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FACTORIZATION OF SPARSE POLYNOMIALS OVER A FUNCTION FIELD

FRANCESCO AMOROSO AND MARTÍN SOMBRA

ABSTRACT. We present a structure theorem for the non-constant irreducible factors appearing in the family of all univariate polynomials with a given set of coefficients in a function field and varying exponents. Roughly speaking, this result shows that the non-constant irreducible factors of these sparse polynomials, are also sparse.

This result is based on a refinement of Zannier's toric Bertini theorem.

1. INTRODUCTION

Let $\mathbf{x} = (x_1, \dots, x_n)$ be a group of n variables, and t a further variable.

Given a matrix $A = (a_{i,j})_{i,j} \in \mathbb{Z}^{m \times n}$, we consider the family of m monomials in the variables \mathbf{x} given by

$$\mathbf{x}^A = \left(\prod_{j=1}^n x_j^{a_{1,j}}, \dots, \prod_{j=1}^n x_j^{a_{m,j}} \right).$$

If $B \in \mathbb{Z}^{l \times m}$ is another matrix, then $\mathbf{x}^{BA} = (\mathbf{x}^A)^B$.

Given $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we can consider it as a row vector, that is, as a matrix in $\mathbb{Z}^{1 \times n}$. In this case,

$$\mathbf{x}^{\mathbf{a}} = \prod_{j=1}^n x_j^{a_j}$$

is an n -variate monomial. Else, we can also consider it as a column vector, that is, as a matrix in $\mathbb{Z}^{n \times 1}$ and, in this case,

$$t^{\mathbf{a}} = (t^{a_1}, \dots, t^{a_n})$$

is a collection of n univariate monomials.

Let $\mathbb{Q}[\mathbf{x}^{\pm 1}] = \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the algebra of n -variate Laurent polynomials over the rationals. Its units are the monomials.

Definition 1.1. A Laurent polynomial $F \in \mathbb{Q}[\mathbf{x}^{\pm 1}] \setminus \{0\}$ is (*generalized*) *cyclotomic* if it is a factor of a Laurent polynomial of the form $\mathbf{x}^{\mathbf{a}} - 1$ for a vector $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

Equivalently, a nonzero Laurent polynomial F is cyclotomic if $V(F)$ is a torsion hypersurface of the torus \mathbb{G}_m^n . In the univariate case, $f \in \mathbb{Q}[t^{\pm 1}] \setminus \{0\}$ is cyclotomic in the sense of Definition 1.1 when it factors as a product of a monomial and a product of cyclotomic (in the usual sense) univariate polynomials.

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Definition 1.2. The *cyclotomic part* of a Laurent polynomial $F \in \mathbb{Q}[\mathbf{x}^{\pm 1}] \setminus \{0\}$, denoted by $\text{cyc}(F)$, is defined as its maximal cyclotomic factor. It is well-defined up to a unit of $\mathbb{Q}[\mathbf{x}^{\pm 1}]$.

In [Sch65], Schinzel proposed a conjecture that can be reformulated as follows:

Conjecture 1.3. *Given a non-cyclotomic irreducible polynomial $F \in \mathbb{Q}[\mathbf{x}^{\pm 1}] \setminus \{0\}$, there is a finite collection $\Omega_F^0 \subset \mathbb{Z}^{n \times n}$ of nonsingular matrices and a finite collection $\Gamma_F \subset \mathbb{Z}^n$ of nonzero vectors satisfying the following property. Let $\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}$ and set $f_{\mathbf{a}} = F(t^{\mathbf{a}}) \in \mathbb{Q}[t^{\pm 1}]$. Then one of the next alternatives holds:*

- (1) *there exists $\mathbf{c} \in \Gamma_F$ verifying $\langle \mathbf{c}, \mathbf{a} \rangle = 0$.*
- (2) *there exist $A \in \Omega_F^0$ and $\mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{a} = A\mathbf{b}$ such that, if*

$$F(\mathbf{x}^A) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\mathbf{x}^A)$, then the irreducible factorization of $f_{\mathbf{a}}/\text{cyc}(f_{\mathbf{a}})$ is given by

$$\frac{f_{\mathbf{a}}}{\text{cyc}(f_{\mathbf{a}})} = \prod_P \left(\frac{P(t^{\mathbf{b}})}{\text{cyc}(P(t^{\mathbf{b}}))} \right)^{e_P};$$

Schinzel has proven this conjecture for the cases when $n = 1$ in *loc. cit.* and, under some restrictive hypothesis (non-symmetry), when $n \geq 2$ [Sch70], see also [Sch00, §6.2].

Remark 1.4. In Conjecture 1.3, taking out the cyclotomic parts in (2) both in $f_{\mathbf{a}}$ and the P 's is necessary when $n \geq 2$. For instance, set $n = 2$ and $F = x_1 + x_2 - 2 \in \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}]$. Then, for every nonsingular $A \in \mathbb{Z}^{2 \times 2}$, the Laurent polynomial

$$P := F(\mathbf{x}^A) = x_1^{a_{1,1}} x_2^{a_{1,2}} + x_1^{a_{2,1}} x_2^{a_{2,2}} - 2$$

is irreducible. However, for all $\mathbf{b} \in \mathbb{Z}^2$, we have that $t - 1$ divides $P(t^{\mathbf{b}})$, and so this univariate Laurent polynomial is not irreducible. Also, for all $\mathbf{a} \in \mathbb{Z}^2$, we have that $t - 1 | f_{\mathbf{a}}$, but this factor $t - 1$ cannot be accounted for from the irreducible factorization of the $F(\mathbf{x}^A)$'s.

Conjecture 1.3 implies that, for a given nonzero multivariate Laurent polynomial F , the irreducible factorizations of the non-cyclotomic parts of the univariate Laurent polynomials $f_{\mathbf{a}} = F(t^{\mathbf{a}})$, $\mathbf{a} \in \mathbb{Z}^n$, can be obtained from the irreducible factorizations of Laurent polynomials of the form $F(\mathbf{x}^A)$ for a *finite* number of matrices A .

