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Sensitivity of the mixing current technique to detect nano-mechanical motion

Yue Wang¹ and Fabio Pistolesi¹

¹*Univ. Bordeaux, LOMA, UMR 5798, Talence, France.
CNRS, LOMA, UMR 5798, F-33400 Talence, France.*

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Detection of nano-mechanical displacement by transport techniques has reached high level of sensitivity and versatility. In order to detect the amplitude of oscillation of nano-mechanical oscillator a widely used technique consists to couple this motion capacitively to a single-electron transistor and to detect the high-frequency modulation of the current through the non-linear mixing with an electric signal at a slightly detuned frequency. The method known as current-mixing technique is employed in particular for the detection of suspended carbon nanotubes. In this paper we study theoretically the limiting conditions on the sensitivity of this method. The sensitivity is increased by increasing the response function to the signal, but also by reducing the noise. For these reasons we study systematically the response function, the effect of current- and displacement-fluctuations, and finally the case where the tunnelling rate of the electrons are of the same order or larger of the resonating frequency. We find thus upper bounds to the sensitivity of the detection technique.

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I. INTRODUCTION

Nano-electromechanical systems have great potentials as ultra-sensitive detectors for several physical quantities. Recent advances allowed to reach record sensitivity in mass sensing.¹⁻³ This has been possible by the detection of the frequency shift of ultralight oscillators when an additional mass is attached to it. Other examples concern the detection of the tiny magnetic field generated by nuclear spins. This can be done by the opto-mechanical detection of the force generated by the magnetic dipoles,⁴ but also with electro-mechanical means,⁵ or by coupling to two-level systems.⁶⁻⁸

The force sensitivity of the device is then the limiting factor for the sensitivity, and again recent advances showed that it is possible to obtain record force sensing with carbon-nanotube oscillators.^{9,10} At the same time nano-mechanical oscillators can be so small that interaction between electronic and mechanical degrees of freedom may lead to new and unexpected phenomena¹¹⁻¹⁵ like the blockade of the current¹⁶⁻¹⁹, cooling²⁰⁻²² or unusual mechanical response.^{23,24}

In order to exploit nanomechanical resonators, or to study their properties, detection of mechanical motion is crucial. Most detection methods exploiting electronic transport are based on the high sensitivity of single-electron transistors (SET) to a variation of the gate charge. By coupling capacitively the oscillator to the gate of the SET it is possible to detect the motion of the oscillator with a high accuracy.²⁵ The method has been used also to cool the oscillator by the back-action of the electronic transport.²⁶ The main difficulty of the method stems from the high frequency character of the oscillator motion that is typically in the 100 MHz-1 GHz range. Due to the high impedance of the SET, it is more convenient to down-convert the signal to lower frequency before extracting it. This can be achieved by non-linear mixing the mechanically generated modulation with a

second high-frequency signal injected between source and drain. The signal at the difference of the two frequencies can be extracted and measured. To our knowledge, for nanomechanical resonators this method was implemented in metallic SET by the group of A. Cleland back in 2003.²⁷ It was later adapted to the detection of carbon nanotube by the group of P.L. McEuen.²⁸ It then became the method of choice for carbon nanotubes, leading to several breakthroughs: the observation of the first single-electron backaction effects in carbon nanotubes,^{29,30} ultrasensitive mass detection,^{2,3} the detection of the charge response function in quantum dot,³¹ the detection of magnetic molecules^{32,33} and the observation of decoherence of mechanical motion.³⁴ The same method can also be implemented by frequency modulation.³⁵ It is clear that the technique is powerful and that it will continue to be used both for fundamental research and for applications. The question we want to address in this paper is which is the ultimate resolution that can be reached with this kind of detection. In order to do this we investigated three main issues. The first one is how to optimize the response function, that is the quantity $\partial I_{\text{mx}}/\partial x_m$, where I_{mx} is the measured signal, the mixing current, and x_m the amplitude of the mechanical oscillation. The second one is to study the effect of current and mechanical fluctuations. These contribute to the fluctuation of the measured signal and in the end are at the origin of the signal to noise ratio. The third is to consider the case of a mechanical oscillator with a resonating frequency ω_m faster than the typical tunneling rate of the electrons Γ . We will develop a theory of transport to obtain the mixing current for any ratio ω_m/Γ . The case of a metallic and single-electronic level SET will be considered in details and explicit expressions will be given.

The paper is structured as follows: Section II gives an introduction to the mixing technique and provides the expression of the detector gain λ . Section III analyze current and mechanical fluctuations giving general ex-

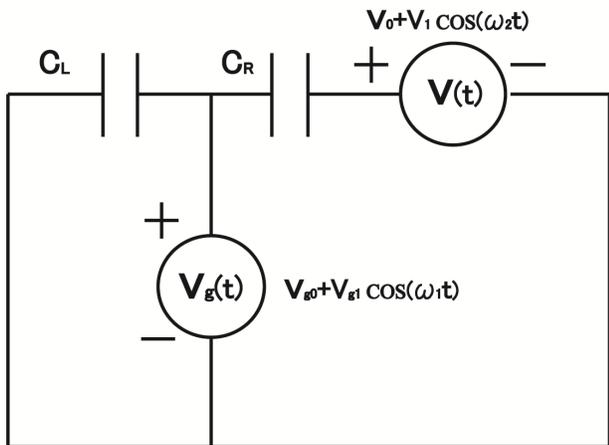


Figure 1. Schematic of the typical experimental set-up used to measure the displacement of a mechanical oscillator by detection of the mixing current (adapted from Ref. 28)

pressions. Section IV provides a general theory for the detector gain when the resonator frequency is comparable or larger than the typical tunnelling time. Section V and VI gives the explicit expressions for the response function and for the noise in the case of a metallic and a single electron SET. Finally Section VI gives our conclusions.

II. MIXING TECHNIQUE AND RESPONSE FUNCTION

Let us begin by describing the typical system used to measure the oscillation amplitude of a mechanical oscillator by detection of the mixing current.^{27–30} As shown in Fig. 1 a conducting oscillator is capacitively coupled to the central island of a single-electron transistor: its displacement modulates thus the gate capacitance $C_g(x)$, where $x(t)$ is the displacement of the oscillator. We assume the presence of a single mechanical mode whose displacement is parametrized by x , a generalized coordinate with the dimensions of a length. We will consider that the SET is operated in the incoherent transport regime valid for $\hbar\Gamma \ll k_B T$, where Γ is the electron tunnelling rate and T the temperature (\hbar and k_B are the reduced Planck constant and the Boltzmann constant, respectively). This is the standard case for nano-mechanical devices. The current I through the device can be obtained by using the Master equation and in general it can be expressed as a function of the source-drain bias voltage V and on the gate charge $n_g = C_g(x)V_g/e$, where V_g is the gate voltage (see Appendix A for a short derivation). The current reads thus:

$$I = I(V, n_g). \quad (1)$$

In this section we want to obtain the current response of the system when both V and V_g are modulated at

two slightly different frequencies ω_1 and ω_2 , both much smaller than Γ . We write

$$V_g(t) = V_{g0} + V_{g1}(t), \quad V(t) = V_0 + V_1(t), \quad (2)$$

where $V_{g1}(t) = V_{g1} \cos(\omega_1 t)$ and $V_1(t) = V_1 \cos(\omega_2 t)$. Choosing ω_1 close to the mechanical resonating frequency ω_m allows to drive the resonator, since the modulation of the gate voltage modulates the charge on the suspended part and thus induces an oscillating force [see also Eq. (18) in the following]. For small driving amplitude the oscillator responds linearly to the external drive:

$$x(t) = x_m \cos(\omega_1 t + \phi), \quad (3)$$

where we always measure x from its equilibrium position. (Note that in general x_m and ϕ depend on the driving frequency ω_1 .) The modulation of V_g induces thus the following modulation of n_g at linear order in the driving:

$$n_g(t) = n_{g0} + \frac{C_g V_{g1}}{e} \cos(\omega_1 t) + \frac{C'_g x_m V_{g0}}{e} \cos(\omega_1 t + \phi), \quad (4)$$

where $C'_g \equiv dC_g/dx$. It is convenient to introduce a length scale by defining $L = C_g/C'_g$. From geometric considerations L has to be of the order of the distance of the gate from the oscillator, thus typically undreds of nm. The fluctuating part of n_g can then be written as

