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Abstract. In this article we define intersection Floer homology for exact Lagrangian cobordisms between Legendrian submanifolds in the contactisation of a Liouville manifold, provided that the Chekanov-Eliashberg algebras of the negative ends of the cobordisms admit augmentations. From this theory we derive several long exact sequences relating the Morse homology of an exact Lagrangian cobordism with the bilinearised contact homologies of its ends. These are then used to investigate the topological properties of exact Lagrangian cobordisms.

Contents

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1. Introduction

Lagrangian cobordism is a natural relation between Legendrian submanifolds, and it is a crucial ingredient in the definition of the functorial properties of invariants of Legendrian submanifolds in the spirit of symplectic field theory as introduced by Eliashberg, Givental and Hofer in [38]. This relation is at the heart of many recent developments in the study of Legendrian submanifolds and its properties have been investigated by the authors and many others over the past years.

In the present paper we study rigidity phenomena in the topology of exact Lagrangian cobordisms in the symplectisation of the contactisation of a Liouville manifold. In [40], Eliashberg and Murphy showed that exact Lagrangian cobordisms are flexible when their negative ends are loose (in the sense of Murphy [57]). On the contrary, we will show that they become rigid if we restrict our attention to cobordisms whose negative ends admit augmentations (or more generally finite-dimensional representations) of their Chekanov-Eliashberg algebras.

In order to study the topology of such cobordisms, we introduce a version of Lagrangian Floer homology (originally defined for closed Lagrangian submanifolds

by Floer in [46]) for pairs of exact Lagrangian cobordisms. This construction finds its inspiration in the work of Ekholm in [31], which gives a symplectic field theory point of view on wrapped Floer homology of Abouzaid and Seidel from [3].

The definition of this new Floer theory requires the use of augmentations of the Chekanov-Eliashberg algebras of the negative ends as bounding cochains in order to algebraically cancel certain "bad" degenerations of the holomorphic curves at the negative ends of the cobordisms. Bounding cochains have been introduced, in the closed case, by Fukaya, Oh, Ohta and Ono in [48], while augmentations, which play a similar role in the context of Legendrian contact homology, have been introduced by Chekanov in [20].

For a pair of exact Lagrangian cobordisms obtained by a suitable small Hamiltonian push-off, our invariant gives rise to various long exact sequences relating the singular homology of the cobordism with the Legendrian contact homology of its ends. We then use these long exact sequences to give restrictions on the topology of exact Lagrangian cobordisms under various hypotheses on the topology of the Legendrian ends. In the context of generating family homology for Legendrian submanifolds in jet spaces, Sabloff and Traynor in [63] describe exact sequences similar to ours for cobordisms which admit compatible generating families.

The notion of Lagrangian cobordism between Legendrian submanifolds studied in this article is (in general) different from the notion of Lagrangian cobordisms between Lagrangian submanifolds introduced by Arnol'd in [7] and [6] and recently popularised by Biran and Cornea in [10] and [11]. Lagrangian cobordisms in the sense of Arnol'd between Lagrangian submanifolds of a symplectic manifold M are Lagrangian submanifolds of $M \times \mathbb{C}$ which project to horizontal half-lines of $\mathbb{C}$ outside of a compact set. The main difference between the two theories is that Arnol'd-type cobordisms do not distinguish between positive and negative ends and therefore are closer in spirit to the notion of cobordism in classical topology. Despite the differences, for Lagrangian cobordisms between Legendrian submanifolds with no Reeb chords, some of the results we obtain resemble some of the results obtained by Biran and Cornea [10, 11] and Suárez [66].

Remark 1.1. *In fact, under the strong assumption that the Legendrian submanifolds $\Lambda_\pm \subset (P \times \mathbb{R}, dz + \theta)$ have no Reeb chords, an exact Lagrangian cobordism from Λ_- to Λ_+ inside the symplectisation*

$$(\mathbb{R} \times P \times \mathbb{R}, d(e^t(dz + \theta))) \cong (P \times \mathbb{C}, d\theta \oplus d(xdy))$$

can be deformed to yield an exact Lagrangian cobordism between the exact Lagrangian embeddings $\Pi_{\mathrm{Lag}}(\Lambda_-), \Pi_{\mathrm{Lag}}(\Lambda_+) \subset (P, d\theta)$ in the sense of Arnol'd, and vice versa. In some sense, these two notions of cobordisms thus coincide in this case. For readers familiar with the language of [36]*, this can be explained as follows: one can go between Lagrangian cobordism in the sense studied here and so-called Morse cobordisms. The latter are embedded in the case when the Legendrian ends have no Reeb chords, and they are thus Lagrangian cobordisms in the sense of Arnol'd.*

1.1. **Main results.** Let (P, θ) be a Liouville manifold and $(Y, \alpha) := (P \times \mathbb{R}, dz + \theta)$ its contactisation. We consider a pair of exact Lagrangian embeddings $\Sigma_0, \Sigma_1 \hookrightarrow X$, where $(X, \omega) = (\mathbb{R} \times Y, d(e^t\alpha))$ is the symplectisation of (Y, α). We assume that the positive and negative ends of Σ_i $i = 0, 1$ are cylindrical over Legendrian submanifolds Λ_i^- and Λ_i^+ respectively, and thus Σ_i is a Lagrangian cobordisms from Λ_i^- to Λ_i^+; see Figure 1 for a schematic representation and Section 2 for the

precise formulation of our geometrical setup. We assume that Σ_0 and Σ_1 intersect transversely and that their Legendrian ends are chord-generic in the sense of Section 2.1. The first rigidity phenomena for Lagrangian submanifolds in this setting were proven by Gromov [51], who showed that there are no closed exact Lagrangian submanifolds in a symplectisation as above (note that they automatically would be displaceable). Differently put, this means that an exact Lagrangian cobordism must have at least one non-empty end.

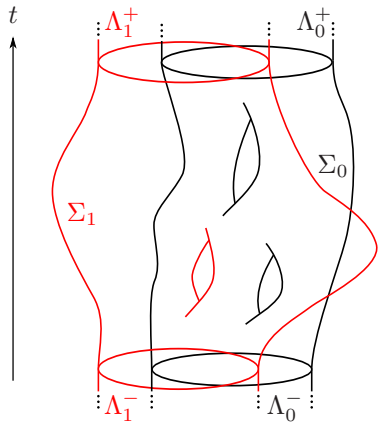

FIGURE 1. Two Lagrangian cobordisms inside a symplectisation $\mathbb{R} \times Y$, where the vertical axis corresponds to the $\mathbb{R}$-coordinate.

We denote by $\mathcal{R}(\Lambda_i^\pm)$ the set of Reeb chords of $\Lambda_i^\pm$ for $i = 0, 1$, and by $\mathcal{R}(\Lambda_1^\pm, \Lambda_0^\pm)$ the set of Reeb chords from $\Lambda_1^\pm$ to $\Lambda_0^\pm$. Let R be a field of characteristic 2 or, if all Σ_i's and $\Lambda_i^\pm$'s are (relatively) spin, any commutative ring. (See Section 1.1.1.) We denote by $C(\Lambda_0^\pm, \Lambda_1^\pm)$ the free R-module spanned by $\mathcal{R}(\Lambda_1^\pm, \Lambda_0^\pm)$.

Remark 1.2. *In fact the commutativity of R can be dropped – Chekanov's linearisation can be generalised to arbitrary rings, and our long exact sequences exist in this setting as well. See Section 11.2 for more details as well as* [18]*. However, we point out that in order to use arguments involving ranks, one must impose additional requirements on R, for example commutativity, or finite dimensionality as an algebra over a field.*

We assume that the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda_i^-; R)$ of Λ_i^- admits an augmentation ε_i^- over R for $i = 0, 1$ (see Section 5 for the definitions). It follows from the results of Ekholm, Honda and Kálmán in [36] that $\mathcal{A}(\Lambda_i^+; R)$ also admits an augmentation $\varepsilon_i^+ = \varepsilon_i^- \circ \Phi_{\Sigma_i}$, where $\Phi_{\Sigma_i} \colon \mathcal{A}(\Lambda_i^+; R) \to \mathcal{A}(\Lambda_i^-; R)$ is the unital DGA morphism induced by the cobordism Σ_i. Thus the bilinearised contact cohomologies $LCH_{\varepsilon_0^\pm, \varepsilon_1^\pm}(\Lambda_0^\pm, \Lambda_1^\pm)$ are defined. See Chekanov [20] and Bourgeois and Chantraine [12] for the notions of linearisation and bilinearisation of a differential graded algebra.

We denote by $CF(\Sigma_0, \Sigma_1)$ the free R-module spanned by the intersection points $\Sigma_0 \cap \Sigma_1$. In Section 3.3.2 we define the notion of the action of an intersection point and use it to filter $CF(\Sigma_0, \Sigma_1)$: we denote by $CF_\pm(\Sigma_0, \Sigma_1)$ the submodule of $CF(\Sigma_0, \Sigma_1)$ generated by intersection points of positive (respectively, negative) action.

