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# EXPECTED UTILITY MAXIMISATION FOR EXPONENTIAL LEVY MODELS WITH OPTION AND INFORMATION PROCESSES

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ABSTRACT. We consider expected utility maximisation problem for exponential Levy models and HARA utilities in presence of illiquid asset in portfolio . This illiquid asset is modelled by an option of European type on another risky asset which is correlated with the first one. Under some hypothesis on Levy processes, we give the expressions of information processes figured in maximum utility formula. As applications, we consider Black-Scholes models with correlated Brownian Motions, and also Black-Scholes models with jump part represented by Poisson process.

KEY WORDS AND PHRASES: utility maximisation, exponential Levy model, f-divergence minimal martingale measure, dual approach, entropy, Kullback-Leibler information, information processes.

MSC 2010 subject classifications: 60G07, 60G51, 91B24

## 1. INTRODUCTION

Levy processes was used in Mathematical Finance since a long time. These models contain a number of popular jump models including General Hyperbolic models and Variance-Gamma models. The use of such processes for modelling allows an excellent fit both for daily log return and intra-day data. The class of Levy processes is also flexible enough to allow the processes with finite and infinite variation, and also with finite and infinite activity. Levy models are not only excellent to fit

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the data but also mathematically tractable (see [8], [9] and references there).

Let  $X = (X_t)_{t \geq 0}$  be a  $d$ -dimensional Levy process,  $d \geq 1$ , with generating triplet  $(b, c, K)$  where  $b \in \mathbb{R}^d$  is drift parameter,  $c$  is  $d \times d$  matrix related with continuous martingale part of  $X$  and  $K$  is Levy measure which satisfies:

$$(1) \quad \int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) K(dx) < \infty.$$

As known, the law of such process is entirely determined by its one-dimensional distributions and the characteristic function of

$$X_t = {}^\top (X_t^1, X_t^2, \dots, X_t^d)$$

at  $\lambda \in \mathbb{R}^d$  is given by:

$$\phi_{X_t}(\lambda) = \exp\{t\psi(\lambda)\}$$

where the characteristic exponent of  $X$

$$\psi(\lambda) = \exp \left\{ i {}^\top \lambda b - \frac{1}{2} {}^\top \lambda c \lambda + \int_{\mathbb{R}^d} (e^{i {}^\top \lambda x} - 1 - i {}^\top \lambda l(x)) K(dx) \right\}$$

with the truncation function  $l$ . In general, truncation function  $l : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a bounded function with compact support such that  $l(x) = x$  in the neighbourhood of zero. The classical choice of  $l$  is  $l(x) = x \mathbf{1}_{\{\|x\| \leq 1\}}$  where  $\mathbf{1}_{\{\cdot\}}$  is indicator function and  $\|\cdot\|$  is euclidean norm in  $\mathbb{R}^d$  (for more information on Levy processes see [2],[19]).

For given Levy process  $X$ , the modelling of risky asset can be made by the exponential process  $S = (S_t)_{t \geq 0}$  with

$$S_t = {}^\top (\mathcal{E}(X^1)_t, \mathcal{E}(X^2)_t, \dots, \mathcal{E}(X^d)_t)$$

where  $\mathcal{E}(\cdot)$  is Doléan-Dade exponential and  $X^i, 1 \leq i \leq d$ , are the components of  $X$ . We recall that for each one-dimensional semi-martingale  $X^i$ ,

$$\mathcal{E}(X^i)_t = \exp \left\{ X_t^i - \frac{1}{2} \langle X^{i,c} \rangle_t \right\} \prod_{0 \leq s \leq t} \exp\{-\Delta X_s^i\} (1 + \Delta X_s^i).$$

Here  $\langle X^{i,c} \rangle$  is quadratic variation of continuous martingale part of  $X^i$  and  $\Delta X^i$  are jumps of  $X^i$  (see [16] for more details).

Utility maximisation of exponential Levy models with single Levy process was considered in a number of articles (see for instance [6], [7] and references there). However, the same questions in presence of illiquid assets in portfolio was not completely studied.

Utility maximisation in mentioned situation was considered in a number of books and papers, see for instance [3], [4], [5], [13], [17], [18]. Some explicit formulas for maximum of expected utility were obtained for Brownian motion models, where the incompleteness on the market comes from the non-traded asset (see [13], [17], [18]). The formulas for maximum of expected utility in complete markets was derived in [1]. But the case of correlated Levy models with jumps was not considered up to now.

To model dependent assets of Levy type, we denote by  $X^{(1)}$  and  $X^{(2)}$  two  $d$ -dimensional independent integrable Levy processes with generating triplets  $(b_1, c_1, K_1)$  and  $(b_2, c_2, K_2)$  respectively where  $K_1$  and  $K_2$  verify the condition of type (1). For two invertible matrix  $\rho_1$  and  $\rho_2$  with real valued components, we introduce the process  $X = (X_t)_{0 \leq t \leq T}$  as a linear map of  $X^{(1)}$  and  $X^{(2)}$ , namely

$$X_t = \rho_1 X_t^{(1)} + \rho_2 X_t^{(2)}$$

We suppose that our two risky assets can be modelled by the processes  $S^{(1)} = (S_t^{(1)})_{0 \leq t \leq T}$  and  $S^{(2)} = (S_t^{(2)})_{0 \leq t \leq T'}$  with  $T' > T$  and

$$S_t^{(1)} = \top(\mathcal{E}(X^1)_t, \mathcal{E}(X^2)_t, \dots, \mathcal{E}(X^d)_t)$$

and

$$S_t^{(2)} = \top(\mathcal{E}(X^{(2),1})_t, \mathcal{E}(X^{(2),2})_t, \dots, \mathcal{E}(X^{(2),d})_t).$$

To ensure that the components of  $S^{(1)}$  and  $S^{(2)}$  are positive, we assume, that for  $1 \leq i \leq d$ , the jumps of  $X^i$  and  $X^{(2),i}$  verify:  $\Delta X_t^i > -1$ ,  $\Delta X_t^{(2),i} > -1$ . Without loss of generality and up to now we assume that the interest rate  $r$  of non-risky asset is equal to zero.

In our setting, the investor, which has two assets  $S^{(1)}$  and  $S^{(2)}$ , can trade the first asset  $S^{(1)}$  with maturity time  $T$ , but the second asset with maturity time  $T' > T$ , can not be traded because of lack of liquidity or legal restrictions. At the same time the investor has an European option  $g(X_{T'}^{(2)})$  on risky asset  $S^{(2)}$ , where  $g$  is some non-negative real valued Borel function on  $\mathbb{R}^d$ . In such situation the investor, who would like to sell the option, would like also to evaluate the corresponding maximal expected utility of the portfolio with option.

Let us denote by  $\Pi(\mathbb{F})$  the set of self-financing admissible strategies with respect to the filtration  $\mathbb{F}$ , generated by  $X$ . Then, for utility function  $u$  and initial capital  $x_0$ , the optimal expected utility  $U_T(x_0, 0)$

related with the first asset  $S^{(1)}$  only, verify

$$U_T(x_0, 0) = \sup_{\phi \in \Pi(\mathbb{F})} \mathbf{E}[u(x_0 + \int_0^T \phi_s \cdot dS_s^{(1)})]$$

and if we add the mentioned option, then the optimal expected utility will be equal to

$$U_T(x_0, g) = \sup_{\phi \in \Pi(\mathbb{G})} \mathbf{E}[u(x_0 + \int_0^T \phi_s \cdot dS_s^{(1)} + g(X_{T'}^{(2)}))]$$

where  $\Pi(\mathbb{G})$  is a set of self-financing admissible strategies with respect to the enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$  with, for  $0 \leq t < T$ ,

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \otimes \sigma(X_{T'}^{(2)}) \quad \text{and} \quad \mathcal{G}_T = \mathcal{F}_T \otimes \sigma(X_{T'}^{(2)}).$$

This approach coincide with so called utility maximisation with distortion . In the case of Levy processes the distortion is  $\delta = X_{T'}^{(2)} - X_T^{(2)}$ , and the information contained in  $\mathcal{G}_T$  coincide with the one's of the  $\sigma$ -algebra  $\mathcal{F}_T^{(1)} \otimes \mathcal{F}_T^{(2)} \vee \sigma(X_{T'}^{(2)} - X_T^{(2)})$ , i.e. with progressive filtration at time  $T$  augmented by  $\sigma$ -algebra generated by distortion.

In this note we concentrate ourselves on non-complete market case modelled by correlated exponential Lévy models. We recall that very often the utility maximisation analysis is carried out for the hyperbolic absolute risk utilities (in short HARA utilities). HARA utilities can be defined trough the coefficient of absolute risk aversion:

$$R(x) = -\frac{u''(x)}{u'(x)}$$

In HARA utility case,

$$R(x) = \frac{1}{A + Bx}$$

with  $A$  and  $B$  positive constants. The solutions of such differential equation for  $u$  are known, and they are logarithmic, power and exponential utilities given below:

$$\begin{aligned} u(x) &= \ln x, \text{ with } x \in \mathbb{R}^{+,*}, \\ u(x) &= \frac{x^p}{p}, \text{ with } x \in \mathbb{R}^{+,*} \text{ and } p \in (-\infty, 0) \cup (0, 1), \\ u(x) &= 1 - e^{-\gamma x}, \text{ with } x \in \mathbb{R} \text{ and } \gamma > 0. \end{aligned}$$

The problem of utility maximisation with option, when  $X$  and  $X^{(2)}$  are semimartingales, was considered in [10]. The method applied was based on enlargement of filtration, combined with the conditioning with respect to the variable  $X_{T'}^{(2)}$  and, then, with dual approach. In dual

approach we replace the problem of maximisation of expected utility by finding so-called  $f$ -divergence minimal martingale measure where  $f$  is dual to  $u$  function, namely

$$(2) \quad f(x) = \sup_{y \in \mathbb{R}} (u(y) - xy),$$

As known, if  $u$  is logarithmic, then

$$f(x) = -\ln x - 1,$$

if  $u$  is power, then

$$f(x) = \frac{1-p}{p} x^{\frac{p}{p-1}},$$

and if  $u$  is exponential,

$$f(x) = 1 - \frac{x}{\gamma}(1 + \ln \gamma) + \frac{1}{\gamma} x \ln x.$$

In Section 2 we give, for convenience of the reader, some results about utility maximisation with option for semimartingale models. The main results of this section are the formulas for maximum of utility. These formulas contain the corresponding information quantities, like Kulback-Leibler information and Hellinger type integrals. In turn, these information quantities can be recovered from respective information processes.