$$n_{g1}(t) = n_{g0} \left[\frac{V_{g1}}{V_{g0}} \cos(\omega_1 t) + \frac{x_m}{L} \cos(\omega_1 t + \phi) \right], \quad (5)$$

where $n_{g0} = C_g V_{g0}/e$. The mechanical term (x_m/L) has a strong frequency dependence close to the mechanical resonance, and can thus be distinguished by the background electrostatic term (V_{g1}/V_{g0}). The two contributions to the modulation of the gate charge can be combined in a single cosine term:

$$n_{g1}(t) = n_{g1} \cos(\omega_1 t + \varphi). \quad (6)$$

Assuming now that the oscillator frequency ω_m , and thus also ω_1 and ω_2 , are much smaller than the typical tunnelling rate Γ , one can use Eq. (1) to obtain the time dependent current in presence of time-dependent V and n_g . For small modulation amplitude we Taylor expand Eq. (1) to second order in V_1 and n_{g1} obtaining

$$I(t) = I(V_0, V_{g0}) + \frac{\partial I}{\partial V} V_1(t) + \frac{\partial I}{\partial n_g} n_{g1}(t) + \frac{1}{2} \frac{\partial^2 I}{\partial V^2} V_1^2(t) + \frac{\partial^2 I}{\partial V \partial n_g} V_1(t) n_{g1}(t) + \frac{1}{2} \frac{\partial^2 I}{\partial n_g^2} n_{g1}^2(t) + \dots \quad (7)$$

Only the term proportional to $\partial^2 I / \partial V \partial n_g$ has a component that oscillates at the frequency $\omega_\Delta = \omega_1 - \omega_2$. This signal can be extracted by a standard lock-in technique that essentially allows to measure the quantity I_{mx} :

$$I_{\text{mx}}^c = \int_0^{T_m} \frac{dt}{T_m} I(t) \cos[\omega_\Delta t]. \quad (8)$$

The other quadrature I_{mx}^s with $\sin(\omega_\Delta t)$ is defined in a similar way. Averaging over a long measurement time $T_m \gg 1/\omega_\Delta$ one obtains:

$$I_{\text{mx}}^c = \frac{V_1}{4} \frac{\partial^2 I}{\partial V \partial n_g} [C_g V_{g1} + C'_g V_{g0} x_m \cos \phi], \quad (9)$$

$$I_{\text{mx}}^s = -\frac{V_1}{4} \frac{\partial^2 I}{\partial V \partial n_g} C'_g V_{g0} x_m \sin \phi. \quad (10)$$

The detector gain with respect to the two quadrature of x_m is thus:

$$\lambda = \frac{1}{4e} \frac{\partial^2 I}{\partial V \partial n_g} C'_g V_{g0} V_1. \quad (11)$$

It measures the sensitivity of the mixing current signal with respect to the two quadratures of x . This quantity depends on the particular bias conditions of the SET, and will be studied in some details in Section V and VI for two explicit models. Note also that in order to obtain λ we need only the static expression for the current. This assumes that the electronic mechanism is much faster than the time dependence of the driving. In order to describe the case of a fast oscillator (to be discussed in Section IV) we will need a detailed description of the charge dynamics, and the response function will be no more expressed only in terms of derivatives of the static non-linear current voltage characteristics.

III. EFFECT OF CURRENT AND DISPLACEMENT FLUCTUATIONS

Expression (11) assumes a deterministic evolution of both the current and the displacement of the oscillator $x(t)$. In practice both quantity fluctuate, the first due to shot or thermal noise, and the second due to stochastic fluctuations induced either by the bias voltage or by the thermal fluctuations. In general one can then write the value of I_{mx} in a specific time region as follows:

$$(I_{\text{mx}}^c)_n = \int_{nT_m}^{(n+1)T_m} [I(t) + \delta I(t)] \cos(\omega_\Delta t) dt, \quad (12)$$

(we write the expression for I_{mx}^c , the one for I_{mx}^s is similar) where $\delta I(t)$ and $I(t)$ are the stochastic and deterministic (in phase with the external drive) part, respectively. We can define the time dependent mixing current as $I_{\text{mx}}^c(t) = (I_{\text{mx}}^c)_{[t/T_m]}$, where $[\alpha]$ stands here for the integer part of α . In terms of that the spectral density of the fluctuation of I_{mx}^c reads:

$$S_{\text{mx}}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} [\langle I_{\text{mx}}^c(t) I_{\text{mx}}^c(0) \rangle - \langle I_{\text{mx}}^c \rangle^2]. \quad (13)$$

We assume that the measuring time is much longer than any correlation time of the quantity $\delta I(t)$. Different sections of the measurement time are thus uncorrelated and

we can write:

$$S_{\text{mx}}(\omega) = \int_0^{T_m} dt e^{i\omega t} \int_0^{T_m} \frac{dt_1}{T_m} \int_0^{T_m} \frac{dt_2}{T_m} \cos(\omega_\Delta t_1) \cos(\omega_\Delta t_2) \langle \delta I(t_1) \delta I(t_2) \rangle. \quad (14)$$

Defining $S_I(\omega = 0) = 2 \int_{-\infty}^{+\infty} dt \langle \delta I(t) \delta I(0) \rangle$ (the numerical factor 2 is conventional for the current-noise spectrum) we have

$$S_{\text{mx}}(\omega) = \frac{1}{4} S_I(\omega = 0) \frac{(e^{i\omega T_m} - 1)}{i\omega T_m} \approx \frac{1}{4} S_I(\omega = 0). \quad (15)$$

Thus the mixing-current low-frequency noise is given simply by the low-frequency current noise spectrum S_{II} . The factor of 4 comes from a different definition of the correlation functions and from the fact that we are collecting a single quadrature. The current noise can have different sources, we consider in the following the two main ones.

A. Shot-noise and thermal current fluctuations

The current fluctuates due to the discrete nature of the the charge. This is characterized by the current-spectral function (for time-independent bias and gate voltages):

$$S_I^{\text{shot}}(\omega) = 2 \int dt e^{i\omega t} \langle \delta I(t) \delta I(0) \rangle, \quad (16)$$

where $\delta I(t) = I(t) - \langle I \rangle$. For the case of a SET the current spectral function is well known.³⁶ As shown there it has a frequency dependent part at low frequency on the scale of the typical tunneling rate Γ . This implies that the correlation function is short ranged with respect to the measuring time T_m . Actually it is typically even short ranged with respect to the time dependence of x and of the V or V_g potentials. Its value can thus be obtained adiabatically, by assuming these parameters to be static. We only need its low frequency part that can, in general, be expressed in terms of the Fano factor \mathcal{F} and the current I :

$$S_I^{\text{shot}}(\omega = 0) = 2\mathcal{F}Ie \quad (17)$$

where \mathcal{F} depends on the details of the SET. In the tunnelling limit of uncorrelated tunneling $\mathcal{F} = 1$, in most other cases the Fano factor is typically of the order of 1.

B. Displacement fluctuations

The electrons that cross the structure modify the charge on the gate that in turn modifies the force acting on the oscillator. This stochastic force, that has the same origin of the current-shot noise, induces fluctuations of the displacement, that changes in a much slower way, since the oscillator responds to an external force on the time scale given by its damping coefficient

γ .^{13,18,37,38} In order to keep the assumption that different averages over the measuring times are uncorrelated one needs $T_m\gamma \gg 1$. In principle, for very high- Q resonators the approximation should be reconsidered.

Let's begin by considering the force acting on the oscillator as a consequence of a variation of the charge on the gate. A recall of the basic expressions for the electrostatic energy is given in the Appendix A and Fig.3 there shows the electrical scheme. The force acting on the oscillator is given by the derivative of the electrostatic energy performed at constant charge:

$$F = -Q_g^2 \frac{\partial}{\partial x} \frac{1}{2C_g(x)} = \frac{Q_g^2 C'_g}{2C_g^2}, \quad (18)$$

where Q_g is the charge on the gate voltage (Fig.3). The fluctuation of the force $\delta F(t)$ due to fluctuation of Q_g reads thus:

$$\delta F(t) = \frac{Q_g C'_g}{C_g^2} \delta Q_g(t). \quad (19)$$

In general the variation of the charge on the gate is proportional to the variation of the charge on the central island of the SET. By an elementary electrostatic calculation (see Appendix A) $\delta Q_g/e = (C_g/C_\Sigma)\delta n$, where $C_\Sigma = C_g + C_L + C_R$ is the sum of the capacitances of the central island to all the electrodes and $-ne$ is the total charge on the island. In conclusion one finds that

$$\delta F(t) = F_0 \delta n(t) \quad (20)$$

with

$$F_0 = \frac{Q_g e C'_g}{C_g C_\Sigma} = 2 \frac{Q_g}{e} \frac{E_C}{L} \quad (21)$$

the force acting on the oscillator when an electron is added to the dot and with $E_C = e^2/(2C_\Sigma)$ the Coulomb energy of the SET. Note that F_0 is a crucial parameter, since it constitutes the electro-mechanical coupling constant.^{18,23} One can estimate the typical value of F_0 : $Q_g/e = 10-100$, $E_C = 1$ K, $L = 100$ nm, thus $F_0 \approx 10^{-11}-10^{-12}$ N.

