triangular*. The family $(A_{mn})_{m,n\in\mathbf{N}^2}$ defines an infinite lower triangular matrix and composition of triangular maps is then simply given by formal matrix multiplication, which only ever involves finite sums thanks to the triangular structure of these matrices.

Remark 2.12 The reason for introducing the notion of bigraded spaces as above is that it allows us to build a family of combinatorial Hopf algebras where the coproduct of each canonical basis vector is given by an infinite formal series. In order to define this, we could of course simply have considered spaces of arbitrary infinite formal series. However the dual of such a space contains only finite sequences, so that it typically contains no non-zero multiplicative functional at all! The notion of bigrading introduced here gives us enough flexibility to construct spaces that are sufficiently large to contain our coproducts and whose dual is still sufficiently large to contain a large class of multiplicative linear functionals.

Remark 2.13 One important remark is that this construction behaves quite nicely under duality in the sense that if V and W are two bigraded spaces, then it is still the case that one has a canonical inclusion $V^* \otimes W^* \subset (V \hat{\otimes} W)^*$. Indeed, the dual V^* consists of formal sums $\sum_n v_n^*$ with $v_n^* \in V_n^*$ such that, for every $k \in \mathbf{N}$ there exists $f(k)$ such that $v_n^* = 0$ for every $n \in \mathbf{N}^2$ with $n_1 \geq f(n_2)$.

The set $\mathfrak{F}$, see Definition 2.5, admits a number of different useful gradings and bigradings. One bigrading that is well adapted to the construction we give below is given by setting

$$|(F, \hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}|_{\mathrm{bi}} \stackrel{\mathrm{def}}{=} (|\mathfrak{e}|, |F \setminus (\hat{F} \cup \varrho_F)|) \,, \tag{2.4}$$

where

$$|\mathfrak{e}| = \sum_{e \in E_F} |\mathfrak{e}(e)|, \qquad |a| = \sum_{i=1}^{d} a_i, \qquad \forall\, a \in \mathbf{N}^d,$$

and $|F \setminus (\hat{F} \cup \varrho_F)|$ denotes the number of edges and vertices on which $\hat{F}$ vanishes that aren't roots of F. Then

$$|\mathscr{F} \cdot \mathscr{G}|_{\mathrm{bi}} = |\mathscr{F}|_{\mathrm{bi}} + |\mathscr{G}|_{\mathrm{bi}} \,, \tag{2.5}$$

for any $\mathscr{F}, \mathscr{G} \in \mathfrak{F}$.

For any subset $A \subseteq \mathfrak{F}$ let now $\langle A \rangle$ denote the space built from A with this grading, namely

$$\langle A \rangle \stackrel{\mathrm{def}}{=} \bigboxplus_{n \in \mathbf{N}^2} \mathrm{Vec}\{\mathscr{F} \in A \,:\, |\mathscr{F}|_{\mathrm{bi}} = n\} \,, \tag{2.6}$$

where Vec S denotes the free vector space generated by a set S. Whenever M is a submonoid of $\mathfrak{F}$, as a consequence of (2.5) the forest product $\cdot$ can be interpreted as a triangular linear map from $\langle M \rangle \hat{\otimes} \langle M \rangle$ into $\langle M \rangle$, thus turning $(\langle M \rangle, \cdot)$ into an algebra in the category of bigraded spaces as in Definition 2.9. This is in particular the case for $M = \mathfrak{F}$. We emphasise once again that in general $\langle M \rangle$ is larger than Vec M. The following simple fact will be used several times in the sequel. Here and throughout this article, we use as usual the notation $f \restriction A$ for the restriction of a map f to some subset A of its domain.

Lemma 2.14 *Let $V = \bigboxplus_n V_n$ be a bigraded space and let $P : V \to V$ be a triangular map preserving the bigrading of V (in the sense that there exist linear maps $P_n \colon V_n \to V_n$ such that $P \restriction V_n = P_n$ for every n) and satisfying $P \circ P = P$. Then, the quotient space $\hat{V} = V/\ker P$ is again bigraded and one has canonical identifications*

$$\hat{V} = \bigboxplus_n (V_n / \ker P_n) = \bigboxplus_n (P_n V_n) \,.$$

3 A general construction

In this section we want to introduce a general class of operators on spaces of decorated forests and show that, under suitable assumptions, one can construct

in this way bialgebras, Hopf algebras and comodules. We start with a simplified setting.

3.1 Incidence coalgebras of subforests

Denote by $\mathscr{P}$ the set of all pairs $(G; F)$ such that F is a typed forest and G is a subforest of F and by $\mathfrak{C}$ the free vector space generated by $\mathscr{P}$. Suppose that for all $(G; F) \in \mathscr{P}$ we are given a (finite) collection $\mathfrak{A}(G; F)$ of subforests A of F such that $G \subseteq A \subseteq F$. Then we define the linear map $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}$ by

$$\Delta(G; F) \stackrel{\text{def}}{=} \sum_{A \in \mathfrak{A}(G;F)} (G; A) \otimes (A; F). \tag{3.1}$$

If $\mathfrak{A}(G; F)$ is equal to the set of all subforests A of F containing G, then it is a simple exercise to show that Δ is *coassociative*, namely that

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \qquad \text{on } \mathfrak{C}.$$

In particular, since the inclusion $G \subseteq F$ endows the set of typed forests with a partial order, $(\mathfrak{C}, \Delta)$ is an example of an *incidence coalgebra*, see [Sch87, Sch94]. However, if $\mathfrak{A}(F; G)$ is a more general class of subforests, then coassociativity is not granted in general and holds only under certain assumptions.

Suppose now that, given a typed forest F, we want to consider not one but several disjoint subforests $G_1, \dots, G_n$ of F. A natural way to code $(G_1, \dots, G_n; F)$ is to use a coloured forest $(F, \hat{F})$ where

$$\hat{F}(x) = \sum_k k \, \mathbb{1}_{x \in G_k}, \qquad x \in N_F \sqcup E_F.$$

Then, in the notation of Definition 2.3, we have $\hat{F}_i = G_i$ for $i > 0$ and $\hat{F}^{-1}(0) = F \setminus (\cup_i G_i)$.

In order to define a generalisation of the operator Δ of formula (3.1) to this setting, we fix $i > 0$ and assume the following.

Assumption 1 *Let $i > 0$. For each coloured forest $(F, \hat{F})$ as in Definition 2.3 we are given a collection $\mathfrak{A}_i(F, \hat{F})$ of subforests of F such that for every $A \in \mathfrak{A}_i(F, \hat{F})$*

1. *$\hat{F}_i \subset A$ and $\hat{F}_j \cap A = \emptyset$ for every $j > i$,*
2. *for all $0 < j < i$ and every connected component T of $\hat{F}_j$, one has either $T \subset A$ or $T \cap A = \emptyset$.*

We also assume that $\mathfrak{A}_i$ is compatible with the equivalence relation $\sim$ given by tree isomorphisms described above in the sense that if $A \in \mathfrak{A}_i(F, \hat{F})$ and $\iota\colon (F, \hat{F}) \to (G, \hat{G})$ is a tree isomorphism, then $\iota(A) \in \mathfrak{A}_i(G, \hat{G})$.

It is important to note that colours are denoted by positive integer numbers and are therefore ordered, so that the forests $\hat{F}_j$, $\hat{F}_i$ and $\hat{F}_k$ can play different roles in Assumption 1 if $j < i < k$. This becomes crucial in our construction below, see Proposition 3.22 and Remark 3.24.

Lemma 3.1 *Let $(F, \hat{F})$ be a coloured forest and $A \in \mathfrak{A}_i(F, \hat{F})$. Write*

- *$\hat{F}\restriction A$ for the restriction of $\hat{F}$ to $N_A \sqcup E_A$*
- *$\hat{F} \cup_i A$ for the function on $E_F \sqcup N_F$ given by*

$$(\hat{F} \cup_i A)(x) = \begin{cases} i & \text{if } x \in E_A \sqcup N_A, \\ \hat{F}(x) & \text{otherwise.} \end{cases}$$

Then, under Assumption 1, $(A, \hat{F}\restriction A)$ and $(F, \hat{F} \cup_i A)$ are coloured forests.

Proof. The claim is elementary for $(A, \hat{F}\restriction A)$; in particular, setting $\hat{G} \stackrel{\text{def}}{=} \hat{F}\restriction A$, we have $\hat{G}_j = \hat{F}_j \cap A$ for all $j > 0$. We prove it now for $(F, \hat{F} \cup_i A)$. We must prove that, setting $\hat{G} \stackrel{\text{def}}{=} \hat{F} \cup_i A$, the sets $\hat{G}_j \stackrel{\text{def}}{=} \hat{G}^{-1}(j)$ define subforests of F for all $j > 0$. We have by the definitions

$$\hat{G}_i = \hat{F}_i \cup A, \qquad \hat{G}_j = \hat{F}_j \backslash A, \qquad j \neq i,\ j > 0,$$

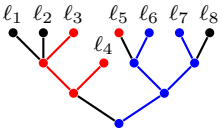
and these are subforests of F by the properties 1 and 2 of Assumption 1. □

This allows to define the following operator for fixed $i > 0$

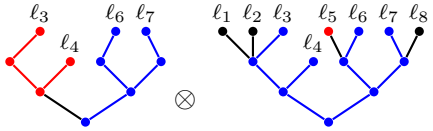
$$\Delta_i(F, \hat{F}) \stackrel{\text{def}}{=} \sum_{A \in \mathfrak{A}_i(F, \hat{F})} (A, \hat{F}\restriction A) \otimes (F, \hat{F} \cup_i A). \tag{3.2}$$

Note that if $i = 1$ and $\hat{F} \leq 1$ then we can identify the coloured forest $(F, \hat{F})$ with the pair of subforests $(\hat{F}_1; F) \in \mathcal{P}$, $\mathfrak{A}(\hat{F}_1; F)$ with $\mathfrak{A}_1(F, \hat{F})$ and Δ in (3.1) with Δ_1 in (3.2).

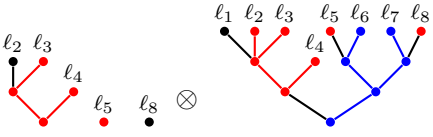
Example 3.2 Consider the following example of a forest with two colours: red for 1 and blue for 2 (and black for 0)

$(F, \hat{F}) =$ 

A valid example of $A \in \mathfrak{A}_2(F, \hat{F})$ could be such that

$(A, \hat{F}\restriction A) \otimes (F, \hat{F} \cup_2 A) =$ 

Note that in this example, one has $\hat{F}_2 \subset A$, so that $A \notin \mathfrak{A}_1(F, \hat{F})$ since A violates the first condition of Assumption 1. A valid example of $B \in \mathfrak{A}_1(F, \hat{F})$ could be such that

$(B, \hat{F}\restriction B) \otimes (F, \hat{F} \cup_1 B) =$ 

In the rest of this section we state several assumptions on the family $\mathfrak{A}_i(F, \hat{F})$ yielding nice properties for the operator Δ_i such as coassociativity, see e.g. Assumption 2. However, one of the main results of this article is the fact that such properties then automatically also hold at the level of decorated forests with a non-trivial action on the decorations which will be defined in the next subsection.

3.2 Operators on decorated forests

We generalise now the construction (3.2) to decorated forests.

Definition 3.3 The triangular linear maps $\Delta_i \colon \langle \mathfrak{F} \rangle \to \langle \mathfrak{F} \rangle \hat{\otimes} \langle \mathfrak{F} \rangle$ are given for $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ by

$$\Delta_i \tau = \sum_{A \in \mathfrak{A}_i(F,\hat{F})} \sum_{\varepsilon_A^F, \mathfrak{n}_A} \frac{1}{\varepsilon_A^F!} \binom{\mathfrak{n}}{\mathfrak{n}_A} (A, \hat{F} \restriction A, \mathfrak{n}_A + \pi \varepsilon_A^F, \mathfrak{o} \restriction N_A, \mathfrak{e} \restriction E_A) \tag{3.3}$$
$$\otimes (F, \hat{F} \cup_i A, \mathfrak{n} - \mathfrak{n}_A,\ \mathfrak{o} + \mathfrak{n}_A + \pi(\varepsilon_A^F - \mathfrak{e}_{\emptyset}^A), \mathfrak{e}_A^F + \varepsilon_A^F) \,,$$

where

a) For $A \subseteq B \subseteq F$ and $f : E_F \to \mathbf{N}^d$, we use the notation $f_A^B \stackrel{\text{def}}{=} f \, \mathbb{1}_{E_B \setminus E_A}$.
b) The sum over $\mathfrak{n}_A$ runs over all maps $\mathfrak{n}_A : N_F \to \mathbf{N}^d$ with $\operatorname{supp} \mathfrak{n}_A \subset N_A$.
c) The sum over ε_A^F runs over all $\varepsilon_A^F : E_F \to \mathbf{N}^d$ supported on the set of edges

$$\partial(A, F) \stackrel{\text{def}}{=} \{(e_+, e_-) \in E_F \setminus E_A \,:\, e_+ \in N_A\}, \tag{3.4}$$

that we call the *boundary of A in F*. This notation is consistent with point a).
d) For all $\varepsilon : E_F \to \mathbf{Z}^d$ we denote

$$\pi\varepsilon : N_F \to \mathbf{Z}^d, \qquad \pi\varepsilon(x) \stackrel{\text{def}}{=} \sum_{e=(x,y) \in E_F} \varepsilon(e).$$

We will henceforth use these notational conventions for sums over node / edge decorations without always spelling them out in full.

Example 3.4 Let $(F, \hat{F})$ and A as in Example 3.2. Then the boundary of A in F is given by $\partial(A, F) = \{(x, \ell_1), (y, \ell_5), (z, \ell_8)\}$ for some $x, y, z \in N_A$.

Remark 3.5 It may not be obvious why Definition 3.3 is natural, so let us try to offer an intuitive explanation of where it comes from. First note that (3.3) reduces to (3.2) if we drop the decorations and the combinatorial coefficients.

If we go back to Remark 2.7, and we recall that a decorated forest encodes a function of a set of variables in $\mathbf{R}^d$ indexed by the nodes of the underlying forest, then we can realise that the operator Δ_i in (3.3) is naturally motivated by Taylor expansions.

Let us consider first the particular case of $\tau = (F, \hat{F}, 0, \mathfrak{o}, \mathfrak{e})$. Then $\mathfrak{n}_A$ has to vsnish because of the constraint $0 \leq \mathfrak{n}_A \leq \mathfrak{n}$ and (3.3) becomes

$$\Delta_i \tau = \sum_{A \in \mathfrak{A}_i(F,\hat{F})} \sum_{\varepsilon_A^F} \frac{1}{\varepsilon_A^F!} (A, \hat{F} \restriction A, \pi \varepsilon_A^F, \mathfrak{o}, \mathfrak{e}) \tag{3.5}$$

$$\otimes\, (F, \hat{F} \cup_i A, 0,\ \mathfrak{o} + \pi(\varepsilon_A^F - \mathfrak{e}_\emptyset^A), \mathfrak{e}_A^F + \varepsilon_A^F)\ .$$

Consider a single term in this sum and fix an edge $e = (v, w) \in \partial(A, F)$. Then, in the expression

$$(F, \hat{F} \cup_i A, 0,\ \mathfrak{o} + \pi(\varepsilon_A^F - \mathfrak{e}_\emptyset^A), \mathfrak{e}_A^F + \varepsilon_A^F)\ ,$$

the decoration of e is changing from $\mathfrak{e}(e)$ to $\mathfrak{e}(e) + \varepsilon_A^F(e)$. Recalling (2.2), this should be interpreted as differentiating $\varepsilon_A^F(e)$ times the kernel encoded by the edge e. At the same time, in the expression

$$(A, \hat{F} \restriction A, \pi\varepsilon_A^F, \mathfrak{o} \restriction N_A, \mathfrak{e} \restriction E_A)\ ,$$

the term $\pi\varepsilon_A^F(v)$ is a sum of several contributions, among which $\varepsilon_A^F(e)$. If we take into account the factor $1/\varepsilon_A^F(e)!$, we recognise a (formal) Taylor sum

$$\sum_{k \in \mathbf{N}^d} \frac{(x_v)^k}{k!} \partial_{x_v}^{\mathfrak{e}(e)+k} \varphi_{\mathfrak{t}(e)}(x_v - x_w), \qquad e = (v, w) \in \partial(A, F).$$

If $\mathfrak{n}$ is not zero, then we have a similar Taylor sum given by

$$\sum_{k \in \mathbf{N}^d} \frac{(x_v)^k}{k!} \partial_{x_v}^{\mathfrak{e}(e)+k} \big[(x_v)^{\mathfrak{n}(v)} \varphi_{\mathfrak{t}(e)}(x_v - x_w)\big], \qquad e = (v, w) \in \partial(A, F).$$

The role of the decoration $\mathfrak{o}$ is still mysterious at this stage: we ask the reader to wait until the Remarks 3.15, 5.38 and 6.25 below for an explanation. The connection between our construction and Taylor expansions (more precisely, Taylor *remainders*) will be made clear in Lemma 6.9 and Remark 6.10 below.

Remark 3.6 Note that, in (3.3), for each fixed A the decoration $\mathfrak{n}_A$ runs over a finite set because of the constraint $0 \le \mathfrak{n}_A \le \mathfrak{n}$.

On the other hand, ε_A^F runs over an infinite set, but the sum is nevertheless well defined as an element of $\langle \mathfrak{F} \rangle \hat{\otimes} \langle \mathfrak{F} \rangle$, even though it does *not* belong to the algebraic tensor product $\langle \mathfrak{F} \rangle \otimes \langle \mathfrak{F} \rangle$. Indeed, since $|\mathfrak{e} \restriction A| + |\mathfrak{e}_A^F + \varepsilon_A^F| = |\mathfrak{e}| + |\varepsilon_A^F| \ge |\mathfrak{e}|$ and

$$|A \setminus ((\hat{F} \restriction A) \cup \varrho_A)| + |F \setminus ((\hat{F} \cup_i A) \cup \varrho_F)| \le |F \setminus (\hat{F} \cup \varrho_F)|\ ,$$

it is the case that if $|\tau|_{\mathsf{bi}} = n$, then the degree of each term appearing on the right hand side of (3.3) is of the type $(n_1 + k_1, n_2 - k_2)$ with $k_i \ge 0$. Since furthermore the sum is finite for any given value of $|\varepsilon_A^F|$, this is indeed a triangular map on $\langle \mathfrak{F} \rangle$.

There are many other ways of bigrading $\mathfrak{F}$ to make the Δ_i triangular, but the one chosen here has the advantage that it behaves nicely with respect to the various quotient operations of Sections 3.5 and 4.1 below.

Remark 3.7 The coproduct Δ_i defined in (3.3) does not look like that of a combinatorial Hopf algebra since for ε_A^F the coefficients are not necessarily integers. This could in principle be rectified easily by a simple change of basis: if we set

$$(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})_\circ \stackrel{\text{def}}{=} \frac{1}{\mathfrak{e}!} (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})\ ,$$

then we can write (3.3) equivalently as

$$\Delta_i \tau = \sum_{A \in \mathfrak{A}_i(F,\hat{F})} \sum_{\varepsilon_A^F, \mathfrak{n}_A} \binom{\mathfrak{e} + \varepsilon_A^F}{\varepsilon_A^F} \binom{\mathfrak{n}}{\mathfrak{n}_A} (A, \hat{F}\restriction A, \mathfrak{n}_A + \pi\varepsilon_A^F, \mathfrak{o}, \mathfrak{e})_\circ \\ \otimes (F, \hat{F} \cup_i A, \mathfrak{n} - \mathfrak{n}_A,\ \mathfrak{o} + \mathfrak{n}_A + \pi(\varepsilon_A^F - \mathfrak{e}_\emptyset^A), \mathfrak{e}_A^F + \varepsilon_A^F)_\circ \,,$$

for $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})_\circ$. Note that with this notation it is still the case that

$$(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})_\circ \cdot (G, \hat{G}, \mathfrak{n}', \mathfrak{o}', \mathfrak{e}')_\circ = (F \sqcup G, \hat{F} + \hat{G}, \mathfrak{n} + \mathfrak{n}', \mathfrak{o} + \mathfrak{o}', \mathfrak{e} + \mathfrak{e}')_\circ \,.$$

However, since this lengthens some expressions, does not seem to create any significant simplifications, and completely destroys compatibility with the notations of [Hai14], we prefer to stick to (3.3).

Remark 3.8 As already remarked, the grading $|\cdot|_{\text{bi}}$ defined in (2.5) is not preserved by the Δ_i. This should be considered a feature, not a bug! Indeed, the fact that the first component of our bigrading is not preserved is precisely what allows us to have an infinite sum in (3.3). A more natural integer-valued grading in that respect would have been given for example by

$$|(F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}|_- = |E_F| - |\hat{E}| + |\mathfrak{n}| - |\mathfrak{e}| \,,$$

which would be preserved by both the forest product $\cdot$ and Δ_i. However, since $\mathfrak{e}$ can take arbitrarily large values, this grading is no longer positive. A grading very similar to this will play an important role later on, see Definition 5.3 below.

3.3 Coassociativity

Assumption 2 *For each coloured forest $(F, \hat{F})$ as in Definition 2.3, the collection $\mathfrak{A}_i(F, \hat{F})$ of subforests of F satisfies the following properties.*

1. *One has*

$$\mathfrak{A}_i(F \sqcup G, \hat{F} + \hat{G}) = \{C \sqcup D \,:\, C \in \mathfrak{A}_i(F, \hat{F}) \,\&\, D \in \mathfrak{A}_i(G, \hat{G})\} \,. \quad (3.6)$$

2. *One has*

$$A \in \mathfrak{A}_i(F, \hat{F}) \quad \& \quad B \in \mathfrak{A}_i(F, \hat{F} \cup_i A) \,, \quad (3.7a)$$

if and only if

$$B \in \mathfrak{A}_i(F, \hat{F}) \quad \& \quad A \in \mathfrak{A}_i(B, \hat{F}\restriction B) \quad (3.7b)$$

where $F_A = (E_A, N_A)$ denotes the subforest of F with edge set $E_A = \{(x, y) \in E \,:\, \{x, y\} \subset A\}$.

Assumption 2 is precisely what is required so that the "undecorated" versions of the maps Δ_i, as defined in (3.2), are both multiplicative and coassociative. The next proposition shows that the definition (3.3) is such that this automatically carries over to the "decorated" counterparts.

Proposition 3.9 *Under Assumptions 1 and 2, the maps Δ_i are coassociative and multiplicative on $\langle\mathfrak{F}\rangle$, namely the identities*

$$(\Delta_i \otimes \mathrm{id})\Delta_i \mathscr{F} = (\mathrm{id} \otimes \Delta_i)\Delta_i \mathscr{F}\,, \tag{3.8a}$$

$$\Delta_i(\mathscr{F} \cdot \mathscr{G}) = (\Delta_i \mathscr{F}) \cdot (\Delta_i \mathscr{G})\,, \tag{3.8b}$$

hold for all $\mathscr{F}, \mathscr{G} \in \langle\mathfrak{F}\rangle$.

Proof. Multiplicativity is an immediate consequence of property 1 in Assumption 2 and the fact that the factorial factorises for functions with disjoint supports, so we only need to verify (3.8b).

Applying the definition (3.3) twice yields the identity

$$\begin{aligned}
&(\Delta_i \otimes \mathrm{id})\Delta_i(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) = \\
&= \sum_{B \in \mathfrak{A}_i(F,\hat{F})} \sum_{\varepsilon_B^F, \mathfrak{n}_B} \sum_{A \in \mathfrak{A}_i(B, \hat{F}\restriction B)} \sum_{\varepsilon_A^B, \mathfrak{n}_A} \frac{1}{\varepsilon_B^F!}\binom{\mathfrak{n}}{\mathfrak{n}_B} \frac{1}{\varepsilon_A^B!}\binom{\mathfrak{n}_B + \pi\varepsilon_B^F}{\mathfrak{n}_A} \\
&\quad (A, \hat{F}\restriction A, \mathfrak{n}_A + \pi\varepsilon_A^B, \mathfrak{o}, \mathfrak{e}) \otimes \\
&\quad (B, (\hat{F}\restriction B) \cup_i A, \mathfrak{n}_B + \pi\varepsilon_B^F - \mathfrak{n}_A,\ \mathfrak{o} + \mathfrak{n}_A + \pi(\varepsilon_A^B - \mathfrak{e}_\emptyset^A), \mathfrak{e}_A^B + \varepsilon_A^B) \otimes \\
&\quad (F, \hat{F} \cup_i B, \mathfrak{n} - \mathfrak{n}_B,\ \mathfrak{o} + \mathfrak{n}_B + \pi(\varepsilon_B^F - \mathfrak{e}_\emptyset^B), \mathfrak{e}_B^F + \varepsilon_B^F)\,.
\end{aligned} \tag{3.9}$$

Note that we should write for instance $(A, \hat{F}\restriction A, \mathfrak{n}_A + \pi\varepsilon_A^B, \mathfrak{o}\restriction N_A, \mathfrak{e}\restriction E_A)$ rather than $(A, \hat{F}\restriction A, \mathfrak{n}_A + \pi\varepsilon_A^B, \mathfrak{o}, \mathfrak{e})$, but in this as in other cases we prefer the lighter notation if there is no risk of confusion. Analogously, one has

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta_i)\Delta_i(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) = \\
&= \sum_{A \in \mathfrak{A}_i(F,\hat{F})} \sum_{\varepsilon_A^F, \mathfrak{n}_A} \sum_{C \in \mathfrak{A}_i(F, \hat{F} \cup_i A)} \sum_{\varepsilon_C^F, \mathfrak{n}_C} \frac{1}{\varepsilon_A^F!}\binom{\mathfrak{n}}{\mathfrak{n}_A} \frac{1}{\varepsilon_C^F!}\binom{\mathfrak{n} - \mathfrak{n}_A}{\mathfrak{n}_C} \\
&(A, \hat{F}\restriction A, \mathfrak{n}_A + \pi\varepsilon_A^F, \mathfrak{o}, \mathfrak{e}) \otimes \\
&(C, (\hat{F} \cup_i A)\restriction C, \mathfrak{n}_C + \pi\varepsilon_C^F, \mathfrak{o} + \mathfrak{n}_A + \pi(\varepsilon_A^F - \mathfrak{e}_\emptyset^A), \mathfrak{e}_A^C + (\varepsilon_A^F)_A^C) \otimes \\
&(F, \hat{F} \cup_i C, \mathfrak{n} - \mathfrak{n}_A - \mathfrak{n}_C, \mathfrak{o} + \mathfrak{n}_A + \mathfrak{n}_C + \pi((\varepsilon_A^F)_C^F + \varepsilon_C^F - \mathfrak{e}_\emptyset^C), \mathfrak{e}_C^F + (\varepsilon_A^F)_C^F + \varepsilon_C^F),
\end{aligned} \tag{3.10}$$

where we recall that, by Definition 3.3, for $A \subseteq B \subseteq F$ and $f : E_F \to \mathbf{N}^d$, we use the notation $f_A^B \stackrel{\text{def}}{=} f\,\mathbb{1}_{E_B \setminus E_A}$; in particular

$$(\varepsilon_A^F)_C^F \stackrel{\text{def}}{=} \varepsilon_A^F\,\mathbb{1}_{E_F \setminus E_C}\,, \qquad (\varepsilon_A^F)_A^C \stackrel{\text{def}}{=} \varepsilon_A^F\,\mathbb{1}_{E_C}\,. \tag{3.11}$$

By this definition it is clear that $(\varepsilon_A^F)_C^F$ and $(\varepsilon_A^F)_A^C$ have disjoint supports and moreover

$$(\varepsilon_A^F)_C^F + (\varepsilon_A^F)_A^C = \varepsilon_A^F.$$

This is the reason, in particular, why the term $\pi((\varepsilon_A^F)_C^F)$ appears in the last line of (3.10). In the proof of (3.10) we also make use of the fact that, since $A \subset C$, one has

$$(\hat{F} \cup_i A) \cup_i C = \hat{F} \cup_i C\,.$$

We now make the following changes of variables. First, we set

$$\bar\varepsilon^F_C \stackrel{\text{def}}{=} (\varepsilon^F_A)^F_C + \varepsilon^F_C\;, \qquad \bar\varepsilon^C_A \stackrel{\text{def}}{=} (\varepsilon^F_A)^C_A\;, \qquad \bar\varepsilon^F_{A,C} \stackrel{\text{def}}{=} \bar\varepsilon^F_C - \varepsilon^F_C = (\varepsilon^F_A)^F_C \tag{3.12}$$

with the naming conventions (3.11). Note that the support of $\bar\varepsilon^F_{A,C}$ is contained in $\partial(A,F)\cap\partial(C,F)$. Now the map

$$(\varepsilon^F_A,\varepsilon^F_C)\mapsto(\bar\varepsilon^F_C,\bar\varepsilon^C_A,\bar\varepsilon^F_{A,C})$$

given by (3.12) is invertible on its image, with inverse given by

$$(\bar\varepsilon^F_C,\bar\varepsilon^C_A,\bar\varepsilon^F_{A,C})\mapsto(\varepsilon^F_A,\varepsilon^F_C)=(\bar\varepsilon^C_A+\bar\varepsilon^F_{A,C},\bar\varepsilon^F_C-\bar\varepsilon^F_{A,C}). \tag{3.13}$$

Furthermore, the only restriction on its image besides the constraints on the supports is the fact that $\bar\varepsilon^F_{A,C}\le\bar\varepsilon^F_C$, which is required to guarantee that, with $\varepsilon^F_C=\bar\varepsilon^F_C-\bar\varepsilon^F_{A,C}$ as in (3.13), one has $\varepsilon^F_C\ge 0$.

Now, the supports of $\bar\varepsilon^C_A$ and $\bar\varepsilon^F_{A,C}$ are disjoint, since

$$\operatorname{supp}\bar\varepsilon^C_A\subset\partial(A,F)\cap E_C,\qquad \operatorname{supp}\bar\varepsilon^F_{A,C}\subset\partial(A,F)\setminus E_C.$$

Since the factorial factorises for functions with disjoint supports, we can rewrite the combinatorial prefactor as

$$\frac{1}{\varepsilon^F_A!}\frac{1}{\varepsilon^F_C!}=\frac{1}{\bar\varepsilon^C_A!\bar\varepsilon^F_{A,C}!}\frac{1}{(\bar\varepsilon^F_C-\bar\varepsilon^F_{A,C})!}=\frac{1}{\bar\varepsilon^C_A!\bar\varepsilon^F_C!}\binom{\bar\varepsilon^F_C}{\bar\varepsilon^F_{A,C}}\;. \tag{3.14}$$

In this way, the constraint $\bar\varepsilon^F_{A,C}\le\bar\varepsilon^F_C$ is automatically enforced by our convention for binomial coefficients, so that (3.10) can be written as

$$\begin{aligned}
&(\mathrm{id}\otimes\Delta_i)\Delta_i(F,\hat F,\mathfrak{n},\mathfrak{o},\mathfrak{e})=\\
&=\sum_{A\in\mathfrak{A}_i(F,\hat F)}\sum_{C\in\mathfrak{A}_i(F,\hat F\cup_i A)}\sum_{\bar\varepsilon^C_A,\bar\varepsilon^F_C,\bar\varepsilon^F_{A,C}}\sum_{\mathfrak{n}_A,\mathfrak{n}_C}\frac{1}{\bar\varepsilon^F_C!\bar\varepsilon^C_A!}\binom{\bar\varepsilon^F_C}{\bar\varepsilon^F_{A,C}}\binom{\mathfrak{n}}{\mathfrak{n}_A}\binom{\mathfrak{n}-\mathfrak{n}_A}{\mathfrak{n}_C}\\
&(A,\hat F\restriction A,\mathfrak{n}_A+\pi\varepsilon^F_A,\mathfrak{o},\mathfrak{e})\otimes\\
&(C,(\hat F\cup_i A)\restriction C,\mathfrak{n}_C+\pi\varepsilon^F_C,\mathfrak{o}+\mathfrak{n}_A+\pi(\varepsilon^F_A-\mathfrak{e}^A_\emptyset),\mathfrak{e}^C_A+\bar\varepsilon^C_A)\otimes\\
&(F,\hat F\cup_i C,\mathfrak{n}-\mathfrak{n}_A-\mathfrak{n}_C,\mathfrak{o}+\mathfrak{n}_A+\mathfrak{n}_C+\pi(\bar\varepsilon^F_C-\mathfrak{e}^C_\emptyset),\mathfrak{e}^F_C+\bar\varepsilon^F_C)\;,
\end{aligned}\tag{3.15}$$

where ε^F_A and ε^F_C are determined by (3.13).

We now make the further change of variables

$$\bar{\mathfrak{n}}_C=\mathfrak{n}_A+\mathfrak{n}_C\;,\qquad \bar{\mathfrak{n}}_A=\mathfrak{n}_A+\pi\bar\varepsilon^F_{A,C}\;.$$

It is clear that, given $\bar\varepsilon^F_{A,C}$, this is again a bijection onto its image and that the latter is given by those functions with the relevant supports such that furthermore

$$\bar{\mathfrak{n}}_A\ge\pi\bar\varepsilon^F_{A,C}\;. \tag{3.16}$$

With these new variables, (3.13) immediately yields

$$\mathfrak{n}_A + \pi\varepsilon_A^F = \bar{\mathfrak{n}}_A + \pi\bar{\varepsilon}_A^C\,, \qquad \mathfrak{n}_C + \pi\varepsilon_C^F = \bar{\mathfrak{n}}_C - \bar{\mathfrak{n}}_A + \pi\bar{\varepsilon}_C^F\,. \tag{3.17}$$

Furthermore, we have

$$\binom{\mathfrak{n}}{\mathfrak{n}_A}\binom{\mathfrak{n}-\mathfrak{n}_A}{\mathfrak{n}_C} = \binom{\mathfrak{n}}{\mathfrak{n}_A+\mathfrak{n}_C}\binom{\mathfrak{n}_A+\mathfrak{n}_C}{\mathfrak{n}_A} = \binom{\mathfrak{n}}{\bar{\mathfrak{n}}_C}\binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_A-\pi\bar{\varepsilon}_{A,C}^F}\,. \tag{3.18}$$

Rewriting the combinatorial factor in this way, our convention on binomial coefficients once again enforces the condition (3.16), so that (3.15) can be written as

$$\begin{aligned}
&(\mathrm{id}\otimes\Delta_i)\Delta_i(F,\hat{F},\mathfrak{n},\mathfrak{o},\mathfrak{e}) = \qquad\qquad (3.19)\\
&= \sum_{A\in\mathfrak{A}_i(F,\hat{F})}\sum_{C\in\mathfrak{A}_i(F,\hat{F}\cup_i A)}\sum_{\bar{\varepsilon}_A^C,\bar{\varepsilon}_C^F,\bar{\varepsilon}_{A,C}^F}\sum_{\bar{\mathfrak{n}}_A,\bar{\mathfrak{n}}_C}\frac{1}{\bar{\varepsilon}_C^F!\bar{\varepsilon}_A^C!}\binom{\mathfrak{n}}{\bar{\mathfrak{n}}_C}\binom{\bar{\varepsilon}_C^F}{\bar{\varepsilon}_{A,C}^F}\binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_A-\pi\bar{\varepsilon}_{A,C}^F}\\
&(A,\hat{F}\restriction A,\bar{\mathfrak{n}}_A+\pi\bar{\varepsilon}_A^C,\mathfrak{o},\mathfrak{e})\otimes\\
&(C,(\hat{F}\cup_i A)\restriction C,\bar{\mathfrak{n}}_C-\bar{\mathfrak{n}}_A+\pi\bar{\varepsilon}_C^F,\mathfrak{o}+\bar{\mathfrak{n}}_A+\pi\bar{\varepsilon}_A^C-\pi\mathfrak{e}_\emptyset^A,\mathfrak{e}_A^C+\bar{\varepsilon}_A^C)\otimes\\
&(F,\hat{F}\cup_i C,\mathfrak{n}-\bar{\mathfrak{n}}_C,\mathfrak{o}+\bar{\mathfrak{n}}_C+\pi(\bar{\varepsilon}_C^F-\mathfrak{e}_\emptyset^C),\mathfrak{e}_C^F+\bar{\varepsilon}_C^F)\,,
\end{aligned}$$

with the summation only restricted by the conditions on the supports implicit in the notations. At this point, we note that the right hand side depends on $\bar{\varepsilon}_{A,C}^F$ only via the combinatorial factor and that, as a consequence of Chu-Vandermonde, one has

$$\begin{aligned}
\sum_{\bar{\varepsilon}_{A,C}^F}\binom{\bar{\varepsilon}_C^F}{\bar{\varepsilon}_{A,C}^F}\binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_A-\pi\bar{\varepsilon}_{A,C}^F} &= \sum_{\bar{\varepsilon}_{A,C}^F}\binom{\bar{\varepsilon}_C^F}{\bar{\varepsilon}_{A,C}^F}\binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_A-\pi\bar{\varepsilon}_{A,C}^F}\\
&= \sum_{\pi\bar{\varepsilon}_{A,C}^F}\binom{\pi\bar{\varepsilon}_C^F}{\pi\bar{\varepsilon}_{A,C}^F}\binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_A-\pi\bar{\varepsilon}_{A,C}^F} = \binom{\bar{\mathfrak{n}}_C+\pi\bar{\varepsilon}_C^F}{\bar{\mathfrak{n}}_A}.
\end{aligned} \tag{3.20}$$

Inserting (3.20) into (3.19), using the fact that $(\hat{F}\restriction C)\cup_i A = (\hat{F}\cup_i A)\restriction C$ and comparing to (3.9) (with B replaced by C) completes the proof. □

3.4 Bialgebra structure

Fix throughout this section $i > 0$.

