Wilf's conjecture for numerical semigroups

Mariam Dhayni

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Abstract: Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity m, embedding dimension ν and conductor $c = f + 1 = qm - \rho$ for some $q, \rho \in \mathbb{N}$ with $\rho < m$. Let $\mathrm{Ap}(S, m) = \{w_0 < w_1 < \ldots < w_{m-1}\}$ be the Apéry set of S. The aim of this paper is to prove Wilf's Conjecture in some special cases. First, we prove that if $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m-1$, then S satisfies Wilf's Conjecture. Then, we prove the conjecture in the following cases: $(2 + \frac{1}{q})\nu \geq m$, $m - \nu \leq 5$ and $m = 9$. Finally, the conjecture is proved if $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$ for some $1 < \alpha < m-1$.

1 Introduction and notations

Let $\mathbb{N}$ denote the set of natural numbers, including 0. A *numerical semigroup* S is an additive submonoid of $(\mathbb{N}, +)$ of finite complement in $\mathbb{N}$, that is $0 \in S$, if $a, b \in S$ then $a + b \in S$ and $\mathbb{N} \setminus S$ is a finite set. The elements of $\mathbb{N} \setminus S$ are called the *gaps* of S. The largest gap is denoted by $f = f(S) = \max(\mathbb{N} \setminus S)$ and is called the *Frobenius number* of S. The smallest non zero element $m = m(S) = \min(S^*)$ is called the *multiplicity* of S and $n = |\{s \in S, s < f(S)\}|$ is also denoted by $n(S)$. Every numerical semigroup S is finitely generated, i.e. is of the form

$$S = < g_1, \ldots, g_\nu > = \mathbb{N}g_1 + \ldots + \mathbb{N}g_\nu$$

for suitable unique coprime integers $g_1, \ldots, g_\nu$. The number of generators of S is denoted by $\nu = \nu(S)$ and is called the *embedding dimension* of S. An integer $x \in \mathbb{N} \setminus S$ is called a *pseudo-Frobenius number* if $x + S^* \subseteq S$. The *type* of the semigroup, denoted by $t(S)$ is the cardinality of set of pseudo-frobenius numbers. The *Apéry set* of S with respect to $a \in S$ is defined as $\mathrm{Ap}(S, a) = \{s \in S; s - a \notin S\}$.

The *Diophantine Frobenius Problem*, also called the *Coin Problem*, is to find the largest number f that cannot be written in the form $\sum_{i=1}^{n} a_i x_i$; $x_i \in \mathbb{N}$ for given coprime positive integers $a_1, \cdots, a_n$. This problem is related to the theory of numerical semigroups in the following way: let S be the numerical semigroup generated by $g_1, \ldots, g_\nu$, then f is simply the largest integer not belonging to S. Hence the problem is to find a formula for f in terms of the set of generators of S. For $\nu = 2$, the formula of $f(< g_1, g_2 >)$ is given by Sylvester ([10]), for $\nu \geq 3$ the problem is much harder. It has been proved in ([7]), that in general $f(S)$ is not algebraic in the set of generators of S. In ([11]) 1978 H. S. Wilf proposed a lower bound for the number of generators of S in terms of the Frobenius number as follows: $f(S) + 1 \leq \nu(S)n(S)$.

Although the problem has been considered by several authors (cf. [1], [2], [3], [4], [5], [6], [9] , [12]), only special cases have been solved and it remains wide open. In ([3]), D. Dobbs and G. Matthews proved Wilf's Conjecture for $\nu \leq 3$. In ([6]) N. Kaplan proved it for $f + 1 \leq 2m$ and in ([4]) S. Eliahou extended Kaplan's work for $f + 1 \leq 3m$.

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†Université d'Angers, Mathématiques, 49045 Angers ceded 01, France, e-mail:dhayni@math.univ-angers.fr

This work is a generalisation of the case covered by A. Sammartano ([9]), who showed that Wilf's Conjecture holds for $2\nu \geq m$ and $m \leq 8$, based on the idea of counting the elements of S in some intervals of length m. We use different intervals in order to get an equivalent form of Wilf's Conjecture and then to prove it in more cases. We also cover the case where $2\nu \geq m$.

Here are few more details on the contents of this paper. Section 2 is devoted to give some notations that will enable us in the same section to give an equivalent form of Wilf's Conjecture. Section 3 is the heart of the paper. Let $\mathrm{Ap}(S,m) = \{0 = w_0 < w_1 < \cdots < w_{m-1}\}$. First, we show that Wilf's conjecture holds for numerical semigroups that satisfy $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m-1$ where $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. Then we prove Wilf's Conjecture for numerical semigroups with $m - \nu \leq 4$ in order to cover the case where $2\nu \geq m$, prove by Sammartano in ([9]). We also show that a numerical semigroup with $m - \nu = 5$ verify Wilf's Conjecture in order to prove the conjecture for $m = 9$. Finally, we show in this section, using the previous cases, that Wilf's conjecture holds for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$. In section 4 we prove Wilf's Conjecture for numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$ for some $1 < \alpha < m-1$.

A good reference on numerical semigroups is [8].

2 Preliminaries

Let the notations be as in the introduction. For the sake of clarity we shall use the notations $\nu, f, n, ...$ for $\nu(S), f(S), n(S)....$ In this section we will introduce some notations and family of numbers that will enable us to give an equivalent form of Wilf's conjecture.

Definition. Let S be a numerical semigroup and let $c = C(S) = f + 1$ be the conductor of S. Denote by

$$q = \lceil \frac{c}{m} \rceil,$$

where $\lceil x \rceil$ denote the smallest integer greater than or equal to x. Thus, $qm \geq c$ and $c = qm - \rho$ with $0 \leq \rho < m$. Given a non negative integer k, we define the kth interval of length m,

$$I_k = [km - \rho, (k+1)m - \rho[= \{km - \rho, km - \rho + 1, \ldots, (k+1)m - \rho - 1\}.$$

We denote by

$$n_k = |\{s \in S \cap I_k\}|.$$

For $j \in \{1, \ldots, m-1\}$, we define η_j to be the number of intervals I_k with $n_k = j$.

$$\eta_j = |\{k \in \mathbb{N}; |I_k \cap S| = j\}|.$$

Proposition 2.1 Under the previous notations, we have:

i) $1 \leq n_k \leq m-1$ for all $k = 0, \ldots, q-1$.

ii) $n_k = m$ for all $k \geq q$.

iii) $n = n(S) = \sum_{k=0}^{q-1} n_k$.

iv) $\sum_{j=1}^{m-1} \eta_j = q$.

v) $\sum_{j=1}^{m-1} j\eta_j = \sum_{k=0}^{q-1} n_k = n$.

Proof. *i), ii), iii)* are obvious. Now using *i)*, *ii)* we will prove *iv)* and *v)*.

iv) $\sum_{j=1}^{m-1} \eta_j = \sum_{j=1}^{m-1} |\{k \in \mathbb{N}; |I_k \cap S| = j\}| = \sum_{j=1}^{m-1} |\{k \in \mathbb{N}; n_k = j; k = 0, \ldots, q-1\}| = q.$

v) $\sum_{j=1}^{m-1} j\eta_j = \sum_{j=1}^{m-1} j|\{k \in \mathbb{N}; |I_k \cap S| = j\}| = \sum_{j=1}^{m-1} j|\{k \in \mathbb{N}; n_k = j; k = 0, \ldots, q-1\}| = \sum_{k=0}^{q-1} n_k = n.$ $\square$

Remark: We shall use the notation $\lfloor x \rfloor$ for the largest integer smaller than or equal to x.

Next we will express η_j in terms of th Apéry set.