Conjecture 1.5. *Given $F \in \mathbb{Q}[\mathbf{x}^{\pm 1}] \setminus \{0\}$, there are finite collections $\Omega_F^k \subset \mathbb{Z}^{n \times (n-k)}$, $k = 0, \dots, n-1$, of full-rank matrices satisfying the following property. Let $\mathbf{a} \in \mathbb{Z}^n$ and set $f_{\mathbf{a}} = F(t^{\mathbf{a}}) \in \mathbb{Q}[t^{\pm 1}]$. Then there exist $k \in \{0, \dots, n-1\}$, $A \in \Omega_F^k$ and $\mathbf{b} \in \mathbb{Z}^{n-k}$ with $\mathbf{a} = A\mathbf{b}$ such that, if*

$$(1.1) \quad F(\mathbf{x}^A) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\mathbf{x}^A)$, then the irreducible factorization of $f_{\mathbf{a}}/\text{cyc}(f_{\mathbf{a}})$ is given by

$$\frac{f_{\mathbf{a}}}{\text{cyc}(f_{\mathbf{a}})} = \prod_{P \in I(\mathbf{b})} \left(\frac{P(t^{\mathbf{b}})}{\text{cyc}(P(t^{\mathbf{b}}))} \right)^{e_P},$$

the product being over the set $I(\mathbf{b})$ of irreducible Laurent polynomials P in (1.1) such that $P(t^{\mathbf{b}})$ is not cyclotomic.

An immediate consequence of Conjecture 1.5 would be that the non-cyclotomic irreducible factors of the univariate Laurent polynomials in the family $f_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z} \setminus \{0\}$ are *sparse*, in the sense that they all obtained by restricting a finite family of Laurent polynomial to 1-parameter subgroups. In particular, the number of non-cyclotomic irreducible factors of the $f_{\mathbf{a}}$'s is bounded above by a constant independent of \mathbf{a} .

In this paper, we prove an analogue of these conjectures over a function field, namely, for Laurent polynomials with coefficients in the field $\mathbb{C}(z)$ where z is a variable.

Definition 1.6. A Laurent polynomial $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$ is *constant* if there is an scalar $\lambda \in \mathbb{C}(z)^{\times}$ such that $\lambda F \in \mathbb{C}[\mathbf{x}^{\pm 1}]$.

Remark 1.7. The analogy between cyclotomic Laurent polynomials over \mathbb{Q} and constant Laurent polynomials over $\mathbb{C}(z)$ stems from height theory. Let \mathbb{K} denote either \mathbb{Q} or $\mathbb{C}(z)$ and $F \in \mathbb{K}[\mathbf{x}^{\pm 1}] \setminus \{0\}$. Let \mathbb{G}_m^n be the n -dimensional torus over \mathbb{K} , equipped with the canonical height of subvarieties induced by the standard inclusion $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$, and denoted by h . Let $V(F)$ be the hypersurface of \mathbb{G}_m^n defined by F . Then $h(V(F)) = 0$ if and only if F is cyclotomic, when $\mathbb{K} = \mathbb{Q}$, or if and only if F is constant, when $\mathbb{K} = \mathbb{C}(z)$.

Definition 1.8. The *constant part* of a Laurent polynomial $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$, denoted by $\text{ct}(F)$, is defined as its maximal constant factor. It is well-defined up to a unit of $\mathbb{C}(z)[\mathbf{x}^{\pm 1}]$. We also call *non-constant part* of F the polynomial $F/\text{ct}(F)$.

We prove the following analogue of Schinzel's conjecture 1.3.

Theorem 1.9. *Given a Laurent polynomial $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$ without nontrivial constant factors, there is a finite collection $\Omega_F^0 \subset \mathbb{Z}^{n \times n}$ of nonsingular matrices and a finite collection $\Gamma_F^0 \subset \mathbb{Z}^n$ of nonzero vectors satisfying the following property. Let $\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}$ and set $f_{\mathbf{a}} = F(t^{\mathbf{a}}) \in \mathbb{C}(z)[t^{\pm 1}]$. Then one of the next alternatives holds:*

- (1) *there exists $\mathbf{c} \in \Gamma_F^0$ verifying $\langle \mathbf{c}, \mathbf{a} \rangle = 0$.*
- (2) *there exist $A \in \Omega_F^0$ and $\mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{a} = A\mathbf{b}$ such that, if*

$$F(\mathbf{x}^A) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\mathbf{x}^A)$, then the irreducible factorization of $f_{\mathbf{a}}/\text{ct}(f_{\mathbf{a}})$ is given by

$$\frac{f_{\mathbf{a}}}{\text{ct}(f_{\mathbf{a}})} = \prod_P \left(\frac{P(t^{\mathbf{b}})}{\text{ct}(P(t^{\mathbf{b}}))} \right)^{e_P};$$

This theorem, whose proof rests on a slight generalization (Theorem 2.2) of the Toric Bertini Theorem of Zannier [Zan10], implies the following factorization result for families of sparse univariate Laurent polynomials over $\mathbb{C}(z)$.

Corollary 1.10. *Given $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$, there are finite collections $\Omega_F^k \subset \mathbb{Z}^{n \times (n-k)}$, $k = 0, \dots, n-1$, of full-rank matrices satisfying the following property. Let $\mathbf{a} \in \mathbb{Z}^n$ and set $f_{\mathbf{a}} = F(t^{\mathbf{a}}) \in \mathbb{C}(z)[t^{\pm 1}]$. Then there exist $k \in \{0, \dots, n-1\}$, $A \in \Omega_F^k$ and $\mathbf{b} \in \mathbb{Z}^{n-k}$ with $\mathbf{a} = A\mathbf{b}$ such that, if*

$$(1.2) \quad F(\mathbf{x}^A) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\mathbf{x}^A)$, then the irreducible factorization of $f_{\mathbf{a}}/\text{ct}(f_{\mathbf{a}})$ is given by

$$\frac{f_{\mathbf{a}}}{\text{ct}(f_{\mathbf{a}})} = \prod_{P \in I(\mathbf{b})} \left(\frac{P(t^{\mathbf{b}})}{\text{ct}(P(t^{\mathbf{b}}))} \right)^{e_P},$$

the product being over the set $I(\mathbf{b})$ of irreducible Laurent polynomials P in (1.2) such that $P(t^{\mathbf{b}})$ is not constant.