Definition 3.10 For $\mathfrak{A}_i$ a family satisfying Assumptions 1 and 2, we set

$$\mathfrak{F}_i = \{(F,\hat{F},\mathfrak{n},\mathfrak{o},\mathfrak{e}) \in \mathfrak{F} \,:\, \hat{F}\le i \;\&\; \{F,\hat{F}_i\}\subset\mathfrak{A}_i(F,\hat{F})\}\,.$$

We also define the set $\mathfrak{U}_i$ of all $(F,i,0,\mathfrak{o},0)\in\mathfrak{F}_i$, where (F,i) denotes the coloured forest $(F,\hat{F})$ such that either F is empty or $\hat{F}\equiv i$ on the whole forest F. In particular, one has $|\tau|_{\mathrm{bi}} = 0$ for every $\tau\in\mathfrak{U}_i$. Finally we define $\mathbf{1}_i^\star\colon\mathfrak{F}\to\mathbf{R}$ by setting

$$\mathbf{1}_i^\star(\tau) \stackrel{\text{def}}{=} \mathbb{1}_{(\tau\in\mathfrak{U}_i)}. \tag{3.21}$$

Assumption 3 *For every coloured forest $(F, \hat{F})$ such that $\hat{F}_i \in \mathfrak{A}_i(F, \hat{F})$ and for all $A \in \mathfrak{A}_i(F, \hat{F})$, we have*
1. $\{A, \hat{F}_i\} \subset \mathfrak{A}_i(A, \hat{F} \restriction A)$
2. *if $\hat{F} \leq i$ then $\{F, A\} \subset \mathfrak{A}_i(F, \hat{F} \cup_i A)$.*

Under Assumptions 1 and 3 it immediately follows from (3.3) that, setting

$$\langle \mathfrak{F}_i \rangle = \bigsqcup_{n \in \mathbf{N}^2} \mathrm{Vec}\{\mathscr{F} \in \mathfrak{F}_i \,:\, |\mathscr{F}|_{\mathrm{bi}} = n\}$$

as in (2.6), Δ_i maps $\langle \mathfrak{F}_i \rangle$ into $\langle \mathfrak{F}_i \rangle \hat{\otimes} \langle \mathfrak{F}_i \rangle$.

Lemma 3.11 *Under Assumptions 1, 2 and 3, $(\langle \mathfrak{F}_i \rangle, \cdot, \Delta_i, \mathbf{1}, \mathbf{1}_i^\star)$ is a bialgebra in the category of bigraded spaces as in Definition 2.9.*

Proof. It follows from the first part of Assumption 2 that $\mathfrak{F}_i$ is closed under the product, so that $(\langle \mathfrak{F}_i \rangle, \cdot, \mathbf{1})$ is indeed an algebra.

Since we already argued that $\Delta_i \colon \langle \mathfrak{F}_i \rangle \to \langle \mathfrak{F}_i \rangle \hat{\otimes} \langle \mathfrak{F}_i \rangle$ and since Δ_i is coassociative by (3.8a), in order to show that $(\langle \mathfrak{F}_i \rangle, \Delta_i, \mathbf{1}_i^\star)$ is a coalgebra, it remains to show that

$$(\mathbf{1}_i^\star \otimes \mathrm{id})\Delta_i = (\mathrm{id} \otimes \mathbf{1}_i^\star)\Delta_i = \mathrm{id}, \quad \text{on} \quad \langle \mathfrak{F}_i \rangle \,.$$

For $A \in \mathfrak{A}_i(F, \hat{F})$, we have $(A, \hat{F} \restriction A, \mathfrak{n}', \mathfrak{o}', \mathfrak{e}') \in \mathfrak{U}_i$ if and only if $\hat{F} \equiv i$ on A, i.e. $A \subseteq \hat{F}_i$; since $\hat{F}_i \subseteq A$ by Assumption 1, then the only possibility is $A = \hat{F}_i$. Analogously, we have $(F, \hat{F} \cup_i A, \mathfrak{n}', \mathfrak{o}', \mathfrak{e}') \in \mathfrak{U}_i$ if and only if $A = F$. The definition (3.3) of Δ_i yields the result.

The required compatibility between the algebra and coalgebra structures is given by (3.8b), thus concluding the proof. □

In general, the bialgebra $(\langle \mathfrak{F}_i \rangle, \cdot, \Delta_i, \mathbf{1}, \mathbf{1}_i^\star)$ does not admit an antipode. However, there is a simple way of turning it into a Hopf algebra (again in the category of bigraded spaces as in Definition 2.9) by taking a suitable quotient, which is what we are going to show now.

3.5 Contraction of coloured subforests and Hopf algebra structure

In order to obtain an antipode for Δ_i, we can restrict to decorated forests s.t. $\hat{F}$ is non-zero only on a set of isolated nodes. An additional requirement is that an isolated node in F which is coloured by i must have a non-zero label $\mathfrak{n}$.

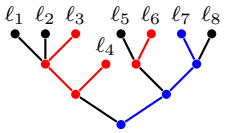
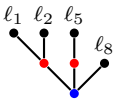
To formalise this, we introduce a *contraction operator* on coloured forests. Given a coloured forest $(F, \hat{F})$, we recall that $\hat{E}$, defined in Definition 2.3, is the union of all edges in $\hat{F}_j$ over all $j > 0$.

Definition 3.12 For any coloured forest $(F, \hat{F})$, we write $\mathscr{K}_{\hat{E}} F$ for the typed forest obtained in the following way. We use the equivalence relation $\sim$ on the node set N_F defined in Definition 2.6, namely $x \sim y$ if x and y are connected in $\hat{E}$. Then $\mathscr{K}_{\hat{E}} F$ is the quotient graph of $(N_F, E_F \setminus \hat{E})$ by $\sim$. By the definition of $\sim$, each

equivalence class is connected so that $\mathscr{K}_{\hat{E}}F$ is again a typed forest. Finally, $\hat{F}$ is constant on equivalence classes with respect to $\sim$, so that the coloured forest $(\mathscr{K}_{\hat{E}}F, \hat{F})$ is well defined.

Note that in $(\mathscr{K}_{\hat{E}}F, \hat{F})$ all non-empty coloured subforests are reduced to single nodes.

Example 3.13 With the notation of Definition 3.12, we have the following example of contraction of a coloured forest

$(F, \hat{F}) =$  $\longrightarrow$ $(\mathscr{K}_{\hat{E}}F, \hat{F}) =$  .

The construction in Definition 3.12 defines a map $\mathscr{K}^{\sharp}_{\hat{E}}$ from subforests of $\mathscr{K}_{\hat{E}}F$ to subforests of F. We are going to restrict our attention to collections $\mathfrak{A}_i$ satisfying the following assumption.

Assumption 4 *For all coloured forests $(F, \hat{F})$, the map $\mathscr{K}^{\sharp}_{\hat{E}}$ is a bijection between $\mathfrak{A}_i(\mathscr{K}_{\hat{E}}F, \hat{F})$ and $\mathfrak{A}_i(F, \hat{F})$.*

We now extend the contraction operator of Definition 3.12 to decorated forests.

Definition 3.14 Let $\mathscr{K} : \langle \mathfrak{F} \rangle \to \langle \mathfrak{F} \rangle$ be the triangular map given by

$$\mathscr{K}(F, \hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \stackrel{\text{def}}{=} (\mathscr{K}_{\hat{E}}F, \hat{F})^{[\mathfrak{n}],[\mathfrak{o}]}_{[\mathfrak{e}]}\,, \qquad (F, \hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \mathfrak{F},$$

where the decorations $[\mathfrak{n}]$, $[\mathfrak{o}]$ and $[\mathfrak{e}]$ are defined as follows:

- if x is an equivalence class of $\sim$ as in Definition 3.12, then $[\mathfrak{n}](x) = \sum_{y \in x} \mathfrak{n}(y)$.
- $[\mathfrak{e}]$ is defined by simple restriction of $\mathfrak{e}$ on $E_F \setminus \hat{E}$.
- $[\mathfrak{o}](x)$ is defined by

$$[\mathfrak{o}](x) = \sum_{y \in x} \mathfrak{o}(y) + \sum_{e \in E_F \cap x^2} \mathfrak{t}(e). \tag{3.22}$$

Recall that, as assumed in Definition 2.5, $\mathfrak{o}$ and $[\mathfrak{o}]$ take values in $\mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$, so that this expression makes sense.

Remark 3.15 The contraction of a subforest entails a loss of information. We use the decoration $\mathfrak{o}$ in order to retain part of the lost information, namely the types of the edges which are contracted. This plays an important role in the degree $|\cdot|_+$ introduced in Definition 5.3 and is the key to one of the main results of this paper, see Remark 5.38.

Let now $\mathfrak{M}_i \subset \mathfrak{F}_i$ be the set of decorated forests which are of type $(F, i, \mathfrak{n}, \mathfrak{o}, 0)$. This includes the case $F = \emptyset$ so that $\mathfrak{U}_i \subset \mathfrak{M}_i$, where $\mathfrak{U}_i$ is defined in Definition 3.10. We define then an operator $k_i : \mathfrak{M}_i \to \mathfrak{M}_i$ by setting

$$k_i(\nu) \stackrel{\text{def}}{=} (\bullet, i, \Sigma\mathfrak{n}, 0, 0) \,,$$

for any $\nu = (F, i, \mathfrak{n}, \mathfrak{o}, 0)$ with $\Sigma\mathfrak{n} \stackrel{\text{def}}{=} \sum_{N_F} \mathfrak{n}$. Note that, for every $\tau \in \mathfrak{F}_i$, there is a unique couple (μ, ν) such that $\tau = \mu \cdot \nu$, $\nu \in \mathfrak{M}_i$, and such that every connected component of $\mu = (G, \hat{G}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ in the sense of Remark 2.8 contains at least one node x on which $\hat{G}(x) \neq i$. Using this factorisation, we define $\Phi_i : \mathfrak{F}_i \to \mathfrak{F}_i$ by

$$\Phi_i(\tau) \stackrel{\text{def}}{=} \mu \cdot k_i(\nu) \,. \tag{3.23}$$

We also define $\hat{\Phi}_i : \mathfrak{F}_i \to \mathfrak{F}_i$ by $\hat{\Phi}_i = \hat{P}_i \circ \Phi_i = \Phi_i \circ \hat{P}_i$, where $\hat{P}_i(G, \hat{G}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ sets $\mathfrak{o}$ to 0 on every connected component of $\hat{G}_i$ that contains a root of G. Finally, we define $\mathscr{K}_i, \hat{\mathscr{K}}_i : \mathfrak{F}_i \to \mathfrak{F}_i$

$$\mathscr{K}_i \stackrel{\text{def}}{=} \Phi_i \circ \mathscr{K} \,, \qquad \hat{\mathscr{K}}_i \stackrel{\text{def}}{=} \hat{\Phi}_i \circ \mathscr{K} \,. \tag{3.24}$$

Since $\mathscr{K}$ commutes with Φ_i (as well as with $\hat{\Phi}_i$), is multiplicative, and is the identity on the image of k_i in $\mathfrak{M}_i$, it follows that for $\tau = \mu \cdot \nu$ as above, we have

$$\mathscr{K}_i(\tau) = \mathscr{K}(\mu) \cdot k_i(\nu) \,.$$

Moreover $\mathscr{K}_i$ and $\hat{\mathscr{K}}_i$ are idempotent and extend to triangular maps on $\langle \mathfrak{F}_i \rangle$ since $\mathscr{K}$, Φ_i and $\hat{\Phi}_i$ are all idempotent and preserve our bigrading. We then have the following.

Lemma 3.16 *Under Assumptions 1–4, the spaces $\mathcal{I}_i = \ker \mathscr{K}_i \subset \langle \mathfrak{F}_i \rangle$ and $\hat{\mathcal{I}}_i = \ker \hat{\mathscr{K}}_i$ are bialgebra ideals, i.e.*

$$\langle \mathfrak{F}_i \rangle \cdot \mathcal{I}_i \subset \mathcal{I}_i, \qquad \Delta_i \mathcal{I}_i \subset \mathcal{I}_i \,\hat{\otimes}\, \langle \mathfrak{F}_i \rangle + \langle \mathfrak{F}_i \rangle \,\hat{\otimes}\, \mathcal{I}_i \,,$$

and similarly for $\hat{\mathcal{I}}_i$.

Proof. Although $\mathscr{K}_i$ is not quite an algebra morphism of $(\langle \mathfrak{F}_i \rangle, \cdot)$, it has the property $\mathscr{K}_i(a \cdot b) = \mathscr{K}_i(a \cdot \mathscr{K}_i(b))$ for all $a, b \in \mathfrak{F}_i$, from which the first property follows for $\mathcal{I}_i$. Since $\hat{P}_i$ is an algebra morphism, the same holds for $\hat{\mathcal{I}}_i$. To show the second claim, we first recall that for all coloured forests $(F, \hat{F})$, the map $\mathscr{K}^\sharp_{\hat{E}}$ defined in Definition 3.12 is, by the Assumption 4, a bijection between $\mathfrak{A}_i(\mathscr{K}_{\hat{E}} F, \hat{F})$ and $\mathfrak{A}_i(F, \hat{F})$. Combining this with Chu-Vandermonde, one can show that $\mathscr{K}$ satisfies

$$(\mathscr{K} \otimes \mathscr{K})\Delta_i \mathscr{K} = (\mathscr{K} \otimes \mathscr{K})\Delta_i \,. \tag{3.25}$$

The same can easily be verified for Φ_i and $\hat{P}_i$, so that it also holds for $\mathscr{K}_i$ and $\hat{\mathscr{K}}_i$, whence the claim follows. □

If we define

$$\mathscr{H}_i \stackrel{\text{def}}{=} \langle \mathfrak{F}_i \rangle / \mathcal{I}_i, \qquad \mathbf{1}_i \stackrel{\text{def}}{=} \mathbf{1} + \mathcal{I}_i \in \mathscr{H}_i, \tag{3.26}$$

then, as a consequence of Lemma 3.16, $(\mathscr{H}_i, \cdot, \Delta_i, \mathbf{1}_i, \mathbf{1}_i^\star)$ defines a bialgebra.

Remark 3.17 Using Lemma 2.14, we have a canonical isomorphism

$$(\mathscr{H}_i, \cdot, \Delta_i, \mathbf{1}_i, \mathbf{1}_i^\star) \quad \longleftrightarrow \quad (\langle H_i \rangle, \mathcal{K}_i \mathscr{M}, (\mathcal{K}_i \otimes \mathcal{K}_i)\Delta_i, \mathbf{1}_i, \mathbf{1}_i^\star)\,,$$

where $H_i = \{\mathcal{F} \in \mathfrak{F}_i : \mathcal{K}_i \mathcal{F} = \mathcal{F}\} = \mathcal{K}_i \mathfrak{F}_i$ and $\mathscr{M}$ denotes the product. This can be useful if one wants to work with explicit representatives rather than with equivalence classes.

Proposition 3.18 *Under Assumptions 1–4, the space $(\mathscr{H}_i, \cdot, \Delta_i, \mathbf{1}, \mathbf{1}_i^\star)$ is a Hopf algebra.*

Proof. By Lemma 3.11, $\mathbf{1}_i^\star$ is a counit in $\mathscr{H}_i$. We only need now to show that this space admits an antipode $\mathscr{A}_i$, that we are going to construct recursively.

For $k \in \mathbf{N}^d$, we denote by $X^k \in \mathscr{H}_i$ the equivalence class of the element $(\bullet, i, k, 0, 0)$. It then follows from

$$\Delta_i X^k = \sum_{j \in \mathbf{N}^d} \binom{k}{j} X^j \otimes X^{k-j} \tag{3.27}$$

that the subspace spanned by $(X^k, k \in \mathbf{N}^d)$ is isomorphic to the Hopf algebra of polynomials in d commuting variables, provided that we set

$$\mathscr{A}_i X^k = (-1)^{|k|} X^k\,. \tag{3.28}$$

For any $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F}_i$, let $|\tau|_i = |F \setminus \hat{F}_i|$ and recall the definition (2.4) of the bigrading $|\tau|_{\text{bi}}$. Note that $|\mathcal{K}_i \tau|_i = |\tau|_i$ and, as we have already remarked, $|\mathcal{K}_i \tau|_{\text{bi}} = |\tau|_{\text{bi}}$, so that both these gradings make sense on $\mathscr{H}_i$. We now extend $\mathscr{A}_i$ to $\mathscr{H}_i$ by induction on $|\tau|_i$.

If $|\tau|_i = 0$ then, by definition, one has $\tau \in \mathfrak{M}_i$ so that $\tau = X^k$ for some k and (3.28) defines $\mathscr{A}_i \tau$. Let now $N > 0$ and assume that $\mathscr{A}_i \tau$ has been defined for all $\tau \in \mathscr{H}_i$ with $|\tau|_i < N$. Assume also that it is such that if $|\tau|_{\text{bi}} = m$, then $(\mathscr{A}_i \tau)_n \neq 0$ only if $n \geq m$, which is indeed the case for (3.28) since all the terms appearing there have degree $(0, 0)$. (This latter condition is required if we want $\mathscr{A}_i$ to be a triangular map.)

For $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ and $k \colon N_F \to \mathbf{N}^d$, we define $R_k \tau \stackrel{\text{def}}{=} (F, \hat{F}, k, \mathfrak{o}, \mathfrak{e})$. For such a τ with $|\tau|_i = N$ and $|\tau|_{\text{bi}} = M$, we then note that one has (with the obvious modification of Sweedler's notation)

$$\Delta_i \tau = \sum_{k \leq \mathfrak{n}} \binom{\mathfrak{n}}{k} R_k \tau \otimes X^{\mathfrak{n}-k} + \sum_{\ell + m \geq M} \tau_{(1)}^\ell \otimes \tau_{(2)}^m\,,$$

where for $j = 1, 2$ and $\ell \in \mathbf{N}^2$

$$\tau_{(j)}^{\ell} \in \mathrm{Vec}\{\sigma \in \mathscr{H}_i : |\sigma|_{\mathrm{bi}} = \ell, |\sigma|_i < N\}.$$

Here, the property $|\sigma|_i < N$ holds because these terms come from terms with $A \neq F$ in (3.3). Since for $\tau \neq \mathbf{1}_i$ we want to have

$$\mathscr{M}(\mathscr{A}_i \otimes \mathrm{id})\Delta_i \tau = 0 \,,$$

this forces us to choose $\mathscr{A}_i\tau$ in such a way that

$$\mathscr{A}_i\tau = -\sum_{k \neq \mathfrak{n}} \binom{\mathfrak{n}}{k} \mathscr{A}_i(R_k\tau) \cdot X^{\mathfrak{n}-k} - \sum_{\ell+m \geq M} \mathscr{A}_i(\tau_{(1)}^{\ell}) \cdot \tau_{(2)}^{m} \,. \tag{3.29}$$

In the case $\mathfrak{n} = 0$, this uniquely defines $\mathscr{A}_i\tau$ by the induction hypothesis since every one of the terms $\tau_{(1)}^{\ell}$ appearing in this expression satisfies $|\tau_{(1)}^{\ell}|_i < N$. In the case where $\mathfrak{n} \neq 0$, it is also easily seen to be uniquely defined by performing a second inductive step over $|\mathfrak{n}| \in \mathbf{N}$.

All terms appearing in the right hand side of (3.29) do indeed satisfy that their total $|\cdot|_{\mathrm{bi}}$-degree is at least M by using the induction hypothesis. Furthermore, our definition immediately guarantees that $\mathscr{M}(\mathscr{A}_i \otimes \mathrm{id})\Delta_i = \mathbf{1}_i\mathbf{1}_i^{\star}$. It remains to verify that one also has $\mathscr{M}(\mathrm{id} \otimes \mathscr{A}_i)\Delta_i = \mathbf{1}_i\mathbf{1}_i^{\star}$. For this, it suffices to verify that $\mathscr{A}_i$ is multiplicative, whence the claim follows by mimicking the proof of the fact that a semigroup with left identity and left inverse is a group.

Multiplicativity of $\mathscr{A}_i$ also follows by induction over $N = |\tau|_i$. Indeed, it follows from (3.28) that it is the case for $N = 0$. It is also easy to see from (3.29) that if τ is of the form $\tau' \cdot X^k$ for some τ' and some $k > 0$, then one has $\mathscr{A}_i\tau = (\mathscr{A}_i\tau') \cdot (\mathscr{A}_i X^k)$. Assuming that it is the case for all values less than some N, it therefore suffices to verify that $\mathscr{A}_i$ is multiplicative for elements of the type $\tau = \sigma \cdot \bar{\sigma}$ with $|\sigma|_i \wedge |\bar{\sigma}|_i > 0$. If we extend $\mathscr{A}_i$ multiplicatively to elements of this type then, as a consequence of the multiplicativity of Δ_i, one has

$$\mathscr{M}(\mathscr{A}_i \otimes \mathrm{id})\Delta_i\tau = (\mathscr{M}(\mathscr{A}_i \otimes \mathrm{id})\Delta_i\sigma) \cdot (\mathscr{M}(\mathscr{A}_i \otimes \mathrm{id})\Delta_i\bar{\sigma}) = 0 \,,$$

as required. Since the map $\mathscr{A}_i$ satisfying this property was uniquely defined by our recursion, this implies that $\mathscr{A}_i$ is indeed multiplicative. □

3.6 Characters group

Recall that an element $g \in \mathscr{H}_i^*$ is a character if $g(\tau \cdot \bar{\tau}) = g(\tau)g(\bar{\tau})$ for any $\tau, \bar{\tau} \in \mathscr{H}_i$. Denoting by $\mathscr{G}_i$ the set of all such characters, the Hopf algebra structure described above turns $\mathscr{G}_i$ into a group by

$$(f \circ g)(\tau) = (f \otimes g)\,\Delta_i\tau \,, \qquad g^{-1}(\tau) = g(\mathscr{A}_i\tau) \,, \tag{3.30}$$

where the former operation is guaranteed to make sense by Remark 2.13.

Definition 3.19 Denote by $\mathfrak{P}_i$ the set of elements $\mathfrak{F} = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in H_i$ as in Remark 3.17, such that

- F has exactly one connected component
- either $\hat{F}$ is not identically equal to i or $\mathfrak{F} = (\bullet, i, \delta_n, 0, 0)$ for some $n \in \{1, \dots, d\}$, where $(\delta_n(\bullet))_j = \delta_{nj}$.

It is then easy to see that for every $\tau \in H_i$ there exists a unique (possibly empty) collection $\{\tau_1, \dots, \tau_N\} \subset \mathfrak{P}_i$ such that $\tau = \mathcal{K}_i(\tau_1 \cdot \ldots \cdot \tau_N)$. As a consequence, a multiplicative functional on $\mathcal{H}_i$ is uniquely determined by the collection of values $\{g(\tau) : \tau \in \mathfrak{P}_i\}$. The following result gives a complete characterisation of the class of functions $g\colon \mathfrak{P}_i \to \mathbf{R}$ which can be extended in this way to a multiplicative functional on $\mathcal{H}_i$.

Proposition 3.20 *A function $g\colon \mathfrak{P}_i \to \mathbf{R}$ determines an element of $\mathcal{G}_i$ as above if and only if there exists $m\colon \mathbf{N} \to \mathbf{N}$ such that $g(\tau) = 0$ for every $\tau \in \mathfrak{P}_i$ with $|\tau|_{\text{bi}} = n$ such that $n_1 > m(n_2)$.*

Proof. We first show that, under this condition, the unique multiplicative extension of g defines an element of $\mathcal{H}_i^*$. By Remark 2.13, we thus need to show that there exists a function $\tilde{m}\colon \mathbf{N} \to \mathbf{N}$ such that $g(\tau) = 0$ for every $\tau \in H_i$ with $|\tau|_{\text{bi}} = n$ and $n_1 > \tilde{m}(n_2)$.

If $\sigma = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{P}_i$ satisfies $n_2 = 0$, then $\hat{F}$ is nowhere equal to 0 on F by the definition (2.4); by property 2 in Definition 2.3, $\hat{F}$ is constant on F, since we also assume that F has a single connected component; in this case $\mathfrak{e} \equiv 0$ by property 3 in Definition 2.5; therefore, if $n_2 = 0$ then $n_1 = 0$ as well. Therefore we can set $\tilde{m}(0) = 0$.

Let now $k \geq 1$. We claim that $\tilde{m}(k) \stackrel{\text{def}}{=} k \sup_{1 \leq \ell \leq k} m(\ell)$ has the required property. Indeed, for $\tau = \mathcal{K}_i(\tau_1 \cdot \ldots \cdot \tau_N)$, one has $g(\tau) = 0$ unless $g(\tau_j) \neq 0$ for every j; in this case, setting $n^j = (n_1^j, n_2^j) = |\tau_j|_{\text{bi}}$, we have $m(n_2^j) \geq n_1^j$ for all $j = 1, \dots, N$. Since $n = (n_1, n_2) \stackrel{\text{def}}{=} |\tau|_{\text{bi}} = \sum_j |\tau_j|_{\text{bi}}$, this implies that $n_k = \sum_j n_k^j$, $k = 1, 2$. Then

$$\tilde{m}(n_2) \geq n_2 \max_{1 \leq \ell \leq n_2} m(\ell) \geq n_2 \max_{1 \leq \ell \leq N} n_1^j \geq n_1.$$

The converse is elementary. □

3.7 Comodule bialgebras

Let us fix throughout this section $0 < i < j$. We want now to study the possible interaction between the structures given by the operators Δ_i and Δ_j.

Assumption 5 *Let $0 < i < j$. For every coloured forest $(F, \hat{F})$ such that $\hat{F} \leq j$ and $\{F, \hat{F}_j\} \subset \mathfrak{A}_j(F, \hat{F})$, one has $\hat{F}_i \in \mathfrak{A}_i(F, \hat{F})$.*

Lemma 3.21 *Let $0 < i < j$. Under Assumptions 1–4 for i and under Assumption 5 we have*

$$\Delta_i\colon \langle \mathfrak{F}_j \rangle \to \langle \mathfrak{F}_i \rangle \hat{\otimes} \langle \mathfrak{F}_j \rangle\,, \qquad (\mathbf{1}_i^\star \otimes \mathrm{id})\Delta_i = \mathrm{id}\,,$$

which endows $\langle \mathfrak{F}_j \rangle$ with the structure of a left-comodule over the bialgebra $\langle \mathfrak{F}_i \rangle$.

Proof. Let $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F}_j$ and $A \in \mathfrak{A}_i(F, \hat{F})$; by Definition 3.10, we have $\hat{F} \leq j$ and $\{F, \hat{F}_j\} \subset \mathfrak{A}_j(F, \hat{F})$, so that by Assumption 5 we have $\hat{F}_i \in \mathfrak{A}_i(F, \hat{F})$. Then, by property 1 in Assumption 3, we have $\hat{F}_i \cap A = \hat{F}_i \in \mathfrak{A}_i(A, \hat{F}\restriction A)$. Now, since $A \cap \hat{F}_j = \emptyset$ by property 1 in Assumption 1, we have $(\hat{F} \cup_i A)_j = \hat{F}_j \setminus A = \hat{F}_j \in \mathfrak{A}_j(F, \hat{F} \cup_i A)$ by the Definition 3.10 of $\mathfrak{F}_j$; all this shows that $\Delta_i \colon \langle \mathfrak{F}_j \rangle \to \langle \mathfrak{F}_i \rangle \hat{\otimes} \langle \mathfrak{F}_j \rangle$.

For $A \in \mathfrak{A}_i(F, \hat{F})$, we have $(A, \hat{F}\restriction A, \mathfrak{n}', \mathfrak{o}', \mathfrak{e}') \in \mathfrak{U}_i$ if and only if $\hat{F} \equiv i$ on A, i.e. $A \subseteq \hat{F}_i$; since $\hat{F}_i \subseteq A$ by Assumption 1, then the only possibility is $A = \hat{F}_i$. By Assumption 5 we have $\hat{F}_i \in \mathfrak{A}_i(F, \hat{F})$ and therefore $(\mathbf{1}_i^\star \otimes \mathrm{id})\Delta_i = \mathrm{id}$.

Finally, the co-associativity (3.8a) of Δ_i on $\mathfrak{F}$ shows the required compatibility between the coaction $\Delta_i \colon \langle \mathfrak{F}_j \rangle \to \langle \mathfrak{F}_i \rangle \hat{\otimes} \langle \mathfrak{F}_j \rangle$ and the coproduct $\Delta_i \colon \langle \mathfrak{F}_i \rangle \to \langle \mathfrak{F}_i \rangle \hat{\otimes} \langle \mathfrak{F}_i \rangle$. □

We now introduce an additional structure which will yield as a consequence a co-interaction property between the maps Δ_i and Δ_j.

Assumption 6 *Let $0 < i < j$. For every coloured forest $(F, \hat{F})$, one has*

$$A \in \mathfrak{A}_i(F, \hat{F}) \qquad \& \qquad B \in \mathfrak{A}_j(F, \hat{F} \cup_i A)\,, \tag{3.31a}$$

if and only if

$$B \in \mathfrak{A}_j(F, \hat{F}) \qquad \& \qquad A \in \mathfrak{A}_i(F, \hat{F} \cup_j B) \sqcup \mathfrak{A}_i(B, \hat{F}\restriction B)\,, \tag{3.31b}$$

where $\mathfrak{A} \sqcup \bar{\mathfrak{A}}$ is a shorthand for $\{A \sqcup \bar{A} \,:\, A \in \mathfrak{A} \;\&\; \bar{A} \in \bar{\mathfrak{A}}\}$.

We then have the following result.

Proposition 3.22 *Under Assumptions 1 and 6 for some $0 < i < j$, the identity*

$$\mathscr{M}^{(13)(2)(4)}(\Delta_i \otimes \Delta_i)\Delta_j = (\mathrm{id} \otimes \Delta_j)\Delta_i \tag{3.32}$$

holds on $\mathfrak{F}$, where we used the notation

$$\mathscr{M}^{(13)(2)(4)}(\tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_4) = (\tau_1 \cdot \tau_3 \otimes \tau_2 \otimes \tau_4)\,. \tag{3.33}$$

Proof. The proof is very similar to that of Proposition 3.9, but using (3.31) instead of (3.7). Using (3.31) and our definitions, for $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F}$ one has

$$\begin{aligned}
&\mathscr{M}^{(13)(2)(4)}(\Delta_i \otimes \Delta_i)\Delta_j\, \tau = \\
&= \sum_{B \in \mathfrak{A}_j(F,\hat{F})} \sum_{A_1 \in \mathfrak{A}_i(B, \hat{F}\restriction B)} \sum_{A_2 \in \mathfrak{A}_i(F, \hat{F} \cup_j B)} \sum_{\varepsilon_B^F, \varepsilon_{A_1}^B, \varepsilon_{A_2}^F} \sum_{\mathfrak{n}_B, \mathfrak{n}_{A_1}, \mathfrak{n}_{A_2}} \\
&\frac{1}{\varepsilon_B^F! \varepsilon_{A_1}^B! \varepsilon_{A_2}^F!} \binom{\mathfrak{n}}{\mathfrak{n}_B} \binom{\mathfrak{n} - \mathfrak{n}_B}{\mathfrak{n}_{A_2}} \binom{\mathfrak{n}_B + \pi\varepsilon_B^F}{\mathfrak{n}_{A_1}}
\end{aligned} \tag{3.34}$$

$$
\begin{aligned}
&(A_1 \sqcup A_2, \hat{F}\restriction A, \mathfrak{n}_{A_1} + \mathfrak{n}_{A_2} + \pi(\varepsilon^B_{A_1} + \varepsilon^F_{A_2}), \mathfrak{o}, \mathfrak{e}) \\
&\otimes (B, (\hat{F}\restriction B) \cup_i A_1, \mathfrak{n}_B + \pi\varepsilon^F_B - \mathfrak{n}_{A_1}, \mathfrak{o} + \mathfrak{n}_{A_1} + \pi(\varepsilon^B_{A_1} - \mathfrak{e}^{A_1}_{\emptyset}), \mathfrak{e}^B_{A_1} + \varepsilon^B_{A_1}) \\
&\otimes (F, (\hat{F} \cup_j B) \cup_i A_2, \mathfrak{n} - \mathfrak{n}_B - \mathfrak{n}_{A_2}, \mathfrak{o} + \mathfrak{n}_B + \mathfrak{n}_{A_2} + \pi(\varepsilon^F_B + \varepsilon^F_{A_2} - \mathfrak{e}^{A_2 \sqcup B}_{\emptyset}) \\
&\qquad , \mathfrak{e}^F_{A_2 \sqcup B} + (\varepsilon^F_B)^F_{A_2} + \varepsilon^F_{A_2}) \; .
\end{aligned}
$$

We claim that $A_2 \cap B = \emptyset$. Indeed, as noted in the proof of Lemma 3.1, since $B \in \mathfrak{A}_j(F, \hat{F})$ one has $(\hat{F} \cup_j B)^{-1}(j) = B$ and since $A_2 \in \mathfrak{A}_i(F, \hat{F} \cup_j B)$ one has $A_2 \cap (\hat{F} \cup_j B)^{-1}(j) = \emptyset$ by property 1 in Assumption 1. This implies that

$$
(\varepsilon^F_B)^F_{A_2} = \varepsilon^F_B \; ,
$$

since ε^F_B has support in $\partial(B, F)$ which is disjoint from E_{A_2}. This is because, for $e = (e_+, e_-) \in \partial(B, F)$ we have by definition $e_+ \in N_B \subset N_F \setminus N_{A_2}$ and therefore $e \notin E_{A_2}$.

Similarly, one has

$$
\begin{aligned}
&(\mathrm{id} \otimes \Delta_j)\Delta_i \tau = \\
&\quad = \sum_{A \in \mathfrak{A}_i(F, \hat{F})} \sum_{C \in \mathfrak{A}_j(F, \hat{F} \cup_i A)} \sum_{\varepsilon^F_C, \varepsilon^F_A} \sum_{\mathfrak{n}_C, \mathfrak{n}_A} \frac{1}{\varepsilon^F_A! \varepsilon^F_C!} \binom{\mathfrak{n}}{\mathfrak{n}_A} \binom{\mathfrak{n} - \mathfrak{n}_A}{\mathfrak{n}_C} \qquad (3.35) \\
&\quad (A, \hat{F}\restriction A, \mathfrak{n}_A + \pi\varepsilon^F_A, \mathfrak{o}, \mathfrak{e}) \\
&\quad \otimes (C, (\hat{F} \cup_i A)\restriction C, \mathfrak{n}_C + \pi\varepsilon^F_C, \mathfrak{o} + \mathfrak{n}_A + \pi(\varepsilon^F_A - \mathfrak{e}^C_{\emptyset}), \mathfrak{e}^F_A + \varepsilon^F_A) \\
&\quad \otimes (F, (\hat{F} \cup_i A) \cup_j C, \mathfrak{n} - \mathfrak{n}_A - \mathfrak{n}_C, \mathfrak{o} + \mathfrak{n}_A + \mathfrak{n}_C + \pi((\varepsilon^F_A)^F_C + \varepsilon^F_C - \mathfrak{e}^{C \cup A}_{\emptyset}) \\
&\qquad , \mathfrak{e}^F_{C \cup A} + (\varepsilon^F_A)^F_C + \varepsilon^F_C) \; .
\end{aligned}
$$

By Assumption 6, there is a bijection between the outer sums of (3.34) and (3.35) given by $(A, C) \leftrightarrow (A_1 \sqcup A_2, B)$, with inverse $(A_1, A_2, B) \leftrightarrow (A \cap C, A \backslash C, C)$. Then one then has indeed $(\hat{F}\restriction B) \cup_i A_1 = (\hat{F} \cup_i A)\restriction C$. Similarly, since $i < j$ and $A_2 \cap C = \emptyset$, one has $(\hat{F} \cup_j B) \cup_i A_2 = (\hat{F} \cup_i A) \cup_j C$, so we only need to consider the decorations and the combinatorial factors.