In Section 3, we consider the exponential Levy models. More precisely, we verify the assumptions of Section 2 and we give the expressions for Girsanov parameters of  $f$ -divergence minimal conditional martingale measures. These expressions permit us to write the corresponding information processes, and, then use the results of Section 2.

In Section 4 we give the expressions of the information quantities for Black-Scholes models with correlated Brownian Motions.

In Section 5 we consider Black-Scholes models with correlated Brownian Motions and jumps represented by Poisson process, in order to derive the mentioned information quantities.

## 2. SOME KNOWN RESULTS ABOUT UTILITY MAXIMISATION WITH OPTION FOR EXPONENTIAL SEMIMARTINGALE MODELS.

**2.1. Modelling and assumptions.** We suppose that the process  $X = (X_t)_{0 \leq t \leq T}$  is given on canonical probability space  $(\Omega, \mathcal{F}, P)$  with filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  satisfying usual properties. This process represents

stochastic logarithms of a  $d$ -dimensional liquid asset  $S^{(1)} = (S_t^{(1)})_{0 \leq t \leq T}$  with

$$S_t^{(1)} = \top(\mathcal{E}(X^1)_t, \mathcal{E}(X^2)_t, \dots, \mathcal{E}(X^d)_t).$$

At the same time, we have also a  $d$ -dimensional semimartingale  $X^{(2)}$ , which represents stochastic logarithms of another risky, but illiquid asset. This illiquid asset, in turn, is represented in portfolio by European type option  $g(X_{T'}^{(2)})$  where  $g$  is a positive measurable function on  $\mathbb{R}^d$  and  $T' > T$ .

To perform utility maximisation, we introduce a product space

$$(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \sigma(X_{T'}^{(2)}), P \times \alpha)$$

with  $P$  "historical" law of  $X$  and  $\alpha$  "historical" law of  $X_{T'}^{(2)}$ , endowed with enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$  with

$$(3) \quad \mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \otimes \sigma(X_{T'}^{(2)}) \quad \text{and} \quad \mathcal{G}_T = \mathcal{F}_T \otimes \sigma(X_{T'}^{(2)}).$$

We remark that  $X$  remains a semimartingale on product space equipped with filtration  $\mathbb{G}$  since  $X$  and  $X^{(2)}$  are independent under the probability measure  $P \times \alpha$ .

Now, we denote by  $\mathbb{P}$  the law of the couple  $(X, X_{T'}^{(2)})$  and by  $P^v$  the regular conditional law of  $X$  given  $\{X_{T'}^{(2)} = v\}$ , i.e. for all  $A \in \mathcal{F}$  and  $v \in \mathbb{R}^d$

$$P^v(A) = \mathbb{P}(A | X_{T'}^{(2)} = v).$$

To preserve semimartingale property of  $X$  under conditioning, we suppose that the following assumption holds.

**Assumption 1.** *For each  $v \in \mathbb{R}^d$  the probability  $P^v$  is locally absolutely continuous with respect to  $P$ , i.e*

$$P^v \stackrel{loc}{\ll} P.$$

Under the Assumption 1 and according to [15] and [16], a semimartingale  $X$  will remain a semimartingale under each measure  $P^v$ ,  $v \in \mathbb{R}^d$ . Of course, the characteristics of a semimartingale  $X$  under  $P^v$  will be changed as it was proved in [16] (cf. Theorem 3.24, p. 159).

For  $0 \leq t \leq T$ , we denote by  $P_t^v$  and  $P_t$  the restrictions of the measures  $P^v$  and  $P$  on the  $\sigma$ -algebra  $\mathcal{F}_t$ . To avoid measurability problems in semimartingale calculus depending on a parameter  $v$  (cf. [20]), we need the optional versions of likelihood processes  $(\frac{dP_t^v}{dP_t})_{0 \leq t \leq T}$  with respect to

the filtration  $\mathbb{G}$ . For that, we introduce conditional distribution of  $X_{T'}^{(2)}$  given  $\mathcal{F}_t$ , i.e.

$$\alpha^t(\omega, dv) = \mathbb{P}(X_{T'}^{(2)} \in dv | \mathcal{F}_t)(\omega).$$

We make the following assumption

**Assumption 2.** *The conditional distributions of random variable  $X_{T'}^{(2)}$  given  $\mathcal{F}_t$  are absolutely continuous with respect to its law, i.e.*

$$\alpha^t \ll \alpha, \quad \forall t \in ]0, T].$$

According to Jacod's lemma (see[14]), under the Assumption 2, there exists an optional version of density process  $(\frac{d\alpha^t}{d\alpha})_{0 \leq t \leq T}$ .

**Remark 1.** *It should be noticed that the Assumption 2 can be replaced by the assumption that  $\frac{dP_T^v}{dP_T}$  considered as a map of  $(\omega, v)$  is  $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Then we can construct an optional version of density process using the results of [20].*

As it was mentioned, the next step consists to solve conditional utility maximisation problem using dual approach (see, for example, [11]). Let us denote by  $f$  dual conjugate of utility function  $u$ . Let  $\mathcal{M}_T^v$  be the set of equivalent martingale measures on probability space  $(\Omega, \mathcal{F}_T, P_T^v)$  for exponential semimartingale  $S^{(1)}$  and let

$$\mathcal{K}_T^v = \left\{ Q_T^v \in \mathcal{M}_T^v : \mathbf{E}_{P^v} \left| f \left( \frac{dQ_T^v}{dP_T^v} \right) \right| < \infty \right\}.$$

We recall that  $Q_T^{v,*}$  is an equivalent f-divergence minimal martingale measure if

$$\mathbf{E}_{P^v} \left[ f \left( \frac{dQ_T^{v,*}}{dP_T^v} \right) \right] = \min_{Q_T^v \in \mathcal{K}_T^v} \mathbf{E}_{P^v} \left[ f \left( \frac{dQ_T^v}{dP_T^v} \right) \right].$$

To use dual approach we introduce the following assumption.

**Assumption 3.** *For each  $v \in \mathbb{R}^d$ , there exists f-divergence minimal equivalent martingale measure  $Q_T^{v,*}$ , which belongs to the set  $\mathcal{K}_T^v$  and such that  $z_T^*(v) = \frac{dQ_T^{v,*}}{dP_T^v}$  considered as a map of  $(\omega, v)$ , is  $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and such that  $\mathbf{E}_{P^v} f(z_T^*(v))$  is integrable in  $v$  with respect to  $\alpha$ .*

## 2.2. Existence of f-divergence minimal martingale measure.

We recall the results about the existence of global f-divergence minimal martingale measure. For that we denote by  $\mathbb{P}_T$  the restrictions of the measure  $\mathbb{P}$  to the  $\sigma$ -algebra  $\mathcal{G}_T$  and let  $\mathcal{M}_T$  be the set of equivalent martingale measures for semimartingale  $(S_t^{(1)})_{0 \leq t \leq T}$  considered as



an application on probability space  $(\Omega_T^{(1)} \times \mathbb{R}^d, \mathcal{F}^{(1)} \otimes \mathcal{B}(\mathbb{R}^d), \mathbb{P}_T)$  with the filtration  $\mathbb{G}$ . Let

$$\mathcal{K}_T = \left\{ \mathbb{Q}_T \in \mathcal{M}_T \mid \mathbf{E}_{\mathbb{P}} \left| f \left( \frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \right) \right| < \infty \right\}.$$

We remark that  $\mathcal{K}_T \neq \emptyset$ . In fact, as Radon-Nikodym density of a measure  $\mathbb{Q}_T$  with respect to  $\mathbb{P}_T$ , we can take  $z_T^*(v)$  from Assumption 3 with replacement of  $v$  by  $X_{T'}^{(2)}$ . We recall that  $\mathbb{Q}_T^*$  is  $f$ -divergence minimal measure if

$$\inf_{\mathbb{Q}_T \in \mathcal{K}_T} \mathbf{E}_{\mathbb{P}} f \left( \frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \right) = \mathbf{E}_{\mathbb{P}} f \left( \frac{d\mathbb{Q}_T^*}{d\mathbb{P}_T} \right).$$

**Theorem 1.** (cf. [10]) *Under the Assumptions 1,2,3 there exists  $\mathbb{Q}_T^* \in \mathcal{K}_T$  which is  $f$ -divergence minimal martingale measure and*

$$(4) \quad U_T(x_0, g) = \int_{\mathbb{R}^d} \mathbf{E}_{P^v} [u(-f'(\lambda_g(v) z_T^*(v)))] d\alpha(v),$$

where  $\lambda_g(v)$  is a solution of the equation

$$(5) \quad -\mathbf{E}_{P^v} [f'(\lambda_g(v) z_T^*(v))] = x_0 + g(v).$$

**2.3. Conditional information quantities and maximal expected utility.** Let us assume the existence of  $f$ -divergence minimal martingale measure  $Q_T^{v,*} \in \mathcal{K}_T^v$  and denote

$$z_T^*(v) = \frac{Q_T^{v,*}}{P_T^v}, \quad p_T^v = \frac{dP_T^v}{dP_T}.$$

Now, we introduce three important quantities related with  $P_T^v$  and  $Q_T^{v,*}$  namely the entropy of  $P_T^v$  with respect to  $Q_T^{v,*}$ ,

$$\mathbf{I}(P_T^v | Q_T^{v,*}) = -\mathbf{E}_{P^v} [\ln z_T^*(v)] = -\mathbf{E}_P [p_T^v \ln z_T^*(v)],$$

then, the entropy of  $Q_T^{v,*}$  with respect to  $P_T^v$  or Kulback-Leibler information

$$\mathbf{I}(Q_T^{v,*} | P_T^v) = \mathbf{E}_{P^v} [z_T^*(v) \ln z_T^*(v)] = \mathbf{E}_P [p_T^v z_T^*(v) \ln z_T^*(v)],$$

and also Hellinger type integrals

$$\mathbf{H}_T^{(q),*}(v) = \mathbf{E}_{P^v} [(z_T^*(v))^q] = \mathbf{E}_P [p_T^v (z_T^*(v))^q],$$

where  $q = \frac{p}{p-1}$  with  $p < 1$ .