The main construction in this article provides a differential on the modules $CF_{\pm}(\Sigma_0, \Sigma_1)$, leading to homology groups $HF_{\pm}(\Sigma_0, \Sigma_1)$ of Floer type. The differentials and the resulting homology groups depend on the choice of the augmentations ε_i^-. In order to define a graded theory, we need that $2c_1(X) = 0$ and that all Lagrangian cobordisms have vanishing Maslov classes. This implies that all Lagrangian cobordisms admit Maslov potentials (as defined in Section 4.2); a particular choice of such a potential leads to the notion of a *graded* Lagrangian cobordisms, for which $HF(\Sigma_0, \Sigma_1)$ has a well-defined grading in $\mathbb{Z}$. In general the grading must be taken in a (possibly trivial) cyclic group.

Our main result is the following relation between the Floer homology of a pair (Σ_0, Σ_1) of exact Lagrangian cobordisms and the bilinearised Legendrian contact homologies of their ends; see Section 9.

Theorem 1.3. *Let Σ_i, $i = 0, 1$, be a graded exact Lagrangian cobordisms from the Legendrian submanifold Λ_i^- to Λ_i^+ inside the symplectisation of the contactisation of a Liouville manifold, and assume that there are augmentations ε_i^- of $\mathcal{A}(\Lambda_i^-)$ for $i = 0, 1$. Then there exists a spectral sequence whose first page is*

$$\begin{array}{c} E_1^{3,\bullet} \oplus E_1^{2,\bullet} \oplus E_1^{1,\bullet} \oplus E_1^{0,\bullet} \\ \| \\ LCH^{\bullet-2}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \oplus HF_+^\bullet(\Sigma_0, \Sigma_1) \oplus LCH^{\bullet-1}_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda_0^-, \Lambda_1^-) \oplus HF_-^\bullet(\Sigma_0, \Sigma_1) \end{array} \tag{1}$$

and which collapses to 0 at the fourth page.

This theorem follows from the acyclicity of a complex $(\mathrm{Cth}(\Sigma_0, \Sigma_1), \mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ associated to a pair of Lagrangian cobordisms which we call the *Cthulhu complex* (see Section 6). Its underlying R-module is

$$\mathrm{Cth}(\Sigma_0, \Sigma_1) = C(\Lambda_0^+, \Lambda_1^+) \oplus CF_+(\Sigma_0, \Sigma_1) \oplus C(\Lambda_0^-, \Lambda_1^-) \oplus CF_-(\Sigma_0, \Sigma_1).$$

The spectral sequence is induced by the filtration of length four given by

$$C(\Lambda_0^+, \Lambda_1^+) > CF_+(\Sigma_0, \Sigma_1) > C(\Lambda_0^-, \Lambda_1^-) > CF_-(\Sigma_0, \Sigma_1),$$

and the acyclicity of the complex $(\mathrm{Cth}(\Sigma_0, \Sigma_1), \mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ follows from its invariance properties with respect to a large class of Hamiltonian deformations which, in the contactisation of a Liouville manifold, allow us to displace any pair of Lagrangian cobordisms.

When the negative ends are empty, this complex recovers the wrapped Floer cohomology complex as described by Ekholm in [31]. When the positive ends are empty and there are no homotopically trivial Reeb chords of both Λ_i^-'s, this complex is similar to the Floer complex sketched in the work of Akaho in [5, Section 8].

Remark 1.4. *The latter situation cannot occur in the symplectisation of a contactisation of a Liouville manifold: an exact Lagrangian cobordism with no positive end cannot have a negative end admitting an augmentation by Corollary 1.9 below. This also follows from an even stronger result due to the second author in* [23]*, where it is shown that such a Legendrian submanifold must have an acyclic Chekanov-Eliashberg algebra.*

We will introduce two classes of pairs (Σ_0, Σ_1) of exact Lagrangian cobordisms for which this filtration is of length three: directed and V-shaped pairs (see Section 9). A pair is directed when there are no intersection points of positive action, and

V-shaped when there are no intersection points of negative action (see Section 9 for more details). In these situations, the spectral sequence collapses to 0 at the third page, giving rise to the following long exact sequences.

Corollary 1.5. *Let (Σ_0, Σ_1) be a pair of exact Lagrangian cobordisms satisfying the assumptions of Theorem 1.3.*

- *If (Σ_0, Σ_1) is directed, then there exists a long exact sequence*

(2)
$$\cdots \to LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \downarrow HF^k_-(\Sigma_0, \Sigma_1) \longrightarrow LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda_0^-, \Lambda_1^-) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \to \cdots$$

- *If (Σ_0, Σ_1) is V-shaped, then there exists a long exact sequence*

(3)
$$\cdots \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \to LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda_0^-, \Lambda_1^-) \longrightarrow HF^{k+2}_+(\Sigma_0, \Sigma_1) \downarrow LCH^{k+1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \to \cdots$$

1.1.1. *Remarks about grading and orientation.* Most of the result here are stated for graded Lagrangian cobordisms. However, our methods apply in the ungraded cases as well. The only difference is that the long exact sequences in Corollary 1.5 and in Section 1.2 become exact triangles (the maps are ungraded or, alternatively, graded modulo the Maslov number).

For the results to holds using coefficients in a ring R different from a field of characteristic two, as well for the results in Section 1.3, we need to be able to define the theory using integer coefficients. In order to perform counts with signs, one has to define coherent orientations for the relevant moduli spaces of pseudoholomorphic curves. This can be done in the case when the Legendrian submanifolds and Lagrangian cobordisms are relatively pin (following Ekholm, Etnyre and Sullivan in [33] and Seidel in [64, Section 11]).

1.2. **Long exact sequences for LCH induced by a Lagrangian cobordism.** If Σ_1 is a Hamiltonian deformation of Σ_0 for some suitable and sufficiently small Hamiltonian, the Floer homology groups $HF_\pm(\Sigma_0, \Sigma_1)$ can be identified with Morse homology groups of Σ_0. Similarly, the bilinearised Legendrian contact homology groups $LCH_{\varepsilon_0^\pm,\varepsilon_1^\pm}(\Lambda_0^\pm, \Lambda_1^\pm)$ can be identified with the bilinearised contact homology groups $LCH_{\varepsilon_0^\pm,\varepsilon_1^\pm}(\Lambda_0^\pm)$ (as defined in Subsection 5.3) following [35]. Thus the long exact sequences in Corollary 1.5 can be reinterpreted as long exact sequences relating the singular homology of a Lagrangian cobordism and the Legendrian contact homology of its ends. These results are proved in Section 10.1.

Analogous long exact sequences have previously been found by Sabloff and Traynor in [63] in the setting of generating family homology under the additional assumption that the cobordism admits a compatible generating family, and by the fourth author in [49] in the case when the negative end of the cobordism admits an exact Lagrangian filling. The latter results have been put in a much more general framework in recent work by Cieliebak-Oancea [21].

In the rest of this introduction, Λ^+ and Λ^- will always denote closed Legendrian submanifolds of dimension n in the contactisation of a Liouville manifold, and every Lagrangian cobordism between them, as well as any Lagrangian filling of them, will always live in the corresponding symplectisation. We will denote by $\overline{\Sigma}$ the natural compactification of Σ obtained by adjoining its Legendrian ends $\Lambda_\pm$. Note that $\overline{\Sigma}$ is diffeomorphic to $\Sigma \cap [-T, +T] \times Y$ for some $T \gg 0$ sufficiently large. We will also use the notation $\partial_\pm \overline{\Sigma} := \Lambda_\pm \subset \overline{\Sigma}$, which implies that $\partial \overline{\Sigma} = \partial_+ \overline{\Sigma} \sqcup \partial_- \overline{\Sigma}$.

1.2.1. *A generalisation of the long exact sequence of a pair.* The first exact sequence we produce from a Lagrangian cobordism (see Section 10.1.1) is given by the following:

Theorem 1.6. *Let Σ be a graded exact Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. There is a long exact sequence*

$$\begin{array}{l} \cdots \longrightarrow LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \\ \qquad\qquad \downarrow \\ \qquad H_{n+1-k}(\overline{\Sigma}, \partial_- \overline{\Sigma}; R) \to LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to \cdots, \end{array} \tag{4}$$

where the map $\Phi_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} : LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+)$ is the adjoint of the bilinearised DGA morphism Φ_Σ induced by Σ (see Section 5.3).

When the negative end $\Lambda_- = \emptyset$ is empty, i.e. when Σ is an exact Lagrangian filling of Λ_+, and ε_i^+, $i = 0, 1$ both are augmentations induced by this filling, the resulting long exact sequence simply becomes the isomorphism

$$LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \xrightarrow{\simeq} H_{n+1-k}(\overline{\Sigma}; R)$$

appearing in the work of Ekholm in [31]. This isomorphism was first observed by Seidel, and is sometimes called *Seidel's isomorphism*. (See the map $G_\Sigma^{\varepsilon_0^-,\varepsilon_1^-}$ in Section 10.2 for another incarnation.) Its proof was completed by the second author in [25], also see [17] by the authors for an analogous isomorphism induced by a *pair* of fillings.