For this purpose, we define

$$
\begin{aligned}
\bar{\varepsilon}^C_{A_1} &= \varepsilon^F_A \, \mathbb{1}_{E_C} \; , \qquad & \bar{\varepsilon}^F_{A_2} &= (\varepsilon^F_A) \, \mathbb{1}_{\partial(A_2, F)} \; , \\
\bar{\varepsilon}^F_{A_1, C} &= \varepsilon^F_A \, \mathbb{1}_{\partial(C, F)} \; , \qquad & \bar{\varepsilon}^F_C &= \varepsilon^F_C + \bar{\varepsilon}^F_{A_1, C} \; ,
\end{aligned}
$$

as well as

$$
\bar{\mathfrak{n}}_{A_1} = (\mathfrak{n}_A\restriction C) + \pi\bar{\varepsilon}^F_{A_1, C} \; , \quad \bar{\mathfrak{n}}_{A_2} = \mathfrak{n}_A\restriction(F \setminus C) \; , \quad \bar{\mathfrak{n}}_C = \mathfrak{n}_C + (\mathfrak{n}_A\restriction C) \; .
$$

As before, the supports of these functions are consistent with our notations, with the particular case of $\bar{\varepsilon}^F_{A_1, C}$ whose support is contained in $\partial(A, F) \cap \partial(C, F) = \partial(A_1, F) \cap \partial(C, F)$, where we use again the fact that $A_2 \cap C = \emptyset$. Moreover the map

$$
(\varepsilon^F_A, \varepsilon^F_C, \mathfrak{n}_A, \mathfrak{n}_C) \mapsto (\bar{\varepsilon}^C_{A_1}, \bar{\varepsilon}^C_{A_2}, \bar{\varepsilon}^F_{A_1, C}, \bar{\varepsilon}^F_C, \bar{\mathfrak{n}}_{A_1}, \bar{\mathfrak{n}}_{A_2}, \bar{\mathfrak{n}}_C)
$$

is invertible on its image, given by the functions with the correct supports and the additional constraint

$$\bar{\mathfrak{n}}_{A_1} \geq \pi \bar{\varepsilon}^F_{A_1,C} \,.$$

Its inverse is given by

$$\begin{aligned} \varepsilon^F_C &= \bar{\varepsilon}^F_C - \bar{\varepsilon}^F_{A_1,C} \,, & \varepsilon^F_A &= \bar{\varepsilon}^C_{A_1} + \bar{\varepsilon}^F_{A_2} + \bar{\varepsilon}^F_{A_1,C} \,, \\ \mathfrak{n}_A &= \bar{\mathfrak{n}}_{A_1} + \bar{\mathfrak{n}}_{A_2} - \pi \bar{\varepsilon}^F_{A_1,C} \,, & \mathfrak{n}_C &= \bar{\mathfrak{n}}_C - \bar{\mathfrak{n}}_{A_1} + \pi \bar{\varepsilon}^F_{A_1,C} \,. \end{aligned}$$

Following a calculation virtually identical to (3.14) and (3.18), combined with the fact that $\mathfrak{n}_A + \mathfrak{n}_C = \bar{\mathfrak{n}}_C + \bar{\mathfrak{n}}_{A_2}$, we see that

$$\begin{aligned} \frac{1}{\varepsilon^F_A! \, \varepsilon^F_C!} &= \frac{1}{\bar{\varepsilon}^C_{A_1}! \, \bar{\varepsilon}^F_{A_2}! \, \bar{\varepsilon}^F_{A_1,C}!} \frac{1}{(\bar{\varepsilon}^F_C - \bar{\varepsilon}^F_{A_1,C})!} = \frac{1}{\bar{\varepsilon}^F_C! \, \bar{\varepsilon}^C_{A_1}! \, \bar{\varepsilon}^F_{A_2}!} \binom{\bar{\varepsilon}^F_C}{\bar{\varepsilon}^F_{A_1,C}} \,, \\ \binom{\mathfrak{n}}{\mathfrak{n}_A} \binom{\mathfrak{n} - \mathfrak{n}_A}{\mathfrak{n}_C} &= \binom{\bar{\mathfrak{n}}_C + \bar{\mathfrak{n}}_{A_2}}{\bar{\mathfrak{n}}_{A_1} + \bar{\mathfrak{n}}_{A_2} - \pi \bar{\varepsilon}^F_{A_1,C}} \binom{\mathfrak{n}}{\bar{\mathfrak{n}}_C + \bar{\mathfrak{n}}_{A_2}} \,. \end{aligned}$$

Since $A_2 \cap C = \emptyset$ and $A_1 \subset C$, we can simplify this expression further and obtain

$$\binom{\bar{\mathfrak{n}}_C + \bar{\mathfrak{n}}_{A_2}}{\bar{\mathfrak{n}}_{A_1} + \bar{\mathfrak{n}}_{A_2} - \pi \bar{\varepsilon}^F_{A_1,C}} = \binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_{A_1} - \pi \bar{\varepsilon}^F_{A_1,C}} \,.$$

Following the same argument as (3.20), we conclude that

$$\sum_{\bar{\varepsilon}^F_{A_1,C}} \binom{\bar{\varepsilon}^F_C}{\bar{\varepsilon}^F_{A_1,C}} \binom{\bar{\mathfrak{n}}_C}{\bar{\mathfrak{n}}_{A_1} - \pi \bar{\varepsilon}^F_{A_1,C}} = \binom{\bar{\mathfrak{n}}_C + \pi \bar{\varepsilon}^F_C}{\bar{\mathfrak{n}}_{A_1}} \,,$$

so that (3.35) can be rewritten as

$$\begin{aligned} (\mathrm{id} \otimes \Delta_j) \Delta_i \tau = \sum_{C \in \mathfrak{A}_j(F,\hat{F})} \sum_{A \in \mathfrak{A}_i(F,\hat{F} \restriction C)} \sum_{A \in \mathfrak{A}_i(F,\hat{F} \cup_j C)} \sum_{\bar{\varepsilon}^C_{A_1}, \bar{\varepsilon}^C_{A_2}, \bar{\varepsilon}^F_C} \sum_{\bar{\mathfrak{n}}_{A_1}, \bar{\mathfrak{n}}_{A_2}, \bar{\mathfrak{n}}_C} \\ \frac{1}{\bar{\varepsilon}^F_C! \bar{\varepsilon}^C_{A_1}! \bar{\varepsilon}^F_{A_2}!} \binom{\mathfrak{n}}{\bar{\mathfrak{n}}_C + \bar{\mathfrak{n}}_{A_2}} \binom{\bar{\mathfrak{n}}_C + \pi \bar{\varepsilon}^F_C}{\bar{\mathfrak{n}}_{A_1}} \\ (A_1 \sqcup A_2, \hat{F} \restriction A, \bar{\mathfrak{n}}_{A_1} + \bar{\mathfrak{n}}_{A_2} + \pi(\bar{\varepsilon}^C_{A_1} + \bar{\varepsilon}^F_{A_2}), \mathfrak{o}, \mathfrak{e}) \\ \otimes (C, (\hat{F} \restriction C) \cup_i A_1, \bar{\mathfrak{n}}_C + \pi \bar{\varepsilon}^F_C - \bar{\mathfrak{n}}_{A_1}, \mathfrak{o} + \bar{\mathfrak{n}}_{A_1} + \pi(\bar{\varepsilon}^C_{A_1} - \mathfrak{e}^{A_1}_{\emptyset}), \mathfrak{e}^C_{A_1} + \bar{\varepsilon}^C_{A_1}) \\ \otimes (F, (\hat{F} \cup_j C) \cup_i A_2, \mathfrak{n} - \bar{\mathfrak{n}}_C - \bar{\mathfrak{n}}_{A_2}, \mathfrak{o} + \bar{\mathfrak{n}}_C + \bar{\mathfrak{n}}_{A_2} + \pi(\bar{\varepsilon}^F_C + \bar{\varepsilon}^F_{A_2} - \mathfrak{e}^{A_2 \sqcup C}_{\emptyset}) \\ \quad , \mathfrak{e}^F_{A_2 \sqcup C} + \bar{\varepsilon}^F_C + \bar{\varepsilon}^F_{A_2}) \,. \end{aligned} \tag{3.36}$$

We have also used the fact that

$$(\pi \varepsilon^F_A) \restriction N_C = \pi(\varepsilon^F_A \mathbb{1}_{E_C}) + \pi(\varepsilon^F_A \mathbb{1}_{\partial(C,F)}) = \pi \bar{\varepsilon}^C_{A_1} + \pi \bar{\varepsilon}^F_{A_1,C}.$$

On the other hand, since A_2 and B are disjoint, one has

$$\binom{\mathfrak{n}}{\mathfrak{n}_B} \binom{\mathfrak{n} - \mathfrak{n}_B}{\mathfrak{n}_{A_2}} = \frac{\mathfrak{n}!}{\mathfrak{n}_B! \, \mathfrak{n}_{A_2}! \, (\mathfrak{n} - \mathfrak{n}_B - \mathfrak{n}_{A_2})!} = \binom{\mathfrak{n}}{\mathfrak{n}_B + \mathfrak{n}_{A_2}} \,,$$

so that (3.34) can be rewritten as

$$\begin{aligned}
&\mathscr{M}^{(13)(2)(4)}(\Delta_i \otimes \Delta_i)\Delta_j \tau = \\
&= \sum_{B \in \mathfrak{A}_j(F,\hat{F})} \sum_{A_1 \in \mathfrak{A}_i(B,\hat{F}\restriction B)} \sum_{A_2 \in \mathfrak{A}_i(F,\hat{F}\cup_j B)} \sum_{\varepsilon_B^F, \varepsilon_{A_1}^B, \varepsilon_{A_2}^F} \sum_{\mathfrak{n}_B, \mathfrak{n}_{A_1}, \mathfrak{n}_{A_2}} \\
&\frac{1}{\varepsilon_B^F! \varepsilon_{A_1}^B! \varepsilon_{A_2}^F!} \binom{\mathfrak{n}}{\mathfrak{n}_B + \mathfrak{n}_{A_2}} \binom{\mathfrak{n}_B + \pi\varepsilon_B^F}{\mathfrak{n}_{A_1}} \\
&(A_1 \sqcup A_2, \hat{F}\restriction A, \mathfrak{n}_{A_1} + \mathfrak{n}_{A_2} + \pi(\varepsilon_{A_1}^B + \varepsilon_{A_2}^F), \mathfrak{o}, \mathfrak{e}_\emptyset^{A_1 \sqcup A_2}) \\
&\otimes (B, (\hat{F}\restriction B) \cup_i A_1, \mathfrak{n}_B + \pi\varepsilon_B^F - \mathfrak{n}_{A_1}, \mathfrak{o} + \mathfrak{n}_{A_1} + \pi(\varepsilon_{A_1}^B - \mathfrak{e}_\emptyset^{A_1}), \mathfrak{e}_{A_1}^B + \varepsilon_{A_1}^B) \\
&\otimes (F, (\hat{F} \cup_j B) \cup_i A_2, \mathfrak{n} - \mathfrak{n}_B - \mathfrak{n}_{A_2}, \mathfrak{o} + \mathfrak{n}_B + \mathfrak{n}_{A_2} + \pi(\varepsilon_B^F + \varepsilon_{A_2}^F - \mathfrak{e}_\emptyset^{A_2}) \\
&\qquad , \mathfrak{e}_{A_2}^F + \varepsilon_B^F + \varepsilon_{A_2}^F) \, .
\end{aligned} \tag{3.37}$$

Comparing this with (3.36) we obtain the desired result. □

Corollary 3.23 *Let $0 < i < j$. If Assumptions 1–6 hold, then the space $\langle \mathfrak{F}_j \rangle$ is a comodule bialgebra over the bialgebra $\langle \mathfrak{F}_i \rangle$ with coaction Δ_i, in the sense of [Mol77, Def 2.1(e)]. In the terminology of [Foi16, Def. 1], $\langle \mathfrak{F}_j \rangle$ and $\langle \mathfrak{F}_i \rangle$ are in cointeraction.*

Remark 3.24 Note that the roles of i and j are asymmetric for $0 < i < j$: $\langle \mathfrak{F}_i \rangle$ is in general *not* a comodule bialgebra over $\langle \mathfrak{F}_j \rangle$. This is a consequence of the asymmetry between the roles played by i and j in Assumption 1. In particular, every $A \in \mathfrak{A}_i(F, \hat{F})$ has empty intersection with $\hat{F}_j$, while any $B \in \mathfrak{A}_j(F, \hat{F})$ can contain connected components of $\hat{F}_i$.

3.8 Skew products and group actions

We assume throughout this subsection that $0 < i < j$ and that Assumptions 1–6 hold. Following [Mol77], we define a space $\mathscr{H}_{ij} = \mathscr{H}_i \ltimes \mathscr{H}_j$ as follows. As a vector space, we set $\mathscr{H}_{ij} = \mathscr{H}_i \hat{\otimes} \mathscr{H}_j$, and we endow it with the product and coproduct

$$\begin{aligned}
(a \otimes b) \cdot (\bar{a} \otimes \bar{b}) &= (a \cdot \bar{a}) \otimes (b \cdot \bar{b}) \, , \\
\Delta_{ij}(a \otimes b) &= \mathscr{M}^{(14)(3)(2)(5)}(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \Delta_i)(\Delta_i \otimes \Delta_j)(a \otimes b) \, .
\end{aligned} \tag{3.38}$$

We also define $\mathbf{1}_{ij} \stackrel{\text{def}}{=} \mathbf{1}_i \otimes \mathbf{1}_j$, $\mathbf{1}_{ij}^\star \stackrel{\text{def}}{=} \mathbf{1}_i^\star \otimes \mathbf{1}_j^\star$.

Proposition 3.25 *The 5-tuple $(\mathscr{H}_{ij}, \cdot, \Delta_{ij}, \mathbf{1}_{ij}, \mathbf{1}_{ij}^\star)$ is a Hopf algebra.*

Proof. We first note that, for every $\tau \in \mathfrak{M}_j$, one has $\Delta_i \tau = \mathbf{1} \otimes \tau$ since one has $\mathfrak{A}_i(F, j) = \{\emptyset\}$ by Assumptions 1 and 5. It follows that one has the identity

$$(\mathscr{K}_i \otimes \mathscr{K}_j)\Delta_i = (\mathscr{K}_i \otimes \mathscr{K}_j)\Delta_i \mathscr{K}_j \, ,$$

see also (3.25). Combining this with Lemma 3.21, we conclude that one can indeed view Δ_i as a map $\Delta_i \colon \mathscr{H}_j \to \mathscr{H}_i \hat{\otimes} \mathscr{H}_j$, so that (3.38) is well-defined.

By Proposition 3.22, Δ_{ij} is coassociative, and it is multiplicative with respect to the product, see also [Mol77, Thm 2.14]. Note also that on $\mathscr{H}_j$ one has the identity

$$(\mathrm{id} \otimes \mathbf{1}_j^\star)\Delta_i = \mathbf{1}_i\, \mathbf{1}_j^\star \ ,$$

where $\mathbf{1}_i$ is the unit in $\mathscr{H}_i$. As a consequence, $\mathbf{1}_{ij}^\star$ is the counit for $\mathscr{H}_{ij}$, and one can verify that

$$\mathscr{A}_{ij} = (\mathscr{A}_i \mathscr{M} \otimes \mathscr{A}_j)(\mathrm{id} \otimes \Delta_i) \ ,$$

is the antipode turning $\mathscr{H}_{ij}$ into a Hopf algebra. □

Let us recall that $\mathscr{G}_i$ denotes the character group of $\mathscr{H}_i$.

Lemma 3.26 *Let us set for $\ell \in \mathscr{G}_i$, $g \in \mathscr{G}_j$, the element $\ell g \in \mathscr{H}_j^*$*

$$(\ell g)\tau \stackrel{def}{=} (\ell \otimes g)\, \Delta_i \tau, \qquad \tau \in \mathscr{H}_j.$$

Then this defines a left action of $\mathscr{G}_i$ onto $\mathscr{G}_j$ by group automorphisms.

Proof. The dualization of Proposition 3.22 yields that $\ell(g_1 g_2) = (\ell g_1)(\ell g_2)$, which means that this is indeed an action. □

Proposition 3.27 *The semi-direct product $\mathscr{G}_{ij} \stackrel{def}{=} \mathscr{G}_i \ltimes \mathscr{G}_j$, with group multiplication*

$$(\ell_1, g_1)(\ell_2, g_2) = (\ell_1 \ell_2, g_1(\ell_1 g_2)), \qquad \ell_1, \ell_2 \in \mathscr{G}_i,\ g_1, g_2 \in \mathscr{G}_j, \tag{3.39}$$

defines a sub-group of the group of characters of $\mathscr{H}_{ij}$.

Proof. Note that (3.39) is the dualisation of Δ_{ij} in (3.38). The inverse is given by

$$(\ell, g)^{-1} = (\ell^{-1}, \ell^{-1} g^{-1}) \ ,$$

since $(\ell, g) \cdot (\ell^{-1}, \ell^{-1} g^{-1}) = (\ell\ell^{-1}, g(\ell\ell^{-1} g^{-1})) = (\mathbf{1}_i^\star, \mathbf{1}_j^\star)$. □

The space $\langle \mathfrak{F}_j \rangle$ is a left comodule over $\mathscr{H}_{ij}$ with coaction given by $\Delta_{ij} \colon \langle \mathfrak{F}_j \rangle \to \mathscr{H}_{ij} \otimes \langle \mathfrak{F}_j \rangle$ with

$$\Delta_{ij} = \sigma^{(132)}(\Delta_i \otimes \mathscr{A}_j)\Delta_j \ , \tag{3.40}$$

where $\sigma^{(132)}(a \otimes b \otimes c) \stackrel{\text{def}}{=} a \otimes c \otimes b$. More generally, we have:

Proposition 3.28 *Let V be a vector space such that $\mathscr{G}_i$ acts on V on the left and $\mathscr{G}_j$ acts on V on the right, and we assume that*

$$\ell(hg) = (\ell h)(\ell g) \ , \qquad \ell \in \mathscr{G}_i,\ g \in \mathscr{G}_j,\ h \in V.$$

Then $\mathscr{G}_{ij}$ acts on the left on V by

$$(\ell, g)h = (\ell h)g^{-1} \ , \qquad \ell \in \mathscr{G}_i,\ g \in \mathscr{G}_j,\ h \in V. \tag{3.41}$$

Proof. Note that (3.41) is the dualisation of (3.40). Now we have

$$\begin{aligned}(\ell_1, g_1)((\ell_2, g_2)h) &= (\ell_1, g_1)((\ell_2 h)g_2^{-1}) = (\ell_1((\ell_2 h)g_2^{-1}))g_1^{-1} \\ &= (\ell_1 \ell_2 h)(\ell_1 g_2^{-1})g_1^{-1} = (\ell_1 \ell_2, g_1(\ell_1 g_2))h = ((\ell_1, g_1)(\ell_2, g_2))h \ ,\end{aligned}$$

which is exactly what we wanted. □

4 A specific setting suitable for renormalisation

We now specialise the framework described in the previous section to the situation of interest to us. We define two collections $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as follows.

Definition 4.1 For any coloured forest $(F, \hat{F})$ as in Definition 2.3 we define the collection $\mathfrak{A}_1(F, \hat{F})$ of all subforests A of F such that $\hat{F}_1 \subset A$ and $\hat{F}_2 \cap A = \emptyset$. We also define $\mathfrak{A}_2(F, \hat{F})$ to consist of all subforests A of F with the following properties:

1. A contains $\hat{F}_2$
2. for every non-empty connected component T of F, $T \cap A$ is connected and contains the root of T
3. for every connected component S of $\hat{F}_1$, one has either $S \subset A$ or $S \cap A = \emptyset$.

The images in Example 3.2 above are compatible with these definitions. We recall from Definition 3.10 that $\mathfrak{F}_i$ is given for $i = 1, 2$ by

$$\mathfrak{F}_i = \{(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F} \,:\, \hat{F} \leq i \;\&\; \{F, \hat{F}_i\} \subset \mathfrak{A}_i(F, \hat{F})\} \,.$$

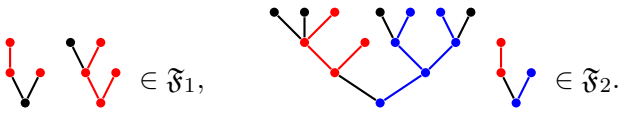
Lemma 4.2 *For $\tau = (F, \hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \mathfrak{F}$ we have*

- *$\tau \in \mathfrak{F}_1$ if and only if $\hat{F} \leq 1$*
- *$\tau \in \mathfrak{F}_2$ if and only if $\hat{F} \leq 2$ and, for every non-empty connected component T of F, $\hat{F}_2 \cap T$ is a subtree of T containing the root of T.*

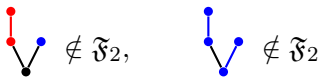
Proof. Let $(F, \hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \mathfrak{F}$. If $\hat{F} \leq 1$ then $\hat{F}_2 = \emptyset$ and therefore $F \in \mathfrak{A}_1(F, \hat{F})$; moreover $A = \hat{F}_1$ clearly satisfies $\hat{F}_1 \subset A$ and $A \cap \hat{F}_2 = \emptyset$, so that $\hat{F}_1 \in \mathfrak{A}_1(F, \hat{F})$ and therefore $(F, \hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \mathfrak{F}_1$. The converse is obvious.

Let us suppose now that $\hat{F} \leq 2$ and for every connected component T of F, $\hat{F}_2 \cap T$ is a subtree of T containing the root of T. Then $A = F$ clearly satisfies the properties 1-3 of Definition 4.1. If now $A = \hat{F}_2$, then A satisfies the properties 1 and 2 since for every non-empty connected component T of F, $\hat{F}_2 \cap T$ is a subtree of T containing the root of T, while property 3 is satisfied since $\hat{F}_1 \cap \hat{F}_2 = \emptyset$. The converse is again obvious. □

Example 4.3 As in previous examples, red stands for 1 and blue for 2 (and black for 0):



On the other hand,



because $\hat{F}_2$ does not contain the root in the first case, and $\hat{F}_2$ is not connected in the second.

Lemma 4.4 *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be given by Definition 4.1.*
- *$\mathfrak{A}_1$ satisfies Assumptions 1, 2, 3 and 4.*
- *$\mathfrak{A}_2$ satisfies Assumptions 1, 2, 3 and 4.*
- *The pair $(\mathfrak{A}_1, \mathfrak{A}_2)$ satisfies Assumptions 5 and 6.*

Proof. The first statement concerning $\mathfrak{A}_1$ is elementary. The only non-trivial property to be checked about $\mathfrak{A}_2$ is (3.7); note that $\mathfrak{A}_2$ has the stronger property that for any two subtrees $B \subset A \subset F$, one has $A \in \mathfrak{A}_2(F, \hat{F})$ if and only if $A \in \mathfrak{A}_2(F, \hat{F} \cup_2 B)$ and $B \in \mathfrak{A}_2(F, \hat{F})$ if and only if $B \in \mathfrak{A}_2(A, \hat{F}\restriction A)$, so that property (3.7) follows at once.

Assumption 5 is easily seen to hold, since for every coloured forest $(F, \hat{F})$ such that $\hat{F} \leq 2$ and $\{F, \hat{F}_2\} \subset \mathfrak{A}_2(F, \hat{F})$, for $A \stackrel{\text{def}}{=} \hat{F}_1$ one has $\hat{F}_1 \subset A$ and $\hat{F}_2 \cap A = \emptyset$, so that $\hat{F}_1 \in \mathfrak{A}_1(F, \hat{F})$.

We check now that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ satisfy Assumption 6. Let $A \in \mathfrak{A}_1(F, \hat{F})$ and $B \in \mathfrak{A}_2(F, \hat{F} \cup_1 A)$; then $A \cap \hat{F}_2 = \emptyset$ and therefore $B \in \mathfrak{A}_2(F, \hat{F})$; moreover every connected component of A is contained in a connected component of $\hat{F}_1$ and therefore is either contained in B or disjoint from B, i.e. $A \in \mathfrak{A}_1(F, \hat{F} \cup_2 B) \sqcup \mathfrak{A}_1(B, \hat{F}\restriction B)$. Conversely, let $B \in \mathfrak{A}_2(F, \hat{F})$ and $A \in \mathfrak{A}_1(F, \hat{F} \cup_2 B) \sqcup \mathfrak{A}_1(B, \hat{F}\restriction B)$; then $\hat{F}_1 = (\hat{F} \cup_2 B)_1 \sqcup (\hat{F}\restriction B)_1$ and $\hat{F}_2 \subset (\hat{F} \cup_2 B)_2$ so that A contains $\hat{F}_1$ and is disjoint from $\hat{F}_2$ and therefore $A \in \mathfrak{A}_1(F, \hat{F})$; moreover $(\hat{F} \cup_1 A)_2 \subseteq \hat{F}_2$ so that B contains $(\hat{F} \cup_1 A)_2$; finally $(\hat{F} \cup_1 A)_1 = A$ and by the assumption on A we have that every connected component of $(\hat{F} \cup_1 A)_1$ is either contained in B or disjoint from B. The proof is complete. □

In view of Propositions 3.18 and 3.22, we have the following.

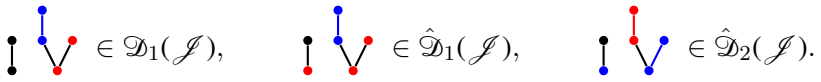
Corollary 4.5 *The space $(\mathscr{H}_2, \cdot, \Delta_2, \mathbf{1}_2, \mathbf{1}_2^\star)$ is a Hopf algebra and a comodule bialgebra over the Hopf algebra $(\mathscr{H}_1, \cdot, \Delta_1, \mathbf{1}_1, \mathbf{1}_1^\star)$ with coaction Δ_1 and counit $\mathbf{1}_1^\star$.*

4.1 Joining roots

While the product given by "disjoint unions" considered so far is very natural when considering forests, it is much less natural when considering spaces of trees. There, the more natural thing to do is to join trees together by their roots. Given a typed forest F, we then define the typed tree $\mathscr{J}(F)$ by joining all the roots of F together. In other words, we set $\mathscr{J}(F) = F/\sim$, where $\sim$ is the equivalence relation on nodes in N_F given by $x \sim y$ if and only if either $x = y$ or both x and y belong to the set ϱ_F of nodes of F.

When considering coloured or decorated trees as we do here, such an operation cannot in general be performed unambiguously since different trees may have roots of different colours. This justifies the definition of the subset $\mathscr{D}_i(\mathscr{J}) \subset \mathfrak{F}$ as the set of all forests $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ such that $\hat{F}(\varrho) \in \{0, i\}$ for every root ϱ of F. We also write $\mathscr{D}(\mathscr{J}) = \bigcup_{i \geq 0} \mathscr{D}_i(\mathscr{J})$ and $\hat{\mathscr{D}}_i(\mathscr{J}) \subset \mathscr{D}_i(\mathscr{J})$ for the set of forests such that *every* root has colour i.

Example 4.6 Using as usual red for 1 and blue for 2, we have



We can then extend $\mathscr{J}$ to $\mathscr{D}(\mathscr{J})$ in a natural way as follows.

Definition 4.7 For $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathscr{D}(\mathscr{J})$, we define the decorated tree $\mathscr{J}(\tau) \in \mathfrak{F}$ by

$$\mathscr{J}(\tau) = (\mathscr{J}(F), [\hat{F}], [\mathfrak{n}], [\mathfrak{o}], \mathfrak{e}) ,$$

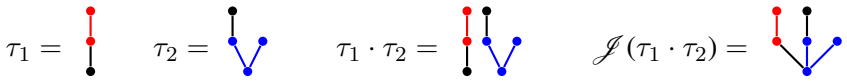
where $[\mathfrak{n}](x) = \sum_{y \in x} \mathfrak{n}(y)$, $[\mathfrak{o}](x) = \sum_{y \in x} \mathfrak{o}(y)$, and $[\hat{F}](x) = \sup_{y \in x} \hat{F}(y)$.

It is clear that the $\mathscr{D}_i$'s are closed under multiplication and that one has

$$\mathscr{J}(\tau \cdot \bar{\tau}) = \mathscr{J}\big(\tau \cdot \mathscr{J}(\bar{\tau})\big) , \qquad \tau, \bar{\tau} \in \mathscr{D}_i(\mathscr{J}) \tag{4.1}$$

for every $i \geq 0$. Furthermore, $\mathscr{J}$ is idempotent and preserves our bigrading.

Example 4.8 The following forests belong to $\mathscr{D}_2(\mathscr{J})$



The following fact is also easy to verify, where $\mathscr{K}$, $\hat{\mathscr{K}}_i$, Φ_i, $\hat{\Phi}_i$ and $\hat{P}_i$ were defined in Section 3.5.

Lemma 4.9 *For $i \geq 0$, the sets $\mathscr{D}_i(\mathscr{J})$ and $\hat{\mathscr{D}}_i(\mathscr{J})$ are invariant under $\mathscr{K}$, Φ_i, $\hat{P}_i$ and $\mathscr{J}$. Furthermore, $\mathscr{J}$ commutes with both $\mathscr{K}$ and $\hat{P}_i$ on $\mathscr{D}_i(\mathscr{J})$ and satisfies the identity*

$$\hat{\mathscr{K}}_i \mathscr{J} = \hat{\mathscr{K}}_i \mathscr{J} \hat{\mathscr{K}}_i , \qquad \text{on } \hat{\mathscr{D}}_i(\mathscr{J}). \tag{4.2}$$

In particular $\hat{\mathscr{K}}_i \mathscr{J}$ is idempotent on $\hat{\mathscr{D}}_i(\mathscr{J})$.

Proof. The spaces $\mathscr{D}_i(\mathscr{J})$ and $\hat{\mathscr{D}}_i(\mathscr{J})$ are invariant under $\mathscr{K}$, Φ_i and $\hat{P}_i$ because these operations never change the colours of the roots. The invariance under $\mathscr{J}$ follows in a similar way.

The fact that $\mathscr{J}$ commutes with $\mathscr{K}$ is obvious. The reason why it commutes with $\hat{P}_i$ is that $\mathfrak{o}$ vanishes on colourless nodes by the definition of $\mathfrak{F}$. Regarding (4.2), since $\hat{\mathscr{K}}_i = \hat{P}_i \Phi_i \mathscr{K}$, and all three operators are idempotent and commute with each other, we have

$$\hat{\mathscr{K}}_i \mathscr{J} = \Phi_i \hat{P}_i \mathscr{J} \mathscr{K} , \qquad \hat{\mathscr{K}}_i \mathscr{J} \hat{\mathscr{K}}_i = \Phi_i \hat{P}_i \mathscr{J} \Phi_i \mathscr{K}$$

so that it suffices to show that

$$\hat{P}_i \mathscr{J} \mathscr{K} = \hat{P}_i \mathscr{J} \Phi_i \mathscr{K} . \tag{4.3}$$

For this, consider an element $\tau \in \hat{\mathcal{D}}_i(\mathscr{J})$ and write $\tau = \mu \cdot \nu$ as in (3.23). By the definition of this decomposition and of $\mathscr{K}$, there exist $k \geq 0$ and labels $n_j \in \mathbf{N}^d$, $o_j \in \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$ with $j \in \{1, \dots, k\}$ such that

$$\mathscr{K}\tau = (\mathscr{K}\mu) \cdot x^{(i)}_{n_1,o_1} \cdots x^{(i)}_{n_k,o_k} ,$$

where $x^{(i)}_{n,o} = (\bullet, i, n, o, 0)$. It follows that

$$\Phi_i \mathscr{K}\tau = (\mathscr{K}\mu) \cdot x^{(i)}_{n,0} \tag{4.4}$$

with $n = \sum_{j=1}^k n_j$. On the other hand, by (4.1), one has

$$\mathscr{J}\mathscr{K}\tau = \mathscr{J}\left((\mathscr{K}\mu) \cdot x^{(i)}_{n,o}\right) ,$$

with o defined from the o_i similarly to n. Comparing this to (4.4), it follows that $\mathscr{J}\mathscr{K}\tau$ differs from $\mathscr{J}\Phi_i\mathscr{K}\tau$ only by its $\mathfrak{o}$-decoration at the root of one of its connected components in the sense of Remark 2.8. Since these are set to 0 by $\hat{\Phi}_i$, (4.3) follows. □

Finally, we show that the operation of joining roots is well adapted to the definitions given in the previous subsection. In particular, we assume from now on that the $\mathfrak{A}_i$ for $i = 1, 2$ are given by Definition 4.1. Our definitions guarantee that

- $\mathfrak{F}_1 \subset \mathfrak{D}_1(\mathscr{J})$
- $\mathfrak{F}_2 \subset \hat{\mathfrak{D}}_2(\mathscr{J})$.

We then have the following, where $\mathscr{J}$ is extended to the relevant spaces as a triangular map.

Proposition 4.10 *One has the identities*

$$\begin{aligned} \Delta_2\mathscr{J} &= (\mathscr{J} \otimes \mathscr{J})\Delta_2 = (\mathscr{J} \otimes \mathscr{J})\Delta_2\mathscr{J} , \quad \text{on} \quad \mathfrak{D}(\mathscr{J}), \\ \Delta_1\mathscr{J} &= (\mathrm{id} \otimes \mathscr{J})\Delta_1 = (\mathrm{id} \otimes \mathscr{J})\Delta_1\mathscr{J} , \quad \text{on} \quad \mathfrak{F}_2. \end{aligned}$$

Proof. Extend $\mathscr{J}$ to coloured trees by $\mathscr{J}(F, \hat{F}) = (\mathscr{J}(F), [\hat{F}])$ with $[\hat{F}]$ as in Definition 4.7. The first identity then follows from the following facts. By the definition of $\mathfrak{A}_2$, one has

$$\mathfrak{A}_2(\mathscr{J}(F, \hat{F})) = \{\mathscr{J}_F A \,:\, A \in \mathfrak{A}_2(F, \hat{F})\} , \tag{4.5}$$

where $\mathscr{J}_F A$ is the subforest of $\mathscr{J}F$ obtained by the image of the subforest A of F under the quotient map. The map $\mathscr{J}_F$ is furthermore injective on $\mathfrak{A}_2(F, \hat{F})$, thus yielding a bijection between $\mathfrak{A}_2(\mathscr{J}(F, \hat{F}))$ and $\mathfrak{A}_2(F, \hat{F})$. Finally, as a consequence of the fact that each connected component of A contains a root of F, there is a natural tree isomorphism between $\mathscr{J}_F A$ and $\mathscr{J}A$. Combining this with an application of the Chu-Vandermonde identity on the roots allows to conclude.

The identity (4.5) fails to be true for $\mathfrak{A}_1$ in general. However, if $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F}_2$, then each of the roots of F is covered by $\hat{F}^{-1}(2)$, so that (4.5) with $\mathfrak{A}_2$ replaced by $\mathfrak{A}_1$ does hold in this case. Furthermore, one then has a natural forest isomorphism between $\mathscr{J}_F A$ and A (as a consequence of the fact that A does not contain any of the roots of F), so that the second identity follows immediately. □

We now use the "root joining" map $\mathscr{J}$ to define

$$\hat{\mathscr{H}}_2 \stackrel{\text{def}}{=} \langle \mathfrak{F}_2 \rangle / \ker(\mathscr{J}\hat{\mathcal{K}}_2) \simeq \mathscr{H}_2 / \ker(\mathscr{J}\hat{P}_2) \,. \tag{4.6}$$

Note here that $\mathscr{J}\hat{P}_2$ is well-defined on $\mathscr{H}_2$ by (4.2), so that the last identity makes sense. The identity (4.2) also implies that $\ker(\mathscr{J}\hat{\mathcal{K}}_2) = \ker(\hat{\mathcal{K}}_2\mathscr{J})$, so the order in which the two operators appear here does not matter. Furthermore, the Hopf algebra structure of $\mathscr{H}_2$ turns $\hat{\mathscr{H}}_2$ into a Hopf algebra as well by the first part of Proposition 4.10 and (4.1), combined with [Nic78, Thm 1 (iv)], which states that if H is a Hopf algebra over a field and I a bi-ideal of H such that H/I is commutative, then H/I is a Hopf algebra.

4.2 Algebraic renormalisation

We set

$$\mathfrak{F}_\circ \stackrel{\text{def}}{=} \{(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F} : \hat{F} \le 1,\ F \text{ is a tree}\} \,, \quad \mathscr{H}_\circ \stackrel{\text{def}}{=} \langle \mathfrak{F}_\circ \rangle / \ker(\mathcal{K}) \,. \tag{4.7}$$

Then, $\mathscr{H}_\circ$ is an algebra when endowed with the *tree product*

$$\mathcal{D}_i(\mathscr{J}) \times \mathcal{D}_i(\mathscr{J}) \ni (\tau, \bar{\tau}) \mapsto \tau\bar{\tau} \stackrel{\text{def}}{=} \mathscr{J}(\tau \cdot \bar{\tau}) \tag{4.8}$$

in the special case $i = 1$. Note that this product is well-defined on $\mathscr{H}_\circ$ since $\mathcal{K}$ is multiplicative and $\mathscr{J}$ commutes with $\mathcal{K}$. Furthermore, one has $\tau \cdot \bar{\tau} \in \mathcal{D}_1(\mathscr{J})$ for any $\tau, \bar{\tau} \in \mathfrak{F}_\circ$. As a consequence of (4.1) and the fact that $\cdot$ is associative, we see that the tree product is associative, thus turning $\mathscr{H}_\circ$ into a commutative algebra with unit $(\bullet, 0, 0, 0, 0)$.

Remark 4.11 The main reason why we do not define $\mathscr{H}_\circ$ similarly to $\hat{\mathscr{H}}_2$ by setting $\mathscr{H}_\circ = \langle \mathfrak{F}_1 \rangle / \ker(\mathscr{J}\mathcal{K})$ is that Δ_1 is not well-defined on that quotient space, while it is well-defined on $\mathscr{H}_\circ$ as given by (4.7), see Proposition 4.14.

Remark 4.12 Using Lemma 2.14 as in Remark 3.17, we have canonical isomorphisms

$$\begin{aligned} \mathscr{H}_\circ &\simeq \langle H_\circ \rangle, \qquad H_\circ \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathfrak{F}_\circ : \mathcal{K}\mathcal{F} = \mathcal{F}\} \,, \\ \mathscr{H}_1 &\simeq \langle H_1 \rangle, \qquad H_1 \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathfrak{F}_1 : \mathcal{K}_1\mathcal{F} = \mathcal{F}\} \,, \\ \hat{\mathscr{H}}_2 &\simeq \langle \hat{H}_2 \rangle, \qquad \hat{H}_2 \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathfrak{F}_2 : \mathscr{J}\hat{\mathcal{K}}_2\mathcal{F} = \mathcal{F}\} \,. \end{aligned} \tag{4.9}$$

In particular, we can view $\mathscr{H}_\circ$ and $\hat{\mathscr{H}}_2$ as spaces of decorated trees rather than forests. In both cases, the original forest product $\cdot$ can (and will) be interpreted as the tree product (4.8) with, respectively, $i = 1$ and $i = 2$.

We obtain from Corollary 4.5 the following extension of Theorem 8 in [CEFM11]:

Corollary 4.13 *The space $(\hat{\mathscr{H}}_2, \mathscr{M}, \Delta_2, \mathbf{1}_2, \mathbf{1}_2^\star)$, where $\mathscr{M}$ is the tree product* (4.8) *with $i = 2$, is a Hopf algebra and a comodule bialgebra over the Hopf algebra $(\mathscr{H}_1, \cdot, \Delta_1, \mathbf{1}_1, \mathbf{1}_1^\star)$ with coaction Δ_1 and counit $\mathbf{1}_1^\star$.*

Moreover, combining all the results we obtained so far, we see that we have constructed the following structure.

Proposition 4.14 *We have*

1. *$\mathcal{H}_\circ$ is a left comodule over $\mathcal{H}_1$ with coaction Δ_1 and counit $\mathbf{1}_1^\star$.*
2. *$\mathcal{H}_\circ$ is a right comodule algebra over $\hat{\mathcal{H}}_2$ with coaction Δ_2 and counit $\mathbf{1}_2^\star$.*
3. *$\mathcal{H}_\circ$ is a left comodule over the Hopf algebra $\hat{\mathcal{H}}_{12} = \mathcal{H}_1 \ltimes \hat{\mathcal{H}}_2 = (\mathcal{H}_1 \ltimes \mathcal{H}_2)/\ker(\mathrm{id} \otimes \mathscr{J})$, with counit $\mathbf{1}_{12}^\star$ and coaction*

$$\Delta_\circ : \mathcal{H}_\circ \to \hat{\mathcal{H}}_{12} \hat{\otimes} \mathcal{H}_\circ, \qquad \Delta_\circ v \stackrel{def}{=} \sigma^{(132)}(\Delta_1 \otimes \hat{\mathcal{A}}_2)\Delta_2$$

where $\sigma^{(132)}(a \otimes b \otimes c) \stackrel{def}{=} a \otimes c \otimes b$ and $\hat{\mathcal{A}}_2$ is the antipode of $\hat{\mathcal{H}}_2$.

Proof. The first and the second assertions follow from the coassociativity of Δ_1, respectively Δ_2, proved in Proposition 3.9, combined with Proposition 4.10 to show that these maps are well-defined on the relevant quotient spaces. The multiplicativity of Δ_2 with respect to the tree product (4.8) follows from the first identity of Proposition 4.10, combined with the fact that $\hat{\mathcal{H}}_2$ is a quotient by $\ker \mathscr{J}$.