Proposition 2.2 Let $\mathrm{Ap}(S, m) = \{w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}\}$. Under the previous notations, we have for all $j \in \{1, \ldots, m-1\}$

$$\eta_j = \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor.$$

Proof. Fix $0 \leq k \leq q-1$, and let $j \in \{1, \ldots, m-1\}$. We will show that the interval I_k contains exactly j elements of S if and only if $w_{j-1} < (k+1)m - \rho \leq w_j$. Recall to this end that for all $s \in S$, there exist $0 \leq i \leq m-1, a \in \mathbb{N}$ such that $s = w_i + am$.

Suppose that I_k contains j elements. Suppose, by contradiction, that $w_{j-1} \geq (k+1)m - \rho$. We have $w_{m-1} > \ldots > w_{j-1} \geq (k+1)m - \rho$, thus $w_{m-1}, \ldots, w_{j-1} \in \cup_{t=k+1}^{q} I_t$. Hence, I_k contains at most $j-1$ elements of S (namely $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_{j-2} + k_{j-2} m$ for some $k_1, \ldots, k_{j-2} \in \{0, \ldots, k-1\}$). This contradicts the fact that I_k contains exactly j elements of S.
If $w_j < (k+1)m - \rho$, then $w_0 < \ldots < w_j < (k+1)m - \rho$, thus $w_0, \ldots, w_j \in \cup_{t=0}^{k} I_t$. Hence, I_k contains at least $j+1$ elements of S which are : $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_j + k_j m$ for some $k_1, \ldots, k_j \in \{0, \ldots, k-1\}$, which contradicts the fact that I_k contains exactly j elements of S.

Conversely, $w_{j-1} < (k+1)m - \rho$ implies that $w_0 < \ldots < w_{j-1} < (k+1)m - \rho$, so $w_0, \ldots, w_{j-1} \in \cup_{t=0}^{k} I_t$. Thus I_k contains at least j elements which are : $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_{j-1} + k_{j-1} m$ for some $k_1, \ldots, k_{j-1} \in \{0, \ldots, k-1\}$.
On the other hand $w_j \geq (k+1)m - \rho$ implies that $w_{m-1} > \ldots > w_j \geq (k+1)m - \rho$, so $w_{m-1}, \ldots, w_j \in \cup_{t=k+1}^{q} I_t$. Thus I_k contains at most j elements which are : $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_{j-1} + k_{j-1} m$ for some $k_1, \ldots, k_{j-1} \in \{0, \ldots, k-1\}$. Hence, if $w_{j-1} < (k+1)m - \rho \leq w_j$, then I_k contains exactly j elements of S and this proves our assertion.
Consequently,

$$\begin{aligned}
\eta_j &= |\{k \in \mathbb{N} \text{ such that } |I_k \cap S| = j\}| \\
&= |\{k \in \mathbb{N} \text{ such that } w_{j-1} < (k+1)m - \rho \leq w_j\}| \\
&= |\{k \in \mathbb{N} \text{ such that } \tfrac{w_{j-1}+\rho}{m} < (k+1) \leq \tfrac{w_j+\rho}{m}\}| \\
&= |\{k \in \mathbb{N} \text{ such that } \tfrac{w_{j-1}+\rho}{m} - 1 < k \leq \tfrac{w_j+\rho}{m} - 1\}| \\
&= |\{k \in \mathbb{N} \text{ such that } \lfloor \tfrac{w_{j-1}+\rho}{m} \rfloor \leq k \leq \lfloor \tfrac{w_j+\rho}{m} \rfloor - 1\}| \\
&= \lfloor \tfrac{w_j+\rho}{m} \rfloor - \lfloor \tfrac{w_{j-1}+\rho}{m} \rfloor. \quad \square
\end{aligned}$$

Proposition 2.3 gives an equivalent form of Wilf's Conjecture using Propositions 2.1 and 2.2.

Proposition 2.3 Let S be a numerical semigroup with multiplicity m, embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$ and $0 \leq \rho \leq m-1$. Let $w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S, m)$. Then S satisfies Wilf's Conjecture *if and only if*

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

Proof. By Proposition 2.1, we have

$$f+1 \leq n\nu \Leftrightarrow qm-\rho \leq \nu\sum_{k=0}^{q-1} n_k \Leftrightarrow \sum_{k=0}^{q-1} m-\rho \leq \sum_{k=0}^{q-1} n_k\nu \Leftrightarrow \sum_{k=0}^{q-1}(n_k\nu-m)+\rho \geq 0 \Leftrightarrow \sum_{j=1}^{m-1}\eta_j(j\nu-m)+\rho \geq 0.$$

Using Proposition 2.2, we have

$$\sum_{j=1}^{m-1}\eta_j(j\nu-m)+\rho \geq 0 \Leftrightarrow \sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \geq 0.$$

Thus the proof is complete. □

Remark 2.4 Let $\mathrm{Ap}(S,m)=\{w_0=0<w_1<\ldots<w_{m-1}\}$. The following technical results will be used through the paper:

1. $\lfloor\frac{w_0+\rho}{m}\rfloor=0$ ($w_0=0$ and $0\leq\rho<m$).
2. For all $1\leq i\leq m-1$, $\lfloor\frac{w_i+\rho}{m}\rfloor\geq 1$ ($w_i>m$).
3. For all $1\leq i\leq m-1$, either $\lfloor\frac{w_i+\rho}{m}\rfloor=\lfloor\frac{w_i}{m}\rfloor$ or $\lfloor\frac{w_i+\rho}{m}\rfloor=\lfloor\frac{w_i}{m}\rfloor+1$. In the second case $\lfloor\frac{w_i+\rho}{m}\rfloor\geq 2$ and $\rho\geq 1$.
4. If $w_i<w_j$ for some $0\leq i<j\leq m-1$, then $\lfloor\frac{w_i+\rho}{m}\rfloor\leq\lfloor\frac{w_j+\rho}{m}\rfloor$.
5. $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor=\lfloor\frac{qm-\rho+m-1+\rho}{m}\rfloor=q$.

3 Main Results

In this section, we show that Wilf's Conjecture holds for numerical semigroups in the following cases:

1. $w_{m-1}\geq w_1+w_\alpha$ and $(2+\frac{\alpha-3}{q})\nu\geq m$ for some $1<\alpha<m-1$.
2. $m-\nu\leq 5$. (Note that the case $m-\nu\leq 4$ results from the fact that Wilf's Conjecture holds for $2\nu\geq m$ ([9]), however we shall give the proof for $m-\nu\leq 3$ in order to cover this result through our techniques).

We then deduce the conjecture for $m=9$ and for $(2+\frac{1}{q})\nu\geq m$.

The following technical Lemma will be used through the paper:

Lemma 3.1 Let $\mathrm{Ap}(S,m)=\{w_0=0<w_1<\ldots<w_{m-1}\}$. Suppose that $w_i\geq w_j+w_k$, then $\lfloor\frac{w_i+\rho}{m}\rfloor\geq\lfloor\frac{w_j+\rho}{m}\rfloor+\lfloor\frac{w_k+\rho}{m}\rfloor-1$. If furthermore, $\lfloor\frac{w_i+\rho}{m}\rfloor-\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_k+\rho}{m}\rfloor=-1$, then $\lfloor\frac{w_j+\rho}{m}\rfloor=\lfloor\frac{w_j}{m}\rfloor+1$, $\lfloor\frac{w_k+\rho}{m}\rfloor=\lfloor\frac{w_k}{m}\rfloor+1$ and $\rho\geq 1$. In particular, $\lfloor\frac{w_j+\rho}{m}\rfloor\geq 2$, $\lfloor\frac{w_k+\rho}{m}\rfloor\geq 2$ and $\rho\geq 1$.