Hence, the non-constant irreducible factors of the univariate Laurent polynomials in the family $f_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z} \setminus \{0\}$ are *sparse*, in the sense that they all obtained by restricting a finite family of Laurent polynomial to 1-parameter subgroups. In particular, the number of non-constant irreducible factors of the $f_{\mathbf{a}}$'s is bounded.

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2. COVERS, FIBER PRODUCTS AND ZANNIER'S TORIC BERTINI THEOREM

Let W, Y, X be varieties with maps $\psi: Y \rightarrow X$ and $\varphi: W \rightarrow X$. We identify the underlying set of the fiber product $W \times_{\varphi, \psi} Y$ with the subset

$$\{(w, y) \in W \times Y \mid \varphi(w) = \psi(y)\}$$

of the cartesian product $W \times Y$.

In the sequel of this section we fix a geometrically irreducible variety Y defined over \mathbb{C} and a cover $\pi: Y \rightarrow X$, with $X = \mathbb{G}_m^n$. The following lemma which is implicit in the proof of [Zan10, Proposition 2.1].

Lemma 2.1. *Let us assume that π factors as $\pi = \lambda \circ \rho$ for some isogeny λ of X of degree $s > 1$. Then the fiber product $X \times_{\lambda, \pi} Y$ is reducible.*

Proof. We have $\lambda(x) = \pi(y) = \lambda(\rho(y))$ if and only if there exists $\zeta \in \ker \lambda$ such that $x = \zeta \cdot \rho(y)$. This gives a decomposition

$$\bigcup_{\zeta \in \ker \lambda} X \times_{\text{Id}, \zeta, \rho} Y$$

of $X \times_{\lambda, \pi} Y$ into the union of $s > 1$ closed proper subsets. Thus $X \times_{\lambda, \pi} Y$ is reducible. \square

We can now state our version of [Zan10, Theorem 3].

Theorem 2.2 (Toric Bertini Theorem). *Let Y be a geometrically irreducible variety (defined over \mathbb{C}) and $\pi: Y \rightarrow \mathbb{G}_m^n$ be a cover. Then there exist a finite union \mathcal{E} of proper torsion cosets and a finite set Ω of isogenies λ of \mathbb{G}_m^n with $\mathbb{G}_m^n \times_{(\lambda, \pi)} Y$ reducible, such that the following holds.*

Let $T \not\subseteq \mathcal{E}$ be a torsion coset. Then either the horizontal part of the set-theoretic fiber $\pi^{-1}(T)$ is irreducible or there exists $\lambda \in \Omega$ such that $\lambda^{-1}(T)$ splits as a union of $\deg(\lambda)$ distinct torsion cosets.

Proof. By [Zan10, Proposition 2.1] we can choose a factorisation $\pi = \lambda' \circ \rho$ where $\lambda': X \rightarrow X$ is an isogeny and $\rho: Y \rightarrow X$ is a cover satisfying (PB) (this means in Zannier terminology that for all $m \in \mathbb{Z}$ the fiber product $X \times_{[m], \rho} Y$ is irreducible).

For a subgroup H of $\ker \lambda'$ we consider the projection $\mu_H: X \rightarrow X/H$. We have a factorisation $\lambda' = \lambda_H \circ \mu_H$. Note that we can identify the n -dimensional torus X/H with X . We define Ω as the (finite) set of isogeny $\lambda: X \rightarrow X$ of *positive degree* such that $\lambda = \lambda_H$ for some subgroup H of $\ker \lambda'$.

Let $\lambda \in \Omega$. By definition of Ω , we have $\deg(\lambda) > 1$ and π factors as $\lambda \circ (\mu \circ \rho)$. By Lemma 2.1 the fiber product $X \times_{\lambda, \pi} Y$ is reducible.

Since the cover $\rho: Y \rightarrow X$ satisfies (PB), by [Zan10, Theorem 3] there exists a finite union \mathcal{E} of proper torsion cosets such that the horizontal part of $\rho^{-1}(T)$ is irreducible for any torsion cosets $T \not\subseteq \mathcal{E}$. We define $\mathcal{E} = \lambda'(\mathcal{E}')$.

Let us show that Ω, \mathcal{E} satisfy the requirement of Theorem 2.2. We have already show that $X \times_{\lambda, \pi} Y$ is reducible for any $\lambda \in \Omega$. Let $T \not\subseteq \mathcal{E}$ be a torsion coset and write $\lambda'^{-1}(T) = T_1 \cup \dots \cup T_s$ for some distinct torsion cosets T_1, \dots, T_s .

If $s = 1$ then $\lambda'^{-1}(T) = T_1$ is an irreducible torsion coset not contained in \mathcal{E} , hence the horizontal part of $\pi^{-1}(T) = \rho^{-1}(T_1)$ is irreducible.

Let us assume $s > 1$. The group $\ker \lambda'$ acts on the set $\{T_1, \dots, T_s\}$. Let $H = \text{Stab}(T_1)$ and consider the decomposition $\lambda' = \lambda_H \circ \mu_H$ as before. Then $\lambda_H^{-1}(T)$ still splits as an union of s distinct torsion cosets and $\lambda_H \in \Omega$ since $\deg \lambda_H = s > 1$. □

We conclude this section with a reformulation of the last assertion of Theorem 2.2, which is more suitable for the applications to the reducibility of lacunary polynomials of the next section.

Remark 2.3. Let $\pi: Y \rightarrow X$ be a cover and λ an isogeny of X . We consider an injective morphism $\tau: \mathbb{G}_m^k \rightarrow X$ with image a subtorus T . Then τ factorizes through λ (that is: $\exists \tau^*: \mathbb{G}_m^k \rightarrow X$ such that $\tau = \lambda \circ \tau^*$) if and only if $\lambda^{-1}(T)$ splits as an union of $\deg(\lambda)$ distinct torsion cosets.

Proof. If $\lambda^{-1}(T)$ splits as an union of $\deg(\lambda)$ distinct torsion cosets, then one of these is a subtorus T' and the restriction $\lambda|_{T'}: T' \rightarrow T$ is an isomorphism. Thus $\tau = \lambda \circ \tau^*$ with $\tau^* = \lambda|_{T'}^{-1} \circ \tau$. Conversely, let us assume that $\tau = \lambda \circ \tau^*$ for some morphism $\tau^*: \mathbb{G}_m^k \rightarrow X$. Let $\tau^*: \mathbb{G}_m^k \rightarrow X$ such that $\tau = \lambda \circ \tau^*$ and set $G' := \tau^*(X)$. Then $\lambda^{-1}(T)$ is the union of the $\deg(\lambda)$ distinct torsion cosets $\theta G'$ for $\theta \in \ker(\lambda)$. □

3. PROOF OF THE MAIN THEOREM

In this section we deduce our main Theorem 1.9 from the Toric Bertini Theorem 2.2.

