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▶ To cite this version:

Aymen Braghtha. DARBOUX INTEGRABLE SYSTEM WITH A TRIPLE POINT AND PSEUDO-ABELIAN INTEGRALS. 2016. hal-01388054v2

HAL Id: hal-01388054 https://hal.science/hal-01388054v2

Preprint submitted on 31 Oct 2016 (v2), last revised 2 Feb 2018 (v3)

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DARBOUX INTEGRABLE SYSTEM WITH A TRIPLE POINT AND PSEUDO-ABELIAN INTEGRALS

AYMEN BRAGHTHA

ABSTRACT. In this paper we consider the degeneracies of the third type. More exact, the perturbations of the Darboux integrable foliation with a triple point, i.e. the case where three of the curves $\{P_i = 0\}$ meet at one point, are considered. Assuming that this is the only non-genericity, we prove that the number of zeros of the corresponding pseudo-abelian integrals is bounded uniformly for close Darboux integrable foliations. Let \mathcal{F} denote the foliation with triple point (assume it to be at the origin), and let $\mathcal{F}_{\lambda} = \{M_{\lambda} \frac{dH_{\lambda}}{H_{\lambda}} = 0\}, M_{\lambda}$ is a integrating factor, be the close foliation. The main problem is that \mathcal{F}_{λ} can have a small nest of cycles which shrinks to the origin as $\lambda \to 0$. A particular case of this situation, namely $H_{\lambda} = (x - \lambda)^{\epsilon} (y - x)^{\epsilon_+} (y + x)^{\epsilon_-} \Delta$ with Δ non-vanishing at the origin (and generic in appropriate sense).

Mathematics subject classification: 34C07, 34C08 Keywords: Abelian integrals, Limit cycles, Integrable systems

1. INTRODUCTION AND MAIN RESULT

This paper represent an amelioration of [Braghtha] in the sense to eleminate the technic condition on the perturbative one-form η . Precisely, we consider an unfolding ω_{λ} of the one-form ω_0 , where λ is a small parameter and ω_{λ} is a family of meromorphic one-forms

(1)
$$\omega_{\lambda} = M_{\lambda} \frac{dH_{\lambda}}{H_{\lambda}}, \quad H_{\lambda} = P_{\lambda}^{\epsilon} \prod_{i=1}^{k} P_{i}^{\epsilon_{i}}, \quad M_{\lambda} = P_{\lambda} \prod_{i=1}^{k} P_{i}$$

with $\epsilon, \epsilon_i > 0, P_0, P_\lambda, P_i \in \mathbb{R}[x, y].$

We assume that $P_0(0,0) = P_1(0,0) = P_2(0,0) = 0$ and $P_i(0,0) \neq 0, i = 3, \ldots, k$. Generically, the triple point unfolds into three saddles $p_0^{\lambda}, p_1^{\lambda}$ and p_2^{λ} correspond to the transversal intersections of level curves $P_1^{-1}(0)$ and $P_{\lambda}^{-1}(0), P_1^{-1}(0)$ and $P_2^{-1}(0)$, and $P_2^{-1}(0)$ and $P_{\lambda}^{-1}(0)$. Here also appears a center p_c^{λ} in the triangular region bounded by these levels curves.

Consider a polynomial perturbation $\omega_{\lambda,\kappa} = \omega_{\lambda} + \kappa \eta$, $\kappa > 0$ of the system $\omega_{\lambda} = M_{\lambda} \frac{\mathrm{d}H_{\lambda}}{H_{\lambda}}$, where

$$\eta = R\mathrm{d}x + S\mathrm{d}y,$$

and $R, S \in \mathbb{R}[x, y]$ are a polynomials of degree n. The foliation $\omega_{\lambda} = 0$ has a maximal nest of cycles $\gamma(\lambda, h) \subseteq \{H_{\lambda}(x, y) = h\}, h \in (0, n(\lambda))$ filling a connected component of $\mathbb{R}^2 \setminus \{P_{\lambda} \prod_{i=1}^k P_i = 0\}$, which we denote $D_{(\lambda,h)}$, whose boundary is a polycycle $\gamma(\lambda, 0)$.