Proof. $w_i\geq w_j+w_k$ implies that $w_i+\rho\geq w_j+w_k+\rho$, hence $\frac{w_i+\rho}{m}\geq\frac{w_j+w_k+\rho}{m}$. Consequently, $\lfloor\frac{w_i+\rho}{m}\rfloor\geq\lfloor\frac{w_j+w_k+\rho}{m}\rfloor$. Therefore, $\lfloor\frac{w_i+\rho}{m}\rfloor\geq\lfloor\frac{w_j+\rho}{m}\rfloor+\lfloor\frac{w_k}{m}\rfloor$. Hence, by Remark 2.4 (3), $\lfloor\frac{w_i+\rho}{m}\rfloor\geq\lfloor\frac{w_j+\rho}{m}\rfloor+\lfloor\frac{w_k+\rho}{m}\rfloor-1$. Suppose that $w_i\geq w_j+w_k$ and that $\lfloor\frac{w_i+\rho}{m}\rfloor-\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_k+\rho}{m}\rfloor=-1$. Suppose by the way of contradiction that either $\lfloor\frac{w_j+\rho}{m}\rfloor\neq\lfloor\frac{w_j}{m}\rfloor+1$ or $\lfloor\frac{w_k+\rho}{m}\rfloor\neq\lfloor\frac{w_k}{m}\rfloor+1$ or $\rho<1$. Then, by Remark 2.4 (3) and the fact that $\rho\geq 0$, we have either $\lfloor\frac{w_j+\rho}{m}\rfloor=\lfloor\frac{w_j}{m}\rfloor$ or $\lfloor\frac{w_k+\rho}{m}\rfloor=\lfloor\frac{w_k}{m}\rfloor$ or $\rho=0$. Since $w_i\geq w_j+w_k$, then $\lfloor\frac{w_i+\rho}{m}\rfloor\geq\lfloor\frac{w_j+w_k+\rho}{m}\rfloor$. In this case $\lfloor\frac{w_i+\rho}{m}\rfloor\geq\lfloor\frac{w_j+\rho}{m}\rfloor+\lfloor\frac{w_k+\rho}{m}\rfloor$, which is impossible. Hence, $\lfloor\frac{w_j+\rho}{m}\rfloor=\lfloor\frac{w_j}{m}\rfloor+1$, $\lfloor\frac{w_k+\rho}{m}\rfloor=\lfloor\frac{w_k}{m}\rfloor+1$ and $\rho\geq 1$. Therefore, by Remark 2.4 (2), $\lfloor\frac{w_j+\rho}{m}\rfloor=\lfloor\frac{w_j}{m}\rfloor+1\geq 2$, $\lfloor\frac{w_k+\rho}{m}\rfloor=\lfloor\frac{w_k}{m}\rfloor+1\geq 2$ and $\rho\geq 1$. □

Next we will show that Wilf's Conjecture holds for numerical semigroups with $w_{m-1}\geq w_1+w_\alpha$ and $(2+\frac{\alpha-3}{q})\nu\geq m$.

Theorem 3.2 Let S be a numerical semigroup with multiplicity m, embedding dimension ν and conductor $f+1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. Let $w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S,m)$. Suppose that $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m-1$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. We are going to show that S satisfies Wilf's Conjecture by means of Proposition 2.3. We have,

$$
\begin{aligned}
\sum_{j=1}^{\alpha}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu - m) &= \sum_{j=1}^{\alpha}\lfloor\frac{w_j+\rho}{m}\rfloor(j\nu-m) - \sum_{j=1}^{\alpha}\lfloor\frac{w_{j-1}+\rho}{m}\rfloor(j\nu-m)\\
&= \sum_{j=1}^{\alpha}\lfloor\frac{w_j+\rho}{m}\rfloor(j\nu-m) - \sum_{j=0}^{\alpha-1}\lfloor\frac{w_j+\rho}{m}\rfloor((j+1)\nu-m)\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_0+\rho}{m}\rfloor(\nu-m) - \sum_{j=1}^{\alpha-1}\lfloor\frac{w_j+\rho}{m}\rfloor\nu\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_1+\rho}{m}\rfloor\nu - \sum_{j=2}^{\alpha-1}\lfloor\frac{w_j+\rho}{m}\rfloor\nu \qquad (3.1)\\
&\geq \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_1+\rho}{m}\rfloor\nu - \sum_{j=2}^{\alpha-1}\lfloor\frac{w_\alpha+\rho}{m}\rfloor\nu\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_1+\rho}{m}\rfloor\nu - \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha-2)\nu\\
&= -\lfloor\frac{w_1+\rho}{m}\rfloor\nu + \lfloor\frac{w_\alpha+\rho}{m}\rfloor(2\nu-m).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=\alpha+1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m) &\geq \sum_{j=\alpha+1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)((\alpha+1)\nu-m)\\
&= ((\alpha+1)\nu-m)\sum_{j=\alpha+1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)\\
&= ((\alpha+1)\nu-m)(\sum_{j=\alpha+1}^{m-1}\lfloor\frac{w_j+\rho}{m}\rfloor - \sum_{j=\alpha+1}^{m-1}\lfloor\frac{w_{j-1}+\rho}{m}\rfloor) \qquad (3.2)\\
&= ((\alpha+1)\nu-m)(\sum_{j=\alpha+1}^{m-1}\lfloor\frac{w_j+\rho}{m}\rfloor - \sum_{j=\alpha}^{m-2}\lfloor\frac{w_j+\rho}{m}\rfloor)\\
&= (\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m).
\end{aligned}
$$

Since $w_{m-1} \geq w_1 + w_\alpha$, by Lemma 3.1, it follows that $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor \geq \lfloor\frac{w_1+\rho}{m}\rfloor + \lfloor\frac{w_\alpha+\rho}{m}\rfloor - 1$. Let $x = \lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_1+\rho}{m}\rfloor - \lfloor\frac{w_\alpha+\rho}{m}\rfloor$. Then, $\lfloor\frac{w_1+\rho}{m}\rfloor + \lfloor\frac{w_\alpha+\rho}{m}\rfloor = q - x$ and $x \geq -1$. Now using (3.1) and (3.2), we have

$$\begin{aligned}
\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho &\geq -\lfloor\frac{w_1+\rho}{m}\rfloor\nu+\lfloor\frac{w_\alpha+\rho}{m}\rfloor(2\nu-m)\\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m)+\rho\\
&=\lfloor\frac{w_1+\rho}{m}\rfloor(-\nu+((\alpha+1)\nu-m)-((\alpha+1)\nu-m))+\lfloor\frac{w_\alpha+\rho}{m}\rfloor(2\nu-m)\\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m)+\rho\\
&=\lfloor\frac{w_1+\rho}{m}\rfloor(\alpha\nu-m)+\lfloor\frac{w_\alpha+\rho}{m}\rfloor(2\nu-m)\\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)((\alpha+1)\nu-m)+\rho\\
&=(\lfloor\frac{w_1+\rho}{m}\rfloor+\lfloor\frac{w_\alpha+\rho}{m}\rfloor)(2\nu-m)+\lfloor\frac{w_1+\rho}{m}\rfloor(\alpha-2)\nu\\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)((\alpha+1)\nu-m)+\rho\\
&=(q-x)(2\nu-m)+\lfloor\frac{w_1+\rho}{m}\rfloor(\alpha-2)\nu+x((\alpha+1)\nu-m)+\rho\\
&\geq (q-x)(2\nu-m)+(\alpha-2)\nu+x((\alpha+1)\nu-m)+\rho\\
&=\nu(2q-2x+\alpha-2+x\alpha+x)-qm+\rho\\
&=\nu(2q+(\alpha-2)(x+1)+x)-qm+\rho\\
&\geq \nu(2q+\alpha-3)-qm+\rho \qquad (x\geq -1)\\
&=q(\nu(2+\frac{\alpha-3}{q})-m)+\rho\\
&\geq 0.
\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture. $\square$

Example 3.3 Consider the following numerical semigroup $S =< 19, 21, 23, 25, 27, 28 >$. Note that $3\nu < m$. We have $w_1 = 21$, $w_{14} = 56$ and $w_{m-1} = 83$ i.e. $w_{m-1} \geq w_1 + w_{14}$. In addition, $(2+\frac{\alpha-3}{q})\nu = (2+\frac{14-3}{4})6 \geq 19 = m$. Thus the conditions of Theorem 3.2 are valid, so S satisfies Wilf's Conjecture.

