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# Stability analysis of coupled linear ODE-hyperbolic PDE systems with two time scales

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## Abstract

This paper is concerned with a class of coupled ODE/PDE systems with two time scales. The fast constant time scale is modeled by a small positive perturbation parameter. First, we state a general sufficient stability condition for such systems. Next, we study the stability for ODE/fast PDE and PDE/fast ODE systems based on the singular perturbation method respectively. In the first case, we consider a linear ODE coupled with a fast hyperbolic PDE system. The stability of both reduced and boundary-layer subsystems implies the stability of the full system. On the contrary, a counter-example shows that the full system can be unstable even though the two subsystems are stable for a PDE coupled with a fast ODE system. Numerical simulations on academic examples are proposed. Moreover, an application to boundary control of a gas flow transport system is used to illustrate the theoretical result.

*Key words:* Stability, Coupled ODE/PDE, Singular perturbation, Hyperbolic systems,

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## 1 Introduction

The control problem for coupled ordinary differential equations (ODEs)-partial differential equations (PDEs) systems has been studied by different methods in research works. For example, Krstic and Smyshlyaev (2008) dealt with a coupled first-order hyperbolic PDE and second-order (in space) ODE system which was stabilized by backstepping approach. Lyapunov technique was used to prove the stability for a hyperbolic system with an integral actions at the boundary in Dos Santos et al. (2008). In Bastin et al. (2015), the authors considered a Proportional Integral (PI) boundary controller to stabilize an open-loop unstable hyperbolic system, and the result is proved in the frequency domain.

Singular perturbation was introduced in control engineering in late 1960s. It has rapidly developed and has become a tool for analysis and design of control systems due to the decomposition of the full system into lower order subsystems, namely the reduced subsystem and the boundary-layer subsystem (Kokotović and Haddad (1975), Kokotović and Sannuti (1968), Kokotović and Yackel (1972)). From late 1980s, singularly perturbed

partial differential equations have been considered from a mathematical view point in the literature (see Kadalbajoo and Patidar (2003) as a survey). In Tang et al. (2015), systems modeled by singularly perturbed hyperbolic equations have been studied from a control theoretical view point.

Singularly perturbed coupled ODE/PDE systems are interesting to analyze due to their applications to numerous physical and engineering problems. For instance, elastic beams linked to rigid bodies in Littman and Markus (1988), power converters connected to transmission lines in Daafouz et al. (2014) etc.. In the present paper, we focus on a class of coupled linear time invariant (LTI) ordinary differential equation and linear hyperbolic partial differential equation with different time scales. Physical motivation comes from the gas flow transport setup presented in Castillo et al. (2012), in which a heating column governed by an ordinary differential equation is coupled with a tube described by a transport equation. The dynamics of this model has two time scales since the cross section area of the heating column is much bigger than that of the tube.

A first contribution of the present work is that we propose a general sufficient stability condition for coupled ODE/PDE systems, the perturbation parameter intro-

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duced either in the dynamics of the ODE or of the PDE. A second contribution is that we study the link of the stability between the full system and both subsystems via the singular perturbation method. For the ODE/fast PDE case, the stability of the two subsystems implies that of the full system which is consistent with the result for linear finite dimensional systems modeled by linear ODEs (e.g. (Kokotović et al., 1986, Chapter 2)). For the PDE/fast ODE system, the stability result of the first case is not valid in this context. Precisely, the exponential stability of the two subsystems does not guarantee the stability of the full system. This is consistent with the result for singularly perturbed hyperbolic systems in Tang et al. (2015).

The paper is organized as follows. The coupled ODE/PDE systems under consideration are given in Section 2.1. A sufficient stability condition for such systems is stated in the following Section 2.2. We study the stability property for ODE/fast PDE system via the singular perturbation method in Section 3.1. More precisely, two subsystems are formally computed in Section 3.1.1. Sections 3.1.2-3.1.3 discuss the link of the stability between the two subsystems and the full system. In Section 3.2, we consider the PDE/fast ODE system. A counter-example is used to show that the link of the stability between the full system and both subsystems is not valid for the context in Section 3.2.2. Section 4 proposes numerical simulations on academic examples. A boundary control of a gas flow transport model is shown in Section 5. Concluding remarks end of this paper.

**Notation.** Given a matrix  $M$  in  $\mathbb{R}^{n \times n}$ ,  $M^{-1}$  and  $M^\top$  represent the inverse and the transpose matrix of  $M$  respectively. The transpose of the inverse matrix of  $M$  is denoted by  $M^{-\top}$ . The minimum and maximum eigenvalues of matrix  $M$  are noted by  $\underline{\lambda}(M)$  and  $\bar{\lambda}(M)$ . The symbol  $\star$  in partitioned symmetric matrix stands for the symmetric block. For a positive integer  $n$ ,  $I_n$  is the identity matrix in  $\mathbb{R}^{n \times n}$ .  $|\cdot|$  denotes the usual Euclidean norm in  $\mathbb{R}^n$  and  $\|\cdot\|$  is associated with the usual 2-norm of matrices in  $\mathbb{R}^{n \times n}$ .  $\|\cdot\|_{L^2}$  denotes the associated norm in

$L^2(0, 1)$  space, defined by  $\|h\|_{L^2(0,1)} = \sqrt{\left(\int_0^1 |h(x)|^2 dx\right)}$

for all functions  $h \in L^2(0, 1)$ . Given a real interval  $I$  and a normed space  $J$ ,  $C^0(I, J)$  denotes the set of continuous functions from  $I$  to  $J$ . Let us adopt the following notation introduced in Coron et al. (2008), for all matrices  $M \in \mathbb{R}^{n \times n}$ ,

$$\rho_1(M) = \inf\{\|\Delta M \Delta^{-1}\|, \Delta \in D_{n,+}\},$$

where  $D_{n,+}$  denotes the set of diagonal positive matrices in  $\mathbb{R}^{n \times n}$ .

## 2 Singularly perturbed ODE/PDE systems

We consider coupled ODE/PDE systems with two time scales. The fast time scale is modeled by a small per-

turbation parameter  $\epsilon$ . This perturbation can be introduced into either the dynamics of the PDE system or the dynamics of the ODE system.

### 2.1 System description

*Linear ODE coupled with fast hyperbolic PDE system* is given by, for all  $x \in [0, 1]$ ,  $t \geq 0$ ,

$$\begin{cases} \dot{Z}(t) = AZ(t) + By(1, t), & (1a) \\ \epsilon y_t(x, t) + \Lambda y_x(x, t) = 0, & (1b) \\ y(0, t) = G_1 y(1, t) + G_2 Z(t), & (1c) \\ Z(0) = Z_0, & (1d) \\ y(x, 0) = y_0(x). & (1e) \end{cases}$$

Similarly, *linear hyperbolic PDE coupled with fast ODE system* is given by, for all  $x \in [0, 1]$ ,  $t \geq 0$ ,

$$\begin{cases} \epsilon \dot{Z}(t) = AZ(t) + By(1, t), & (2a) \\ y_t(x, t) + \Lambda y_x(x, t) = 0, & (2b) \\ y(0, t) = G_1 y(1, t) + G_2 Z(t), & (2c) \\ Z(0) = Z_0, & (2d) \\ y(x, 0) = y_0(x). & (2e) \end{cases}$$

In (1) and (2), it holds  $Z : [0, +\infty) \rightarrow \mathbb{R}^n$ ,  $y : [0, 1] \times [0, +\infty) \rightarrow \mathbb{R}^m$ , and  $\Lambda$  is a diagonal positive matrix in  $\mathbb{R}^{m \times m}$ . The perturbation parameter  $\epsilon$  is small and positive. The matrices  $A$  and  $B$  are in appropriate dimensions. The boundary condition matrices  $G_1$  and  $G_2$  are constant matrices of appropriate dimensions.

