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# DARBOUX INTEGRABLE SYSTEM WITH A TRIPLE POINT AND PSEUDO-ABELIAN INTEGRALS

AYMEN BRAGHTHA

ABSTRACT. In this paper we consider the degeneracies of the third type. More exact, the perturbations of the Darboux integrable foliation with a triple point, i.e. the case where three of the curves  $\{P_i = 0\}$  meet at one point, are considered. Assuming that this is the only non-genericity, we prove that the number of zeros of the corresponding pseudo-abelian integrals is bounded uniformly for close Darboux integrable foliations. Let  $\mathcal{F}$  denote the foliation with triple point (assume it to be at the origin), and let  $\mathcal{F}_\lambda = \{M_\lambda \frac{dH_\lambda}{H_\lambda} = 0\}$ ,  $M_\lambda$  is a integrating factor, be the close foliation. The main problem is that  $\mathcal{F}_\lambda$  can have a small nest of cycles which shrinks to the origin as  $\lambda \rightarrow 0$ . A particular case of this situation, namely  $H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon+} (y + x)^{\epsilon-} \Delta$  with  $\Delta$  non-vanishing at the origin (and generic in appropriate sense).

Mathematics subject classification: 34C07, 34C08

Keywords: Abelian integrals, Limit cycles, Integrable systems

## 1. INTRODUCTION AND MAIN RESULT

This paper represent an amelioration of [Braghtha] in the sense to eliminate the technic condition on the perturbative one-form  $\eta$ . Precisely, we consider an unfolding  $\omega_\lambda$  of the one-form  $\omega_0$ , where  $\lambda$  is a small parameter and  $\omega_\lambda$  is a family of meromorphic one-forms

$$(1) \quad \omega_\lambda = M_\lambda \frac{dH_\lambda}{H_\lambda}, \quad H_\lambda = P_\lambda^\epsilon \prod_{i=1}^k P_i^{\epsilon_i}, \quad M_\lambda = P_\lambda \prod_{i=1}^k P_i$$

with  $\epsilon, \epsilon_i > 0, P_0, P_\lambda, P_i \in \mathbb{R}[x, y]$ .

We assume that  $P_0(0, 0) = P_1(0, 0) = P_2(0, 0) = 0$  and  $P_i(0, 0) \neq 0, i = 3, \dots, k$ . Generically, the triple point unfolds into three saddles  $p_0^\lambda, p_1^\lambda$  and  $p_2^\lambda$  correspond to the transversal intersections of level curves  $P_1^{-1}(0)$  and  $P_\lambda^{-1}(0)$ ,  $P_1^{-1}(0)$  and  $P_2^{-1}(0)$ , and  $P_2^{-1}(0)$  and  $P_\lambda^{-1}(0)$ . Here also appears a center  $p_c^\lambda$  in the triangular region bounded by these levels curves.

Consider a polynomial perturbation  $\omega_{\lambda, \kappa} = \omega_\lambda + \kappa\eta$ ,  $\kappa > 0$  of the system  $\omega_\lambda = M_\lambda \frac{dH_\lambda}{H_\lambda}$ , where

$$\eta = Rdx + Sdy,$$

and  $R, S \in \mathbb{R}[x, y]$  are a polynomials of degree  $n$ . The foliation  $\omega_\lambda = 0$  has a maximal nest of cycles  $\gamma(\lambda, h) \subseteq \{H_\lambda(x, y) = h\}, h \in (0, n(\lambda))$  filling a connected component of  $\mathbb{R}^2 \setminus \{P_\lambda \prod_{i=1}^k P_i = 0\}$ , which we denote  $D_{(\lambda, h)}$ , whose boundary is a polycycle  $\gamma(\lambda, 0)$ .

To such perturbation one can associate the pseudo-Abelian integral

$$(2) \quad I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda},$$

which is the principal part of the Poincaré displacement function

$$D(\kappa, \lambda, h) = \kappa h \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda} + O(\kappa)$$

of the perturbation  $\omega_{\lambda, \kappa}$  along  $\gamma(\lambda, h)$ .