We first settle in the following technical lemma some questions concerning, degrees, multiple and constant factors of a specialization of a polynomial $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}]$.

Lemma 3.1.

- (1) Let $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}] \setminus \{0\}$ without non trivial constant factors, of degree $d \geq 1$ in z . There exists a finite collection $\Gamma_F^{(0)}$ of nonzero vectors $\mathbf{c} \in \mathbb{Z}^n$ having the following property. Let $\mathbf{a} \in \mathbb{Z}^n$ such that $\langle \mathbf{c}, \mathbf{a} \rangle \neq 0$ for $\mathbf{c} \in \Gamma_F^{(0)}$. Then the polynomial $F(z, t^{\mathbf{a}})$ has degree d in z .
- (2) Let $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}] \setminus \{0\}$ be irreducible. Then there exists a finite collection $\Gamma_F^{(1)}$ of non-zero vectors $\mathbf{c} \in \mathbb{Z}^n$ having the following property. Let $\mathbf{a} \in \mathbb{Z}^n$ such

that $\mathbf{c} \cdot \mathbf{a} \neq 0$ for $\mathbf{c} \in \Gamma_F^{(1)}$. Then the polynomial $F(z, t^{\mathbf{a}})$ does not have multiple factors.

- (3) Let $f \in \mathbb{C}[z, t^{\pm 1}]$ irreducible, of degree $d \geq 1$ in z . Let us assume that $f(z, t^m)$ is reducible for some $m \in \mathbb{N}$. Then, there exists $s \in \mathbb{N}$ dividing $\gcd(m, d)$ such that $f(z, t^s)$ is reducible.
- (4) Let $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$ without nontrivial constant factors, and $A \in \mathbb{Z}^{n \times n}$ be a nonsingular matrix. Then the Laurent polynomial $F(z, \mathbf{x}^A)$ has no nontrivial constant factors.

Proof. For the proof of the first assertion, let $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ be the coefficient of z^d in F . Up to a unit in $\mathbb{C}[\mathbf{x}^{\pm 1}]$ we can assume $f(\mathbf{x}) = \sum_{i=0}^r f_i \mathbf{x}^{c_i}$ for some distinct vectors $\mathbf{c}_i \in \mathbb{Z}^n$. We choose $\Gamma_F^{(0)} = \{\mathbf{c}_j - \mathbf{c}_i \mid 1 \leq i < j \leq r\}$.

To prove the second assertion, remark that the discriminant Δ of F with respect to the variable z is a non zero Laurent polynomial in $\mathbb{C}[\mathbf{x}^{\pm 1}]$. For a $\mathbf{a} \in \mathbb{Z}^n$ which is not orthogonal to the vectors in a finite set $\Gamma_F^{(1)}$, we thus have $\Delta(t^{\mathbf{a}}) \neq 0$ which in turn implies that $F(z, t^{\mathbf{a}})$ does not have multiple factors.

Let now prove the third assertion. This is a consequence of the proof of [Zan10, Proposition 2.1] applied to the cover $Y \rightarrow \mathbb{G}_m$ defined by f . For the convenience of the reader, we reproduce here the argument.

The group of the m -th roots of units μ_m acts on the set of irreducible factors P_1, \dots, P_s of $f(z, t^m)$ by $P(z, t) \mapsto P(z, \zeta t)$. This action is transitive. Indeed, let us consider an orbit of cardinality s' , say $\{P_1, \dots, P_{s'}\}$. Then $P_1 \cdots P_{s'}$ is stabilized by μ_m , thus it is a polynomial in t^m , say $g(z, t^m)$. Hence $g \mid f$ and, since f is irreducible, $f = cg$ for a non-zero monomial c , showing that $s' = s$. Let $P = P_j$ be one of the irreducible factors. Then $\text{Stab}(P)$ is a subgroup of μ_m , say $\text{Stab}(P) = \mu_\delta$. Since the action is transitive and μ_m is abelian, δ does not depend on P .

Since the action is transitive, the number s of irreducible factors of $f(z, t^m)$ is the index of the stabilizer, that is $s = m/\delta$. If $f(z, t^m)$ is reducible, then $f(z, t^s)$ is also reducible, as we now show. Let $f(z, t^m) = g_1(z, t)g_2(z, t)$ be a non trivial factorization. Since μ_δ stabilizes all the irreducible factors of $f(z, t^m)$, these factors are polynomials in t^δ . Thus $g_1(z, t)$ and $g_2(z, t)$ are also polynomials in t^δ , say $g_j(z, t) = f_j(z, t^\delta)$. Let $u = t^\delta$. Since $u^s = t^m$, we have $f(z, u^s) = f_1(z, u)f_2(z, u)$ and $f(z, u^s)$ is reducible, as required.

Finally the irreducible factors of $f(z, t^m)$ have the same degree d' in z and $d = sd'$. Thus $s \mid d$.

We finally prove the last assertion. We can assume without loss of generality that F is irreducible. Let

$$F(z, \mathbf{x}^A) = \prod_{i=1}^s P_i^{e_i}$$

be the irreducible factorization of this Laurent polynomial. As in the proof of the previous assertion, the finite group $G = \{\mathbf{x} \in \mathbb{G}_m^n \mid \mathbf{x}^A = 1\}$ acts transitively on the irreducible factors. It follows that, for $i, j = 1, \dots, s$,

$$\deg_z(P_i) = \deg_z(P_j) \quad \text{and} \quad e_i = e_j.$$

Hence

$$\deg(P_i) = \frac{\deg_z(F)}{e_{is}} > 0,$$

which proves the statement. \square

Remark 3.2. A statement similar to 2) of this lemma for Laurent polynomial in $\mathbb{Q}[\mathbf{x}^{\pm 1}]$ is still true, but the proof is deeper. See [ASZ] for details.

