



HAL
open science

DARBOUX INTEGRABLE SYSTEM WITH A TRIPLE POINT AND PSEUDO-ABELIAN INTEGRALS

Aymen Braghtha

► **To cite this version:**

Aymen Braghtha. DARBOUX INTEGRABLE SYSTEM WITH A TRIPLE POINT AND PSEUDO-ABELIAN INTEGRALS. 2016. hal-01388054v2

HAL Id: hal-01388054

<https://hal.science/hal-01388054v2>

Preprint submitted on 31 Oct 2016 (v2), last revised 2 Feb 2018 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DARBOUX INTEGRABLE SYSTEM WITH A TRIPLE POINT AND PSEUDO-ABELIAN INTEGRALS

AYMEN BRAGHTHA

ABSTRACT. In this paper we consider the degeneracies of the third type. More exact, the perturbations of the Darboux integrable foliation with a triple point, i.e. the case where three of the curves $\{P_i = 0\}$ meet at one point, are considered. Assuming that this is the only non-genericity, we prove that the number of zeros of the corresponding pseudo-abelian integrals is bounded uniformly for close Darboux integrable foliations. Let \mathcal{F} denote the foliation with triple point (assume it to be at the origin), and let $\mathcal{F}_\lambda = \{M_\lambda \frac{dH_\lambda}{H_\lambda} = 0\}$, M_λ is a integrating factor, be the close foliation. The main problem is that \mathcal{F}_λ can have a small nest of cycles which shrinks to the origin as $\lambda \rightarrow 0$. A particular case of this situation, namely $H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon+} (y + x)^{\epsilon-} \Delta$ with Δ non-vanishing at the origin (and generic in appropriate sense).

Mathematics subject classification: 34C07, 34C08

Keywords: Abelian integrals, Limit cycles, Integrable systems

1. INTRODUCTION AND MAIN RESULT

This paper represent an amelioration of [Braghtha] in the sense to eliminate the technic condition on the perturbative one-form η . Precisely, we consider an unfolding ω_λ of the one-form ω_0 , where λ is a small parameter and ω_λ is a family of meromorphic one-forms

$$(1) \quad \omega_\lambda = M_\lambda \frac{dH_\lambda}{H_\lambda}, \quad H_\lambda = P_\lambda^\epsilon \prod_{i=1}^k P_i^{\epsilon_i}, \quad M_\lambda = P_\lambda \prod_{i=1}^k P_i$$

with $\epsilon, \epsilon_i > 0, P_0, P_\lambda, P_i \in \mathbb{R}[x, y]$.

We assume that $P_0(0, 0) = P_1(0, 0) = P_2(0, 0) = 0$ and $P_i(0, 0) \neq 0, i = 3, \dots, k$. Generically, the triple point unfolds into three saddles p_0^λ, p_1^λ and p_2^λ correspond to the transversal intersections of level curves $P_1^{-1}(0)$ and $P_\lambda^{-1}(0)$, $P_1^{-1}(0)$ and $P_2^{-1}(0)$, and $P_2^{-1}(0)$ and $P_\lambda^{-1}(0)$. Here also appears a center p_c^λ in the triangular region bounded by these levels curves.

Consider a polynomial perturbation $\omega_{\lambda, \kappa} = \omega_\lambda + \kappa\eta$, $\kappa > 0$ of the system $\omega_\lambda = M_\lambda \frac{dH_\lambda}{H_\lambda}$, where

$$\eta = Rdx + Sdy,$$

and $R, S \in \mathbb{R}[x, y]$ are a polynomials of degree n . The foliation $\omega_\lambda = 0$ has a maximal nest of cycles $\gamma(\lambda, h) \subseteq \{H_\lambda(x, y) = h\}, h \in (0, n(\lambda))$ filling a connected component of $\mathbb{R}^2 \setminus \{P_\lambda \prod_{i=1}^k P_i = 0\}$, which we denote $D_{(\lambda, h)}$, whose boundary is a polycycle $\gamma(\lambda, 0)$.