Let us impose the following generic assumptions:

**A<sub>1</sub>** :  $\frac{\partial P_\lambda}{\partial \lambda}|_{(0,0,0)} \neq 0$ .

**A<sub>2</sub>** :  $P_1^{-1}(0), P_2^{-1}(0)$  and  $P_0^{-1}(0)$  intersect transversally two by two at the origin which is the only triple point. The level curves  $P_i^{-1}(0), i = 3, \dots, k$  intersect transversally and two by two.

**Theorem 1.** *Let  $I(\lambda, h)$  be the family of pseudo-Abelian integrals as defined above. Under assumptions **A<sub>1</sub>** and **A<sub>2</sub>**, there exists a bound for the number of isolated zeros of  $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}$ . The bound depends only on  $n_i = \deg P_i, n = \max(\deg R, \deg S)$  and is uniform in the coefficients of the polynomials  $P_\lambda, P_i, R, S$ , the exponents  $\epsilon, \epsilon_i, i = 1, \dots, k$  and the parameter  $\lambda$ .*

**1.1. Bobiński result.** We recall also a similar result of Bobiński [1] which he prove the existence of a local upper bound for the number of zeros of pseudo-abelian integrals. He consider a one-parameter unfolding of the singular codimension one case. The difference relies in the fact that in this work the Darboux first integrals is more general ( $\epsilon_1 \neq \epsilon_2$ ) and the proof of our main result is purely geometric: we use the blow-up in families. This approach gives directly uniform validity of our study of the pseudo-abelian integrals.

## 2. DARBOUX INTEGRABLE FOLIATION

Let us a stablish a local normal form near the triple point  $(0, 0, 0)$  for the unfolding of the degenerate polycycle  $H_0$ .

**Proposition 1.** *Under above assumptions **A<sub>1</sub>**, **A<sub>2</sub>**. There exists a local analytic coordinate system  $(x, y, \lambda)$  at  $(0, 0, 0)$  such that  $H_\lambda$  takes the form*

$$(3) \quad H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon_+} (y + x)^{\epsilon_-} \Delta, \quad \lambda > 0$$

where  $\Delta$  is an analytic unity function  $\Delta(0, 0, 0) \neq 0$ .

Let  $\mathcal{F}_1, \mathcal{F}_2$  are two foliations of dimension two in space of the total complex space  $\mathbb{C}^3$  with coordinates  $(x, y, \lambda)$  such that

$$\mathcal{F}_1 : \{H(x, y, \lambda) = P_\lambda^\epsilon \prod_{i=1}^k P_i^{\epsilon_i} = h\}, \quad \mathcal{F}_2 : \{\lambda = \text{constant}\}.$$

Consider the Darboux foliation  $\mathcal{F} := \{\mathcal{F}_1, \mathcal{F}_2\}$  of dimension one in  $\mathbb{C}^3$  with coordinates  $(x, y, \lambda)$  which is given by the intersection of the leaves of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . This foliation has a non-elementary singular point at the origin  $(0, 0, 0)$ . This foliation has a complicated singularity at the origin  $(0, 0, 0)$ . To reduce this singularity, we perform a directional blowing-up in the family  $\omega_\lambda$ . Note that blow-up in a family was introduced in [5], see also [6].

**2.1. Directional Blow-up.** The blow-up of  $\mathbb{C}^3$  at the origin is defined as the incidence three dimensional manifold  $W = \{(p, q) \in \mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3 : q \in p\}$ . The blow down  $\sigma : W \rightarrow \mathbb{C}^3$  is just the restriction to  $W$  of the projection  $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}^3$ . The inverse map  $\sigma^{-1} : \mathbb{C}^3 \rightarrow W$  is called blow-up and  $\sigma^{-1}(0) = \mathbb{C}\mathbb{P}^2$  is called exceptional divisor. The projective space  $\mathbb{C}\mathbb{P}^2$  is covered by three canonical charts:  $W_1 = \{x \neq 0\}$  with coordinates  $(v_1, w_1)$ ,  $W_2 = \{y \neq 0\}$  with coordinates  $(u_2, w_2)$  and  $W_3 = \{\lambda \neq 0\}$  with coordinates  $(u_3, v_3)$ .  $W_1, W_2$  and  $W_3$  define canonical charts on  $W$ , with coordinates  $(u_1, v_1, w_1)$ ,