We now offer a dictionary between the polynomial setting of Theorem 1.9 and the language of covers and fiber products of the last section. Let $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}]$ irreducible and non constant of degree $d \geq 1$ in z , defining a hypersurface Y of \mathbb{G}_m^{n+1} . Consider the cover $\pi: Y \rightarrow \mathbb{G}_m^n$ (of degree d) given by $(z, \mathbf{x}) \mapsto \mathbf{x}$ and let λ be an isogeny λ defined by a non-singular $A \in \mathrm{GL}_n(\mathbb{Z})$.

Then the irreducibility of $F(z, \mathbf{x}^A) \in \mathbb{K}[z, \mathbf{x}^{\pm 1}]$ translates into the irreducibility of the fiber product $\mathbb{G}_m^n \times_{\lambda, \pi} Y$, as we see considering the map

$$\{(z, \mathbf{x}) \in \mathbb{G}_m \times \mathbb{G}_m^n \mid F(z, \mathbf{x}^A) = 0\} \rightarrow \mathbb{G}_m^n \times_{\lambda, \pi} Y$$

which sends (z, \mathbf{x}) to $(\mathbf{x}, (z, \mathbf{x}^A))$.

Similarly, let $\tau: \mathbb{G}_m \rightarrow \mathbb{G}_m^n$ be a finite morphism, say $\tau(\mathbf{t}) = \mathbf{t}^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{Z}^n$, and set $f_{\mathbf{a}} = F(z, \mathbf{t}^{\mathbf{a}})$. Then the irreducibility of its non-constant part $f_{\mathbf{a}}/\mathrm{ct}(f_{\mathbf{a}})$ is equivalent to the irreducibility of the horizontal part of the fiber product $\mathbb{G}_m \times_{\tau, \pi} Y$.

Let now suppose that τ is injective, that is \mathbf{a} primitive. Then it is easily see that the irreducibility of the horizontal part of the set-theoretic fiber $\pi^{-1}(T)$ implies the irreducibility of the horizontal part of the *scheme* $\mathbb{G}_m \times_{\tau, \pi} Y$ for $\mathbf{a} \in \mathbb{Z}^n$ outside a finite union of proper subgroups (by Lemma 3.1(2), $F(z, \mathbf{t}^{\mathbf{a}})$ does not have multiple factors).

Moreover, by Remark 2.3, $\lambda^{-1}(T)$ splits as an union of $\deg(\lambda)$ distinct torsion cosets if and only if $\mathbf{a} \in \mathrm{Im}(A)$.

The next theorem allow us to prove Theorem 1.9 by induction.

Theorem 3.3 (Polynomial Toric Bertini Theorem). *Let F be a Laurent polynomial in $\mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$ without nontrivial constant factors. Then there exist a finite collection Ω_F of non singular matrices $A \in \mathbb{Z}^{n \times n}$ and a finite collection Γ_F of non-zero vectors $\mathbf{c} \in \mathbb{Z}^n$ having the following properties. Let $\mathbf{a} \in \mathbb{Z}^n$ and set $f_{\mathbf{a}} = F(\mathbf{t}^{\mathbf{a}}) \in \mathbb{C}(z)[\mathbf{t}^{\pm 1}]$. Then one of the following assertions hold:*

- (1) *There exists a vector $\mathbf{c} \in \Gamma_F$ such that $\langle \mathbf{c}, \mathbf{a} \rangle = 0$.*
- (2) *There exists $A \in \Omega_F$ such that $\mathbf{a} \in \mathrm{Im}(A)$ and $F(\mathbf{x}^A)$ is reducible.*
- (3) *The polynomial $f_{\mathbf{a}}/\mathrm{ct}(f_{\mathbf{a}})$ is irreducible.*

Proof of Theorem 3.3. If F is reducible, we can simply take for Ω_F the set whose only element is the identity matrix and (2) of that theorem trivially holds. By clearing the denominators, we can assume that $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}]$ is a non-constant irreducible polynomial. If we restrict to *primitive* vectors $\mathbf{a} \in \mathbb{Z}^n$, our result then follows by Theorem 2.2 using the dictionary at the beginning of this section. To remove this restriction we shall use Lemma 3.1(3).

For $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}] \setminus \{0\}$ without non-constant nontrivial factors, we denote by Ω_F a finite collection of nonsingular matrices and let Γ_F a finite collection of nonzero vectors such that the statement of Theorem 3.3 holds when we restrict to primitive vectors $\mathbf{a} \in \mathbb{Z}^n$.

Let $F \in \mathbb{C}[z, x^{\pm 1}] \setminus \{0\}$ be a Laurent polynomial without non-constant nontrivial factors. For $s \geq 1$ set $F_s(z, \mathbf{x}) = F(z, x_1^s, \dots, x_n^s)$. By Lemma 3.1(4), this Laurent polynomial is also without non-constant nontrivial factors. Set then

$$\Omega'_F = \bigcup_{s|d} (\Omega_{F_s} \cup \{sI_n\}), \quad \Gamma'_F = \bigcup_{s|d} \Gamma_{F_s}$$

where I_n is the identity matrix and

Let $\mathbf{a} \in \mathbb{Z}^n$ and write $\mathbf{a} = m\mathbf{a}'$ with \mathbf{a}' primitive and $m \in \mathbb{N}$. We have already remarked that Theorem 3.3 holds if we restrict ourself to primitive vectors. Thus either there exists $\mathbf{c} \in \Gamma_F \subseteq \Gamma'_F$ such that $\langle \mathbf{c}, \mathbf{a}' \rangle = 0$ (and thus $\langle \mathbf{c}, \mathbf{a} \rangle = 0$), or there exists $A \in \Omega_F \subseteq \Omega'_F$ such that $\mathbf{a}' \in \text{Im}(A)$ (which implies $\mathbf{a} \in \text{Im}(A)$) and $F(z, \mathbf{x}^A)$ is reducible, or $F(z, t^{\mathbf{a}'})$ is irreducible.