To such perturbation one can associate the pseudo-Abelian integral

(2)
$$I(\lambda,h) = \int_{\gamma(\lambda,h)} \frac{\eta}{M_{\lambda}},$$

which is the principal part of the Poincaré displacement function

$$D(\kappa, \lambda, h) = \kappa h \int_{\gamma(\lambda, h)} \frac{\eta}{M_{\lambda}} + O(\kappa)$$

of the perturbation $\omega_{\lambda,\kappa}$ along $\gamma(\lambda,h)$.

Let us impose the following generic assumptions:

 $\mathbf{A}_1: \frac{\partial P_{\lambda}}{\partial \lambda}|_{(0,0,0)} \neq 0.$ $\mathbf{A}_2: P_1^{-1}(0), P_2^{-1}(0) \text{ and } P_0^{-1}(0) \text{ intersect transversally two by two at the origin which is the only triple point. The level curves <math>P_i^{-1}(0), i = 3, \cdots, k$ intersect transversally and two by two.

Theorem 1. Let $I(\lambda, h)$ be the family of pseudo-Abelian integrals as defined above. Under assumptions A_1 and A_2 , there exists a bound for the number of isolated zeros of $I(\lambda, h) =$ $\int_{\gamma(\lambda,h)} \frac{\eta}{M_{\lambda}}$. The bound depends only on $n_i = \deg P_i$, $n = \max(\deg R, \deg S)$ and is uniform in the coefficients of the polynomials P_{λ} , P_i , R, S, the exponents $\epsilon, \epsilon_i, i = 1, \ldots, k$ and the parameter λ.

1.1. Bobieński result. We recall also a similar result of Bobiénski [1] which he prove the existence of a local upper bound for the number of zeros of pseudo-abelian integrals. He consider a one-parameter unfolding of the singular codimension one case. The difference relies in the fact that in this work the Darboux first integrals is more general $(\epsilon_1 \neq \epsilon_2)$ and the proof of our main result is purely geometric: we use the blow-up in families. This approach gives directly uniform validity of our study of the pseudo-abelian integrals.

2. DARBOUX INTEGRABLE FOLIATION

Let us a stablish a local normal form near the triple point (0,0,0) for the unfolding of the degenerate polycyle H_0 .

Proposition 1. Under above assumptions A_1, A_2 . There exists a local analytic coordinate system (x, y, λ) at (0, 0, 0) such that H_{λ} takes the form

(3)
$$H_{\lambda} = (x - \lambda)^{\epsilon} (y - x)^{\epsilon_{+}} (y + x)^{\epsilon_{-}} \Delta, \quad \lambda > 0$$

where Δ is an analytic unity function $\Delta(0,0,0) \neq 0$.

Let $\mathcal{F}_1, \mathcal{F}_2$ are two foliations of dimension two in space of the total complex space \mathbb{C}^3 with coordinates (x, y, λ) such that

$$\mathcal{F}_1: \{H(x, y, \lambda) = P_{\lambda}^{\epsilon} \prod_{i=1}^k P_i^{\epsilon_i} = h\}, \quad \mathcal{F}_2: \{\lambda = \text{constant}\}.$$

Consider the Darboux foliation $\mathcal{F} := \{\mathcal{F}_1, \mathcal{F}_2\}$ of dimension one in \mathbb{C}^3 with coordinates (x, y, λ) which is given by the intersection of the leaves of \mathcal{F}_1 and \mathcal{F}_2 . This foliation has a non-elementary singular point at the origin (0,0,0). This foliation has a complicated singularity at the origin (0,0,0). To reduce this singularity, we perform a directional blowing-up in the family ω_{λ} . Note that blow-up in a family was introduced in [5], see also [6].