$(u_2, v_2, w_2)$  and  $(u_3, v_3, w_3)$  respectively. The blow-up  $\sigma^{-1}$  is written as:

$$(4) \quad \sigma_1^{-1} = \sigma^{-1}|_{W_1} : x = u_1 \quad y = u_1 w_1 \quad \lambda = w_1 u_1$$

$$(5) \quad \sigma_2^{-1} = \sigma^{-1}|_{W_2} : x = u_2 v_2 \quad y = v_2 \quad \lambda = w_2 v_2$$

$$(6) \quad \sigma_3^{-1} = \sigma^{-1}|_{W_3} : x = u_3 w_3 \quad y = v_3 w_3 \quad \lambda = w_3$$

Let  $\sigma^{-1}\mathcal{F}$  be the lift of the foliation  $\mathcal{F}$  to the complement of the exceptional divisor  $\mathbb{C}\mathbb{P}^2$ . This foliation is regular outside of the preimage of the hypersurface  $\{H_\lambda = 0, \lambda = 0\}$ .

**Proposition 2.** *Let  $a = \epsilon + \epsilon_- + \epsilon_+$*

- (1) *The foliation  $\sigma^{-1}\mathcal{F}$  extends in a unique way to a holomorphic singular foliation  $\sigma^*\mathcal{F}$  on  $W$  which we call the blow-up of the original codimension two foliation  $\mathcal{F}$  by the map  $\sigma$ .*
- (2) *Let  $\sigma_1^*\mathcal{F}$  be the restriction of the blown-up foliation  $\sigma^*\mathcal{F}$  to the chart  $W_1$ . The singularities of  $\sigma_1^*\mathcal{F}$  are located at the points  $p_+ = (0, 1, 0), p_- = (0, -1, 0), q_+ = (0, 1, 1)$  and  $q_- = (0, -1, 1)$ . All these singular points are linearisable saddles, with eigenvalues  $\mu_+ = (\epsilon_+, -a, -\epsilon_-), \mu_- = (-\epsilon_-, a, \epsilon_-), \nu_+ = (0, -\epsilon, \epsilon_+)$  and  $\nu_- = (0, -\epsilon, \epsilon_-)$  respectively.*

*Proof.* We prove the second item of Proposition. Since  $\sigma : W \rightarrow \mathbb{C}^3$  is a biholomorphism outside  $\mathbb{C}\mathbb{P}^2$ , all singularities of  $\sigma_1^*\mathcal{F}$  on  $W_1 \setminus \{u_1 = 0\}$  correspond to singularities of  $\mathcal{F}$ .

Thus, it suffices to compute the singularities of  $\sigma_1^*\mathcal{F}$  on the exceptional divisor  $\{u_1 = 0\}$ . On the exceptional divisor, the foliation is given by the levels of

$$G := \frac{(\pi \circ \sigma_1)^a}{H \circ \sigma_1} = w_1^a (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1},$$

where  $\tilde{\Delta}$  is unit of the form  $\tilde{\Delta} = c + u_1 f$ ,  $f$  is a holomorphic function. Let us compute the eigenvalues at  $p_+, p_-, q_+$  and  $q_-$ . Near the exceptional divisor  $\{u_1 = 0\}$ , the foliation  $\sigma_1^*\mathcal{F}$  is given by

$$\begin{aligned} H \circ \sigma_1(u_1, v_1, w_1) &= u_1^a (1 - w_1)^\epsilon (v_1 - 1)^{\epsilon_+} (v_1 + 1)^{\epsilon_-} \tilde{\Delta} = h, \\ \pi \circ \sigma_1(u_1, v_1, w_1) &= u_1 w_1 = \lambda. \end{aligned}$$

Near  $p_+$  and after the respective changes of variable  $v = (v_1 - 1)(v_1 + 1)^{\frac{\epsilon_-}{\epsilon_+}} \Delta$ , the blown-up foliation  $\sigma_1^*\mathcal{F}$  is given by the two first integrals  $u_1^a v_+^{\epsilon_+} = h$  and  $u_1 w_1 = s$ . Then, near this point the vector field generating the foliation  $\sigma_1^*\mathcal{F}$  is given by

$$X(u_1, v, w_1) = \mu_1^+ u_1 \frac{\partial}{\partial u_1} + \mu_2^+ v \frac{\partial}{\partial v} + \mu_3^+ w_1 \frac{\partial}{\partial w_1},$$

where the vector  $\mu_\pm = (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm)$  satisfies the following equations

$$\langle (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm), (a, \epsilon_\pm, 0) \rangle = 0, \quad \langle (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm), (1, 0, 1) \rangle = 0.$$

