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► **To cite this version:**

Marc Bonnet, Rémi Cornaggia. Higher-order expansion of misfit functional for defect identification in elastic solids. WAVES 2015 The 12th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jul 2015, Karlsruhe, Germany. hal-01388731

HAL Id: hal-01388731

<https://hal.science/hal-01388731v1>

Submitted on 27 Oct 2016

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Higher-order expansion of misfit functional for defect identification in elastic solids

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Abstract. In this work, least-squares functionals commonly used for defect identification are expanded in powers of the small radius of a trial inclusion, in the context of time-harmonic elastodynamics, generalizing to higher orders the concept of topological derivative. Such expansion, whose derivation and evaluation are facilitated by using an adjoint state, provides a basis for the quantitative estimation of flaws whereby a region of interest may be exhaustively probed at reasonable computational cost.

Keywords: Topological derivative, identification, elastodynamics, asymptotic analysis

Problem statement. We consider a reference (i.e defect-free) 3D elastic solid Ω characterized by Hooke's tensor \mathbf{C} and mass density ρ . The time-harmonic background displacement \mathbf{u} then solves

$$\langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\mathbf{C}} - \omega^2 \langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\rho} = \mathcal{F}(\mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{W}, \quad (1)$$

where $\langle \cdot, \cdot \rangle_D^{\mathbf{C}}$ and $\langle \cdot, \cdot \rangle_D^{\rho}$ denote the stiffness and mass bilinear forms associated to a given domain D characterized by (\mathbf{C}, ρ) , $\mathcal{W} \subset H^1(\Omega)$ is the function space incorporating the relevant essential boundary conditions (if any), the linear form $\mathcal{F} \in \mathcal{W}'$ defines the applied time-harmonic loading and ω is the angular frequency.

Assuming the presence of a defect inside Ω , and that we can measure the resulting displacement \mathbf{u}^{ex} on a surface Γ , we define the least-squares cost functional $J(\mathbf{w})$, with the elastodynamic displacement \mathbf{w} associated to the given excitation and a known trial defect, by:

$$J(\mathbf{w}) = \frac{1}{2} \int_{\Gamma} |\mathbf{w}(\mathbf{x}) - \mathbf{u}^{\text{ex}}(\mathbf{x})|^2 dS_{\mathbf{x}} \quad (2)$$

We now consider a specific trial defect $B_a = \mathbf{z} + a\mathcal{B}$ (Fig 1), centered at $\mathbf{z} \in \Omega$, of small size a and reference shape \mathcal{B} . It is a perfectly bonded inclusion filled with a material characterized by its Hooke tensor $\mathbf{C}^* = \mathbf{C} + \Delta\mathbf{C}$ and mass density $\rho^* = \rho + \Delta\rho$. We denote \mathbf{u}_a the displacement in the perturbed domain, and $\mathbf{v}_a = \mathbf{u}_a - \mathbf{u}$ the displacement perturbation. $J(\mathbf{u}_a)$ admits the

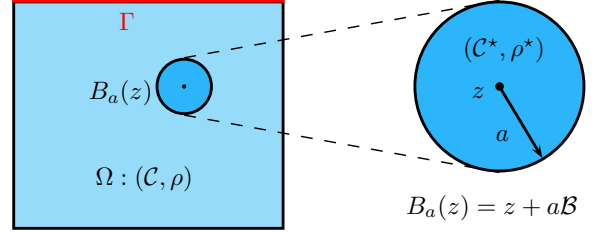


Figure 1: Computational domain and inclusion.

exact expansion about \mathbf{u} :

$$\begin{aligned} J(\mathbf{u}_a) &= J(\mathbf{u}) + J'(\mathbf{u}; \mathbf{v}_a) + J''(\mathbf{u}; \mathbf{v}_a, \mathbf{v}_a) \\ &= J(\mathbf{u}) + \Re \int_{\Gamma} \overline{(\mathbf{u} - \mathbf{u}^{\text{ex}})} \cdot \mathbf{v}_a + \frac{1}{2} \int_{\Gamma} |\mathbf{v}_a|^2 \quad (3) \end{aligned}$$

The goal is now to expand $J(\mathbf{u}_a)$ in powers of a . Similar expansions have been studied in e.g. [2] for rigid obstacles in 3D acoustic media and [5] for holes in 2D elastic bodies.