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2.1. **Directional Blow-up.** The blow-up of \mathbb{C}^3 at the origin is defined as the incidence three dimensional manifold $W = \{(p,q) \in \mathbb{CP}^2 \times \mathbb{C}^3 : q \in p\}$. The blow down $\sigma : W \to \mathbb{C}^3$ is just the restriction to W of the projection $\mathbb{CP}^2 \times \mathbb{C}^3$. The inverse map $\sigma^{-1} : \mathbb{C}^3 \to W$ is called blow-up and $\sigma^{-1}(0) = \mathbb{CP}^2$ is called exceptional divisor. The projective space \mathbb{CP}^2 is covered by three canonical charts: $W_1 = \{x \neq 0\}$ with coordinates $(v_1, w_1), W_2 = \{y \neq 0\}$ with coordinates (u_2, w_2) and $W_3 = \{\lambda \neq 0\}$ with coordinates (u_3, v_3) . W_1, W_2 and W_3 define canonical charts on W, with coordinates (u_1, v_1, w_1) ,

 (u_2, v_2, w_2) and (u_3, v_3, w_3) respectively. The blow-up σ^{-1} is written as:

(4)
$$\sigma_1^{-1} = \sigma^{-1}|_{W_1} : x = u_1 \qquad y = u_1 w_1 \qquad \lambda = w_1 u_1$$

(5)
$$\sigma_2^{-1} = \sigma^{-1}|_{W_2} : x = u_2 v_2 \qquad y = v_2 \qquad \lambda = w_2 v_2$$

(6)
$$\sigma_3^{-1} = \sigma^{-1}|_{W_3} : x = u_3 w_3 \qquad y = v_3 w_3 \qquad \lambda = w_3$$

Let $\sigma^{-1}\mathcal{F}$ be the lift of the foliation \mathcal{F} to the complement of the exceptional divisor \mathbb{CP}^2 . This foliation is regular outside of the preimage of the hypersurface $\{H_{\lambda} = 0, \lambda = 0\}$.

Prposition 2. Let $a = \epsilon + \epsilon_{-} + \epsilon_{+}$

- (1) The foliation $\sigma^{-1}\mathcal{F}$ extends in a unique way to a holomorphic singular foliation $\sigma^*\mathcal{F}$ on W which we call the blow-up of the original codimension two foliation \mathcal{F} by the map σ .
- (2) Let $\sigma_1^* \mathcal{F}$ be the restriction of the blown-up foliation $\sigma^* \mathcal{F}$ to the chart W_1 . The singularities of $\sigma_1^* \mathcal{F}$ are located at the points $p_+ = (0, 1, 0), p_- = (0, -1, 0), q_+ = (0, 1, 1)$ and $q_- = (0, -1, 1)$. All these singular points are linearisable saddles, with eigenvalues $\mu_+ = (\epsilon_+, -a, -\epsilon_-), \mu_- = (-\epsilon_-, a, \epsilon_-), \nu_+ = (0, -\epsilon, \epsilon_+)$ and $\nu_- = (0, -\epsilon, \epsilon_-)$ respectively.

Proof. We prove the second item of Proposition. Since $\sigma : W \to \mathbb{C}^3$ is a biholomorphism outside \mathbb{CP}^2 , all singularities of $\sigma_1^* \widetilde{\mathcal{F}}$ on $W_1 \setminus \{u_1 = 0\}$ correspond to singularities of \mathcal{F} .

Thus, it suffices to compute the singularities of $\sigma_1^* \mathcal{F}$ on the exceptional divisor $\{u_1 = 0\}$. On the exceptional divisor, the foliation is given by the levels of

$$G := \frac{(\pi \circ \sigma_1)^a}{H \circ \sigma_1} = w_1^a (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1},$$

where $\tilde{\Delta}$ is unit of the form $\tilde{\Delta} = c + u_1 f, f$ is a holomorphic function. Let us compute the eigenvalues at p_+, p_-, q_+ and q_- . Near the exceptional divisor $\{u_1 = 0\}$, the foliation $\sigma_1^* \mathcal{F}$ is given by

$$H \circ \sigma_1(u_1, v_1, w_1) = u_1^a (1 - w_1)^{\epsilon} (v_1 - 1)^{\epsilon_+} (v_1 + 1)^{\epsilon_-} \tilde{\Delta} = h,$$