We denote by $\hat{\mathcal{G}}_2$ the group of characters of $\hat{\mathcal{H}}_2$. In order to prove the third assertion, we show an equivalent statement: that the semi-direct product $\mathcal{G}_1 \ltimes \hat{\mathcal{G}}_2$ acts on the left on the dual space $\mathcal{H}_\circ^*$ by the formula

$$(\ell, g)h(\tau) \stackrel{\text{def}}{=} (\ell \otimes h \otimes g\hat{\mathcal{A}}_2)(\Delta_1 \otimes \mathrm{id})\Delta_2\tau,$$

for $\ell \in \mathcal{G}_1$, $g \in \hat{\mathcal{G}}_2$, $h \in \mathcal{H}_\circ^*$, $\tau \in \mathcal{H}_\circ$. In order words, we want to show that with this action $\mathcal{H}_\circ^*$ is a left module on $\mathcal{G}_1 \ltimes \hat{\mathcal{G}}_2$. In order to use Proposition 3.28, we define a left action of $\mathcal{G}_1$ on $\mathcal{H}_\circ^*$ by

$$\ell h(\tau) \stackrel{\text{def}}{=} (\ell \otimes h)\Delta_1\tau, \qquad \ell \in \mathcal{G}_1,\ h \in \mathcal{H}_\circ^*,\ \tau \in \mathcal{H}_\circ,$$

and a right action of $\hat{\mathcal{G}}_2$ on $\mathcal{H}_\circ^*$ by

$$hg(\tau) \stackrel{\text{def}}{=} (h \otimes g)\Delta_2\tau, \qquad g \in \hat{\mathcal{G}}_2,\ h \in \mathcal{H}_\circ^*,\ \tau \in \mathcal{H}_\circ.$$

These are indeed actions since by the coassociativity of Δ_1 and Δ_2 proved in Proposition 3.9

$$\begin{aligned}\ell_1(\ell_2 h) &= (\ell_1 \otimes (\ell_2 \otimes h)\Delta_1)\Delta_1 = (\ell_1 \otimes \ell_2 \otimes h)(\mathrm{id} \otimes \Delta_1)\Delta_1 \\ &= (\ell_1 \otimes \ell_2 \otimes h)(\Delta_1 \otimes \mathrm{id})\Delta_1 = ((\ell_1 \otimes \ell_2)\Delta_1 \otimes h)\Delta_1 = (\ell_1\ell_2)h,\end{aligned}$$

and

$$\begin{aligned}(hg_1)g_2 &= ((h \otimes g_1)\Delta_2 \otimes g_2)\Delta_2 = (h \otimes g_1 \otimes g_2)(\Delta_2 \otimes \mathrm{id})\Delta_2 \\ &= (h \otimes g_1 \otimes g_2)(\mathrm{id} \otimes \Delta_2)\Delta_2 = (h \otimes (g_1 \otimes g_2)\Delta_2)\Delta_2 = h(g_1 g_2).\end{aligned}$$

Following (3.41), the natural definition is for $(\ell, g) \in \mathcal{G}_1 \times \hat{\mathcal{G}}_2$ and $h \in \mathcal{H}_\circ^*$

$$(\ell, g)h \stackrel{\text{def}}{=} (\ell h)g^{-1} = (\ell h \otimes g\mathcal{A}_2)\Delta_2 = (\ell \otimes h \otimes g\mathcal{A}_2)(\Delta_1 \otimes \mathrm{id})\Delta_2.$$

By Proposition 3.28, it is now enough to prove that

$$\ell(hg) = (\ell h)(\ell g)\,, \qquad \ell \in \mathcal{G}_1,\ g \in \hat{\mathcal{G}}_2,\ h \in \mathcal{H}_\circ^*. \tag{4.10}$$

By the definitions, we have

$$\begin{aligned}\ell(hg) &= (\ell \otimes (h \otimes g)\Delta_2)\Delta_1 = (\ell \otimes h \otimes g)(\mathrm{id} \otimes \Delta_2)\Delta_1 \\ &= (\ell \otimes h \otimes g)(\mathrm{id} \otimes \Delta_2)\Delta_1,\end{aligned}$$

while

$$\begin{aligned}(\ell h)(\ell g) &= ((\ell \otimes h)\Delta_1 \otimes (\ell \otimes g)\Delta_1)\Delta_2 \\ &= (\ell \otimes h \otimes \ell \otimes g)(\Delta_1 \otimes \Delta_1)\Delta_2 \\ &= (\ell \otimes h \otimes g)\mathcal{M}^{(13)(2)(4)}(\Delta_1 \otimes \Delta_1)\Delta_2,\end{aligned}$$

and we conclude by Proposition 3.22. □

4.3 Recursive formulae

We now show how the formalism developed so far in this article links to the one developed in [Hai14, Sec. 8]. For that, we use the canonical identifications

$$\mathcal{H}_\circ = \langle H_\circ \rangle, \qquad \mathcal{H}_1 = \langle H_1 \rangle, \qquad \hat{\mathcal{H}}_2 = \langle \hat{H}_2 \rangle,$$

given in Remarks 3.17 and 4.12. We furthermore introduce the following notations.

1. For $k \in \mathbf{N}^d$, we write X^k as a shorthand for $(\bullet, 0)_0^{k,0} \in H_\circ$. We also interpret this as an element of $\hat{\mathcal{H}}_2$, although its canonical representative there is $(\bullet, 2)_0^{k,0} \in \hat{H}_2$. As usual, we also write $\mathbf{1}$ instead of X^0, and we write X_i with $i \in \{1, \dots, d\}$ as a shorthand for X^k with k equal to the ith canonical basis element of $\mathbf{N}^d$.
2. For every type $\mathfrak{t} \in \mathfrak{L}$ and every $k \in \mathbf{N}^d$, we define the linear operator

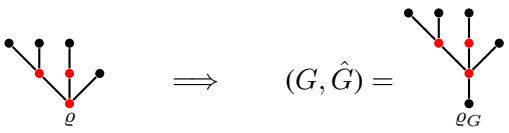
$$\mathcal{I}_k^{\mathfrak{t}} \colon \mathcal{H}_\circ \to \mathcal{H}_\circ \tag{4.11}$$

in the following way. Let $\tau = (F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in H_\circ$, so that we can assume that F consists of a single tree with root ϱ. Then, $\mathcal{I}_k^{\mathfrak{t}}(\tau) = (G, \hat{G})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}},\bar{\mathfrak{o}}} \in H_\circ$ is given by

$$N_G = N_F \sqcup \{\varrho_G\}\,, \qquad E_G = E_F \sqcup \{(\varrho_G, \varrho)\}\,,$$

the root of G is ϱ_G, the type of the edge (ϱ_G, ϱ) is $\mathfrak{t}$. For instance

$$(F, \hat{F}) = \quad \Longrightarrow \quad (G, \hat{G}) =$$

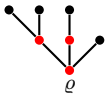
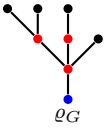


The decorations of $\mathcal{I}_k^{\mathfrak{t}}(\tau)$, as well as $\hat{G}$, coincide with those of τ, except on the newly added edge / vertex where $\hat{G}$, $\bar{\mathfrak{n}}$ and $\bar{\mathfrak{o}}$ vanish, while $\bar{\mathfrak{e}}(\varrho_G, \varrho) = k$. This gives a triangular operator and $\mathcal{I}_k^{\mathfrak{t}} \colon \mathcal{H}_\circ \to \mathcal{H}_\circ$ is therefore well defined.

3. Similarly, we define operators

$$\mathcal{I}_k^{\mathfrak{t}} \colon \mathscr{H}_\circ \to \hat{\mathscr{H}}_2 \tag{4.12}$$

in exactly the same way as the operators $\mathcal{I}_k^{\mathfrak{t}}$ defined in (4.11), except that the root of $\mathcal{I}_k^{\mathfrak{t}}(\tau)$ is coloured with the colour 2, for instance

$(F, \hat{F}) =$  ϱ $\Longrightarrow$ $(G, \hat{G}) =$  ϱ_G

4. For $\alpha \in \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$, we define linear triangular maps $\mathcal{R}_\alpha \colon \mathscr{H}_\circ \to \mathscr{H}_\circ$ in such a way that if $\tau = (T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in H_\circ$ with root $\varrho \in N_T$, then $\mathcal{R}_\alpha(\tau) \in H_\circ$ coincides with τ, except for $\mathfrak{o}(\varrho)$ to which we add α and $\hat{T}(\varrho)$ which is set to 1. In particular, one has $\mathcal{R}_\alpha \circ \mathcal{R}_\beta = \mathcal{R}_{\alpha+\beta}$.

Remark 4.15 With these notations, it follows from the definition of the sets $H_\circ$, H_1 and $\hat{H}_2$ that they can be constructed as follows.

- Every element of $H_\circ \setminus \{\mathbf{1}\}$ can be obtained from elements of the type X^k by successive applications of the maps $\mathcal{I}_k^{\mathfrak{t}}$, $\mathcal{R}_\alpha$, and the tree product (4.8).
- Every element of H_1 is the forest product of a finite number of elements of $H_\circ$.
- Every element of $\hat{H}_2$ is of the form

$$X^k \prod_i \mathcal{I}_{k_i}^{\mathfrak{t}_i}(\tau_i)\,, \tag{4.13}$$

for some finite collection of elements $\tau_i \in H_\circ \setminus \{\mathbf{1}\}$, $\mathfrak{t}_i \in \mathfrak{L}$ and $k_i \in \mathbf{N}^d$.

Then, one obtains a simple recursive description of the coproduct Δ_2.

Proposition 4.16 *With the above notations, the operator $\Delta_2 \colon \mathscr{H}_\circ \to \mathscr{H}_\circ \hat{\otimes} \hat{\mathscr{H}}_2$ is multiplicative and satisfies the identities*

$$\begin{aligned} \Delta_2 X_i &= X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i\,, \qquad \Delta_2 \mathbf{1} = \mathbf{1} \otimes \mathbf{1}\,, \\ \Delta_2 \mathcal{I}_k^{\mathfrak{t}}(\tau) &= \big(\mathcal{I}_k^{\mathfrak{t}} \otimes \mathrm{id}\big)\Delta_2 \tau + \sum_\ell \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{k+\ell}^{\mathfrak{t}}(\tau), \\ \Delta_2 \mathcal{R}_\alpha(\tau) &= (\mathcal{R}_\alpha \otimes \mathrm{id})\Delta_2 \tau\,. \end{aligned} \tag{4.14}$$

Furthermore, the action $\Delta_2 \colon \hat{\mathscr{H}}_2 \to \hat{\mathscr{H}}_2 \otimes \hat{\mathscr{H}}_2$ is completely determined by the fact that it satisfies the identities on the first line of (4.14) *and that, similarly to above,*

$$\Delta_2 \mathcal{I}_k^{\mathfrak{t}}(\tau) = \big(\mathcal{I}_k^{\mathfrak{t}} \otimes \mathrm{id}\big)\Delta_2 \tau + \sum_\ell \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{k+\ell}^{\mathfrak{t}}(\tau)\,. \tag{4.15}$$

Proof. The operator Δ_2 is multiplicative on $\mathscr{H}_\circ$ as a consequence of the first identity of Proposition 4.10 and its action on X^k was already mentioned in (3.27). It remains to verify that the recursive identities hold as well.

We first consider $\Delta_2\sigma$ with $\sigma = \mathcal{I}_k^{\mathfrak{t}}(\tau)$ and $\tau = (T,\hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$. We write $\sigma = (F,\hat{F})_{\mathfrak{e}+k\mathbf{1}_e}^{\mathfrak{n},\mathfrak{o}}$, where e is the "trunk" of type $\mathfrak{t}$ created by $\mathcal{I}_k^{\mathfrak{t}}$. Writing ϱ for the root of F, it follows from the definitions that

$$\mathfrak{A}_2(F,\hat{F}) = \{\{\varrho\}\} \cup \{A \cup \{\varrho, e\} \,:\, A \in \mathfrak{A}_2(T,\hat{T})\} \,.$$

Indeed, if e does not belong to an element A of $\mathfrak{A}_2(F,\hat{F})$ then, since A has to contain ϱ and be connected, one necessarily has $A = \{\varrho\}$. If on the other hand $e \in A$, then one also has $\varrho \in A$ and the remainder of A is necessarily a connected subtree of T containing its root, namely an element of $\mathfrak{A}_2(T,\hat{T})$.

Given $A \in \mathfrak{A}_2(T,\hat{T})$, since the root-label of σ is 0, the set of all possible node-labels $\mathfrak{n}_A$ for σ appearing in (3.3) for $\Delta_2\sigma$ coincides with those appearing in the expression for $\Delta_2\tau$, so that we have the identity

$$\begin{aligned}\Delta_2\sigma &= (\mathcal{I}_k^{\mathfrak{t}} \otimes \mathrm{id})\Delta_2\tau + \sum_{\mathfrak{e}_\varrho,\mathfrak{n}_\varrho} \frac{1}{\mathfrak{e}_\varrho!}\binom{\mathfrak{n}}{\mathfrak{n}_\varrho}(\bullet, 0, \mathfrak{n}_\varrho + \pi\mathfrak{e}_\varrho, \mathfrak{o}, 0) \\ &\qquad \otimes (F, \hat{F} + 2\mathbb{1}_e, \mathfrak{n} - \mathfrak{n}_\varrho, \mathfrak{o}, \mathfrak{e}) \\ &= (\mathcal{I}_k^{\mathfrak{t}} \otimes \mathrm{id})\Delta_2\tau + \sum_\ell \frac{1}{\ell!} X^\ell \otimes \mathcal{I}_{k+\ell}^{\mathfrak{t}}(\tau) \,.\end{aligned}$$

This is because $\mathfrak{n}(\varrho) = 0$, so that the sum over $\mathfrak{n}_\varrho$ contains only the zero term.

We now consider $\Delta_2\sigma$ with $\sigma = \mathcal{R}_\alpha(\tau)$. In this case, we write $\tau = (T,\hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$ so that, denoting by ϱ the root of T, one has $\sigma = (T, \hat{T} \vee \mathbb{1}_\varrho, \mathfrak{n}, \mathfrak{o} + \alpha\mathbb{1}_\varrho, \mathfrak{e})$. We claim that in this case one has

$$\mathfrak{A}_2(T,\hat{T}) = \mathfrak{A}_2(T, \hat{T} \vee \mathbb{1}_\varrho) \,.$$

This is non-trivial only in the case $\hat{T}(\varrho) = 0$. In this case however, it is necessarily the case that $\hat{T}(e) = 0$ for every edge e incident to the root. This in turn guarantees that the family $\mathfrak{A}_2(T,\hat{T})$ remains unchanged by the operation of colouring the root. This implies that one has

$$\Delta_2\mathcal{R}_\alpha(\tau) = (\mathcal{R}_\alpha \otimes \mathcal{R}_\alpha)\Delta_2\tau \,.$$

This appears slightly different from the desired identity, but the latter then follows by observing that, for every $\bar{\tau} \in \hat{\mathscr{H}}_2$, one has $\mathcal{R}_\alpha\bar{\tau} = \bar{\tau}$ as elements of $\hat{\mathscr{H}}_2$, thanks to the fact that we quotiented by the kernel of $\hat{\mathcal{R}}_2$ which sets the value of $\mathfrak{o}$ to 0 on the root. □

We finally have the following results on the antipode of $\hat{\mathscr{H}}_2$:

Proposition 4.17 *Let $\hat{\mathcal{A}}_2 : \hat{\mathscr{H}}_2 \to \hat{\mathscr{H}}_2$ be the antipode of $\hat{\mathscr{H}}_2$. Then*

- *The algebra morphism $\hat{\mathcal{A}}_2 : \hat{\mathcal{H}}_2 \to \hat{\mathcal{H}}_2$ is defined uniquely by the fact that $\hat{\mathcal{A}}_2 X_i = -X_i$ and for all $\mathcal{I}_k^{\mathfrak{t}}(\tau) \in \hat{\mathcal{H}}_2$ with $\tau \in \mathcal{H}_\circ$*

$$\hat{\mathcal{A}}_2 \mathcal{I}_k(\tau) = -\sum_{\ell \in \mathbf{N}^d} \frac{(-X)^\ell}{\ell!} \mathcal{M}(\mathcal{I}_{k+\ell} \otimes \hat{\mathcal{A}}_2) \Delta_2 \tau \,, \tag{4.16}$$

  where $\mathcal{M} : \hat{\mathcal{H}}_2 \otimes \hat{\mathcal{H}}_2 \to \hat{\mathcal{H}}_2$ denotes the (tree) product.
- *On $\hat{\mathcal{H}}_2$, one has the identity*

$$\Delta_1 \hat{\mathcal{A}}_2 = (\mathrm{id} \otimes \hat{\mathcal{A}}_2) \Delta_1 \,. \tag{4.17}$$

Proof. By (4.13) and by induction over the number of edges in τ, this uniquely determines a morphism $\hat{\mathcal{A}}_2$ of $\hat{\mathcal{H}}_2$, so it only remains to show that

$$\mathcal{M}(\mathrm{id} \otimes \hat{\mathcal{A}}_2) \Delta_2 \tau = \mathbf{1}_{\hat{\mathcal{H}}_2} \mathbf{1}^\star_{\hat{\mathcal{H}}_2}(\tau) \,.$$

The formula is true for $\tau = X^k$, so that, since both sides are multiplicative, it is enough to consider elements of the form $\mathcal{I}_k^{\mathfrak{t}}(\tau)$ for some $\tau \in \mathcal{H}_\circ$. Exploiting the identity (4.16), one then has

$$\begin{aligned}
&\mathcal{M}(\mathrm{id} \otimes \hat{\mathcal{A}}_2) \Delta_2 \mathcal{I}_k^{\mathfrak{t}}(\tau) = \\
&= \mathcal{M}(\mathrm{id} \otimes \hat{\mathcal{A}}_2) \left[\big(\mathcal{I}_k^{\mathfrak{t}} \otimes \mathrm{id}\big) \Delta_2 \tau + \sum_\ell \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{k+\ell}^{\mathfrak{t}}(\tau) \right] \\
&= \mathcal{M} \left[\Big(\mathcal{I}_k^{\mathfrak{t}} \otimes \hat{\mathcal{A}}_2\Big) - \sum_{\ell,i} \frac{X^\ell}{\ell!} \otimes \frac{(-X)^i}{i!} \mathcal{M}(\mathcal{I}_{k+\ell+i}^{\mathfrak{t}} \otimes \hat{\mathcal{A}}_2) \right] \Delta_2 \tau \\
&= \mathcal{M} \left[\Big(\mathcal{I}_k^{\mathfrak{t}} \otimes \hat{\mathcal{A}}_2\Big) - \sum_{\ell} \frac{(X-X)^\ell}{\ell!} \mathcal{M}(\mathcal{I}_{k+\ell}^{\mathfrak{t}} \otimes \hat{\mathcal{A}}_2) \right] \Delta_2 \tau \\
&= \Big[\mathcal{M}\Big(\mathcal{I}_k^{\mathfrak{t}} \otimes \hat{\mathcal{A}}_2\Big) - \mathcal{M}(\mathcal{I}_k^{\mathfrak{t}} \otimes \hat{\mathcal{A}}_2) \Big] \Delta_2 \tau = 0 \,,
\end{aligned}$$

as required.

A similar proof by induction yields (4.17): see the proof of Lemma 6.4 for an analogous argument. Note that (4.17) is also a direct consequence of Proposition 3.22 and more precisely of the fact that the bialgebras $\mathcal{H}_1$ and $\hat{\mathcal{H}}_2$ are in cointeraction, as follows from Remark 3.23: see [Foi16, Prop. 2] for a proof. Having this property, the antipode $\hat{\mathcal{A}}_2$ is a morphism of the $\mathcal{H}_1$-comodule $\hat{\mathcal{H}}_2$. □

5 Renormalisation group in SPDEs

We recall the definition of a regularity structure from [Hai14, Def. 2.1]

Definition 5.1 A *regularity structure* $\mathscr{T} = (A, T, G)$ consists of the following elements:

- An index set $A \subset \mathbf{R}$ such that A is bounded from below, and A is locally finite.
- A *model space* T, which is a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, with each T_α a Banach space.
- A *structure group* G of linear operators acting on T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $a \in T_\alpha$, one has

$$\Gamma a - a \in \bigoplus_{\beta < \alpha} T_\beta \, . \tag{5.1}$$

The aim of this section is to relate the construction of the previous section to the theory of regularity structures as exposed in [Hai14, Hai16b]. For this, we first assign real-valued degrees to each element of $\mathfrak{F}$.

Definition 5.2 A *scaling* is a map $\mathfrak{s} : \{1, \dots d\} \sqcup \mathfrak{L} \to \mathbf{R} \setminus \{0\}$ such that $\mathfrak{s}(i) \geq 1$ for $i \in \{1, \dots d\}$. By additivity, we then assign a *degree* to each $(k, v) \in \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$ by setting

$$|(k,v)|_{\mathfrak{s}} \stackrel{\text{def}}{=} |k|_{\mathfrak{s}} + |v|_{\mathfrak{s}} \in \mathbf{R}, \qquad |k|_{\mathfrak{s}} \stackrel{\text{def}}{=} \sum_{i=1}^{d} k_i \mathfrak{s}_i, \qquad |v|_{\mathfrak{s}} \stackrel{\text{def}}{=} \sum_{\mathfrak{t} \in \mathfrak{L}} v_{\mathfrak{t}} \, \mathfrak{s}(\mathfrak{t}), \tag{5.2}$$

if $v = \sum_{\mathfrak{t} \in \mathfrak{L}} v_{\mathfrak{t}} \mathfrak{t}$ with $v_{\mathfrak{t}} \in \mathbf{Z}$.

Definition 5.3 Given a scaling $\mathfrak{s}$ as above, for $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F}_2$, we define two different notions of degree $|\tau|_-, |\tau|_+ \in \mathbf{R}$ by

$$|\tau|_- = \sum_{e \in E_F \setminus \hat{E}} (|\mathfrak{t}(e)|_{\mathfrak{s}} - |\mathfrak{e}(e)|_{\mathfrak{s}}) + \sum_{x \in N_F} |\mathfrak{n}(x)|_{\mathfrak{s}} \, ,$$

$$|\tau|_+ = \sum_{e \in E_F \setminus E_{\hat{F}_2}} (|\mathfrak{t}(e)|_{\mathfrak{s}} - |\mathfrak{e}(e)|_{\mathfrak{s}}) + \sum_{x \in N_F} |\mathfrak{n}(x)|_{\mathfrak{s}} + \sum_{x \in N_F \setminus N_{\hat{F}_2}} |\mathfrak{o}(x)|_{\mathfrak{s}} \, ,$$

where we recall that $\mathfrak{o}$ takes values in $\mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$ and $\mathfrak{t} : E_F \to \mathfrak{L}$ is the map assigning to an edge its type in F, see Section 2.1.

Note that both of these degrees are compatible with the contraction operator $\mathcal{K}$ of Definition 3.14, as well as the operator $\mathscr{J}$, in the sense that $|\tau|_\pm = |\bar{\tau}|_\pm$ if and only if $|\mathcal{K}\tau|_\pm = |\mathcal{K}\bar{\tau}|_\pm$ and similarly for $\mathscr{J}$. In the case of $|\cdot|_+$, this is true thanks to the definition (3.22), while the coloured part of the tree is simply ignored by $|\cdot|_-$. We furthermore have

Lemma 5.4 *The degree $|\cdot|_-$ is compatible with the operator $\mathcal{K}_1$ of (3.24). Similarly, $|\cdot|_+$ is compatible with both $\mathcal{K}_2$ and $\hat{\mathcal{K}}_2$. Furthermore, both degrees are compatible with $\mathscr{J}$ and $\mathcal{K}$, so that $\mathcal{H}_1$ is $|\cdot|_-$-graded, $\hat{\mathcal{H}}_2$ is $|\cdot|_+$-graded, and $\mathcal{H}_\circ$ is both $|\cdot|_-$ and $|\cdot|_+$-graded.*

Proof. For the first statement, we only need to verify that $|\cdot|_-$ is compatible with Φ_1 as defined in (3.23), as long as the argument is in $\mathfrak{M}_1$. This is indeed the case since for such elements only the second term in the definition of $|\cdot|_-$ contributes. For the second statement, we need to verify that $|\cdot|_+$ is compatible with $\hat{\Phi}_2$ as defined just below (3.23). which is the case when acting on a tree with $\varrho \in \hat{F}_2$ since the $\mathfrak{o}$-decoration of nodes in $\hat{F}_2$ does not contribute to the definition of $|\cdot|_+$. □

As a consequence, $|\cdot|_-$ yields a grading for $\mathscr{H}_1$, $|\cdot|_+$ yields a grading for $\hat{\mathscr{H}}_2$, and both of them yield gradings for $\mathscr{H}_\circ$. With these definitions, we see that we obtain a structure resembling a regularity structure by taking $\mathscr{H}_\circ$ to be our model space, with grading given by $|\cdot|_+$ and structure group given by the character group $\hat{\mathscr{G}}_2$ of $\hat{\mathscr{H}}_2$ acting on $\mathscr{H}_\circ$ via

$$\Gamma_g \colon \langle \mathfrak{F}_\circ \rangle \to \langle \mathfrak{F}_\circ \rangle \ , \quad \Gamma_g \tau = (\mathrm{id} \otimes g)\Delta_2 \tau \ .$$

The second statement of Proposition 4.14 then guarantees that this action is multiplicative with respect to the tree product (4.8) on $\mathscr{H}_\circ$, so that we are in the context of [Hai14, Sec. 4]. There are however two conditions that are not met:

1. The action of $\hat{\mathscr{G}}_2$ on $\mathscr{H}_\circ$ is not of the form "identity plus terms of strictly lower degree", as required for regularity structures.
2. The possible degrees appearing in $\mathscr{H}_\circ$ have no lower bound and might have accumulation points.

We will fix the first problem by encoding in our context what we mean by considering a "subcritical problem". Such problems will allow us to prune our structure in a natural way so that we are left with a subspace of $\mathscr{H}_\circ$ that has the required properties. The second problem will then be addressed by quotienting a suitable subspace of $\hat{\mathscr{H}}_2$ by the terms of negative degree. The group of characters of the resulting Hopf algebra will then turn out to act on $\mathscr{H}_\circ$ in the desired way.

5.1 Simple decorated trees

In most of Section 5, the colourings and the labels $\mathfrak{o}$ will be ignored. It is therefore convenient to consider the space

$$\mathfrak{T} \stackrel{\text{def}}{=} \{(T, \hat{T}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F} : \ T \text{ is a tree}, \ \hat{T} \equiv 0, \ \mathfrak{o} \equiv 0\}. \tag{5.3}$$

In order to lighten notations, we write elements of $\mathfrak{T}$ as $(T, \mathfrak{n}, \mathfrak{e}) = T^{\mathfrak{n}}_{\mathfrak{e}}$ with T a typed tree (for some set of types $\mathfrak{L}$) and $\mathfrak{n} \colon N_T \to \mathbf{N}^d$, $\mathfrak{e} \colon E_T \to \mathbf{N}^d$ as above. Similarly to before, $\mathfrak{T}$ is a monoid for the tree product (4.8). Again, this product is associative and commutative, with unit $(\bullet, 0, 0)$.

Definition 5.5 We say that an element $T^{\mathfrak{n}}_{\mathfrak{e}} \in \mathfrak{T}$ is *trivial* if T consists of a single node $\bullet$. It is *elementary* if T has exactly one edge incident to its root ϱ and furthermore $\mathfrak{n}(\varrho) = 0$.

In other words, an elementary $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}$ is necessarily of the form $\mathcal{I}_k^{\mathfrak{l}}(\tau)$ with $\tau \in \mathfrak{T}$, see (4.11).

With this definition, each $\tau \in \mathfrak{T}$ has by (4.13) a unique (up to permutations) factorisation with respect to the tree product (4.8)

$$\tau = \bullet_n \tau_1 \tau_2 \cdots \tau_k \,, \tag{5.4}$$

for some $n \in \mathbf{N}^d$, where each τ_i is elementary and $\bullet_n$ denotes the trivial element $(\bullet, n, 0) \in \mathfrak{T}$.

5.2 Trees generated by rules

In order to define a suitable substructure of the structure described in Proposition 4.14, we introduce the notion of "rules". Essentially, a "rule" describes what behaviour we allow for a tree in the vicinity of any one of its nodes.

In order to formalise this, we first define the set of *edge types* $\mathscr{E}$ and the set of *node types* $\mathcal{N}$ by

$$\mathscr{E} = \mathfrak{L} \times \mathbf{N}^d \,, \qquad \mathcal{N} = \hat{\mathcal{P}}(\mathscr{E}) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} [\mathscr{E}]^n \,, \tag{5.5}$$

where $[\mathscr{E}]^n$ denotes the set of *unordered* $\mathscr{E}$-valued n-uples, namely $[\mathscr{E}]^n = \mathscr{E}^n / S_n$, with the natural action of the symmetric group S_n on $\mathscr{E}^n$. In other words, given any set A, $\hat{\mathcal{P}}(A)$ consists of all finite multisets whose elements are elements of A.

Remark 5.6 The fact that we consider multisets and not just n-uples is a reflection of the fact that we always consider the situation where the tree product (4.8) is commutative. This condition could in principle be dropped, thus leading us to consider forests consisting of planar trees instead, but this would lead to additional complications and does not seem to bring any advantage.

Given two sets $A \subset B$, we have a natural inclusion $\hat{\mathcal{P}}(A) \subset \hat{\mathcal{P}}(B)$. We will usually write elements of $[\mathscr{E}]^n$ as n-uples with the understanding that this is just an arbitrary representative of an equivalence class. In particular, we write $()$ for the unique element of $[\mathscr{E}]^0$.

Given any $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}$, we then associate to each node $x \in N_T$ a node type $\mathcal{N}(x) \in \mathcal{N}$ by

$$\mathcal{N}(x) = (s(e_1), \dots, s(e_n)), \qquad s(e) \stackrel{\text{def}}{=} (\mathfrak{t}(e), \mathfrak{e}(e)) \in \mathscr{E}, \quad e \in E_T, \tag{5.6}$$

where $(e_1, \dots, e_n)$ denotes the collection of edges leaving x, i.e. edges of the form (x, y) for some node y. We will sometimes use set-theoretic notations. In particular, given $N = (s_1, \dots, s_n) \in \mathcal{N}$ and $M = (r_1, \dots, r_\ell) \in \mathcal{N}$, we write

$$M \sqcup N \stackrel{\text{def}}{=} (r_1, \dots, r_\ell, s_1, \dots, s_n) \,,$$

and we say that $M \subset N$ if there exists $\bar{N}$ such that $N = M \sqcup \bar{N}$. When we write a sum of the type $\sum_{M \subset N}$, we take multiplicities into account. For example (a, b) is

contained twice in (a, b, b), so that such a sum always contains 2^n terms if N is an n-tuple. Similarly, we write $t \in N$ if $(t) \subset N$ and we also count sums of the type $\sum_{t \in N}$ with the corresponding multiplicities.

Definition 5.7 Denoting by $\mathcal{P}\mathcal{N}$ the powerset of $\mathcal{N}$, a *rule* is a map $R\colon \mathfrak{L} \to \mathcal{P}\mathcal{N} \setminus \{\emptyset\}$ such that, defining the subsets $R_n \subset \mathfrak{L}$ by

$$R_0 = \emptyset\,, \qquad R_{n+1} = R_n \cup \{\mathfrak{t} \in \mathfrak{L} : \exists N \in R(\mathfrak{t}) \text{ with } N \in \hat{\mathcal{P}}(R_n \times \mathbf{N}^d)\}\,,$$

one has $\bigcup_{n>0} R_n = \mathfrak{L}$, with the convention $\hat{\mathcal{P}}(\emptyset) = \{()\}$. A rule is said to be *normal* if $R(\mathfrak{t}) = \{()\}$ for every $\mathfrak{t} \in \mathfrak{L}_-$ and, whenever $M \subset N \in R(\mathfrak{t})$, one also has $M \in R(\mathfrak{t})$.

See Remark 5.9 below for a discussion of the condition $\bigcup_{n>0} R_n = \mathfrak{L}$.

Definition 5.8 Let R be a rule and $\tau = T^{\mathfrak{n}}_{\mathfrak{e}} \in \mathfrak{T}$. We say that

- τ *conforms to* R if, for every edge $e = (x, y) \in E_T$, one has $\mathcal{N}(y) \in R(\mathfrak{t}(e))$, with $\mathcal{N}(y)$ defined as in (5.6)
- τ *strongly conforms to* R if it conforms to R and there exists $\mathfrak{t} \in \mathfrak{L}$ such that $\mathcal{N}(\varrho_\tau) \in R(\mathfrak{t})$, where ϱ_τ is the root of τ.

In particular, the trivial tree $\bullet$ strongly conforms to every rule since, as a consequence of Definition 5.7, there exists at least one $\mathfrak{t} \in \mathfrak{L}$ with $() \in R(\mathfrak{t})$.

Remark 5.9 The condition $\bigcup_{n>0} R_n = \mathfrak{L}$ in Definition 5.7 guarantees that $\mathfrak{L}$ contains no *useless* labels in the sense that, for every $\mathfrak{t} \in \mathfrak{L}$, there exists a tree conforming to R containing an edge of type $\mathfrak{t}$. More importantly, this condition also guarantees that if we start building a tree conforming to R from the root upwards (start with an edge of type $\mathfrak{t}$, add to it a node of some type in $R(\mathfrak{t})$, then restart the construction for each of the outgoing edges of that node), such a construction can always be terminated after finitely many steps by choosing for every branch a final edge of type $\mathfrak{t}$ such that $() \in R(\mathfrak{t})$.

Remark 5.10 A rule R can be represented by a directed bipartite multigraph $\mathcal{G}(R) = (V(R), E(R))$ as follows. Take as the vertex set $V(R) = \mathcal{E} \sqcup \mathcal{N}$. Then, connect $N \in \mathcal{N}$ to $t \in \mathcal{E}$ if $t \in N$. If t is contained in N multiple times, repeat the connection the corresponding number of times. Conversely, connect $(\mathfrak{t}, k) \in \mathcal{E}$ to $N \in \mathcal{N}$ if $N \in R(\mathfrak{t})$. The conditions then guarantee that $() \in \mathcal{N}$ can be reached from every vertex in the graph. Given a tree $\tau \in \mathfrak{T}$, every edge of τ corresponds to an element of $\mathcal{E}$ and every node corresponds to an element of $\mathcal{N}$ via the map $x \mapsto \mathcal{N}(x)$ defined above. A tree then conforms to R if, for every path joining the root to one of the leaves, the corresponding path in V always follows directed edges in $\mathcal{G}(R)$. It strongly conforms to R if the root corresponds to a vertex in V with at least one incoming edge.

Definition 5.11 Given $\mathfrak{s}$ as in Definition 5.2, we assign a degree $|\tau|_{\mathfrak{s}}$ to any $\tau \in \mathfrak{T}$ by setting

$$|T^{\mathfrak{n}}_{\mathfrak{e}}|_{\mathfrak{s}} = \sum_{e \in E_T} \bigl(|\mathfrak{t}(e)|_{\mathfrak{s}} - |\mathfrak{e}(e)|_{\mathfrak{s}}\bigr) + \sum_{x \in N_T} |\mathfrak{n}(x)|_{\mathfrak{s}} \; . \tag{5.7}$$

This definition is compatible with both notions of degree given in Definition 5.3, since we view $\mathfrak{T}$ as a subset of $\mathfrak{F}$ with $\hat{F}$ and $\mathfrak{o}$ identically 0. This also allows us to give the following definition.

Definition 5.12 Given a rule R, we write

- $\mathfrak{T}_\circ(R) \subset \mathfrak{T}$ for the set of trees that strongly conform to R
- $\mathfrak{T}_1(R) \subset \mathfrak{F}$ for the submonoid of $\mathfrak{F}$ (for the forest product) generated by $\mathfrak{T}_\circ(R)$
- $\mathfrak{T}_2(R) \subset \mathfrak{T}$ for the set of trees that conform to R.

Moreover, we write $\mathfrak{T}_-(R) \subset \mathfrak{T}_\circ(R)$ for the set of trees $\tau = T^{\mathfrak{n}}_{\mathfrak{e}}$ such that

- $|\tau|_{\mathfrak{s}} < 0$, $\mathfrak{n}(\varrho_\tau) = 0$,
- if τ is elementary, namely $\tau = \mathcal{I}^{\mathfrak{t}}_k(\bar\tau)$ with $\bar\tau \in \mathfrak{T}$, see (4.11), then $|\mathfrak{t}|_{\mathfrak{s}} < 0$.

The second restriction on the definition of $\tau \in \mathfrak{T}_-(R)$ is related to the definition (5.20) of the Hopf algebra $\mathcal{T}^{\mathrm{ex}}_-$ and of its characters group $\mathcal{G}^{\mathrm{ex}}_-$, that we call the *renormalisation group* and which plays a fundamental role in the theory, see e.g. Theorem 6.15.

5.3 Subcriticality

Given a map $\mathrm{reg}\colon \mathfrak{L} \to \mathbf{R}$ we will henceforth interpret it as maps $\mathrm{reg}\colon \mathcal{E} \to \mathbf{R}$ and $\mathrm{reg}\colon \mathcal{N} \to \mathbf{R}$ as follows: for $(\mathfrak{t}, k) \in \mathcal{E}$ and $N \in \mathcal{N}$

$$\mathrm{reg}(\mathfrak{t}, k) \stackrel{\text{def}}{=} \mathrm{reg}(\mathfrak{t}) - |k|_{\mathfrak{s}}, \qquad \mathrm{reg}(N) \stackrel{\text{def}}{=} \sum_{(\mathfrak{t},k) \in N} \mathrm{reg}(\mathfrak{t}, k), \tag{5.8}$$

with the convention that the sum over the empty word $() \in \mathcal{N}$ is 0.

Definition 5.13 A rule R is *subcritical* with respect to a fixed scaling $\mathfrak{s}$ if there exists a map $\mathrm{reg}\colon \mathfrak{L} \to \mathbf{R}$ such that

$$\mathrm{reg}(\mathfrak{t}) < |\mathfrak{t}|_{\mathfrak{s}} + \inf_{N \in R(\mathfrak{t})} \mathrm{reg}(N) \; , \qquad \forall\, \mathfrak{t} \in \mathfrak{L}, \tag{5.9}$$

where we use the notation (5.8).

We will see in Section 5.5 below that classes of stochastic PDEs generate rules. In this context, the notion of subcriticality given here formalises the one given somewhat informally in [Hai14]. In particular, we have the following result which is essentially a reformulation of [Hai14, Lem. 8.10] in this context.

Proposition 5.14 *If R is a subcritical rule, then, for every $\gamma \in \mathbf{R}$, the set $\{\tau \in \mathfrak{T}_\circ(R) \,:\, |\tau|_{\mathfrak{s}} \le \gamma\}$ is finite.*

Proof. Fix $\gamma \in \mathbf{R}$ and let $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}_\circ(R)$ with $|T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} \le \gamma$. Since there exists $c > 0$ such that

$$|T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} \ge |T_{\mathfrak{e}}^{0}|_{\mathfrak{s}} + c|\mathfrak{n}|$$

and there exist only finitely many trees in $\mathfrak{T}_\circ(R)$ of the type $|T_{\mathfrak{e}}^{0}|$ for a given number of edges, it suffices to show that the number $|E_T|$ of edges of T is bounded by some constant depending only on γ.