In the following theorem we give the expressions of the maximal expected utility involving relative entropies and Hellinger-type integrals.

**Theorem 2.** (cf. [10]) *Under the Assumptions 1, 2, 3, there exist a  $\mathcal{B}(\mathbb{R}^d)$ -measurable versions of the information quantities. Moreover, we have the following expressions for  $U_T(x_0, g)$  :*

(i) *If  $u(x) = \ln x$  then*

$$(6) \quad U_T(x_0, g) = \int_{\mathbb{R}^d} [\ln(x_0 + g(v)) + \mathbf{I}(P_T^v | Q_T^{v,*})] d\alpha(v).$$

(ii) *If  $u(x) = \frac{x^p}{p}$  with  $p < 1, p \neq 0$  then*

$$(7) \quad U_T(x_0, g) = \frac{1}{p} \int_{\mathbb{R}^d} (x_0 + g(v))^p \left( \mathbf{H}_T^{(q),*}(v) \right)^{1-p} d\alpha(v).$$

(iii) *If  $u(x) = 1 - e^{-\gamma x}$  with  $\gamma > 0$  then*

$$(8) \quad U_T(x_0, g) = 1 - \int_{\mathbb{R}^d} \exp\{-[\gamma(x_0 + g(v)) + \mathbf{I}(Q_T^{v,*} | P_T^v)]\} d\alpha(v).$$

#### 2.4. Conditional information processes and conditional information quantities.

In this subsection we recall that the conditional information quantities can be recovered from conditional information processes. To simplify the expression for information processes we suppose during this subsection that the process  $X$  is quasi-left continuous. We recall that  $(P, \mathbb{F})$ -semimartingale  $X$  is a quasi-left continuous, if for any predictable stopping time  $\tau$ , the jump  $\Delta X_\tau = 0$  ( $P$ -a.s.) on the set  $\{\tau < \infty\}$ . We remark that since  $P^v \stackrel{loc}{\ll} P$ ,  $(P^v, \mathbb{F})$  semi-martingale  $X$  will be also quasi-left continuous.

Let us denote by  $\beta^{v,*}$  and  $Y^{v,*}$  two  $(P^v, \mathbb{F})$ -predictable processes known as Girsanov parameters for the change of measure  $P^v$  into  $Q^{v,*}$  such that:  $\forall t \geq 0$  and  $P^v$ -a.s.

$$\int_0^t \int_{\mathbb{R}^d} \|l(x)\| |(Y_s^{v,*}(x) - 1)| \nu^v(ds, dx) < \infty,$$

and

$$\int_0^t \|c_s \beta_s^{v,*}\| ds < \infty, \quad \int_0^t {}^\top \beta_s^{v,*} c_s \beta_s^{v,*} ds < \infty.$$

Here  $\nu^v$  stands for the compensator of the jump measure of  $X$  with respect to  $(P^v, \mathbb{F})$ ,  $l$  is the truncation function and  $c$  is the density of the predictable variation of continuous martingale part of  $X$ , w.r.t. Lebesgue measure.

In the case of logarithmic utility we consider the entropy  $\mathbf{I}(P_T^v | Q_T^{v,*})$  and also the corresponding information process  $\mathcal{I}^*(v) = (\mathcal{I}_t^*(v))_{t \in [0, T]}$  with

$$(9) \quad \mathcal{I}_t^*(v) = \frac{1}{2} \int_0^t \top \beta_s^{v,*} c_s \beta_s^{v,*} ds - \int_0^t \int_{\mathbb{R}^d} [\ln(Y_s^{v,*}(x)) - Y_s^{v,*}(x) + 1] \nu^v(ds, dx).$$

**Proposition 1.** *Let  $Q_T^{v,*} \in \mathcal{K}_T^v$ . Then the corresponding relative entropy is well-defined and*

$$(10) \quad \mathbf{I}(P_T^v | Q_T^{v,*}) = \mathbf{E}_{P^v} \mathcal{I}_T^*(v).$$

In the case of exponential utility we consider Kullback-Leiber information  $\mathbf{I}(Q_T^{v,*} | P_T^v)$  and we introduce the corresponding Kullback-Leiber process  $I^*(v) = (I_t^*(v))_{t \in [0, T]}$  with

$$(11) \quad I_t^*(v) = \frac{1}{2} \int_0^t \top \beta_s^{v,*} c_s \beta_s^{v,*} ds + \int_0^t \int_{\mathbb{R}^d} [Y_s^{v,*}(x) \ln(Y_s^{v,*}(x)) - Y_s^{v,*}(x) + 1] \nu^v(ds, dx).$$

**Proposition 2.** *Let  $Q_T^{v,*} \in \mathcal{K}_T^v$ . Then, the corresponding Kullback-Leiber information is well defined and*

$$(12) \quad \mathbf{I}(Q_T^{v,*} | P_T^v) = \mathbf{E}_{P^v} \left[ \int_0^T z_{s-}^*(v) dI_s^*(v) \right] = \mathbf{E}_{Q^{v,*}} (I_T^*(v)).$$

For the case of power utility we consider Hellinger types integrals

$$\mathbf{H}_T^{(q),*}(v) = \mathbf{E}_{P^v} [(z_T^*(v))^q],$$

where  $q = \frac{p}{p-1} < 1$ .

We introduce the corresponding predictable process called Hellinger type process  $h^{(q),*}(v) = (h_t^{(q),*}(v))_{t \in [0, T]}$

$$(13) \quad h_t^{(q),*}(v) = \frac{1}{2} q(1-q) \int_0^t \top \beta_s^{v,*} c_s \beta_s^{v,*} ds - \int_0^t \int_{\mathbb{R}^d} [(Y_s^{v,*}(x))^q - q(Y_s^{v,*}(x) - 1) - 1] \nu^v(ds, dx).$$

**Proposition 3.** *Let  $Q_T^{v,*} \in \mathcal{K}_T^v$ . Then respective Hellinger type integral  $\mathbf{H}_T^{(q),*}(v)$  is well defined and*

$$(14) \quad \mathbf{H}_T^{(q),*}(v) = 1 - \mathbf{E}_{P^v} \left[ \int_0^T (z_{s-}^*(v))^q dh_s^{(q),*}(v) \right]$$

or, in the terms of the stochastic exponential,

$$(15) \quad \mathbf{H}_T^{(q),*}(v) = \mathbf{E}_{R^v} \left[ \mathcal{E} \left( -h^{(q),*}(v) \right)_T \right]$$

where  $R^v$  is some locally absolutely continuous w.r.t.  $P^v$  measure.

### 3. UTILITY MAXIMISATION WITH OPTION FOR EXPONENTIAL LÉVY MODELS

We begin with some basic notations for the exponential Lévy models involved in the utility maximisation calculus.

**3.1. Description of the model.** Let  $X^{(1)} = (X_t^{(1)})_{0 \leq t \leq T}$  and  $X^{(2)} = (X_t^{(2)})_{0 \leq t \leq T'}$  be two independent  $d$ -dimensional Lévy processes starting from zero with generating triplets  $(b_1, c_1, K_1)$  and  $(b_2, c_2, K_2)$  respectively. Each process is given on its own filtered canonical probability space  $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{F}^{(1)}, P^{(1)})$  and  $(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{F}^{(2)}, P^{(2)})$  respectively where  $\mathbb{F}^{(1)} = (\mathcal{F}_t^{(1)})_{0 \leq t \leq T}$  and  $\mathbb{F}^{(2)} = (\mathcal{F}_t^{(2)})_{0 \leq t \leq T'}$  are the corresponding filtrations verifying usual properties. Let  $X = (X_t)_{0 \leq t \leq T}$  be the linear map of the processes  $X^{(1)}$  and  $X^{(2)}$ , namely, for  $t \in [0, T]$

$$(16) \quad X_t = \rho_1 X_t^{(1)} + \rho_2 X_t^{(2)}$$

involving non-random invertible matrices  $\rho_1$  and  $\rho_2$ . As it was mentioned, the process  $X$  is considered on canonical probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  with filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  which satisfy usual properties.