In the following we shall deduce some cases where Wilf's Conjecture holds. We start with the following technical Lemma.

Lemma 3.4 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. Let $w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S, m)$. If $m - \nu > \binom{\alpha}{2} = \dfrac{\alpha(\alpha-1)}{2}$ for some $\alpha \in \mathbb{N}^*$, then $w_{m-1} \geq w_1 + w_\alpha$.

Proof. Recall that an element x of the Apéry set of S belongs to $\min(\mathrm{Ap}(S, m))$ if and only if $x \neq w_i + w_j$ for all $w_i, w_j \in \mathrm{Ap}(S, m) \setminus \{0\}$, in particular $m - \nu = |\mathrm{Ap}(S, m) \setminus \min(\mathrm{Ap}(S, m))|$. Suppose by the way of contradiction that $w_{m-1} < w_1 + w_\alpha$, and let $w \in \mathrm{Ap}(S, m) \setminus \min(\mathrm{Ap}(S, m))$. Then $w \leq w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in$

$\mathrm{Ap}(S,m)\backslash\{0\}$. Hence, $w \leq w_{m-1} < w_1+w_\alpha$. Thus the only possible values for w are $\{w_i+w_j; 1 \leq i \leq j \leq \alpha-1\}$. Therefore, $m-\nu \leq \binom{\alpha}{2} = \dfrac{\alpha(\alpha-1)}{2}$, which is impossible. Hence, $w_{m-1} \geq w_1 + w_\alpha$. $\square$

Next we will deduce Wilf's Conjecture for numerical Semigroups with $m-\nu > \frac{\alpha(\alpha-1)}{2}$ and $(2+\frac{\alpha-3}{q})\nu \geq m$. It will be used later to show that the conjecture holds for those with $(2+\frac{1}{q})\nu \geq m$, and inorder also to cover the result in [9] saying that the conjecture is true for $2\nu \geq m$.

Corollary 3.5 Let S be a numerical semigroup with multiplicity m, embedding dimension ν and conductor $f+1 = qm-\rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. Suppose that $m-\nu > \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ for some $1 < \alpha < m-1$. If $(2+\frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. It follows from Lemma 3.4 that if $m-\nu > \frac{\alpha(\alpha-1)}{2}$, then $w_{m-1} \geq w_1 + w_\alpha$. Now use Theorem 3.2. $\square$

As a direct consequence of Theorem 3.2, we get the following Corollary.

Corollary 3.6 Let S be a numerical semigroup with a given multiplicity m and conductor $f+1 = qm-\rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. Let $w_0 = 0 < w_1 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S,m)$. If $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m-1$ and $m \leq 8 + 4(\frac{\alpha-3}{q})$ then S satisfies Wilf's Conjecture.

Proof. By Theorem 3.2, we may assume that $(2+\frac{\alpha-3}{q})\nu < m$. Therefore, $\nu < \frac{qm}{2q+\alpha-3} \leq \frac{8q+\alpha-12}{2q+\alpha-3}$. Hence $\nu < 4$, consequently S satisfies Wilf's Conjecture ([3]). $\square$

In the following Lemma, we will show that Wilf's Conjecture holds for numerical semigroups with $m-\nu \leq 3$. This will enable us later to prove the conjecture for numerical semigroups with $(2+\frac{1}{q})\nu \geq m$ and cover the result in [9] saying that the conjecture is true for $2\nu \geq m$.

Lemma 3.7 Let S be a numerical Semigroup with multiplicity m and embedding dimension ν. If $m-\nu \leq 3$, then S satisfies Wilf's Conjecture.

Proof. We my assume that $\nu \geq 4$ ($\nu \leq 3$ is solved [3]). We are going to show that S satisfies Wilf's Conjecture by means of Proposition 2.3.

i) If $m-\nu = 1$, then we may assume that $m = \nu+1 \geq 5$. By taking $\alpha = 1$ in (3.2), we get
$\sum_{j=2}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m) \geq (\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)$. Hence,

$$\begin{aligned}
\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho &= (\lfloor\frac{w_1+\rho}{m}\rfloor - \lfloor\frac{w_0+\rho}{m}\rfloor)(\nu-m) \\
&\quad + \sum_{j=2}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \\
&\geq \lfloor\frac{w_1+\rho}{m}\rfloor(\nu-m) \\
&\quad + (\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)+\rho \\
&= \lfloor\frac{w_1+\rho}{m}\rfloor(\nu-m+(2\nu-m)-(2\nu-m)) \\
&\quad + (\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)+\rho
\end{aligned}$$

$$= \lfloor \frac{w_1+\rho}{m} \rfloor (3\nu - 2m)$$
$$+(\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)(2\nu - m) + \rho$$

$$= \lfloor \frac{w_1+\rho}{m} \rfloor (m-3)$$
$$+(\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)(m-2) + \rho.$$

Since $m - \nu = 1 > 0 = \frac{1(0)}{2}$, then by Lemma 3.4, it follows that $w_{m-1} \geq w_1 + w_1$. Consequently, by Lemma 3.1, we have $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor \geq \lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_1+\rho}{m} \rfloor - 1$.

- If $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor = -1$. Then by Lemma 3.1, we have $\lfloor \frac{w_1+\rho}{m} \rfloor \geq 2$. Since $m \geq 5$, then
  $\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \quad \geq 2(m-3) - (m-2) + \rho \geq 0.$
- If $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor \geq 0$. Since $m \geq 5$, then
  $\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \quad \geq (m-3) + \rho \geq 0.$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture if $m - \nu = 1$.

ii) If $m - \nu \in \{2, 3\}$. By taking $\alpha = 2$ in (3.2), we get
$\sum_{j=3}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) \quad \geq (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor)(3\nu - m)$. Hence,

$$\begin{aligned}
\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \quad &= (\lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_0+\rho}{m} \rfloor)(\nu - m) \\
&+(\lfloor \frac{w_2+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)(2\nu - m) \\
&+ \sum_{j=3}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \\
&\geq \lfloor \frac{w_1+\rho}{m} \rfloor (-\nu) + \lfloor \frac{w_2+\rho}{m} \rfloor (2\nu - m) \\
&+(\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor)(3\nu - m) + \rho \\
&= \lfloor \frac{w_1+\rho}{m} \rfloor (-\nu + (3\nu - m) - (3\nu - m)) \\
&+\lfloor \frac{w_2+\rho}{m} \rfloor (2\nu - m) \\
&+(\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor)(3\nu - m) + \rho \\
&= \lfloor \frac{w_1+\rho}{m} \rfloor (2\nu - m) + \lfloor \frac{w_2+\rho}{m} \rfloor (2\nu - m) \\
&+(\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor)(3\nu - m) \\
&+\rho.
\end{aligned} \tag{3.3}$$

Since $m - \nu \in \{2, 3\} > 1$, by Lemma 3.4, we have $w_{m-1} \geq w_1 + w_2$. It follows from Lemma 3.1 that $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor \geq \lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_2+\rho}{m} \rfloor - 1$.