Since the set $\mathfrak{L}$ is finite, (5.9) implies that there exists a constant $\kappa > 0$ such that the bound

$$\mathrm{reg}(\mathfrak{t}) + \kappa \le |\mathfrak{t}|_{\mathfrak{s}} + \inf_{N \in R(\mathfrak{t})} \mathrm{reg}(N)\,, \tag{5.10}$$

holds for every $\mathfrak{t} \in \mathfrak{L}$ with the notation (5.8). We claim that for every elementary $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}_\circ(R)$ such that the edge type of its trunk $e = (\varrho, x)$ is $(\mathfrak{t}, k) \in \mathfrak{E}$, we have

$$\mathrm{reg}(\mathfrak{t}, k) \le |T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} - \kappa |E_T|. \tag{5.11}$$

We verify (5.11) by induction on the number of edges $|E_T|$ of T. If $|E_T| = 1$, namely the unique element of E_T is the trunk $e = (\varrho, x)$, then $\mathcal{N}(x) = () \in R(\mathfrak{t})$ in the notation of (5.6) and by (5.10)

$$\mathrm{reg}(\mathfrak{t}) + \kappa \le |\mathfrak{t}|_{\mathfrak{s}} \quad \Longrightarrow \quad \mathrm{reg}(\mathfrak{t}, k) \le |\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} - \kappa \le |T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} - \kappa.$$

For an elementary $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}_\circ(R)$ with $|E_T| > 1$, then $\mathcal{N}(x) = (s(e_1), \dots, s(e_n)) \in R(\mathfrak{t})$ and by (5.10) and the induction hypothesis

$$\mathrm{reg}(\mathfrak{t}) - |k|_{\mathfrak{s}} + \kappa \le |\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} + \sum_{i=1}^{n} [\mathrm{reg}(\mathfrak{t}_i) - |k_i|_{\mathfrak{s}}] \le |T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} - \kappa(|E_T| - 1)\,,$$

where $s(e_i) = (\mathfrak{t}_i, k_i)$. Therefore (5.11) is proved for elementary trees.

Given an arbitrary tree $T_{\mathfrak{e}}^{\mathfrak{n}}$ of degree at most γ strongly conforming to the rule R, there exists $\mathfrak{t}_0 \in \mathfrak{L}$ such that $e \in \mathcal{N}(\varrho_T) = R(\mathfrak{t}_0)$. We can therefore consider the elementary tree $\bar{T}_{\mathfrak{e}}^{\mathfrak{n}}$ containing a trunk of type $\mathfrak{t}_0$ connected to the root of T, and with vanishing labels on the root and trunk respectively. It then follows that

$$\begin{aligned} \kappa|E_T| < \kappa|E_{\bar{T}}| &\le |\bar{T}_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} - \mathrm{reg}(\mathfrak{t}_0) = |T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} + |\mathfrak{t}_0|_{\mathfrak{s}} - \mathrm{reg}(\mathfrak{t}_0) \\ &\le \gamma + \inf_{\mathfrak{t} \in \mathfrak{L}} (|\mathfrak{t}|_{\mathfrak{s}} - \mathrm{reg}(\mathfrak{t}))\,, \end{aligned}$$

and the latter expression is finite since $\mathfrak{L}$ is finite. The claim follows at once. □

Remark 5.15 The fact that the inequality in (5.9) is strict is crucial, otherwise the conclusion of Proposition 5.14 may fail to hold.

Remark 5.16 Assuming that there exists a map reg satisfying (5.10) for a given $\kappa > 0$, the optimal such map can be constructed as follows. Set $\mathrm{reg}_\kappa^0(\mathfrak{t}) = +\infty$ for every $\mathfrak{t} \in \mathfrak{L}$ and then define recursively

$$\mathrm{reg}_\kappa^{n+1}(\mathfrak{t}) = |\mathfrak{t}|_{\mathfrak{s}} - \kappa + \inf_{N \in R(\mathfrak{t})} \mathrm{reg}_\kappa^n(N)\,. \tag{5.12}$$

Since $\mathrm{reg}_\kappa^n(\mathfrak{t})$ is decreasing in n and the function reg from (5.10) is a lower bound for reg_κ^n, the limit

$$\mathrm{reg}_\kappa(\mathfrak{t}) = \lim_{n\to\infty} \mathrm{reg}_\kappa^n(\mathfrak{t})$$

exists and has the required properties. If we extend reg_κ^n to $\mathcal{C} \sqcup \mathcal{N}$ by (5.8), the iteration (5.12) can be interpreted as a min-plus network on the graph $\mathcal{G}(R)$ with arrows reversed, see Remark 5.10.

5.4 Completeness

Given an arbitrary rule (subcritical or not), there is no reason in general to expect that the actions of the analogues of the groups $\mathcal{G}_1$ and $\hat{\mathcal{G}}_2$ constructed in Section 4 leave the linear span of $\mathcal{T}_\circ(R)$ invariant. We now introduce a notion of completeness, which will guarantee later on that the actions of $\mathcal{G}_1$ and $\hat{\mathcal{G}}_2$ do indeed leave the span of $\mathcal{T}_\circ(R)$ (or rather an extension of it involving again labels $\mathfrak{o}$ on nodes) invariant. This eventually allows us to build, for large classes of subcritical stochastic PDEs, regularity structures allowing to formulate them, endowed with a large enough group of automorphisms to perform the renormalisation procedures required to give them canonical meaning.

Definition 5.17 Given $N = ((\mathfrak{t}_1, k_1), \dots, (\mathfrak{t}_n, k_n)) \in \mathcal{N}$ and $m \in \mathbf{N}^d$, we define $\partial^m N \subset \mathcal{N}$ as the set of all n-tuples of the form $((\mathfrak{t}_1, k_1 + m_1), \dots, (\mathfrak{t}_n, k_n + m_n))$ where the $m_i \in \mathbf{N}^d$ are such that $\sum_i m_i = m$.

Furthermore, we introduce the following *substitution* operation on $\mathcal{N}$. Assume that we are given $N \in \mathcal{N}$, $M \subset N$ and an element $\tilde{M} \in \hat{\mathcal{P}}(\mathcal{N})$ which has the same size as M. In other words, if $M = (r_1, \dots, r_\ell)$, one has $\tilde{M} = (\tilde{M}_1, \dots, \tilde{M}_\ell)$ with $\tilde{M}_i \in \mathcal{N}$. Then, writing $N = M \sqcup \bar{N}$, we define

$$\mathscr{R}_M^{\tilde{M}} N \stackrel{\text{def}}{=} \bar{N} \sqcup \tilde{M}_1 \sqcup \ldots \sqcup \tilde{M}_\ell \,. \tag{5.13}$$

Definition 5.18 Given a rule R, for any tree $T_{\mathfrak{c}}^{\mathfrak{n}} \in \mathcal{T}_\circ(R)$ we associate to each edge $e \in E_T$ a set $\bar{\mathcal{N}}(e) \subset \mathcal{N}$ in the following recursive way. If $e = (x, y)$ and y is a leaf, namely the node-type $\mathcal{N}(y)$ of the vertex y is equal to the empty word $() \in \mathcal{N}$, then we set

$$\bar{\mathcal{N}}(e) \stackrel{\text{def}}{=} R(\mathfrak{t}(e)) \,.$$

Otherwise, writing $(e_1, \dots, e_\ell)$ the incoming edges of y, namely $e_i = (y, v_i)$, we define

$$\bar{\mathcal{N}}(e) \stackrel{\text{def}}{=} \{\mathscr{R}_{\mathcal{N}(y)}^M N \,:\, \mathcal{N}(y) \subset N \in R(\mathfrak{t}(e)),\ M \in \bar{\mathcal{N}}(e_1) \times \cdots \times \bar{\mathcal{N}}(e_\ell)\} \,.$$

- Finally, we define for every node $y \in N_T$ a set $\mathscr{M}(y) \subset \hat{\mathcal{P}}(\mathcal{N})$ by $\mathscr{M}(y) \stackrel{\text{def}}{=} \{()\}$ if y is a leaf, and

$$\mathscr{M}(y) \stackrel{\text{def}}{=} \bar{\mathcal{N}}(e_1) \times \cdots \times \bar{\mathcal{N}}(e_\ell)$$

if $(e_1, \dots, e_\ell)$ are the outgoing edges of y.

It is easy to see that, if we explore the tree *from the leaves down*, this specifies $\bar{\mathcal{N}}(e)$ and $\mathscr{M}(y)$ uniquely for all edges and nodes of T.

Definition 5.19 A rule R is $\ominus$-*complete* with respect to a fixed scaling $\mathfrak{s}$ if, whenever $\tau \in \mathfrak{T}_-(R)$ and $\mathfrak{t} \in \mathfrak{L}$ are such that there exists $N \in R(\mathfrak{t})$ with $\mathcal{N}(\varrho_\tau) \subset N$, one also has

$$\partial^m(\mathscr{R}^M_{\mathcal{N}(\varrho_\tau)}N) \subset R(\mathfrak{t}) \,,$$

for every $M \in \mathscr{M}(\varrho_\tau)$ and for every multiindex m with $|m|_{\mathfrak{s}} + |\tau|_{\mathfrak{s}} < 0$.

At first sight, the notion of $\ominus$-completeness might seem rather tedious to verify and potentially quite restrictive. Our next result shows that this is fortunately not the case, at least when we are in the subcritical situation.

Proposition 5.20 *Let R be a normal subcritical rule. Then, there exists a normal subcritical rule $\bar{R}$ which is $\ominus$-complete and extends R in the sense that $R(\mathfrak{t}) \subset \bar{R}(\mathfrak{t})$ for every $\mathfrak{t} \in \mathfrak{L}$.*

Proof. Given a normal subcritical rule R, we define a new rule $\mathbb{Q}R$ by setting

$$(\mathbb{Q}R)(\mathfrak{t}) = R(\mathfrak{t}) \cup \bigcup_{\tau \in \mathfrak{T}_-(R)} R_-(\mathfrak{t};\tau) \,, \tag{5.14}$$

where $R_-(\mathfrak{t};\tau)$ is the union of all collections of node types of the type

$$\hat{N} \in \partial^m(\mathscr{R}^M_{\mathcal{N}(\varrho_\tau)}N) \,,$$

for some $N \in R(\mathfrak{t})$ with $\mathcal{N}(\varrho_\tau) \subset N$, some $M \in \mathscr{M}(\varrho_\tau)$, and some multiindex m with $|m|_{\mathfrak{s}} + |\tau|_{\mathfrak{s}} \le 0$. Since $(\mathbb{Q}R)(\mathfrak{t}) \supset R(\mathfrak{t})$ and $\mathfrak{T}_-(R)$ is finite by Proposition 5.14, this is again a valid rule. Furthermore, by definition, a rule R is $\ominus$-complete if and only if $\mathbb{Q}R = R$.

We claim that the desired rule $\bar{R}$ can be obtained by setting

$$\bar{R}(\mathfrak{t}) = \bigcup_{n \ge 0} (\mathbb{Q}^n R)(\mathfrak{t}) \,.$$

It is straightforward to verify that $\bar{R}$ is $\ominus$-complete. (This follows from the fact that the sequence of rules $\mathbb{Q}^n R$ is increasing and $\mathbb{Q}$ is closed under increasing limits.)

It remains to show that $\bar{R}$ is again normal and subcritical. To show normality, we note that if R is normal, then $\mathbb{Q}R$ is again normal. This is because, by Definition 5.18, the sets $\bar{\mathcal{N}}(e)$ used to build $\mathscr{M}(\varrho_\tau)$ also have the property that if $N \in \bar{\mathcal{N}}(e)$ and $M \subset N$, then one also has $M \in \bar{\mathcal{N}}(e)$. As a consequence, $\mathbb{Q}^n R$ is normal for every n, from which the normality of $\bar{R}$ follows.

To show that $\bar{R}$ is subcritical, we first recall that by Remark 5.16, for κ as in (5.10), we can find a maximal function $\mathrm{reg}_\kappa \colon \mathfrak{L} \to \mathbf{R}$ such that

$$\mathrm{reg}_\kappa(\mathfrak{t}) = |\mathfrak{t}|_{\mathfrak{s}} - \kappa + \inf_{N \in R(\mathfrak{t})} \mathrm{reg}_\kappa(N) \,. \tag{5.15}$$

Furthermore, the extension of reg_κ to node types given by (5.8) is such that, for every node type N and every multiindex m, one has

$$\mathrm{reg}_\kappa(\partial^m N) = \mathrm{reg}_\kappa(N) - |m|_{\mathfrak{s}} \,. \tag{5.16}$$

(We used a small abuse of notation here since $\partial^m N$ is really a collection of node types. Since reg_κ takes the same value on each of them, this creates no ambiguity.)

We claim that the same function reg_κ also satisfies (5.9) for the larger rule $\mathbb{Q}R$. In view of (5.15) and of the definition (5.14) of $\mathbb{Q}R$, it is enough to prove that

$$\mathrm{reg}_\kappa(\mathfrak{t}) \le |\mathfrak{t}|_{\mathfrak{s}} - \kappa + \mathrm{reg}(N), \qquad \forall\, N \in \bigcup_{\tau \in \mathfrak{T}_-(R)} R_-(\mathfrak{t};\tau). \tag{5.17}$$

Arguing by induction as in the proof of (5.11), one can first show the following. Let $\sigma \in \mathfrak{T}_\circ(R)$ any every elementary tree whose trunk e has edge type $(\mathfrak{t}, k)$. Then one has the bound

$$\mathrm{reg}_\kappa(\mathfrak{t}, k) \le |\sigma|_{\mathfrak{s}} + \mathrm{reg}_\kappa(G) \,, \qquad \forall\, G \in \bar{\mathcal{N}}(e). \tag{5.18}$$

Indeed, if e is the only edge of σ, then $\bar{\mathcal{N}}(e) = R(\mathfrak{t})$ and by (5.15)

$$\mathrm{reg}_\kappa(\mathfrak{t}, k) \le |\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} + \mathrm{reg}_\kappa(G) = |\sigma|_{\mathfrak{s}} + \mathrm{reg}_\kappa(G).$$

If now $e = (x, y)$ and $(e_1, \dots, e_\ell)$ are the outgoing edges of y, then $\bar{\mathcal{N}}(e)$ is the set of all $\mathcal{R}^M_{\mathcal{N}(y)} N$ with $\mathcal{N}(y) \subset N \in R(\mathfrak{t}(e))$ and $M = (M_1, \dots, M_\ell)$ with $M_i \in \bar{\mathcal{N}}(e_i)$. By the induction hypothesis,

$$\mathrm{reg}_\kappa(\mathcal{N}(y)) \le \sum_{i=1}^{\ell} \Big[|\sigma_i|_{\mathfrak{s}} + \mathrm{reg}_\kappa(M_i) \Big]$$

where σ_i is the largest elementary subtree of σ with trunk e_i. Then

$$\mathrm{reg}_\kappa(\mathcal{R}^M_{\mathcal{N}(y)} N) = \mathrm{reg}_\kappa(N) - \mathrm{reg}_\kappa(\mathcal{N}(y)) + \sum_{i=1}^{\ell} \mathrm{reg}_\kappa(M_i) \ge \mathrm{reg}_\kappa(N) - \sum_{i=1}^{\ell} |\sigma_i|_{\mathfrak{s}}.$$

Combining this with (5.15) we obtain, since $|\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} + \sum_{i=1}^{\ell} |\sigma_i|_{\mathfrak{s}} = |\sigma|_{\mathfrak{s}}$,

$$\mathrm{reg}_\kappa(\mathfrak{t}, k) \le |\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} + \mathrm{reg}(N) \le |\sigma|_{\mathfrak{s}} + \mathrm{reg}_\kappa(\mathcal{R}^M_{\mathcal{N}(y)} N)$$

and (5.18) is proved.

We prove now (5.17). Let $\tau \in \mathfrak{T}_-(R)$, $N \in R(\mathfrak{t})$ with $\mathcal{N}(\varrho_\tau) \subset N$, $M = (M_1, \dots, M_\ell) \in \mathscr{M}(\varrho_\tau)$, and $m \in \mathbf{N}^d$ with $|m|_{\mathfrak{s}} + |\tau|_{\mathfrak{s}} \le 0$. Let $\tau = \tau_1 \dots \tau_\ell$ be the decomposition of τ into elementary trees. Recalling (5.16) and Definitions 5.18 and 5.17, we have

$$\mathrm{reg}_\kappa(\partial^m(\mathcal{R}^M_{\mathcal{N}(\varrho_\tau)} N)) \;= \mathrm{reg}_\kappa\big(\mathcal{R}^M_{\mathcal{N}(\varrho_\tau)} N\big) - |m|_{\mathfrak{s}}$$

$$= \mathrm{reg}_\kappa(N) + \sum_{i=1}^{\ell} \Big[\mathrm{reg}_\kappa(M_i) - \mathrm{reg}_\kappa(s_i)\Big] - |m|_{\mathfrak{s}} \,,$$

where s_i is the edge type of the trunk of τ_i. Combining this with (5.18) yields

$$\mathrm{reg}_\kappa(\partial^m(\mathcal{R}^M_{N(\varrho_\tau)}N)) \geq \mathrm{reg}_\kappa(N) - |m|_{\mathfrak{s}} - |\tau|_{\mathfrak{s}} \geq \mathrm{reg}_\kappa(N) \,,$$

with the last inequality a consequence of the condition $|m|_{\mathfrak{s}} + |\tau|_{\mathfrak{s}} \leq 0$. This proves (5.17).

We conclude that (5.15) also holds when considering $N \in (\mathbb{Q}R)(\mathfrak{t})$, thus yielding the desired claim. Iterating this, we conclude that reg_κ satisfies (5.9) for each of the rules $\mathbb{Q}^n R$ and therefore also for $\bar{R}$ as required. □

Definition 5.21 We say that a subcritical rule R is *complete* (with respect to a fixed scaling $\mathfrak{s}$) if it is both normal and $\ominus$-complete. If R is only normal, we call the rule $\bar{R}$ constructed in the proof of Proposition 5.20 the completion of R.

5.5 Three prototypical examples

Let us now show how, concretely, a given stochastic PDE (or system thereof) gives rise to a rule in a natural way. Let us start with a very simple example, the KPZ equation formally given by

$$\partial_t u = \Delta u + (\partial_x u)^2 + \xi \,.$$

One then chooses the set $\mathfrak{L}$ so that it has one element for each noise process and one for each convolution operator appearing in the equation. In this case, using the variation of constants formula, we rewrite the equation in integral form as

$$u = Pu_0 + P * \mathbf{1}_{t>0}((\partial_x u)^2 + \xi) \,,$$

where P denotes the heat kernel and $*$ is space-time convolution. We therefore need two types in $\mathfrak{L}$ in this case, which we call $\{\Xi, \mathcal{I}\}$ in order to be consistent with [Hai14].

We assign degrees to these types just as in [Hai14]. In our example, the underlying space-time dimension is $d = 2$ and the equation is parabolic, so we fix the parabolic scaling $\mathfrak{s} = (2, 1)$ and then assign to Ξ a degree just below the exponent of self-similarity of white noise under the scaling $\mathfrak{s}$, namely $|\Xi|_{\mathfrak{s}} = -\frac{3}{2} - \kappa$ for some small $\kappa > 0$. We also assign to each type representing a convolution operator the degree corresponding to the amount by which it improves regularity in the sense of [Hai14, Sec. 4]. In our case, this is given by $|\mathcal{I}|_{\mathfrak{s}} = 2$.

It then seems natural to assign to such an equation a rule $\tilde{R}$ by

$$\tilde{R}(\Xi) = \{()\}, \qquad \tilde{R}(\mathcal{I}) = \{(\Xi), (\mathcal{I}_1, \mathcal{I}_1)\} \,,$$

where $\mathcal{I}_1$ is a shorthand for the edge type $(\mathcal{I}, (0, 1))$ and we simply write $\mathfrak{t}$ as a shorthand for the edge type $(\mathfrak{t}, 0)$. In other words, for every noise type $\mathfrak{t}$, we set

$\tilde{R}(\mathfrak{t}) = \{()\}$ and for every kernel type $\mathfrak{t}$ we include one node type into $\tilde{R}(\mathfrak{t})$ for each of the monomials in our equation that are convolved with the corresponding kernel. The problem is that such a rule is not normal. Therefore we define rather

$$R(\Xi) = \{()\}, \quad R(\mathcal{I}) = \{(), (\Xi), (\mathcal{I}_1), (\mathcal{I}_1, \mathcal{I}_1)\} ,$$

which turns out to be normal and complete. It is simple to see that the function $\mathrm{reg}_\kappa : \{\Xi, \mathcal{I}\} \to \mathbf{R}$

$$\mathrm{reg}_\kappa(\Xi) = -\frac{3}{2} - 2\kappa, \qquad \mathrm{reg}_\kappa(\mathcal{I}) = \frac{1}{2} - 3\kappa,$$

makes R subcritical for sufficiently small $\kappa > 0$.

One can also consider systems of equations. Consider for example the system of coupled KPZ equations formally given by

$$\begin{aligned} \partial_t u_1 &= \Delta u_1 + (\partial_x u_1)^2 + \xi_1 , \\ \partial_t u_2 &= \nu \Delta u_2 + (\partial_x u_2)^2 + \Delta u_1 + \xi_2 . \end{aligned}$$

In this case, we have two noise types $\Xi_{1,2}$ as well as two kernel types, which we call $\mathcal{I}$ for the heat kernel with diffusion constant 1 and $\mathcal{I}^\nu$ for the heat kernel with diffusion constant ν. There is some ambiguity in this case whether the term Δu_1 appearing in the second equation should be considered part of the linearisation of the equation or part of the nonlinearity. In this case, it turns out to be more convenient to consider this term as part of the nonlinearity, and we will see that the corresponding rule is still subcritical thanks to the triangular structure of this system.

Using the same notations as above, the normal and complete rule R naturally associated with this system of equations is given by

$$\begin{gathered} R(\Xi_i) = \{()\}, \quad R(\mathcal{I}) = \{(), (\Xi^1), (\mathcal{I}_1), (\mathcal{I}_1, \mathcal{I}_1)\} \\ R(\mathcal{I}^\nu) = \{(), (\Xi_2), (\mathcal{I}_1^\nu), (\mathcal{I}_1^\nu, \mathcal{I}_1^\nu),\ (\mathcal{I}_2)\}. \end{gathered}$$

In this case, we see that R is again subcritical for sufficiently small $\kappa > 0$ with

$$\mathrm{reg}_\kappa(\Xi_i) = -3/2 - 2\kappa , \quad \mathrm{reg}_\kappa(\mathcal{I}) = 1/2 - 3\kappa , \quad \mathrm{reg}_\kappa(\mathcal{I}^\nu) = 1/2 - 4\kappa .$$

Our last example is given by the following generalisation of the KPZ equation:

$$\partial_t u = \Delta u + g(u)(\partial_x u)^2 + h(u)\partial_x u + k(u) + f(u)\xi ,$$

which is motivated by (1.6) above, see [Hai16a]. In this case, the set $\mathfrak{L}$ is again given by $\{\Xi,\ \mathcal{I}\}$, just as in the case of the standard KPZ equation. Writing $[\mathcal{I}]_\ell$ as a shorthand for $\mathcal{I}, ..., \mathcal{I}$ where $\mathcal{I}$ is repeated ℓ times, the rule R associated to this equation is given by

$$R(\Xi) = \{()\},\ R(\mathcal{I}) = \{([\mathcal{I}]_\ell),\ ([\mathcal{I}]_\ell, \mathcal{I}_1),\ ([\mathcal{I}]_\ell, \mathcal{I}_1, \mathcal{I}_1),\ ([\mathcal{I}]_\ell, \Xi),\ \ell \in \mathbf{N}\} .$$

Again, it is straightforward to verify that R is subcritical and that one can use the same map reg_κ as in the case of the standard KPZ equation. Even though in this case there are infinitely many node types appearing in $R(\mathcal{I})$, this is not a problem because $\mathrm{reg}_\kappa(\mathcal{I}) > 0$, so that repetitions of the symbol $\mathcal{I}$ in a node type only increase the corresponding degree.

5.6 Regularity structures determined by rules

Throughout this section, we assume that we are given

- a *finite* type set $\mathfrak{L}$ together with a scaling $\mathfrak{s}$ and degrees $|\cdot|_\mathfrak{s}$ as in Definition 5.2,
- a normal rule R for $\mathfrak{L}$ which is both subcritical and complete, in the sense of Definition 5.21,
- the integer $d \geq 1$ which has been fixed at the beginning of the paper.

We henceforth write $\mathfrak{L} = \mathfrak{L}_- \sqcup \mathfrak{L}_+$ where $\mathfrak{L}_+ \stackrel{\text{def}}{=} \{\mathfrak{t} \in \mathfrak{L} : |\mathfrak{t}|_\mathfrak{s} > 0\}$ and $\mathfrak{L}_- \stackrel{\text{def}}{=} \{\mathfrak{t} \in \mathfrak{L} : |\mathfrak{t}|_\mathfrak{s} < 0\}$.

We show that the above choices, when combined with the structure built in Sections 3 and 4, yield a natural substructure with the same algebraic properties (the only exception being that the subspace of $\mathscr{H}_\circ$ we consider is not an algebra in general), but which is sufficiently small to yield a regularity structure. Furthermore, this regularity structure contains a very large group of automorphisms, unlike the slightly smaller structure described in [Hai14]. The reason for this is the additional flexibility granted by the presence of the decoration $\mathfrak{o}$, which allows to keep track of the degrees of the subtrees contracted by the action of $\mathscr{G}_1$.

Definition 5.22 Given $\mathfrak{L}$, R, $\mathfrak{s}$ and d as above, we define for every possible pair $(\mathfrak{t}, N)$ with $\mathfrak{t} \in \mathfrak{L}$ and $N \in R(\mathfrak{t})$, a set $D(\mathfrak{t}, N) \subset \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$ by postulating that $\alpha \in D(\mathfrak{t}, N)$ if and only if there exist

- $\tau = (F, \mathfrak{n}, \mathfrak{e}) \in \mathfrak{T}_-(R)$,
- $M \in R(\mathfrak{t})$ with $\mathcal{N}(\varrho_\tau) \subset M$ and $\bar{M} \in \mathscr{M}(\varrho_\tau)$,
- $m \in \mathbf{N}^d$ with $|\tau|_\mathfrak{s} + |m|_\mathfrak{s} < 0$,

such that

$$N \in \partial^m(\mathscr{R}^{\bar{M}}_{\mathcal{N}(\varrho_\tau)} M) \qquad \text{and} \qquad \alpha = \sum_{e \in E_F} (\mathfrak{t}(e) - \mathfrak{e}(e)) + \sum_{x \in N_F} \mathfrak{n}(x) + m\,.$$

A crucial fact is the following.

Lemma 5.23 *For every pair* $(\mathfrak{t}, N)$*, the set* $D(\mathfrak{t}, N)$ *is finite.*

Proof. The set $\mathfrak{T}_-(R) \subset \mathfrak{T}_\circ(R)$ is finite by Proposition 5.14. □

We now use the family of sets $D(\mathfrak{t}, N)$ in order to constrain the possible values of the decoration $\mathfrak{o}$ in our forests.

Definition 5.24 We denote by $\Lambda = \Lambda(\mathfrak{L}, R, \mathfrak{s}, d)$ the set of all $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F}$ such that, for all $x \in N_F$, exactly one of the following three mutually exclusive statements holds.

- One has $\hat{F}(x) \in \{0,2\}$ and $\mathfrak{o}(x) = 0$.
- One has $\hat{F}(x) = 1$ and there exists a node $y \in N_F$ and an edge $e = (y,x) \in E_F$ such that $\mathfrak{o}(x) \in D(\mathfrak{t}(e), \mathcal{N}(x))$, where $\mathcal{N}(x)$ is as in Definition 5.8.
- One has $\hat{F}(x) = 1$, x is a root of F, and there exists $\mathfrak{t} \in \mathfrak{L}$ such that $\mathfrak{o}(x) \in D(\mathfrak{t}, \mathcal{N}(x))$.

The reason why we define $D(\mathfrak{t}, N)$ in this way is that it guarantees the following characterisation of Λ which follows immediately from the previous definitions.

Lemma 5.25 *One has $\tau \in \Lambda$ if and only if there exists $(F, \hat{F}, \mathfrak{n}, 0, \mathfrak{e}) \in \mathfrak{F}$ with $\hat{F}(x) \in \{0,2\}$ for all x, a subforest $A \in \mathfrak{A}_1(F, \hat{F})$, i.e. $A \subset F \setminus \hat{F}^{-1}(2)$, and functions $\mathfrak{n}_A \colon N_A \to \mathbf{N}^d$ and $\varepsilon_A^F \colon \partial(A,F) \to \mathbf{N}^d$ with $\mathfrak{n}_A \le \mathfrak{n}$ such that*

- *One has*

$$\tau = \mathscr{K}_1(F, \hat{F} \cup_1 A, \mathfrak{n} - \mathfrak{n}_A, \mathfrak{n}_A + \pi(\varepsilon_A^F - \mathfrak{e}_{\emptyset}^A), \mathfrak{e}_A^F + \varepsilon_A^F) \,. \tag{5.19}$$

- *Define $\mathfrak{L}_x \subset \mathfrak{L}$ as $\mathfrak{L}_x = \{\mathfrak{t}(e)\}$ if there exists an edge $e = (z,x)$ terminating at x and $\mathfrak{L}_x = \mathfrak{L}$ otherwise (i.e. if x is a root of F). Then, for every $x \in N_A$, the node type $\mathcal{N}(x)$ of x in F belongs to $R(\mathfrak{t})$ for some $\mathfrak{t} \in \mathfrak{L}_x$.*
- *For each connected component B of A, we have $(B, 0, \mathfrak{n}_A + \pi\varepsilon_A^F, 0, \mathfrak{e}) \in \mathfrak{T}_-(R)$, see Definition 5.12.*

We now define spaces of coloured forests $\tau = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ such that $(F, 0, \mathfrak{n}, 0, \mathfrak{e})$ is compatible with the rule R in a suitable sense, and such that $\tau \in \Lambda$.

Definition 5.26 We define $\mathcal{S} : \mathfrak{F} \to \mathfrak{F}$ by $\mathcal{S}(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \stackrel{\text{def}}{=} (F, 0, \mathfrak{n}, 0, \mathfrak{e})$. Recalling Definition 5.12 and Remark 4.12, we define the bigraded spaces

$$\begin{aligned}
\hat{\mathcal{T}}_+^{\text{ex}} &= \langle B_+ \rangle \subset \hat{\mathscr{H}}_2 \,, \qquad B_+ \stackrel{\text{def}}{=} \{\tau \in \hat{H}_2 \,:\, \tau \in \Lambda \,\&\, \mathcal{S}\tau \in \mathfrak{T}_2(R)\} \,, \\
\hat{\mathcal{T}}_-^{\text{ex}} &= \langle B_- \rangle \subset \mathscr{H}_1 \,, \qquad B_- \stackrel{\text{def}}{=} \{\tau \in H_1 \,:\, \tau \in \Lambda \,\&\, \mathcal{S}\tau \in \mathfrak{T}_1(R)\} \,, \\
\mathcal{T}^{\text{ex}} &= \langle B_\circ \rangle \subset \mathscr{H}_\circ \,, \qquad B_\circ \stackrel{\text{def}}{=} \{\tau \in H_\circ \,:\, \tau \in \Lambda \,\&\, \mathcal{S}\tau \in \mathfrak{T}_\circ(R)\} \,.
\end{aligned}$$

Remark 5.27 The superscript "ex" stands for "extended", see Section 6.4 below for an explanation of the reason why we choose this terminology. The identification of these spaces as suitable subspaces of $\hat{\mathscr{H}}_2$, $\mathscr{H}_1$ and $\mathscr{H}_\circ$ is done via (4.9).

Note that both $\hat{\mathcal{T}}_-^{\text{ex}}$ and $\hat{\mathcal{T}}_+^{\text{ex}}$ are algebras for the products inherited from $\mathscr{H}_1$ and $\hat{\mathscr{H}}_2$ respectively. On the other hand, $\mathcal{T}^{\text{ex}}$ is in general not an algebra anymore.

Lemma 5.28 *We have*

$$\Delta_1 \colon \mathcal{T}^{\text{ex}} \to \hat{\mathcal{T}}_-^{\text{ex}} \hat{\otimes} \mathscr{H}_\circ \,, \quad \Delta_1 \colon \hat{\mathcal{T}}_-^{\text{ex}} \to \hat{\mathcal{T}}_-^{\text{ex}} \hat{\otimes} \mathscr{H}_1 \,, \quad \Delta_1 \colon \hat{\mathcal{T}}_+^{\text{ex}} \to \hat{\mathcal{T}}_-^{\text{ex}} \hat{\otimes} \hat{\mathscr{H}}_2 \,,$$

as well as $\Delta_2 \colon \mathscr{H} \to \mathscr{H} \hat{\otimes} \hat{\mathcal{T}}_+^{\text{ex}}$ for $\mathscr{H} \in \{\mathcal{T}^{\text{ex}}, \hat{\mathcal{T}}_+^{\text{ex}}\}$. Moreover, $\hat{\mathcal{T}}_+^{\text{ex}}$ is a Hopf subalgebra of $\hat{\mathscr{H}}_2$ and $\mathcal{T}^{\text{ex}}$ is a right Hopf-comodule over $\hat{\mathcal{T}}_+^{\text{ex}}$ with coaction Δ_2.

Proof. By the normality of the rule R, if a tree conforms to R then any of its subtrees does too. On the other hand, contracting subforests can generate non-conforming trees in the case of Δ_1, while, since Δ_2 extracts only subtrees at the root, completeness of the rule implies that this can not happen in the case of Δ_2, thus showing that the maps Δ_i do indeed behave as claimed.

The fact that $\hat{\mathcal{T}}_+^{\mathrm{ex}}$ is in fact a Hopf algebra, namely that the antipode $\hat{\mathcal{A}}_2$ of $\hat{\mathscr{H}}_2$ leaves $\hat{\mathcal{T}}_+^{\mathrm{ex}}$ invariant, can be shown by induction using (4.16) and Remark 4.15. □

Note that $\hat{\mathcal{T}}_-^{\mathrm{ex}}$ is a sub-algebra but in general *not* a sub-coalgebra of $\mathscr{H}_1$ (and *a fortiori* not a Hopf algebra). Recall also that, by Lemma 5.4, the grading $|\cdot|_-$ of Definition 5.3 is well defined on $\hat{\mathcal{T}}_-^{\mathrm{ex}}$ and that $|\cdot|_+$ is well defined on both $\hat{\mathcal{T}}_+^{\mathrm{ex}}$ and $\mathcal{T}^{\mathrm{ex}}$. Furthermore, these gradings are preserved by the corresponding products and coproducts.

Definition 5.29 Let $\mathcal{J}_\mp \subset \hat{\mathcal{T}}_\pm^{\mathrm{ex}}$ be the ideals given by

$$\begin{aligned}\mathcal{J}_- =&\langle\{\tau \in B_+ \,:\, \tau = \not{\mathcal{J}}\hat{\mathcal{K}}_2(\sigma\cdot\bar\sigma)\,,\quad \sigma,\bar\sigma\in B_+,\ \sigma\neq\mathbf{1}_2,\ |\sigma|_+\le 0\}\rangle\,,\\ \mathcal{J}_+ =&\langle\{\tau \in B_- \,:\, \tau = \mathcal{K}_1(\sigma\cdot\bar\sigma)\,,\quad \sigma,\bar\sigma\in B_-,\\ &\qquad (\sigma\neq\mathbf{1}_1\quad\&\quad |\sigma|_-\ge 0)\quad\text{or}\quad(\sigma=\mathcal{I}_k^{\mathfrak{t}}(\sigma')\quad\&\quad |\mathfrak{t}|_{\mathfrak{s}}>0)\}\rangle\,.\end{aligned}\tag{5.20}$$

Then, we set

$$\mathcal{T}_-^{\mathrm{ex}} \stackrel{\mathrm{def}}{=} \hat{\mathcal{T}}_-^{\mathrm{ex}}/\mathcal{J}_+\,,\qquad \mathcal{T}_+^{\mathrm{ex}} \stackrel{\mathrm{def}}{=} \hat{\mathcal{T}}_+^{\mathrm{ex}}/\mathcal{J}_-\,,$$

with canonical projections $\mathfrak{p}_\pm^{\mathrm{ex}} : \hat{\mathcal{T}}_\pm^{\mathrm{ex}} \to \mathcal{T}_\pm^{\mathrm{ex}}$.

With these definitions at hand, it turns out that the map $(\mathfrak{p}_-^{\mathrm{ex}}\otimes\mathrm{id})\Delta_1$ is much better behaved. Indeed, we have the following.

Lemma 5.30 *The map* $\Delta_{\mathrm{ex}}^- = (\mathfrak{p}_-^{\mathrm{ex}}\otimes\mathrm{id})\Delta_1$ *satisfies*

$$\Delta_{\mathrm{ex}}^-\colon \mathscr{H} \to \mathcal{T}_-^{\mathrm{ex}}\,\hat\otimes\,\mathscr{H}\,,\qquad \text{for } \mathscr{H}\in\{\hat{\mathcal{T}}_-^{\mathrm{ex}},\mathcal{T},\hat{\mathcal{T}}_+^{\mathrm{ex}}\}.$$

Proof. This follows immediately from Lemma 5.25. □

Analogously to Lemma 3.16 we have

Lemma 5.31 *We have*

$$(\mathfrak{p}_-^{\mathrm{ex}}\otimes\mathfrak{p}_-^{\mathrm{ex}})\Delta_1\mathcal{J}_+ = 0\,,\quad (\mathfrak{p}_-^{\mathrm{ex}}\otimes\mathfrak{p}_+^{\mathrm{ex}})\Delta_1\mathcal{J}_- = 0\,,\quad (\mathfrak{p}_+^{\mathrm{ex}}\otimes\mathfrak{p}_+^{\mathrm{ex}})\Delta_2\mathcal{J}_- = 0\,.\tag{5.21}$$

Proof. We note that the degrees $|\cdot|_\pm$ have the following compatibility properties with the operators Δ_i. For $0<i\le j\le 2$, $\tau\in\mathfrak{F}_j$ and $\Delta_i\tau = \sum \tau_i^{(1)}\otimes\tau_i^{(2)}$ (with the summation variable suppressed), one has

$$|\tau_1^{(1)}|_- + |\tau_1^{(2)}|_- = |\tau|_-\,,\quad |\tau_1^{(2)}|_+ = |\tau|_+\,,\quad |\tau_2^{(1)}|_+ + |\tau_2^{(2)}|_+ = |\tau|_+\,.\tag{5.22}$$

The first identity of (5.21) then follows from the first identity of (5.22) and from the following remark: if $B_-\ni\tau = \mathcal{I}_k^{\mathfrak{t}}(\sigma)$, then for each term appearing in the sum over $A\in\mathfrak{A}_1$ in the expression (3.3) for $\Delta_1\tau$, one has two possibilities:

- either A does not contain the edge incident to the root of τ, and then the second factor is a tree with only one edge incident to its root,
- or A does contain the edge incident to the root, in which case the first factor contains one connected component of that type.