1.2.2. *A generalisation of the duality long exact sequence and fundamental class.* A Legendrian submanifold Λ is horizontally displaceable if there exists a Hamiltonian isotopy ϕ_t of $(P, d\theta)$ which displaces the Lagrangian projection $\Pi_{\mathrm{Lag}}(\Lambda) \subset P$ from itself. In Section 10.1.1 we obtain the following:

Theorem 1.7. *Let Σ be an exact graded Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. Assume that Λ^- is horizontally displaceable; then there is a long exact sequence*

$$\begin{array}{l} \cdots \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to LCH^{\varepsilon_0^-,\varepsilon_1^-}_{n-k-1}(\Lambda^-) \to H_{n-k-1}(\Sigma; R) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad LCH^{k+1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to \cdots, \end{array} \tag{5}$$

where the map $G_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} : H_{n-k-1}(\Sigma; R) \to LCH^{k+1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+)$ is defined in Section 10.3.

When $\Sigma = \mathbb{R} \times \Lambda$ then $H_\bullet(\Sigma) = H_\bullet(\Lambda)$, and hence the above long exact sequence recovers the duality long exact sequence for Legendrian contact homology, which was proved by Sabloff in [62] for Legendrian knots and later generalised to arbitrary Legendrian submanifolds in [35] by Ekholm, Etnyre and Sabloff. In the bilinearised setting, the duality long exact sequence was introduced by Bourgeois and the first author in [12]. In Section 11.3 we use Exact Sequence (5) to prove that the fundamental class in LCH defined by Sabloff in [62] and Ekholm, Etnyre and Sabloff in [35] is functorial with respect to the maps induced by exact Lagrangian cobordisms.

1.2.3. *A generalisation of the Mayer-Vietoris long exact sequence.* The last exact sequence that we will extract from Corollary 1.5 generalises the Mayer-Vietoris exact sequence (see Section 10.1.3).

Theorem 1.8. *Let Σ be an exact graded Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. Then there is a long exact sequence*
(6)

$$\cdots \to LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \downarrow H_{n-k}(\partial_-\overline{\Sigma};R) \to LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \oplus H_{n-k}(\overline{\Sigma};R) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to \cdots,$$

where the component

$$H_{n-k}(\partial_-\overline{\Sigma};R) \to H_{n-k}(\overline{\Sigma};R)$$

of the left map is induced by the topological inclusion of the negative end.

If $\varepsilon_0^- = \varepsilon_1^- = \varepsilon$, it moreover follows that the image of the fundamental class under the component $H_n(\partial_-\overline{\Sigma};R) \to LCH^0_{\varepsilon,\varepsilon}(\Lambda^-)$ of the above morphism vanishes. Under the additional assumption that Λ^- is horizontally displaceable, it is moreover the case that the image of a generator under $H_0(\partial_-\overline{\Sigma};R) \to LCH^n_{\varepsilon,\varepsilon}(\Lambda^-)$ is equal to the fundamental class in Legendrian contact homology.

In particular we get that the fundamental class in $H_n(\partial_-\overline{\Sigma};R)$ either is non-zero in $H_n(\overline{\Sigma})$, or is the image of a class in $LCH^{-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+)$. In both cases, $\Lambda^+ \neq \emptyset$. Thus we obtain a new proof of the following result.

Corollary 1.9 ([23])**.** *If $\Lambda \subset P \times \mathbb{R}$ admits an augmentation, then there is no exact Lagrangian cobordism from Λ to $\emptyset$, i.e. there is no exact Lagrangian "cap" of Λ.*

Remark 1.10. *Assume that Λ_- admits an exact Lagrangian filling L inside the symplectisation, and that ε^- is the augmentation induced by this filling. It follows that ε^+ is the augmentation induced by the filling $L \odot \Sigma$ of Λ_+ obtained as the concatenation of L and Σ. Using Seidel's isomorphisms*

$$LCH^k_{\varepsilon^-,\varepsilon^-}(\Lambda^-) \simeq H_{n-k}(L;R),$$
$$LCH^k_{\varepsilon^+,\varepsilon^+}(\Lambda^+) \simeq H_{n-k}(L \odot \Sigma;R)$$

to replace the relevant terms in the long exact sequences (4) and (6), we obtain the long exact sequence for the pair $(L \odot \Sigma, L)$ and the Mayer-Vietoris long exact sequence for the decomposition $L \odot \Sigma = L \cup \Sigma$, respectively. This fact was already observed and used by the fourth author in [49].

1.3. **Topological restrictions on Lagrangian cobordisms.** Using the long exact sequences from the previous subsection and their refinements to coefficients twisted by the fundamental group, as defined in Section 11, we find strong topological restrictions on exact Lagrangian cobordisms between certain classes of Legendrian submanifolds.

1.3.1. *The homology of an exact Lagrangian cobordism from a Legendrian submanifold to itself.* We recall that Λ always will denote a Lagrangian submanifold of the contactisation of a Liouville manifold. In the following results we study the homology of any exact Lagrangian cobordism from Λ to itself in the symplectisation. One of the consequences of Theorem 1.8 is the following theorem, proved in Section 12.1. A similar statement has been proven by the second and the fourth author in [26, Theorem 1.6] under the more restrictive assumption that Λ bounds an exact Lagrangian filling.

Theorem 1.11. *Let Σ be an exact Lagrangian cobordism from Λ to Λ and $\mathbb{F}$ a field (of characteristic two if Λ is not spin). If the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda;\mathbb{F})$ admits an augmentation, then:*

(i) There is an equality $\dim_{\mathbb{F}} H_\bullet(\Sigma;\mathbb{F}) = \dim_{\mathbb{F}} H_\bullet(\Lambda;\mathbb{F})$;

(ii) The map

$$(i_*^-, i_*^+) : H_\bullet(\Lambda;\mathbb{F}) \to H_\bullet(\Sigma;\mathbb{F}) \oplus H_\bullet(\Sigma;\mathbb{F})$$

is injective; and

(iii) The map

$$i_*^+ \oplus i_*^- : H_\bullet(\Lambda \sqcup \Lambda) \to H_\bullet(\Sigma)$$

is surjective.

Here i^+ is the inclusion of Λ as the positive end of Σ, while i^- is the inclusion of Λ as the negative end of Σ.

Remark 1.12. *The above equalities hold for the $\mathbb{Z}$-graded singular homology groups without assuming that the cobordism Σ is graded.*

An immediate corollary of Theorem 1.6 is the following result, which had already appeared in [26, Theorem 1.7] under the stronger assumption that the negative end is fillable.

Theorem 1.13. *If Λ is a homology sphere which admits an augmentation over $\mathbb{Z}$, then any exact Lagrangian cobordism Σ from Λ to itself is a homology cylinder (i.e. $H_\bullet(\Sigma,\Lambda) = 0$).*

Inspired by the work of Capovilla-Searle and Traynor [14], in Section 12.2 we prove the following restriction on the characteristic classes of an exact Lagrangian cobordism from a Legendrian submanifold to itself. Given a manifold M, we denote by $w_i(M)$ the i-th Stiefel-Whitney class of TM.

Theorem 1.14. *Let Σ be an exact Lagrangian cobordism from Λ to itself, and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Assume that $\mathcal{A}(\Lambda;\mathbb{F})$ admits an augmentation. If, for some $i \in \mathbb{N}$, $w_i(\Lambda) = 0$, then $w_i(\Sigma) = 0$.*

If Λ is spin, the same holds for the Pontryagin classes.

By specialising to w_1 we obtain the following corollary, which extends the main result in [14]; in particular we partially answer Question 6.1 of the same article.

Corollary 1.15. *If Λ is an orientable Legendrian submanifold admitting an augmentation, then any exact Lagrangian cobordism from Λ to itself is orientable.*

1.3.2. *Restrictions on the fundamental group of certain exact Lagrangian fillings and cobordisms.* Since Theorem 1.13 shows that an exact Lagrangian cobordism from a Legendrian homology sphere to itself is a homology cylinder, it is natural to ask under what conditions this cobordism in fact is an h-cobordism. We therefore need to incorporate the fundamental group in our constructions. To that end, following ideas of Sullivan in [67] and Damian in [22], we define a "twisted" version of the Floer homology groups $HF_{\pm}(\Sigma_0, \Sigma_1)$ with coefficient ring $R[\pi_1(\Sigma, *)]$ in Section 11.1. We also establish a result analogous to Theorem 1.5, as well as the long exact sequences in Section 1.2, with twisted coefficients in $R[\pi_1(\Sigma, *)]$. In the setting of Legendrian contact homology, these techniques were introduced by Eriksson-Östman in [42].