The correlation function of the stochastic force acting on the resonator [$S_F(t) = \langle \delta F(t) \delta F(0) \rangle$] is thus simply proportional to the correlation function of the charge on the island [$S_{\delta n}(t) = \langle \delta n(t) \delta n(0) \rangle$]:

$$S_F(t) = F_0^2 S_n(t), \quad (22)$$

that can be calculated by the standard method of the master equation. For the case of a metallic dot see for instance Refs. 18 and 39. Its Fourier transform has a Lorentzian form with a width on the scale of Γ . Thus this force act as a white noise on the slow oscillator.

Let's now turn to the displacement correlation function. In order to evaluate it we use a simple Langevin approach.^{11,16} We neglect the driving, since we are interested in the low frequency response. The Langevin equation reads

$$m\ddot{x} + m\gamma\dot{x} + kx = \delta F(t), \quad (23)$$

where m is the (effective mass) of the oscillator mode considered, γ the damping coefficient, and k the effective spring constant. The stochastic force generated by the electrons is also at the origin of the damping coefficient. In general other effects participate, but close to the degeneracy point of the SET, when the current is maximal, the electronic contribution to the damping can dominate, as observed experimentally in Ref. 32. We will assume thus that γ is due only to the electronic damping. In equilibrium the fluctuation-dissipation theorem gives

$$S_{\delta F}(\omega = 0) = 2\gamma m k_B T. \quad (24)$$

For finite $eV \gg k_B T$, the system is out of equilibrium and one has to evaluate explicitly γ and $S_{\delta F}$ from a direct calculation of $S_F(\omega)$. As shown in Ref. 40 $2m\hbar\gamma = dS_F(\omega)/d\omega|_{\omega=0}$. One can then always define an effective temperature by the relation $S_F(\omega = 0) = 2\gamma m k_B T_{\text{eff}}$, since the oscillator has a very sharp response in frequency and the correlation functions are flat on that scale, one can always interpret the ratio of the fluctuation and the dissipation as an effective temperature. In the case of the SET it has been shown that the typical value of $k_B T_{\text{eff}}$ is of the order of eV .¹³

The Langevin equation (23) can then be solved by Fourier transform giving

$$S_x(\omega) = \langle x(\omega)x(-\omega) \rangle = \frac{F_0^2 S_n(\omega)}{m^2 |\omega_m^2 - \omega^2 - i\gamma\omega|^2} \quad (25)$$

and in particular in the low-frequency limit:

$$S_x(\omega = 0) = \frac{F_0^2 S_n(\omega = 0)}{m^2 \omega_m^4}. \quad (26)$$

We can now use the expansion (7) to find the lowest order contribution of the stochastic fluctuations of $x(t)$ to the current. We denote these fluctuations $\delta x(t)$ to distinguish them from the time-dependent average induced by the external driving:

$$\delta I(t) = \frac{\partial I}{\partial n_g} \frac{V_{g0} C'_g}{e} \delta x(t) + \dots \quad (27)$$

The back-action current noise is then

$$S_I^{\text{ba}}(\omega) = 2 \left(\frac{\partial I}{\partial n_g} \frac{F_0 n_g}{kL} \right)^2 S_n(\omega = 0). \quad (28)$$

As discussed in Refs. 19 and 41 the mechanical back-action noise can be very strong and induce effective giant Fano factors.

Finally the measurement added noise can be obtained as is done for the amplifiers,⁴² by dividing the fluctuation of the current signal by the amplifier gain squared. This gives:

$$S_x^{\text{add}} = \frac{S_{\text{mx}}}{\lambda^2} = \frac{S_I^{\text{shot}} + S_I^{\text{ba}}}{4\lambda^2}. \quad (29)$$

This quantity gives the upper bound on the detection sensibility, since the limitations considered are intrinsic to the detection method. We will evaluate explicitly these quantities for two specific models in sections V and VI.

IV. FAST OSCILLATOR

In this section we relax the condition $\omega_m \ll \Gamma$ for the calculation of the mixing current. We assume $\hbar\Gamma, \hbar\omega_m \ll k_B T$, the electronic transport is then described by sequential transport and we will find the mixing current to lowest non-vanishing order in the amplitude of the oscillating field by making use of a master equation description.

Let's begin by introducing in some details the electron tunnelling description. We assume that the only available charge states on the island are those associated with two charge states Ne and $(N+1)e$. We will call these two states 0 and 1. The state of the SET is thus fully described by the probabilities of one of these two state to be realized: P_n , with $n = 0, 1$. We define $\Gamma^{L+(-)}$ as the rate for adding (subtracting) one electron on (from) the central island through the left tunnel junction. Similarly we define $\Gamma^{R+(-)}$ for the right junction. We define also $\Gamma^\alpha = \Gamma^{L\alpha} + \Gamma^{R\alpha}$, with $\alpha = \pm$, $\Gamma^L = \Gamma^{L+} + \Gamma^{L-}$, $\Gamma^R = \Gamma^{R+} + \Gamma^{R-}$, and $\Gamma^T = \Gamma^+ + \Gamma^-$. The master equation for the the probability reads ($\dot{P} \equiv dP/dt$):

$$\dot{P}_0 = -\Gamma^+ P_0 + \Gamma^- P_1 \quad (30)$$

$$\dot{P}_1 = \Gamma^+ P_0 - \Gamma^- P_1. \quad (31)$$

Using the conservation of probability ($P_0 + P_1 = 1$) we are left with

$$\dot{P}_0 = -\Gamma^T P_0 + \Gamma^- . \quad (32)$$

We consider now that the rate equations are modulated by two oscillating parameters, in our specific case V and n_g . We expand in power series of the amplitude of oscillation the rates keeping only the lowest orders:

$$\Gamma^\alpha(t) = \Gamma^{\alpha(0)}(t) + \Gamma^{\alpha(1)}(t) + \Gamma^{\alpha(2)}(t) + \dots \quad (33)$$

where α stands for any of the previously introduced labels, and the term into parenthesis indicates the order in the expansion. As far as the driving frequency is smaller than the temperature, $\hbar\omega_i \ll k_B T$, the explicit expression of the time-dependent rates can be obtained by that for the static case by substituting the time-dependent fields:⁴³ for instance $\Gamma^\alpha(t) = \Gamma^\alpha(a(t), b(t))$, where $a = a_0 + a_1(t)$, $b = b_0 + b_1(t)$, and $a_1(t) = a_1 \cos(\omega_1 t)$, $b_1(t) = b_1 \cos(\omega_2 t)$. One can then expand to second order in the time dependent part of the two parameters to obtain:

$$\begin{aligned} \Gamma^\alpha(t) = & \Gamma^\alpha + \frac{\partial \Gamma^\alpha}{\partial a} a_1(t) + \frac{\partial \Gamma^\alpha}{\partial b} b_1(t) + \frac{1}{2} \frac{\partial^2 \Gamma^\alpha}{\partial a^2} a_1^2(t) \\ & + \frac{\partial^2 \Gamma^\alpha}{\partial a \partial b} a_1(t) b_1(t) + \frac{1}{2} \frac{\partial^2 \Gamma^\alpha}{\partial b^2} b_1^2(t) + \dots \end{aligned}$$

The expansion up to second order can then be rearranged in a Fourier series:

$$\begin{aligned} \Gamma^\alpha(t) = & \Gamma_{00}^\alpha + \sum_{n=-1,1} \left[\Gamma_{n,0}^{\alpha(1)} e^{in\omega_1 t} + \Gamma_{0,n}^{\alpha(1)} e^{in\omega_2 t} \right] \\ & + \left[\Gamma_{1,-1}^{\alpha(2)} e^{i(\omega_1 - \omega_2)t} + \text{cc} \right] + \dots \end{aligned} \quad (34)$$

where the static part Γ_{00}^α has contributions of zero and second order in the driving fields. The notation $\Gamma_{n,m}^{\alpha(p)}$ indicates a contribution of order p in the driving intensity. Concerning the time dependent second order terms, we keep only the interesting part at the mixing-current frequency ω_Δ .