- If $m-\nu=2$. then we may assume that $m=\nu+2\geq 6$. Now (3.3) gives,

$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \quad \geq \lfloor\frac{w_1+\rho}{m}\rfloor(m-4)+\lfloor\frac{w_2+\rho}{m}\rfloor(m-4)$$
$$+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor)(2m-6)+\rho.$$

  - If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor=-1$. Then by Lemma 3.1 we have, $\lfloor\frac{w_1+\rho}{m}\rfloor\geq 2$ and $\lfloor\frac{w_2+\rho}{m}\rfloor\geq 2$. Since $m\geq 6$, then
  $$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \quad \geq 2(m-4)+2(m-4)-(2m-6)+\rho\geq 0.$$
  - If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor\geq 0$. Since $m\geq 6$, then
  $$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \quad \geq (m-4)+(m-4)+\rho\geq 0.$$

  Using Proposition 2.3, we get that S satisfies Wilf's Conjecture if $m-\nu=2$.

- If $m-\nu=3$, then we may assume that $m=\nu+3\geq 7$. Now (3.3) gives,
$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \quad \geq \lfloor\frac{w_1+\rho}{m}\rfloor(m-6)+\lfloor\frac{w_2+\rho}{m}\rfloor(m-6)$$
$$+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor)(2m-9)+\rho.$$

  - If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor=-1$. Then by Lemma 3.1 we have, $\lfloor\frac{w_1+\rho}{m}\rfloor\geq 2, \lfloor\frac{w_2+\rho}{m}\rfloor\geq 2$ and $\rho\geq 1$. Since $m\geq 7$, then
  $$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \quad \geq 2(m-6)+2(m-6)-(2m-9)+1\geq 0.$$
  - If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor\geq 0$. Since $m\geq 7$, then
  $$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho \quad \geq (m-6)+(m-6)+\rho\geq 0.$$

  Using Proposition 2.3, we get that S satisfies Wilf's Conjecture if $m-\nu=3$.

Thus Wilf's Conjecture holds if $m-\nu\leq 3$. $\square$

The next Corollary covers the result of Sammartano for numerical semigroups with $2\nu\geq m$ ([9]) using Corollary 3.5 and Lemma 3.7.

Corollary 3.8 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. If $2\nu\geq m$, then S satisfies Wilf's Conjecture.

Proof. If $m-\nu>3$ and $2\nu\geq m$, then by Corollary 3.5 Wilf's Conjecture holds. If $m-\nu\leq 3$, by Lemma 3.7, S satisfies Wilf's Conjecture. $\square$

In the following Corollary we will deduce Wilf's Conjecture for numerical semigroups with $m-\nu=4$. This will enable us later to prove the conjecture for those with $(2+\frac{1}{q})\nu\geq m$.

Corollary 3.9 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. If $m-\nu=4$, then S satisfies Wilf's Conjecture.

Proof. Since Wilf's conjecture holds for $\nu \leq 3$ ([3]), then we may assume that $\nu \geq 4$. Hence, $\nu \geq m - \nu$. Consequently, $2\nu \geq m$. Hence, S satisfies Wilf's Conjecture. $\square$

The following technical Lemma will be used through the paper.

Lemma 3.10 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. Let $w_0 = 0 < w_1 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S, m)$. If $m - \nu \geq \binom{\alpha}{2} - 1 = \frac{\alpha(\alpha-1)}{2} - 1$ for some $3 \leq \alpha \leq m-2$, then $w_{m-1} \geq w_1 + w_\alpha$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$.

Proof. Suppose by the way of contradiction that $w_{m-1} < w_1 + w_\alpha$ and $w_{m-1} < w_{\alpha-2} + w_{\alpha-1}$. Let $w \in \mathrm{Ap}(S,m) \backslash \min(\mathrm{Ap}(S,m))$, then $w \leq w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in \mathrm{Ap}(S,m) \setminus \{0\}$. In this case, the only possible values of w are $\{w_i + w_j; 1 \leq i \leq j \leq \alpha - 1\} \setminus \{w_{\alpha-2} + w_{\alpha-1}, w_{\alpha-1} + w_{\alpha-1}\}$. Consequently, $m - \nu = |\mathrm{Ap}(S,m) \backslash \min(\mathrm{Ap}(S,m))| \leq \frac{\alpha(\alpha-1)}{2} - 2$. But $\frac{\alpha(\alpha-1)}{2} - 2 < \frac{\alpha(\alpha-1)}{2} - 1$, which contradicts the hypothesis. Hence, $w_{m-1} \geq w_1 + w_\alpha$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$. $\square$

In the next theorem, we will show that Wilf's Conjecture holds for numerical semigroups with $m - \nu = 5$.

Theorem 3.11 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. If $m - \nu = 5$, then S satisfies Wilf's Conjecture.

Proof. Let $m - \nu = 5$. Since Wilf's Conjecture holds for $2\nu \geq m$, then we may assume that $2\nu < m$. This implies that $\nu < 5$. Since the case $\nu \leq 3$ is known ([3]), then we shall assume that $\nu = 4$. This also implies that $m = 9$.

Since $m - \nu = 5 = \frac{4(3)}{2} - 1$, by Lemma 3.10, it follows that $w_8 \geq w_2 + w_3$ or $w_8 \geq w_1 + w_4$.

i) If $w_8 \geq w_2 + w_3$. By taking $\alpha = 3$ in (3.2) ($m = 9$, $\nu = 4$), we get

$$\sum_{j=4}^{8} (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) \quad \geq (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(16 - 9) \quad = (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7).$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{8} (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \quad &= (\lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_0 + \rho}{9} \rfloor)(-5) \\
&+ (\lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor)(-1) \\
&+ (\lfloor \frac{w_3 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor)(3) \\
&+ \sum_{j=4}^{8} (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\
&\geq \lfloor \frac{w_1 + \rho}{9} \rfloor (-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor (-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor (3) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho \\
&\geq \left(\lfloor \frac{w_2 + \rho}{m} \rfloor ((\frac{-3}{4})4) + \lfloor \frac{w_3 + \rho}{9} \rfloor ((\frac{-1}{4})4) \right) \\
&+ \lfloor \frac{w_2 + \rho}{9} \rfloor (-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor (3) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho
\end{aligned} \tag{3.4}$$

$$= \lfloor \frac{w_2+\rho}{9} \rfloor(-7) + \lfloor \frac{w_3+\rho}{9} \rfloor(2)$$
$$+(\lfloor \frac{w_8+\rho}{9} \rfloor - \lfloor \frac{w_3+\rho}{9} \rfloor)(7) + \rho$$

$$= \lfloor \frac{w_3+\rho}{9} \rfloor(2)$$
$$+(\lfloor \frac{w_8+\rho}{9} \rfloor - \lfloor \frac{w_2+\rho}{9} \rfloor - \lfloor \frac{w_3+\rho}{9} \rfloor)(7) + \rho.$$

Since $w_8 \geq w_2 + w_3$, by Lemma 3.1, it follows that $\lfloor \frac{w_8+\rho}{9} \rfloor \geq \lfloor \frac{w_2+\rho}{9} \rfloor + \lfloor \frac{w_3+\rho}{9} \rfloor - 1$.