Using generalisations of the long exact sequence from Theorem 1.6 and the functoriality of the fundamental class from Proposition 11.7 (see Section 12.3.1) we prove the following theorem:

Theorem 1.16. *Let Σ be a graded exact Lagrangian cobordism from Λ^- to Λ^+. Assume that $\mathcal{A}(\Lambda^-; R)$ admits an augmentation and that Λ^+ has no Reeb chords in degree zero. If Λ^- and Λ^+ both are simply connected, then Σ is simply connected as well.*

Remark 1.17. *The seemingly unnatural condition that Λ_+ has no Reeb chords in degree zero is used to ensure that the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda^+; A)$ has* at most *one augmentation in A for every unital R-algebra A. (This algebraic condition does not ensure the* existence *of an augmentation, but rather it states that Λ^+ admits exactly one augmentation in the case when Λ^- admits an augmentation.) This condition is clearly not invariant under Legendrian isotopy, but the conclusion of Theorem 1.16 can be extended to every Legendrian submanifold which is Legendrian isotopic to Λ^+ because Legendrian isotopies induce Lagrangian cylinders by* [39, 4.2.5] *(also, see* [15]*). Ideally, one should replace the algebraic condition with one that only depends on the DGA homotopy type.*

We now present another result which imposes constraints on the fundamental group of an exact Lagrangian cobordism from a Legendrian submanifold to itself (see Section 12.3.2). Its proof uses an L^2-completion of the Floer homology groups with twisted coefficients and the L^2-Betti numbers of the universal cover (using results of Cheeger and Gromov in [19]).

Theorem 1.18. *Let Λ be a simply connected Legendrian submanifold which is spin, and let Σ be an exact Lagrangian cobordism from Λ to itself. If $\mathcal{A}(\Lambda; \mathbb{C})$ admits an augmentation, then Σ is simply connected as well.*

Combining Theorem 1.13 with Theorem 1.18, we get the following result.

Corollary 1.19. *Let Σ be an n-dimensional Legendrian homotopy sphere and assume that $\mathcal{A}(\Lambda; \mathbb{Z})$ admits an augmentation. Then any exact Lagrangian cobordism Σ from Λ to itself is an h-cobordism. In particular:*

(1) If $n \neq 3, 4$, then Σ is diffeomorphic to a cylinder;
(2) If $n = 3$, then Σ is homeomorphic to a cylinder; and
(3) If $n = 4$ and Λ is diffeomorphic to S^4, then Σ is diffeomorphic to a cylinder.

When $n = 1$, a stronger result is known. Namely, in [17, Section 4] we proved that any exact Lagrangian cobordism Σ from the standard Legendrian unknot Λ_0 to itself is compactly supported Hamiltonian isotopic to the trace of a Legendrian isotopy of Λ_0 which is induced by the complexification of a rotation by $k\pi$, $k \in \mathbb{Z}$. This classification makes use of the uniqueness of the exact Lagrangian filling of Λ_0 up to compactly supported Hamiltonian isotopy, which was proved in [41] by Eliashberg and Polterovich. In contrast, the methods we develop in this article give restrictions only on the **smooth** type of the cobordisms and little information is known about their symplectic knottedness in higher dimension.

1.3.3. *Obstructions to the existence of a Lagrangian concordance.* A *Lagrangian concordance* from Λ^- to Λ^+ is a symplectic cobordism from Λ^- to Λ^+ which is diffeomorphic to a product. In particular this implies that Λ^- and Λ^+ are diffeomorphic as smooth manifolds. Note that a Lagrangian concordance automatically is exact.

If Σ is a Lagrangian concordance, then $H_\bullet(\overline{\Sigma}, \partial_- \overline{\Sigma}; R) = 0$, and thus Theorem 1.6 implies the following corollary.

Corollary 1.20. *Let Σ be an exact Lagrangian concordance from Λ^- to Λ^+. If, for $i = 0, 1$, ε_i^- is an augmentation of $\mathcal{A}(\Lambda^-; R)$ and ε_i^+ is the pull-back of ε_i^- under the DGA morphism induced by Σ, then the map*

$$\Phi_\Sigma^{\varepsilon_0^-, \varepsilon_1^-} : LCH^\bullet_{\varepsilon_0^-, \varepsilon_1^-}(\Lambda^-) \to LCH^\bullet_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+)$$

is an isomorphism. Consequently, there is an inclusion

$$\{LCH^\bullet_{\varepsilon_0^-, \varepsilon_1^-}(\Lambda^-)\}/\mathit{isom.} \hookrightarrow \{LCH^\bullet_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+)\}/\mathit{isom.}$$

of the sets consisting of isomorphism classes of bilinearised Legendrian contact cohomologies, for all possible pairs of augmentations.

This corollary can be used to obstruct the existence of Lagrangian concordances. For example, it can be applied to the computation of the linearised Legendrian contact homologies given by Chekanov in [20, Theorem 5.8] to prove that there is no exact Lagrangian concordance from either of the two Chekanov-Eliashberg knots to the other. We also use Corollary 1.20 to deduce new examples of non-symmetric concordances in the spirit of the example given by the first author in [16]. We refer to Section 12.4.3 for a simply connected example in high dimensions.

We recall that a Legendrian isotopy induces a Lagrangian concordance. Since Legendrian isotopies are invertible, two isotopic Legendrian submanifolds thus admit Lagrangian concordances going in either direction. On the other hand, we have now many examples of non-symmetric Lagrangian concordances, and hence the following natural question can be asked.

Question 1.21. Assume that there exists Lagrangian concordances from Λ_0 to Λ_1 as well as from Λ_1 to Λ_0. Does this imply that the Legendrian submanifolds Λ_0 and Λ_1 are Legendrian isotopic? Are such Lagrangian concordances moreover Hamiltonian isotopic to one induced by a Legendrian isotopy (as constructed by [39, 4.2.5])?

We argue that this question will not be easily answered by Legendrian contact homology. Chekanov showed in [20] that the set of isomorphism classes of all linearised Legendrian contact homology groups is invariant under Legendrian isotopy. Later

Bourgeois and the first author in [12] extended this result to *bilinearised* Legendrian contact homology. However, Corollary 1.20 says that bilinearised Legendrian contact homology in fact is an invariant of Lagrangian concordances, rather than of Legendrian isotopies. This means that, every time two Legendrian submanifolds have been proved not to be Legendrian isotopic by exhibiting two non-isomorphic bilinearised Legendrian contact homology groups, what has in fact been proved is that there cannot exist Lagrangian concordances between the Legendrian submanifolds going both directions.

1.4. **Remarks about the hypotheses.**

1.4.1. *Restrictions on the ambient manifolds.* The reasons for restricting our attention to Lagrangian cobordisms in the symplectisation of the contactisation of a Liouville manifold are two-fold. First, the analytic framework to have a well defined complex $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ is vastly simplified from the fact that the Reeb flow has no periodic Reeb orbits. Using recent work of Pardon in [60] (or the polyfold technology being developed by Hofer, Wysocki and Zehnder), it is possible to extend the construction of the complex $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ to more general symplectic cobordisms. Second, our applications use exact sequences arising from the acyclicity of the complex $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$, which is a consequence of the fact that that any Lagrangian cobordism can be displaced in the symplectisation of a contactisation. Floer theory for Lagrangian cobordisms in more general symplectic cobordisms will be investigated in a future article.

1.4.2. *Restrictions on the Lagrangian submanifolds.* Now we describe some examples showing that many of the hypotheses we made in Subsection 1.3 are in fact essential, and not merely artefacts of the techniques used. First, an exact Lagrangian cobordism having a negative end whose Chekanov-Eliashberg algebra admits no augmentation can be a quite flexible object: in fact Eliashberg and Murphy proved in [40] that exact Lagrangian cobordisms with a loose negative end satisfy an h-principle, and therefore one cannot hope for a result in the spirit of Theorem 1.13 to hold in complete generality. Indeed, we refer to the work of the second and fourth authors in [26] for examples of exact Lagrangian cobordisms from a loose Legendrian sphere to itself having arbitrarily big Betti numbers.

Second, the condition that Λ is a homology sphere in the statement of Theorem 1.13 was shown to be essential already in [26, Section 2.3].

Finally, the importance of the condition on the Reeb chords of the positive end in Theorem 1.16 is emphasised by the following example, which will be detailed in Section 12.4.

Proposition 1.22. *There exists a non-simply connected exact Lagrangian cobordism from the two-dimensional standard Legendrian sphere to a Legendrian sphere inside the symplectisation of standard contact* $(\mathbb{R}^5,\xi_{\mathrm{std}})$.

As a converse to Theorem 1.16, the existence of a non-simply connected exact Lagrangian cobordism can be used to show the existence of degree zero Reeb chords on the positive end of the cobordism.