The second identity of (5.21) follows from the second identity of (5.22) combined with the fact that, for $\tau \in \mathfrak{F}_2$, $\Delta_1\tau$ contains no term of the form $\sigma \otimes \mathbf{1}_2$, even when quotiented by $\ker(\mathscr{J}\hat{\mathcal{R}}_2)$. The third identity of (5.21) finally follows from the third identity of (5.22), combined with the fact that if $\tau \in B_+ \setminus \{\mathbf{1}_2\}$ with $|\tau|_+ \leq 0$, then the term $\mathbf{1}_2 \otimes \mathbf{1}_2$ does not appear in the expansion for $\Delta_2\tau$. □

As a corollary, we have the following.

Corollary 5.32 *The operator $\Delta^-_{\mathrm{ex}} = (\mathfrak{p}^{\mathrm{ex}}_- \otimes \mathrm{id})\Delta_1$ is well-defined as a map*

$$\Delta^-_{\mathrm{ex}} \colon \mathcal{H} \to \mathcal{T}^{\mathrm{ex}}_- \hat{\otimes} \mathcal{H}\,, \qquad \text{for } \mathcal{H} \in \{\hat{\mathcal{T}}^{\mathrm{ex}}_-, \mathcal{T}^{\mathrm{ex}}_-, \mathcal{T}^{\mathrm{ex}}, \mathcal{T}^{\mathrm{ex}}_+, \hat{\mathcal{T}}^{\mathrm{ex}}_+\}.$$

Similarly, the operator $\Delta^+_{\mathrm{ex}} = (\mathrm{id} \otimes \mathfrak{p}^{\mathrm{ex}}_+)\Delta_2$ is well-defined as a map

$$\Delta^+_{\mathrm{ex}} \colon \mathcal{H} \to \mathcal{H} \hat{\otimes} \mathcal{T}^{\mathrm{ex}}_+\,, \qquad \text{for } \mathcal{H} \in \{\mathcal{T}^{\mathrm{ex}}, \mathcal{T}^{\mathrm{ex}}_+, \hat{\mathcal{T}}^{\mathrm{ex}}_+\}.$$

Remark 5.33 The operators $\Delta^{\pm}_{\mathrm{ex}}$ of Corollary 5.32 are now given by *finite sums* so that for all of these choices of $\mathcal{H}$, the operators Δ^-_{ex} and Δ^+_{ex} actually map $\mathcal{H}$ into $\mathcal{T}^{\mathrm{ex}}_- \otimes \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{T}^{\mathrm{ex}}_+$ respectively.

Proposition 5.34 *There exists an algebra morphism $\mathcal{A}^{\mathrm{ex}}_+ \colon \mathcal{T}^{\mathrm{ex}}_+ \to \mathcal{T}^{\mathrm{ex}}_+$ so that $(\mathcal{T}^{\mathrm{ex}}_+, \mathcal{M}, \Delta^+_{\mathrm{ex}}, \mathbf{1}_2, \mathbf{1}^\star_2, \mathcal{A}^{\mathrm{ex}}_+)$, where $\mathcal{M}$ is the tree product* (4.8)*, is a Hopf algebra. Moreover the map $\Delta^+_{\mathrm{ex}} : \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}} \otimes \mathcal{T}^{\mathrm{ex}}_+$, turns $\mathcal{T}^{\mathrm{ex}}$ into a right comodule for $\mathcal{T}^{\mathrm{ex}}_+$ with counit $\mathbf{1}^\star_2$.*

Proof. We already know that $\hat{\mathcal{T}}^{\mathrm{ex}}_+$ is a Hopf sub-algebra of $\hat{\mathcal{H}}_2$ with antipode $\hat{\mathcal{A}}_2$ satisfying (4.16). Since $\mathcal{J}_-$ is a bialgebra ideal by Lemma 5.31, the first claim follows from [Nic78, Theorem 1.(iv)].

The fact that $\Delta^+_{\mathrm{ex}} : \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}} \otimes \mathcal{T}^{\mathrm{ex}}_+$ is a co-action and turns $\mathcal{T}^{\mathrm{ex}}$ into a right comodule for $\mathcal{T}^{\mathrm{ex}}_+$ follows from the coassociativity of Δ_2. □

Proposition 5.35 *There exists an algebra morphism $\mathcal{A}^{\mathrm{ex}}_- \colon \mathcal{T}^{\mathrm{ex}}_- \to \mathcal{T}^{\mathrm{ex}}_-$ so that $(\mathcal{T}^{\mathrm{ex}}_-, \cdot, \Delta^-_{\mathrm{ex}}, \mathbf{1}_1, \mathbf{1}^\star_1, \mathcal{A}^{\mathrm{ex}}_-)$ is a Hopf algebra. Moreover the map $\Delta^-_{\mathrm{ex}} : \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}}_- \otimes \mathcal{T}^{\mathrm{ex}}$ turns $\mathcal{T}^{\mathrm{ex}}$ into a left comodule for $\mathcal{T}^{\mathrm{ex}}_-$ with counit $\mathbf{1}^\star_1$.*

Proof. One difference between $\mathcal{T}^{\mathrm{ex}}_-$ and $\mathcal{T}^{\mathrm{ex}}_+$ is that $\hat{\mathcal{T}}^{\mathrm{ex}}_-$ is not in general a sub-coalgebra of $\mathcal{H}_1$ and therefore it does not possess an antipode. However we can see that the antipode $\mathcal{A}_1$ of $\mathcal{H}_1$ satisfies for all $\tau \neq \mathbf{1}$

$$\mathcal{A}_1\tau = -\tau - \mathcal{M}(\mathcal{A}_1 \otimes \mathrm{id})(\Delta_1\tau - \tau \otimes \mathbf{1} - \mathbf{1} \otimes \tau),$$

where $\mathcal{M}$ is the product map. By the second formula of (5.22), it follows that if $|\tau|_- > 0$ then $\mathcal{A}_1\tau \in \mathcal{J}_+$ and therefore, since $\mathcal{A}_1$ is an algebra morphism, $\mathcal{A}_1(\mathcal{J}_+) \subseteq \mathcal{J}_+$. We obtain that $\mathcal{A}_1$ defines a unique algebra morphism $\mathcal{A}^{\mathrm{ex}}_- \colon \mathcal{T}^{\mathrm{ex}}_- \to \mathcal{T}^{\mathrm{ex}}_-$ which is an antipode for $\mathcal{T}^{\mathrm{ex}}_-$. □

Definition 5.36 We call $\mathcal{G}_{\pm}^{\mathrm{ex}}$ the character group of $\mathcal{T}_{\pm}^{\mathrm{ex}}$.

We have therefore obtained the following analogue of Proposition 4.14:

Theorem 5.37 *We have*
1. *On $\mathcal{T}^{\mathrm{ex}}$, the identity*

$$\mathcal{M}^{(13)(2)(4)}(\Delta_{\mathrm{ex}}^{-} \otimes \Delta_{\mathrm{ex}}^{-})\Delta_{\mathrm{ex}}^{+} = (\mathrm{id} \otimes \Delta_{\mathrm{ex}}^{+})\Delta_{\mathrm{ex}}^{-}\,, \tag{5.23}$$

   holds, with $\mathcal{M}^{(13)(2)(4)}$ as in (3.33)*. The same is also true on $\mathcal{T}_{+}^{\mathrm{ex}}$.*
2. *$\mathcal{T}^{\mathrm{ex}}$ is a left comodule over the Hopf algebra $\hat{\mathcal{T}}_{12}^{\mathrm{ex}} \stackrel{def}{=} \mathcal{T}_{-}^{\mathrm{ex}} \ltimes \mathcal{T}_{+}^{\mathrm{ex}}$, with coaction*

$$\Delta_{\circ} : \mathcal{T}^{\mathrm{ex}} \to \hat{\mathcal{T}}_{12}^{\mathrm{ex}} \hat{\otimes} \mathcal{T}^{\mathrm{ex}}, \qquad \Delta_{\circ} v \stackrel{def}{=} \sigma^{(132)}(\Delta_{\mathrm{ex}}^{-} \otimes \mathcal{A}_{+}^{\mathrm{ex}})\Delta_{\mathrm{ex}}^{+}$$

   where $\sigma^{(132)}(a \otimes b \otimes c) \stackrel{def}{=} a \otimes c \otimes b$ and $\mathcal{A}_{+}^{\mathrm{ex}}$ is the antipode of $\mathcal{T}_{+}^{\mathrm{ex}}$.

Proof. By the second identity of (5.22), the action of Δ_{ex}^{-} preserves the degree $|\cdot|_{+}$. In particular we have

$$\Delta_{\mathrm{ex}}^{-}\mathfrak{p}_{+}^{\mathrm{ex}} = \big(\mathrm{id} \otimes \mathfrak{p}_{+}^{\mathrm{ex}}\big)\Delta_{\mathrm{ex}}^{-}. \tag{5.24}$$

From this property, one has:

$$\begin{aligned}\mathcal{M}^{(13)(2)(4)}(\Delta_{\mathrm{ex}}^{-} \otimes \Delta_{\mathrm{ex}}^{-})\Delta_{\mathrm{ex}}^{+} &= \mathcal{M}^{(13)(2)(4)}(\Delta_{\mathrm{ex}}^{-} \otimes \big(\mathrm{id} \otimes \mathfrak{p}_{+}^{\mathrm{ex}}\big)\Delta_{\mathrm{ex}}^{-})\Delta_{2} \\ &= \big(\mathfrak{p}_{-}^{\mathrm{ex}} \otimes \mathrm{id} \otimes \mathfrak{p}_{+}^{\mathrm{ex}}\big)\mathcal{M}^{(13)(2)(4)}(\Delta_{1} \otimes \Delta_{1})\Delta_{2}\end{aligned}$$

and we conclude by applying the Proposition 3.22. □

Remark 5.38 We can finally see here the role played by the decoration $\mathfrak{o}$: were it not included, the cointeraction property (5.23) of Theorem 5.37 would fail, since it is based upon (5.24), which itself depends on the second identity of (5.22). Now recall that $|\cdot|_{+}$ takes the decoration $\mathfrak{o}$ into account, and this is what makes the second identity of (5.22) true. See also Remark 6.25 below.

We define $A^{\mathrm{ex}} \stackrel{\mathrm{def}}{=} \{|\tau|_{+} : \tau \in B_{\circ}\}$, where $\mathcal{T}^{\mathrm{ex}} = \langle B_{\circ} \rangle$ as in Definition 5.26.

Proposition 5.39 *The above construction yields a regularity structure $\mathscr{T}^{\mathrm{ex}} = (A^{\mathrm{ex}}, \mathcal{T}^{\mathrm{ex}}, \mathcal{G}_{+}^{\mathrm{ex}})$ in the sense of Definition 5.1.*

Proof. By Lemma 5.25, every element $\tau \in \Lambda$ has a representation of the type (5.19) for some $\sigma = (T, \mathfrak{n}, \mathfrak{e}) \in \mathfrak{T}$. Furthermore, it follows from the definitions of $|\cdot|_{+}$ and $|\cdot|_{\mathfrak{s}}$ that one has $|\tau|_{+} = |\sigma|_{\mathfrak{s}}$. The fact that, for all $\gamma \in \mathbf{R}$, the set $\{a \in A^{\mathrm{ex}} : a \leq \gamma\}$ is finite then follows from Proposition 5.14.

The space $\mathcal{T}^{\mathrm{ex}}$ is graded by $|\cdot|_{+}$ and $\mathcal{G}_{+}^{\mathrm{ex}}$ acts on it by $\Gamma_{g} \stackrel{\mathrm{def}}{=} (\mathrm{id} \otimes g)\Delta_{\mathrm{ex}}^{+}$. The property (5.1) then follows from the fact the action of $\mathcal{G}_{+}^{\mathrm{ex}}$ preserves the $|\cdot|_{+}$-degree by the second identity in (5.22) and all terms appearing in the second factor of $\Delta_{\mathrm{ex}}^{+}\tau - \tau \otimes \mathbf{1}$ have strictly positive $|\cdot|_{+}$-degree by Definition 5.29. □

5.7 Link to previous constructions

Recall that in [Hai14, Sec. 8], we considered a regularity structure $\mathscr{T}$ generated by symbols X^k and Ξ, as well as operations $\tilde{\mathcal{I}}_k$ and a product. (We call the integration map appearing in [Hai14] $\tilde{\mathcal{I}}_k$ rather than $\mathcal{I}_k$ in order to distinguish the definitions there from the ones above.) Similarly to the present situation, the structure group $\mathscr{G}$ considered in [Hai14] is described by a comodule structure of $\mathscr{T}$ on a Hopf algebra $\mathscr{T}_+$, with basis vectors of $\mathscr{T}_+$ of the form (4.13). In order to distinguish the two, we write $\tilde{\mathcal{I}}_{k_i}^{\mathfrak{t}_i}(\tau_i)$ for the elements of $\mathscr{T}_+$ as described in [Hai14]. The operators $\Delta\colon \mathscr{T} \to \mathscr{T} \otimes \mathscr{T}_+$ and $\Delta^+\colon \mathscr{T}_+ \to \mathscr{T}_+ \otimes \mathscr{T}_+$ defined in [Hai14, Eq. (8.8)–(8.9)] are then given recursively by

$$\begin{aligned}\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}\,, \quad \Delta \tilde{\Xi} = \tilde{\Xi} \otimes \mathbf{1}\,, \quad \Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i\,,\\ \Delta \tilde{\mathcal{I}}_k(\tau) = (\tilde{\mathcal{I}}_k \otimes 1)\Delta\tau + \sum_{\ell,m} \frac{X^\ell}{\ell!} \otimes \frac{X^m}{m!}\tilde{\mathcal{I}}_{k+\ell+m}(\tau)\,,\end{aligned} \tag{5.25}$$

as well as

$$\begin{aligned}\Delta^+ \mathbf{1} = \mathbf{1} \otimes \mathbf{1}\,, \quad \Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i\,,\\ \Delta^+ \tilde{\mathcal{I}}_k(\tau) = 1 \otimes \tilde{\mathcal{I}}_k\tau + \sum_{\ell} \Big(\tilde{\mathcal{I}}_{k+\ell} \otimes \frac{(-X)^\ell}{\ell!}\Big)\Delta\tau\,,\end{aligned} \tag{5.26}$$

with the additional property that both maps are multiplicative. Even if we disregard the additional presence of the maps $\mathscr{R}_\alpha$ in our present context, these identities appear at first sight somewhat different from those described in (4.14) and (4.15).

The first line of (5.25) is similar to the first line of (4.14), with the exception of the appearance of the symbol $\tilde{\Xi}$. This can however easily be linked to our construction. Indeed, assuming that we have $\mathfrak{L} = \{\mathfrak{t}, \mathfrak{i}\}$ with $|\mathfrak{t}|_{\mathfrak{s}} < 0$ and $|\mathfrak{i}|_{\mathfrak{s}} > 0$, we set $\Xi = \mathcal{I}_0^{\mathfrak{t}}(\mathbf{1})$ and $\mathcal{I}_k = \mathcal{I}_k^{\mathfrak{i}}$. The second line of (4.14) then yields indeed

$$\Delta^+_{\mathrm{ex}} \Xi = \Xi \otimes \mathbf{1}\,,$$

since $|\mathcal{I}_k^{\mathfrak{t}}(\mathbf{1})|_{\mathfrak{s}} < 0$ for every k, and therefore the corresponding terms vanish in $\mathscr{T}_+^{\mathrm{ex}}$.

The second line of (5.25) on the other hand looks quite different from the second line of (4.14). This can be remedied by a simple change of basis. Performing the identifications

$$\tilde{\mathcal{I}}_k(\tau) \;\leftrightarrow\; \mathcal{I}_k(\tau)\,, \qquad \mathcal{I}_k(\tau) \;\leftrightarrow\; \sum_m \frac{X^m}{m!}\tilde{\mathcal{I}}_{k+m}(\tau)\,, \tag{5.27}$$

we see that (5.25) turns into

$$\Delta \mathcal{I}_k(\tau) = (\mathcal{I}_k \otimes \mathrm{id})\Delta\tau + \sum_\ell \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{k+\ell}(\tau)\,,$$

which is the same recursion as (4.14). As far as Δ^+ is concerned, exploiting the multiplicativity of Δ^+ and the fact that $\Delta^+ \frac{X^k}{k!} = \sum_{m+n=k} \frac{X^m}{m!} \otimes \frac{X^n}{n!}$, one has the identity

$$\begin{aligned}\Delta^+ \mathcal{J}_k(\tau) &= \sum_{\ell,m,n} \Big(\frac{X^m}{m!} \tilde{\mathcal{J}}_{k+\ell+m+n} \otimes \frac{(-X)^\ell}{\ell!} \frac{X^n}{n!} \Big) \Delta\tau + \sum_{\ell,m} \frac{X^\ell}{\ell!} \otimes \frac{X^m}{m!} \tilde{\mathcal{J}}_{k+\ell+m}(\tau) \\ &= \sum_{m} \Big(\frac{X^m}{m!} \tilde{\mathcal{J}}_{k+m} \otimes 1 \Big) \Delta\tau + \sum_{\ell,m} \frac{X^\ell}{\ell!} \otimes \frac{X^m}{m!} \tilde{\mathcal{J}}_{k+\ell+m}(\tau) \\ &= (\mathcal{J}_k \otimes 1)\Delta\tau + \sum_{\ell} \frac{X^\ell}{\ell!} \otimes \mathcal{J}_{k+\ell}(\tau) \,,\end{aligned}$$

which is nothing but (4.15). This shows that the structure constructed in [Hai14] is included in the one constructed in the present article, modulo a simple change of coordinates. Note that this change of coordinates is "harmless" as far as the link to the analytical part of [Hai14] is concerned since it does not mix basis vectors of different degrees.

6 Renormalisation of models

We now show how the construction of the previous sections can be applied to the theory of regularity structures to show that the "contraction" operations one would like to perform in order to renormalise models are "legitimate" in the sense that they give rise to automorphisms of the regularity structures built in Section 5.6. Throughout this section, we are in the framework set at the beginning of Section 5.6.

Note now that we have a natural identification of $\mathcal{T}_\pm^{\mathrm{ex}}$ with the subspaces

$$\langle \{\tau \in B_\pm \,:\, \tau \notin \mathcal{J}_\mp^{\mathrm{ex}}\} \rangle \subset \hat{\mathcal{T}}_\pm^{\mathrm{ex}} \,.$$

Denote by $\mathfrak{i}_\pm^{\mathrm{ex}} \colon \mathcal{T}_\pm^{\mathrm{ex}} \to \hat{\mathcal{T}}_\pm^{\mathrm{ex}}$ the corresponding inclusions, so that we have direct sum decompositions

$$\hat{\mathcal{T}}_\pm^{\mathrm{ex}} = \mathcal{T}_\pm^{\mathrm{ex}} \oplus \mathcal{J}_\mp^{\mathrm{ex}} \,. \tag{6.1}$$

For instance, with this identification, the map $\mathcal{J}_k^{\mathfrak{t}} \colon \mathcal{T}^{\mathrm{ex}} \to \hat{\mathcal{T}}_+^{\mathrm{ex}}$ defined in (4.12) associates to $\tau \in \mathcal{T}^{\mathrm{ex}}$ an element $\mathcal{J}_k^{\mathfrak{t}}(\tau) \in \hat{\mathcal{T}}_+^{\mathrm{ex}}$ which belongs to $\mathcal{T}_+^{\mathrm{ex}}$ if and only if its degree $|\mathcal{J}_k^{\mathfrak{t}}(\tau)|_+$ is positive, namely $|\tau|_+ + |\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} > 0$.

Proposition 6.1 *Let $\mathcal{A}_+^{\mathrm{ex}} : \mathcal{T}_+^{\mathrm{ex}} \to \mathcal{T}_+^{\mathrm{ex}}$ be the antipode of $\mathcal{T}_+^{\mathrm{ex}}$. Then*

- *$\mathcal{A}_+^{\mathrm{ex}}$ is defined uniquely by the fact that $\mathcal{A}_+^{\mathrm{ex}} X_i = -X_i$ and for all $\mathcal{J}_k^{\mathfrak{t}}(\tau) \in \mathcal{T}_+^{\mathrm{ex}}$*

$$\mathcal{A}_+^{\mathrm{ex}} \mathcal{J}_k(\tau) = -\sum_{\ell \in \mathbf{N}^d} \frac{(-X)^\ell}{\ell!} \mathcal{M}_+^{\mathrm{ex}} (\mathfrak{p}_+^{\mathrm{ex}} \mathcal{J}_{k+\ell} \otimes \mathcal{A}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^+ \tau \,, \tag{6.2}$$

where $\mathcal{M}_+^{\mathrm{ex}} \colon \mathcal{T}_+^{\mathrm{ex}} \otimes \mathcal{T}_+^{\mathrm{ex}} \to \mathcal{T}_+^{\mathrm{ex}}$ denotes the (tree) product and $\Delta_{\mathrm{ex}}^+ \colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}} \otimes \mathcal{T}_+^{\mathrm{ex}}$.

- *On $\mathcal{T}_+^{\mathrm{ex}}$, one has the identity*

$$\Delta_{\mathrm{ex}}^- \mathcal{A}_+^{\mathrm{ex}} = (\mathrm{id} \otimes \mathcal{A}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^- \,. \tag{6.3}$$

Proof. The claims follow easily from Propositions 4.17 and 5.34. □

6.1 Twisted antipodes

We define now the operator $P_+ : \hat{\mathcal{T}}_+^{\mathrm{ex}} \to \hat{\mathcal{T}}_+^{\mathrm{ex}}$ given on $\tau \in B_+$ by

$$P_+(\tau) \stackrel{\mathrm{def}}{=} \begin{cases} \tau & \text{if } |\tau|_+ \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is quite different from the projection $\mathfrak{i}_+^{\mathrm{ex}} \circ \mathfrak{p}_+^{\mathrm{ex}}$. However, for elements of the form $\mathcal{I}_k^{\mathfrak{t}}(\tau) \in \hat{\mathcal{T}}_+^{\mathrm{ex}}$ for some $\tau \in \mathcal{T}^{\mathrm{ex}}$, we have $P_+ \mathcal{I}_k^{\mathfrak{t}}(\tau) = (\mathfrak{i}_+^{\mathrm{ex}} \circ \mathfrak{p}_+^{\mathrm{ex}})(\mathcal{I}_k^{\mathfrak{t}}(\tau))$. The difference is that $\mathfrak{i}_+^{\mathrm{ex}} \circ \mathfrak{p}_+^{\mathrm{ex}}$ is multiplicative under the tree product, while P_+ is not.

Proposition 6.2 *There exists a unique algebra morphism $\tilde{\mathcal{A}}_+^{\mathrm{ex}} : \mathcal{T}_+^{\mathrm{ex}} \to \hat{\mathcal{T}}_+^{\mathrm{ex}}$, which we call the "positive twisted antipode", such that $\tilde{\mathcal{A}}_+^{\mathrm{ex}} X_i = -X_i$ and furthermore for all $\mathcal{I}_k^{\mathfrak{t}}(\tau) \in \mathcal{T}_+^{\mathrm{ex}}$*

$$\tilde{\mathcal{A}}_+^{\mathrm{ex}} \mathcal{I}_k^{\mathfrak{t}}(\tau) = -\sum_{\ell \in \mathbf{N}^d} \frac{(-X)^\ell}{\ell!} P_+ \hat{\mathcal{M}}_+^{\mathrm{ex}} (\mathcal{I}_{k+\ell}^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^+ \tau \,, \tag{6.4}$$

where $\mathcal{I}_k^{\mathfrak{t}} : \mathcal{T}^{\mathrm{ex}} \to \hat{\mathcal{T}}_+^{\mathrm{ex}}$ is defined in (4.12)*, similarly to above $\hat{\mathcal{M}}_+^{\mathrm{ex}}$ is the product in $\hat{\mathcal{T}}_+^{\mathrm{ex}}$ and $\Delta_{\mathrm{ex}}^+ : \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}} \otimes \mathcal{T}_+^{\mathrm{ex}}$ is as in Corollary 5.32.*

Proof. Proceeding by induction over the number of edges appearing in τ, one easily verifies that such a map exists and is uniquely determined by the above properties. □

Comparing this to the recursion for $\mathcal{A}_+^{\mathrm{ex}}$ given in (6.2), we see that they are very similar, but the projection $\mathfrak{p}_+^{\mathrm{ex}}$ in (6.2) is *inside* the multiplication $\mathcal{M}_+^{\mathrm{ex}}$, while P_+ in (6.4) is *outside* $\hat{\mathcal{M}}_+^{\mathrm{ex}}$.

We recall now that the antipode $\mathcal{A}_+^{\mathrm{ex}}$ is characterised among algebra-morphisms of $\mathcal{T}_+^{\mathrm{ex}}$ by the identity

$$\mathcal{M}_+^{\mathrm{ex}} (\mathrm{id} \otimes \mathcal{A}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^+ = \mathbf{1}_2 \mathbf{1}_2^\star \qquad \text{on} \quad \mathcal{T}_+^{\mathrm{ex}} \,, \tag{6.5}$$

where $\Delta_{\mathrm{ex}}^+ : \mathcal{T}_+^{\mathrm{ex}} \to \mathcal{T}_+^{\mathrm{ex}} \otimes \mathcal{T}_+^{\mathrm{ex}}$ is as in Corollary 5.32. The following result shows that $\tilde{\mathcal{A}}_+^{\mathrm{ex}}$ satisfies a property close to (6.5), which is where the name "twisted antipode" comes from.

Proposition 6.3 *The map $\tilde{\mathcal{A}}_+^{\mathrm{ex}} : \mathcal{T}_+^{\mathrm{ex}} \to \hat{\mathcal{T}}_+^{\mathrm{ex}}$ satisfies the equation*

$$\hat{\mathcal{M}}_+^{\mathrm{ex}} (\mathrm{id} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^+ \mathfrak{i}_+^{\mathrm{ex}} = \mathbf{1}_2 \mathbf{1}_2^\star \qquad \textit{on} \quad \mathcal{T}_+^{\mathrm{ex}} \,, \tag{6.6}$$

where $\Delta_{\mathrm{ex}}^+ : \hat{\mathcal{T}}_+^{\mathrm{ex}} \to \hat{\mathcal{T}}_+^{\mathrm{ex}} \otimes \mathcal{T}_+^{\mathrm{ex}}$ is as in Corollary 5.32.

Proof. Since both sides of (6.6) are multiplicative and since the identity obviously holds when applied to elements of the type X^k, we only need to verify that the left hand side vanishes when applied to elements of the form $\mathcal{I}_k^{\mathfrak{t}}(\tau)$ for some $\tau \in \mathcal{T}^{\mathrm{ex}}$ with $|\tau|_+ + |\mathfrak{t}|_{\mathfrak{s}} - |k|_{\mathfrak{s}} > 0$, and then use Remark 4.15. Similarly to the proof of (4.16), we have

$$\begin{aligned}
&\hat{\mathcal{M}}_+^{\mathrm{ex}}(\mathrm{id} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^+ \mathcal{I}_k^{\mathfrak{t}}(\tau) = \\
&= \hat{\mathcal{M}}_+^{\mathrm{ex}}(\mathrm{id} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \left[\big(\mathcal{I}_k^{\mathfrak{t}} \otimes \mathrm{id}\big)\Delta_{\mathrm{ex}}^+ \tau + \sum_\ell \frac{X^\ell}{\ell!} \otimes \mathfrak{p}_+^{\mathrm{ex}} \mathcal{I}_{k+\ell}^{\mathfrak{t}}(\tau) \right] \\
&= \hat{\mathcal{M}}_+^{\mathrm{ex}} \left[\big(\mathcal{I}_k^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}\big) - \sum_{\ell, i : |\ell|_{\mathfrak{s}} \le |\mathcal{I}_k^{\mathfrak{t}}(\tau)|_+} \frac{X^\ell}{\ell!} \otimes \frac{(-X)^i}{i!} P_+ \hat{\mathcal{M}}_+^{\mathrm{ex}} (\mathcal{I}_{k+\ell+i}^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \right] \Delta_{\mathrm{ex}}^+ \tau \\
&= \Big[\hat{\mathcal{M}}_+^{\mathrm{ex}} \big(\mathcal{I}_k^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}\big) - P_+ \hat{\mathcal{M}}_+^{\mathrm{ex}} (\mathcal{I}_k^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \Big] \Delta_{\mathrm{ex}}^+ \tau = 0 \,,
\end{aligned}$$

since $|\hat{\mathcal{M}}_+^{\mathrm{ex}}(\mathcal{I}_k^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^+ \tau|_+ = |\mathcal{I}_k^{\mathfrak{t}}(\tau)|_+ > 0$. □

A very useful property of the positive twisted antipode $\tilde{\mathcal{A}}_+^{\mathrm{ex}}$ is that its action is intertwined with that of Δ_{ex}^- in the following way.

Lemma 6.4 *The identity*

$$\Delta_{\mathrm{ex}}^- \tilde{\mathcal{A}}_+^{\mathrm{ex}} = (\mathrm{id} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^-$$

holds between linear maps from $\mathcal{T}_+^{\mathrm{ex}}$ *to* $\mathcal{T}_-^{\mathrm{ex}} \otimes \hat{\mathcal{T}}_+^{\mathrm{ex}}$.

Proof. Since both sides of the identity are multiplicative, by using Remark 4.15 it is enough to prove the result on X_i and on elements of the form $\mathcal{I}_k(\tau) \in \mathcal{T}_+^{\mathrm{ex}}$. The identity clearly holds on the linear span of X^k since Δ_{ex}^- acts trivially on them and $\tilde{\mathcal{A}}_+^{\mathrm{ex}}$ preserves that subspace.

Using the recursion (6.4) for $\tilde{\mathcal{A}}_+^{\mathrm{ex}}$, the identity $\Delta_{\mathrm{ex}}^- P_+ = (\mathrm{id} \otimes P_+)\Delta_{\mathrm{ex}}^-$ on $\hat{\mathcal{T}}_+^{\mathrm{ex}}$, followed by the fact that Δ_{ex}^- is multiplicative, we obtain

$$\begin{aligned}
\Delta_{\mathrm{ex}}^- \tilde{\mathcal{A}}_+^{\mathrm{ex}} \mathcal{I}_k^{\mathfrak{t}}(\tau) &= -\sum_{\ell \in \mathbf{N}^d} \Big(\mathrm{id} \otimes \frac{(-X)^\ell}{\ell!}\Big) \Delta_{\mathrm{ex}}^- P_+ \hat{\mathcal{M}}_+^{\mathrm{ex}} (\mathcal{I}_{k+\ell}^{\mathfrak{t}} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^+ \tau \\
&= -\sum_{\ell \in \mathbf{N}^d} \Big(\mathrm{id} \otimes \frac{(-X)^\ell}{\ell!} P_+ \hat{\mathcal{M}}_+^{\mathrm{ex}}\Big) \mathcal{M}^{(13)(2)(4)} (\Delta_{\mathrm{ex}}^- \mathcal{I}_{k+\ell}^{\mathfrak{t}} \otimes \Delta_{\mathrm{ex}}^- \tilde{\mathcal{A}}_+^{\mathrm{ex}}) \Delta_{\mathrm{ex}}^+ \tau \,.
\end{aligned}$$

Using the fact that $\Delta_{\mathrm{ex}}^- \mathcal{I}_k^{\mathfrak{t}} = \big(\mathrm{id} \otimes \mathcal{I}_k^{\mathfrak{t}}\big)\Delta_{\mathrm{ex}}^-$, as well as (5.23), we have

$$\begin{aligned}
\Delta_{\mathrm{ex}}^- \tilde{\mathcal{A}}_+^{\mathrm{ex}} \mathcal{I}_k^{\mathfrak{t}}(\tau) = -\sum_{\ell \in \mathbf{N}^d} &\Big(\mathrm{id} \otimes \frac{(-X)^\ell}{\ell!} P_+ \hat{\mathcal{M}}_+^{\mathrm{ex}}\Big) \\
&\times \mathcal{M}^{(13)(2)(4)} ((\mathrm{id} \otimes \mathcal{I}_{k+\ell}^{\mathfrak{t}})\Delta_{\mathrm{ex}}^- \otimes (\mathrm{id} \otimes \tilde{\mathcal{A}}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^-)\Delta_{\mathrm{ex}}^+ \tau
\end{aligned}$$

$$= -\sum_{\ell\in\mathbf{N}^d}\Big(\mathrm{id}\otimes\frac{(-X)^\ell}{\ell!}P_+\hat{\mathscr{M}}_+^{\mathrm{ex}}(\mathcal{I}_{k+\ell}^{\mathrm{t}}\otimes\tilde{\mathscr{A}}_+^{\mathrm{ex}})\Big)(\mathrm{id}\otimes\Delta_{\mathrm{ex}}^+)\Delta_{\mathrm{ex}}^-\tau$$
$$= (\mathrm{id}\otimes\tilde{\mathscr{A}}_+^{\mathrm{ex}}\mathcal{I}_k^{\mathrm{t}})\Delta_{\mathrm{ex}}^-\tau = (\mathrm{id}\otimes\tilde{\mathscr{A}}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^-\mathcal{I}_k^{\mathrm{t}}\tau\ .$$

Here, the passage from the penultimate to the last line crucially relies on the fact that the action of $\mathcal{G}_{\mathrm{ex}}^-$ onto $\mathcal{T}_+^{\mathrm{ex}}$ preserves the $|\cdot|_+$-degree, i.e. on the second formula in (5.22). □

We have now a similar construction of a *negative twisted antipode*.

Proposition 6.5 *There exists a unique algebra morphism $\tilde{\mathscr{A}}_-^{\mathrm{ex}}\colon\mathcal{T}_-^{\mathrm{ex}}\to\hat{\mathcal{T}}_-^{\mathrm{ex}}$, that we call the "negative twisted antipode", such that for $\tau\in\mathcal{T}_-^{\mathrm{ex}}\cap\ker\mathbf{1}_1^\star$*

$$\tilde{\mathscr{A}}_-^{\mathrm{ex}}\tau = -\hat{\mathscr{M}}_-^{\mathrm{ex}}(\tilde{\mathscr{A}}_-^{\mathrm{ex}}\otimes\mathrm{id})(\Delta_{\mathrm{ex}}^-\mathrm{i}_-^{\mathrm{ex}}\tau-\tau\otimes\mathbf{1}_1). \tag{6.7}$$

Similarly to (6.6)*, the morphism $\tilde{\mathscr{A}}_-^{\mathrm{ex}}\colon\mathcal{T}_-^{\mathrm{ex}}\to\hat{\mathcal{T}}_-^{\mathrm{ex}}$ satisfies*

$$\hat{\mathscr{M}}_-^{\mathrm{ex}}(\tilde{\mathscr{A}}_-^{\mathrm{ex}}\otimes\mathrm{id})\Delta_{\mathrm{ex}}^-\mathrm{i}_-^{\mathrm{ex}} = \mathbf{1}_1\mathbf{1}_1^\star \qquad \textit{on}\quad \mathcal{T}_-^{\mathrm{ex}}\ , \tag{6.8}$$

where $\Delta_{\mathrm{ex}}^-\colon\hat{\mathcal{T}}_-^{\mathrm{ex}}\to\mathcal{T}_-^{\mathrm{ex}}\otimes\hat{\mathcal{T}}_-^{\mathrm{ex}}$ is as in Corollary 5.32.

Proof. Proceeding by induction over the number of colourless edges appearing in τ, one easily verifies that such a morphism exists and is uniquely determined by (6.7). The property (6.8) is a trivial consequence of (6.7). □

6.2 Models

We now recall (a simplified version of) the definition of a model for a regularity structure given in [Hai14, Def. 2.17]. Given a scaling $\mathfrak{s}$ as in Definition 5.2 and interpreting our constant $d\in\mathbf{N}$ as a space(-time) dimension, we define a metric $d_{\mathfrak{s}}$ on $\mathbf{R}^d$ by

$$\|x-y\|_{\mathfrak{s}} \stackrel{\mathrm{def}}{=} \sum_{i=1}^d |x_i-y_i|^{1/\mathfrak{s}_i}\ . \tag{6.9}$$

Note that $\|\cdot\|_{\mathfrak{s}}$ is not a norm since it is not 1-homogeneous, but it is still a distance function since $\mathfrak{s}_i\geq 1$. It is also homogeneous with respect to the (inhomogeneous) scaling in which the ith component is multiplied by $\lambda^{\mathfrak{s}_i}$.

Definition 6.6 A *smooth model* for a given regularity structure $\mathscr{T}=(A,T,G)$ on $\mathbf{R}^d$ with scaling $\mathfrak{s}$ consists of the following elements:

- A map $\Gamma\colon\mathbf{R}^d\times\mathbf{R}^d\to G$ such that $\Gamma_{xx}=\mathrm{id}$, the identity operator, and such that $\Gamma_{xy}\Gamma_{yz}=\Gamma_{xz}$ for every x,y,z in $\mathbf{R}^d$.
- A collection of continuous linear maps $\Pi_x\colon T\to\mathscr{C}^\infty(\mathbf{R}^d)$ such that $\Pi_y=\Pi_x\circ\Gamma_{xy}$ for every $x,y\in\mathbf{R}^d$.

Furthermore, for every $\ell \in A$ and every compact set $\mathfrak{K} \subset \mathbf{R}^d$, we assume the existence of a constant $C_{\ell,\mathfrak{K}}$ such that the bounds

$$|\Pi_x\tau(y)| \le C_{\ell,\mathfrak{K}}\|\tau\|_\ell \, \|x-y\|_{\mathfrak{s}}^\ell, \qquad \|\Gamma_{xy}\tau\|_m \le C_{\ell,\mathfrak{K}}\|\tau\|_\ell \, \|x-y\|_{\mathfrak{s}}^{\ell-m} , \quad (6.10)$$

hold uniformly over all $x, y \in \mathfrak{K}$, all $m \in A$ with $m < \ell$ and all $\tau \in T_\ell$.

Here, recalling that the space T in Definitions 5.1 and 6.6 is a direct sum of Banach spaces $(T_\alpha)_{\alpha\in A}$, the quantity $\|\sigma\|_m$ appearing in (6.10) denotes the norm of the component of $\sigma \in T$ in the Banach space T_m for $m \in A$. We also note that Definition 6.6 does not include the general framework of [Hai14, Definition 2.17], where Π_x takes values in $\mathscr{D}'(\mathbf{R}^d)$ rather than $\mathscr{C}^\infty(\mathbf{R}^d)$; however this simplified setting is sufficient for our purposes, at least for now. The condition (6.10) on Π_x is of course relevant only for $\ell > 0$ since $\Pi_x\tau(\cdot)$ is assumed to be a smooth function at this stage.

As in Section 5.6, we consider a label set of the type $\mathfrak{L} = \mathfrak{L}_- \sqcup \mathfrak{L}_+$. We fix a collection of kernels $\{K_{\mathfrak{t}}\}_{\mathfrak{t}\in\mathfrak{L}_+}$, $K_{\mathfrak{t}} : \mathbf{R}^d \setminus \{0\} \to \mathbf{R}$, satisfying the conditions of [Hai14, Ass. 5.1] with $\beta = |\mathfrak{t}|_{\mathfrak{s}}$. We use extensively the notations of Section 4.3.