We introduce also the enlarged space  $(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \sigma(X_{T'}^{(2)}), \mathbb{G})$ , corresponding to the couple  $(X, X_{T'}^{(2)})$  with enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$  where for  $0 \leq t < T$

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \otimes \sigma(X_{T'}^{(2)}) \quad \text{and} \quad \mathcal{G}_T = \mathcal{F}_T \otimes \sigma(X_{T'}^{(2)}).$$

We remark that on the space  $(\Omega, \mathcal{F}, P)$  the process  $X$ , is, evidently, a Lévy process. Now, if we equip  $(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d), \mathbb{G})$  with the probability  $P \times \alpha$ , where  $\alpha$  is the law of  $X_{T'}^{(2)}$ , then the process  $X$  will remain a Lévy process with the same triplet. We recall that, as before,

we use the notation  $\mathbb{P}$  for joint law of  $(X, X_{T'}^{(2)})$  and  $P^v$  for conditional law of  $X$  given  $X_{T'}^{(2)} = v$ .

**3.2. Assumptions 1 and 2.** In this subsection we show that the Assumptions 1 and 2 of Section 2 will be verified under the following hypothesis on Lévy processes.

**Hypothesis H1:** *The processes  $X$  and  $X^{(2)}$  are integrable on  $[0, T]$  and  $[0, T']$  respectively.*

**Hypothesis H2:** *The process  $(X, X^{(2)})$  has a transition density w.r.t. a product  $\eta = \eta_1 \times \eta_2$  of two  $\sigma$ -finite measures  $\eta_1$  and  $\eta_2$  on  $\mathbb{R}^d$ , and the marginal transition densities  $f$  and  $f^{(2)}$  of  $X$  and  $X^{(2)}$  w.r.t.  $\eta$  and  $\eta_2$  respectively, are strictly positive.*

**Remark 2.** It should be noticed that in the case when  $\eta_1$  and  $\eta_2$  are Lebesgue measures, the Hypothesis H2 is equivalent to the existence of marginal strictly positive transition densities  $f_1$  and  $f_2$  of the processes of  $X^{(1)}$  and  $X^{(2)}$ . This fact follows from the independence of  $X^{(1)}$  and  $X^{(2)}$ .

**Proposition 4.** *Under hypotheses (H1) and (H2), the Assumptions 1 and 2 are satisfied and there exists a function  $F_v : [T' - T, T'] \times \mathbb{R}^d \rightarrow \mathbb{R}^+$  depending on a parameter  $v \in \mathbb{R}^d$ , such that*

$$(17) \quad \frac{d\alpha^t}{d\alpha}(v) = \frac{F_v(T' - t, X_t)}{F_v(T', 0)}$$

Moreover,

$$\frac{d\alpha^t}{d\alpha} = \mathcal{E}(M)_t$$

with  $M = (M_t)_{0 \leq t \leq T}$  which is a  $(P, \mathbb{F})$ -martingale such that

$$M_t = \int_0^t \top \beta_s^{v,P} X_s^c + \int_0^t \int_{\mathbb{R}^d} l(x)(Y_s^{v,P}(x) - 1) dK(x) ds$$

where  $(\beta^{v,P}, Y^{v,P})$  are the Girsanov parameters for the change the measure  $P$  into  $P^v$ , and  $K$  is Levy measure of  $X$ .

If  $F_v \in C_b^{1,2}([T' - T, T'] \times \mathbb{R}^d)$  and  $c$  is invertible, then the mentioned Girsanov parameters  $(\beta^{v,P}, Y^{v,P})$  can be calculated by the following formulas:

$$\top \beta_s^{v,P} = \left( \frac{\partial \ln F_v}{\partial x_1}(T' - s, X_{s-}), \dots, \frac{\partial \ln F_v}{\partial x_d}(T' - s, X_{s-}) \right)$$

and for  $x \in \mathbb{R}^d \setminus \{0\}$

$$Y_s^{v,P}(x) = \frac{F_v(T-s, X_{s-} + x)}{F_v(T'-s, X_{s-})}.$$

**Proof:** Conditionally to  $X_{T'}^{(2)} = v$ , the process  $X$  is distributed as  $\rho_1 X^{(1)} + \rho_2 V^{(2)}$  where  $V^{(2)}$  is a Levy bridge of  $X^{(2)}$  starting at  $(0,0)$  and ending at  $(v, T')$ . Under the hypothesis (H2) and according to [12], the law of  $(V_t^{(2)})_{0 \leq t \leq T}$  is absolutely continuous w.r.t. the law of  $(X_t^{(2)})_{0 \leq t \leq T}$  and

$$(18) \quad \frac{dP_{V^{(2)}}(T, v)}{dP_{X^{(2)}}(T, v)} = \frac{f^{(2)}(T' - T, v - X_T^{(2)})}{f^{(2)}(T', v)}.$$

Since the process  $X^{(1)}$  is independent from  $X^{(2)}$  and also from  $V^{(2)}$ , the conditional distributions of  $X$  given  $X^{(2)}$  and the conditional distributions of  $X$  given  $V^{(2)}$  coincide as maps, under the measure  $P$ . Let us denote this map by  $q(A, x)$ ,  $A \in \mathcal{F}_T$ ,  $x \in \mathbb{R}^d$ . Then,

$$P(A) = \int_{\mathbb{R}^d} P(\rho_1 X^{(1)} + \rho_2 X^{(2)} \in A \mid X^{(2)} = x) dP_{X^{(2)}}(x) = \int_{\mathbb{R}^d} q(A, x) dP_{X^{(2)}}(x)$$

and

$$\begin{aligned} P^v(A) &= P(\rho_1 X^{(1)} + \rho_2 V^{(2)} \in A) \\ &= \int_{\mathbb{R}^d} P(\rho_1 X^{(1)} + \rho_2 V^{(2)} \in A \mid V^{(2)} = x) dP_{V^{(2)}}(x) \\ &= \int_{\mathbb{R}^d} q(A, x) \frac{dP_{V^{(2)}}(T, v)}{dP_{X^{(2)}}(x)} dP_{X^{(2)}}(x). \end{aligned}$$

Finally, if  $P(A) = 0$  then  $q(A, x) = 0$  ( $P_{X^{(2)}}$ -a.s.) and, hence  $P^v(A) = 0$ . Hence, the Assumption 1 is verified.

The Assumption 1 and Bayes formula for conditional densities gives us:

$$\begin{aligned} P(X_{T'}^{(2)} \in dv \mid X_t = y) &= \frac{P(X_{T'}^{(2)} \in dv, X_t \in dy)}{P(X_t \in dy)} = \\ &= \frac{P(X_t \in dy \mid X_{T'}^{(2)} = v) P(X_{T'}^{(2)} \in dv)}{P(X_t \in dy)}. \end{aligned}$$

This means that the Assumption 2 is verified. Using Markov property we write:

$$\begin{aligned} \alpha^t(dv) &= P(X_{T'}^{(2)} \in dv \mid \mathcal{F}_t) = P(X_{T'}^{(2)} \in dv \mid X_t) = \\ &= P(X_{T'}^{(2)} - X_t^{(2)} + X_t^{(2)} \in dv \mid X_t) = P(\tilde{X}_{T'-t}^{(2)} + X_t^{(2)} \in dv \mid X_t) \end{aligned}$$

where  $\tilde{X}^{(2)}$  is a process, which is independent from  $X^{(1)}$  and  $X^{(2)}$ , and is distributed as  $X^{(2)}$ . Then, we see that  $\alpha^t(dv)$  is a function of  $T' - t$ ,  $X_t$  and the parameter  $v$ , denoted  $F_v(T' - t, X_t)$ . At the same time  $\alpha^0(dv) = \alpha(dv) = F_v(T', 0)$  since  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ . It gives us (17).

Now, we use Ito formula to obtain that

$$\begin{aligned} F_v(T' - t, X_t) &= F_v(T', 0) \\ &- \int_0^t \frac{\partial F_v}{\partial s}(T' - s, X_{s-}) ds + \sum_{i=1}^d \int_0^t \frac{\partial F_v}{\partial x_i}(T' - s, X_{s-}) dX_s^i \\ &+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 F_v}{\partial x_i \partial x_j}(T' - s, X_{s-}) d \langle X^{i,c}, X^{j,c} \rangle_s \\ &+ \sum_{0 < s \leq t} F_v(T' - s, X_s) - F_v(T' - s, X_{s-}) - \sum_{i=1}^d \frac{\partial F_v}{\partial x_i}(T' - s, X_{s-}) \Delta X_s^i. \end{aligned}$$

Under the conditions  $P_t^v \ll P_t$  and  $\alpha^t \ll \alpha$  for  $t \in ]0, T]$ , we know from Jacod's lemma (cf.[14]) that  $\frac{d\alpha^t}{d\alpha}_{0 \leq t \leq T}$  is a  $(P, \mathbb{F})$  martingale. Let us put for  $t \in [0, T]$ ,  $p_t^v = \frac{d\alpha^t}{d\alpha}(v)$ . Then, we divide the above expression for  $F_v(T' - t, X_t)$  by  $F_v(T', 0)$  and we identify its continuous martingale part. We get that

$$p_t^{v,c} = \frac{1}{F_v(T', 0)} \sum_{i=1}^d \int_0^t \frac{\partial F_v}{\partial x_i}(T' - s, X_{s-}) dX_s^{i,c}$$

and, hence,

$$\langle p^{v,c}, X^{j,c} \rangle_t = \frac{1}{F_v(T', 0)} \sum_{i=1}^d \int_0^t c_{i,j} \frac{\partial F_v}{\partial x_i}(T' - s, X_{s-}) ds.$$

In addition, according to Girsanov theorem,

$$\langle p^{v,c}, X^{j,c} \rangle_t = \sum_{i=1}^d \int_0^t c_{i,j} (\beta_s^{v,P})^i p_{s-}^v ds.$$

Since  $c$  is invertible, this implies the formula for  $\beta_t^{v,P}$ .