1.5. **Outline of the article.** This article is organised as follows. In sections from 2 to 5 we collect some basic material: in Section 2 we describe our geometric setup and give a precise definition of the class of Lagrangian cobordisms that we consider,

in Section 3 we introduce the moduli spaces used to define $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}$ and discuss their compactification, in Section 4 we study their Fredholm theory (e.g. indices and transversality), and in Section 5 we briefly review the definitions of Legendrian contact homology and augmentations.

Sections 6 to 8 are the theoretical core of the paper: in Section 6 we define the complex $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$, in Section 7 we show how $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ behaves under concatenations of cobordisms, and in Section 8 we show that the complex $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ is acyclic.

The last part of the article is devoted to applications. In Section 9 we prove Theorem 1.3 and in Section 10 we deduce the long exact sequences of Section 1.2. In Section 11 we lift the coefficient ring to the group ring $R[\pi_1(\Sigma)]$ and to its L^2-completion. Finally, in Section 12 we prove the results concerning the topology of Lagrangian cobordisms stated in Section 1.3, and then describe examples which exhibit the necessity of their hypotheses.

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2. Geometric preliminaries

2.1. **Basic definitions.** A *contact manifold* (Y,ξ) is a $(2n+1)$-manifold Y equipped with a smooth maximally non-integrable field of hyperplanes $\xi \subset TY$, which is called a *contact structure*. Non-integrability implies that locally ξ can be written as the kernel of a 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. We will be interested only in *coorientable* contact structures, which are *globally* the kernel of a 1-form α called a *contact form*. A contact form α defines a canonical vector field R_α, called the *Reeb vector field*, via the equations

$$\begin{cases} i_{R_\alpha} d\alpha = 0, \\ \alpha(R_\alpha) = 1. \end{cases}$$

We will use $\phi^t\colon (Y,\alpha) \to (Y,\alpha)$ to denote the flow of the Reeb vector field R_α, which can be seen to preserve α. Also, in the following we will always assume the contact form to be fixed.

An n-dimensional submanifold $\Lambda \subset Y$ which is everywhere tangent to ξ is called *Legendrian*. The *Reeb chords* of Λ are the trajectories of the Reeb flow starting and ending on Λ. We denote the set of the Reeb chords of Λ by $\mathcal{R}(\Lambda)$.

Let γ be a periodic orbit of R_α of length T. It is *non-degenerate* if $d\phi_q^T - \mathrm{Id}$ is invertible for one (and thus all) $q \in \gamma$. Let γ be a Reeb chord of length T. The flow ϕ^t of the Reeb vector field preserves ξ, and therefore $d\phi^T_{\gamma(0)}(T_{\gamma(0)}\Lambda) \subset \xi_{\gamma(T)}$. We say that a Reeb chord γ is *non-degenerate* if $d\phi^T_{\gamma(0)}(T_{\gamma(0)}\Lambda)$ is transverse to $T_{\gamma(T)}\Lambda$ in $\xi_{\gamma(T)}$. We say that Λ is *chord-generic* if all its Reeb chords are non-degenerate. From now on we assume that all Legendrian submanifolds are chord-generic. This is not a restrictive assumption because chord genericity is a property which can be achieved by a generic Legendrian perturbation of Λ, provided that all periodic Reeb orbits are non-degenerate.

We will here restrict ourselves to the case when (Y, α) is the contactisation of a Liouville manifold. We recall that a *Liouville manifold* is a pair (P, θ), where P is a $2n$-dimensional open manifold and θ is a one-form on P such that $d\theta$ is symplectic. The Liouville vector filed v, which is defined by the equation

$$\iota_v d\theta = \theta,$$

is moreover required to be a pseudo-gradient for an exhausting function $f\colon P \to \mathbb{R}_{\geq 0}$ outside of a compact set. For simplicity we will assume that f has a finite number of critical points. We define the *contactisation* (Y, α) of (P, θ) to be $Y = P \times \mathbb{R}$ and $\alpha := dz + \theta$, where z is the coordinate on the $\mathbb{R}$-factor. Note that in this case $R_\alpha = \partial_z$ and then there are no periodic Reeb orbits. This implies, in particular, that if Λ is a chord-generic closed Legendrian submanifold, then $|\mathcal{R}(\Lambda)| < \infty$. The prototypical example of contactisation of a Liouville manifold is the standard contact structure on $\mathbb{R}^{2n+1}$ defined by the contact form

$$\alpha_0 = dz - \sum_{i=1}^{n} y_i dx_i.$$

There is a natural projection $\Pi_{\mathrm{Lag}} : P \times \mathbb{R} \to P$ defined by $\Pi_{\mathrm{Lag}}(x, z) := x$ which is called the *Lagrangian projection*. Given a Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$, $\Pi_{\mathrm{Lag}}|_\Lambda : \Lambda \to P$ is an exact Lagrangian immersion. In this situation, there is a one-to-one correspondence between the Reeb chords of Λ and the double points of $\Pi_{\mathrm{Lag}}(\Lambda)$. Furthermore, Λ is chord-generic if and only if the only self-intersections of $\Pi_{\mathrm{Lag}}(\Lambda)$ are transverse double points.

2.2. **Lagrangian cobordisms.** The main object of study in this article are exact Lagrangian cobordisms in the symplectisation of a contactisation. Recall that, for a general contact manifold (Y, α), its *symplectisation* is the exact symplectic manifold

$$(X, \omega) := (\mathbb{R} \times Y, d(e^t \alpha)),$$

where t denotes the standard coordinate on the $\mathbb{R}$-factor. In the case when $\dim Y = 2n + 1$, a $(n + 1)$-dimensional submanifold of the above symplectisation is *exact Lagrangian* if the pull-back of the one-form $e^t \alpha$ is exact.

Definition 2.1. Let Λ^- and Λ^+ be two closed Legendrian submanifolds of (Y, α). An *exact Lagrangian cobordism from* Λ^- *to* Λ^+ in $(\mathbb{R}\times Y, d(e^t\alpha))$ is a properly embedded submanifold $\Sigma \subset \mathbb{R} \times Y$ without boundary satisfying the following conditions:

(1) for some $T \gg 0$,
   (a) $\Sigma \cap ((-\infty, -T) \times Y) = (-\infty, -T) \times \Lambda^-$,
   (b) $\Sigma \cap ((T, +\infty) \times Y) = (T, +\infty) \times \Lambda^+$, and
   (c) $\Sigma \cap ([-T, T] \times Y)$ is compact;
(2) There exists a smooth function $f_\Sigma : \Sigma \to \mathbb{R}$ for which
   (a) $e^t \alpha|_{T\Sigma} = df_\Sigma$,
   (b) $f_\Sigma|_{(-\infty,-T)\times\Lambda^-}$ is constant, and
   (c) $f_\Sigma|_{(T,\infty)\times\Lambda^+}$ is constant.

We will call $(T, +\infty) \times \Lambda_+ \subset \Sigma$ and $(-\infty, -T) \times \Lambda_- \subset \Sigma$ the *positive end* and the *negative end* of Σ, respectively. We will call a cobordism from a submanifold to itself an *endocobordism*.

Conditions (2b) and (2c) are equivalent to saying that for any smooth paths $\gamma_- : ([0, 1], \{0, 1\}) \to (\Sigma, (-\infty, -T) \times \Lambda^-)$ and $\gamma_+ : ([0, 1], \{0, 1\}) \to (\Sigma, (T, \infty) \times \Lambda^+)$,

we have $\int_{\gamma_\pm} e^t\alpha = 0$. Condition (2b) will later be used to rule out certain bad breakings of pseudoholomorphic curves. Condition (2c) is used to ensure that the concatenation of two exact Lagrangian cobordisms (as in Defintion 2.3) still is an exact Lagrangian cobordism. If one does not care about concatenations, then this condition can be dropped.

Example 2.2. If Λ is a closed Legendrian submanifold of (Y,ξ), then $\mathbb{R}\times\Lambda$ is an exact Lagrangian cobordism inside $(\mathbb{R}\times Y, d(e^t\alpha))$ from Λ to itself. Cobordisms of this type are called *(trivial) Lagrangian cylinders*.

In the case when there exists an exact Lagrangian cobordism from Λ^- to Λ^+ we say that Λ^- *is exact Lagrangian cobordant to* Λ^+. If Σ is an exact Lagrangian cobordism from the empty set to Λ, we call Σ an *exact Lagrangian filling* of Λ. In the latter case we also say that Λ is *exactly fillable*.

The group $\mathbb{R}$ acts on $\mathbb{R}\times Y$ by translations in the first factor. For any $s\in\mathbb{R}$ we define

$$\tau_s\colon \mathbb{R}\times Y\to\mathbb{R}\times Y,$$
$$\tau_s(t,p) = (t+s,p).$$

It is easy to check that the translate of an exact Lagrangian cobordism still is an exact Lagrangian cobordism.

Definition 2.3. Given exact Lagrangian cobordisms Σ_a from Λ^- to Λ and Σ_b from Λ to Λ^+, their *concatenation* $\Sigma_a\odot\Sigma_b$ is defined as follows.