Here  $\langle, \rangle$  is the usual scalar product on  $\mathbb{C}^3$ . By simple calculations, we obtain

$$X_{\pm}(u_{\pm}, v_{\pm}, w_{\pm}) = \pm \epsilon_{\pm} u_{\pm} \frac{\partial}{\partial u_{\pm}} \mp a v_{\pm} \frac{\partial}{\partial v_{\pm}} \mp \epsilon_{\pm} w_{\pm} \frac{\partial}{\partial w_{\pm}}.$$

Similar computation shows that there are local coordinates near  $q_{\pm}$  in which the vector field generating the foliation is given by

$$X_{\pm}(u_{\pm}, v_{\pm}, w_{\pm}) = -\epsilon w_{\pm} \frac{\partial}{\partial v_{\pm}} + \epsilon_{\pm} w_{\pm} \frac{\partial}{\partial w_{\pm}}.$$

□

### 3. PROOF OF THE THEOREM

Let  $t = \frac{\lambda^{\alpha}}{h}$ . The blown-up foliation  $\sigma_1^* \mathcal{F}$  is given by the two first integrals

$$\begin{aligned} G &= w_1^{\alpha} (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1} = t, \\ F &= u_1 w_1 = \lambda. \end{aligned}$$

Consider the two-dimensional square  $Q \subset \mathbb{C}\mathbb{P}^2$  with vertices  $p_+, p_-, q_+$  and  $q_-$ . All levels curves  $\{G = t\}$  inside  $Q$  correspond to values of  $t \in [0, +\infty]$ . We consider the family of hyperbolic polycycles  $\delta^t$ , i.e. at each intersection of two consecutive curve we have a saddle point.,

$$\delta^t = (\sigma_1^{-1}(\gamma(0, 0) \setminus (0, 0, 0)) \cup (Q \cap \{G = t\}))^{\mathbb{R}}, t \in [0, +\infty].$$

where  $(\dots)^{\mathbb{R}}$  denotes the real part of a complex analytic set. All polycycle  $\delta^t$  satisfies the genericity assumptions from [2].

Let  $\delta(\lambda, t) = \sigma^{-1}(\gamma(\lambda, h)) \subset W$  be the pull-back of the cycle  $\gamma(\lambda, h)$  by the blowing-up map. Let  $\delta^t$  be the polycycle corresponding to the cycle  $\delta(\lambda, t)$ .

Let

$$J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_{\lambda}}.$$

The integral  $J(\lambda, t)$  can be analytically continued to the universal cover of  $\mathbb{C}^2 \setminus \{\lambda t = 0\}$ .

**3.1. Variation operator.** Given any multivalued function  $F$  defined in a neighborhood of the origin in  $\mathbb{C}$  i.e. a holomorphic function defined on the universal covering  $\widetilde{\mathbb{C}^*}$  of  $\mathbb{C}^*$ . We define the rescaled monodromy as

$$\text{Mon}_{(t, \alpha)} F(t) = F(te^{i\pi\alpha}).$$

The variation is given as the difference between the counterclockwise and clockwise continuation

$$\begin{aligned} \text{Var}_{(t, \alpha)} F(t) &= \text{Mon}_{(t, \alpha)} J(t) - \text{Mon}_{(t, -\alpha)} J(t) \\ &= F(te^{i\pi\alpha}) - F(te^{-i\pi\alpha}). \end{aligned}$$