Define the adjoint field \mathbf{p} as the solution of

$$\langle \mathbf{p}, \mathbf{w} \rangle_{\Omega}^{\mathbf{C}} - \omega^2 \langle \mathbf{p}, \mathbf{w} \rangle_{\Omega}^{\rho} = J'(\mathbf{u}; \mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{W}. \quad (4)$$

We can then compute $J'(\mathbf{u}; \mathbf{v}_a)$ as

$$J'(\mathbf{u}; \mathbf{v}_a) = -\langle \mathbf{p}, \mathbf{u}_a \rangle_{B_a}^{\Delta\mathbf{C}} + \omega^2 \langle \mathbf{p}, \mathbf{u}_a \rangle_{B_a}^{\Delta\rho} \quad (5)$$

Expanding \mathbf{v}_a in powers of a is now needed. As we will see, $J''(\mathbf{u}; \mathbf{v}_a, \mathbf{v}_a)$ requires only the leading contribution of $\mathbf{v}_a|_{\Gamma}$ whereas a higher-order expansion of $\mathbf{v}_a|_{B_a}$ is needed for evaluating (5).

Expansion of the solution perturbation.

Following e.g. [3], \mathbf{v}_a solves the integro-differential Lippmann-Schwinger equation:

$$\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = -\langle \mathbf{u}, \mathbf{G} \rangle_{B_a}^{\Delta\mathbf{C}} + \omega^2 \langle \mathbf{u}, \mathbf{G} \rangle_{B_a}^{\Delta\rho} \quad (6)$$

with $\mathcal{L}_a[\mathbf{v}](\mathbf{x}) := \mathbf{v}(\mathbf{x}) + \langle \mathbf{v}, \mathbf{G} \rangle_{B_a}^{\Delta\mathbf{C}} - \omega^2 \langle \mathbf{v}, \mathbf{G} \rangle_{B_a}^{\Delta\rho}$ and $\mathbf{G} = \mathbf{G}(\cdot, \mathbf{x})$ is the elastodynamic Green's tensor for a unit point force applied at \mathbf{x} and satisfying homogeneous boundary conditions consistent with problem (1) on $\partial\Omega$. Substituting the ansatz

$$\begin{aligned} \mathbf{v}_a(\mathbf{x}) &= a\mathbf{V}_1(\bar{\mathbf{x}}) + a^2\mathbf{V}_2(\bar{\mathbf{x}}) + \frac{1}{2}a^3\mathbf{V}_3(\bar{\mathbf{x}}) \\ &\quad + \frac{1}{6}a^4\mathbf{V}_4(\bar{\mathbf{x}}) + \delta_a(\mathbf{x}), \quad \mathbf{x} \in B_a \quad (7) \end{aligned}$$

(with $\bar{\mathbf{x}} := (\mathbf{x} - \mathbf{z})/a \in \mathcal{B}$) into (6) and expanding the resulting equation in powers of a (in particular using that $\mathbf{G}(\boldsymbol{\xi}, \mathbf{x}) = a^{-1}\mathbf{G}_\infty(\bar{\boldsymbol{\xi}} - \bar{\mathbf{x}}) + O(1)$, with \mathbf{G}_∞ denoting the *static* full-space Kelvin fundamental solution) yields a sequence of integral equations for the \mathbf{V}_j . These equations correspond to *elastostatic* problems for the normalized inclusion \mathcal{B} embedded in an unbounded reference medium, and are solved with the help of Eshelby's equivalent inclusion method [4].

The remainder $\boldsymbol{\delta}_a$ in (7) solves an integro-differential equation of the form $\mathcal{L}_a[\boldsymbol{\delta}_a] = \boldsymbol{\gamma}_a$. The operator $\mathcal{L}_a : H^1(B_a) \rightarrow H^1(B_a)$ is shown to be invertible with bounded inverse, while $\boldsymbol{\gamma}_a$ can be estimated as $\|\boldsymbol{\gamma}_a\|_{H^1(B_a)} = O(a^{11/2})$. Consequently, there exists a constant $C > 0$ independent of a such that

$$\|\boldsymbol{\delta}_a\|_{H^1(B_a)} \leq Ca^{11/2}. \quad (8)$$

For $\mathbf{x} \notin B_a$, plugging (7) in the form $\mathbf{v}_a(\mathbf{x}) \approx a\mathbf{V}_1(\bar{\mathbf{x}})$ into (6) yields the outer expansion

$$\begin{aligned} \mathbf{v}_a(\mathbf{x}) = & -a^3 [\boldsymbol{\nabla} \mathbf{u}(\mathbf{z}) : \boldsymbol{\mathcal{A}} : \boldsymbol{\nabla} \mathbf{G}(\mathbf{z}, \mathbf{x}) \\ & - \omega^2 \Delta \rho |\mathcal{B}| \mathbf{u}(\mathbf{z}) \cdot \mathbf{G}(\mathbf{z}, \mathbf{x})] + o(a^3), \end{aligned} \quad (9)$$

$\boldsymbol{\mathcal{A}}$ being the elastic moment tensor associated to \mathcal{B} , $\boldsymbol{\mathcal{C}}$ and $\Delta \boldsymbol{\mathcal{C}}$ [1, 3].