First, translate Σ_a and Σ_b so that

$$\Sigma_a\cap((-1,+\infty)\times Y) = (-1,+\infty)\times\Lambda,$$
$$\Sigma_b\cap((-\infty,1)\times Y) = (-\infty,1)\times\Lambda.$$

Then we define

$$\Sigma_a\odot\Sigma_b := (\Sigma_a\cap((-\infty,0]\times Y))\cup(\Sigma_b\cap([0,+\infty)\times Y)).$$

Conditions (2b) and (2c) of Definition 2.1 imply that $\Sigma_a\odot\Sigma_b$ is an exact Lagrangian cobordism from Λ^- to Λ^+.

Lemma 2.4. *The compactly supported Hamiltonian isotopy class of* $\Sigma_a\odot\Sigma_b$ *is independent of the above choices of translations.*

Proof. In order to prove the lemma, it is enough to show that any translation of an exact Lagrangian cobordism can be realised by a compactly supported Hamiltonian isotopy. It is a standard fact that a smooth isotopy of exact Lagrangian submanifolds can be realised by a Hamiltonian isotopy. In this case, the exact Lagrangian submanifolds considered are obviously non-compact and translations act on them in a non compactly supported way. However, given an exact symplectic cobordism $\Sigma\subset\mathbb{R}\times Y$ and $S\geq 0$, we can find a compactly supported smooth isotopy $\psi_s\colon\mathbb{R}\times Y\to\mathbb{R}\times Y$ such that $\psi_s(\Sigma) = \tau_s(\Sigma)$ for all $s\in[-S,S]$. The isotopy ψ_s can be defined by integrating the vector field $\widetilde{\chi}\partial_t$, where $\widetilde{\chi}\colon\mathbb{R}\times Y\to\mathbb{R}$ is the pull-back of a bump function $\chi\colon\mathbb{R}\to\mathbb{R}$ such that $\chi(t)=1$ for $t\in[-S-T,S+T]$, while $\chi(t)=0$ for $t\notin[-S-T-1,S+T+1]$. (Here $T\geq 0$ is the constant in Definition 2.1.) □

2.3. **Deligne-Mumford spaces and labels.**

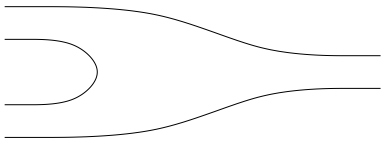

FIGURE 2. A surface with strip-like ends. The incoming end is on the right, the outgoing ends are on the left.

2.3.1. *Universal families of pointed discs.* For fixed $d \in \mathbb{N}$, we denote by $\mathcal{R}^{d+1}$ the moduli space of Riemann discs having a number $d+1$ of ordered marked points on its boundary. One realisation of $\mathcal{R}^{d+1}$ is given by

$$\mathcal{R}^{d+1} = \left\{(a_0, \dots, a_d) : a_j = e^{i\theta_j} \text{ with } \theta_0 < \dots < \theta_d < \theta_0 + 2\pi\right\} / \operatorname{Aut}(D^2),$$

where the action of $\operatorname{Aut}(D^2)$ is defined as $f \cdot (a_0, \dots, a_d) := (f(a_0), \dots, f(a_d))$ for $f \in \operatorname{Aut}(D^2)$.

We denote by $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ the universal curve over $\mathcal{R}^{d+1}$. One of its realisations is

$$\mathcal{S}^{d+1} = \left\{(z, a_0, \dots, a_d) : z \in D^2,\, a_j = e^{i\theta_j}, \theta_0 < \dots < \theta_d < \theta_0 + 2\pi\right\} / \operatorname{Aut}(D^2),$$

where the action of $\operatorname{Aut}(D^2)$ is defined as $f \cdot (z, a_0, \dots, a_d) := (f(z), f(a_0), \dots, f(a_d))$ for $f \in \operatorname{Aut}(D^2)$. The map $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ is induced by the projection

$$(z, a_0, \dots, a_d) \mapsto (a_0, \dots, a_d),$$

and therefore is a fibre bundle with $d+1$ canonical sections $\sigma_0, \dots, \sigma_d$,

$$\sigma_i \colon \mathcal{R}^{d+1} \to \mathcal{S}^{d+1},$$

where σ_i is induced by $(a_0, \dots, a_d) \mapsto (a_i, a_0, \dots, a_d)$. By an abuse of notation, when r is clear from the context, we will denote $\sigma_i(r) = a_i$.

For $r \in \mathcal{R}^{d+1}$, we denote by $\overline{S}_r \subset \mathcal{S}^{d+1}$ its preimage under the projection and write $S_r = \overline{S}_r \setminus \{a_0, \dots, a_d\}$. The connected components of ∂S_r are oriented arcs, and we let $\partial_i S_r$, $i = 0, \dots, d$, denote the arc whose closure has a_i as a starting point.

2.3.2. *Strip-like ends.* We denote by Z^+ and Z^- the Riemann surfaces $(0, +\infty) \times [0,1]$ and $(-\infty, 0) \times [0,1]$ respectively, with coordinates (s,t) and conformal structure induced by the complex coordinate $s + it$.

A *universal choice of strip-like ends* is given by $d+1$ disjoint neighbourhoods ν_i of the images of the sections σ_i together with identifications

$$\varepsilon_0 : \mathcal{R}^{d+1} \times Z^+ \to \nu_0 \setminus \sigma_0 \quad \text{and} \quad \varepsilon_i : \mathcal{R}^{d+1} \times Z^- \to \nu_i \setminus \sigma_i, \text{ for } i \geq 1,$$

such that for each $r \in \mathcal{R}^{d+1}$ we have:

- The maps $\varepsilon_i|_{\{r\} \times Z^\pm} : \{r\} \times Z^\pm \to \nu_i \cap S_r$ are holomorphic,
- $\lim_{s \to \pm\infty} \varepsilon_i(s,t) = a_i$, and

The point a_0 is called an *incoming end*, all the other a_i's are called *outgoing ends* (see Remark 2.5). See Figure 2.

We still denote by (s,t) the coordinates on the strip-like ends $S_r \cap (\nu_i \setminus \sigma_i)$ induced by the identifications ε_i and the coordinates (s,t) on $Z^\pm$. In [64, Section 9a], it is shown that such universal choices of strip-like ends exist.

2.3.3. *Deligne-Mumford compactification of $\mathcal{R}^{d+1}$.* In this Section, we describe a compactification of the space $\mathcal{R}^{d+1}$ into a manifold with corners $\overline{\mathcal{R}^{d+1}}$. We describe the faces of this compactification using the language of stable trees following [64, Section 9]. A *stable rooted tree* is a tree with one distinguished exterior vertex (i.e. of valence one), called the *root*, and whose interior vertices all have valence at least three. Exterior vertices other than the root are called *leaves*.

For a stable rooted tree with d leaves T, we denote by $\mathcal{R}^T$ the product $\Pi_{v\in \mathrm{Ve}(T)}\mathcal{R}^{|v|}$, where $\mathrm{Ve}(T)$ is the set of interior vertices and $|v|$ denotes the valence of the vertex v, which is at least 3 by the stability condition.

Remark 2.5. *The root induces a natural orientation of each edges of a rooted tree T by the following convention: the edge at the root is oriented so that it* leaves *the root. And at each other vertex there is exactly one incoming edge.*

If a stable rooted tree T' is obtained from the stable rooted tree T by collapsing k edges, in [64, Sections 9e and 9f] there is a description of a gluing map

$$\gamma^{T,T'} : \mathcal{R}^T \times (-1,0)^k \to \mathcal{R}^{T'}. \tag{7}$$

Naively, for an element $(r_v)_{v\in \mathrm{Ve}(T)}$ of $\mathcal{R}^T$ and a k-tuple (l_e) of positive numbers indexed by the collapsed edges, gluing is performed by cutting the strip-like ends of $S_{r_{v_1}}$ and $S_{r_{v_2}}$ corresponding to a collapsed edge e from v_1 to v_2 at time $|t| = e^{-\pi l_e}$, identifying the remaining part of the strip-like ends, and then uniformising the resulting disc.

Remark 2.6. *Actually,* [64] *defines only the map γ^{T,T_d}, where T_d is the tree with only one interior vertex, which corresponds to $\mathcal{R}^{d+1}$.*

The gluing maps satisfy the cocycle relation:

$$\begin{aligned}\gamma^{T',T''}(\gamma^{T,T'}(\{r_v\},\rho_1,\dots,\rho_{k_1}),\rho_{k_1+1},\dots,\rho_{k_1+k_2}) \\ = \quad \gamma^{T,T''}(\{r_v\},\rho_1,\dots,\rho_{k_1+k_2}).\end{aligned} \tag{8}$$

We define $\overline{\mathcal{R}^{d+1}} = \sqcup_T \mathcal{R}^T$ (as a set). The gluing maps in Equation (7) allow us to build an atlas on $\overline{\mathcal{R}^{d+1}}$, and therefore one can prove the following.