Now, let  $G$  be a multivalued function in two variables  $\lambda$  and  $t$  defined in universal covering  $\widetilde{\mathbb{C}^2 \setminus \{st = 0\}}$  of  $\mathbb{C}^2 \setminus \{st = 0\}$ . We define the mixed variation as

$$\begin{aligned} \text{Var}_{(\beta, \lambda)} \circ \text{Var}_{\alpha, t} G(\lambda, t) &= \text{Var}_{(\beta, \lambda)} (G(\lambda, te^{i\pi\alpha}) - G(\lambda, te^{-i\pi\alpha})) = \\ &= G(\lambda e^{i\pi\beta}, te^{i\pi\alpha}) - G(\lambda e^{-i\pi\beta}, te^{i\pi\alpha}) - G(\lambda e^{i\pi\beta}, te^{-i\pi\alpha}) + G(\lambda e^{-i\pi\beta}, te^{-i\pi\alpha}). \end{aligned}$$

**Lemma 1** *The variations  $\text{Var}_{(\lambda, \beta)}$  and  $\text{Var}_{(t, \alpha)}$  commute*

$$\text{Var}_{(\lambda, \beta)} \circ \text{Var}_{(t, \alpha)} = \text{Var}_{(t, \alpha)} \circ \text{Var}_{(\lambda, \beta)}.$$

*Proof.* The proof is a direct consequence of monodromy theorem. □

**3.2. The analytic properties of the integral  $J$  near the polycycle  $\delta^t$ .** Using the partition of unity of the blown-up space we can decompose our cycle of integration  $\delta(\lambda, t)$  in a relative cycles and we can check that any relative cycle can be chosen as a lift of a base path (for more details see [3]). Then, we have

**Proposition 3 .** *The integral  $J(\lambda, t)$  satisfies the rescaled iterated variation equations*

$$(7) \quad \mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_k)} J(\lambda, t) = 0,$$

$$(8) \quad \mathcal{V}ar_{(\lambda, 1)} \circ \mathcal{V}ar_{(\lambda, 1)} J(\lambda, t) = 0,$$

where  $\beta_i$  are analytic functions in  $\epsilon, \epsilon_+, \epsilon_-, \dots, \epsilon_k$ .

**3.3. Increment of argument of  $J$ .** The estimation of a local bound of the number of isolated zeros of the pseudo-abelian integral  $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}$  is analogous to estimate a local  $\lambda$ -uniform bound of isolated zeros of the integral  $J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_\lambda}$ . As consequence of the equations (7) and (8) the integral  $J(s, t)$  has the following expansion

$$(9) \quad J(\lambda, t) = J_1(\lambda, t) + J_2(\lambda, t) \log \lambda,$$

where

$$(10) \quad J_2(\lambda, t) = \mathcal{V}ar_{(\lambda, 1)} J(\lambda, t) = \int_{\text{eight loop}} \sigma_1^* \frac{\eta}{M_\lambda},$$

$$(11) \quad \mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_k)} J_i(\lambda, t) = 0, \quad i = 1, 2,$$

$$(12) \quad \mathcal{V}ar_{(\lambda, 1)} J_i(\lambda, t) = 0, \quad i = 1, 2.$$

**Theorem 2.** *Let  $\epsilon > 0$  be sufficiently small. Then, for all  $|\lambda| < \epsilon$  the number of zeros*

$$\#\{t \in [0, +\infty] : J(\lambda, t) = 0\}$$

*is uniformly bounded with respect to  $\lambda$ .*

The integral  $J(\lambda, t)$  has analytic prolongation to the complex argument  $t$ . This is a multi-valued function with unique ramification point  $t = 0$ .

Let  $\partial\Omega$  be the boundary of a domain complex  $\Omega$  which consists of a big circular arc  $C_{R_1} = \{|t| = R_1, |\arg t| \leq \alpha\pi\}$ , a two segments  $C^\pm = \{r_1 \leq |t| \leq R_1, |\arg t| = \pm\alpha\pi\}$  and the small circular arc  $C_{r_1} = \{|t| = r_1, |\arg t| \leq \alpha\pi\}$ .

To count the number of zeros of the function  $J(s, t)$ , we estimate the increment of argument of the function  $J(\lambda, t)$  we apply the argument principle which says that

$$\#Z(J_\Omega) \leq \frac{1}{2\pi} \Delta \arg(J_{\partial\Omega}) = \frac{1}{2\pi} (\Delta \arg(J_{C_{R_1}}) + \Delta \arg(J_{C_{r_1}}) + \Delta \arg(J_{C^\pm}))$$

The increment of arguments  $\Delta \arg(J_{C_{R_1}})$  and  $\Delta \arg(J_{C^\pm})$  are locally uniformly bounded, for more details see [3]. The problem consist to estimate the increment of argument of the integral  $J(s, t)$  along the small circular arc  $C_{r_1}$ .