- If $\lfloor \frac{w_8+\rho}{9} \rfloor - \lfloor \frac{w_2+\rho}{9} \rfloor - \lfloor \frac{w_3+\rho}{9} \rfloor \geq 0$, then (3.4) gives $\sum_{j=1}^{8} (\lfloor \frac{w_j+\rho}{9} \rfloor - \lfloor \frac{w_{j-1}+\rho}{9} \rfloor)(4j-9) + \rho \geq 0$.
- If $\lfloor \frac{w_8+\rho}{9} \rfloor - \lfloor \frac{w_2+\rho}{9} \rfloor - \lfloor \frac{w_3+\rho}{9} \rfloor = -1$. By Lemma 3.1, we have $\rho \geq 1$.
  Since for $q \leq 3$ Wilf's Conjecture is solved ([4], [6]), then may assume that $q \geq 4$. Since $\lfloor \frac{w_2+\rho}{9} \rfloor \leq \lfloor \frac{w_3+\rho}{9} \rfloor$ and $\lfloor \frac{w_2+\rho}{9} \rfloor + \lfloor \frac{w_3+\rho}{9} \rfloor = \lfloor \frac{w_8+\rho}{9} \rfloor + 1 = q+1$, in this case it follows that $\lfloor \frac{w_3+\rho}{9} \rfloor \geq 3$.
  Now (3.4) gives, $\sum_{j=1}^{8} (\lfloor \frac{w_j+\rho}{9} \rfloor - \lfloor \frac{w_{j-1}+\rho}{9} \rfloor)(4j-m) + \rho \geq 3(2) - 7 + 1 \geq 0$.

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture in this case.

ii) If $w_8 \geq w_1 + w_4$. We may assume that $w_8 < w_2 + w_3$, since otherwise we are back to case 1. Hence, the possible values of $w \in \mathrm{Ap}(S,9) \backslash \min(\mathrm{Ap}(S,9))$ are $\{w_1 + w_j; 1 \leq j \leq 7\} \cup \{w_2 + w_2\}$.

- Recall that an element x of the Apéry set of S belongs to $\max(\mathrm{Ap}(S,m))$ if and only if $w_i \neq x + w_j$ for all $w_i, w_j \in \mathrm{Ap}(S,m) \setminus \{0\}$. If $\mathrm{Ap}(S,9) \backslash \min(\mathrm{Ap}(S,9)) \subseteq \{w_1 + w_j; 1 \leq j \leq 7\}$, then there exists at least five elements in $\mathrm{Ap}(S,9)$ that are not maximal, hence $t(S) = |\{\max(\mathrm{Ap}(S,9)) - 9\}| \leq 3 = \nu - 1$. Consequently, S satisfies Wilf's Conjecture ([3] Proposition 2.3).
- If $w_2 + w_2 \in \mathrm{Ap}(S,9) \backslash \min(\mathrm{Ap}(S,9))$, then $w_8 \geq w_2 + w_2$. By Lemma 3.1 we have $\lfloor \frac{w_8+\rho}{9} \rfloor \geq 2\lfloor \frac{w_2+\rho}{9} \rfloor - 1$. In particular,

$$\lfloor \frac{w_2+\rho}{9} \rfloor \leq \frac{q+1}{2}. \tag{3.5}$$

By taking $\alpha = 4$ in (3.2) ($m = 9$, $\nu = 4$), we get
$\sum_{j=5}^{8} (\lfloor \frac{w_j+\rho}{9} \rfloor - \lfloor \frac{w_{j-1}+\rho}{9} \rfloor)(4j-9) \geq (\lfloor \frac{w_8+\rho}{9} \rfloor - \lfloor \frac{w_4+\rho}{9} \rfloor)(11)$. Now using (3.5), we get

$$\begin{aligned} \sum_{j=1}^{8} (\lfloor \frac{w_j+\rho}{9} \rfloor - \lfloor \frac{w_{j-1}+\rho}{9} \rfloor)(4j-9) + \rho &= (\lfloor \frac{w_1+\rho}{9} \rfloor - \lfloor \frac{w_0+\rho}{9} \rfloor)(-5) \\ &+(\lfloor \frac{w_2+\rho}{9} \rfloor - \lfloor \frac{w_1+\rho}{9} \rfloor)(-1) \\ &+(\lfloor \frac{w_3+\rho}{9} \rfloor - \lfloor \frac{w_2+\rho}{9} \rfloor)(3) \\ &+(\lfloor \frac{w_4+\rho}{9} \rfloor - \lfloor \frac{w_3+\rho}{9} \rfloor)(7) \\ &+\sum_{j=5}^{8} (\lfloor \frac{w_j+\rho}{9} \rfloor - \lfloor \frac{w_{j-1}+\rho}{9} \rfloor)(4j-9) + \rho \end{aligned} \tag{3.6}$$

$$\geq \lfloor\frac{w_1+\rho}{9}\rfloor(-4)+\lfloor\frac{w_2+\rho}{9}\rfloor(-4)+\lfloor\frac{w_3+\rho}{9}\rfloor(-4)$$
$$+\lfloor\frac{w_4+\rho}{9}\rfloor(7)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho$$

$$\geq \lfloor\frac{w_1+\rho}{9}\rfloor(-4)+(\frac{q+1}{2})(-4)+\lfloor\frac{w_4+\rho}{9}\rfloor(-4)$$
$$+\lfloor\frac{w_4+\rho}{9}\rfloor(7)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho$$

$$=\lfloor\frac{w_1+\rho}{9}\rfloor(-4)-2(q+1)+\lfloor\frac{w_4+\rho}{9}\rfloor(3)$$
$$+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho$$

$$=\lfloor\frac{w_1+\rho}{9}\rfloor(-4+11-11)-2(q+1)$$
$$+\lfloor\frac{w_4+\rho}{9}\rfloor(3)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho$$

$$=\lfloor\frac{w_1+\rho}{9}\rfloor(7)-2(q+1)+\lfloor\frac{w_4+\rho}{9}\rfloor(3)$$
$$+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor)(11)+\rho$$

$$=(\lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor)(3)+\lfloor\frac{w_1+\rho}{9}\rfloor(4)-2(q+1)$$
$$+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho.$$

We have $w_8 \geq w_1+w_4$, then by Lemma 3.1 $\lfloor\frac{w_8+\rho}{9}\rfloor \geq \lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor-1$.

- If $\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor \geq 0$. Let $x=\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor$. Hence, $x\geq 0$ and $\lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor=q-x$. Then (3.6) gives,
$$\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho \quad \geq (q-x)(3)+4-2(q+1)+11x+\rho = q+8x+2+\rho \geq 0.$$
- If $\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor=-1$. Then $\lfloor\frac{w_1+\rho}{m}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor=q+1$. By Lemma 3.1, we have $\lfloor\frac{w_1+\rho}{9}\rfloor\geq 2$ and $\rho\geq 1$. Since $q\geq 1$, then (3.6) gives,
$$\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho \quad \geq (q+1)(3)+8-2(q+1)-11+1 = q-1 \geq 0.$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture in this case.

Thus, Wilf's Conjecture holds if $m-\nu=5$. □

In the next corollary, we will deduce the conjecture for $m=9$.

Corollary 3.12 If S is a numerical Semigroup with multiplicity $m=9$, then S satisfies Wilf's Conjecture.

Proof. By Lemma 3.7, Corollary 3.9 and Theorem 3.11, we may assume that $m-\nu>5$, hence $\nu<m-5=4$. By ([3]) S satisfies Wilf's Conjecture. □

The following Lemma will enable us later to show that Wilf's Conjecture holds for numerical semigroups with $(2+\frac{1}{q})\nu \geq m$.