Lemma 2.7. [64, Lemma 9.2] *The set $\overline{\mathcal{R}^{d+1}}$ has a structure of manifold with corners which is induced by the gluing maps* (7)*. For every stable rooted tree T with d leaves, $\mathcal{R}^T$ is a stratum of the stratification of $\overline{\mathcal{R}^{d+1}}$ defined as follows: if T' is obtained from T by collapsing some interior edges, then $\mathcal{R}^T$ is a face of the compactification of $\mathcal{R}^{T'}$.*

Elements of $\overline{\mathcal{R}^{d+1}}$ are called *nodal stable $(d+1)$-punctured discs.*

Remark 2.8. *Note that the definition of the maps $\gamma^{T,T'}$ depends on the choice of strip-like ends for all $\mathcal{R}^{d'}$ for $d' < d$. However, the structure of a manifold with corners on $\overline{\mathcal{R}^{d+1}}$ turns out to be independent of these choices.*

An element $\{r_v\}$ of $\mathcal{R}^T$ is called a *nodal stable $(d+1)$-punctured disc.* In this situation, marked points of r_v corresponding to interior edges are called *boundary nodes*, while those corresponding to edges connected to leaves or to the root are called *ends*. A nodal stable $(d+1)$-punctured disc in $\mathcal{R}^{T'}$ which is in the image of a map $\gamma^{T,T'}$ inherits two strip-like ends: one coming from the universal choices of strip-like ends in the moduli spaces $\mathcal{R}^{|v|}$ for $v \in \mathrm{Ve}(T')$, and one coming from

the universal choices of strip-like ends in the moduli spaces $\mathcal{R}^{|v|}$ for $v \in \mathrm{Ve}(T)$. If there exists an $\epsilon > 0$ such that, for all gluing parameters smaller than ϵ and for all nodal stable $(d+1)$-punctured discs, those two strip-like ends agree, then we call the universal choices of strip-like ends in the moduli spaces $\mathcal{R}^{d+1}$ *consistent*. Consistent universal choices of strip-like ends exist because the space of strip-like ends is contractible, and the relation (8) allows an inductive argument. From now on, we assume that such consistent choices have been made.

2.3.4. *Asymptotics labels for Deligne-Mumford spaces.* In the present paper we will consider various moduli spaces of punctured holomorphic discs whose punctures are asymptotic to either Reeb chords or intersection points between Lagrangian cobordisms. We therefore make the following definitions.

Let Σ_0 and Σ_1 be two Lagrangian cobordisms from Legendrian submanifolds Λ_0^- and Λ_1^- at the negative ends to Legendrian submanifolds Λ_0^+ and Λ_1^+ at the positive ends, respectively. A *Lagrangian label* $\underline{L}$ for $\mathcal{R}^{d+1}$ is a map from $\mathbb{Z}/(d+1)\mathbb{Z}$ to $\{\Sigma_0, \Sigma_1\}$. If $\underline{L}(i-1) \neq \underline{L}(i)$, then the marked point a_i for $i \in \{0, \dots, d\}$ is called a *jump* of $\underline{L}$. Note that a Lagrangian label has an even number of jumps.

A Lagrangian label is *simple* if it satisfies the following conditions:

- If a_0 is not a jump, then none of the a_i's are jumps.
- It has at most two jumps.
- If a_0 is a jump, then $\underline{L}(0) = \Sigma_0$.

Remark 2.9. *A Lagrangian label for $\mathcal{R}^{d+1}$ induces compatible Lagrangian labels for all faces of $\overline{\mathcal{R}^{d+1}}$. If the Lagrangian label is simple, then the induced labels are also simple.*

The set of *asymptotics* of a given pair of Lagrangian cobordisms Σ_0, Σ_1 is the union

$$A(\Sigma_0, \Sigma_1) := (\Sigma_0 \cap \Sigma_1) \cup \mathcal{R}(\Lambda_0^+ \sqcup \Lambda_1^+) \cup \mathcal{R}(\Lambda_0^- \sqcup \Lambda_1^-).$$

The main definition in this section is the following.

Definition 2.10.

- An *asymptotics label* for $\mathcal{R}^{d+1}$ is an assignment of an asymptotic in $A(\Sigma_0, \Sigma_1)$ to each section σ_i, $i = 0, \dots, d$ (thus inducing asymptotics for each marked point a_i given any $r \in \mathcal{R}^{d+1}$).
- A Lagrangian label is *compatible* with a given asymptotics label if:
  - Every marked point asymptotic to a double point is a jump;
  - In the case when the asymptotic of a_i is a Reeb chord starting on $\Lambda_j^\pm$ and ending on $\Lambda_k^\pm$ (here $\{j,k\} \subset \{0,1\}$), it follows that $\underline{L}(i-1) = \Sigma_j$ while $\underline{L}(i) = \Sigma_k$.

Remark 2.11. *It is an obvious fact that there can be at most one* simple *Lagrangian label compatible with a given asymptotics label, thus we will always only specify a set of asymptotics assuming that the implicit compatible label is chosen.*

2.3.5. *Semi-stable nodal discs.* In the paper we will also consider more general domains called semi stable whose combinatorics is still described by decorated trees similarly as in Section 2.3.3, but possibly containing semi-stable vertices. We use the convention that $\mathcal{R}^{d+1}$ consists of a single point when $d = 0, 1$. Elements of $\mathcal{S}^{0+1}$ or $\mathcal{S}^{1+1}$ is called a *semi-stable* punctured disc. In other word a punctured disc is semi-stable if it has either one or two punctures on the boundary. After removing the punctures, they can be uniformised either as the *strip* $Z = \mathbb{R} \times [0,1]$ or the

half-plane $H = \{\operatorname{im} z > 0\} \in \mathbb{C}$ with their standard Riemann structures. We call (s,t) the global coordinates on Z, and we require that any strip-like end on Z is a restriction of these coordinates.

A *semi-stable rooted tree with d leaves* is a tree T with one chosen valence one vertex (the root) and d chosen valence one vertices (the leaves). All other vertices, including the remaining valence one vertices, will be called *internal vertices*.

The *stabilisation* of T, denoted $S(T)$, is the stable tree obtained by performing the following two operations:

- promoting valence one interior vertices to leaves, and
- suppressing all vertices of valence 2 and merging the corresponding edges.

Note that the interior vertices of $S(T)$ are in natural bijection with the vertices of T of valence at least three and, therefore, $S(T)$ is empty in the case when every vertex of T has valence one or two.

To any semi-stable tree T together with an element of $\mathcal{R}^{S(T)}$, we associate a punctured nodal disc as follows.

- To any interior vertex of T we associate a disc;
- For any edge of T we put a boundary node between the discs corresponding to the vertices joined by the edge; and
- For any exterior vertex we put a marked point in the disc associated to the closest interior vertex.

The element of $\mathcal{R}^{S(T)}$ specifies a conformal structure for any disc associated to a vertex of valence at least three (there are no moduli for discs with one or two punctures). Finally, the boundary marked points associated to the root will become an incoming puncture and the marked points associated to the leaves will become outgoing punctures.

For any semi-stable tree T with k interior vertices and d leaves, there is a gluing map

$$\gamma^{T,T_d} \colon \mathcal{R}^{S(T)} \to \mathcal{R}^{d+1}$$

similar to the one described in Section 2.3.3, but since H and Z have nontrivial automorphism groups, these gluing maps are not local embeddings.

Lagrangian and asymptotics labels can be defined for unstable curves in the same way. Obviously, the Lagrangian label on H must have no jumps.

To a (stable or semi-stable) nodal discs $S = \{r_v\} \in \mathcal{R}^T$ one associates its *normalisation* which is simply the quadruple $(\widehat{S}, \mathfrak{m}, \mathfrak{n}, \iota)$ with

- $\widehat{S} = \sqcup_{v \in \mathrm{Ve}(T)}^{k} S_{r_v}$.
- $\mathfrak{m}$ is the union of the marked points of $\widehat{S}$ corresponding to edges of T connecting to leaves or to the root of T.
- $\mathfrak{n}$ is the union of the marked point of $\widehat{S}$ corresponding to interior edges of T. It comes equipped with a fixed point free involution ι determined by $\iota(n_0) = n_1$ if n_0 and n_1 are connected by an edge e.

Elements of $\mathfrak{m}$ are called *ends* of S and elements of $\mathfrak{n}$ are called *nodes*. Note that one can rebuild the gluing tree out of the data $(\widehat{S}, \mathfrak{m}, \mathfrak{n}, \iota)$.

3. Analytic preliminaries

3.1. **Almost complex structures.** Before defining the moduli spaces of pseudo-holomorphic curves that are relevant for the theory (see Section 3.2), it is necessary

to describe the almost complex structures. For technical reasons it will be necessary to make certain additional assumptions on them that we here describe.