**Remark 1** Due to Theorem A.6. in (Bastin and Coron (2016)), for every  $Z_0 \in \mathbb{R}^n$ , for every  $y_0 \in L_2(0, 1)$ , the Cauchy problems (1) and (2) have a unique solution (in the classical weak sense)  $Z \in C^0([0, +\infty), \mathbb{R}^n)$ ,  $y \in C^0([0, +\infty), L^2((0, 1), \mathbb{R}^m))$ .  $\circ$

### 2.2 Sufficient stability condition

Let us state a preliminary result dealing with the stability of systems (1) and (2) based on a  $L^2$  Lyapunov function.

**Proposition 1** For all  $\epsilon > 0$ , systems (1) and (2) are exponentially stable, if there exist diagonal positive matrix  $Q$ , symmetric positive matrix  $P$  and positive constant  $\mu$  such that the following holds

$$\begin{pmatrix} e^{-\mu} Q \Lambda - G_1^\top Q \Lambda G_1 & -(G_1^\top Q \Lambda G_2 + B^\top P) \\ \star & -(A^\top P + P A) - G_2^\top Q \Lambda G_2 \end{pmatrix} > 0. \quad (3)$$

Moreover, there exists a strict Lyapunov function for system (1)

$$V_{1\epsilon}(Z, y) = Z^\top PZ + \epsilon \int_0^1 e^{-\mu x} y^\top Qy \, dx, \quad (4)$$

and a strict Lyapunov function for system (2)

$$V_{2\epsilon}(Z, y) = \epsilon Z^\top PZ + \int_0^1 e^{-\mu x} y^\top Qy \, dx. \quad (5)$$

**Proof.** Computing the time derivative of  $V_{1\epsilon}(Z, y)$  defined by (4) along the solution to system (1) yields

$$\begin{aligned} \dot{V}_{1\epsilon}(Z, y) &= 2Z^\top P\dot{Z} + 2\epsilon \int_0^1 e^{-\mu x} y^\top Qy_t \, dx \\ &= 2Z^\top P(AZ + By(1)) - [e^{-\mu x} y^\top(x) Q\Lambda y(x)]_{x=0}^{x=1} \\ &\quad - \mu \int_0^1 e^{-\mu x} y^\top Q\Lambda y \, dx \\ &= Z^\top (A^\top P + PA)Z + 2Z^\top PBy(1) \\ &\quad - [e^{-\mu} y^\top(1) Q\Lambda y(1) - y^\top(0) Q\Lambda y(0)] \\ &\quad - \mu \int_0^1 e^{-\mu x} y^\top Q\Lambda y \, dx, \end{aligned} \quad (6)$$

substituting the boundary condition (1c) into (6), we obtain

$$\begin{aligned} \dot{V}_{1\epsilon}(Z, y) &= Z^\top (A^\top P + PA)Z + 2Z^\top PBy(1) \\ &\quad - \left[ y^\top(1) (e^{-\mu} Q\Lambda - G_1^\top Q\Lambda G_1) y(1) - 2y^\top(1) G_1^\top Q\Lambda G_2 Z \right. \\ &\quad \left. - Z^\top G_2^\top Q\Lambda G_2 Z \right] - \mu \int_0^1 e^{-\mu x} y^\top Q\Lambda y \, dx \\ &= -\mu \int_0^1 e^{-\mu x} y^\top Q\Lambda y \, dx - \begin{pmatrix} y(1) \\ Z \end{pmatrix}^\top R \begin{pmatrix} y(1) \\ Z \end{pmatrix}, \end{aligned} \quad (7)$$

with  $R = \begin{pmatrix} e^{-\mu} Q\Lambda - G_1^\top Q\Lambda G_1 & -(G_1^\top Q\Lambda G_2 + B^\top P) \\ * & -(A^\top P + PA) - G_2^\top Q\Lambda G_2 \end{pmatrix}$ . There exist symmetric positive matrix  $P$  and diagonal positive matrix  $Q$  and positive constant  $\mu$  sufficiently small such that condition (3) is satisfied. Then it is deduced from (7)

$$\dot{V}_{1\epsilon}(Z, y) \leq -\gamma V_{1\epsilon}(Z, y),$$

where  $\gamma$  is positive constant. This concludes that system (1) is exponentially stable.

Similarly, we consider the Lyapunov function  $V_{2\epsilon}(Z, y)$  defined by (5) for system (2). Similar computations show the exponential stability of system (2). This concludes the proof of Proposition 1.  $\square$

### 3 Stability analysis via the singular perturbation method

Since the dynamics of systems (1) and (2) evolve in two time scales, the stability of these two systems is studied in this section based on the singular perturbation method.

#### 3.1 ODE/fast hyperbolic PDE system

System (1) is an ODE/fast PDE system where the perturbation parameter is introduced into the dynamics of the PDE.

##### 3.1.1 Reduced and boundary-layer subsystems

Let us adopt the computations of the two subsystems for finite dimensional system modeled by ODEs (Kokotović et al. (1986)). The reduced and the boundary-layer subsystems are formally computed as follows. By setting  $\epsilon = 0$  in equation (1b), we obtain

$$y_x(x, t) = 0. \quad (8)$$

It implies  $y(\cdot, t) = y(1, t)$ . Using this fact in the boundary condition (1c) and assuming  $(I_m - G_1)$  invertible yield

$$y(\cdot, t) = G_r Z(t), \quad (9)$$

where  $G_r = (I_m - G_1)^{-1} G_2$ . Using the right-hand side of (9) to replace  $y(1, t)$  in (1a), the reduced subsystem is computed as follows

$$\begin{cases} \dot{\bar{Z}}(t) = A_r \bar{Z}(t), & (10a) \\ \bar{Z}_0 = Z_0. & (10b) \end{cases}$$

where  $A_r = A + BG_r$ . The bar is used to indicate that the variables belong to the system with  $\epsilon = 0$ .

Performing the following change of variable  $\bar{y} = y - G_r Z$  and using a new time scale  $\tau = t/\epsilon$ , it is computed

$$\bar{y}_\tau(x, \tau) + \Lambda \bar{y}_x(x, \tau) = -\epsilon G_r (AZ(t) + By(1, t)).$$

The boundary-layer subsystem is formally computed with  $\epsilon = 0$

$$\begin{cases} \bar{y}_\tau(x, \tau) + \Lambda \bar{y}_x(x, \tau) = 0, & (11a) \\ \bar{y}(0, \tau) = G_1 \bar{y}(1, \tau), & (11b) \\ \bar{y}_0(x) = y_0(x) - G_r Z_0. & (11c) \end{cases}$$

##### 3.1.2 Stability analysis between the full system and both subsystems

In Proposition 1, we have stated a sufficient stability condition for system (1). In the following Proposition 2, we show that this condition (3) also implies the stability of the two subsystems (10) and (11).

**Proposition 2** *Condition (3) implies*

$$A_r^\top P + PA_r < 0, \quad (12)$$

which is equivalent to the stability of the reduced subsystem (10), and

$$e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1 > 0, \quad (13)$$

which implies the stability of the boundary-layer subsystem (11).