Let

$$\Lambda(\lambda; n_0, n_1, \dots, n_k; n) = \left\{ M_\lambda \frac{dH_\lambda}{H_\lambda} + \kappa(Rdx + Rdy) : H_\lambda = P_\lambda \prod_{i=1}^k P_i^{\epsilon_i}, \right. \\ \left. \deg P_\lambda \leq n_0, \deg P_i \leq n_i, \deg(R, S) \leq n \right\}.$$

be the parameters space. Consider the following functional space  $\mathcal{P}$

$$\mathcal{P}(v, V; \alpha_1, \dots, \alpha_k; \lambda) := \left\{ \sum \sum c_{jln}(\lambda) t^{\alpha_j n} \log^n t, c_{jln} \in \mathbb{C}, v \leq \alpha_j n \leq V, 0 \leq l \leq k \right\}.$$

3.3.1. *Finite order approximation to  $J(\lambda, t)$ .* Our goal is to obtain an asymptotic expansion for  $J(\lambda, t)$ . This will allow us to prove the existence of an upper bound for the increment of the argument  $\Delta \arg(J_{C_{r_1}})$ . The problem is proposed in the fact that the existence of the term  $\log(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow 0$  in the expression of the function  $J(\lambda, t)$ . To resolve the problem we make a weighted quasi-homogeneous blowing-up. Let

$$c = (\log \lambda)^{-1}, \quad a = J_1(\lambda, t), \quad b = J_2(\lambda, t).$$

Then, we have

$$J(\lambda, t) = J_1(\lambda, t) + \log(\lambda)J_2(\lambda, t) = c^{-1}(ca + b) = c^{-1}\psi(a, b, c).$$

As  $c^{-1} = \log \lambda \in \mathbb{R}$ , we have

$$\arg(J(\lambda, t)) = \arg(c^{-1}\psi(a, b, c)) = \arg(\psi(a, b, c)).$$

**Proposition 4.** *There exist a some meromorphic function  $\phi$  such that*

$$\Delta \arg(\psi(a, b, c)) = \Delta \arg(\phi).$$

*Proof.* To estimate the increment of argument  $\arg(q(a, b, c))$  of polynomial  $q$  we make a quasi-homogeneous blowing-up of weight  $(\frac{1}{2}, 1, \frac{1}{2})$  with the center  $C_1 = \{a = b = c = 0\}$ . The explicit formulae of the blowing-up  $\pi_1^{-1}$  in the affine charts  $\tau_1 = \{a \neq 0\}$ ,  $\tau_2 = \{b \neq 0\}$  and  $\tau_3 = \{c \neq 0\}$  is written respectively as

$$\begin{aligned} \pi_1^{-1}|_{\tau_1} &= \pi_{11}^{-1} : a = \sqrt{a_1}, \quad b = b_1 a_1, \quad c = c_1 \sqrt{a_1}, \\ \pi_1^{-1}|_{\tau_2} &= \pi_{12}^{-1} : a = a_2 \sqrt{b_2}, \quad b = b_2, \quad c = c_2 \sqrt{b_2}, \\ \pi_1^{-1}|_{\tau_3} &= \pi_{13}^{-1} : a = a_3 \sqrt{c_3}, \quad b = b_3 c_3, \quad c = \sqrt{c_3}. \end{aligned}$$

The total transform  $\pi_1^* q(a, b, c)$  of  $q(a, b, c)$  is given, in different charts, by

$$\begin{aligned} \pi_{11}^*(ca + b) &= a_1(c_1 + b_1) = a_1 P_1(a_1, b_1, c_1), \\ \pi_{12}^*(ca + b) &= b_2(a_2 c_2 + 1) = b_2 P_2(a_2, b_2, c_2), \\ \pi_{13}^*(ca + b) &= c_3(a_3 + b_3) = c_3 P_3(a_3, b_3, c_3). \end{aligned}$$

where  $\{a_1 = 0\}$ ,  $\{b_2 = 0\}$  and  $\{c_3 = 0\}$  are local equations of the exceptional divisor and  $\{P_1 = 0\}$ ,  $\{P_2 = 0\}$  and  $\{P_3 = 0\}$  local equations of strict transform of  $\{ca + b = 0\}$ .