Lemma 3.13 Let S be a numerical Semigroup with multiplicity m, embedding dimension ν and conductor $f+1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. If $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. Since $m - \nu = 6 \geq \frac{4(3)}{2} - 1$, by Lemma 3.10, it follows that $w_{m-1} \geq w_1 + w_4$ or $w_{m-1} \geq w_2 + w_3$.

i) If $w_{m-1} \geq w_1 + w_4$. By hypothesis $(2 + \frac{1}{q})\nu \geq m$ and Theorem 3.2 Wilf's Conjecture holds in this case.

ii) If $w_{m-1} \geq w_2 + w_3$. We may assume that $w_{m-1} < w_1 + w_4$, since otherwise we are back to case 1. Hence, $\mathrm{Ap}(S,m)\backslash\min(\mathrm{Ap}(S,m)) = \{w_1+w_1, w_1+w_2, w_1+w_3, w_2+w_2, w_2+w_3, w_3+w_3\}$.

By taking $\alpha = 3$ in (3.2), we get $\displaystyle\sum_{j=4}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu - m) \quad \geq (\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu - m)$.

Hence,

$$
\begin{aligned}
\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu - m) + \rho \quad &= (\lfloor\frac{w_1+\rho}{m}\rfloor - \lfloor\frac{w_0+\rho}{m}\rfloor)(\nu - m) \\
&+(\lfloor\frac{w_2+\rho}{m}\rfloor - \lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu - m) \\
&+(\lfloor\frac{w_3+\rho}{m}\rfloor - \lfloor\frac{w_2+\rho}{m}\rfloor)(3\nu - m) \\
&+\sum_{j=4}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu - m) + \rho \\
&\geq \lfloor\frac{w_1+\rho}{m}\rfloor(-\nu) + \lfloor\frac{w_2+\rho}{m}\rfloor(-\nu) \\
&+\lfloor\frac{w_3+\rho}{m}\rfloor(3\nu - m) \\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu - m) + \rho \\
&\geq \big(\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{-\nu}{2}) + \lfloor\frac{w_3+\rho}{m}\rfloor(\frac{-\nu}{2})\big) \\
&+\lfloor\frac{w_2+\rho}{m}\rfloor(-\nu) + \lfloor\frac{w_3+\rho}{m}\rfloor(3\nu - m) \\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu - m) + \rho \\
&= \lfloor\frac{w_2+\rho}{m}\rfloor(\frac{-3\nu}{2}) + \lfloor\frac{w_3+\rho}{m}\rfloor(\frac{5\nu}{2} - m) \\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu - m) + \rho \\
&= \lfloor\frac{w_2+\rho}{m}\rfloor(\frac{-3\nu}{2} + (4\nu - m) - (4\nu - m)) \\
&+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{5\nu}{2} - m) \\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu - m) + \rho \\
&= \lfloor\frac{w_2+\rho}{m}\rfloor(\frac{5\nu}{2} - m) + \lfloor\frac{w_3+\rho}{m}\rfloor(\frac{5\nu}{2} - m) \\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor - \lfloor\frac{w_2+\rho}{m}\rfloor - \lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu - m) \\
&+\rho
\end{aligned}
\tag{3.7}
$$

$$= \lfloor \frac{w_2+\rho}{m} \rfloor (\frac{3\nu}{2} - 6) + \lfloor \frac{w_3+\rho}{m} \rfloor (\frac{3\nu}{2} - 6)$$
$$+ (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor - \lfloor \frac{w_3+\rho}{m} \rfloor)(3\nu - 6)$$
$$+\rho.$$

We have $w_{m-1} \geq w_2 + w_3$, by Lemma 3.1, it follows that $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor \geq \lfloor \frac{w_2+\rho}{m} \rfloor + \lfloor \frac{w_3+\rho}{m} \rfloor - 1$.

- If $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor - \lfloor \frac{w_3+\rho}{m} \rfloor \geq 0$, using $\nu \geq 4$ in (3.7) ($\nu \leq 3$ is solved [3]), we get $\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \geq 0$.
- If $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor - \lfloor \frac{w_3+\rho}{m} \rfloor = -1$. Then,

$$\lfloor \frac{w_2+\rho}{m} \rfloor + \lfloor \frac{w_3+\rho}{m} \rfloor = q + 1. \tag{3.8}$$

We have $w_3 + w_3 \in \mathrm{Ap}(S,m) \backslash \min(\mathrm{Ap}(S,m))$, then $w_{m-1} \geq w_3 + w_3$. By Lemma 3.1, we have $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor \geq 2\lfloor \frac{w_3+\rho}{m} \rfloor - 1$. In particular,

$$\lfloor \frac{w_3+\rho}{m} \rfloor \leq \frac{q+1}{2}. \tag{3.9}$$

Since Wilf's Conjecture holds for $q \leq 3$ ([4], [6]), so we may assume that $q \geq 4$. Since $\lfloor \frac{w_2+\rho}{m} \rfloor \leq \lfloor \frac{w_3+\rho}{m} \rfloor$, by (3.8) and (3.9), it follows that $\lfloor \frac{w_2+\rho}{m} \rfloor = \lfloor \frac{w_3+\rho}{m} \rfloor = \frac{q+1}{2}$, in particular q is odd, so we have to assume that $q \geq 5$. Now using Now using (3.8), $q \geq 5$ and the hypothesis $(2 + \frac{1}{q})\nu \geq m = \nu + 6$ in (3.7), we get

$$\begin{aligned}
\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \quad &\geq (\lfloor \frac{w_2+\rho}{m} \rfloor + \lfloor \frac{w_3+\rho}{m} \rfloor)(\frac{3\nu}{2} - 6) \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_2+\rho}{m} \rfloor - \lfloor \frac{w_3+\rho}{m} \rfloor)(3\nu - 6) + \rho \\
&= (q+1)(\frac{3\nu}{2} - 6) - (3\nu - 6) + \rho \\
&= \nu(\frac{3q}{2} + \frac{3}{2} - 3) - 6q + \rho \\
&\geq \nu(\frac{3q}{2} - \frac{3}{2}) - q\nu - \nu + \rho \qquad (6q \leq q\nu + \nu) \\
&= \nu(\frac{q}{2} - \frac{5}{2}) + \rho \geq 0.
\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture in this case.

Thus, Wilf's Conjecture holds if $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$. $\square$

Next we will generalize a result for Sammartano ([9]) and show that Wilf's Conjecture holds for numerical semigroups satisfying $(2 + \frac{1}{q})\nu \geq m$, using Lemma 3.7, Corollary 3.9, Theorem 3.11, Lemma 3.13 and Corollary 3.5.

Theorem 3.14 Let S be a numerical semigroup with multiplicity m, embedding dimension ν and conductor $f+1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. If $(2+\frac{1}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof.

- If $m - \nu \leq 3$, then by Lemma 3.7 Wilf's Conjecture holds.
- If $m - \nu = 4$, then by Corollary 3.9 Wilf's Conjecture holds.
- If $m - \nu = 5$, then by Theorem 3.11 Wilf's Conjecture holds.
- If $m - \nu = 6$ and $(2+\frac{1}{q})\nu \geq m$, then by Lemma 3.13 Wilf's Conjecture holds.
- If $m - \nu > 6$ and $(2+\frac{1}{q})\nu \geq m$, then by Corollary 3.5 Wilf's Conjecture holds. $\square$

Example 3.15 Consider the following numerical semigroup $S =< 13, 15, 17, 19, 21, 27 >$. Note that $2\nu < m$. We have $(2+\frac{1}{q})\nu = (2+\frac{1}{4})6 \geq 13 = m$. Thus the conditions of Theorem 3.14 are valid, so S satisfies Wilf's Conjecture.

Corollary 3.16 Let S be a numerical semigroup with multiplicity m and conductor $f+1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. If $m \leq 8 + \frac{4}{q}$, then S satisfies Wilf's Conjecture.