We look for a solution of the master equation in terms of the stationary Fourier components

$$P_n(t) = \sum_{n,m} A_{nm} e^{i(n\omega_1 + m\omega_2)t}. \quad (35)$$

This gives for each Fourier component the equation:

$$(in\omega_1 + im\omega_2)A_{nm} + \sum_{n',m'} \Gamma_{n',m'}^T A_{n-n',m-m'} - \Gamma_{nm}^- = 0. \quad (36)$$

We further expand the A coefficients writing:

$$A_{nm} = \sum_{p=0}^{\infty} A_{nm}^{(p)}, \quad (37)$$

where again p indicates the order in the driving fields. This leads to a set of equations that can be solved recursively. The zeroth-order one reads:

$$(in\omega_1 + im\omega_2)A_{nm}^{(0)} + \Gamma_{00}^{T(0)} A_{n,m}^{(0)} - \Gamma_{00}^{-(0)} \delta_{n,0} \delta_{m,0} = 0. \quad (38)$$

It gives immediately the static solution:

$$A_{nm}^{(0)} = \delta_{n,0} \delta_{m,0} \frac{\Gamma_{00}^{-(0)}}{\Gamma_{00}^{T(0)}}. \quad (39)$$

For the next two orders we obtain:

$$A_{nm}^{(1)} = \frac{\Gamma_{00}^T \Gamma_{nm}^{-(1)} - \Gamma_{nm}^{T(1)} \Gamma_{00}^{-(0)}}{\Gamma_{00}^{T(0)} (in\omega_1 + im\omega_2 + \Gamma_{00}^{T(0)})}, \quad (40)$$

and

$$A_{nm}^{(2)} = \frac{\Gamma_{nm}^{-(2)} - \Gamma_{00}^{T(2)} A_{n,m}^{(0)} - \sum_{n',m'} \Gamma_{n',m'}^{T(1)} A_{n-n',m-m'}^{(1)}}{in\omega_1 + im\omega_2 + \Gamma_{00}^{T(0)}}. \quad (41)$$

The non-vanishing terms up to order two are $A_{0,0}^{(0)}$, $A_{\pm 1,0}^{(1)}$, $A_{0,\pm 1}^{(1)}$, $A_{0,0}^{(2)}$, $A_{\pm 2,0}^{(2)}$, $A_{\pm 1,\pm 1}^{(2)}$, and $A_{0,\pm 2}^{(2)}$. As usual for the Fourier transform of real functions the following relation holds: $A_{n,m}^* = A_{-n,-m}$.

Let us now consider the particle current. It can be expressed in terms of P and Γ , for instance, on the left junction (note that this expression does not include the displacement current):

$$I(t)/e = \Gamma^{L+} P_0 - \Gamma^{L-} P_1 = \Gamma^L P_0 - \Gamma^{L-}. \quad (42)$$

Substituting the expansion (35) into Eq. (42) we obtain for I a similar expansion to Eq. (35). The first three

orders read:

$$I_{nm}^{(0)}/e = \left[\Gamma_{00}^{L(0)} A_{00}^{(0)} - \Gamma_{00}^{L-(0)} \right] \delta_{nm} \delta_{n0} \quad (43)$$

$$I_{nm}^{(1)}/e = \Gamma_{nm}^{L(1)} A_{00}^{(0)} + \Gamma_{00}^{L(0)} A_{nm}^{(1)} - \Gamma_{nm}^{L-(1)} \quad (44)$$

$$I_{nm}^{(2)}/e = \Gamma_{00}^{L(0)} A_{nm}^{(2)} + \sum_{n'm'} \Gamma_{n-n', m-m'}^{L(1)} A_{n'm'}^{(1)} \\ + \Gamma_{nm}^{L(2)} A_{00}^{(0)} - \Gamma_{nm}^{L-(2)}. \quad (45)$$

The mixing current is given by

$$I_{\text{mix}}^c = \text{Re} I_{1,-1}/2 \quad , \quad I_{\text{mix}}^s = -\text{Im} I_{1,-1}/2. \quad (46)$$

In order to simplify the expressions obtained above we use the fact that in general $\omega_1 \approx \omega_2 \equiv \omega_D$ so that even in the fast oscillator limit $|\omega_1 - \omega_2| \ll \Gamma_{00}^{T(0)}$. This gives the approximate expressions:

$$A_{10}^{(1)} = \frac{\Gamma_{00}^T \Gamma_{10}^{--(1)} - \Gamma_{10}^{T(1)} \Gamma_{00}^{--(0)}}{\Gamma_{00}^{T(0)} (i\omega_D + \Gamma_{00}^{T(0)})} \quad (47)$$

$$A_{1,-1}^{(2)} = \frac{\Gamma_{1,-1}^{-(2)} - \Gamma_{1,0}^{T(1)} A_{0,-1}^{(1)} - \Gamma_{0,-1}^{T(1)} A_{1,0}^{(1)}}{\Gamma_{00}^{T(0)}} \quad (48)$$

One can see that the residual ω_D -dependence is due to the relaxation time of the charge in the island. As expected it disappears for $\omega_D \ll \Gamma_{00}^{T(0)}$. The contribution from $I_{1,-1}^{(1)}$ vanishes since $\Gamma_{1,-1}^{\alpha(1)} = 0$. The interesting part is the contribution of second order which reads:

$$I_{1,-1}^{(2)} = \Gamma_{00}^{L(0)} A_{1,-1}^{(2)} + \Gamma_{1,0}^{L(1)} A_{0,-1}^{(1)} + \Gamma_{0,-1}^{L(1)} A_{1,0}^{(1)} \\ + \Gamma_{1,-1}^{L(2)} A_{00}^{(0)} - \Gamma_{1,-1}^{L-(2)}. \quad (49)$$

One can verify that for $\omega \ll \Gamma_{00}^{T(0)}$ expression Eq. (49) reduces to $\partial I^2 / \partial a \partial b$ recovering the standard results for the mixing-current [cf. expressions (10) and (9)].

In the opposite limit of $\omega \gg \Gamma_{00}^{T(0)}$ the first order correction to the charge variation vanishes ($A_{1,0}^{(1)} \rightarrow 0$): the charge has not the time to follow the driving. Only a second order correction survives $A_{1,-1}^{(2)} = \Gamma_{1,-1}^{-(2)} / \Gamma_{00}^{T(0)}$. The residual time dependence at the mixing frequency is only due to the direct modulation of the tunneling rates ($\Gamma_{1,-1}^{\alpha(2)}$). The final expression for $I_{1,-1}$ in the limit $\omega \rightarrow \infty$ reads:

$$I_{1,-1}^{(2)} = \Gamma_{00}^{L(0)} \frac{\Gamma_{1,-1}^{-(2)}}{\Gamma_{00}^{T(0)}} + \Gamma_{1,-1}^{L(2)} A_{00}^{(0)} - \Gamma_{1,-1}^{L-(2)} \quad (50)$$

In the following two sections we consider explicitly the case of a metallic dot and of a single electronic level dot and we derive explicit expressions for the mixing current, its fluctuation and the response function in the high-frequency regime.

V. THE METALLIC DOT SINGLE-ELECTRON TRANSISTOR

The expression for the tunnelling rate are well known for a metallic dot in the Coulomb blockade regime.⁴⁴ For

convenience of the reader, we report in the appendix a very short derivation of the electrostatic relations. We consider only the two states with N and $N+1$ electrons.

A. Low temperature case

We begin by discussing the low temperature case $k_B T \ll eV \ll E_C$ where $E_C = e^2/2C_\Sigma$ is the Coulomb energy. In this case there are only two non-vanishing rates (for $V > 0$)

$$\Gamma_L^+(N) = \Gamma_o(v + \tilde{n}_g) \theta(v + \tilde{n}_g) \quad (51)$$

$$\Gamma_R^-(N+1) = \Gamma_o(v - \tilde{n}_g) \theta(v - \tilde{n}_g) \quad (52)$$

where $\Gamma_o = 1/RC_\Sigma$, $v = (C + C_g/2)V/e$ and $\tilde{n}_g = C_g(x)V_g/e - N - 1/2$, we assume a symmetric device with tunneling resistance R . The stationary solution to the master equation (32) and the stationary current (42) read

$$P_1^{\text{st}} = \frac{\tilde{n}_g + v}{2v}, \quad I = e\Gamma_o \frac{v^2 - \tilde{n}_g^2}{2v}, \quad (53)$$

both equations valid for $|\tilde{n}_g| < v$. The current vanishes continuously for $|\tilde{n}_g| \geq v$ while the probability is 1 for $\tilde{n}_g > v$ and 0 for $\tilde{n}_g < -v$.