Now, we compute the jumps of  $p^v$ :

$$\Delta p_t^v = \frac{F_v(T' - t, X_t) - F_v(T' - t, X_{t-})}{F_v(T', 0)}$$

and

$$\frac{\Delta p_t^v}{p_{t-}^v} = \frac{F_v(T' - t, X_{t-} + \Delta X_t)}{F_v(T' - t, X_{t-})} - 1.$$

Then, according to the Theorem 3.24, p.159, Chapter 3 in [16]

$$Y_t^{v,P} = \frac{F_v(T' - t, X_{t-} + x)}{F_v(T' - t, X_{t-})}$$

and the proposition is proved.  $\square$

**3.3. Conditional locally equivalent martingale measures.** The main difficulty related with the application of the results of Section 2 is the verification of the Assumption 3. The first step for this verification, consists in the complete description of the set of conditional locally equivalent martingale measures. This step can be done by use of semimartingale calculus.

We recall that the process  $X$  is defined by (16). As before we denote by  $(\beta^{v,P}, Y^{v,P})$  the Girsanov parameters for the change of the measure  $P$  into  $P^v$ . We denote also by  $\mathcal{M}(P^v)$  the set of locally equivalent to  $P^v$  martingale measures  $Q^v$ . We denote by  $(\beta^v, Y^v)$  the Girsanov parameters for the change of the measure  $P^v$  into  $Q^v$ . We notice that  $X$  is  $(P^v, \mathbb{F})$ - semimartingale, and hence,  $(Q^v, \mathbb{F})$  semimartingale. In the following proposition we give the triplet of predictable characteristics of  $X$  w.r.t.  $Q^v$ .

**Proposition 5.** *The triplet of predictable characteristics  $(B^v, C^v, \nu^v)$  of  $X$  with respect to  $(Q^v, \mathbb{F})$  is given by the expressions:*

$$\begin{aligned} B_t^v &= (\rho_1 b_1 + \rho_2 b_2)t + \rho_2 c_2 \int_0^t \beta_s^{v,P} ds \\ &+ \rho_2 \int_0^t \int_{\mathbb{R}^d} \vec{l}_2(x) (Y_s^{v,P}(\rho_2^{-1}x) - 1)(K_2 \circ \rho_2^{-1})(dx) ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \vec{l}(x) (Y_s^v(x) - 1) K_s^{v,P}(dx) ds + (\rho_1 c_1^\top \rho_1 + \rho_2 c_2^\top \rho_2) \int_0^t \beta_s^v ds, \\ C_t^v &= (\rho_1 c_1^\top \rho_1 + \rho_2 c_2^\top \rho_2)t, \end{aligned}$$

$$d\nu^v(x, s) = Y_s^v K_s^{v,P}(dx) ds,$$

where  $K_s^{v,P}(dx) = (K_1 \circ \rho_1^{-1})(dx) + Y_s^{v,P}(\rho_2^{-1}x) (K_2 \circ \rho_2^{-1})(dx)$ . Moreover, an equivalent to  $P_T^v$  martingale measure  $Q_T^v$  satisfy : for  $s \in [0, T]$

$$(19) \quad \rho_1 b_1 + \rho_2 b_2 + \rho_2 c_2 \beta_s^{v,P} + \rho_2 \int_{\mathbb{R}^d} \vec{l}_2(x) (Y_s^{v,P}(\rho_2^{-1}x) - 1)(K_2 \circ \rho_2^{-1})(dx)$$



$$+ \int_{\mathbb{R}^d} \vec{l}(x) (Y_s^v(x) - 1) K_s^{v,P}(dx) + (\rho_1 c_1^\top \rho_1 + \rho_2 c_2^\top \rho_2) \beta_s^v = 0.$$

**Proof:** We use Girsanov theorem for successive change of the measures:  $P \rightarrow P^v \rightarrow Q^v$ . For that we write first a semimartingale decomposition of  $X$ :

$$X_t = B_t + X_t^c + \int_0^t \int_{\mathbb{R}^d} \vec{l}(x) (\mu_X(dx, ds) - \nu_X(dx, ds)) + \sum_{s \leq t} (\Delta X_s - l(\Delta X_s))$$

Here  $B$  is the drift part of semimartingale decomposition,  $X^c$  is its continuous martingale part,  $\mu_X$  and  $\nu_X$  are the measure of jumps and its compensator, and  $l$  is the truncation function,  $l(x) = x \mathbf{1}_{\{\|x\| \leq 1\}}$ ,  $x \in \mathbb{R}^d$ . It should be noticed that the integral on  $\mathbb{R}^d$  in previous expression is taken in component by component way, namely for each  $x \in \mathbb{R}^d$  and  $l(x) = {}^\top (l^1(x), \dots, l^d(x))$  the integral

$$\int_0^t \int_{\mathbb{R}^d} \vec{l}(x) (\mu_X(dx, ds) - \nu_X(dx, ds))$$

is a vector with components

$$\int_0^t \int_{\mathbb{R}^d} l^i(x) (\mu_X(dx, ds) - \nu_X(dx, ds))$$

where  $1 \leq i \leq d$ . We use the notation  $\vec{l}(x)$  to underline this particular integration.

At the same time we write a semi-martingale decompositions of the processes  $X^{(1)}$  and  $X^{(2)}$ :

$$\begin{aligned} X_t^{(1)} &= b_1 t + X_t^{(1),c} + \int_0^t \int_{\mathbb{R}^d} \vec{l}_1(x) (\mu_{X^{(1)}}(dx, ds) - K_1(dx) ds) \\ &\quad + \sum_{s \leq t} (\Delta X_s^{(1)} - l_1(\Delta X_s^{(1)})) \\ X_t^{(2)} &= b_2 t + X_t^{(2),c} + \int_0^t \int_{\mathbb{R}^d} \vec{l}_2(x) (\mu_{X^{(2)}}(dx, ds) - K_2(dx) ds) \\ &\quad + \sum_{s \leq t} (\Delta X_s^{(2)} - l_2(\Delta X_s^{(2)})) \end{aligned}$$

with truncation functions  $l_1(x) = x \mathbf{1}_{\{\|\rho_1 x\| \leq 1\}}$  and  $l_2(x) = x \mathbf{1}_{\{\|\rho_2 x\| \leq 1\}}$  respectively.

We compare now the linear combination of the canonical decompositions of the processes  $X^{(1)}$  and  $X^{(2)}$  given above with canonical decomposition of  $X$ . We can easily identify a drift part of  $X$ , which is

$(\rho_1 b_1 + \rho_2 b_2)t$ ,  $t \geq 0$ , and a continuous martingale part of  $X$ , which is equal to  $\rho_1 X^{(1),c} + \rho_2 X^{(2),c}$ . Since  $X_t^{(1),c}$  and  $X_t^{(2),c}$  are independent with quadratic variations  $c_1 t$ ,  $t \geq 0$  and  $c_2 t$ ,  $t \geq 0$ , the quadratic variation of continuous martingale part of  $X$  is equal to  $(\rho_1 c_1^\top \rho_1 + \rho_2 c_2^\top \rho_2)t$ ,  $t \geq 0$ .

For jump-part we write the measure of jumps of the process  $X$ :

$$\mu_X(\omega, dt, dx) = \sum_s \mathbf{1}_{\{\Delta X_s(\omega) \neq 0\}} \delta_{\{(s, \Delta X_s(\omega))\}}(dt, dx)$$

where  $\delta$  is Dirac delta-function in  $\mathbb{R}^{d+1}$ . In addition,

$$\Delta X = \rho_1 \Delta X^{(1)} + \rho_2 \Delta X^{(2)}, \quad l(\Delta X) = \rho_1 l_1(\Delta X^{(1)}) + \rho_2 l_2(\Delta X^{(2)}).$$

We know that two independent Levy processes can not jump at the same time. In fact, the jumps of Levy processes are totally inaccessible stopping times. If we suppose that the jumps of the processes  $X^{(1)}$  and  $X^{(2)}$  happen at the times  $\tau_1$  and  $\tau_2$  with  $\tau_1 = \tau_2$  (P-a.s.) then for all  $A \in \mathbb{R}^d$

$$P(\{\tau_1 \in A\} \cap \{\tau_2 \in A\}) = P(\{\tau_1 \in A\}) = P^2(\{\tau_1 \in A\}).$$

Then,  $P(\{\tau_1 \in A\}) = 0$  or  $1$ , and the law of  $\tau_1$  can be only Dirac measure. Then, there exists  $t_0 \in \mathbb{R}^+$  such that  $P(\tau_1 = t_0) = 1$ , but this contradicts with the fact that  $\tau_1$  is inaccessible. This fact gives us that  $P - a.s.$

$$\begin{aligned} \{\Delta X_s(\omega) \neq 0\} &= \{\rho_1 \Delta X_s^{(1)}(\omega) \neq 0\} \cup \{\rho_2 \Delta X_s^{(2)}(\omega) \neq 0\} \\ &= \{\rho_1 \Delta X_s^{(1)}(\omega) \neq 0\} \cap \{\Delta X_s^{(2)}(\omega) = 0\} \cup \{\Delta X_s^{(1)}(\omega) = 0\} \cap \{\rho_2 \Delta X_s^{(2)}(\omega) \neq 0\}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \mu_X(\omega, dt, dx) &= \\ &= \sum_s \mathbf{1}_{\{\Delta X_s^{(1)}(\omega) \neq 0\}} \delta_{\{(s, \rho_1 \Delta X_s^{(1)}(\omega))\}}(dt, dx) + \sum_s \mathbf{1}_{\{\Delta X_s^{(2)}(\omega) \neq 0\}} \delta_{\{(s, \rho_2 \Delta X_s^{(2)}(\omega))\}}(dt, dx) \\ &= \mu_{\rho_1 X^{(1)}}(\omega, dt, dx) + \mu_{\rho_2 X^{(2)}}(\omega, dt, dx). \end{aligned}$$