We observe that the exceptional divisor  $\{a_1 = 0\}$  (resp  $\{c_3 = 0\}$ ) has not a normal crossing with the strict transform  $\{P_1 = 0\}$  (resp  $\{P_3 = 0\}$ ). Conversely in the chart  $\tau_2$ , the exceptional divisor  $\{b_2 = 0\}$  has a normal crossing with the strict transform  $\{P_2 = 0\}$ . To resolve this problem we make a second blowing-up  $\pi_2^{-1}$  with center  $C_2$  such that

- (1) in the chart  $\tau_1$  with coordinates  $(a_1, b_1, c_1)$ , the center  $C_2$  is given by  $C_2 = \{c_1 = b_1 = 0\}$ ,
- (2) in the chart  $\tau_2$  the blowing-up  $\pi_2$  is a biholomorphism,
- (3) in the chart  $\tau_3$  with coordinates  $(a_3, b_3, c_3)$ , the center  $C_2$  is given by  $C_2 = \{a_3 = b_3 = 0\}$ ,

and

- (1) in the chart  $\tau_1$ , we have  $\pi_2^{-1}(C_2)$  is covered by two coordinates charts  $V_{b_1}$  and  $V_{c_1}$  with coordinate  $(\tilde{a}_1, \tilde{b}_1, \tilde{c}_1)$  such that the blowing-up  $\pi_2^{-1}$  is given in the charts  $V_{b_1}$  and  $V_{c_1}$  respectively, by  $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1, c_1 = \tilde{b}_1 \tilde{c}_1$  and  $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1 \tilde{c}_1, c_1 = \tilde{c}_1$ ,
- (2) in the chart  $\tau_3$ , we have  $\pi_2^{-1}(C_2)$  is covered by two coordinates charts  $V_{a_3}$  and  $V_{b_3}$  with coordinates  $(\tilde{a}_3, \tilde{b}_3, \tilde{c}_3)$  such that the blowing-up  $\pi_2^{-1}$  is given in  $V_{a_3}$  and  $V_{b_3}$  respectively, by  $a_3 = \tilde{a}_3, b_3 = \tilde{a}_3 \tilde{b}_3, c_3 = \tilde{c}_3$  and  $a_3 = \tilde{a}_3 \tilde{b}_3, b_3 = \tilde{a}_3, c_3 = \tilde{c}_3$ .

Then, we have

(1) in the chart  $V_{b_1}$ ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{b}_1 = J_2(\lambda, t),$$

(2) in the chart  $V_{c_1}$ ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{c}_1 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(3) in the chart  $V_{a_3}$ ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(4) in the chart  $V_{b_3}$ ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{b}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

(5) in the chart  $\tau_2$ , the exceptional divisor  $\{b_2 = 0\}$  has a normal crossings with the strict transform  $P_2 = 0$  and the blowing-up  $\pi_2^{-1}$  is a biholomorphism, so the function

$$\frac{J_1(\lambda, t)}{\log \lambda} + J_2(\lambda, t) = \frac{J(\lambda, t)}{\log \lambda}$$

is meromorphic function.

Then, we conclude that  $\phi \in \left\{ \frac{J(\lambda, t)}{\log \lambda}, \frac{J_1(\lambda, t)}{\log \lambda}, J_2(\lambda, t) \right\}$ . □

3.3.2. *Proof of the theorem 2.* Using lemma 4.8 of [2] for  $\phi \in \left\{ \frac{J_1(\lambda, t)}{\log \lambda}, J_2(\lambda, t) \right\}$ , there exist a  $\tilde{\phi} \in \mathcal{P}(\dots)$  such that  $|\phi(\lambda, t) - \tilde{\phi}(\lambda, t)| \leq N$ . For  $\phi = \frac{J(\lambda, t)}{\log \lambda}$ , we can conclude using Gabrielov's theorem [4].

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