**Proof.** First, condition (13) comes directly from the first diagonal component of condition (3). Let us choose  $Q = \Delta^2 \Lambda^{-1}$ , where  $\Delta$  is a diagonal positive matrix. Condition (13) implies that  $\Delta^2 - G_1^\top \Delta^2 G_1 > 0$ , which is equivalent to  $\|\Delta G_1 \Delta^{-1}\| < 1$ . Thus, according to (Coron et al., 2008, Theorem 2.3), the boundary-layer subsystem is exponentially stable. Next, let us to prove condition (12). It holds from (3)

$$\begin{pmatrix} (I_m - G_1)^{-1} G_2 \\ I_n \end{pmatrix}^\top \begin{pmatrix} e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1 & -(G_1^\top Q\Lambda G_2 + B^\top P) \\ * & -(A^\top P + PA) - G_2^\top Q\Lambda G_2 \end{pmatrix} \times \begin{pmatrix} (I_m - G_1)^{-1} G_2 \\ I_n \end{pmatrix} > 0.$$

Developing the left-hand side of the above inequality yields

$$\begin{aligned} 0 < & \left( \begin{pmatrix} G_2^\top (I_m - G_1)^{-\top} (e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1) \\ -(G_1^\top Q\Lambda G_2 + B^\top P)^\top \end{pmatrix} (I_m - G_1)^{-1} G_2 - G_2^\top (I_m - G_1)^{-\top} \right. \\ & \left. \times (G_1^\top Q\Lambda G_2 + B^\top P) - (A^\top P + PA) - G_2^\top Q\Lambda G_2 \right). \end{aligned}$$

Developing and reorganizing the above inequality, we have

$$\begin{aligned} 0 < & \left( G_2^\top (I_m - G_1)^{-\top} e^{-\mu}Q\Lambda (I_m - G_1)^{-1} G_2 \right. \\ & - (A^\top + G_2^\top (I_m - G_1)^\top B^\top) P - P(A + B(I_m - G_1)^{-1} G_2) \\ & - G_2^\top (I_m - G_1)^{-\top} G_1^\top Q\Lambda G_1 (I_m - G_1)^{-1} G_2 \\ & - G_2^\top Q\Lambda G_1 (I_m - G_1)^{-1} G_2 - G_2^\top (I_m - G_1)^{-\top} G_1^\top Q\Lambda G_2 \\ & \left. - G_2^\top Q\Lambda G_2 \right). \end{aligned} \quad (14)$$

The last three lines of (14) are equivalent to the following terms

$$\begin{aligned} & -(G_2 + G_1(I_m - G_1)^{-1} G_2)^\top Q\Lambda (G_2 + G_1(I_m - G_1)^{-1} G_2) \\ & = -G_2^\top (I_m - G_1)^{-\top} Q\Lambda (I_m - G_1)^{-1} G_2. \end{aligned} \quad (15) \left( |Z(t)| + \|y(\cdot, t)\|_{L^2(0,1)} \right) \leq C e^{-\nu t} \left( |Z_0| + \|y_0\|_{L^2(0,1)} \right).$$

We recall that  $A_r = A + B(I_m - G_1)^{-1} G_2$ . Replacing the last three lines of (14) by (15) yields

$$0 < \left( G_2^\top (I_m - G_1)^{-\top} e^{-\mu}Q\Lambda (I_m - G_1)^{-1} G_2 - G_2^\top (I_m - G_1)^{-\top} Q\Lambda (I_m - G_1)^{-1} G_2 - (A_r^\top P + PA_r) \right).$$

Since  $e^{-\mu}Q\Lambda - Q\Lambda < 0$ , this implies that condition (12) holds, which is equivalent to the stability of the reduced subsystem. This concludes the proof of Proposition 2.  $\square$

Proposition 2 indicates that the sufficient stability condition (3) implies the stability of both subsystems. On the other hand, the converse may not be true, as stated in the following proposition.

**Proposition 3** *There exists system (1) such that conditions (12) and (13) are verified, however, it does not satisfy condition (3).*

**Proof.** We consider system (1) with  $A = 2$ ,  $B = -\frac{3}{2}$ ,  $\Lambda = 1$ ,  $G_1 = \frac{1}{2}$  and  $G_2 = 1$ . It is computed  $A_r = -1$ , which verifies condition (12) for any  $P > 0$ . Condition (13) is satisfied since  $\rho_1(G_1) < 1$ . We write condition (3) as follows

$$\begin{pmatrix} (e^{-\mu} - 0.25)Q & -(0.5Q - 1.5P) \\ * & -4P - Q \end{pmatrix} > 0. \quad (16)$$

Without loss of generality, we choose  $P = 1$ . It is checked that the trace and the determinant of the left-hand side of (16) are negative, which means there does not exist  $Q > 0$  such that (16) is satisfied. This concludes the proof of this proposition.  $\square$

### 3.1.3 Stability analysis for $\epsilon$ sufficiently small

Proposition 1 does not apply to ODE-fast PDE system (1) in the context of Proposition 3, since condition (3) is not verified. If we consider the effect of the perturbation parameter  $\epsilon$ , which is sufficiently small, the stability of system (1) can be guaranteed by the stability of its subsystems. This result is stated in the next theorem.

**Theorem 1** *Under conditions (12) and (13), there exists  $\epsilon^* > 0$ , such that for all  $\epsilon \in (0, \epsilon^*)$ , system (1) is exponentially stable, that is there exist positive values  $C$  and  $\nu$ , and for any  $Z_0 \in \mathbb{R}^n$ ,  $y_0 \in L^2(0, 1)$ , it holds for all  $t \geq 0$ ,*

Moreover, it has a strict Lyapunov function

$$V(Z, y) = Z^\top PZ + \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q (y - G_r Z) dx. \quad (17)$$

**Proof.** Let us consider the candidate Lyapunov function  $V$  defined by (17) and write it as  $V = V_1 + V_2$ , with

$$V_1 = Z^\top PZ,$$

$$V_2 = \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q (y - G_r Z) dx,$$

where  $P$  and  $Q$  will be specified later.

Computing the time derivative of  $V_1$  along the solution to system (1a) yields

$$\begin{aligned} \dot{V}_1 &= 2Z^\top P\dot{Z} \\ &= 2Z^\top P(AZ + By(1)) \\ &= 2Z^\top P\left(\left(A + BG_r\right)Z + B\left(y(1) - G_r Z\right)\right) \\ &= Z^\top\left(PA_r + A_r^\top P\right)Z + 2Z^\top PB\left(y(1) - G_r Z\right). \end{aligned}$$

Under condition (12), there exists symmetric positive matrix  $P$  such that

$$PA_r + A_r^\top P < -I_n. \quad (18)$$

Due to Cauchy Schwarz inequality, it follows

$$\dot{V}_1 \leq -|Z|^2 + 2\|PB\| |y(1) - G_r Z| |Z|. \quad (19)$$

Similarly, computing the time derivative of  $V_2$  along the solution to system (1b) yields

$$\begin{aligned} \dot{V}_2 &= 2 \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q (y_t - G_r \dot{Z}) dx \\ &= 2 \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q \left( -\frac{\Lambda y_x}{\epsilon} - G_r (AZ + By(1)) \right) dx \\ &= -\frac{2}{\epsilon} \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q \Lambda y_x dx \\ &\quad - 2 \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q G_r (AZ + By(1)) dx. \end{aligned}$$