Proof. If $\nu < 4$, then S satisfies Wilf's Conjecture ([3]). Hence, we can suppose that $\nu \geq 4$. Thus, $(2+\frac{1}{q})\nu \geq (2+\frac{1}{q})4 \geq m$. By using Theorem 3.14 S satisfies Wilf's Conjecture . $\square$

4 Numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$

In this section, we will show that if S is a numerical Semigroup such that $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$, then S satisfies Wilf's Conjecture.

Theorem 4.1 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. Let $w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S,m)$. Suppose that $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ for some $1 < \alpha < m-1$. If $(\frac{\alpha+3}{3})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. We may assume that $\rho \geq \frac{(3-q)\alpha m}{2\alpha+6}$. Indeed, if $0 \leq \rho < \frac{(3-q)\alpha m}{2\alpha+6}$, then $q < 3$ and Wilf's conjecture holds for this case ([6]). We are going to show that S satisfies Wilf's Conjecture by means of Proposition 2.3. We have,

$$\begin{aligned}
\sum_{j=1}^{\alpha}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu - m) &= \sum_{j=1}^{\alpha}\lfloor\frac{w_j+\rho}{m}\rfloor(j\nu-m) - \sum_{j=1}^{\alpha}\lfloor\frac{w_{j-1}+\rho}{m}\rfloor(j\nu-m)\\
&= \sum_{j=1}^{\alpha}\lfloor\frac{w_j+\rho}{m}\rfloor(j\nu-m) - \sum_{j=0}^{\alpha-1}\lfloor\frac{w_j+\rho}{m}\rfloor((j+1)\nu-m)\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_0+\rho}{m}\rfloor(\nu-m) - \sum_{j=1}^{\alpha-1}\lfloor\frac{w_j+\rho}{m}\rfloor\nu\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor\nu - \sum_{j=1}^{\alpha-2}\lfloor\frac{w_j+\rho}{m}\rfloor\nu\\
&\geq \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor\nu - \sum_{j=1}^{\alpha-2}\left(\frac{\lfloor\frac{w_\alpha+\rho}{m}\rfloor + \lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor}{2}\right)\nu\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor(\alpha\nu-m) - \lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor\nu - (\lfloor\frac{w_\alpha+\rho}{m}\rfloor + \lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor)\frac{(\alpha-2)\nu}{2}\\
&= \lfloor\frac{w_\alpha+\rho}{m}\rfloor((\frac{\alpha+2}{2})\nu-m) - \lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor(\frac{\alpha\nu}{2}).
\end{aligned}$$

By (3.2), we have $\sum_{j=\alpha+1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)\geq(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m)$.

Since $w_{m-1}\geq w_{\alpha-1}+w_\alpha$, by Lemma 3.1, it follows that $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor\geq\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor+\lfloor\frac{w_\alpha+\rho}{m}\rfloor-1$. Let $x=\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor$. Then, $\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor+\lfloor\frac{w_\alpha+\rho}{m}\rfloor=q-x$ and $x\geq-1$. Now using $\rho\geq\frac{(3-q)\alpha m}{2\alpha+6}$ and $(\frac{\alpha+3}{3})\nu\geq m$, we get

$$\begin{aligned}
\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho &\geq \lfloor\frac{w_\alpha+\rho}{m}\rfloor((\frac{\alpha+2}{2})\nu-m)-\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor(\frac{\alpha\nu}{2})\\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m)+\rho\\
&=\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor(\frac{-\alpha\nu}{2}+(\alpha+1)\nu-m-((\alpha+1)\nu-m))\\
&+\lfloor\frac{w_\alpha+\rho}{m}\rfloor((\frac{\alpha+2}{2})\nu-m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m)\\
&+\rho\\
&=(\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor+\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\frac{\alpha+2}{2})\nu-m)\\
&+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_{\alpha-1}+\rho}{m}\rfloor-\lfloor\frac{w_\alpha+\rho}{m}\rfloor)((\alpha+1)\nu-m)+\rho\\
&=(q-x)((\tfrac{\alpha+2}{2})\nu-m)+x((\alpha+1)\nu-m)+\rho\\
&=\nu(q+\tfrac{q\alpha}{2}+\tfrac{x\alpha}{2})-qm+\rho\\
&\geq\nu(q+\tfrac{q\alpha}{2}-\tfrac{\alpha}{2})-qm+\tfrac{(3-q)\alpha m}{2\alpha+6}\\
&=\nu(q+\tfrac{q\alpha}{2}-\tfrac{\alpha}{2})-m(\tfrac{q(2\alpha+6)+(q-3)\alpha}{2\alpha+6})\\
&=\nu(q+\tfrac{q\alpha}{2}-\tfrac{\alpha}{2})-m(\tfrac{3q}{\alpha+3}+\tfrac{3q\alpha}{2(\alpha+3)}-\tfrac{3\alpha}{2(\alpha+3)})\\
&=(q+\tfrac{q\alpha}{2}-\tfrac{\alpha}{2})(\tfrac{3}{\alpha+3})((\tfrac{\alpha+3}{3})\nu-m)\\
&\geq 0.
\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture. □

Example 4.2 Consider the following numerical semigroup $S=<22,23,25,27,29,31,33>$. Note that $3\nu<m$. We have $w_6=33$, $w_7=46$ and $w_{m-1}=87$ and i.e. $w_{m-1}\geq w_6+w_7$. Moreover, $(\frac{\alpha+3}{3})\nu=(\frac{7+3}{3})7\geq 22=m$, thus the conditions of Theorem 4.1 are valid. Hence, S satisfies Wilf's Conjecture. □

Th following Corollary 4.3 is an extension for Corollary 3.5 using Theorems 3.2 and 4.1.

Corollary 4.3 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. Suppose that $m-\nu\geq\frac{\alpha(\alpha-1)}{2}-1$ for some $7\leq\alpha\leq m-2$. If $(2+\frac{\alpha-3}{q})\nu\geq m$, then S satisfies Wilf's Conjecture.

Proof. Since $m-\nu\geq\frac{\alpha(\alpha-1)}{2}-1$, then by Lemma 3.10 we have $w_{m-1}\geq w_1+w_\alpha$ or $w_{m-1}\geq w_{\alpha-2}+w_{\alpha-1}$. Suppose that $w_{m-1}\geq w_1+w_\alpha$. Since $(2+\frac{\alpha-3}{q})\nu\geq m$, by applying Theorem 3.2, S satisfies wilf's Conjecture. Now suppose that $w_{m-1}\geq w_{\alpha-2}+w_{\alpha-1}$. We may assume that $q\geq 4$ ($q\leq 3$ is solved [6], [4]). Then for $\alpha\geq 7$

we have, $(\frac{\alpha-1+3}{3})\nu \geq (2+\frac{\alpha-3}{q})\nu$. Consequently, $(\frac{\alpha-1+3}{3})\nu \geq m$. Now by applying Theorem 4.1, S satisfies Wilf's Conjecture. $\square$

As a direct consequence of Theorem 4.1, we get the following Corollary.

Corollary 4.4 Let S be a numerical semigroup with multiplicity m and embedding dimension ν. Let $w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}$ be the elements of $\mathrm{Ap}(S,m)$. Suppose that $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ for some $1 < \alpha < m-1$. If $m \leq \frac{4(\alpha+3)}{3}$, then S satisfies Wilf's Conjecture.

Proof. If $\nu < 4$, then S satisfies Wilf's Conjecture ([3]). Hence, we can suppose that $\nu \geq 4$. Thus, $(\frac{\alpha+3}{3})(\nu) \geq \frac{4(\alpha+3)}{3} \geq m$. By applying Theorem 4.1 S satisfies Wilf's Conjecture. $\square$

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