Now, the processes  $\rho_1 X^{(1)}$  and  $\rho_2 X^{(2)}$  are Levy processes with Levy measures  $K_1(\rho_1^{-1}A)$  and  $K_2(\rho_2^{-1}A)$  respectively where  $A \in \mathcal{B}(\mathbb{R}^d)$ . As a consequence, the triplet of predictable characteristics  $(B, C, \nu)$  of  $X$  is given by:

$$\begin{aligned} B_t &= (\rho_1 b_1 + \rho_2 b_2)t, \\ C_t &= (\rho_1 c_1^\top \rho_1 + \rho_2 c_2^\top \rho_2)t, \\ d\nu(x, t) &= ((K_1 \circ \rho_1^{-1})(dx) + (K_2 \circ \rho_2^{-1})(dx)) dt. \end{aligned}$$

Next, we write the triplet  $(B^V, C^V, \nu^V)$  of Levy bridge  $V^{(2)}$ :

$$\begin{aligned} B_t^V &= b_2 t + c_2 \int_0^t \beta_s^{v,P} ds + \int_0^t \int_{\mathbb{R}^d} l_2(\vec{x}) (Y_s^{v,P}(x) - 1) K_2(dx) ds, \\ C_t^V &= c_2 t, \\ d\nu^V(x, t) &= Y_t^{v,P}(x) K_2(dx) dt. \end{aligned}$$

To write the characteristics for linear combination of  $X^{(1)}$  and  $V^{(2)}$ , we take in account the fact that the processes  $X^{(1)}$  and  $V^{(2)}$  remain independent under  $P^v$ . Then, we add the additional drift coming from the change of the measure  $P^v$  into  $Q^v$  and we multiply the corresponding Levy measure by the factor  $Y_s^v$ . This give us the formulas for the characteristics.

The process  $X$  is a  $(Q^v, \mathbb{F})$ -martingale if and only if its drift term under  $Q^v$  is identically equal to zero, and it gives us mentioned identity.  $\square$

**3.4. Conditional information processes.** To simplify the expression for finding of the Girsanov parameters  $(\beta^{v,*}, Y^{v,*})$  of the  $f$ -divergence minimal equivalent martingale measure  $Q^{v,*}$ , we use the notations:

$$b = \rho_1 b_1 + \rho_2 b_2, \quad c = \rho_1 c_1^\top \rho_1 + \rho_2 c_2^\top \rho_2$$

We recall that  $(b, c, K)$  are the parameters of Levy process  $X$  under "historical" measure  $P$ .

**Theorem 3.** *Let  $u(x) = \ln(x)$  and the hypothesis (H1) and (H2) hold. If there exists a predictable process  $\lambda^v = (\lambda_s^v)_{0 \leq s \leq T}$  with the values in  $\mathbb{R}^d$  such that for all  $s \in [0, T]$*

$$\begin{aligned} (20) \quad b + c \lambda_s^v + \rho_2 c_2 \beta_s^{v,P} + \rho_2 \int_{\mathbb{R}^d} l_2(\vec{x}) [Y_s^{v,P}(\rho_2^{-1}x) - 1] (K_2 \circ \rho_2^{-1})(dx) \\ + \int_{\mathbb{R}^d} l(x) \frac{\lambda_s^v l(x)}{1 - \lambda_s^v l(x)} K_s^{v,P}(dx) = 0, \end{aligned}$$

and such that  $1 - \lambda_s^v l(x) > 0$  ( $K^{v,P}$ -a.s.), then the Girsanov parameters of  $f$ -divergence minimal martingale measure  $Q_T^{v,*}$  verify:

$$\beta_s^{v,*} = \lambda_s^v, \quad Y_s^{v,*}(x) = \frac{1}{1 - \lambda_s^v l(x)}.$$

The corresponding information process  $\mathcal{I}^*(v)$  is given by (9) and the corresponding entropy is equal to (10). If this entropy is finite, the corresponding measure will be  $f$ -divergence minimal equivalent martingale measure.

**Proof:** To find the Girsanov parameters of the corresponding  $f$ -divergence minimal martingale measure we minimise the relative entropy of  $P_T^v$  given  $Q_T^v$ :

$$\mathbf{I}(P_T^v | Q_T^v) = \mathbf{E}_{P_T^v}(\mathcal{I}_T(v))$$

with

$$\mathcal{I}_T(v) = \frac{1}{2} \int_0^T \top \beta_s^v c \beta_s^v ds - \int_0^T \int_{\mathbb{R}^d} (\ln Y_s^v(x) - Y_s^v + 1) K_s^{v,P}(dx) ds,$$

under constraint: for  $s \in [0, T]$

$$(21) \quad R(\beta_s^v, Y_s^v) = 0.$$

In this constraint, the function  $R(\beta_s^v, Y_s^v)$  is defined as follows:

$$\begin{aligned} R(\beta_s^v, Y_s^v) &= b + \rho_2 c_2 \beta_s^{v,P} + \int_{\mathbb{R}^d} \vec{l}(x) [Y_s^{v,P}(\rho_2^{-1}x) - 1] (K_2 \circ \rho_2^{-1})(dx) \\ &\quad + c \beta_s^v + \int_{\mathbb{R}^d} \vec{l}(x) (Y_s^v(x) - 1) K_s^{v,P}(dx). \end{aligned}$$

According to the traditional procedure of minimisation, we introduce the function  $G$  with

$$G(\beta_s^v, Y_s^v) = \frac{1}{2} \top \beta_s^v c \beta_s^v - \int_{\mathbb{R}^d} (\ln(Y_s^v(x)) - Y_s^v + 1) K_s^{v,P}(dx) - \top \lambda_s^v R(\beta_s^v, Y_s^v),$$

where  $\lambda_s^v$  is the Lagrangian factor. This function is convex continuously differentiable function, its extreme points are stationary points, which are the solutions of the equations:

$$\begin{cases} \top \left( \frac{\partial G}{\partial \beta_1}(\beta_s^v, Y_s^v), \dots, \frac{\partial G}{\partial \beta_d}(\beta_s^v, Y_s^v) \right) = c(\beta_s^v - \lambda_s^v) = 0, \\ \frac{\partial G}{\partial Y}(\beta_s^v, Y_s^v) = \int_{\mathbb{R}^d} \left( 1 - \frac{1}{Y_s^v(x)} - \top \lambda_s^v l(x) \right) K_s^{v,P}(dx) = 0. \end{cases}$$

It is clear that  $\beta_s^v = \lambda_s^v$  is a solution of the first equation. In general, second equation has multiple solutions, but due to the convexity of  $G$ , the corresponding value of the information process will be the same. One of the solutions of the second equation is given by

$$Y_s^v(x) = \frac{1}{1 - \top \lambda_s^v l(x)}$$

and we assume that it is positive. Finally, we put the expression for  $\beta_s^v$  and  $Y_s^v$  into the martingale condition (20), to find  $\lambda_s^v$ , and, hence,  $\beta^{v,*}$  and  $Y^{v,*}$ .

The convexity of the function  $G$  gives

$$G(\beta_s^v, Y_s^v) - G(\beta_s^{v,*}, Y_s^{v,*}) \geq \top \left( \frac{\partial G}{\partial \beta_1}(\beta_s^{v,*}, Y_s^{v,*}), \dots, \frac{\partial G}{\partial \beta_d}(\beta_s^{v,*}, Y_s^{v,*}) \right) (\beta_s^v - \beta_s^{v,*}) + \frac{\partial G}{\partial Y}(\beta_s^{v,*}, Y_s^{v,*})(Y_s^v - Y_s^{v,*}) = 0.$$

To prove that the corresponding measure is  $f$ -divergence minimal, i.e.

$$\mathbf{I}(P_T^v | Q_T^v) \geq \mathbf{I}(P_T^v | Q_T^{v,*}),$$

we integrate the above inequality w.r.t.  $s$  and we take expectation with respect to the measure  $P_T^v$ .  $\square$

**Theorem 4.** *Let  $u(x) = x \ln(x) + x - 1$  and the hypothesis (H1) and (H2) are valid. If there exists predictable process  $\lambda^v = (\lambda_s^v)_{0 \leq s \leq T}$  with the values in  $\mathbb{R}^d$  such that for all  $s \in [0, T]$*

$$(22) \quad b + c\lambda_s^v + \rho_2 c_2 \beta_s^{v,P} + \rho_2 \int_{\mathbb{R}^d} l_2(\vec{x}) [Y_s^{v,P}(\rho_2^{-1}x) - 1](K_2 \circ \rho_2^{-1})(dx) + \int_{\mathbb{R}^d} l(\vec{x}) [\exp(\top \lambda_s^v l(x)) - 1] K_s^{v,P}(dx) = 0$$

then the Girsanov parameters of the  $f$ -divergence minimal martingale measure  $Q_T^{v,*}$  verify:

$$\beta_s^{v,*} = \lambda_s^v, \quad Y_s^{v,*}(x) = \exp(\top \lambda_s^v l(x)).$$

Moreover, the corresponding information process  $I^*(v)$  is given by (11) and the Kullback-Leibler information is given by (12). If this Kullback-Leibler information is finite, the corresponding measure will be  $f$ -divergence minimal equivalent martingale measure.