To such perturbation one can associate the pseudo-Abelian integral

$$(2) \quad I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda},$$

which is the principal part of the Poincaré displacement function

$$D(\kappa, \lambda, h) = \kappa h \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda} + O(\kappa)$$

of the perturbation $\omega_{\lambda, \kappa}$ along $\gamma(\lambda, h)$.

Let us impose the following generic assumptions:

A₁ : $\frac{\partial P_\lambda}{\partial \lambda}|_{(0,0,0)} \neq 0$.

A₂ : $P_1^{-1}(0), P_2^{-1}(0)$ and $P_0^{-1}(0)$ intersect transversally two by two at the origin which is the only triple point. The level curves $P_i^{-1}(0), i = 3, \dots, k$ intersect transversally and two by two.

Theorem 1. *Let $I(\lambda, h)$ be the family of pseudo-Abelian integrals as defined above. Under assumptions **A₁** and **A₂**, there exists a bound for the number of isolated zeros of $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}$. The bound depends only on $n_i = \deg P_i, n = \max(\deg R, \deg S)$ and is uniform in the coefficients of the polynomials P_λ, P_i, R, S , the exponents $\epsilon, \epsilon_i, i = 1, \dots, k$ and the parameter λ .*

1.1. Bobiński result. We recall also a similar result of Bobiński [1] which he prove the existence of a local upper bound for the number of zeros of pseudo-abelian integrals. He consider a one-parameter unfolding of the singular codimension one case. The difference relies in the fact that in this work the Darboux first integrals is more general ($\epsilon_1 \neq \epsilon_2$) and the proof of our main result is purely geometric: we use the blow-up in families. This approach gives directly uniform validity of our study of the pseudo-abelian integrals.

2. DARBOUX INTEGRABLE FOLIATION

Let us a stablish a local normal form near the triple point $(0, 0, 0)$ for the unfolding of the degenerate polycycle H_0 .

Proposition 1. *Under above assumptions **A₁**, **A₂**. There exists a local analytic coordinate system (x, y, λ) at $(0, 0, 0)$ such that H_λ takes the form*

$$(3) \quad H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon_+} (y + x)^{\epsilon_-} \Delta, \quad \lambda > 0$$

where Δ is an analytic unity function $\Delta(0, 0, 0) \neq 0$.

Let $\mathcal{F}_1, \mathcal{F}_2$ are two foliations of dimension two in space of the total complex space \mathbb{C}^3 with coordinates (x, y, λ) such that

$$\mathcal{F}_1 : \{H(x, y, \lambda) = P_\lambda^\epsilon \prod_{i=1}^k P_i^{\epsilon_i} = h\}, \quad \mathcal{F}_2 : \{\lambda = \text{constant}\}.$$

Consider the Darboux foliation $\mathcal{F} := \{\mathcal{F}_1, \mathcal{F}_2\}$ of dimension one in \mathbb{C}^3 with coordinates (x, y, λ) which is given by the intersection of the leaves of \mathcal{F}_1 and \mathcal{F}_2 . This foliation has a non-elementary singular point at the origin $(0, 0, 0)$. This foliation has a complicated singularity at the origin $(0, 0, 0)$. To reduce this singularity, we perform a directional blowing-up in the family ω_λ . Note that blow-up in a family was introduced in [5], see also [6].

2.1. Directional Blow-up. The blow-up of \mathbb{C}^3 at the origin is defined as the incidence three dimensional manifold $W = \{(p, q) \in \mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3 : q \in p\}$. The blow down $\sigma : W \rightarrow \mathbb{C}^3$ is just the restriction to W of the projection $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3$. The inverse map $\sigma^{-1} : \mathbb{C}^3 \rightarrow W$ is called blow-up and $\sigma^{-1}(0) = \mathbb{C}\mathbb{P}^2$ is called exceptional divisor. The projective space $\mathbb{C}\mathbb{P}^2$ is covered by three canonical charts: $W_1 = \{x \neq 0\}$ with coordinates (v_1, w_1) , $W_2 = \{y \neq 0\}$ with coordinates (u_2, w_2) and $W_3 = \{\lambda \neq 0\}$ with coordinates (u_3, v_3) . W_1, W_2 and W_3 define canonical charts on W , with coordinates (u_1, v_1, w_1) , (u_2, v_2, w_2) and (u_3, v_3, w_3) respectively. The blow-up σ^{-1} is written as:

$$(4) \quad \sigma_1^{-1} = \sigma^{-1}|_{W_1} : x = u_1 \quad y = u_1 w_1 \quad \lambda = w_1 u_1$$

$$(5) \quad \sigma_2^{-1} = \sigma^{-1}|_{W_2} : x = u_2 v_2 \quad y = v_2 \quad \lambda = w_2 v_2$$

$$(6) \quad \sigma_3^{-1} = \sigma^{-1}|_{W_3} : x = u_3 w_3 \quad y = v_3 w_3 \quad \lambda = w_3$$

Let $\sigma^{-1}\mathcal{F}$ be the lift of the foliation \mathcal{F} to the complement of the exceptional divisor $\mathbb{C}\mathbb{P}^2$. This foliation is regular outside of the preimage of the hypersurface $\{H_\lambda = 0, \lambda = 0\}$.

Proposition 2. *Let $a = \epsilon + \epsilon_- + \epsilon_+$*

- (1) *The foliation $\sigma^{-1}\mathcal{F}$ extends in a unique way to a holomorphic singular foliation $\sigma^*\mathcal{F}$ on W which we call the blow-up of the original codimension two foliation \mathcal{F} by the map σ .*
- (2) *Let $\sigma_1^*\mathcal{F}$ be the restriction of the blown-up foliation $\sigma^*\mathcal{F}$ to the chart W_1 . The singularities of $\sigma_1^*\mathcal{F}$ are located at the points $p_+ = (0, 1, 0), p_- = (0, -1, 0), q_+ = (0, 1, 1)$ and $q_- = (0, -1, 1)$. All these singular points are linearisable saddles, with eigenvalues $\mu_+ = (\epsilon_+, -a, -\epsilon_-), \mu_- = (-\epsilon_-, a, \epsilon_-), \nu_+ = (0, -\epsilon, \epsilon_+)$ and $\nu_- = (0, -\epsilon, \epsilon_-)$ respectively.*

Proof. We prove the second item of Proposition. Since $\sigma : W \rightarrow \mathbb{C}^3$ is a biholomorphism outside $\mathbb{C}\mathbb{P}^2$, all singularities of $\sigma_1^*\mathcal{F}$ on $W_1 \setminus \{u_1 = 0\}$ correspond to singularities of \mathcal{F} .

Thus, it suffices to compute the singularities of $\sigma_1^*\mathcal{F}$ on the exceptional divisor $\{u_1 = 0\}$. On the exceptional divisor, the foliation is given by the levels of

$$G := \frac{(\pi \circ \sigma_1)^a}{H \circ \sigma_1} = w_1^a (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1},$$

where $\tilde{\Delta}$ is unit of the form $\tilde{\Delta} = c + u_1 f$, f is a holomorphic function. Let us compute the eigenvalues at p_+, p_-, q_+ and q_- . Near the exceptional divisor $\{u_1 = 0\}$, the foliation $\sigma_1^*\mathcal{F}$ is given by

$$\begin{aligned} H \circ \sigma_1(u_1, v_1, w_1) &= u_1^a (1 - w_1)^\epsilon (v_1 - 1)^{\epsilon_+} (v_1 + 1)^{\epsilon_-} \tilde{\Delta} = h, \\ \pi \circ \sigma_1(u_1, v_1, w_1) &= u_1 w_1 = \lambda. \end{aligned}$$

Near p_+ and after the respective changes of variable $v = (v_1 - 1)(v_1 + 1)^{\frac{\epsilon_-}{\epsilon_+}} \Delta$, the blown-up foliation $\sigma_1^*\mathcal{F}$ is given by the two first integrals $u_1^a v_+^{\epsilon_+} = h$ and $u_1 w_1 = s$. Then, near this point the vector field generating the foliation $\sigma_1^*\mathcal{F}$ is given by

$$X(u_1, v, w_1) = \mu_1^+ u_1 \frac{\partial}{\partial u_1} + \mu_2^+ v \frac{\partial}{\partial v} + \mu_3^+ w_1 \frac{\partial}{\partial w_1},$$

where the vector $\mu_\pm = (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm)$ satisfies the following equations

$$\langle (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm), (a, \epsilon_\pm, 0) \rangle = 0, \quad \langle (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm), (1, 0, 1) \rangle = 0.$$