Assume that the first two alternative do not hold and $F(z, t^{\mathbf{a}'})$ not irreducible. Let $f(z, t) := F(z, t^{\mathbf{a}'})$. Thus $f(z, t)$ is irreducible and $f(z, t^m) = F(z, t^{\mathbf{a}'})$ is reducible. By Lemma 3.1(1), we can also assume (by eventually enlarging Γ'_F) that $\deg_z(f) = d$. By Lemma 3.1(3) there exists $s \mid \gcd(m, d)$ such that $f(z, t^s)$ is reducible. If the polynomial $F_s(z, \mathbf{x}) = F(z, x_1^s, \dots, x_n^s)$ is reducible, then for $A = sI_n \in \Omega'_F$ we have $\mathbf{a} = A(\frac{m}{s}\mathbf{a}') \in \text{Im}(A)$ and $F(z, \mathbf{x}^A) = F_s(z, \mathbf{x})$ reducible. Thus assume that $F_s(z, \mathbf{x})$ is irreducible. We apply again Theorem 3.3 (with F replaced by F_s and \mathbf{a} replaced by the primitive vector \mathbf{a}'). Since $F_s(z, t^{\mathbf{a}'}) = f(z, t^s)$ is reducible, either there exists $\mathbf{c} \in \Gamma_{F_s} \subseteq \Gamma'_F$ such that $\langle \mathbf{c}, \mathbf{a}' \rangle = 0$ (and thus $\langle \mathbf{c}, \mathbf{a} \rangle = 0$), or there exists $A \in \Omega_{F_s} \subseteq \Omega'_F$ such that $\mathbf{a}' \in \text{Im}(A)$ (which implies $\mathbf{a} \in \text{Im}(A)$) and $F(z, \mathbf{x}^A)$ is reducible. □

We now deduce by induction Theorem 1.9 from Theorem 3.3.

Proof of Theorem 1.9. Let $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}] \setminus \{0\}$ without nontrivial constant factors. By clearing the denominators, we can assume that $F \in \mathbb{C}[z, \mathbf{x}^{\pm 1}]$.

We first remark that assertion (2) of Theorem 1.9 is equivalent to:

(2') there exist $A \in \Omega_F^0$ and $\mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{a} = A\mathbf{b}$ such that

$$F(z, \mathbf{x}^A) = F_1(z, \mathbf{x}) \cdots F_r(z, \mathbf{x})$$

with the non-constant parts $F_i(z, t^{\mathbf{b}})$ irreducible.

Indeed, since $F(z, \mathbf{x}^A)$ has no nontrivial constant factors by Lemma 3.1(4), if the non-constant parts of $F_i(z, t^{\mathbf{b}})$ are irreducible, $F(z, \mathbf{x}^A) = F_1 \cdots F_r$ must be the irreducible factorization of $F(z, \mathbf{x}^A)$.

Let Ω_F, Γ_F be such that the conclusion of Theorem 3.3 holds. We construct, by induction on $\deg_z(F)$, finite sets $\Omega_F^0 \supseteq \Omega_F$ and $\Gamma_F^0 \supseteq \Gamma_F$ for which the statement of Theorem 1.9 holds. We assume, as we can, that the identity matrix $I_n \in \Omega_F^0$.

Let $\mathbf{a} \in \mathbb{Z}^n$. If assertion 1) of Theorem 3.3 holds, than assertion 1) of Theorem 1.9 still holds (taking for Γ_F^0 any finite set which contains Γ_F). Let us assume that assertion 3) of Theorem 3.3 holds, that is the polynomial $F(z, t^{\mathbf{a}})$ is irreducible. Since $I_n \in \Omega_F^0$, assertion (2') trivially holds (take $A = I_n$ and $\mathbf{b} = \mathbf{a}$).

Assume now that 2) of Theorem 3.3 holds: there exist $A \in \Omega_F$ and $\mathbf{b} \in \mathbb{Z}^n$ such that $\mathbf{a} = A\mathbf{b}$ and $F(z, \mathbf{x}^A)$ is reducible, say

$$(3.1) \quad F(z, \mathbf{x}^A) = F_1(z, \mathbf{x})F_2(z, \mathbf{x})$$

Note that for a fixed F we only have a finite number of decompositions as in (3.1), since A is in the finite set Ω_F . We also remark that $\deg_z(F_i) \geq 1$ by Lemma 3.1(4),

and thus $\deg_z(F_i) < \deg_z(F)$. By induction, Theorem 1.9 holds for F_1 and for F_2 . For each $i = 1, 2$, let $\Omega_{F_i}^0$ and $\Gamma_{F_i}^0$ be the corresponding finite collections whose existence is assured by this theorem. Hence, either there exists a vector $\mathbf{c} \in \Gamma_{F_1}^0 \cup \Gamma_{F_2}^0$ such that $\langle \mathbf{c}, \mathbf{b} \rangle = 0$ or we can find $A_i \in \Omega_i$, $\mathbf{b}_i \in \mathbb{Z}^n$ and a decomposition

$$F_i(z, \mathbf{x}^{A_i}) = F_{i,1}(z, \mathbf{x}) \cdots F_{i,r_i}(z, \mathbf{x})$$

such that $\mathbf{b} = A_i \mathbf{b}_i$ and the non-constant part of $F_{i,j}(z, t^{\mathbf{b}_i})$ is irreducible. Let

$$\Gamma_F = \{ \mathbf{c} \det(A) A^{-1} \mid A \in \Omega_F, \mathbf{c} \in \Gamma_{F_1}^0 \cup \Gamma_{F_2}^0 \}$$

and assume $\langle \mathbf{c}', \mathbf{a} \rangle \neq 0$ for $\mathbf{c}' \in \Gamma_F^0$. Thus

$$\forall \mathbf{c} \in \Gamma_{F_1}^0 \cup \Gamma_{F_2}^0, \quad \langle \mathbf{c}, \mathbf{b} \rangle = \mathbf{c} A^{-1} \mathbf{a} \neq 0$$

and the non-constant part of $F_{i,j}(z, t^{\mathbf{b}_i})$ are irreducible.

Let $K_i = \text{Im}(A_i)$ and $K = K_1 \cap K_2$. Since K is a lattice, $K = \text{Im}(A')$ for some non-singular matrix $A' \in \mathbb{Z}^{n \times n}$ (which depends only on A_1 and A_2). Since $K \subseteq K_i$, we can find a non-singular matrix B_i with integral entries such that $A' = A_i B_i$ for $i = 1, 2$.