The driving amplitudes in terms of the dimensionless variables introduced read v_1 and n_{g1} . Note that the dependence of the rates on v and \tilde{n}_g is non-analytic for $\tilde{n}_g = \pm v$, this gives a constraint on the amplitude of the oscillations since the Taylor expansions are not valid if the parameters cross this values. This gives the constraints $|\tilde{n}_g \pm n_{g1}| < v$ and $v - v_1 > \tilde{n}_g$, that can be written $n_{g1}, v_1 < v - \tilde{n}_g$. Using Eq. (51) and Eq. (52) we can readily obtain the non-vanishing coefficients of the expansion (34): $\Gamma_{00}^{L+} = \Gamma_o(v + \tilde{n}_g)$, $\Gamma_{10}^{L+} = \Gamma_o e^{i\varphi} n_{g1}/2$, $\Gamma_{01}^{L+} = \Gamma_o v_1/2$, $\Gamma_{00}^{R-} = \Gamma_o(v - \tilde{n}_g)$, $\Gamma_{10}^{R-} = -\Gamma_o e^{i\varphi} n_{g1}/2$, $\Gamma_{01}^{R-} = \Gamma_o v_1/2$. For $\omega_1 \approx \omega_2 = \omega_D$ we obtain a very simple expression for the component $I_{1,-1}$:

$$I_{1,-1} = e\Gamma_o \frac{\tilde{n}_g v_1 n_{g1} e^{-i\varphi}}{\tilde{\omega}_D^2 + 4v^2} \quad (54)$$

here we defined $\tilde{\omega}_D = \omega_D/\Gamma_o$. One finds thus a Lorentzian behaviour, the amplification factor decreases quite rapidly for large frequency driving ω_D . The main reason for the reduction of sensitivity is the incapacity of the charge in the dot to follow the driving signal. The crossover value for the frequency is $\omega_D \approx V/Re$, above this value one cannot use anymore the adiabatic approximation for the relaxation of the charge on the dot. It simply coincides with the frequency for which one electron per driving period crosses the device. For instance for $\omega_m = 100$ MHz, $R = 10^5$ Ohm, for voltage below a mV the corrections due to the retardation of the charge on the dot becomes relevant. This regime has been observed in the experiment presented in Ref. 45, where the crossover from slow to fast oscillator has been investigated by a fine tuning of the tunnelling resistances.

The amplification factor for the mechanical quadratures is thus:

$$\lambda = \frac{e\Gamma_o}{L} \frac{n_g \tilde{n}_g v_1}{(\tilde{\omega}_D^2 + 4v^2)}. \quad (55)$$

It is maximum for $\tilde{n}_g = \pm v$, but one should also take into account the constraint on the amplitude of $v_1 < v - |\tilde{n}_g|$. One way to take that into account is to set $v_1 = v - |\tilde{n}_g|$, this is the maximum allowed value for the driving amplitude, and since the signal increases linearly with v_1 , it gives the maximum value for λ . This gives:

$$\lambda = \frac{e\Gamma_o}{L} \frac{n_g \tilde{n}_g (v - |\tilde{n}_g|)}{(\tilde{\omega}_D^2 + 4v^2)}. \quad (56)$$

The maximum of λ as a function of the gate voltage is obtained for $\tilde{n}_g = \pm v/2$ and its value (for $\tilde{\omega}_D \ll v$) is

$$\lambda = \frac{e\Gamma_o n_g}{16L} \quad (57)$$

independently of v . For a typical device one has $n_g \approx 100$ $L \approx 1\mu\text{m}$, $\Gamma_0 = 10^{11}$ Hz leading to $\lambda \sim 0.1$ A/m.^{9,10}

The gain is only a part of the detection, one has also to evaluate the noise. For that we need the two contributions considered in the section III. The Fano factor has been obtained in Ref. 36 (cf. Eq. 41 there):

$$\mathcal{F} = \frac{\Gamma_L^+ + \Gamma_R^-}{(\Gamma_L^+ + \Gamma_R^-)^2} = \frac{v^2 + \tilde{n}_g^2}{2v^2} \quad (58)$$

it varies between 1/2 and 1. The shot noise becomes thus:

$$S_{II}^{\text{shot}} = e^2 \Gamma_o \frac{v^4 - \tilde{n}_g^4}{2v^3}. \quad (59)$$

To obtain the contribution of the displacement fluctuation we need to calculate the charge noise correlation function: $S_n(t) = \langle \delta n(t) \delta n(0) \rangle$. This symmetric correlator can be obtained by the conditional probability $P(1t|10)$ that the dot it occupied at time $t > 0$ with the condition that it was occupied at time 0: $S_n(t) = [P(1t|10) - P_1^{\text{st}}]P_1^{\text{st}}$. Solving the master equation with the initial condition $P_1 = 1$ one finds

$$P(1t|10) = 1 + (P_1^{\text{st}} - 1)(1 - e^{-\Gamma^T t}), \quad (60)$$

with $P_1^{\text{st}} = \Gamma^+/\Gamma^T$. By Fourier transforming we obtain:

$$S_n(\omega) = P_1^{\text{st}}(1 - P_1^{\text{st}}) \frac{2\Gamma^T}{\omega^2 + \Gamma^T} \quad (61)$$

As expected the correlation function is flat for $\omega \ll \Gamma^T$, the required low frequency correlator reads then:

$$S_n(\omega = 0) = 2\Gamma^+ \Gamma^- / (\Gamma^T)^3. \quad (62)$$

In the specific case of low temperature one obtains thus $S_n = (v^2 - \tilde{n}_g^2)/(4\Gamma_o v^3)$.

In the typical working regime of a SET $V \ll V_g$, and $n_g \approx N$. Using the Eq. (A1) one finds that $Q_g/e \approx n_g \approx N$. We thus have $F_0 = 2NE_C/L$. Collecting all the terms we can substitute into Eq. (28) to obtain:

$$S_I^{\text{ba}} = 2e^2 \Gamma_o N^4 \left(\frac{E_c}{kL^2} \right)^2 \frac{(v^2 - \tilde{n}_g^2) \tilde{n}_g^2}{v^5}. \quad (63)$$

The ratio of the mechanical to the shot noise is thus:

$$\frac{S_I^{\text{ba}}}{S_I^{\text{shot}}} = \left(\frac{E_c}{kL^2} \right)^2 \frac{4N^4 \tilde{n}_g^2}{v^2(v^2 + \tilde{n}_g^2)}. \quad (64)$$

For large mechanical coupling (L small and N large) the mechanical noise dominate even if for small \tilde{n}_g it is always suppressed, due to the vanishing of $\partial I/\partial n_g$.

From Eqs (57), (59), and (63), we obtain the seeked added noise as defined by Eq. (65). In order to study its dependence on the different parameters it is convenient to introduce the two dimensionless variables $\nu \equiv \tilde{n}_g/v$ and the dimensionless coupling constant $\delta \equiv (E_c/kL^2)(N^2/v) = \epsilon_P/eV$, where $\epsilon_P = F_0^2/k$ is the energy scale of the electromechanical coupling.^{13,14,18} The added noise then reads:

$$S_x^{\text{add}} = \frac{E_c}{k\Gamma_0} f(\nu, \delta), \quad (65)$$

with

$$f(\nu, \delta) = \frac{2(1+\nu)[4(\delta^2+1)\nu^2+1]}{\nu^2(1-\nu)\delta}, \quad (66)$$

and $0 < \nu < 1$. The function diverges for $\nu \rightarrow 1$ due to the fact that we have to limit the amplitude of the voltage modulation and diverges for $\nu \rightarrow 0$ due to the vanishing of the amplification factor. The minimum added noise is thus always for values of ν between 0 and 1. In the weak coupling limit, for $\delta \ll 1$, one finds that the minimum is at $\nu \approx 0.54$ and reads

$$S_x^{\text{add}} \approx 14.8 \frac{E_c/k}{\Gamma_0} \frac{eV}{\epsilon_P}. \quad (67)$$

For strong coupling, $\delta \gg 1$, instead the minimum is close to $\nu = 0$ with a value for

$$S_x^{\text{add}} \approx 8 \frac{E_c}{k\Gamma_0} \left(\frac{\epsilon_P}{eV} \right)^2. \quad (68)$$

In both cases the noise diverges when δ becomes very small or very large. In the weak coupling limit the added noise is dominated by the current noise (imprecision noise), in the strong coupling it is instead given essentially by the back-action noise. As usual⁴² the optimal situation is in the middle for $\delta \approx 1$.