Performing an integration by parts on the first integral,  $\dot{V}_2$  follows

$$\begin{aligned} \dot{V}_2 &= -\frac{1}{\epsilon} \left[ e^{-\mu x} (y - G_r Z)^\top Q \Lambda (y - G_r Z) \right]_{x=0}^{x=1} \\ &\quad - \frac{\mu}{\epsilon} \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q \Lambda (y - G_r Z) dx \\ &\quad - 2 \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q G_r (AZ + By(1)) dx. \quad (20) \end{aligned}$$

The first term in (20) is noted by  $V_{21}$ . Using the boundary condition (1c) and reorganizing the expression, it follows

$$\begin{aligned} V_{21} &= -\frac{1}{\epsilon} \left[ e^{-\mu} (y(1) - G_r Z)^\top Q \Lambda (y(1) - G_r Z) \right. \\ &\quad \left. - (y(0) - G_r Z)^\top Q \Lambda (y(0) - G_r Z) \right] \\ &= -\frac{1}{\epsilon} \left[ e^{-\mu} (y(1) - G_r Z)^\top Q \Lambda (y(1) - G_r Z) \right. \\ &\quad \left. - (G_1 y(1) + G_2 Z - G_r Z)^\top Q \Lambda (G_1 y(1) + G_2 Z - G_r Z) \right]. \quad (21) \end{aligned}$$

Note that  $G_2 - G_r$  can be computed as follows

$$\begin{aligned} G_2 - G_r &= G_2 - (I_m - G_1)^{-1} G_2 \\ &= \left( (I_m - G_1)(I_m - G_1)^{-1} - (I_m - G_1)^{-1} \right) G_2 \\ &= -G_1 (I_m - G_1)^{-1} G_2 = -G_1 G_r, \quad (22) \end{aligned}$$

substituting the right-hand side of (22) into (21), we obtain

$$V_{21} = -\frac{1}{\epsilon} \left[ (y(1) - G_r Z)^\top (e^{-\mu} Q \Lambda - G_1^\top Q \Lambda G_1) (y(1) - G_r Z) \right].$$

Under condition (13), there exists diagonal positive matrix  $Q$  such that

$$e^{-\mu} Q \Lambda - G_1^\top Q \Lambda G_1 > \underline{\lambda} (e^{-\mu} Q \Lambda - G_1^\top Q \Lambda G_1) > 0. \quad (23)$$

Thus

$$V_{21} \leq -\frac{\underline{\lambda} (e^{-\mu} Q \Lambda - G_1^\top Q \Lambda G_1)}{\epsilon} |y(1) - G_r Z|^2. \quad (24)$$

Let  $V_{22}$  denote the second term in (20), it follows

$$V_{22} \leq -\frac{\mu e^{-\mu} \underline{\lambda} (Q \Lambda)}{\epsilon} \|y - G_r Z\|_{L^2(0,1)}^2. \quad (25)$$

Let  $V_{23}$  denote the last term in (20), it follows

$$\begin{aligned} V_{23} &= -2 \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q G_r \left( (A + BG_r)Z \right. \\ &\quad \left. + B(y(1) - G_r Z) \right) dx \\ &= -2 \int_0^1 e^{-\mu x} (y - G_r Z)^\top Q G_r \left( A_r Z \right. \\ &\quad \left. + B(y(1) - G_r Z) \right) dx. \end{aligned}$$

Due to Cauchy Schwarz inequality,  $V_{23}$  becomes

$$\begin{aligned} V_{23} &\leq 2\|QG_r A_r\| |Z| \int_0^1 |y - G_r Z| dx \\ &\quad + 2\|QG_r B\| |y(1) - G_r Z| \int_0^1 |y - G_r Z| dx \\ &\leq 2\|QG_r A_r\| |Z| \|y - G_r Z\|_{L^2(0,1)} \\ &\quad + 2\|QG_r B\| |y(1) - G_r Z| \|y - G_r Z\|_{L^2(0,1)}. \end{aligned} \quad (26)$$

Combining (24), (25) and (26) yields

$$\begin{aligned} \dot{V}_2 &\leq -\frac{\lambda(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1)}{\epsilon} |y(1) - G_r Z|^2 \\ &\quad - \frac{\mu e^{-\mu} \lambda(Q\Lambda)}{\epsilon} \|y - G_r Z\|_{L^2(0,1)}^2 \\ &\quad + 2\|QG_r A_r\| |Z| \|y - G_r Z\|_{L^2(0,1)} \\ &\quad + 2\|QG_r B\| |y(1) - G_r Z| \|y - G_r Z\|_{L^2(0,1)}. \end{aligned} \quad (27)$$

Combining (19) with (27), we obtain

$$\dot{V} \leq - \begin{pmatrix} |y(1) - G_r Z| \\ |Z| \\ \|y - G_r Z\|_{L^2(0,1)} \end{pmatrix}^\top M \begin{pmatrix} |y(1) - G_r Z| \\ |Z| \\ \|y - G_r Z\|_{L^2(0,1)} \end{pmatrix},$$

where

$$M = \begin{pmatrix} M_1 & M_2 \\ \star & M_4 \end{pmatrix},$$

with

$$M_1 = \begin{pmatrix} M_{11} & M_{12} \\ \star & M_{14} \end{pmatrix} = \begin{pmatrix} \frac{\lambda(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1)}{\epsilon} & -\|PB\| \\ \star & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -\|QG_r B\| \\ -\|QG_r A_r\| \end{pmatrix},$$

$$M_4 = \begin{pmatrix} \frac{\mu e^{-\mu} \lambda(Q\Lambda)}{\epsilon} \end{pmatrix}.$$

Let us first study matrix  $M_1$ . Since  $M_{14} > 0$ , there exists  $\epsilon_1^* > 0$  such that for  $\epsilon \in (0, \epsilon_1^*)$ ,  $M_{11} - M_{12}M_{14}^{-1}M_{12}^\top > 0$ . Due to the Schur complement, it holds  $M_1 > 0$ . The inverse of  $M_1$  is computed as

$$\begin{aligned} M_1^{-1} &= \frac{1}{\lambda(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1) - \epsilon\|PB\|^2} \\ &\quad \times \begin{pmatrix} \epsilon & \epsilon\|PB\| \\ \star & \lambda(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1) \end{pmatrix}. \end{aligned}$$

Since  $M_4 > 0$ , there exists  $\epsilon_2^* > 0$ , such that for all  $0 < \epsilon < \min(\epsilon_1^*, \epsilon_2^*)$ ,

$$M_4 - M_2^\top M_1^{-1} M_2 > 0, \quad (28)$$

using again the Schur complement, we get  $M > 0$ . Thus there exists  $\alpha > 0$  such that, along the solution to (1),

$$\dot{V} \leq -\alpha V.$$

This concludes the proof of Theorem 1.  $\square$

### 3.2 Hyperbolic PDE/fast ODE system

In this section, we consider system (2), in which the fast dynamics is given by the ODE.

#### 3.2.1 Reduced and boundary-layer subsystems

By formally setting  $\epsilon = 0$  in (2a) and assuming  $A$  invertible, we compute

$$Z = -A^{-1}By(1). \quad (29)$$

Substituting (29) into (2c), the reduced subsystem is computed as

$$\begin{cases} \bar{y}_t(x, t) + \Lambda \bar{y}_x(x, t) = 0, & (30a) \\ \bar{y}(0, t) = G_r \bar{y}(1, t), & (30b) \\ \bar{y}(x, 0) = \bar{y}_0(x) = y_0(x), & (30c) \end{cases}$$

where  $G_r = G_1 - G_2 A^{-1} B$ .