Here \langle, \rangle is the usual scalar product on \mathbb{C}^3 . By simple calculations, we obtain

$$X_{\pm}(u_{\pm}, v_{\pm}, w_{\pm}) = \pm \epsilon_{\pm} u_{\pm} \frac{\partial}{\partial u_{\pm}} \mp a v_{\pm} \frac{\partial}{\partial v_{\pm}} \mp \epsilon_{\pm} w_{\pm} \frac{\partial}{\partial w_{\pm}}.$$

Similar computation shows that there are local coordinates near q_{\pm} in which the vector field generating the foliation is given by

$$X_{\pm}(u_{\pm}, v_{\pm}, w_{\pm}) = -\epsilon w_{\pm} \frac{\partial}{\partial v_{\pm}} + \epsilon_{\pm} w_{\pm} \frac{\partial}{\partial w_{\pm}}.$$

□

3. PROOF OF THE THEOREM

Let $t = \frac{\lambda^{\alpha}}{h}$. The blown-up foliation $\sigma_1^* \mathcal{F}$ is given by the two first integrals

$$\begin{aligned} G &= w_1^{\alpha} (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1} = t, \\ F &= u_1 w_1 = \lambda. \end{aligned}$$

Consider the two-dimensional square $Q \subset \mathbb{CP}^2$ with vertices p_+, p_-, q_+ and q_- . All levels curves $\{G = t\}$ inside Q correspond to values of $t \in [0, +\infty]$. We consider the family of hyperbolic polycycles δ^t , i.e. at each intersection of two consecutive curve we have a saddle point.,

$$\delta^t = (\sigma_1^{-1}(\gamma(0, 0) \setminus (0, 0, 0)) \cup (Q \cap \{G = t\}))^{\mathbb{R}}, t \in [0, +\infty].$$

where $(\dots)^{\mathbb{R}}$ denotes the real part of a complex analytic set. All polycycle δ^t satisfies the genericity assumptions from [2].

Let $\delta(\lambda, t) = \sigma^{-1}(\gamma(\lambda, h)) \subset W$ be the pull-back of the cycle $\gamma(\lambda, h)$ by the blowing-up map. Let δ^t be the polycycle corresponding to the cycle $\delta(\lambda, t)$.

Let

$$J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_{\lambda}}.$$

The integral $J(\lambda, t)$ can be analytically continued to the universal cover of $\mathbb{C}^2 \setminus \{\lambda t = 0\}$.

3.1. Variation operator. Given any multivalued function F defined in a neighborhood of the origin in \mathbb{C} i.e. a holomorphic function defined on the universal covering $\widetilde{\mathbb{C}^*}$ of \mathbb{C}^* . We define the rescaled monodromy as

$$\text{Mon}_{(t, \alpha)} F(t) = F(te^{i\pi\alpha}).$$

The variation is given as the difference between the counterclockwise and clockwise continuation

$$\begin{aligned} \text{Var}_{(t, \alpha)} F(t) &= \text{Mon}_{(t, \alpha)} J(t) - \text{Mon}_{(t, -\alpha)} J(t) \\ &= F(te^{i\pi\alpha}) - F(te^{-i\pi\alpha}). \end{aligned}$$

Now, let G be a multivalued function in two variables λ and t defined in universal covering $\widetilde{\mathbb{C}^2 \setminus \{st = 0\}}$ of $\mathbb{C}^2 \setminus \{st = 0\}$. We define the mixed variation as

$$\begin{aligned} \text{Var}_{(\beta, \lambda)} \circ \text{Var}_{\alpha, t} G(\lambda, t) &= \text{Var}_{(\beta, \lambda)} (G(\lambda, te^{i\pi\alpha}) - G(\lambda, te^{-i\pi\alpha})) = \\ &= G(\lambda e^{i\pi\beta}, te^{i\pi\alpha}) - G(\lambda e^{-i\pi\beta}, te^{i\pi\alpha}) - G(\lambda e^{i\pi\beta}, te^{-i\pi\alpha}) + G(\lambda e^{-i\pi\beta}, te^{-i\pi\alpha}). \end{aligned}$$