$$\pi \circ \sigma_1(u_1, v_1, w_1) = u_1 w_1 = \lambda.$$

Near p_+ and after the respective changes of variable $v = (v_1 - 1)(v_1 + 1)^{\frac{c_-}{\epsilon_+}}\Delta$, the blown-up foliation $\sigma_1^* \mathcal{F}$ is given by the two first integrals $u_1^a v_+^{\epsilon_+} = h$ and $u_1 w_1 = s$. Then, near this point the vector field generating the foliation $\sigma_1^* \mathcal{F}$ is given by

$$X(u_1, v, w_1) = \mu_1^+ u_1 \frac{\partial}{\partial u_1} + \mu_2^+ v \frac{\partial}{\partial v} + \mu_3^+ w_1 \frac{\partial}{\partial w_1},$$

where the vector $\mu_{\pm} = (\mu_1^{\pm}, \mu_2^{\pm}, \mu_3^{\pm})$ satisfies the following equations

$$<(\mu_1^{\pm},\mu_2^{\pm},\mu_3^{\pm}),(a,\epsilon_{\pm},0)>=0, <(\mu_1^{\pm},\mu_2^{\pm},\mu_3^{\pm}),(1,0,1)>=0.$$

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Here <,> is the usual scalar product on \mathbb{C}^3 . By simple calculations, we obtain

$$X_{\pm}(u_1, v_{\pm}, w_1) = \pm \epsilon_{\pm} u_1 \frac{\partial}{\partial u_1} \mp a v_{\pm} \frac{\partial}{\partial v_{\pm}} \mp \epsilon_{\pm} w_1 \frac{\partial}{\partial w_1}$$

Similar computation shows that there are local coordinates near q_{\pm} in which the vector field generating the foliation is given by

$$X_{\pm}(u_1, v_{\pm}, w_{\pm}) = -\epsilon w_{\pm} \frac{\partial}{v_{\pm}} + \epsilon_{\pm} w_{\pm} \frac{\partial}{\partial w_{\pm}}.$$

3. Proof of the theorem

Let $t = \frac{\lambda^a}{h}$. The blown-up foliation $\sigma_1^* \mathcal{F}$ is given by the two first integrals

$$G = w_1^a (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1} = t,$$

$$F = u_1 w_1 = \lambda.$$

Consider the two-dimensional square $Q \subset \mathbb{CP}^2$ with vertices p_+, p_-, q_+ and q_- . All levels curves $\{G = t\}$ inside Q correspond to values of $t \in [0, +\infty]$. We consider the family of hyperbolic polycycles δ^t , i.e. at each intersection of two consecutive curve we have a saddle point,.

$$\delta^t = \left(\sigma_1^{-1}\left(\gamma(0,0) \setminus (0,0,0)\right) \cup \left(Q \cap \{G=t\}\right)\right)^{\mathbb{K}}, t \in [0,+\infty]$$

where $(...)^{\mathbb{R}}$ denotes the real part of a complex analytic set. All polycycle δ^t satisfies the generecity assumptions from [2].

Let $\delta(\lambda, t) = \sigma^{-1}(\gamma(\lambda, h)) \subset W$ be the pull-back of the cycle $\gamma(\lambda, h)$ by the blowing-up map. Let δ^t be the polycycle corresponding to the cycle $\delta(\lambda, t)$.

Let

$$J(\lambda,t) = \int_{\delta(\lambda,t)} \sigma_1^* \frac{\eta}{M_\lambda}$$

The integral $J(\lambda, t)$ can be analytically continued to the universal cover of $\mathbb{C}^2 \setminus \{\lambda t = 0\}$.

3.1. Variation operator. Given any multivalued function F defined in a neighborhood of the origin in \mathbb{C} i.e. a holomorphic function defined on the universal covering $\widetilde{\mathbb{C}^*}$ of \mathbb{C}^* . We define the rescaled monodromy as

$$\mathcal{M}on_{(t,\alpha)}F(t) = F(te^{i\pi\alpha}).$$

The variation is given as the difference between the counterclockwise and clockwise continuation

$$\begin{aligned} \mathcal{V}ar_{(t,\alpha}F(t) &= \mathcal{M}on_{(t,\alpha)}J(t) - \mathcal{M}on_{(t,-\alpha)}J(t) \\ &= F(te^{i\pi\alpha}) - F(te^{-i\pi\alpha}). \end{aligned}$$