Performing a change of variable  $\bar{Z} = Z + A^{-1}By(1)$  and using a new time scale  $\tau = t/\epsilon$ , it is computed

$$\begin{aligned} \frac{d\bar{Z}(\tau)}{d\tau} &= \epsilon \frac{dZ}{dt} + \epsilon A^{-1}B\dot{y}(1) \\ &= A(Z + A^{-1}By(1)) - \epsilon A^{-1}B\Lambda y_x(1). \end{aligned}$$

The boundary-layer subsystem is formally computed with  $\epsilon = 0$  as follows

$$\begin{cases} \frac{d\bar{Z}(\tau)}{d\tau} = A\bar{Z}(\tau), & (31a) \\ \bar{Z}(0) = \bar{Z}_0 = Z_0 + A^{-1}By_0(1). & (31b) \end{cases}$$

#### 3.2.2 Stability analysis between the full system and both subsystems

The following Proposition 3 indicates that the stability condition (3) implies the stability of both subsystems (30) and (31).

**Proposition 4** *Condition (3) implies*

$$A^\top P + PA < 0, \quad (32)$$

which is equivalent to the stability of the boundary-layer subsystem (31), and

$$\epsilon^{-\mu}Q\Lambda - G_r^\top Q\Lambda G_r > 0, \quad (33)$$

which implies the stability of the reduced subsystem (30).

**Proof.** Condition (32) comes directly from the second diagonal component of condition (3). Thus, the boundary-layer subsystem is exponentially stable. The proof of condition (33) is similar to the proof of condition (12). The following holds from (3)

$$\begin{pmatrix} I_m \\ -A^{-1}B \end{pmatrix}^\top \begin{pmatrix} e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1 & -(G_1^\top Q\Lambda G_2 + B^\top P) \\ \star & -(A^\top P + PA) - G_2^\top Q\Lambda G_2 \end{pmatrix} \\ \times \begin{pmatrix} I_m \\ -A^{-1}B \end{pmatrix} > 0.$$

Developing the left-hand side of the above inequality, we obtain

$$0 < \begin{pmatrix} e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1 + (A^{-1}B)^\top G_2^\top Q\Lambda G_1 \\ + G_1^\top Q\Lambda G_2 A^{-1}B - (A^{-1}B)^\top G_2^\top Q\Lambda G_2 A^{-1}B \\ + (A^{-1}B)^\top PB + B^\top PA^{-1}B \\ - (A^{-1}B)^\top (A^\top P + PA) A^{-1}B \end{pmatrix}. \quad (34)$$

We recall that  $G_r = G_1 - G_2 A^{-1}B$ . Then, the first two lines of the right-hand side of (34) is written as

$$e^{-\mu}Q\Lambda - G_r^\top Q\Lambda G_r.$$

The last two lines of the right-hand side of (34) equal zero. Thus, we get condition (33). According to (Coron et al., 2008, Theorem 2.3), the reduced subsystem is exponentially stable. This concludes the proof of Proposition 3.  $\square$

The stability condition (3) implies that both subsystems (30) and (31) are stable. However, the following proposition shows that there exists system (2) which verifies the two conditions (32), (33) but not (3).

**Proposition 5** *There exists system (2) such that conditions (32) and (33) are verified, however, it does not satisfy condition (3).*

**Proof.** We consider system (2) with  $A = -0.25$ ,  $B = 1.5$ ,  $\Lambda = 1$ ,  $G_1 = 3$ ,  $G_2 = -0.5$ . We compute that  $G_r = 0$ . Conditions (32) and (33) are verified for any  $P > 0$ ,  $Q > 0$ . Condition (3) is written as

$$\begin{pmatrix} (e^{-\mu} - 9)Q & 1.5(Q - P) \\ \star & 0.5P - 0.25Q \end{pmatrix} > 0.$$

Without loss of generality, we choose  $P = 1$ , it can be checked that there exists no real value  $Q > 0$ , which implies that condition (3) is not satisfied.  $\square$

### 3.2.3 Stability analysis of a counter-example

In the following, a counter-example shows that for PDE/fast ODE system, the stability of the two subsystems do not imply the stability of the full system. This

is in contrast with the result for ODE/fast PDE system stated in Theorem 1.

**Proposition 6** *There exists system (2) that verifies conditions (32) and (33), but which is unstable for all  $\epsilon > 0$ .*

**Proof.** Let us consider system (2) with  $A = -0.1$ ,  $B = -1$ ,  $\Lambda = 1$ ,  $G_1 = 2$ ,  $G_2 = 0.2$ . The full system is written as

$$\begin{cases} \epsilon \dot{Z}(t) = -0.1Z(t) - y(1), & (35a) \\ y_t(x, t) + y_x(x, t) = 0, & (35b) \\ y(0, t) = 2y(1, t) + 0.2Z(t). & (35c) \end{cases}$$

The reduced subsystem is computed as

$$\begin{cases} \bar{y}_t(x, t) + \bar{y}_x(x, t) = 0, & (36a) \\ \bar{y}(0, t) = 0. & (36b) \end{cases}$$

Since condition (33) is satisfied, the reduced subsystem is exponentially stable.

The boundary-layer subsystem is computed by

$$\frac{d\bar{Z}(\tau)}{d\tau} = -0.1\bar{Z}(\tau), \quad (37)$$

which is exponentially stable since condition (32) is satisfied. In the following we consider the stability property of the full system (35). Equation (35b) implies  $y(1, t) = y(0, t - 1)$ . Then, the full system is rewritten as

$$\begin{cases} \epsilon \dot{Z}(t) = -0.1Z(t) - u(t - 1), & (38a) \\ u(t) = 2u(t - 1) + 0.2Z(t), & (38b) \end{cases}$$

where  $u(t) = y(0, t)$ . Differentiating both sides of (38b) with respect to time yields

$$\begin{cases} \epsilon \dot{Z}(t) = -0.1Z(t) - u(t - 1), & (39a) \\ \dot{u}(t) = 2\dot{u}(t - 1) + 0.2\dot{Z}(t). & (39b) \end{cases}$$

Let us denote  $W(t) = (Z(t) \ u(t))^\top$ . System (39) becomes

$$\begin{aligned} \frac{d}{dt} \left( \begin{pmatrix} \epsilon & 0 \\ -0.2 & 1 \end{pmatrix} W(t) + \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} W(t - 1) \right) \\ = \begin{pmatrix} -0.1 & 0 \\ 0 & 0 \end{pmatrix} W(t) + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} W(t - 1). \end{aligned}$$

According to (Henry, 1974, Lemma 4.1) (see also (Mazanti, 2016, Theorem 1.41)), a necessary condition for the exponential stability of system (39) is that the following difference system

$$\begin{pmatrix} \epsilon & 0 \\ -0.2 & 1 \end{pmatrix} W(t) + \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} W(t - 1) = 0,$$



is exponentially stable. Reorganizing the above difference equation, we get

$$W(t) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} W(t-1). \quad (40)$$

Indeed, system (40) is unstable. Therefore, system (39) is unstable. In the following, we deduce the instability of system (38) from that of system (39). Let us consider a solution  $(Z \ u)^\top$  to system (39). By (39b), we obtain

$$u(t) = 2u(t-1) + 0.2Z(t) + C, \quad (41)$$

for suitable  $C \in \mathbb{R}$ .