In Fig. 2 we plot $S_x^{\text{add}}[\delta, \nu_m(\delta)]$, where $\nu_m(\delta)$ is the value of ν that minimizes S_x^{add} for given δ . We thus find that the absolute minimum for the added noise is obtained for $\nu = 0.32$ and $\delta = 1.857$ and reads

$$S_{xx}^{\text{add}} = 57.61 \frac{E_c}{k\Gamma_0} \quad (69)$$

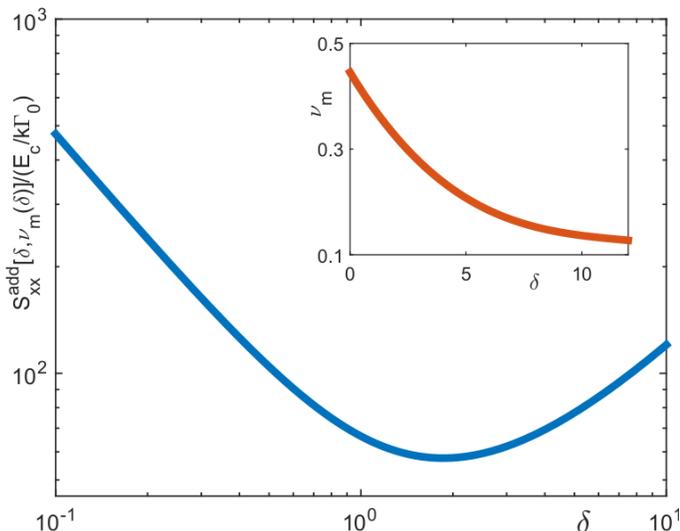


Figure 2. S_{xx}^{add} as a function of $\delta = \epsilon_P/eV$ for $\nu(\delta)$ minimizing the function. In a inset the value of ν that minimizes the function for given δ [$\nu_m(\delta)$].

This is the ultimate sensitivity that can be obtained with this device in ideal conditions, when all other sources of imprecisions have been eliminated. Inserting typical values of $E_c \approx 10\text{K}$, $\Gamma_0 \approx 10^{11}\text{Hz}$, $k = 10^{-5}\text{N/m}$ one obtains the value of $S_{xx}^{\text{add}} \approx 10^{-26}\text{m}^2/\text{Hz}$. One should regard this value with some caution. Let's consider the value of the coupling that is required to obtain this sensitivity. The optimal value of δ is for $eV \approx \epsilon_P$. As discussed in the literature (see for instance Ref. 18, where this energy is called E_E) this scale determines the value at which the system undergoes a current blockade. It is difficult to reach this limit (since one needs also $k_B T \ll \epsilon_P$) in metallic SETs. On the other side ϵ_P of the order of 0.3 K has been observed in suspended carbon nanotubes.⁴⁵ The dramatic effects expected at low temperature on the mechanical resonators have been discussed recently.^{23,24} This extreme limit need to be reconsidered, since the resonating frequency of the resonator is renormalized by the coupling, and the added noise induced by the oscillator is expected to be more effective. In particular the oscillator becomes strongly non-linear close to the transition.

We can estimate in a simple way the effect of the softening of the mechanical resonator following Ref. 18. The correction to the variation of the energy reads¹¹

$$\Delta E^\pm \rightarrow \Delta E^\pm \pm F_0 x \quad (70)$$

this changes the form of P_1^{st} given by Eq. (53) as follows:

$$P_1^{\text{st}} = \frac{\tilde{n}_g + x F_0 / (2E_c) + v}{2v}. \quad (71)$$

Substituting into the equation for the average force $F_0 P_1^{\text{st}}$ and taking the derivative with respect to x one obtains the renormalized spring constant:

$$k' = k(1 - \delta). \quad (72)$$

The instability appears for $\delta = 1$, where two new stable solutions bifurcate. The only change in our previous calculations is the value of k entering Eq. (65):

$$S_{xx}^{\text{add}} = \frac{E_c}{k\Gamma_0} \frac{1}{1 - \delta} f[\nu, \delta/(1 - \delta)]. \quad (73)$$

Repeating the minimization procedure we find that the minimum is now for $\delta = 0.48$ holding the value of $132.7(E_c/k\Gamma_0)$. Thus the renormalization of the resonating frequency reduces the precision of a factor of 2, leaving space for high sensitivity detection.

The actual limitation in current experiments will be the value of the coupling, since in practice the typical temperature reached in experiments on metallic quantum dot is much larger than ϵ_P . In the following section we consider the detection at finite temperature and low voltage.

B. Finite temperature case

Let us now consider the finite temperature case $eV \ll k_B T \ll E_c$. In this case we have to take into account the four possible tunnelling processes that change the charge on the dot from the N to the $N+1$ state (cfr. Appendix). The respective rates read:

$$\Gamma_{L(R)}^+(N) = \Gamma_{\text{Th}} h[(\mp eV - 2\tilde{n}_g E_c)/k_B T], \quad (74)$$

$$\Gamma_{L(R)}^+(N+1) = \Gamma_{\text{Th}} h[(\pm eV + 2\tilde{n}_g E_c)/k_B T], \quad (75)$$

with $h(y) = -y[1 - e^y]$ and $\Gamma_{\text{Th}} = k_B T/e^2 R$. We consider the low bias voltage limit $eV/k_B T \ll 1$. In this limit the expression for the current Eq. (42) becomes:

$$I = e\Gamma_{\text{Th}} \frac{eV}{2k_B T} g(2\tilde{n}_g E_c/k_B T), \quad (76)$$

where

$$g(y) = \frac{h_+ h'_- + h'_+ h_-}{h_+ + h_-} = \frac{e^y y}{e^{2y} - 1} \quad (77)$$

and $h_\pm = h(\pm y)$. From the expression of the current we obtain

$$\frac{\partial^2 I}{\partial V \partial \tilde{n}_g} = -\frac{E_c}{Rk_B T} g'(2\tilde{n}_g E_c/k_B T), \quad (78)$$

(for brevity, we omit in the following the arguments of g and of the other functions of $y = 2\tilde{n}_g E_c/k_B T$) with the amplification factor:

$$\lambda = \frac{n_{g0} \Gamma_0 e V_1}{4Lk_B T} g' \approx \frac{F_0 V_1}{8Rk_B T} g'. \quad (79)$$

The factor $g'(x)$ has a maximum for $y = 1.16$ for which it holds the approximate value 0.154. Thus tuning $\tilde{n}_g = 0.58k_B T/E_c$ allows to obtain the maximum value of the amplification factor. Comparing this value to Eq. (57),

valid for $k_B T \ll eV$, we see that the amplification factor is reduced by the term $eV_1/k_B T \ll 1$.