Cost functional expansion. Substituting (7) into (5) and (9) into (3), $J(\mathbf{u}_a)$ is finally found to have an expansion of the form:

$$J(\mathbf{u}_a) = J_6(a, \mathbf{z}) + o(a^6) \quad (10)$$

$$\begin{aligned} \text{with } J_6(a, \mathbf{z}) = & J(\mathbf{u}) + a^3 \mathcal{T}_3(\mathbf{z}) + a^4 \mathcal{T}_4(\mathbf{z}) \\ & + a^5 \mathcal{T}_5(\mathbf{z}) + a^6 \mathcal{T}_6(\mathbf{z}), \end{aligned}$$

the $o(a^6)$ estimate resulting from (8) and (9).

The $\mathcal{T}_j(\mathbf{z})$ are found to be given in terms of (i) the background field \mathbf{u} and its derivatives at \mathbf{z} , (ii) the adjoint field \mathbf{p} and its derivatives at \mathbf{z} , (iii) $\boldsymbol{\mathcal{A}}$ and other elastic moment tensors that involve the material parameters, the shape \mathcal{B} and the angular frequency ω , and (iv) the complementary part of \mathbf{G} , i.e. $\mathbf{G} - \mathbf{G}_\infty = (\mathbf{G}_{\infty, \omega} - \mathbf{G}_\infty) + \mathbf{G}_C$, where $\mathbf{G}_{\infty, \omega}$ is the elastodynamic full-space fundamental solution and \mathbf{G}_C accounts for the boundedness of Ω . In particular, $\mathcal{T}_3(\mathbf{z})$ is the well-known topological derivative:

$$\mathcal{T}_3(\mathbf{z}) = -[\boldsymbol{\nabla} \mathbf{u} : \boldsymbol{\mathcal{A}} : \boldsymbol{\nabla} \mathbf{p} - \omega^2 \Delta \rho |\mathcal{B}| \mathbf{u} \cdot \mathbf{p}](\mathbf{z}).$$

Moreover, $\mathcal{T}_4(\mathbf{z}) = 0$ for any centrally-symmetric shape \mathcal{B} . The complementary part $\mathbf{G}_{\infty, \omega} - \mathbf{G}_\infty$ (known analytically) is involved in $\mathcal{T}_5(\mathbf{z})$ and

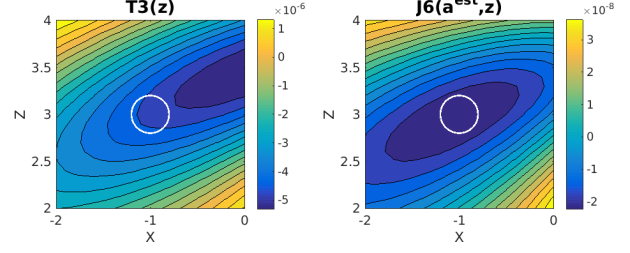


Figure 2: $\mathcal{T}_3(\mathbf{z})$ and $J_6(a^{\text{est}}, \mathbf{z})$ plotted in the (XZ) plane around the obstacle (in white).

$\mathcal{T}_6(\mathbf{z})$, while \mathbf{G}_C appears in $\mathcal{T}_6(\mathbf{z})$ only. Since the exact computation of \mathbf{G}_C would require solving an elastodynamic problem on Ω for each trial location \mathbf{z} , we plan to use an approximation method to save computational time.

Closed-form formulae for the \mathcal{T}_j can be obtained when \mathcal{B} is spherical (for which case we provide explicit expressions) or ellipsoidal.

Identification. Following [2], estimates of the location \mathbf{z}^{est} and size a^{est} of the real defect can then be sought as minimizers of $J_6(a, \mathbf{z})$, with \mathbf{z} spanning a predefined sampling grid. This entails computing the $\mathcal{T}_j(\mathbf{z})$ over the sampling grid and minimizing $a \mapsto J_6(a, \mathbf{z})$ for each \mathbf{z} , the latter step being very fast and straightforward.

A preliminary example is set in free space (so that $\mathbf{G}_C = 0$) for a spherical scatterer of radius $0.1\lambda_S$ illuminated by a plane P-wave travelling along the positive x -direction, with a discrete array of displacement sensors lying behind the scatterer. The above procedure yields the size estimate $a^{\text{est}} \approx 0.105\lambda_S$; moreover, the estimated location \mathbf{z}^{est} is found to be very close to the true center of the scatterer. The contour plot of $J_6(a^{\text{est}}, \mathbf{z})$ (Fig 2) shows improved localisation (relative to the topological derivative $\mathcal{T}_3(\mathbf{z})$) for this partial-aperture configuration.

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