In order to show that there exist real values  $\alpha, \beta$  such that

$$(\bar{Z} \ \bar{u})^\top = (Z + \alpha u + \beta)^\top \quad (42)$$

is a solution to system (38), we rewrite system (39) in view of (39a) and (41),

$$\begin{cases} \epsilon \dot{\bar{Z}}(t) = -0.1\bar{Z}(t) - \bar{u}(t-1) + 0.1\alpha + \beta, \\ \bar{u}(t) = 2\bar{u}(t-1) + 0.2\bar{Z}(t) - 0.2\alpha - \beta + C. \end{cases}$$

Note that (42) is a solution to system (38) as soon as

$$\begin{cases} 0.1\alpha + \beta = 0, \\ -0.2\alpha - \beta + C = 0, \end{cases}$$

hold. The existence of  $\alpha, \beta$  is equivalent to the invertibility of the matrix  $\begin{pmatrix} 0.1 & 1 \\ -0.2 & -1 \end{pmatrix}$  whose determinant is different from 0. This concludes the proof.  $\square$

## 4 Numerical simulations

In this section, firstly, the numerical simulations on an academic example are used to illustrate that under condition (3), the PDE/fast ODE system (2) is exponentially stable as stated in Proposition 1. Moreover, the two subsystems are also stable since the full system is stable. Secondly, we show the numerical results on the ODE/fast PDE system (1) given in the proof of Proposition 3. Although the general stability condition (3) is not satisfied, the stability of the full system can still be guaranteed by the stability of both subsystems when  $\epsilon$  sufficiently small. Finally, the simulations on the counterexample given in Section 3.2.3 indicate that the full PDE/fast ODE system is unstable even though the two subsystems are exponentially stable.

### 4.1 Numerical simulations on a PDE/fast ODE system illustrating Proposition 1

We consider system (2) with  $A = -1$ ,  $B = 0.1$  and  $\Lambda = 1$ . The boundary condition (2c) is given by  $G_1 = G_2 = 0.5$ . The perturbation parameter  $\epsilon$  is selected as  $\epsilon = 0.01$ . The initial conditions (2d)-(2e) are selected as  $Z_0 = 2$  and  $y_0(x) = \cos(4\pi x) - 1$ . By choosing  $P = Q = 1$ , the stability condition (3) is satisfied. Therefore Proposition 1 applies. In Figures 1 and 2, the solutions of the slow and the fast dynamics of the full system tend to zero when time increases, as expected from Proposition 1.

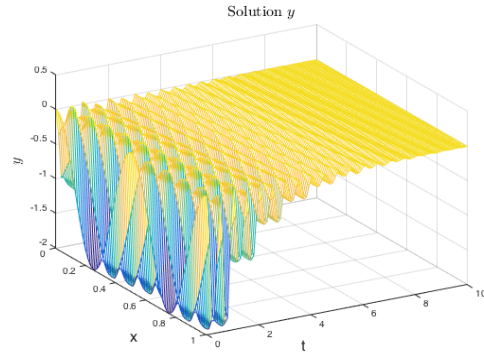


Fig. 1. Solution of the slow dynamics of the full system (2) in Section 4.1

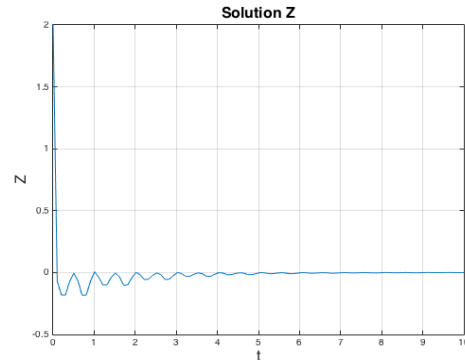


Fig. 2. Solution of the fast dynamics of the full system (2) in Section 4.1

Moreover, we compute the boundary condition matrix for the reduced subsystem as  $G_r = 0.55$ . The initial condition (30c) is chosen as  $\bar{y}_0 = y_0(x) = \cos(4\pi x) - 1$ . The boundary-layer subsystem is  $\frac{d\bar{Z}(\tau)}{d\tau} = -\bar{Z}(\tau)$ . Figures 3 and 4 are the solutions of the boundary-layer and the reduced subsystems respectively. It is observed that the solutions of both subsystems converge to the origin as time increases.

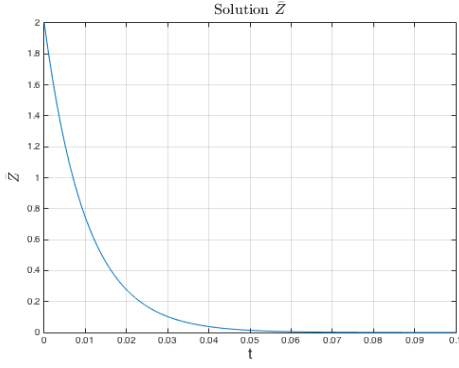


Fig. 3. Solution of the boundary-layer subsystem (31) in Section 4.1

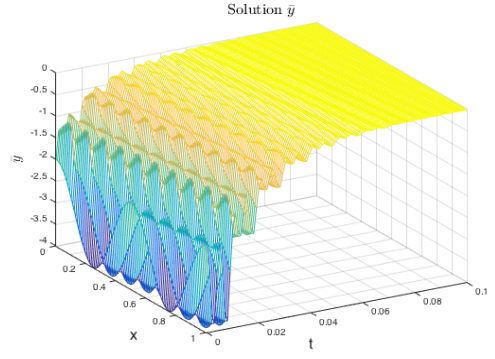


Fig. 5. Solution of the boundary-layer subsystem (11) in Section 4.2

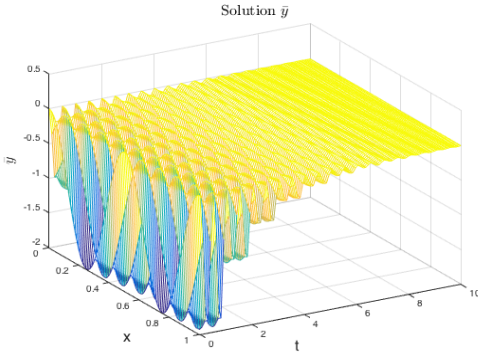


Fig. 4. Solution of the reduced subsystem (30) in Section 4.1

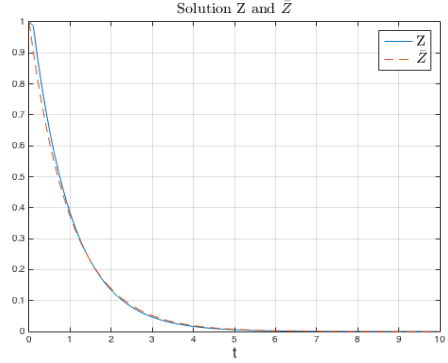


Fig. 6. Solution of the slow dynamics of the full system (1) and the reduced subsystem (10) in Section 4.2

#### 4.2 Numerical simulations on an ODE/fast PDE system illustrating Theorem 1

Let us consider system (1) given in the proof of Proposition 3 with  $A = 2$ ,  $B = -\frac{3}{2}$ ,  $\Lambda = 1$ . The boundary condition (1c) is  $G_1 = \frac{1}{2}$ ,  $G_2 = 1$ . The initial conditions (1d)-(1e) are selected as  $Z_0 = 1$  and  $y_0(x) = \cos(4\pi x) - 1$ . It is computed  $A_r = -1$  for the reduced subsystem (10). The initial condition (10b) is chosen as the same as for the full system  $\bar{Z}_0 = Z_0 = 1$ . The boundary condition for the boundary-layer subsystem is  $G_1 = 0.5$ . The initial condition is chosen as  $\bar{y}_0 = \cos(4\pi x) - 3$ . The perturbation parameter  $\epsilon$  is selected as  $\epsilon = 0.01$ . Since  $A_r = -1$ , condition (12) is satisfied for any  $P > 0$ . By choosing  $Q = 1$ , condition (13) holds. Therefore, Theorem 1 applies. Figure 5 shows that the boundary-layer subsystem converges to the origin as time increases. Figure 6 gives the solution of the reduced subsystem and the slow dynamics of the full system respectively. It is observed that the evolutions of  $Z$  and  $\bar{Z}$  are similar. However, there is a difference at the beginning which is due to the fast dynamics. After the fast dynamics vanish, the evolutions of the solution of the reduced subsystem and the slow dynamics of the full system are almost the same. In Figure 7, the solution of the fast dynamics of the full system tends to zero when time increases.