Now, let G be a multivalued function in two variables λ and t defined in universal covering $\mathbb{C}^2 \setminus \{ st = 0 \}$ of $\mathbb{C}^2 \setminus \{ st = 0 \}$. We define the mixed variation as

$$\begin{aligned} &\mathcal{V}ar_{(\beta,\lambda)}\circ\mathcal{V}ar_{\alpha,t}G(\lambda,t)=\mathcal{V}ar_{(\beta,\lambda)}(G(\lambda,te^{i\pi\alpha})-G(\lambda,te^{-i\pi\alpha}))=\\ &G(\lambda e^{i\pi\beta},te^{i\pi\alpha})-G(\lambda e^{-i\pi\beta},te^{i\pi\alpha})-G(\lambda e^{i\pi\beta},te^{-i\pi\alpha})+G(\lambda e^{-i\pi\beta},te^{-i\pi\alpha})\end{aligned}$$

Lemma 1 The variations $\mathcal{V}ar_{(\lambda,\beta)}$ and $\mathcal{V}ar_{(t,\alpha)}$ commute

$$\mathcal{V}ar_{(\lambda,\beta)} \circ \mathcal{V}ar_{(t,\alpha)} = \mathcal{V}ar_{(t,\alpha)} \circ \mathcal{V}ar_{(\lambda,\beta)}.$$

Proof. The proof is a direct consequence of monodromy theorem.

3.2. The analytic proporties of the integral J near the polycycle δ^t . Using the partition of unity of the blown-up space we can decompose our cycle of integration $\delta(\lambda, t)$ in a relative cycles and we can check that any relative cycle can be chosen as a lift of a base path (for more details see [3]). Then, we have

Proposition 3. The integral $J(\lambda,t)$ satisfies the rescaled iterated variation equations

(7)
$$\mathcal{V}ar_{(t,\beta_1)} \circ \ldots \circ \mathcal{V}ar_{(t,\beta_k)}J(\lambda,t) = 0,$$

(8)
$$\mathcal{V}ar_{(\lambda,1)} \circ \mathcal{V}ar_{(\lambda,1)}J(\lambda,t) = 0,$$

where β_i are analytic functions in $\epsilon, \epsilon_+, \epsilon_-, \ldots, \epsilon_k$.

3.3. Increment of argument of J. The estimation of a local bound of the number of isolated zeros of the pseudo-abelian integral $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_{\lambda}}$ is analoguous to estimate a local λ -uniform bound of isolated zeros of the integral $J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_{\lambda}}$. As consequence of the equations (7) and (8) the integral J(s, t) has the following expansion

(9)
$$J(\lambda, t) = J_1(\lambda, t) + J_2(\lambda, t) \log \lambda,$$

where

(10)
$$J_2(\lambda, t) = \mathcal{V}ar_{(\lambda, 1)}J(\lambda, t) = \int_{\text{eight loop}} \sigma_1^* \frac{\eta}{M_\lambda},$$

(11)
$$\mathcal{V}ar_{(t,\beta_1)} \circ \ldots \circ \mathcal{V}ar_{(t,\beta_k)}J_i(\lambda,t) = 0, \quad i = 1, 2,$$

(12) $\mathcal{V}ar_{(\lambda,1)}J_i(\lambda,t) = 0, i = 1, 2.$

Theorem 2. Let $\epsilon > 0$ be sufficiently small. Then, for all $|\lambda| < \epsilon$ the number of zeros

$$\#\{t \in [0, +\infty] : J(\lambda, t) = 0\}$$

is uniformly bounded with respect to λ .

The integral $J(\lambda, t)$ has analytic prolongation to the complex argument t. This is a multivalued function with unique ramification point t = 0.

Let $\partial\Omega$ be the boundary of a domain complex Ω which consists of a big circular arc $C_{R_1} = \{|t| = R_1, |\arg t| \leq \alpha \pi\}$, a two segments $C^{\pm} = \{r_1 \leq |t| \leq R_1, |\arg t| = \pm \alpha \pi\}$ and the small circular arc $C_{r_1} = \{|t| = r_1, |\arg t| \leq \alpha \pi\}$.