The shot noise at low frequency reads:³⁶

$$S_I^{\text{shot}} = e^2 \left[\frac{\Gamma_L^+ \Gamma_R^- + \Gamma_L^- \Gamma_R^+}{\Gamma_T} - 2 \frac{(\Gamma_L^+ \Gamma_R^- - \Gamma_L^- \Gamma_R^+)^2}{\Gamma_T^3} \right]. \quad (80)$$

For small V the first term (thermal noise) dominates and gives:

$$S_I^{\text{shot}} = e^2 \Gamma_{\text{Th}} \frac{h_+ h_-}{h_+ + h_-}. \quad (81)$$

The charge noise in the same limit reads

$$S_n = \frac{1}{\Gamma_{\text{Th}}} \frac{h_+ h_-}{(h_+ + h_-)^3}. \quad (82)$$

From the expression of the back-action noise (28) we see that for $V \rightarrow 0$ there is (apparently) no back action of the measurement. It is possible to set $V = 0$ and exploit its modulation around 0 to detect the motion of the oscillator. But in this case we need to consider the next order in the expansion (7). For $V = 0$ we have:

$$\delta I = \frac{\partial I}{\partial n_g \partial V} \delta n_g V_1 + \dots \quad (83)$$

From this we have for the current-current correlator:

$$\langle \delta I(t_1) \delta I(t_2) \rangle = \left(\frac{\partial G}{\partial n_g} \right)^2 V_1(t_1) V_1(t_2) \langle \delta n_g(t_1) \delta n_g(t_2) \rangle, \quad (84)$$

where $G = dI/dV$ for $V = 0$ is the conductance. The product of the two V_1 terms gives an oscillating term depending on $t_1 + t_2$ that averages to zero and a second term proportional to $\cos[\omega_2(t_1 - t_2)]$. Using $\delta n_g(t) = (C'_g V_g/e) \delta x(t)$ we have

$$S_I^{\text{ba}} = \frac{1}{2} \left(\frac{\partial G}{\partial n_g} \right)^2 (V_{g0} C'_g V_1/e)^2 S_x(\omega_2). \quad (85)$$

Typically $\omega_2 \approx \omega_m$, we thus assume that it is resonant in order to evaluate the case of maximal back-action:

$$S_I^{\text{ba}} = \frac{g'^2 h_+ h_-}{16(h_+ + h_-)^3} e^2 \Gamma_0 \frac{\epsilon_P^2 Q^2 (eV_1)^2}{(k_B T)^3 E_C}, \quad (86)$$

with $h^{\text{ba}} = (g')^2 h_+ h_- / (h_+ + h_-)^3$ and $Q = \omega_m / \gamma$ the oscillator quality factor.

Adding the two sources of current noise Eq. (86) and Eq. (81) we obtain for the added noise:

$$S_x^{\text{add}} = \frac{E_C}{k\Gamma_0} \left[\alpha^{\text{ba}} \frac{\epsilon_P Q^2}{k_B T} + \alpha^{\text{shot}} \frac{(k_B T)^3}{\epsilon_P (eV_1)^2} \right], \quad (87)$$

with the numerical factors $\alpha^{\text{ba}} = 4h_+ h_- / (h_+ + h_-)^3$ and $\alpha^{\text{shot}} = 32h_+ h_- / [(g')^2 (h_+ + h_-)]$. Choosing the value $\tilde{n}_g = 1.60$ that maximizes λ their values are $\alpha^{\text{ba}} = 0.23$ and $\alpha^{\text{shot}} = 449$.

The minimum of the added noise is obtained for

$$\epsilon_P = \left(\frac{\alpha^{\text{shot}}}{\alpha^{\text{ba}}} \right)^{1/2} \frac{(k_B T)^2}{Q e V_1}, \quad (88)$$

with a minimum noise of

$$S_x^{\text{add}} = 2 \frac{E_C}{k\Gamma_0} (\alpha^{\text{ba}} \alpha^{\text{shot}})^{1/2} Q \frac{k_B T}{e V_1}. \quad (89)$$

Since $eV_1/k_B T \ll 1$, at best we can set this ratio to 0.1. This gives for the optimal value of the coupling

$$\frac{\epsilon_P}{k_B T} \approx \frac{441}{Q} \quad (90)$$

and the minimum of the added noise

$$S_x^{\text{add}} = 203 \frac{Q E_C}{k\Gamma_0}. \quad (91)$$

Some comments are at order. First we assumed that the frequency driving the voltage bias is resonant with the oscillator. This is an upper limit to the back action, in particular if $Q \gg 1$ this condition is not fulfilled and the back action will be reduced. For the non-resonant case it is sufficient to use the above results with $Q \approx \omega_m / \omega_\Delta$, reducing enormously the minimum added noise, to the expenses of finding a much larger coupling constant. The second comment concern the value of the coupling constant ϵ_P necessary to reach the minimum. One can see that even with the assumption of resonant back action it is relatively large. For a typical $Q \approx 10^4$ one finds $\epsilon_P/k_B T \approx 0.04$. To our knowledge the largest value of the ratio $k_B T/\epsilon_P$ is ≈ 0.017 has been reported in Ref. 45. Since as soon as $Q \gg 1$ it is possible to avoid resonant back-action, in most cases the main limitation is to reach large values of ϵ_P .

It is interesting to compare the shot-noise contribution of the added noise with the resonant brownian motion fluctuations:

$$S_x^{\text{B}}(\omega_m) = 2 \frac{k_B T}{k\gamma}. \quad (92)$$

The ratio reads:

$$\frac{S_x^{\text{add}}}{S_x^{\text{B}}} = \frac{\alpha^{\text{shot}}}{2} \frac{E_C \gamma}{\epsilon_P \Gamma_0} \left(\frac{k_B T}{e V_1} \right)^2. \quad (93)$$

Detection of brownian motion can then be done for $\epsilon_P/E_C > 2 \times 10^4 \gamma/\Gamma_0$ (where we assumed as before $eV_1/k_B T = 0.1$). For instance in Ref. 10 $\gamma/\Gamma_0 \approx 10^{-8}$ allowing the detection of the brownian motion fluctuations even for very weak coupling. For a rough estimate of the coupling in that experiment one can use the expression given in Ref. 23 $\epsilon_P/k_B T \approx 2\delta\omega_m/\omega_m$, where $\delta\omega_m$ is the modulation of the resonating frequency near the degeneracy point (see Fig. 3 in Ref. 10). For Ref. 10 one finds $\epsilon_P \approx 16\text{m K}$ to be compared to E_C of the order of 10K. Notwithstanding the low value of the coupling constant, the resolution is largely sufficient to detect the Brownian motion of the carbon nanotube.

VI. THE SINGLE-ELECTRONIC LEVEL SET

When the temperature and the voltage bias is much smaller than the electronic level separation the rates for electron transfer reads:⁴⁶

$$\Gamma_{L(R)}^+ = \Gamma_{L(R)0} f_F[(\epsilon - \mu_{L(R)})/k_B T], \quad (94)$$

$$\Gamma_{L(R)}^- = \Gamma_{L(R)0} [1 - f_F[(\epsilon - \mu_{L(R)})/k_B T]], \quad (95)$$

where $f_F(y) = 1/(1 + e^y)$ if the Fermi function, ϵ is the level position, $\mu_{L(R)}$ is the left (right) chemical potential, and $\Gamma_{L(R)0}$ are the transfer rates. For simplicity in the following we choose $\Gamma_{L0} = \Gamma_{R0} = \Gamma_0$. The modulation of the gate voltage leads to the time-dependence $\epsilon(t) = \epsilon_0 + \epsilon_1(t)$ of the electronic level energy ϵ with

$$\epsilon_0 = \epsilon_{d0} + eC_g V_{g0}/C_\Sigma, \quad (96)$$

$$\epsilon_1(t) = e[C'_g V_{g0} x(t) + C_g V_{g1}(t)]/C_\Sigma, \quad (97)$$

and ϵ_{d0} the position of the electronic level for vanishing V_g . We assume symmetric bias so that the chemical potential read:

$$\mu_{L(R)}(t) = \mu_{L(R)0} + (-)e(V + V_1 \cos \omega_2 t)/2. \quad (98)$$

Following the steps of the previous section we can calculate the current

$$I = \frac{e\Gamma_0}{2} [f_F[(\epsilon - \mu_L)/k_B T] - f_F[(\epsilon - \mu_R)/k_B T]] \quad (99)$$

from which we obtain for vanishing V the amplification factor:

$$\lambda = \frac{en_{g0}\Gamma_0}{4L} \frac{eV_1 E_C}{(k_B T)^2} f_F''(y), \quad (100)$$

where the argument of the Fermi function is $y = (\epsilon_0 - \mu)/k_B T$, and will be omitted in the following. The maximum of f_F'' is obtained for $y = 1.31$ with a value of 0.096. The thermal part of the shot noise and the charge noise read:

$$S_I^{\text{shot}} = e^2 \Gamma_0 f_F(1 - f_F), \quad (101)$$

$$S_n = \frac{f_F(1 - f_F)}{\Gamma_0}. \quad (102)$$

Using Eq. (85) for the back-action noise we obtain

$$S_I^{\text{ba}} = \frac{f_F(1 - f_F) f_F''^2}{8} e^2 \Gamma_0 \left(\frac{eV_1 Q \epsilon_P}{(k_B T)^2} \right)^2. \quad (103)$$

The added noise has thus the form:

$$S_x^{\text{add}} = \frac{k_B T}{k\Gamma_0} \left[\alpha^{\text{ba}} Q^2 \frac{\epsilon_P}{k_B T} + \alpha^{\text{shot}} \left(\frac{k_B T}{eV_1} \right)^2 \frac{k_B T}{\epsilon_P} \right] \quad (104)$$

with $\alpha^{\text{ba}} = 2f_F(1 - f_F)$ and $\alpha^{\text{shot}} = 16f_F(1 - f_F)/f_F''^2$. Their values for $y = 1.31$ are $\alpha^{\text{ba}} = 0.34$ and $\alpha^{\text{shot}} = 289.2$. We find the same value of ϵ_P for the minimum of

the added noise in the metallic case [cf. Eq. (88)], but the minimum of the noise has a different expression:

$$S_x^{\text{add}} = 2 (\alpha^{\text{ba}} \alpha^{\text{shot}})^{1/2} \frac{(k_B T)^2}{k\Gamma_0 eV_1}. \quad (105)$$

Essentially the energy scale of the Coulomb blockade is substituted by the temperature, in principle reducing the added noise. The conclusion is that the single-level SET should allow a better resolution of the metallic SET by a factor $E_C/k_B T$.