**Proof:** To find the Girsanov parameters of the  $f$ -divergence minimal martingale measure  $Q_T^v$ , we minimise relative entropy of  $Q_T^v$  given  $P_T^v$ :

$$\mathbf{I}(Q_T^v | P_T^v) = \mathbf{E}_{Q_T^v}(I_T(v))$$

with

$$I_T(v) = \frac{1}{2} \int_0^T \top \beta_s^v c \beta_s^v ds + \int_0^T \int_{\mathbb{R}^d} (Y_s^v(x) \ln(Y_s^v(x)) - Y_s^v + 1) K_s^{v,P}(dx) ds,$$

under constraint (21). For that we introduce the function  $G$  such that

$$G(\beta_s^v, Y_s^v) = \frac{1}{2} \top \beta_s^v c \beta_s^v + \int_{\mathbb{R}^d} (Y_s^v(x) \ln(Y_s^v(x)) - Y_s^v + 1) K_s^{v,P}(dx) - \top \lambda_s^v R(\beta_s^v, Y_s^v)$$

with the Lagrangian factor  $\lambda_s^v$ . This function is convex continuously differentiable function, so, the minimum of this function is realised on

the set of stationary points, which verify :

$$\begin{cases} \top \left( \frac{\partial G}{\partial \beta_1}(\beta_s^v, Y_s^v), \dots, \frac{\partial G}{\partial \beta_d}(\beta_s^v, Y_s^v) \right) = c(\beta_s^v - \lambda_s^v) = 0, \\ \frac{\partial G}{\partial Y}(\beta_s^v, Y_s^v) = \int_{\mathbb{R}^d} (\ln(Y_s^v(x)) - \top \lambda_s^v l(x)) K_s^{v,P}(dx) = 0. \end{cases}$$

The solution of the first equation is  $\beta_s^v = \lambda_s^v$ . We remark that second equation has multiple solutions, but the corresponding value of the information process will be the same. One of the solutions of the second equation is given by

$$Y_s^v(x) = \exp(\top \lambda_s^v l(x)).$$

To find  $\lambda^v$ , we put the expressions for  $\beta_s^v$  and  $Y_s^v$  into martingale condition (22), and it gives us the expressions for  $\beta^{v,*}$  and  $Y^{v,*}$ .

We clearly have:

$$\begin{aligned} & G(\beta_s^v, Y_s^v) - G(\beta_s^{v,*}, Y_s^{v,*}) \geq \\ & \top \left( \frac{\partial G}{\partial \beta_1}(\beta_s^{v,*}, Y_s^{v,*}), \dots, \frac{\partial G}{\partial \beta_d}(\beta_s^{v,*}, Y_s^{v,*}) \right) (\beta_s^v - \beta_s^{v,*}) + \frac{\partial G}{\partial Y}(\beta_s^{v,*}, Y_s^{v,*})(Y_s^v - Y_s^{v,*}) = 0. \end{aligned}$$

To show that the corresponding measure is  $f$ -divergence minimal, we integrate this inequality w.r.t.  $s$  and we take the expectation with respect to  $Q_T^v$ . Then,

$$\mathbf{I}(Q_T^v | P_T^v) \geq \mathbf{I}(Q_T^{v,*} | P_T^v).$$

□

**Theorem 5.** *Let  $u(x) = x^p$ ,  $p < 1$  and the hypothesis (H1) and (H2) are satisfied. If there exists predictable process  $\lambda^v = (\lambda_s^v)_{0 \leq s \leq T}$  with the values in  $\mathbb{R}^d$  such that for all  $s \in [0, T]$  and  $q = \frac{p}{p-1}$*

$$\begin{aligned} & b + \frac{c\lambda_s^v}{q(1-q)} + \rho_2 c_2 \beta_s^{v,P} + \rho_2 \int_{\mathbb{R}^d} \vec{l}_2(x) [Y_s^{v,P}(\rho_2^{-1}x) - 1](K_2 \circ \rho_2^{-1})(dx) \\ & + \int_{\mathbb{R}^d} \vec{l}(x) \left[ \left( 1 - \frac{\top \lambda_s^v l(x)}{q} \right)^{\frac{1}{q-1}} - 1 \right] K_s^{v,P}(dx) = 0, \end{aligned}$$

and such that  $1 - \frac{\top \lambda_s^v l(x)}{q} > 0$  ( $K^{v,P}$ -a.s.), then the Girsanov parameters of  $f$ -divergence minimal martingale measure  $Q_T^{v,*}$  verify:

$$\beta_s^{v,*} = \frac{1}{q(1-q)} \lambda_s^v, \quad Y_s^{v,*}(x) = \left( 1 - \frac{\top \lambda_s^v l(x)}{q} \right)^{\frac{1}{q-1}}.$$

In addition, the Hellinger type process  $h^{(q),*}(v)$  is defined by (13) and the corresponding Hellinger type integral is given by (15). If this Hellinger integral is finite, the corresponding measure is  $f$ -divergence minimal equivalent martingale measure.

**Proof:** To find the Girsanov parameters of the  $f$ -divergence minimal martingale measure  $Q_T^v$ , we minimise Hellinger integral of  $Q_T^v$  and  $P_T^v$ :

$$\mathbf{H}_T^{(q)}(v) = \mathbf{E}_{R_T^v} \exp(-h_T^{(q)}(v))$$

with

$$h_T^{(q)}(v) = \frac{q(1-q)}{2} \int_0^T \top \beta_s^v c \beta_s^v ds - \int_0^T \int_{\mathbb{R}^d} ((Y_s^v(x))^q - q(Y_s^v - 1) - 1) K_s^{v,P}(dx) ds$$

under constraint (21). For that we introduce the function  $G$  via

$$G(\beta_s^v, Y_s^v) = \frac{q(1-q)}{2} \top \beta_s^v c \beta_s^v ds - \int_{\mathbb{R}^d} ((Y_s^v(x))^q - q(Y_s^v - 1) - 1) K_s^{v,P}(dx) - \top \lambda_s^v R(\beta_s^v, Y_s^v)$$

where  $\lambda_s^v$  is again the Lagrangian factor. This function is convex continuously differentiable function, so, the stationary points verify:

$$\begin{cases} \top \left( \frac{\partial G}{\partial \beta_1}(\beta_s^v, Y_s^v), \dots, \frac{\partial G}{\partial \beta_d}(\beta_s^v, Y_s^v) \right) = c(q(1-q)\beta_s^v - \lambda_s^v) = 0, \\ \frac{\partial G}{\partial Y}(\beta_s^v, Y_s^v) = - \int_{\mathbb{R}^d} [q(Y_s^v(x))^{q-1} - q + \top \lambda_s^v l(x)] K_s^{v,P}(dx) = 0. \end{cases}$$

From the first equation we find that  $\beta_s^v = \frac{1}{q(1-q)} \lambda_s^v$ . One of the solutions of the second equation is given by

$$Y_s^v(x) = \left( 1 - \frac{\top \lambda_s^v l(x)}{q} \right)^{\frac{1}{q-1}}.$$

Next, we put the expression for  $\beta_s^v$  and  $Y_s^v$  in the martingale condition (22) to find  $\lambda_s^v$  and, then,  $\beta_s^{v,*}$  and  $Y_s^{v,*}$ .

Since  $G$  is convex,

$$G(\beta_s^v, Y_s^v) - G(\beta_s^{v,*}, Y_s^{v,*}) \geq \top \left( \frac{\partial G}{\partial \beta_1}(\beta_s^{v,*}, Y_s^{v,*}), \dots, \frac{\partial G}{\partial \beta_d}(\beta_s^{v,*}, Y_s^{v,*}) \right) (\beta_s^v - \beta_s^{v,*}) + \frac{\partial G}{\partial Y}(\beta_s^{v,*}, Y_s^{v,*})(Y_s^v - Y_s^{v,*}) = 0.$$

Then, we integrate this inequality w.r.t.  $s$ , we use the fact that exponential is convex function, and, finally, we take expectation with respect to  $R_T^v$ , in order to prove that

$$\mathbf{H}_T^{(q)} \geq \mathbf{H}_T^{(q),*},$$

i.e. that the measure  $Q_T^{v,*}$  is f-divergence minimal.  $\square$

#### 4. BLACK-SHOLES MODELS WITH CORRELATED BROWNIAN MOTIONS

Let  $(W^{(1)}, W^{(2)})$  be independent standard Brownian motions. Let  $\mu_1, \mu_2 \in \mathbb{R}$  and  $\sigma_1 > 0, \sigma_2 > 0$ . We put

$$X_t^{(1)} = \mu_1 + \sigma_1 W_t^{(1)}, \quad X_t^{(2)} = \mu_2 + \sigma_2 W_t^{(2)},$$

and for the parameter  $|\rho| \leq 1$ , let

$$X_t = \sqrt{1 - \rho^2} X_t^{(1)} + \rho X_t^{(2)}.$$

Then,  $X$  will be Brownian motion with drift coefficient

$$\mu = \sqrt{1 - \rho^2} \mu_1 + \rho \mu_2,$$

diffusion coefficient

$$\sigma^2 = (1 - \rho^2) \sigma_1^2 + \rho^2 \sigma_2^2,$$

and the correlation coefficient between  $X_t$  and  $X_t^{(2)}$  equal to  $\rho$ . We take  $W_{T'}^{(2)}$  for conditioning instead of  $X_{T'}^{(2)}$  since these two variables are in bijection. But this replacement also implies that we should replace  $g(v)$  by  $\tilde{g}(v) = \exp\{(\mu_2 T' + \sigma_2 v)\}$  in maximum utility formula. In this setting, the law  $\alpha$  is, evidently, nothing else as  $\mathcal{N}(0, T')$ .