Lemma 1 *The variations $\text{Var}_{(\lambda, \beta)}$ and $\text{Var}_{(t, \alpha)}$ commute*

$$\text{Var}_{(\lambda, \beta)} \circ \text{Var}_{(t, \alpha)} = \text{Var}_{(t, \alpha)} \circ \text{Var}_{(\lambda, \beta)}.$$

Proof. The proof is a direct consequence of monodromy theorem. □

3.2. The analytic properties of the integral J near the polycycle δ^t . Using the partition of unity of the blown-up space we can decompose our cycle of integration $\delta(\lambda, t)$ in a relative cycles and we can check that any relative cycle can be chosen as a lift of a base path (for more details see [3]). Then, we have

Proposition 3 . *The integral $J(\lambda, t)$ satisfies the rescaled iterated variation equations*

$$(7) \quad \mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_k)} J(\lambda, t) = 0,$$

$$(8) \quad \mathcal{V}ar_{(\lambda, 1)} \circ \mathcal{V}ar_{(\lambda, 1)} J(\lambda, t) = 0,$$

where β_i are analytic functions in $\epsilon, \epsilon_+, \epsilon_-, \dots, \epsilon_k$.

3.3. Increment of argument of J . The estimation of a local bound of the number of isolated zeros of the pseudo-abelian integral $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}$ is analogous to estimate a local λ -uniform bound of isolated zeros of the integral $J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_\lambda}$. As consequence of the equations (7) and (8) the integral $J(s, t)$ has the following expansion

$$(9) \quad J(\lambda, t) = J_1(\lambda, t) + J_2(\lambda, t) \log \lambda,$$

where

$$(10) \quad J_2(\lambda, t) = \mathcal{V}ar_{(\lambda, 1)} J(\lambda, t) = \int_{\text{eight loop}} \sigma_1^* \frac{\eta}{M_\lambda},$$

$$(11) \quad \mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_k)} J_i(\lambda, t) = 0, \quad i = 1, 2,$$

$$(12) \quad \mathcal{V}ar_{(\lambda, 1)} J_i(\lambda, t) = 0, \quad i = 1, 2.$$

Theorem 2. *Let $\epsilon > 0$ be sufficiently small. Then, for all $|\lambda| < \epsilon$ the number of zeros*

$$\#\{t \in [0, +\infty] : J(\lambda, t) = 0\}$$

is uniformly bounded with respect to λ .

The integral $J(\lambda, t)$ has analytic prolongation to the complex argument t . This is a multi-valued function with unique ramification point $t = 0$.

Let $\partial\Omega$ be the boundary of a domain complex Ω which consists of a big circular arc $C_{R_1} = \{|t| = R_1, |\arg t| \leq \alpha\pi\}$, a two segments $C^\pm = \{r_1 \leq |t| \leq R_1, |\arg t| = \pm\alpha\pi\}$ and the small circular arc $C_{r_1} = \{|t| = r_1, |\arg t| \leq \alpha\pi\}$.

To count the number of zeros of the function $J(s, t)$, we estimate the increment of argument of the function $J(\lambda, t)$ we apply the argument principle which says that

$$\#Z(J_\Omega) \leq \frac{1}{2\pi} \Delta \arg(J_{\partial\Omega}) = \frac{1}{2\pi} (\Delta \arg(J_{C_{R_1}}) + \Delta \arg(J_{C_{r_1}}) + \Delta \arg(J_{C^\pm}))$$

The increment of arguments $\Delta \arg(J_{C_{R_1}})$ and $\Delta \arg(J_{C^\pm})$ are locally uniformly bounded, for more details see [3]. The problem consist to estimate the increment of argument of the integral $J(s, t)$ along the small circular arc C_{r_1} .