To count the number of zeros of the function J(s,t), we estimate the increment of argument of the function $J(\lambda,t)$ we apply the argument principle which says that

$$#Z(J_{\Omega}) \leq \frac{1}{2\pi} \Delta \arg(J_{\partial\Omega}) = \frac{1}{2\pi} (\Delta \arg(J_{C_{R_1}}) + \Delta \arg(J_{C_{r_1}}) + \Delta \arg(J_{C_{t_1}}))$$

The increment of arguments $\Delta \arg(J_{C_{R_1}})$ and $\Delta \arg(J_{C^{\pm}})$ are locally uniformly bounded, for more details see [3]. The problem consist to estimate the increment of argument of the integral J(s,t) along the small circular arc C_{r_1} .

Let

$$\Lambda(\lambda; n_0, n_1, \cdots, n_k; n) = \{ M_\lambda \frac{dH_\lambda}{H_\lambda} + \kappa (Rdx + Rdy) : H_\lambda = P_\lambda \prod_{i=1}^{\kappa} P_i^{\epsilon_i}, \deg P_\lambda \le n_0, \deg P_i \le n_i, \deg(R, S) \le n \}.$$

be the parameters space. Consider the following functional space \mathcal{P}

$$\mathcal{P}(v, V; \alpha_1, \dots, \alpha_k; \lambda) := \{ \sum \sum c_{jln}(\lambda) t^{\alpha_j n} \log^n t, c_{jln} \in \mathbb{C}, v \le \alpha_j n \le V, 0 \le l \le k \}.$$

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3.3.1. Finite order approximation to $J(\lambda, t)$. Our goal is to obtain an asymptotic expansion for $J(\lambda, t)$. This will allow us to prove the existence of an upper bound for the increment of the argument $\Delta \arg(J_{C_{r_1}})$. The problem is proposed in the fact that the existence of the term $\log(\lambda) \to \infty$ as $\lambda \to 0$ in the expression of the function $J(\lambda, t)$. To resolve the problem we make a weighted quasi-homogeneous blowing-up. Let

$$c = (\log \lambda)^{-1}, \quad a = J_1(\lambda, t), \quad b = J_2(\lambda, t).$$

Then, we have

$$J(\lambda, t) = J_1(\lambda, t) + \log(\lambda)J_2(\lambda, t) = c^{-1}(ca + b) = c^{-1}\psi(a, b, c).$$

As $c^{-1} = \log \lambda \in \mathbb{R}$, we have

$$\operatorname{rg}(J(\lambda,t)) = \operatorname{arg}(c^{-1}\psi(a,b,c)) = \operatorname{arg}(\psi(a,b,c))$$

Proposition 4. There exist a some meromorphic function ϕ such that

$$\Delta \arg(\psi(a, b, c)) = \Delta \arg(\phi).$$

Proof. To estimate the increment of argument $\arg(q(a, b, c))$ of polynomial q we make a quasihomogeneous blowing-up of weight $(\frac{1}{2}, 1, \frac{1}{2})$ with the center $C_1 = \{a = b = c = 0\}$. The explicit formulae of the blowing-up π_1^{-1} in the affine charts $\tau_1 = \{a \neq 0\}, \tau_2 = \{b \neq 0\}$ and $\tau_3 = \{c \neq 0\}$ is written respectively as

$$\begin{aligned} \pi_1^{-1}|_{\tau_1} &= \pi_{11}^{-1} : a = \sqrt{a_1}, \quad b = b_1 a_1, \quad c = c_1 \sqrt{a_1}, \\ \pi_1^{-1}|_{\tau_2} &= \pi_{12}^{-1} : a = a_2 \sqrt{b_2}, \quad b = b_2, \quad c = c_2 \sqrt{b_2}, \\ \pi_1^{-1}|_{\tau_3} &= \pi_{13}^{-1} : a = a_3 \sqrt{c_3}, \quad b = b_3 c_3, \quad c = \sqrt{c_3}. \end{aligned}$$