VII. CONCLUSIONS

In this work we have studied theoretically the sensitivity of the mixing-current technique. We first found general expressions valid when the oscillator resonating frequency is comparable or larger of the transfer rate of electrons. We find that a reduction of the amplification factor of the order of $(\Gamma_0/\omega_D)^2$ is expected. This effect should be relatively small in most practical experimental realizations. We then analysed the fundamental limitations due to the intrinsic noise present in the (current) signal and the effect of the back-action fluctuations. On general grounds one finds that an optimal value of the electromechanical coupling (ϵ_P) exists that minimizes the added noise. This value is larger than what is realized in the present experiments, showing that increasing the coupling allows to reach higher sensitivity. At finite temperature the relevant parameter is the ratio $\epsilon_P/k_B T$ and values of the order of 1 are needed to reach the optimal minimum added noise. At vanishing temperature the relevant parameter is instead ϵ_P/eV . In all cases the scale of the sensitivity is given by $E_C/\Gamma_0 k$. Optical means can detect CNTs displacement with good accuracy, even if the small size of the object does not allows to reach the spectacular sensitivity obtained with macroscopic mirrors. A sensitivity of $5 \cdot 10^{-22} \text{ m}^2/\text{Hz}$ as been reported⁴⁷ by cavity-enhanced optical detection of CNTs.

We considered only classical fluctuations. It seems difficult to use the mixing technique to reach the quantum limit of detection, since the effective temperature of the oscillator, even at vanishing temperature, is of the order of eV that typically needs to be larger than $\hbar\omega_m$. On the other side it may be instructive to compare the sensitivity found at vanishing temperature with the zero point fluctuations spectrum at resonance: $S_x^{\text{SQL}} = 2\hbar\omega_m/k\gamma$. One sees that the ratio to the typical mixing-current technique added noise at zero temperature is $10^{-2}(\hbar\omega_m/E_C)(\Gamma/\gamma)(\epsilon_P/E_C)$, since $\Gamma/\gamma \gg 1$, for sufficiently large ϵ_P the added noise can be of the same order of the zero-point fluctuations.

We conclude that the sensitivity of the mixing technique can still be improved by increasing the electromechanical coupling till reaching ϵ_P of the order of the temperature or the Coulomb blockade energy where the back-action will be of the same order of the intrinsic current noise of the device.

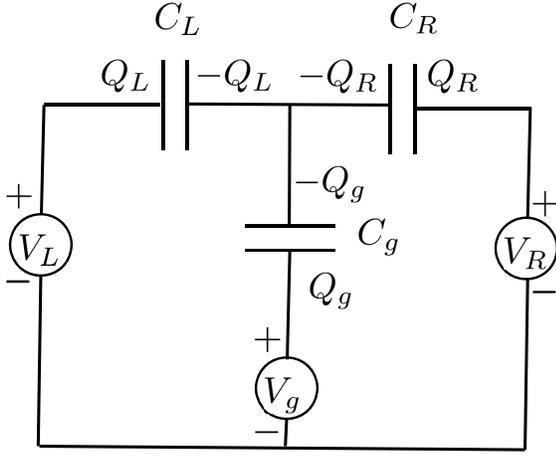


Figure 3. Electric scheme of a single electron transistor

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Appendix A

In this appendix we present, mainly for clarifying the notation, a brief derivation of the electrostatic energy variation for the tunnelling of an electron in a single electron transistor.⁴⁴ The electric scheme is presented in Fig.3 where the potentials of the left, right, and gate leads are defined as V_L , V_R , and V_g , respectively. In the same way the charge on each capacitance (on the leads side) is indicated with Q_i with $i = L, R$, and g . Defining V_I the potential of the island one has

$$Q_i = (V_i - V_I)C_i. \quad (\text{A1})$$

Summing the three equations one obtains immediately the expression for the potential on the island:

$$V_I = \left(\sum_i C_i V_i + Q \right) / C_\Sigma, \quad (\text{A2})$$

where $C_\Sigma = \sum_i C_i$ and $Q = -\sum_i Q_i$ is the total charge on the island. The total electrostatic energy $E_e(Q) = \sum_i Q_i^2 / 2C_i = Q^2 / 2C_\Sigma + \text{constant}$, where the constant term does not depend on Q . From Eq. (A1) and Eq. (A2) one then finds that adding a charge q on the island will change the charge on each capacitor plate of

$$\delta Q_i = -\frac{qC_i}{C_\Sigma}. \quad (\text{A3})$$

The total electrostatic energy variation (final energy minus initial energy) for the transfer of an electron from the left electrode on the island is then given by the variation of the total electrostatic energy plus the the work done by the voltage sources ($-\sum_i V_i \delta Q_i$, with $\delta Q_i = eC_i / C_\Sigma$ for $i \neq L$ and $\delta Q_L = eC_L / C_\Sigma - e$):

$$\Delta E_L^\pm = E_e(Q - e) - E_e(Q) - e \sum_i V_i \frac{C_i}{C_\Sigma} + eV_L. \quad (\text{A4})$$

The general expression reads then:

$$\Delta E_{L,R}^\pm = -e \frac{(-e \pm 2Q)}{2C_\Sigma} \mp \frac{e}{C_\Sigma} \left(\sum_i V_i C_i - C_\Sigma V_{L,R} \right). \quad (\text{A5})$$

The variation of the energy depends only on the difference of the three potentials, we can thus choose to express it in terms of $V = V_R - V_L$ and $V'_g = V_g - (V_L + V_R)/2$. For simplicity we write the expressions in the symmetric case of $C_L = C_R = C$:

$$\Delta E_L^\pm = \frac{e}{2C_\Sigma} (e \mp 2Q) \mp \frac{e}{C_\Sigma} (C'V + C_g V'_g) \quad (\text{A6})$$

$$\Delta E_R^\pm = \frac{e}{2C_\Sigma} (e \mp 2Q) \mp \frac{e}{C_\Sigma} (-C'V + C_g V'_g) \quad (\text{A7})$$

with $C' = C + C_g/2$. Typically V is very small, while V'_g can be very large, in particular $V'_g C_g / e = n_g$ is normally regarded as finite, while $C_g \rightarrow 0$ and $V'_g \rightarrow \infty$. For this reasons we can normally neglect the displacement dependence induced by $C_g(x)$ in C' or C_Σ , while it is necessary to keep the x dependence in $C_g(x)$ that appears in the expression $C_g V'_g$.

Let now focus on the four energy variations associated with the change of the number of electrons in the dot between the two states N and $N + 1$. We need $\Delta E_{L,R}^+(N) = -\Delta E_{L,R}^-(N+1)$ that can be explicitly written as:

$$\Delta E^+(N)_{L,R} = -\frac{e^2}{C_\Sigma} (n_g \pm v - N - 1/2), \quad (\text{A8})$$

with $n_g = C_g V'_g / e$ and $v = C'V / e$. The expression of the tunneling rate is obtained then by the Fermi golden rule:

$$\Gamma^{\alpha\pm} = \frac{k_B T}{e^2 R_\alpha} h(\Delta_\alpha^\pm / k_B T) \quad (\text{A9})$$

with $h(x) = -x / (1 - e^x)$. In particular for $T \rightarrow 0$ the expression for the rate becomes simply $\Gamma^{\alpha\pm} = -\Delta_\alpha^\pm / e^2 R_\alpha \theta(-\Delta_\alpha^\pm)$. These expressions allow to obtain the tunneling rates necessary for the calculations presented in the main text of the paper.

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