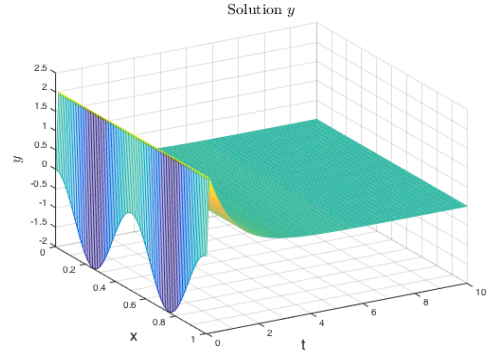


Fig. 7. Solution of the fast dynamics of the full system (1) in Section 4.2

#### 4.3 Numerical simulations of system (35) (counterexample)

The initial conditions are selected as  $Z_0 = \bar{Z}_0 = 1$ ,  $y_0 = \bar{y}_0 = 0$ . We choose the perturbation parameter  $\epsilon = 0.01$ . Figures 8 and 9 show the solutions of the two subsystems. It is observed that the solutions of both subsystems converge to the origin when time increases. Figures 10 and 11 are the solutions of the full system. It is

shown that the full system is divergent as time increases.

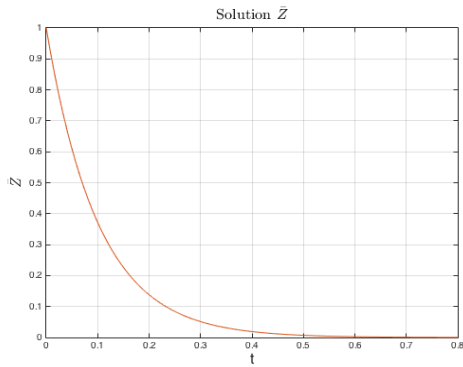


Fig. 8. Solution of the boundary-layer subsystem (37) in Section 4.3

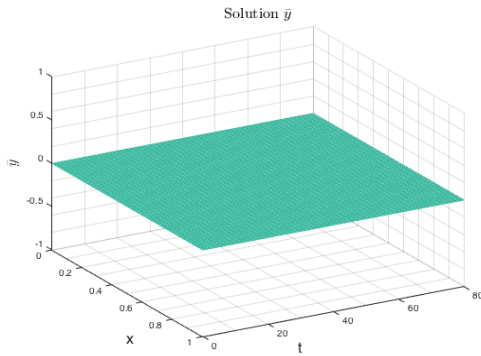


Fig. 9. Solution of the reduced subsystem (36) in Section 4.3

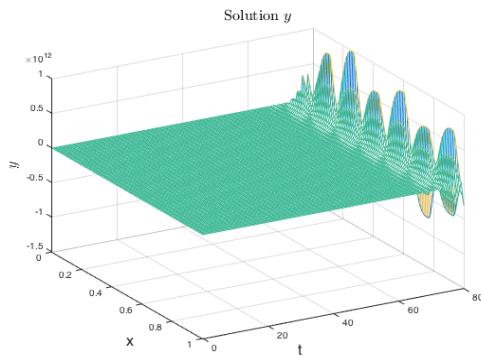


Fig. 10. Solution of the slow dynamics of the full system (35) in Section 4.3

## 5 Application to gas flow transport model

Let us consider the following experimental setup in Figure 12.

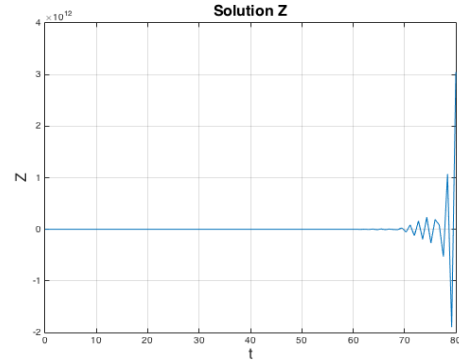


Fig. 11. Solution of the fast dynamics of the full system (35) in Section 4.3

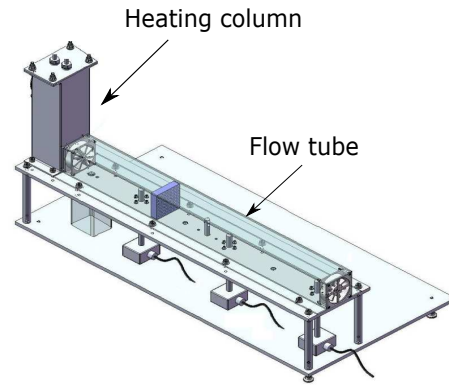


Fig. 12. Experimental Setup

The setup consists of two parts: a heating column and a tube. The gas dynamics in the heating column and in the tube are considered as two subsystems.

### 5.1 System descriptions and control design

**Model of the heating column:** To model the gas dynamics in the heating column, we first consider the following assumptions.

**Assumption 1** *The dynamics of the pressure in the gas control volume is much faster than that of the temperature, the pressure and the mass can be considered as quasi static.*

**Assumption 2** *The pressure losses are neglected because of the low mass flow and of the sufficiently large input output section of gas. This implies  $p_0 \approx p_{in}$ , where  $p_{in}$  is input pressure.*

**Assumption 3** *There is no work done by gas.*

Under the above three assumptions and due to the first law of thermodynamics and ideal gas law, the gas dynamics in the heating column is modeled by (see Castillo

et al. (2012))

$$\dot{\rho}_0 = -\frac{R\gamma T_{in}\dot{m}_{in}}{p_{in}V_0}\rho_0 - \frac{R}{p_{in}V_0C_v}\rho_0 dQ + \frac{\gamma\dot{m}_{in}}{V_0}, \quad (45)$$

where  $\rho_0$  is the gas density in the heating column,  $R$  is the specific gas constant,  $T_{in}$  denotes the gas temperature at input, the input mass flow is given by  $\dot{m}_{in}$ ,  $V_0$  is the volume of the heating column,  $C_v$  and  $C_p$  are the special heat of volume constant gas and of pressure constant gas respectively,  $dQ$  is the heating exchange that can be controlled and  $\gamma = \frac{C_p}{C_v}$ .

**Model of the tube:** To model the gas dynamics in the tube, let us state the following assumptions.

**Assumption 4** *All the heat transfers and friction losses are negligible.*

**Assumption 5** *The gas pressure in the tube is assumed to be constant, which is close to the atmosphere pressure.*

Under Assumptions 4-5, the gas dynamics in the tube is given by (see Castillo et al. (2012)), for  $x \in [0, 1]$  and for  $t \geq 0$ ,

$$\rho_t(x, t) + u_b\rho_x(x, t) = 0, \quad (46)$$

where  $\rho$  represents the gas density in the tube. The propagation speed in the tube is denoted by  $u_b$ . With a scaling of the space domain, it may be assumed that the tube's length equals 1. Due to ideal gas law,  $u_b = \frac{\dot{m}_{in}}{\rho_0 S_b}$ , where  $S_b$  is the cross section of the tube.