The total transform $\pi_1^*q(a, b, c)$ of q(a, b, c) is given, in different charts, by

$$\pi_{11}^*(ca+b) = a_1(c_1+b_1) = a_1P_1(a_1,b_1,c_1),$$

$$\pi_{12}^*(ca+b) = b_2(a_2c_2+1) = b_2P_2(a_2,b_2,c_2),$$

$$\pi_{13}^*(ca+b) = c_3(a_3+b_3) = c_3P_3(a_3,b_3,c_3).$$

where $\{a_1 = 0\}, \{b_2 = 0\}$ and $\{c_3 = 0\}$ are local equations of the exceptional divisor and $\{P_1 = 0\}, \{P_2 = 0\}$ and $\{P_3 = 0\}$ local equations of strict transform of $\{ca + b = 0\}$.

We observe that the exceptional divisor $\{a_1 = 0\}$ (resp $\{c_3 = 0\}$) has not a normal crossing with the strict transform $\{P_1 = 0\}$ (resp $\{P_3 = 0\}$). Conversely in the chart τ_2 , the exceptional divisor $\{b_2 = 0\}$ has a normal crossing with the strict transform $\{P_2 = 0\}$. To resolve this problem we make a second blowing-up π_2^{-1} with center C_2 such that

- (1) in the chart τ_1 with coordinates (a_1, b_1, c_1) , the center C_2 is given by $C_2 = \{c_1 = b_1 = 0\}$,
- (2) in the chart τ_2 the blowing-up π_2 is a biholomorphism,
- (3) in the chart τ_3 with coordinates (a_3, b_3, c_3) , the center C_2 is given by $C_2 = \{a_3 = b_3 = 0\}$, and
- (1) in the chart τ_1 , we have $\pi_2^{-1}(C_2)$ is covred by two coordinates charts V_{b_1} and V_{c_1} with coordinate $(\tilde{a}_1, \tilde{b}_1, \tilde{c}_1)$ such that the blowing-up π_2^{-1} is given in the charts V_{b_1} and V_{c_1}
 - respectively, by $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1, c_1 = \tilde{b}_1 \tilde{c}_1$ and $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1 \tilde{c}_1, c_1 = \tilde{c}_1$, (2) in the chart τ_3 , we have $\pi_2^{-1}(C_2)$ is covred by two coordinates charts V_{a_3} and V_{b_3} with coordinates $(\tilde{a}_3, \tilde{b}_3, \tilde{c}_3)$ such that the blowing-up π_2^{-1} is given in V_{a_3} and V_{b_3} respectively, by $a_3 = \tilde{a}_3, b_3 = \tilde{a}_3 \tilde{b}_3, c_3 = \tilde{c}_3$ and $a_3 = \tilde{a}_3 \tilde{b}_3, b_3 = \tilde{a}_3$.

Then, we have

(1) in the chart V_{b_1} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{b}_1 = J_2(\lambda, t),$$

(2) in the chart V_{c_1} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{c}_1 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(3) in the chart V_{a_3} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(4) in the chart V_{b_3} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{b}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda}$$

(5) in the chart τ_2 , the exceptinal divisor $\{b_2 = 0\}$ has a normal crossings with the strict transform $P_2 = 0$ and the blowing-up π_2^{-1} is a biholomophism, so the function

$$\frac{J_1(\lambda, t)}{\log \lambda} + J_2(\lambda, t) = \frac{J(\lambda, t)}{\log \lambda}$$

is meromorphic function.

Then, we conclude that $\phi \in \{\frac{J(\lambda,t)}{\log \lambda}, \frac{J_1(\lambda,t)}{\log \lambda}, J_2(\lambda,t)\}.$

3.3.2. Proof of the theorem 2. Using lemma 4.8 of [2] for $\phi \in \{\frac{J_1(\lambda,t)}{\log \lambda}, J_2(\lambda,t)\}$, there exist a $\tilde{\phi} \in \mathcal{P}(...)$ such that $|\phi(\lambda,t) - \tilde{\phi}(\lambda,t)| \leq N$. For $\phi = \frac{J(\lambda,t)}{\log \lambda}$, we can conclude using Gabrielov's theorem [4].

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