We see that the hypotheses (H1) and (H2) are verified. In fact, the processes  $X^{(1)}$  and  $X^{(2)}$  are integrable, both have a strictly positive density with respect to Lebesgue measure. In particular, as well known,  $W_t^{(2)}$  has a strictly positive density w.r.t. Lebesgue measure for  $t > 0$ :

$$f(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}$$

which is  $C_b^{1,2}([\epsilon, \infty[)$  for any  $\epsilon > 0$ . Moreover, we use normal correlation theorem to get that

$$p_t^v = \frac{dP_T^v}{dP_T}(X) = \left(\frac{T'}{T' - \rho^2 T}\right)^{1/2} \exp\left\{-\frac{1}{2} \left[\frac{(v - \rho X_T)^2}{T' - \rho^2 T} - \frac{v^2}{T'}\right]\right\}.$$



Then, we write this quantity as a stochastic exponential

$$p_T^v(X) = \exp \left\{ \int_0^T \beta_s^{v,P} dX_s^c - \frac{1}{2} \int_0^T (\beta_s^{v,P})^2 ds \right\},$$

where  $X$  is canonical process, and we deduce that  $P$ -a.s. and for  $t \in [0, T]$

$$(23) \quad \beta_t^{v,P} = \rho \frac{v - \rho X_t^c}{T' - \rho^2 t}.$$

After calculations, we obtain the conditional information quantities.

**Proposition 6.** (cf. [10]) *For entropy, Kullback-Leibler information and Hellinger type integrals we have:*

$$\mathbf{I}(P^v | Q^{v,*}) = \frac{1}{2} \ln \left( \frac{T'}{T' - \rho^2 T} \right) + \frac{T}{2} \left( \frac{\mu}{\sigma} + \frac{\rho v}{T'} \right)^2 - \frac{\rho^2 T}{2T'},$$

$$\mathbf{I}(Q^{v,*} | P^v) = -\frac{1}{2} \ln \left( \frac{T'}{T' - \rho^2 T} \right) + \frac{TT'}{2(T' - \rho^2 T)} \left( \frac{\mu}{\sigma} + \frac{\rho v}{T'} \right)^2 + \frac{\rho^2 T}{2(T' - \rho^2 T)},$$

$$\mathbf{H}_T^{(q)}(v) = \left( \frac{T'}{T' - q\rho^2 T} \right)^{1/2} \left( \frac{T' - \rho^2 T}{T'} \right)^{q/2} \exp \left\{ -\frac{q(1-q)T}{2(T' - q\rho^2 T)} \left( \frac{\mu}{\sigma} + \frac{\rho v}{T'} \right)^2 \right\}.$$

Finally, to know maximum of utility, we use the Theorem 2 with  $\alpha$  being  $\mathcal{N}(0, T')$ .

## 5. SOME JUMP-TYPE MODELS

Let  $(W^{(1)}, W^{(2)})$  be two standard Brownian motion with correlation  $\rho$ ,  $|\rho| \leq 1$ . Let  $N$  be homogeneous Poisson process of intensity  $\lambda > 0$ , independent from  $(W^{(1)}, W^{(2)})$ . We put

$$X_t = \mu_1 t + \sigma_1 W_t^{(1)} + N_t, \quad t \in [0, T],$$

$$X_t^{(2)} = \mu_1 t + \sigma_1 W_t^{(2)}, \quad t \in [0, T']$$

with  $T' > T$ . The option will be supported by  $g(X_{T'}^{(2)})$  where  $g$  is measurable non-negative function on  $\mathbb{R}$ .

Using the same arguments as in Section 4, we take  $W_{T'}^{(2)}$  instead of  $X^{(2)}$  with replacing of  $g(v)$  by  $\tilde{g}(v) = \exp\{(\mu_1 T' + \sigma_1 v)\}$ . We can verify exactly in the same manner as in previous section that the hypothesis (H1) and (H2) are verified. Moreover,

$$p_T^v(X) = \frac{dP_T^v}{dP_T}(X) = \left( \frac{T'}{T' - \rho^2 T} \right)^{1/2} \exp \left\{ -\frac{1}{2} \left[ \frac{(\sigma_1 v - \rho X_T^c)^2}{\sigma_1^2 (T' - \rho^2 T)} - \frac{v^2}{T'} \right] \right\}$$

with  $X$  canonical process corresponding to  $X^{(1)}$  and  $X^c$  being its continuous martingale part. Writing the last expression as stochastic exponential, we find that  $P$ -a.s. and for  $t \in [0, T]$

$$(24) \quad \beta_t^{v,P} = \frac{\rho(v\sigma_1 - \rho X_t^c)}{\sigma_1^2(T - \rho^2 t)}.$$

We remark that  $Y^{v,P} = 1$  here since  $N$  and  $W^{(2)}$  are independent.

In the following lemma we give the equations for the Girsanov parameters  $(\beta^{v,*}, Y^{v,*})$  of the change of the measure  $P^v$  into  $Q^{v,*}$ .

**Lemma 1.** *The Girsanov parameters  $(\beta^{v,*}, Y^{v,*})$  of the equivalent  $f$ -divergence minimal martingale measure  $Q_T^{v,*}$  are the solutions of the following equations:*

(1) *for logarithmic utility and  $f(x) = -\ln(x)$*

$$\frac{\lambda}{\sigma_1^2}(Y_t^{v,*} - 1) + \frac{\mu_1}{\sigma_1^2} + \beta_t^{v,P} - \frac{1}{Y_t^{v,*}} + 1 = 0, \quad \beta_t^{v,*} = 1 - \frac{1}{Y_t^{v,*}},$$

(2) *for exponential utility and  $f(x) = x \ln(x) - x + 1$*

$$\frac{\lambda}{\sigma_1^2}(Y_t^{v,*} - 1) + \frac{\mu_1}{\sigma_1^2} + \beta_t^{v,P} + \ln(Y_t^{v,*}) = 0, \quad \beta_t^{v,*} = \ln(Y_t^{v,*}),$$

(3) *for power utility and  $f(x) = -\frac{x^q}{q}$*

$$\frac{\lambda}{\sigma_1^2}(Y_t^{v,*} - 1) + \frac{\mu_1}{\sigma_1^2} + \beta_t^{v,P} + \frac{1}{1-q}[1 - (Y_t^{v,*})^{q-1}] = 0, \quad \beta_t^{v,*} = \frac{1}{1-q}[1 - (Y_t^{v,*})^{q-1}].$$

**Proof:** The result follows from Theorems 3,4 and 5. For that we express  $\lambda_s^v$  in terms of  $Y_s^{v,*}$ , and we replace  $b$  by  $\mu_1$ ,  $c$  and  $c_2$  by  $\sigma_1^2$ , and we incorporate the compensator of  $N$  which is equal to  $\lambda\delta_1$ , where  $\delta_1$  is delta-function at point 1. We take also in account that  $l(1) = 1$ .  $\square$

We denote by  $\hat{f}$  a new convex function related with the previous one by the relation  $\hat{f}(x) = f(x) + \frac{x^2}{2}$ . Let also  $\hat{I} = (-\hat{f}')^{-1}$  be the derivative of Fenchel-Legendre conjugate  $\hat{u}$  of  $\hat{f}$ .

**Proposition 7.** *Then we have the following expressions for  $Y^{v,*}$  :*

(1) *for logarithmic utility*

$$Y_t^{v,*} = \frac{\sigma_1}{\sqrt{\lambda}} \hat{I} \left( \frac{\sigma_1}{\sqrt{\lambda}} \left( \beta_t^{v,P} + \frac{\mu_1}{\sigma_1^2} + 1 - \frac{\lambda}{\sigma_1^2} \right) \right),$$

(2) for exponential utility

$$Y_t^{v,*} = \frac{\sigma_1^2}{\lambda} \hat{I} \left( \beta_t^{v,P} + \frac{\mu_1}{\sigma_1^2} + \ln\left(\frac{\sigma_1^2}{\lambda}\right) - \frac{\lambda}{\sigma_1^2} \right),$$

(3) for power utility

$$Y_t^{v,*} = \left( \frac{\sigma_1^2}{(1-q)\lambda} \right)^{\frac{1}{2-q}} \hat{I} \left( \left( \frac{\sigma_1^2}{(1-q)\lambda} \right)^{\frac{1-q}{2-q}} \left[ (1-q)(\beta_t^{v,P} + \frac{\mu_1}{\sigma_1^2} - \frac{\lambda}{\sigma_1^2}) + 1 \right] \right).$$

**Proof:** These formulas follows directly from previous lemma. To obtain them, it is sufficient to do scaling of  $Y$ , i.e. introduce a new function  $U$  such that  $Y = cU$ , then choose  $c$  in a way to express the l.h.s. of the equation via the function  $\hat{I}$ .  $\square$

**Proposition 8.** For the information quantities we have the following expressions:

$$\begin{aligned} \mathbf{I}(P_T^v | Q_T^v) &= \int_0^T \mathbf{E}_{P_T^v} \left[ \frac{1}{2} \sigma_1^2 (\beta_t^{v,*})^2 - \lambda (\ln Y_t^{v,*} - Y_t^{v,*} + 1) \right] dt, \\ \mathbf{I}(Q_T^v | P_T^v) &= \int_0^T \mathbf{E}_{Q_T^v} \left[ \frac{1}{2} \sigma_1^2 (\beta_t^{v,*})^2 + \lambda (Y_t^{v,*} \ln Y_t^{v,*} - Y_t^{v,*} + 1) \right] dt, \\ \mathbf{H}^{(q)}(v) &= \mathbf{E}_{R_T^v} \exp \left\{ \int_0^T \left( \frac{1}{2} (q(1-q)(\beta_t^{v,*})^2 - \lambda ((Y_t^{v,*})^q - qY_t^{v,*} + q - 1)) \right) dt \right\}. \end{aligned}$$

**Proof:** The expressions for information quantities can be obtained easily from general expressions via information processes given in Propositions 1, 2 and 3 of Section 2 .  $\square$

Finally, to obtain the maximum expected utility, we use, of course, the Theorem 2 with  $\alpha$  being  $\mathcal{N}(0, T')$ .

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