Definition 6.7 Given a linear map $\mathbf{\Pi} \colon \mathcal{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$, we define for all $z, \bar z \in \mathbf{R}^d$

- a character $g_z^+(\mathbf{\Pi}) \colon \hat{\mathcal{T}}_+^{\mathrm{ex}} \to \mathbf{R}$ by extending multiplicatively

$$g_z^+(\mathbf{\Pi})X_i = (\mathbf{\Pi}X_i)(z), \qquad g_z^+(\mathbf{\Pi})\mathcal{I}_k^{\mathfrak{t}}(\tau) = (D^k K_{\mathfrak{t}} * \mathbf{\Pi}\tau)(z)$$

  for $\mathfrak{t} \in \mathfrak{L}_+$ and setting $g_z^+(\mathbf{\Pi})\mathcal{I}_k^{\mathfrak{l}}(\tau) = 0$ for $\mathfrak{l} \in \mathfrak{L}_-$.
- a linear map $\Pi_z \colon \mathcal{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$ and a character $f_z \in \mathcal{G}_+^{\mathrm{ex}}$ by

$$\Pi_z = (\mathbf{\Pi} \otimes f_z)\Delta_{\mathrm{ex}}^+ , \qquad f_z = g_z^+(\mathbf{\Pi})\tilde{\mathcal{A}}_+^{\mathrm{ex}} , \quad (6.11)$$

  where $\tilde{\mathcal{A}}_+^{\mathrm{ex}}$ is the positive twisted antipode defined in (6.4)
- a linear map $\Gamma_{z\bar z} \colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}}$ and a character $\gamma_{z\bar z} \in \mathcal{G}_+^{\mathrm{ex}}$ by

$$\Gamma_{z\bar z} = (\mathrm{id} \otimes \gamma_{z\bar z})\Delta_{\mathrm{ex}}^+, \quad \gamma_{z\bar z} = (f_z \mathcal{A}_+^{\mathrm{ex}} \otimes f_{\bar z})\Delta_{\mathrm{ex}}^+ . \quad (6.12)$$

Finally, we write $\mathfrak{Z}^{\mathrm{ex}} \colon \mathbf{\Pi} \mapsto (\Pi, \Gamma)$ for the map given by (6.11) and (6.12).

We do not want to consider arbitrary maps $\mathbf{\Pi}$ as above, but we want them to behave in a "nice" way with respect to the natural operations we have on $\mathcal{T}^{\mathrm{ex}}$. We therefore introduce the following notion of admissibility. For this, we note that since the rule R used to construct our structure is normal in the sense of Definition 5.7, the only basis vectors of the type $\mathcal{I}_k^{\mathfrak{t}}(\tau)$ with $\mathfrak{t} \in \mathfrak{L}_-$ belonging to $\mathcal{T}^{\mathrm{ex}}$ are those with $\tau = X^\ell$ for some $\ell \in \mathbf{N}^d$, so we give them a special name by setting $\Xi_{k,\ell}^{\mathfrak{l}} = \mathcal{I}_k^{\mathfrak{l}}(X^\ell)$ and $\Xi^{\mathfrak{l}} = \Xi_{0,0}^{\mathfrak{l}}$.

Definition 6.8 Given a linear map $\mathbf{\Pi} \colon \mathcal{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$, we set $\xi_{\mathfrak{l}} \stackrel{\mathrm{def}}{=} \mathbf{\Pi}\Xi^{\mathfrak{l}}$ for $\mathfrak{l} \in \mathfrak{L}_-$. We then say that $\mathbf{\Pi}$ is *admissible* if it satisfies

$$\begin{aligned} \mathbf{\Pi}\mathbf{1} &= 1 , & \mathbf{\Pi}X^k\tau &= x^k\mathbf{\Pi}\tau , \\ \mathbf{\Pi}\mathcal{I}_k^{\mathfrak{t}}(\tau) &= D^k K_{\mathfrak{t}} * \mathbf{\Pi}\tau , & \mathbf{\Pi}\Xi_{k,\ell}^{\mathfrak{l}} &= D^k(x^\ell \xi_{\mathfrak{l}}) , \end{aligned} \quad (6.13)$$

for all $\tau \in \mathcal{T}^{\mathrm{ex}}$, $k, \ell \in \mathbf{N}^d$, $\mathfrak{t} \in \mathfrak{L}_+$, $\mathfrak{l} \in \mathfrak{L}_-$, where $\mathcal{I}_k^{\mathfrak{t}} : \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}}$ is defined by (4.11), $*$ is the distributional convolution in $\mathbf{R}^d$, and we use the notation

$$D^k = \prod_{i=1}^d \frac{\partial^{k_i}}{\partial y_i^{k_i}}, \qquad x^k : \mathbf{R}^d \to \mathbf{R}, \quad x^k(y) \stackrel{\mathrm{def}}{=} \prod_{i=1}^d y_i^{k_i}.$$

Note that this definition guarantees that the identity $\mathbf{\Pi}\mathcal{I}_k^{\mathfrak{t}}(\tau) = D^k \mathbf{\Pi}\mathcal{I}_0^{\mathfrak{t}}(\tau)$ always holds, whether $\mathfrak{t}$ is in $\mathfrak{L}_-$ or in $\mathfrak{L}_+$.

It is then simple to check that, with these definitions, $\Pi_z \Gamma_{z\bar{z}} = \Pi_{\bar{z}}$ and (Π, Γ) satisfies the algebraic requirements of Definition 6.6. However, (Π, Γ) does not necessarily satisfy the analytical bounds (6.10), although one has the following.

Lemma 6.9 *If $\mathbf{\Pi}$ is admissible then, for every $\mathcal{I}_k^{\mathfrak{t}}(\tau) \in \mathcal{T}^{\mathrm{ex}}$ with $\mathfrak{t} \in \mathfrak{L}_+$, we have*

$$f_z(\mathcal{I}_k^{\mathfrak{t}}(\tau)) = - \sum_{|\ell|_{\mathfrak{s}} < |\mathcal{I}_k^{\mathfrak{t}}(\tau)|_+} \frac{(-z)^\ell}{\ell!} (D^{k+\ell} K_{\mathfrak{t}} * \Pi_z \tau)(z)\,,$$

$$\big(\Pi_z \mathcal{I}_k^{\mathfrak{t}}(\tau)\big)(\bar{z}) = (D^k K_{\mathfrak{t}} * \Pi_z \tau)(\bar{z}) - \sum_{|\ell|_{\mathfrak{s}} < |\mathcal{I}_k^{\mathfrak{t}}(\tau)|_+} \frac{(\bar{z} - z)^\ell}{\ell!} (D^{k+\ell} K_{\mathfrak{t}} * \Pi_z \tau)(z)\,.$$

Proof. In view of the identifications (5.27), this is essentially the same as [Hai14, Eqns (8.19)–(8.20)].

Alternatively, it follows immediately from (4.15) and the admissibility of $\mathbf{\Pi}$ that $\Pi_z \mathcal{I}_k^{\mathfrak{t}}(\tau) - D^k K_{\mathfrak{t}} * \Pi_z \tau$ is a polynomial of degree $|\mathcal{I}_k^{\mathfrak{t}}(\tau)|_+$. On the other hand, it follows from (6.6) that $\Pi_z \mathcal{I}_k^{\mathfrak{t}}(\tau)$ and its derivatives up to the required order (because taking derivatives commutes with the action of the structure group) vanish at z, so there is no choice of what that polynomial is, thus yielding the second identity. The first identity then follows by comparing the second formula to (6.11). □

Remark 6.10 Lemma 6.9 shows that the positive twisted antipode $\tilde{\mathcal{A}}_+^{\mathrm{ex}}$ is intimately related to Taylor remainders, see (6.11).

Lemma 6.9 shows that (Π, Γ) satisfies the analytical property (6.10) on elementary trees of the form $\mathcal{I}_k^{\mathfrak{t}}(\tau) \in \mathcal{T}^{\mathrm{ex}}$. However this is not necessarily the case for products of such trees, since neither $\mathbf{\Pi}$ nor Π_z are assumed to be multiplicative under the tree product (4.8). If, however, we also assume that $\mathbf{\Pi}$ is multiplicative, then the map $\mathfrak{Z}^{\mathrm{ex}}$ always produces a *bona fide* model.

Proposition 6.11 *If $\mathbf{\Pi} \colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ is admissible and such that, for all $\tau, \bar{\tau} \in \mathcal{T}^{\mathrm{ex}}$ with $\tau\bar{\tau} \in \mathcal{T}^{\mathrm{ex}}$ and all $\alpha \in \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$, we have*

$$\mathbf{\Pi}(\tau\bar{\tau}) = (\mathbf{\Pi}\tau) \cdot (\mathbf{\Pi}\bar{\tau})\,, \qquad \mathbf{\Pi}\mathcal{R}_\alpha(\tau) = \mathbf{\Pi}\tau\,, \tag{6.14}$$

then $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi})$ is a model for $\mathcal{T}^{\mathrm{ex}}$.

Proof. The proof of the algebraic properties follows immediately from (6.12). Regarding the analytical bound (6.10) on $\Pi_z\sigma$, it immediately follows from Lemma 6.9 in the case when σ is of the form $\mathcal{I}_k^{\mathfrak{l}}(\tau)$. For products of such elements, it follows immediately from the multiplicative property of $\mathbf{\Pi}$ combined with the multiplicativity of the action of Δ_{ex}^+ on $\mathcal{T}^{\mathrm{ex}}$, which imply that

$$\Pi_x(\sigma\bar\sigma) = (\Pi_x\sigma)\cdot(\Pi_x\bar\sigma)\;.$$

Regarding vectors of the type $\sigma = \mathcal{R}_\alpha(\tau)$, it follows immediately from the last identity in (4.14) combined with (6.14) that $\Pi_x\mathcal{R}_\alpha(\tau) = \Pi_x\tau$.

The proof of the second bound in (6.10) for Γ_{xy} is virtually identical to the one given in [Hai14, Proposition 8.27], combined with Lemma 6.9. Formally, the main difference comes from the change of basis mentioned in Section 5.7, but this does not affect the relevant bounds since it does not mix basis vectors of different $|\cdot|_+$-degree. □

Remark 6.12 If a map $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ is admissible and furthermore satisfies (6.14), then it is uniquely determined by the functions $\xi_{\mathfrak{l}} \stackrel{\text{def}}{=} \mathbf{\Pi}\Xi^{\mathfrak{l}}$ for $\mathfrak{l} \in \mathfrak{L}_-$. In this case, we call $\mathbf{\Pi}$ the *canonical lift* of the functions $\xi_{\mathfrak{l}}$.

6.3 Renormalised Models

We now use the structure built in this article to provide a large class of renormalisation procedures, which in particular includes those used in [Hai14, HP15, HS15]. For this, we first need a topology on the space of all models for a given regularity structure. Given two smooth models (Π,Γ) and $(\bar\Pi,\bar\Gamma)$, for all $\ell \in A$ and $\mathfrak{K} \subset \mathbf{R}^d$ a compact set, we define the pseudo-metrics

$$|\!|\!|(\Pi,\Gamma);(\bar\Pi,\bar\Gamma)|\!|\!|_{\ell;\mathfrak{K}} \stackrel{\text{def}}{=} \|\Pi - \bar\Pi\|_{\ell;\mathfrak{K}} + \|\Gamma - \bar\Gamma\|_{\ell;\mathfrak{K}}\;, \tag{6.15}$$

where

$$\|\Pi - \bar\Pi\|_{\ell;\mathfrak{K}} \stackrel{\text{def}}{=} \sup\left\{\frac{|\langle(\Pi_x - \bar\Pi_x)\tau, \varphi_x^\lambda\rangle|}{\|\tau\|\,\lambda^\ell} :\; x\in\mathfrak{K}, |\tau|_+ = \ell, \lambda\in(0,1], \varphi\in\mathcal{B}\right\},$$
$$\|\Gamma - \bar\Gamma\|_{\ell;\mathfrak{K}} \stackrel{\text{def}}{=} \sup\left\{\frac{\|\Gamma_{xy}\tau - \bar\Gamma_{xy}\tau\|_m}{\|\tau\|\,\|x-y\|_{\mathfrak{s}}^{m-\ell}} : x,y\in\mathfrak{K}, x\neq y, |\tau|_+ = \ell, m<\ell\right\}.$$

Here, the set $\mathcal{B} \in \mathcal{C}_0^\infty(\mathbf{R}^d)$ denotes the set of test functions with support in the centred ball of radius one and all derivatives up to oder $1 + |\inf A|$ bounded by 1. Given $\varphi\in\mathcal{B}$, $\varphi_x^\lambda : \mathbf{R}^d \to \mathbf{R}$ denotes the translated and rescaled function

$$\varphi_x^\lambda(y) \stackrel{\text{def}}{=} \lambda^{-(\mathfrak{s}_1+\cdots+\mathfrak{s}_d)}\,\varphi\Big((y_i - x_i)\lambda^{-\mathfrak{s}_i})_{i=1}^d\Big), \qquad y\in\mathbf{R}^d,$$

for $x\in\mathbf{R}^d$ and $\lambda>0$ as in [Hai14]. Finally, $\langle\cdot,\cdot\rangle$ is the usual L^2 scalar product.

Definition 6.13 We denote by $\mathscr{M}_\infty^{\mathrm{ex}}$ the space of all smooth models of the form $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi})$ for some admissible linear map $\mathbf{\Pi}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$ in the sense of Definition 6.8. We endow $\mathscr{M}_\infty^{\mathrm{ex}}$ with the system of pseudo-metrics $(|\!|\!|\cdot;\cdot|\!|\!|_{\ell;\mathfrak{K}})_{\ell;\mathfrak{K}}$ and we denote by $\mathscr{M}_0^{\mathrm{ex}}$ the completion of this metric space.

We refer to [Hai14, Def. 2.17] for the definition of the space $\mathscr{M}^{\mathrm{ex}}$ of models of a fixed regularity structure. With that definition, $\mathscr{M}_0^{\mathrm{ex}}$ is nothing but the closure of $\mathscr{M}_\infty^{\mathrm{ex}}$ in $\mathscr{M}^{\mathrm{ex}}$.

In many singular SPDEs, one is naturally led to a sequence of models $\mathscr{Z}(\mathbf{\Pi}^{(\varepsilon)})$ which do not converge as $\varepsilon \to 0$. One would then like to be able to "tweak" this model in such a way that it remains an admissible model but has a chance of converging as $\varepsilon \to 0$. A natural way of "tweaking" $\mathbf{\Pi}^{(\varepsilon)}$ is to compose it with some linear map $M^{\mathrm{ex}}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{T}^{\mathrm{ex}}$. This naturally leads to the following question: what are the linear maps M^{ex} which are such that if $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi})$ is an admissible model, then $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}M^{\mathrm{ex}})$ is also a model? We then give the following definition.

Definition 6.14 A linear map $M\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{T}^{\mathrm{ex}}$ is an *admissible renormalisation procedure* if

- for every admissible $\mathbf{\Pi}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$ such that $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}) \in \mathscr{M}_\infty^{\mathrm{ex}}$, $\mathbf{\Pi}M$ is admissible and $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}M) \in \mathscr{M}_\infty^{\mathrm{ex}}$
- the map $\mathscr{M}_\infty^{\mathrm{ex}} \ni \mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}) \mapsto \mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}M) \in \mathscr{M}_\infty^{\mathrm{ex}}$ extends to a continuous map from $\mathscr{M}_0^{\mathrm{ex}}$ to $\mathscr{M}_0^{\mathrm{ex}}$.

Recall that we have a right action of $\mathscr{G}_-^{\mathrm{ex}}$ onto $\mathscr{T}^{\mathrm{ex}}$ by $g \mapsto M_g^{\mathrm{ex}}$ with

$$M_g^{\mathrm{ex}}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{T}^{\mathrm{ex}}, \qquad M_g^{\mathrm{ex}}\tau = (g \otimes \mathrm{id})\Delta_{\mathrm{ex}}^-\tau. \tag{6.16}$$

Since Δ_{ex}^- can also be viewed as a map on $\mathscr{T}_+^{\mathrm{ex}}$ and $\mathscr{T}_-^{\mathrm{ex}}$ by Corollary 5.32, we will also view M_g^{ex} as a map on these spaces by formally using the same definition (6.16). The following is one of the main results of this article.

Theorem 6.15 *For every $g \in \mathscr{G}_-^{\mathrm{ex}}$, the map $M_g^{\mathrm{ex}}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{T}^{\mathrm{ex}}$ is an admissible renormalisation procedure. Moreover the renormalised model $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}M_g^{\mathrm{ex}}) = (\Pi^g, \Gamma^g)$ is described by:*

$$\Pi_z^g = \Pi_z M_g^{\mathrm{ex}}, \quad \gamma_{z\bar{z}}^g = \gamma_{z\bar{z}} M_g^{\mathrm{ex}}\,. \tag{6.17}$$

Proof. Let us fix $g \in \mathscr{G}_-^{\mathrm{ex}}$ and an admissible linear map $\mathbf{\Pi}$ such that $\mathscr{Z}^{\mathrm{ex}}(\mathbf{\Pi}) = (\Pi, \Gamma)$ is a model and set $\mathbf{\Pi}^g \stackrel{\mathrm{def}}{=} \mathbf{\Pi}M_g^{\mathrm{ex}}$. We check first that $\mathbf{\Pi}^g$ is admissible, namely that it satisfies (6.13). First, we note that, in the sum over A in (3.3) defining $\Delta_1\mathcal{I}_k^{\mathfrak{t}}(\tau)$, we have two mutually excluding possibilities:

- A is a subforest of τ
- A contains the edge of type $\mathfrak{t}$ added by the operator $\mathcal{I}_k^{\mathfrak{t}}$ or the root of $\mathcal{I}_k^{\mathfrak{t}}(\tau)$ as an isolated node (which has however positive degree and is therefore killed by the projection $\mathfrak{p}_-^{\mathrm{ex}}$ in Δ_{ex}^-).

Therefore we have

$$(g \otimes \mathrm{id})\Delta^-_{\mathrm{ex}}\mathcal{I}^{\mathfrak{t}}_k(\tau) = (g\mathfrak{p}^{\mathrm{ex}}_- \otimes \mathcal{I}^{\mathfrak{t}}_k)\Delta_1\tau + (g\mathfrak{p}^{\mathrm{ex}}_-\mathcal{I}^{\mathfrak{t}}_k \otimes \mathrm{id})\Delta_1\tau = (g \otimes \mathcal{I}^{\mathfrak{t}}_k)\Delta^-_{\mathrm{ex}}\tau$$

since $\mathfrak{p}^{\mathrm{ex}}_-\mathcal{I}^{\mathfrak{t}}_k = 0$ by the definition (5.20) of $\mathcal{I}_+$. Therefore

$$\begin{aligned}\mathbf{\Pi}^g\mathcal{I}^{\mathfrak{t}}_k(\tau) &= (g \otimes \mathbf{\Pi})\Delta^-_{\mathrm{ex}}\mathcal{I}^{\mathfrak{t}}_k(\tau) = (g \otimes \mathbf{\Pi}\mathcal{I}^{\mathfrak{t}}_k)\Delta^-_{\mathrm{ex}}\tau\\ &= (g \otimes D^k K_{\mathfrak{t}} * \mathbf{\Pi})\Delta^-_{\mathrm{ex}}\tau = D^k K_{\mathfrak{t}} * \mathbf{\Pi}^g\tau.\end{aligned}$$

Since X^k has positive degree, with a similar computation we obtain

$$\begin{aligned}\mathbf{\Pi}^g X^k\tau &= (g \otimes \mathbf{\Pi})\Delta^-_{\mathrm{ex}}X^k\tau = (g \otimes \mathbf{\Pi}X^k)\Delta^-_{\mathrm{ex}}\tau\\ &= (g \otimes x^k\mathbf{\Pi})\Delta^-_{\mathrm{ex}}\tau = x^k\mathbf{\Pi}^g\tau\end{aligned}$$

and this shows that $\mathbf{\Pi}^g$ is admissible.

Now we verify that, writing M^{ex}_g as before and $\mathfrak{X}^{\mathrm{ex}}(\mathbf{\Pi}^g) = (\Pi^g, \Gamma^g)$, we have

$$\gamma^g_{z\bar z} = (g \otimes \gamma_{z\bar z})\Delta^-_{\mathrm{ex}}\,, \qquad \Pi^g_z = (g \otimes \Pi_z)\Delta^-_{\mathrm{ex}}\,.$$

To show this, one first uses (6.3) to show that $f^g_z = (g \otimes f_z)\Delta^-_{\mathrm{ex}}$, where f and f^g are defined from $\mathbf{\Pi}$ and $\mathbf{\Pi}^g$ as in (6.11). Indeed, one has

$$\begin{aligned}f^g_z &= g^+_z(\mathbf{\Pi}M^{\mathrm{ex}}_g)\tilde{\mathcal{A}}^{\mathrm{ex}}_+ = \big(g \otimes g^+_z(\mathbf{\Pi})\big)\Delta^-_{\mathrm{ex}}\tilde{\mathcal{A}}^{\mathrm{ex}}_+\\ &= (g \otimes g^+_z(\mathbf{\Pi})\tilde{\mathcal{A}}^{\mathrm{ex}}_+)\Delta^-_{\mathrm{ex}} = (g \otimes f^g_z)\Delta^-_{\mathrm{ex}} = f_z M^{\mathrm{ex}}_g\,.\end{aligned}$$

One then uses (5.23) on $\mathcal{T}^{\mathrm{ex}}$ to show that the required identity (6.17) for Π^g_z holds. Indeed, it follows that

$$\begin{aligned}\Pi^g_z &= (\mathbf{\Pi}^g \otimes f^g_z)\Delta^+_{\mathrm{ex}} = (g \otimes \mathbf{\Pi} \otimes g \otimes f_z)\big(\Delta^-_{\mathrm{ex}} \otimes \Delta^-_{\mathrm{ex}}\big)\Delta^+_{\mathrm{ex}}\\ &= (g \otimes \mathbf{\Pi} \otimes f_z)\big(\mathrm{id} \otimes \Delta^+_{\mathrm{ex}}\big)\Delta^-_{\mathrm{ex}} = (g \otimes \Pi_z)\Delta^-_{\mathrm{ex}}.\end{aligned} \tag{6.18}$$

Regarding $\gamma_{z\bar z}$, we have

$$\begin{aligned}\gamma^g_{z\bar z} &= \big(f^g_z\tilde{\mathcal{A}}^{\mathrm{ex}}_+ \otimes f^g_{\bar z}\big)\Delta^+_{\mathrm{ex}} = \big(f_z M^{\mathrm{ex}}_g\tilde{\mathcal{A}}^{\mathrm{ex}}_+ \otimes f_{\bar z}M^{\mathrm{ex}}_g\big)\Delta^+_{\mathrm{ex}}\\ &= (f_z\tilde{\mathcal{A}}^{\mathrm{ex}}_+ \otimes f_{\bar z})\big(M^{\mathrm{ex}}_g \otimes M^{\mathrm{ex}}_g\big)\Delta^+_{\mathrm{ex}} = (f_z\tilde{\mathcal{A}}^{\mathrm{ex}}_+ \otimes f_{\bar z})\Delta^+_{\mathrm{ex}}M^{\mathrm{ex}}_g\\ &= (g \otimes \gamma_{z\bar z})\Delta^-_{\mathrm{ex}}\,.\end{aligned} \tag{6.19}$$

Note now that, at the level of the character $\gamma_{z\bar z}$, the bound (6.10) reads $|\gamma_{z\bar z}(\tau)| \le \|z - \bar z\|_{\mathfrak{s}}^{|\tau|_+}$ as a consequence of the fact that Δ^+_{ex} preserves the sum of the $|\cdot|_+$-degrees of each factor. On the other hand, for every character g of $\mathcal{T}^{\mathrm{ex}}_-$ and any τ belonging to either $B_\circ$ or B_+ (see Definition 5.26), the element $(g \otimes \mathrm{id})\Delta^-_{\mathrm{ex}}\tau$ is a linear combination of terms with the *same* $|\cdot|_+$-degree as τ. As a consequence, it is immediate that if a given model (Π, Γ) satisfies the bounds (6.10), then the renormalised model (Π^g, Γ^g) satisfies the same bounds, albeit with different constants, depending on g. We conclude that indeed for every admissible $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ such that $\mathfrak{X}^{\mathrm{ex}}(\mathbf{\Pi}) \in \mathscr{M}^{\mathrm{ex}}_\infty$, $\mathbf{\Pi}M^g$ is admissible and $\mathfrak{X}^{\mathrm{ex}}(\mathbf{\Pi}M) \in \mathscr{M}^{\mathrm{ex}}_\infty$.

The exact same argument also shows that if we extend the action of $\mathcal{G}^{\mathrm{ex}}_-$ to all of $\mathscr{M}^{\mathrm{ex}}$ by (6.18) and (6.19), then this yields a continuous action, which in particular leaves $\mathscr{M}^{\mathrm{ex}}_0$ invariant as required by Definition 6.14. □

Note now that the group $\mathbf{R}^d$ acts on admissible (in the sense of Definition 6.8) linear maps $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ in two different ways. First, we have the natural action by translations T_h, $h \in \mathbf{R}^d$ given by

$$(T_h(\mathbf{\Pi})\tau)(z) \stackrel{\mathrm{def}}{=} (\mathbf{\Pi}\tau)(z-h)\ .$$

However, $\mathbf{R}^d$ can also be viewed as a subgroup of $\mathcal{G}_+^{\mathrm{ex}}$ by setting

$$g_h(X_i) = -h_i\ , \qquad g_h(\mathcal{I}_k^{\mathrm{t}}(\tau)) = 0\ . \tag{6.20}$$

This also acts on admissible linear maps by setting

$$(\tilde{T}_h(\mathbf{\Pi})\tau)(z) \stackrel{\mathrm{def}}{=} ((\mathbf{\Pi} \otimes g_h)\Delta_{\mathrm{ex}}^+\tau)(z)\ . \tag{6.21}$$

Note that if $\mathbf{\Pi}$ is admissible, then one has $T_h(\mathbf{\Pi})X^k = \tilde{T}_h(\mathbf{\Pi})X^k$ for every $k \in \mathbf{N}^d$ and every $h \in \mathbf{R}^d$.

Definition 6.16 We say that a *random* linear map $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ is *stationary* if, for every (deterministic) element $h \in \mathbf{R}^d$, the random linear maps $T_h(\mathbf{\Pi})$ and $\tilde{T}_h(\mathbf{\Pi})$ are equal in law. We also assume that $\mathbf{\Pi}$ and its derivatives, computed at 0 have moments of all orders.

Using the the Definition 5.26 and the Remark 4.15, $\hat{\mathcal{T}}_-^{\mathrm{ex}}$ can be identified canonically with the free algebra generated by $B_\circ$. We write

$$\iota_\circ\colon \mathcal{T}^{\mathrm{ex}} = \langle B_\circ \rangle \to \hat{\mathcal{T}}_-^{\mathrm{ex}}$$

for the associated canonical injection.

Every random stationary map $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ in the sense of Definition 6.16 then naturally determines a (deterministic) character $g^-(\mathbf{\Pi})$ of $\hat{\mathcal{T}}_-^{\mathrm{ex}}$ by setting

$$g^-(\mathbf{\Pi})(\iota_\circ\tau) \stackrel{\mathrm{def}}{=} \mathbf{E}(\mathbf{\Pi}\tau)(0)\ ,$$

for $\tau \in B_\circ$, where the symbol $\mathbf{E}$ on the right hand side denotes expectation over the underlying probability space. This is extended multiplicatively to all of $\hat{\mathcal{T}}_-^{\mathrm{ex}}$. Then we can define a *renormalised* map $\hat{\mathbf{\Pi}}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ by

$$\hat{\mathbf{\Pi}}\tau = (g^-(\mathbf{\Pi})\tilde{\mathcal{A}}_-^{\mathrm{ex}} \otimes \mathbf{\Pi})\Delta_{\mathrm{ex}}^-\tau\ , \tag{6.22}$$

where $\tilde{\mathcal{A}}_-^{\mathrm{ex}}\colon \mathcal{T}_-^{\mathrm{ex}} \to \hat{\mathcal{T}}_-^{\mathrm{ex}}$ is the negative twisted antipode defined in (6.7) and satisfying (6.8).

Let us also denote by $B_\circ^-$ the (finite!) set of basis vectors $\tau \in B_\circ$ such that $|\tau|_- < 0$. The specific choice of $g = g^-(\mathbf{\Pi})\tilde{\mathcal{A}}_-^{\mathrm{ex}}$ used to define $\hat{\mathbf{\Pi}}$ is very natural and canonical in the following sense.

Theorem 6.17 *Let $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ be stationary and admissible such that $\mathcal{L}^{\mathrm{ex}}(\mathbf{\Pi})$ is a model in $\mathcal{M}_\infty^{\mathrm{ex}}$. Then, among all random functions $\mathbf{\Pi}^g\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ of the form*

$$\mathbf{\Pi}^g = \mathbf{\Pi} M_g^{\mathrm{ex}} = (g \otimes \mathbf{\Pi})\Delta_{\mathrm{ex}}^- \tau, \qquad g \in \mathcal{G}_-^{\mathrm{ex}},$$

with M_g^{ex} as in (6.16)*, $\hat{\mathbf{\Pi}}$ is the only one such that, for all $h \in \mathbf{R}^d$, we have*

$$\mathbf{E}(\hat{\mathbf{\Pi}}\tau)(h) = 0\,, \qquad \forall\, \tau \in B_\circ^-\,. \tag{6.23}$$

We call $\hat{\mathbf{\Pi}}$ the BPHZ renormalisation *of $\mathbf{\Pi}$.*

Proof. We first show that $\hat{\mathbf{\Pi}}$ does indeed have the desired property. We first consider $h = 0$ and we write $\mathbf{\Pi}_0\colon \mathcal{T}^{\mathrm{ex}} \to \mathbf{R}$ for the map (not to be confused with Π_0)

$$\mathbf{\Pi}_0\tau = \mathbf{E}(\mathbf{\Pi}\tau)(0)\,.$$

Let us denote by $B_\circ^\sharp$ the set of $\tau \in B_\circ^-$ which are *not* of the form $\mathcal{I}_k^{\mathfrak{t}}(\sigma)$ with $|\mathfrak{t}|_{\mathfrak{s}} > 0$. The main point now is that, thanks to the definitions of $g^-(\mathbf{\Pi})$ and Δ_{ex}^-, we have the identity

$$(\mathrm{id} \otimes \mathbf{\Pi}_0)\Delta_{\mathrm{ex}}^- = (\mathrm{id} \otimes g^-(\mathbf{\Pi}))\Delta_{\mathrm{ex}}^- \iota_\circ\,, \qquad \text{on} \quad \mathcal{T}^{\mathrm{ex}}.$$

Combining this with (6.22), we obtain for all $\tau \in B_\circ^\sharp$

$$\begin{aligned}\mathbf{E}(\hat{\mathbf{\Pi}}\tau)(0) &= (g^-(\mathbf{\Pi})\tilde{\mathcal{A}}_-^{\mathrm{ex}} \otimes \mathbf{\Pi}_0)\Delta_{\mathrm{ex}}^-\tau = (g^-(\mathbf{\Pi})\tilde{\mathcal{A}}_-^{\mathrm{ex}} \otimes g^-(\mathbf{\Pi}))\Delta_{\mathrm{ex}}^-\iota_\circ\tau \\ &= g^-(\mathbf{\Pi})\hat{\mathcal{M}}_-^{\mathrm{ex}}(\tilde{\mathcal{A}}_-^{\mathrm{ex}} \otimes \mathrm{id})\Delta_{\mathrm{ex}}^-\iota_\circ\tau = 0\,,\end{aligned}$$

by the defining property (6.8) of the negative twisted antipode, since $\iota_\circ\tau$ belongs both to the image of $\mathfrak{i}_-^{\mathrm{ex}}$ and to the kernel of $\mathbf{1}_1^\star$.

Let now $\tau \in B_\circ^\sharp$ be of the form $\mathcal{I}_k^{\mathfrak{t}}(\sigma)$ with $|\mathfrak{t}|_{\mathfrak{s}} > 0$. Arguing as in the proof of Theorem 6.15 we see that

$$\Delta_{\mathrm{ex}}^-\iota_\circ\mathcal{I}_k^{\mathfrak{t}}(\sigma) = (\mathrm{id} \otimes \iota_\circ\mathcal{I}_k^{\mathfrak{t}})\Delta_{\mathrm{ex}}^-\sigma\,.$$

It then follows that

$$\mathbf{E}(\hat{\mathbf{\Pi}}\tau)(0) = \hat{\mathcal{M}}_-^{\mathrm{ex}}(g^-(\mathbf{\Pi})\tilde{\mathcal{A}}_-^{\mathrm{ex}} \otimes g^-(\mathbf{\Pi})\iota_\circ\mathcal{I}_k^{\mathfrak{t}})\Delta_{\mathrm{ex}}^-\sigma\,.$$

The definition of $g^-(\mathbf{\Pi})$ combined with the fact that $\mathbf{\Pi}$ is admissible and the definition of $\hat{\mathbf{\Pi}}$ now implies that

$$\mathbf{E}(\hat{\mathbf{\Pi}}\tau)(0) = \int_{\mathbf{R}^d} D^k K_{\mathfrak{t}}(-y)\mathbf{E}(\hat{\mathbf{\Pi}}\sigma)(y)\,dy\,,$$

where $D^k K_{\mathfrak{t}}$ should be interpreted in the sense of distributions. In particular, one has

$$\mathbf{E}(\hat{\mathbf{\Pi}}\tau)(0) = (-1)^{|k|}\int_{\mathbf{R}^d} K_{\mathfrak{t}}(-y)D^k\mathbf{E}(\hat{\mathbf{\Pi}}\sigma)(y)\,dy\,. \tag{6.24}$$

For $\sigma = (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ and $\bar{\mathfrak{n}}\colon N_F \to \mathbf{N}^d$ with $\bar{\mathfrak{n}} \le \mathfrak{n}$, we now write $L_{\bar{\mathfrak{n}}}\sigma = (F, \hat{F}, \mathfrak{n} - \bar{\mathfrak{n}}, \mathfrak{o}, \mathfrak{e})$ and we note that for g_h as in (6.20) one has the identity

$$(\mathrm{id} \otimes g_h)\Delta^+_{\mathrm{ex}}\sigma = \sum_{\bar{\mathfrak{n}}} \binom{\mathfrak{n}}{\bar{\mathfrak{n}}} (-h)^{\Sigma\bar{\mathfrak{n}}} L_{\bar{\mathfrak{n}}}\sigma \;,$$

so that the stationarity of $\mathbf{\Pi}$ implies that

$$\mathbf{E}(\hat{\mathbf{\Pi}}\sigma)(y) = \mathbf{E}(\tilde{T}_{-y}\hat{\mathbf{\Pi}}\sigma)(0) = \sum_{\bar{\mathfrak{n}}} \binom{\mathfrak{n}}{\bar{\mathfrak{n}}} y^{\Sigma\bar{\mathfrak{n}}} \mathbf{E}(\hat{\mathbf{\Pi}} L_{\bar{\mathfrak{n}}}\sigma)(0) \;.$$

Plugging this into (6.24), we conclude that the terms for which there exists i with $k_i > (\Sigma\bar{\mathfrak{n}})_i$ vanish. If on the other hand one has $k_i \le (\Sigma\bar{\mathfrak{n}})_i$ for every i, then $|k|_{\mathfrak{s}} \le |\Sigma\bar{\mathfrak{n}}|_{\mathfrak{s}}$ and one has

$$|L_{\bar{\mathfrak{n}}}\sigma|_{\mathfrak{s}} = |\sigma|_{\mathfrak{s}} - |\Sigma\bar{\mathfrak{n}}|_{\mathfrak{s}} \le |\sigma|_{\mathfrak{s}} - |k|_{\mathfrak{s}} \le |\sigma|_{\mathfrak{s}} - |k|_{\mathfrak{s}} + |\mathfrak{t}|_{\mathfrak{s}} = |\mathcal{I}^{\mathfrak{t}}_k(\sigma)|_{\mathfrak{s}} < 0 \;,$$

so that $L_{\bar{\mathfrak{n}}}\sigma \in B^-_\circ$ and has strictly less colourless edges than $\tau = \mathcal{I}^{\mathfrak{t}}_k(\sigma)$. If σ has only one colourless edge, then σ belongs to $B^\sharp_\circ$; therefore the proof follows by induction over the number of colourless edges of τ.

Let us now turn to the case $h \neq 0$. First, we claim that, setting $\hat{\mathbf{\Pi}}_h = \tilde{T}_h(\hat{\mathbf{\Pi}})$, one has

$$\mathbf{E}(\hat{\mathbf{\Pi}}_h\tau)(h) = 0 \;. \tag{6.25}$$

This follows from the fact that $\hat{\mathbf{\Pi}}$ is stationary since the action $\tilde{T}$ commutes with that of $\mathcal{G}^{\mathrm{ex}}_-$ as a consequence of (5.23), combined with the fact that $(f \otimes g_h)\Delta^-_{\mathrm{ex}}\tau = g_h(\tau)$ for every $f \in \mathcal{G}^{\mathrm{ex}}_-$, every $\tau \in \mathcal{T}^{\mathrm{ex}}_+$ and every g_h of the form (6.20).

On the other hand, we have

$$\hat{\mathbf{\Pi}}\tau = \tilde{T}_{-h}(\hat{\mathbf{\Pi}}_h)\tau \;.$$

It follows immediately from the expression for the action of $\tilde{T}$ that $\hat{\mathbf{\Pi}}\tau$ is a deterministic linear combination of terms of the form $\hat{\mathbf{\Pi}}_h\sigma$ with $|\sigma|_- \le |\tau|_-$, so that the claim (6.23) follows from (6.25).

It remains to show that $\hat{\mathbf{\Pi}}$ is the only function of the type $\mathbf{\Pi}^g$ with this property. For this, note that every such function is also of the form $\hat{\mathbf{\Pi}}^g$ for some different $g \in \mathcal{G}^{\mathrm{ex}}_-$, so that we only need to show that for every element g different from the identity, there exists τ such that $\mathbf{E}(\hat{\mathbf{\Pi}}^g\tau)(0) \neq 0$.

Using the Definitions 5.26 and 5.29, the Remark 4.15 and the identification (6.1), $\mathcal{T}^{\mathrm{ex}}_-$ can be canonically identified with the free algebra generated by $B^\sharp_\circ$. Therefore the character g is completely characterised by its evaluation on $B^\sharp_\circ$ and it is the identity if and only if this evaluation vanishes identically. Fix now such a g different from the identity and let $\tau \in B^\sharp_\circ$ be such that $g(\tau) \neq 0$, and such that $g(\sigma) = 0$ for all $\sigma \in B^\sharp_\circ$ with the property that either $|\sigma|_- < |\tau|_-$ or $|\sigma|_- = |\tau|_-$, but σ has strictly less colourless edges than τ. Since $B^\sharp_\circ$ is finite and g doesn't vanish identically, such a τ exists.