Let

$$\Lambda(\lambda; n_0, n_1, \dots, n_k; n) = \left\{ M_\lambda \frac{dH_\lambda}{H_\lambda} + \kappa(Rdx + Rdy) : H_\lambda = P_\lambda \prod_{i=1}^k P_i^{\epsilon_i}, \right. \\ \left. \deg P_\lambda \leq n_0, \deg P_i \leq n_i, \deg(R, S) \leq n \right\}.$$

be the parameters space. Consider the following functional space \mathcal{P}

$$\mathcal{P}(v, V; \alpha_1, \dots, \alpha_k; \lambda) := \left\{ \sum \sum c_{jln}(\lambda) t^{\alpha_j n} \log^n t, c_{jln} \in \mathbb{C}, v \leq \alpha_j n \leq V, 0 \leq l \leq k \right\}.$$

3.3.1. *Finite order approximation to $J(\lambda, t)$.* Our goal is to obtain an asymptotic expansion for $J(\lambda, t)$. This will allow us to prove the existence of an upper bound for the increment of the argument $\Delta \arg(J_{C_{r_1}})$. The problem is proposed in the fact that the existence of the term $\log(\lambda) \rightarrow \infty$ as $\lambda \rightarrow 0$ in the expression of the function $J(\lambda, t)$. To resolve the problem we make a weighted quasi-homogeneous blowing-up. Let

$$c = (\log \lambda)^{-1}, \quad a = J_1(\lambda, t), \quad b = J_2(\lambda, t).$$

Then, we have

$$J(\lambda, t) = J_1(\lambda, t) + \log(\lambda)J_2(\lambda, t) = c^{-1}(ca + b) = c^{-1}\psi(a, b, c).$$

As $c^{-1} = \log \lambda \in \mathbb{R}$, we have

$$\arg(J(\lambda, t)) = \arg(c^{-1}\psi(a, b, c)) = \arg(\psi(a, b, c)).$$

Proposition 4. *There exist a some meromorphic function ϕ such that*

$$\Delta \arg(\psi(a, b, c)) = \Delta \arg(\phi).$$

Proof. To estimate the increment of argument $\arg(q(a, b, c))$ of polynomial q we make a quasi-homogeneous blowing-up of weight $(\frac{1}{2}, 1, \frac{1}{2})$ with the center $C_1 = \{a = b = c = 0\}$. The explicit formulae of the blowing-up π_1^{-1} in the affine charts $\tau_1 = \{a \neq 0\}$, $\tau_2 = \{b \neq 0\}$ and $\tau_3 = \{c \neq 0\}$ is written respectively as

$$\begin{aligned} \pi_1^{-1}|_{\tau_1} &= \pi_{11}^{-1} : a = \sqrt{a_1}, \quad b = b_1 a_1, \quad c = c_1 \sqrt{a_1}, \\ \pi_1^{-1}|_{\tau_2} &= \pi_{12}^{-1} : a = a_2 \sqrt{b_2}, \quad b = b_2, \quad c = c_2 \sqrt{b_2}, \\ \pi_1^{-1}|_{\tau_3} &= \pi_{13}^{-1} : a = a_3 \sqrt{c_3}, \quad b = b_3 c_3, \quad c = \sqrt{c_3}. \end{aligned}$$

The total transform $\pi_1^* q(a, b, c)$ of $q(a, b, c)$ is given, in different charts, by

$$\begin{aligned} \pi_{11}^*(ca + b) &= a_1(c_1 + b_1) = a_1 P_1(a_1, b_1, c_1), \\ \pi_{12}^*(ca + b) &= b_2(a_2 c_2 + 1) = b_2 P_2(a_2, b_2, c_2), \\ \pi_{13}^*(ca + b) &= c_3(a_3 + b_3) = c_3 P_3(a_3, b_3, c_3). \end{aligned}$$

where $\{a_1 = 0\}$, $\{b_2 = 0\}$ and $\{c_3 = 0\}$ are local equations of the exceptional divisor and $\{P_1 = 0\}$, $\{P_2 = 0\}$ and $\{P_3 = 0\}$ local equations of strict transform of $\{ca + b = 0\}$.