Now recall that for $i = 1, 2$ we have $\mathbf{b} = A_i \mathbf{b}_i$. Thus $\mathbf{b} \in K$ and there exists $\mathbf{b}' \in \mathbb{Z}^n$ such that $\mathbf{b} = A' \mathbf{b}' = A_i B_i \mathbf{b}'$ for $i = 1, 2$. We get

$$\mathbf{b}_i = A_i^{-1} \mathbf{b} = A_i^{-1} A_i B_i \mathbf{b}' = B_i \mathbf{b}'$$

and

$$\begin{aligned} F(z, \mathbf{x}^{AA'}) &= F_1(z, \mathbf{x}^{A'}) F_2(z, \mathbf{x}^{A'}) \\ &= F_1(z, \mathbf{x}^{A_1 B_1}) F_2(z, \mathbf{x}^{A_2 B_2}) \\ (3.2) \quad &= \prod_{i=1}^2 \prod_{j=1}^{r_i} F_{i,j}(z, \mathbf{x}^{B_i}). \end{aligned}$$

Let $A'' = AA'$, $G_{i,j}(z, \mathbf{x}) = F_{i,j}(z, \mathbf{x}^{B_i})$ and consider the decomposition

$$F(z, \mathbf{x}^{A''}) = \prod_{i=1}^2 \prod_{j=1}^{r_i} G_{i,j}(z, \mathbf{x}).$$

We have $\mathbf{a} = A \mathbf{b} = A'' \mathbf{b}'$ and the non-constant part of

$$G_{i,j}(z, t^{\mathbf{b}'}) = F_{i,j}(z, t^{B_i \mathbf{b}'}) = F_{i,j}(z, t^{\mathbf{b}_i})$$

are irreducible. The conjecture for F follows taking for Ω_F^0 a finite set containing Ω and all the above matrices AA' . □

Proof of Corollary 1.10. We can suppose that F is not constant, because otherwise the statement is trivial. When $n = 1$, Corollary 1.10 follows then from the case $n = 1$ of Theorem 1.9. So we suppose that $n \geq 2$ and that Corollary 1.10 holds in dimension $n - 1$. Let Ω_F^0 and Γ_F be as in Theorem 1.9, and take $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. If there is no $\mathbf{c} \in \Gamma_F$ orthogonal to \mathbf{a} , the statement (2) in this conjecture gives the desired conclusion. Otherwise, pick $\mathbf{c}_0 \in \Gamma_F$ such that $\langle \mathbf{c}_0, \mathbf{a} \rangle = 0$. Up to a monomial change of variables given by a matrix in $\text{SL}_n(\mathbb{Z})$ of size depending only on the size of this vector (hence, only on F), we can suppose that $\mathbf{c}_0 = (0, \dots, 0, 1)$, thus $a_n = 0$. Then $F' = F(x_1, \dots, x_{n-1}, 1) \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ and $\mathbf{a}' = (a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$. Hence

$$f_{\mathbf{a}} = F(t^{\mathbf{a}}) = F'(t^{\mathbf{a}'}),$$

and the statement follows from the inductive hypothesis.

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4. A TORIC VERSION OF BERTINI'S THEOREM

In this section, all schemes, varieties, regular and rational maps are over the field of complex numbers.

By a *cover*, we mean a dominant rational map $\pi: Y \dashrightarrow X$ of finite degree between irreducible varieties. Given regular $\varphi: Y \rightarrow S$ and $\psi: Z \rightarrow S$ of schemes, we denote by $Y \times_{\varphi, \psi} Z$ their fibered product.

The proof of Theorem 1.9 reposes on the following generalization of the Zannier's toric version of Bertini's theorem in [Zan10, Theorem 3].

Theorem 4.1. *Given a cover $\pi: Y \dashrightarrow \mathbb{G}_m^n$, there exist a finite union \mathcal{E} of proper torsion cosets of \mathbb{G}_m^n and a finite set \mathcal{I} of isogenies of \mathbb{G}_m^n with*

$$(4.1) \quad \mathbb{G}_m^n \times_{\rho, \pi} Y \text{ reducible}$$

for all $\rho \in \mathcal{I}$, satisfying the following property. Let $\tau: \mathbb{G}_m^m \rightarrow \mathbb{G}_m^n$ be a finite homomorphism and $\theta \in \mathbb{G}_m^n$ a torsion point. If the torsion coset $\theta \operatorname{Im}(\tau)$ of \mathbb{G}_m^n is not contained in \mathcal{E} , then either

$$(4.2) \quad \mathbb{G}_m^m \times_{\theta, \tau, \pi} Y \text{ is irreducible of dimension } m,$$

or

$$(4.3) \quad \text{there exist } \rho \in \mathcal{I} \text{ and a homomorphism } \varsigma: \mathbb{G}_m^m \rightarrow \mathbb{G}_m^n \text{ such that } \tau = \rho \circ \varsigma.$$

Proof: see Section 2.

Remark 4.2. Fibered products of rational maps are not defined. Hence, to properly define those appearing in (4.1) and (4.2), one has to restrict to open subsets where the cover π is regular. The conditions for the obtained fibered products of being or not irreducible, and their dimension, is independent of the choice of these open subsets.

5. PROOF OF THEOREM 1.9

For subvarieties of tori, the fibered products in Theorem 4.1 can be expressed in more concrete terms. The next proposition gives such an expression for the hypersurface case that interests us.

Proposition 5.1. *Let $F \in \mathbb{C}[z^{\pm 1}, \mathbf{x}^{\pm 1}]$ and set $Y = V(F)$ for the hypersurface of $\mathbb{G}_m^1 \times \mathbb{G}_m^n$ defined by F and $\pi: Y \rightarrow \mathbb{G}_m^n$ for the restriction to Y of the projection $\mathbb{G}_m^1 \times \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n$ onto the second factor. Let $\tau: \mathbb{G}_m^m \rightarrow \mathbb{G}_m^n$ be a finite homomorphism defined by $\tau(\mathbf{y}) = \mathbf{y}^B$ for a matrix $B \in \mathbb{Z}^{n \times m}$ of rank m . Then*

$$\mathbb{G}_m^m \times_{\tau, \pi} Y \simeq \operatorname{Spec}(\mathbb{C}[\mathbf{y}^{\pm 1}, z^{\pm 1}]/F(z, \mathbf{y}^B)).$$