The boundary condition is given by

$$\rho(0, t) = \rho_0. \quad (47)$$

**Control problem statement:** In the following we state our control problem. Let us rewrite (45) and (46) as follows

$$\begin{cases} \dot{\rho}_0 = -\frac{1}{\kappa}\rho_0 + U(t), & (48a) \\ \rho_t + u_b\rho_x = 0. & (48b) \end{cases}$$

where  $\kappa$  is the transport time constant in the heating column and  $U(t)$  is the control (it could be defined from  $dQ$ ). The boundary condition is the same as (47).

The control problem is formulated as: for any desired mass density in the tube  $\rho^* \geq 0$ , let the controller be

$$U(t) = c_1\rho(1, t) + c_2\rho^*, \quad (49)$$

such that the system is exponentially stable at the equilibrium point  $\rho = \rho^*$  with an appropriate choice of real values  $c_1$  and  $c_2$ . Replacing  $U(t)$  in (48a) by the right-hand side in (49), the closed-loop system is written as

$$\begin{cases} \dot{\rho}_0 = -\frac{1}{\kappa}\rho_0 + c_1\rho(1, t) + c_2\rho^*, & (50a) \\ \rho_t + u_b\rho_x = 0, & (50b) \end{cases}$$

with the same boundary condition (47).

At the equilibrium point of the gas density inside of the tube  $\rho_0^*$ , from (50a) we get

$$\rho^* = \frac{\frac{1}{\kappa} - c_1}{c_2}\rho_0^*, \quad (51)$$

where  $c_2$  has to be selected such that  $c_2 \neq 0$ . Due to (47), it holds at the equilibrium  $\rho^* = \rho_0^*$ . From (51), the values of  $c_1$  and  $c_2$  should satisfy

$$c_1 + c_2 = \frac{1}{\kappa}.$$

Let us define the state deviations with respect to the equilibrium point

$$\begin{cases} \tilde{\rho}_0 = \rho_0 - \rho^*, \\ \tilde{\rho} = \rho - \rho^*. \end{cases}$$

The linearized system is

$$\begin{cases} \dot{\tilde{\rho}}_0 = -\frac{1}{\kappa}\tilde{\rho}_0 + c_1\tilde{\rho}(1, t), & (52a) \\ \epsilon\tilde{\rho}_t + \frac{1}{\kappa}\tilde{\rho}_x = 0, & (52b) \end{cases}$$

where the perturbation parameter is given by  $\epsilon = \frac{1/\kappa}{u_b^*}$ , due to the transport velocity of gas in the heating column is much smaller than that in the tube. The boundary condition is

$$\tilde{\rho}(0, t) = \tilde{\rho}_0. \quad (53)$$

From (53), recalling the condition in (1c), we compute  $G_1 = 0$ ,  $G_2 = 1$  and  $G_r = 1$ . Moreover, by (1) and (10), condition (12) is satisfied as soon as  $c_1 < \frac{1}{\kappa}$ . Condition (13) is verified since  $\rho_1(G_1) = 0$ .

The reduced subsystem is

$$\dot{\bar{\rho}}_0 = \left(-\frac{1}{\kappa} + c_1\right)\bar{\rho}_0, \quad (54)$$

whereas the boundary-layer subsystem is

$$\begin{cases} \bar{\rho}_\tau + \frac{1}{\kappa}\bar{\rho}_x = 0, & (55a) \\ \bar{\rho}(0, \tau) = 0, & (55b) \end{cases}$$

where  $\tau = \frac{t}{\epsilon}$ .

## 5.2 Numerical simulations of the application to gas flow transport system

Let us take the experimental data from Castillo et al. (2012):  $\gamma = 1.4$ ,  $R = 8.3$ ,  $p_{in} = 1 \times 10^5$ ,  $T_{in} = 300$ ,  $V_0 = 4 \times 10^{-3}$ ,  $\dot{m}_{in} = 0.01$ ,  $S_b = 6.4 \times 10^{-3}$ . We compute  $\kappa = 10$ . By choosing  $\rho^* = 1.5$  we obtain  $\epsilon = 0.1$ . We

choose  $c_1 = 0.03$ . The initial conditions are given by:  $\rho_0(0) = \bar{\rho}_0(0) = 2$ ,  $\rho(0) = \cos(4\pi x) - 1$ ,  $\bar{\rho}(0) = \rho(0) - \rho_0(0) = \cos(4\pi x) - 3$ . Figure 13 is the solution of the boundary-layer subsystem. It converges to the origin. Figure 14 shows that the reduced subsystem and the slow dynamics of the full system are roughly the same. They converge to zero as time increases. In Figure 15, it is shown that the solution of the fast dynamics of the full system tends to zero as time increases, as expected from Theorem 1.

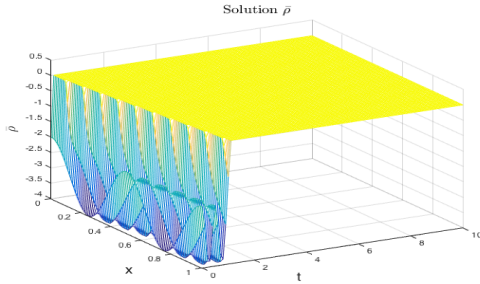


Fig. 13. Solution of the boundary-layer subsystem (55)

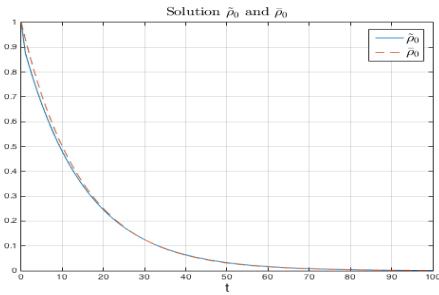


Fig. 14. Solution of the slow dynamics of the full system (52)-(53) and the reduced subsystem (54)

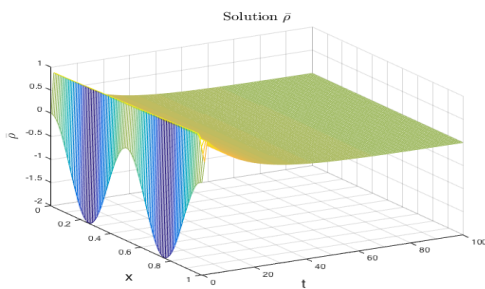


Fig. 15. Solution of the fast dynamics of the full system (52)-(53)

**Remark 2** According to the proof of Theorem 1, by choosing  $P = Q = 1$ , the theoretical value of  $\epsilon^*$  is computed as 0.9 which is consistent with the choice of the perturbation parameter  $\epsilon = 0.1$  in this physical model. ◻

## 6 Conclusion

This work has dealt with linear ODE coupled with linear hyperbolic PDE systems with two time scales. Firstly, a general sufficient stability condition has been stated for such systems. Next, the link of the stability between the full system and the subsystems has been provided based on the singular perturbation method. For ODE/fast PDE system, the stability of the two subsystems guarantees the stability of the full system for  $\epsilon$  sufficiently small. However this is not true for PDE/fast ODE system. Precisely, the full system could be unstable even though both subsystems are stable. Moreover, a new boundary control strategy has been proposed to stabilize a gas flow transport system modeled by an ODE/fast PDE system.

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