We observe that the exceptional divisor $\{a_1 = 0\}$ (resp $\{c_3 = 0\}$) has not a normal crossing with the strict transform $\{P_1 = 0\}$ (resp $\{P_3 = 0\}$). Conversely in the chart τ_2 , the exceptional divisor $\{b_2 = 0\}$ has a normal crossing with the strict transform $\{P_2 = 0\}$. To resolve this problem we make a second blowing-up π_2^{-1} with center C_2 such that

- (1) in the chart τ_1 with coordinates (a_1, b_1, c_1) , the center C_2 is given by $C_2 = \{c_1 = b_1 = 0\}$,
- (2) in the chart τ_2 the blowing-up π_2 is a biholomorphism,
- (3) in the chart τ_3 with coordinates (a_3, b_3, c_3) , the center C_2 is given by $C_2 = \{a_3 = b_3 = 0\}$,

and

- (1) in the chart τ_1 , we have $\pi_2^{-1}(C_2)$ is covered by two coordinates charts V_{b_1} and V_{c_1} with coordinate $(\tilde{a}_1, \tilde{b}_1, \tilde{c}_1)$ such that the blowing-up π_2^{-1} is given in the charts V_{b_1} and V_{c_1} respectively, by $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1, c_1 = \tilde{b}_1 \tilde{c}_1$ and $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1 \tilde{c}_1, c_1 = \tilde{c}_1$,
- (2) in the chart τ_3 , we have $\pi_2^{-1}(C_2)$ is covered by two coordinates charts V_{a_3} and V_{b_3} with coordinates $(\tilde{a}_3, \tilde{b}_3, \tilde{c}_3)$ such that the blowing-up π_2^{-1} is given in V_{a_3} and V_{b_3} respectively, by $a_3 = \tilde{a}_3, b_3 = \tilde{a}_3 \tilde{b}_3, c_3 = \tilde{c}_3$ and $a_3 = \tilde{a}_3 \tilde{b}_3, b_3 = \tilde{a}_3, c_3 = \tilde{c}_3$.

Then, we have

(1) in the chart V_{b_1} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{b}_1 = J_2(\lambda, t),$$

(2) in the chart V_{c_1} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{c}_1 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(3) in the chart V_{a_3} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(4) in the chart V_{b_3} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{b}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(5) in the chart τ_2 , the exceptional divisor $\{b_2 = 0\}$ has a normal crossings with the strict transform $P_2 = 0$ and the blowing-up π_2^{-1} is a biholomorphism, so the function

$$\frac{J_1(\lambda, t)}{\log \lambda} + J_2(\lambda, t) = \frac{J(\lambda, t)}{\log \lambda}$$

is meromorphic function.

Then, we conclude that $\phi \in \left\{ \frac{J(\lambda, t)}{\log \lambda}, \frac{J_1(\lambda, t)}{\log \lambda}, J_2(\lambda, t) \right\}$. \square

3.3.2. *Proof of the theorem 2.* Using lemma 4.8 of [2] for $\phi \in \left\{ \frac{J_1(\lambda, t)}{\log \lambda}, J_2(\lambda, t) \right\}$, there exist a $\tilde{\phi} \in \mathcal{P}(\dots)$ such that $|\phi(\lambda, t) - \tilde{\phi}(\lambda, t)| \leq N$. For $\phi = \frac{J(\lambda, t)}{\log \lambda}$, we can conclude using Gabrielov's theorem [4].

REFERENCES

- [1] Bobieński, Marcin Pseudo-Abelian integrals along Darboux cycles *a codimension one case*. *J. Differential Equations* 246 (2009), no. 3, 1264-1273.
- [2] Bobieński, Marcin; Mardešić, Pavao *Pseudo-Abelian integrals along Darboux cycles*. *Proc. Lond. Math. Soc.* (3) 97 (2008), no. 3, 669-688.
- [3] Braghtha Aymen (2013) *Les zéros des intégrales pseudo-abeliennes: cas non générique*. *PhD thesis. Université de Bourgogne: France*.
- [4] Gabrièlov, A. M *Projections of semianalytic sets. (Russian) Funktsional. Anal. i Priložen.* 2 1968 no. 4, 18-30

**Burgundy University, Burgundy Institute of Mathematics,
U.M.R. 5584 du C.N.R.S., B.P. 47870, 21078 Dijon
Cedex - France.
E-mail address: aymenbraghtha@yahoo.fr**