Proof. Set $R = \mathbb{C}[\mathbf{x}^{\pm 1}]$ and consider the \mathbb{C} -algebras morphisms

$$R \rightarrow R[z^{\pm 1}]/F \quad \text{and} \quad R \rightarrow R[\mathbf{y}^{\pm 1}]/(x_1 - \tau_1(\mathbf{y}), \dots, x_n - \tau_n(\mathbf{y})).$$

By [EH00, Exercice I-46],

$$A \otimes_R B \simeq \mathbb{C}[\mathbf{x}, \mathbf{y}^{\pm 1}, z^{\pm 1}]/(F, x_1 - \tau_1(\mathbf{y}), \dots, x_n - \tau_n(\mathbf{y})) \simeq \mathbb{C}[\mathbf{y}^{\pm 1}, z^{\pm 1}]/(F(z, \mathbf{y}^B)).$$

Hence, the fibered product $\mathbb{G}_m^m \times_{\tau, \pi} Y$ identifies with the subscheme of $\mathbb{G}_m^m \times \mathbb{G}_m^1$ defined by the Laurent polynomial $F(z, \mathbf{y}^B)$, which gives the statement. \square

The following result is a straightforward consequence of Theorem 4.1.

Proposition 5.2. *Given a non-constant (Definition 1.6) irreducible Laurent polynomial $F \in \mathbb{C}(z)[\mathbf{x}^{\pm 1}]$, there exist a finite collection $\Theta_F \subset \mathbb{Z}^{n \times n}$ of nonsingular matrices and a finite collection $\Lambda_F \subset \mathbb{Z}^n$ of nonzero vectors satisfying the following properties. Let $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and set $f_{\mathbf{a}} = F(z, t^{\mathbf{a}}) \in \mathbb{C}(z)[t^{\pm 1}]$. Then one of the next alternatives holds:*

- (1) *there exists $\mathbf{c} \in \Lambda_F$ such that $\langle \mathbf{c}, \mathbf{a} \rangle = 0$;*
- (2) *there exists $A \in \Theta_F$ such that $\mathbf{a} \in \text{Im}(A)$ and $F(z, \mathbf{x}^A)$ is reducible;*
- (3) *$f_{\mathbf{a}}$ is irreducible.*

Proof. Clearing denominators, we suppose without loss of generality that $F \in \mathbb{C}[z^{\pm 1}, \mathbf{x}^{\pm 1}]$. Set $Y = V(F)$ be the hypersurface of $\mathbb{G}_m^1 \times \mathbb{G}_m^n$ defined by F and $\pi: Y \rightarrow \mathbb{G}_m^n$ the restriction to Y of the projection onto the second factor. The hypothesis that F is not constant implies that this map is a cover.

Let \mathcal{E} be the finite union of proper torsion cosets of \mathbb{G}_m^n and \mathcal{I} the finite set of isogenies of \mathbb{G}_m^n given by Theorem 4.1 applied to this cover. Then we pick Λ_F as a finite subset of $\mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that

$$(5.1) \quad \mathcal{E} \subset V\left(\prod_{\mathbf{c} \in \Lambda_F} (\mathbf{x}^{\mathbf{c}} - 1)\right),$$

and $\Theta_F \subset \mathbb{Z}^{n \times n}$ for the set of nonsingular matrices defining the isogenies in \mathcal{I} .

Since $\mathbf{a} \neq \mathbf{0}$, the homomorphism $\tau: \mathbb{G}_m \rightarrow \mathbb{G}_m^n$ given by $\tau(t) = t^{\mathbf{a}}$ is finite. Suppose that (1) does not hold. By (5.1), this implies that $\text{Im}(\tau) \not\subset \mathcal{E}$. Hence by Theorem 4.1, either $\mathbb{G}_m \times_{\tau, \pi} Y$ is irreducible of dimension 1 or there exist $\rho \in \mathcal{I}$ and a homomorphism $\varsigma: \mathbb{G}_m \rightarrow \mathbb{G}_m^n$ such that $\tau = \rho \circ \varsigma$.

By Proposition 5.2, the first case is equivalent to the statement that $f_{\mathbf{a}} = F(z, t^{\mathbf{a}})$ is irreducible, corresponding to (3). The second case translates into the existence of $A \in \Theta_F$ and $\mathbf{b} \in \mathbb{Z}^n$ such that $\mathbf{a} = A\mathbf{b}$ and, by Proposition 5.2, the second case corre statement that $F(z, \mathbf{x}^A)$ is reducible. \square

Proof of Theorem 1.9. We can suppose without loss of generality that F is a primitive Laurent polynomial in $\mathbb{C}[z][\mathbf{x}^{\pm 1}]$. Since F is not constant, $\deg_z(F) \geq 1$.

We proceed by induction on this quantity. Suppose first that $\deg_z(F) = 1$. Then, for all nonsingular $A \in \mathbb{Z}^{n \times n}$, we have that $\deg_z(F(\mathbf{x}^A)) = 1$, and so this Laurent polynomial is irreducible. Hence, the alternative (2) in Proposition 5.2 cannot hold and, in this case, we can set $\Omega_F^0 = \Theta_F$ and $\Gamma_F = \Lambda_F$.

Now suppose that $\deg(F_z) \geq 2$ and let Θ_F and Γ_F be the sets from Proposition 5.2. For each $A \in \Theta_F$, consider the irreducible factorization

$$F(\mathbf{x}^A) = \prod_P P^{e_{P,A}}.$$

We have that $\deg_z(P) < \deg_z(F)$ for each irreducible factor P of $F(\mathbf{x}^A)$. The result follows then by the inductive hypothesis. \square

LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, CNRS UMR 6139, UNIVERSITÉ DE CAEN, BP 5186, 14032 CAEN CEDEX, FRANCE

E-mail address: `francesco.amoroso@unicaen.fr`

URL: `http://www.math.unicaen.fr/~amoroso/`

ICREA. PASSEIG LLUÍS COMPANYYS 23, 08010 BARCELONA, SPAIN

DEPARTAMENT DE MATEMÀTIQUES I INFORMÀTICA, UNIVERSITAT DE BARCELONA. GRAN VIA 585, 08007 BARCELONA, SPAIN

E-mail address: `sombra@ub.edu`

URL: `http://atlas.mat.ub.es/personals/sombra`