We can then also view τ as an element of $\mathcal{T}^{\mathrm{ex}}$ and we write

$$\Delta^-_{\mathrm{ex}}\tau = \tau \otimes \mathbf{1}_1 + \sum_i \tau_i^{(1)} \otimes \tau_i^{(2)} \,,$$

so that

$$\hat{\mathbf{\Pi}}^g\tau = g(\tau) + \sum_i g(\tau_i^{(1)})\hat{\mathbf{\Pi}}\tau_i^{(2)} \,. \tag{6.26}$$

Note now that Δ^-_{ex} preserves the $|\cdot|_-$-degree so that for each of the term in the sum it is either the case that $|\tau_i^{(1)}|_- < |\tau|_-$ or that $|\tau_i^{(2)}| \le 0$. In the former case, the corresponding term in (6.26) vanishes identically by the definition of τ. In the latter case, its expectation vanishes at the origin if $|\tau_i^{(2)}| < 0$ by (6.23). If $|\tau_i^{(2)}| = 0$ then, since $\tau_i^{(2)}$ is not proportional to $\mathbf{1}_1$ (this is the first term which was taken out of the sum explicitly), $\tau_i^{(2)}$ must contain at least one colourless edge. Since Δ^-_{ex} also preserves the number of colourless edges, this implies that again $g(\tau_i^{(1)}) = 0$ by our construction of τ. We conclude that one has indeed $\mathbf{E}(\hat{\mathbf{\Pi}}^g\tau)(0) = g(\tau) \neq 0$, as required. □

Remark 6.18 The rigidity apparent in (6.23) suggests that for a large class of random admissible maps $\mathbf{\Pi}^{(\varepsilon)} \colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ built from some stationary processes $\xi_{\mathfrak{t}}^{(\varepsilon)}$ by (6.13) and (6.14), the corresponding collection of models built from $\hat{\mathbf{\Pi}}^{(\varepsilon)}$ defined as in (6.22) should converge to a limiting model, provided that the $\xi_{\mathfrak{t}}^{(\varepsilon)}$ converge in a suitable sense as $\varepsilon \to 0$. This is indeed the case, as shown in the companion "analytical" article [CH16]. It is also possible to verify that the renormalisation procedures that were essentially "guessed" in [Hai13, Hai14, HP15, HS15] are precisely of BPHZ type.

Remark 6.19 One immediate consequence of Theorem 6.17 is that, for any $g \in \mathcal{G}^{\mathrm{ex}}_-$ and any admissible $\mathbf{\Pi}$, if we set $\mathbf{\Pi}^g = (g \otimes \mathbf{\Pi})\Delta^-_{\mathrm{ex}}$ as in Theorem 6.15, then the BPHZ renormalisation of $\mathbf{\Pi}^g$ is $\hat{\mathbf{\Pi}}$. In particular, the BPHZ renormalisation of the canonical lift of a collection of stationary processes $\{\xi_{\mathfrak{l}}\}_{\mathfrak{l}\in\mathfrak{L}_-}$ as in Remark 6.12 is identical to that of the centred collection $\{\tilde\xi_{\mathfrak{l}}\}_{\mathfrak{l}\in\mathfrak{L}_-}$ where $\tilde\xi_{\mathfrak{l}} = \xi_{\mathfrak{l}} - \mathbf{E}\xi_{\mathfrak{l}}(0)$.

Remark 6.20 Although the map $\mathbf{\Pi} \mapsto \hat{\mathbf{\Pi}}$ selects a "canonical" representative in the class of functions of the form $\mathbf{\Pi}^g$, this does not necessarily mean that every stochastic PDE in the class described by the underlying rule R can be renormalised in a canonical way. The reason is that the kernels $K_{\mathfrak{t}}$ are typically some *truncated* version of the heat kernel and not simply the heat kernel itself. Different choices of the kernels $K_{\mathfrak{t}}$ may then lead to different choices of the renormalisation constants for the corresponding SPDEs.

6.4 The reduced regularity structure

In this section we study the relation between the regularity structure $\mathscr{T}^{\mathrm{ex}}$ introduced in this paper and the one originally constructed in [Hai14, Sec. 8].

Definition 6.21 Let us call an admissible map $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ *reduced* if the second identity in (6.14) holds, namely $\mathbf{\Pi}\mathcal{R}_\alpha(\tau) = \mathbf{\Pi}\tau$ for all $\tau \in \mathcal{T}^{\mathrm{ex}}$ and $\alpha \in \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$. We also define the idempotent map $\mathbb{Q}_1\colon \mathfrak{F} \to \mathfrak{F}$ by

$$\mathbb{Q}_1\colon (F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e}) \mapsto (F, \mathfrak{d}_1 \circ \hat{F}, \mathfrak{n}, 0, \mathfrak{e})\,,$$

with $\mathfrak{d}_1\colon \mathbf{N} \to \mathbf{N}$, $\mathfrak{d}_1(n) = n\mathbb{1}_{(n\neq 1)}$, and set $\mathbb{Q} = \mathbb{Q}_1\mathcal{K}$.

An admissible map is reduced iff $\mathbf{\Pi}\tau = \mathbf{\Pi}\mathbb{Q}\tau$ for every $\tau \in \mathcal{T}^{\mathrm{ex}}$. Moreover $\mathbb{Q}$ commutes with the maps $\mathcal{K}_i$, $\hat{\mathcal{K}}_i$ and $\mathscr{J}$, and preserves the $|\cdot|_-$-degree, so that it is in particular also well-defined on $\mathcal{T}^{\mathrm{ex}}$, $\hat{\mathcal{T}}^{\mathrm{ex}}_+$, $\hat{\mathcal{T}}^{\mathrm{ex}}_-$ and $\mathcal{T}^{\mathrm{ex}}_-$. (It does however *not* preserve the $|\cdot|_+$-degree so that it is *not* well-defined on $\mathcal{T}^{\mathrm{ex}}_+$!) An important remark is that, as a consequence of the induction (6.7), the fact that $\mathbb{Q}$ is multiplicative, and the fact that

$$(\mathbb{Q}\otimes\mathbb{Q})\Delta^-_{\mathrm{ex}}\mathbb{Q} = (\mathbb{Q}\otimes\mathbb{Q})\Delta^-_{\mathrm{ex}}\,, \tag{6.27}$$

for $\Delta^-_{\mathrm{ex}}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}^{\mathrm{ex}}_- \otimes \mathcal{T}^{\mathrm{ex}}$, the twisted antipode $\tilde{\mathcal{A}}^{\mathrm{ex}}_-$ satisfies the identity $\mathbb{Q}\tilde{\mathcal{A}}^{\mathrm{ex}}_- = \mathbb{Q}\tilde{\mathcal{A}}^{\mathrm{ex}}_-\mathbb{Q}$. Therefore, if a stationary admissible $\mathbf{\Pi}$ is (almost surely) reduced, then the character $g^-(\mathbf{\Pi})$ is also reduced in the sense that $g^-(\mathbf{\Pi})(\mathbb{Q}\tau) = g^-(\mathbf{\Pi})(\tau)$. Using again (6.27), it follows immediately that $\hat{\mathbf{\Pi}}$ as given by (6.22) is again reduced, so that the class of reduced models is preserved by the BPHZ renormalisation procedure.

Definition 6.22 Let $\mathcal{T}$ and $\hat{\mathcal{T}}_+$ respectively be the subspaces of $\mathcal{T}^{\mathrm{ex}}$ and $\hat{\mathcal{T}}^{\mathrm{ex}}_+$ given by

$$\mathcal{T} = \{\tau \in \mathcal{T}^{\mathrm{ex}} \,:\, \mathbb{Q}\tau = \tau\}\,, \qquad \hat{\mathcal{T}}_+ = \{\tau \in \hat{\mathcal{T}}^{\mathrm{ex}}_+ \,:\, \mathbb{Q}\tau = \tau\}\,.$$

We also set $\mathcal{T}_+ = \mathfrak{p}^{\mathrm{ex}}_+\hat{\mathcal{T}}_+$.

The reason why we define $\mathcal{T}_+$ in this slightly more convoluted way instead of setting it equal to $\{\tau \in \mathcal{T}^{\mathrm{ex}}_+ \,:\, \mathbb{Q}\tau = \tau\}$ is that although $\mathbb{Q}$ is well-defined on $\hat{\mathcal{T}}^{\mathrm{ex}}_+$, it is not well-defined on $\mathcal{T}^{\mathrm{ex}}_+$ as a consequence of the fact that it doesn't preserve the $|\cdot|_+$-degree.

Since $\mathbb{Q}$ is multiplicative, $\mathcal{T}_+$ is a subalgebra of $\mathcal{T}^{\mathrm{ex}}_+$. Furthermore, it is clear that Δ^+_{ex} maps $\hat{\mathcal{T}}_+$ to $\hat{\mathcal{T}}_+ \otimes \hat{\mathcal{T}}_+$ so that $\mathcal{T}_+$ is a bialgebra. Looking at the recursive definition of the antipode $\mathcal{A}^{\mathrm{ex}}_+$, it is clear that it also maps $\mathcal{T}_+$ into itself, so that $\mathcal{T}_+$ is a Hopf subalgebra of $\mathcal{T}^{\mathrm{ex}}_+$.

This allows to construct a regularity structure $\mathscr{T} = (A, \mathcal{T}, \mathcal{G}_+)$ where $A \stackrel{\mathrm{def}}{=} \{|\tau|_+ : \tau \in B_\circ,\ \tau = \mathbb{Q}\tau\}$, $\mathcal{T}^{\mathrm{ex}} = \langle B_\circ\rangle$ as in Definition 5.26 and $\mathcal{G}_+$ is the group of characters of $\mathcal{T}_+$. The regularity structure $\mathscr{T}$, associated to a subcritical complete rule R, is then isomorphic to the regularity structure associated to a subcritical equation constructed in [Hai14, Sec. 8], as a consequence of the discussion of Section 5.7. As explained in Remark 5.27, the superscript ex stands for extended: the reason is that the regularity structure $\mathscr{T}^{\mathrm{ex}}$ is an extension of $\mathscr{T}$ in the sense that $\mathscr{T} \subset \mathscr{T}^{\mathrm{ex}}$ with the inclusion interpreted as in [Hai14, Sec. 2.1]. By contrast, we call $\mathscr{T}$ the *reduced* regularity structure.

Note that if $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ is such that $\mathcal{Z}^{\mathrm{ex}}(\mathbf{\Pi}) = (\Pi, \Gamma)$ is a model of $\mathscr{T}^{\mathrm{ex}}$, then the restriction $\mathcal{Z}(\mathbf{\Pi})$ of $\mathcal{Z}^{\mathrm{ex}}(\mathbf{\Pi})$ to $\mathscr{T}$ is automatically again a model. This is always the case, irrespective of whether $\mathbf{\Pi}$ is reduced or not. This allows to give the following

Definition 6.23 We denote by $\mathscr{M}_\infty$ the space of all smooth models for $\mathscr{T}$, in the sense of Definition 6.6, obtained by restriction to $\mathcal{T}$ of $\mathcal{Z}^{\mathrm{ex}}(\mathbf{\Pi})$ for some reduced admissible linear map $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$. We endow $\mathscr{M}_\infty$ with the system of pseudo-metrics (6.15) and we denote by $\mathscr{M}_0$ the completion of this metric space.

Remark 6.24 The restriction that $\mathbf{\Pi}$ be reduced may not seem very natural in view of the discussion preceding the definition. It follows however from Theorem 6.30 below that lifting this restriction makes no difference whatsoever since it implies in particular that every smooth admissible model on $\mathscr{T}$ is of the form $\mathcal{Z}(\mathbf{\Pi})$ for some reduced $\mathbf{\Pi}$.

Remark 6.25 By restriction of $\mathcal{Z}^{\mathrm{ex}}(\mathbf{\Pi} M_g)$ to $\mathscr{T}$ for $g \in \mathcal{G}_-^{\mathrm{ex}}$, we get a renormalised model $\mathcal{Z}(\mathbf{\Pi} M_g)$ which covers all the examples treated so far in singular SPDEs. It is however not clear a priori whether we really have an action of a suitable subgroup of $\mathcal{G}_-^{\mathrm{ex}}$ onto $\mathscr{M}_\infty$ or $\mathscr{M}_0$. This is because the coaction of Δ_{ex}^- on $\mathcal{T}^{\mathrm{ex}}$ and $\mathcal{T}_+^{\mathrm{ex}}$ fails to leave the reduced sector invariant. If on the other hand we tweak this coaction by setting $\Delta^- = (\mathrm{id} \otimes \mathbb{Q})\Delta_{\mathrm{ex}}^-$, then it unfortunately *fails* to have the cointeraction property (3.32), which was crucial for our construction, see Remark 5.38.

There turn out to be two natural subgroups of $\mathcal{G}_-^{\mathrm{ex}}$ that are determined by their values on $\mathbb{Q}\mathcal{T}_-^{\mathrm{ex}}$:

- We set $\mathcal{G}_- \stackrel{\mathrm{def}}{=} \{g \in \mathcal{G}_-^{\mathrm{ex}} : g(\tau) = g(\mathbb{Q}\tau), \forall \tau \in \mathcal{T}_-^{\mathrm{ex}}\}$. This is the most natural subgroup of $\mathcal{G}_-^{\mathrm{ex}}$ since it contains the characters $g^-(\mathbf{\Pi})\tilde{\mathcal{A}}_-^{\mathrm{ex}}$ used for the definition of $\hat{\mathbf{\Pi}}$ in (6.22), as soon as $\mathbf{\Pi} = \mathbf{\Pi}\mathbb{Q}$. The fact that $\mathcal{G}_-$ is a subgroup follows from the property (6.27).
- We set $\mathcal{G}_-^a \stackrel{\mathrm{def}}{=} \{g \in \mathcal{G}_-^{\mathrm{ex}} : g(\tau) = 0 \,\forall \tau \in \mathcal{T}_-^c\}$ where $\mathcal{T}_-^c$ is the bialgebra ideal of $\mathcal{T}_-^{\mathrm{ex}}$ generated by $\{\tau \in B_-,\ \mathbb{Q}\tau \neq \tau\}$. Then one can identify $\mathcal{G}_-^a$ with the group of characters of the Hopf algebra $\big(\mathcal{T}_-^{\mathrm{ex}}/\mathcal{T}_-^c, \Delta_{\mathrm{ex}}^-\big)$. It turns out that this is simply the polynomial Hopf algebra with generators $\{\tau \in B_- : |\tau|_- < 0,\ \mathbb{Q}\tau = \tau\}$, so that $\mathcal{G}_-^a$ is abelian.

We then have the following result.

Theorem 6.26 *There is a continuous action R of $\mathcal{G}_-$ onto $\mathscr{M}_0$ with the property that, for every $g \in \mathcal{G}_-$ and every reduced and admissible $\mathbf{\Pi}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ with $\mathcal{Z}^{\mathrm{ex}}(\mathbf{\Pi}) \in \mathscr{M}_0^{\mathrm{ex}}$, one has $R_g\mathcal{Z}(\mathbf{\Pi}) = \mathcal{Z}(\mathbf{\Pi} M_g)$.*

Proof. We already know by Theorem 6.15 that $\mathcal{G}_-$ acts continuously onto $\mathscr{M}_0^{\mathrm{ex}}$. Furthermore, by the definition of $\mathcal{G}_-$, it preserves the subset $\mathscr{M}_0^r \subset \mathscr{M}_0^{\mathrm{ex}}$ of reduced models, i.e. the closure in $\mathscr{M}_0^{\mathrm{ex}}$ of all models of the form $\mathcal{Z}^{\mathrm{ex}}(\mathbf{\Pi})$ for $\mathbf{\Pi}$ admissible and reduced. Since $\mathscr{T} \subset \mathscr{T}^{\mathrm{ex}}$, we already mentioned that we have a natural

projection $\pi^{\mathrm{ex}}\colon \mathscr{M}_0^{\mathrm{ex}} \to \mathscr{M}_0$ given by restriction (so that $\mathfrak{Z}(\mathbf{\Pi}) = \pi^{\mathrm{ex}}\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi})$), and it is straightforward to see that π^{ex} is injective on $\mathscr{M}_0^r$. It therefore suffices to show that there is a continuous map $\iota^{\mathrm{ex}}\colon \mathscr{M}_0 \to \mathscr{M}_0^{\mathrm{ex}}$ which is a right inverse to π^{ex}, and this is the content of Theorem 6.30 below. □

Remark 6.27 In principle, one can show slightly more, namely that the action of $\mathscr{G}_-$ onto $\mathscr{M}_0$ is given by elements of the "renormalisation group" defined in [Hai14, Sec. 8.3].

6.4.1 Construction of extended models

In general if, for some sequence $\mathbf{\Pi}^{(n)}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$, $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}^{(n)}) \in \mathscr{M}_\infty^{\mathrm{ex}}$ converges to a limiting model in $\mathscr{M}_0^{\mathrm{ex}}$, it does *not* follow that the characters $g_+(\mathbf{\Pi}^{(n)})$ of $\hat{\mathscr{T}}_+^{\mathrm{ex}}$ converge to a limiting character. However, we claim that the characters $f_x^{(n)}$ of $\mathscr{T}_+^{\mathrm{ex}}$ given by (6.11) *do* converge, which is not so surprising since our definition of convergence implies that the characters $\gamma_{xy}^{(n)}$ of $\mathscr{T}_+^{\mathrm{ex}}$ given by (6.12) do converge. More surprising is that the convergence of the characters $f_x^{(n)}$ follows already from a seemingly much weaker type of convergence. Writing $\mathscr{D}'$ for the space of distributions on $\mathbf{R}^d$, we have the following.

Proposition 6.28 *Let $\mathbf{\Pi}^{(n)}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$ be an admissible linear map with*

$$\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}^{(n)}) = (\Pi^{(n)}, \Gamma^{(n)}) \in \mathscr{M}_\infty^{\mathrm{ex}}$$

and assume that there exist linear maps $\Pi_x\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{D}'(\mathbf{R}^d)$ such that, with the notation of (6.15)*, $\|\Pi^{(n)} - \Pi\|_{\ell,\mathfrak{K}} \to 0$ for every $\ell \in \mathbf{R}$ and every compact set $\mathscr{K}$. Then, the characters $f_x^{(n)}$ defined as in* (6.11) *converge to a limit f_x. Furthermore, defining Γ_{xy} by* (6.12)*, one has $\mathfrak{Z} = (\Pi, \Gamma) \in \mathscr{M}_0^{\mathrm{ex}}$ and $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}^{(n)}) \to \mathfrak{Z}$ in $\mathscr{M}_0^{\mathrm{ex}}$.*

Finally, one has $\mathbf{\Pi}\colon \mathscr{T}^{\mathrm{ex}} \to \mathcal{S}'(\mathbf{R}^d)$ such that $\Pi_x = (\mathbf{\Pi} \otimes f_x)\Delta_{\mathrm{ex}}^+$ and such that $\mathbf{\Pi}^{(n)}\tau \to \mathbf{\Pi}\tau$ in $\mathcal{S}'(\mathbf{R}^d)$ for every $\tau \in \mathscr{T}^{\mathrm{ex}}$.

Proof. The convergence of the $f_x^{(n)}$ follows immediately from the formula given in Lemma 6.9, combined with the convergence of the $\Pi_x^{(n)}$ and [Hai14, Lem. 5.19]. The fact that (Π, Γ) satisfies the algebraic identities required for a model follows immediately from the fact that this is true for every n. The convergence of the $\Gamma_{xy}^{(n)}$ and the analytical bound on the limit then follow from [Hai14, Sec. 5.1]. □

Remark 6.29 This relies crucially on the fact that the maps $\mathbf{\Pi}$ under consideration are admissible and that the kernels $K_{\mathfrak{t}}$ satisfy the assumptions of [Hai14, Sec. 5]. If one considers different notions of admissibility, as is the case for example in [HQ15], then the conclusion of Proposition 6.28 may fail.

For a linear $\mathbf{\Pi}\colon \mathscr{T} \to \mathscr{C}^\infty$ we define $\mathbf{\Pi}^{\mathrm{ex}}\colon \mathscr{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$ by simply setting $\mathbf{\Pi}^{\mathrm{ex}} = \mathbf{\Pi}\mathbb{Q}$. Then we say that $\mathbf{\Pi}$ is admissible if $\mathbf{\Pi}^{\mathrm{ex}}$ is. We have the following crucial fact

Theorem 6.30 *If $\mathbf{\Pi}\colon \mathcal{T} \to \mathcal{C}^\infty$ is admissible and $\mathfrak{Z}(\mathbf{\Pi}^{\mathrm{ex}})$ belongs to $\mathscr{M}_\infty$, then $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}^{\mathrm{ex}})$ belongs to $\mathscr{M}_\infty^{\mathrm{ex}}$. Furthermore, the map $\mathfrak{Z}(\mathbf{\Pi}^{\mathrm{ex}}) \mapsto \mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}^{\mathrm{ex}})$ extends to a continuous map from $\mathscr{M}_0$ to $\mathscr{M}_0^{\mathrm{ex}}$.*

Proof. Let $\mathbf{\Pi}\colon \mathcal{T} \to \mathcal{C}^\infty$ be such that $\mathfrak{Z}(\mathbf{\Pi}^{\mathrm{ex}}) = (\Pi, \Gamma)$ is a model of $\mathscr{T}$ and write $(\Pi^{\mathrm{ex}}, \Gamma^{\mathrm{ex}}) = \mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}^{\mathrm{ex}})$. Our aim is to find linear maps $L_x\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}$, depending furthermore continuously on the model $\mathfrak{Z}(\mathbf{\Pi}^{\mathrm{ex}})$, with the property that

$$\Pi_x^{\mathrm{ex}}\tau = \Pi_x L_x \tau\;, \qquad \forall \tau \in \mathcal{T}^{\mathrm{ex}}\;. \tag{6.28}$$

If these maps all increase the $|\cdot|_+$-degree, then it follows immediately that Π_x^{ex} satisfies the required analytical bounds, provided that Π_x does. The claim then follows at once from Proposition 6.28.

For every $x \in \mathbf{R}^d$, we define L_x as follows. First, we set

$$L_x \Xi_{k,\ell}^{\mathfrak{l}} = \Xi_{k,\ell}^{\mathfrak{l}}\;, \qquad L_x X^k = X^k\;, \tag{6.29}$$

and we then define recursively

$$L_x \mathcal{R}_\alpha(\tau) = L_x \tau\;, \qquad L_x(\tau\bar\tau) = L_x(\tau) L_x(\bar\tau)\;,$$

where the product on $\mathcal{T}^{\mathrm{ex}}$ is given by (4.8), as well as

$$L_x \mathcal{I}_k^{\mathfrak{t}}(\tau) = \mathcal{I}_k^{\mathfrak{t}}(L_x\tau) - \sum_{|\mathcal{I}_k^{\mathfrak{t}}\tau|_+ \le |m|_{\mathfrak{s}}} \frac{X^m}{m!} f_x(\mathcal{I}_{k+m}^{\mathfrak{t}}(\bar\tau))\;. \tag{6.30}$$

Note that this sum is finite since $\mathcal{I}_{k+m}^{\mathfrak{t}}\bar\tau = 0$ as an element of $\mathcal{T}_+^{\mathrm{ex}}$ as soon as $|m|_{\mathfrak{s}} \ge |\mathcal{I}_k^{\mathfrak{t}}\bar\tau|_+$. It is immediate from these definitions that L_x does indeed increase the $|\cdot|_+$-degree, as a consequence of the fact that elements of the type $\mathcal{R}_\alpha(\tau)$ appear in $\mathcal{T}^{\mathrm{ex}}$ only if $|\alpha|_{\mathfrak{s}} < 0$.

It remains to show that (6.28) does indeed hold. For this, define $\hat{\mathbf{\Pi}}\colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ by

$$\hat{\mathbf{\Pi}}\tau = (\Pi_x L_x \otimes f_x \mathcal{A}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^+ \tau\;.$$

Since one then has

$$\Pi_x L_x \tau = (\hat{\mathbf{\Pi}} \otimes f_x)\Delta_{\mathrm{ex}}^+ \tau\;,$$

it suffices to show that one has $\hat{\mathbf{\Pi}} = \mathbf{\Pi}^{\mathrm{ex}}$. Noting that

$$\hat{\mathbf{\Pi}} = ((\mathbf{\Pi} \otimes f_x)\Delta_{\mathrm{ex}}^+ L_x \otimes f_x \mathcal{A}_+ \mathrm{ex})\Delta_{\mathrm{ex}}^+\;,$$

we actually check a slightly stronger property, namely we show that the linear maps

$$\mathbb{Q}_x \tau \stackrel{\mathrm{def}}{=} ((\mathrm{id} \otimes f_x)\Delta_{\mathrm{ex}}^+ L_x \otimes f_x \mathcal{A}_+^{\mathrm{ex}})\Delta_{\mathrm{ex}}^+ \tau$$

satisfy $\mathbb{Q}_x = \mathbb{Q}$ for every x. This in turn follows immediately from the properties of the antipode $\mathcal{A}_+^{\mathrm{ex}}$ if we can show that

$$(\mathrm{id} \otimes f_x)\Delta_{\mathrm{ex}}^+ L_x \tau = (\mathbb{Q} \otimes f_x)\Delta_{\mathrm{ex}}^+ \tau\;. \tag{6.31}$$

Since both L_x and $\mathbb{Q}$ act as the identity on X^k and $\Xi^{\mathfrak{l}}_{k,\ell}$, it follows that (6.31) does indeed holds for these elements. We then proceed inductively over the number of colourless edges of τ, using (4.14). Note first that since $\Delta^+_{\mathrm{ex}}\mathscr{R}_\alpha = (\mathscr{R}_\alpha \otimes \mathrm{id})\Delta^+_{\mathrm{ex}}$ and $\mathbb{Q}\mathscr{R}_\alpha = \mathbb{Q}$, it follows that if (6.31) holds for τ, then it also holds for $\mathscr{R}_\alpha\tau$. Similarly, since both sides of (6.31) are multiplicative, it follows that if the identity holds for both τ and $\bar\tau$, then it also holds for $\tau\bar\tau$. It therefore remains to show that if (6.31) holds for every τ with at most n colourless edges, then it also holds for every element of $\mathcal{T}^{\mathrm{ex}}$ of the form $\mathcal{I}^{\mathfrak{t}}_k(\tau)$ where τ has exactly n colourless edges.

Note now that by (4.14) and the fact that $\mathbb{Q}$ commutes with $\mathcal{I}^{\mathfrak{t}}_k$ one has

$$(\mathbb{Q}\otimes f_x)\Delta^+_{\mathrm{ex}}\mathcal{I}^{\mathfrak{t}}_k(\tau) = \mathcal{I}^{\mathfrak{t}}_k(\mathbb{Q}\otimes f_x)\Delta^+_{\mathrm{ex}}\tau + \sum_\ell \frac{X^\ell}{\ell!} f_x(\mathcal{I}^{\mathfrak{t}}_{k+\ell}(\tau))\;.$$

On the other hand, it follows from (6.30), (4.14), and the fact that $f_x(X) = -x$ that

$$\begin{aligned}(\mathrm{id}\otimes f_x)\Delta^+_{\mathrm{ex}}L_x\mathcal{I}^{\mathfrak{t}}_k(\tau) = \mathcal{I}^{\mathfrak{t}}_k(\mathrm{id}\otimes f_x)\Delta^+_{\mathrm{ex}}L_x\tau + \sum_\ell \frac{X^\ell}{\ell!} f_x(\mathcal{I}^{\mathfrak{t}}_{k+\ell}(L_x\tau))\\ - \sum_{|\mathcal{I}^{\mathfrak{t}}_k\tau|_+ \le |\ell+m|_{\mathfrak{s}}} \frac{X^\ell}{\ell!}\frac{(-x)^m}{m!} f_x(\mathcal{I}^{\mathfrak{t}}_{k+\ell+m}(L_x\tau))\;.\end{aligned}$$

Comparing these two expressions and exploiting the induction hypothesis, we see that it remains to show that

$$f_x(\mathcal{I}^{\mathfrak{t}}_k(L_x\tau)) - f_x(\mathcal{I}^{\mathfrak{t}}_k(\tau)) = \sum_{|\mathcal{I}^{\mathfrak{t}}_k\tau|_+ \le |m|_{\mathfrak{s}}} \frac{(-x)^m}{m!} f_x(\mathcal{I}^{\mathfrak{t}}_{k+m}(L_x\tau))\;. \tag{6.32}$$

Writing $L_x\tau = \sum_i c_{i,x}\tau_i$ and using the fact that $|\tau_i|_+ \ge |\tau|_+$ for every i, it follows from Lemma 6.9 that

$$\begin{aligned}f_x(\mathcal{I}^{\mathfrak{t}}_k(L_x\tau)) &= -\sum_i c_{i,x} \sum_{|k+\ell|_{\mathfrak{s}} < |\mathcal{I}^{\mathfrak{t}}\tau_i|_+} \frac{(-x)^\ell}{\ell!}(D^{k+\ell}K_{\mathfrak{t}} * \Pi_x\tau_i)(x)\\ &= -\sum_{|k+\ell|_{\mathfrak{s}} < |\mathcal{I}^{\mathfrak{t}}\tau|_+} \frac{(-x)^\ell}{\ell!}(D^{k+\ell}K_{\mathfrak{t}} * \Pi_x L_x\tau)(x)\\ &\quad - \sum_i c_{i,x} \sum_{|\mathcal{I}^{\mathfrak{t}}\tau|_+ \le |k+\ell|_{\mathfrak{s}} < |\mathcal{I}^{\mathfrak{t}}\tau_i|_+} \frac{(-x)^\ell}{\ell!}(D^{k+\ell}K_{\mathfrak{t}} * \Pi_x\tau_i)(x)\;.\end{aligned}$$

Since $\Pi_x L_x\tau = \Pi_x\tau$ by our inductive hypothesis, the first term is indeed equal to $f_x(\mathcal{I}^{\mathfrak{t}}_k(\tau))$. The fact that the second term is equal to the right hand side in (6.32) follows from the binomial identity, thus concluding the proof. □

Appendix A Symbolic index

Here, we collect some of the most used symbols of the article, together with their meaning and the page where they were first introduced.

Symbol	Meaning	Page
$\lvert\cdot\rvert_{\text{bi}}$	Bigrading on coloured decorated forests	13
$\lvert\cdot\rvert_{-}$	Degree not taking into account the label $\mathfrak{o}$	43
$\lvert\cdot\rvert_{+}$	Degree taking into account the label $\mathfrak{o}$	43
$\mathfrak{A}_i$	Subforests appearing in the definition of Δ_i	14
$\mathscr{A}_i$	Antipode of $\mathscr{H}_i$	25
$\mathscr{A}_{\pm}^{\text{ex}}$	Antipode of $\mathscr{T}_{\pm}^{\text{ex}}$	57
$\tilde{\mathscr{A}}_{\pm}^{\text{ex}}$	Twisted antipode $\mathscr{T}_{\pm}^{\text{ex}} \to \hat{\mathscr{T}}_{\pm}^{\text{ex}}$	61
$\hat{\mathscr{A}}_2$	Antipode of $\hat{\mathscr{H}}_2$	38
$B_\circ$	Elements of $H_\circ$ strongly conforming to the rule R	55
$B_\circ^-$	Elements of $B_\circ$ of negative degree	55
$B_\circ^\sharp$	Elements of $B_\circ^-$ that are not elementary	55
B_-	Elements of H_1 strongly conforming to the rule R	55
B_+	Elements of $\hat{H}_2$ conforming to the rule R	55
$\mathscr{D}_i(\mathscr{J})$	All roots of colour in $\{0, i\}$	34
$\hat{\mathscr{D}}_i(\mathscr{J})$	All roots of colour i	34
Δ_i	Coproduct on $\langle\mathfrak{F}\rangle$ turning the $\langle\mathfrak{F}_i\rangle$ into bialgebras	16
$\mathscr{E}$	Edge types given by $\mathscr{E} = \mathfrak{L} \times \mathbf{N}^d$	45
$f\restriction A$	Restriction of the function f to the set A	13
$\mathfrak{F}$	All coloured decorated forests $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$	10
$\mathfrak{F}_i$	All forests compatible with $\mathfrak{A}_i$	21
$\mathfrak{F}_\circ$	Trees with colours in $\{0, 1\}$	37
Φ_i	Collapse of factors in $\mathfrak{M}_i$	24
$g_z^+(\mathbf{\Pi})$	Character on $\hat{\mathscr{T}}_+^{\text{ex}}$ defined by $\mathbf{\Pi}$	64
$g_z^-(\mathbf{\Pi})$	Character on $\hat{\mathscr{T}}_-^{\text{ex}}$ defined by $\mathbf{\Pi}$	69
$\mathscr{G}_i$	Characters of $\mathscr{H}_i$	26
$\mathscr{G}_{\pm}^{\text{ex}}$	Character group of $\mathscr{T}_{\pm}^{\text{ex}}$	58
$\hat{\mathscr{G}}_2$	Characters of $\hat{\mathscr{H}}_2$	38
$\mathscr{H}_\circ$	Algebra given by $\langle\mathfrak{F}_\circ\rangle / \ker\mathscr{K}$	37
$\hat{\mathscr{H}}_2$	Hopf algebra $\mathscr{H}_2 / \ker(\mathscr{J}\hat{P}_2)$	37
$\mathscr{H}_i$	Hopf algebra $\langle\mathfrak{F}_i\rangle / \mathcal{I}_i$	25
$H_\circ$	Representative of $\mathscr{H}_\circ$ given by $H_\circ = \mathscr{K}\mathfrak{F}_\circ$	37
$\hat{H}_2$	Representative of $\hat{\mathscr{H}}_2$ given by $\hat{H}_2 = \mathscr{J}\hat{\mathscr{K}}_2\mathfrak{F}_2$	37
H_i	Representative of $\mathscr{H}_i$ given by $H_i = \mathscr{K}_i\mathfrak{F}_i$	25
$\mathfrak{i}_{\pm}^{\text{ex}}$	Canonical injection $\mathscr{T}_{\pm}^{\text{ex}} \hookrightarrow \hat{\mathscr{T}}_{\pm}^{\text{ex}}$	60

Symbol	Meaning	Page
$\mathcal{I}_i$	Kernel of $\mathcal{K}_i$	24
$\hat{\mathcal{I}}_i$	Kernel of $\hat{\mathcal{K}}_i$	24
$\mathcal{I}_k^{\mathrm{t}}$	Abstract integration map in $\mathcal{H}_\circ$	39
$\mathscr{J}$	Joins the root of all trees together	34
$\hat{\mathcal{I}}_k^{\mathrm{t}}$	Abstract integration map $\mathcal{H}_\circ \to \hat{\mathcal{H}}_2$	40
$\mathcal{J}_+$	Subspace of terms in $\hat{\mathcal{T}}_-^{\mathrm{ex}}$ with a factor of positive degree	56
$\mathcal{J}_-$	Subspace of terms in $\hat{\mathcal{T}}_+^{\mathrm{ex}}$ with a factor of negative degree	56
$\mathcal{K}$	Contraction of coloured portions	23
$\mathcal{K}_i$	Defined by $\mathcal{K}_i = \Phi_i \circ \mathcal{K}$	24
$\hat{\mathcal{K}}_i$	Defined by $\mathcal{K}_i = P_i \circ \Phi_i \circ \mathcal{K}$	24
$\lvert k\rvert$	Unscaled length of a multi-index k	13
$\lvert k\rvert_{\mathfrak{s}}$	Scaled length of a multi-index k	43
$\mathfrak{L}$	Set of all types	9
$\mathfrak{M}_i$	Elements of $\langle \mathfrak{F}_i \rangle$ completely coloured with i	24
$\mathscr{M}$	Space of all models	67
$\mathscr{M}_0$	Closure of smooth models	67
$\mathscr{M}_\infty$	Space of all smooth models	67
$\mathcal{N}$	Node types given by $\mathcal{N} = \hat{\mathcal{P}}(\mathcal{E})$	45
$\mathcal{N}(x)$	Type of the node x	45
$\hat{P}_i$	Sets $\mathfrak{o}$-decoration to 0 on i-coloured roots	24
$\mathcal{P}(A)$	Powerset of the set A	46
$\hat{\mathcal{P}}(A)$	Multisets with elements from the set A	45
$\mathfrak{p}_\pm^{\mathrm{ex}}$	Canonical projection $\hat{\mathcal{T}}_\pm^{\mathrm{ex}} \to \mathcal{T}_\pm^{\mathrm{ex}}$	56
$\mathbf{\Pi}$	Linear map $\mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ specifying a model	64
R	Rule determining a class of trees	46
$\mathcal{R}_\alpha$	Operator adding α to $\mathfrak{o}$ at the root	40
$\mathfrak{s}$	Scaling of $\mathbf{R}^d$	43
$\mathfrak{T}$	Simple decorated trees	44
$\mathfrak{T}_\circ(R)$	Trees strongly conforming to the rule R	47
$\mathfrak{T}_1(R)$	Forests strongly conforming to the rule R	47
$\mathfrak{T}_2(R)$	Trees conforming to the rule R	47
$\mathfrak{T}_-(R)$	Trees strongly conforming to R of negative degree	47
$\hat{\mathcal{T}}_+^{\mathrm{ex}}$	Subspace of $\hat{\mathcal{H}}_2$ determined by a rule R	55
$\hat{\mathcal{T}}_-^{\mathrm{ex}}$	Subspace of $\mathcal{H}_1$ determined by a rule R	55
$\mathcal{T}^{\mathrm{ex}}$	Subspace of $\mathcal{H}_\circ$ determined by a rule R	55
$\mathcal{T}_+^{\mathrm{ex}}$	Quotient space $\hat{\mathcal{T}}_+^{\mathrm{ex}}/\mathcal{J}_-$	56
$\mathcal{T}_-^{\mathrm{ex}}$	Quotient space $\hat{\mathcal{T}}_-^{\mathrm{ex}}/\mathcal{J}_+$	56
$\mathfrak{U}_i$	Units of $\langle \mathfrak{F}_i \rangle$	21

Symbol	Meaning	Page
$\langle V \rangle$	Bigraded space generated from a bigraded set V	13
X^k	Shorthand for $(\bullet, i)_0^{k,0}$ with $i \in \{0, 2\}$ depending on context	39
$\Xi^{\mathfrak{l}}$	Element $\mathcal{I}_0^{\mathfrak{l}}(\mathbf{1})$ representing the noise	64
$\|z\|_{\mathfrak{s}}$	Scaled distance	63
$\mathfrak{L}^{\mathrm{ex}}$	Map turning $\mathbf{\Pi}$ into a model	64

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