3.1.1. *Cylindrical almost complex structures.* Let (Y, α) be a contact manifold with the choice of a contact form. We denote by $\mathcal{J}^{\mathrm{cyl}}(Y)$ the set of *cylindrical* almost complex structures on the symplectisation $(\mathbb{R} \times Y, d(e^t\alpha))$, i.e. almost complex structures J satisfying the following conditions:

- J is invariant under the natural (symplectically conformal) action of $\mathbb{R}$ on $\mathbb{R} \times Y$;
- $J\frac{\partial}{\partial t} = R_\alpha$;
- $J(\xi) = \xi$, where $\xi := \ker \alpha \subset TY$; and
- J is compatible with $d\alpha|_\xi$, i.e. $d\alpha_\xi(\cdot, J\cdot)$ is a metric on ξ.

We will say that an almost complex structure defined on a subset of the symplectisation of the form $I \times Y$ (where I is an interval) comes from $\mathcal{J}^{\mathrm{cyl}}(Y)$ if it is the restriction of an almost complex structure in $\mathcal{J}^{\mathrm{cyl}}(Y)$.

3.1.2. *Almost complex structures on Liouville manifolds.* Let (P, θ) be a Liouville manifold. Recall that there is a subset $P_\infty \subset P$ that is exact symplectomorphic to half a symplectisation

$$([0, +\infty) \times V, d(e^\tau \alpha_V)),$$

and where $P \setminus P_\infty \subset P$ is pre-compact. We say that an almost complex structure J_P compatible with $d\theta$ is *admissible* if the almost complex structure J_P on P comes from $\mathcal{J}^{\mathrm{cyl}}(V)$ outside of a compact subset of P_∞. We denote by $\mathcal{J}^{\mathrm{adm}}(P)$ the set of these almost complex structures.

3.1.3. *Cylindrical lifts.* We will now restrict our attention to the manifolds that we will be considering here; namely, the contactisation of a Liouville manifold (P, θ), i.e.

$$(Y, \alpha) := (P \times \mathbb{R}, dz + \theta),$$

and its symplectisation

$$(X, \omega) := (\mathbb{R} \times P \times \mathbb{R}, d(e^t(dz + \theta))).$$

From now on (Y, α) and (X, ω) will always denote manifolds of this type.

Given a compatible almost complex structure J_P in $\mathcal{J}^{\mathrm{adm}}(P)$ as defined above, there is a unique cylindrical almost complex structure $\widetilde{J}_P$ on $(X, d(e^t\alpha))$ which makes the projection

$$\pi \colon X = \mathbb{R} \times P \times \mathbb{R} \to P$$

a $(\widetilde{J}_P, J_P)$-holomorphic map. We will call this almost complex structure the *cylindrical lift of* J_P. An important feature of the cylindrical lift is that the diffeomorphisms $(t, p, z) \mapsto (t, p, z + z_0)$, $z_0 \in \mathbb{R}$, induced by the Reeb flow all are J-holomorphic.

We denote by $\mathcal{J}^{\mathrm{cyl}}_\pi(Y) \subset \mathcal{J}^{\mathrm{cyl}}(Y)$ the set of cylindrical lifts of almost complex structures in $\mathcal{J}^{\mathrm{adm}}(P)$.

3.1.4. *Compatible almost complex structures with cylindrical ends.* Let J^+ and J^- be almost complex structures in $\mathcal{J}^{\mathrm{cyl}}_\pi(Y)$, and let $T \in \mathbb{R}^+$. We will require that both almost complex structures $J^\pm$ coincide outside of $\mathbb{R} \times K$ for some compact subset $K \subset Y$. We denote by $\mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T}(X)$ the set of almost complex structures on $X = \mathbb{R} \times Y$ that tame $d(e^t\alpha)$ and satisfy the following.

(a) The almost complex structure J is equal to the cylindrical almost complex structures J^- and J^+ on subsets of the form
$$(-\infty, -T] \times P \times \mathbb{R},$$
$$[T, +\infty) \times P \times \mathbb{R},$$
respectively; and

(b) outside of $\mathbb{R}\times K$, the almost complex structure J coincides with a cylindrical lift in $\mathcal{J}^{\mathrm{cyl}}_\pi(X)$.

Condition (b) is needed in order to deal with compactness issues. Recall that the contact manifold $P \times \mathbb{R}$ as as well as the Liouville manifold P are non-compact.

In the case when we do not care about the parameter $T > 0$, we will simply write $\mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(X)$. The union of all $\mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(X)$ over all $J^-, J^+ \in \mathcal{J}^{\mathrm{cyl}}(Y)$ is denoted $\mathcal{J}^{\mathrm{adm}}(X)$, almost complex structure in this set will be called *admissible*.

3.1.5. *Domain dependent almost complex structures.* Let $J^\pm \in \mathcal{J}^{\mathrm{cyl}}(Y)$ be two cylindrical almost complex structures and let $\{J_t\}$ be a smooth path in $\mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(X)$ which is locally constant near $t = 0$ and $t = 1$.

Let Σ_0 and Σ_1 be two Lagrangian cobordisms in the symplectisation $(X, d(e^t\alpha))$ of (Y, ξ) with Legendrian ends $\Lambda_i^\pm$ $i = 0, 1$. To a punctured disc S (either stable or semi-stable) with simple Lagrangian label $\underline{L}$ we associate a domain-dependent almost complex structure $J_{(S,\underline{L})} \colon S \to \mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(X)$ using the path $\{J_t\}$, as described below.

For every punctured disc S, if $\underline{L}$ has no jumps, then $J_{(S,\underline{L})}$ is constant and has value J_0 or J_1 depending if the Lagrangian label $\underline{L}$ is constant at Σ_0 or Σ_1. Note that $S = H$ (i.e. a half-plane) always falls into this case.

If $\underline{L}$ has two jumps (which means, in particular, that S has $d + 1$ boundary punctures with $d \geq 1$), we uniformise S to a strip $\mathbb{R} \times [0, 1]$ (with coordinates (s, t)) with $d - 1$ punctures in the boundary, such that the jumps of $\underline{L}$ correspond to the ends of the strip, and moreover the incoming puncture of S corresponds to the end of the strip at $s \ll 0$. This uniformisation, which is unique up to translations in the s-coordinate of the strip, defines a map $t \colon S \to [0, 1]$ by composition with the t-coordinate on the strip, and we define $J_{(S,\underline{L})} \colon S \to \mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(X)$ by $J_{(S,\underline{L})}(z) = J_{t(z)}$. Note that $J_{(S,\underline{L})}$ is constantly equal to J_0 near the boundary components associated to Σ_0 and constantly equal to J_1 near the boundary components associated to Σ_1.

When S is a stable $(d+1)$-punctured disc, the maps $J_{(S,\underline{L})}$ fit into smooth maps
$$J_{(d,\underline{L})} : \mathcal{S}^{d+1} \to \mathcal{J}^{\mathrm{adm}}_{J^-,J^+}$$
which are obviously compatible with the degenerations of the Deligne-Mumford moduli spaces. In order to uniformise the notation we will denote $J_{(2,\underline{L})} := J_{(Z,\underline{L})}$ and $J_{(1,\underline{L})} := J_{(H,\underline{L})}$, The collection of these maps is called a *universal choice of domain dependent almost complex structures* induced by (J^+, J^-, J_t). For a point $z \in \mathcal{S}^{d+1}$ we denote by $J(z)$ the corresponding almost complex structure on X induced by this construction.

3.2. **Moduli space of holomorphic discs.** We describe now the moduli space of discs used in Section 6 in order to define the differential of the Cthulhu complex. We begin with the general definition of these moduli spaces, and then detail some of them with particular asymptotics.

3.2.1. *General definitions.* In this subsection we set up some terminology which will be used in all subsequent discussions about moduli spaces. Let S be a $d+1$-punctured disc ($d \geq 0$) and $\underline{L}$ a simple Lagrangian label for S with values in a pair of Lagrangian cobordisms (Σ_0, Σ_1). Suppose we have a (possibly domain dependent) almost complex structure J on X. We say that a map $u : S \to X$ is *J-holomorphic with boundary conditions in $\underline{L}$* if $\forall z \in S$

$$d_z u \circ j = J(z) \circ d_z u, \tag{9}$$

where j denotes the standard complex structure on D^2, and $u(\partial_i S) \subset \underline{L}(i)$. If the Lagrangian label $\underline{L}$ is constant, we will say that u is *pure*; if it has jumps, we will say that u is *mixed*. Here we will remain vague about the almost complex structure J, because it will depend on the specific moduli space that we will consider.

Given an intersection point $p \in \Sigma_0 \cap \Sigma_1$, we say that u is *asymptotic* to p at a_i if

- the marked point a_i is a jump, and
- $\lim_{z \to a_i} u(z) = p$.

Let γ be a Reeb chord of $\Lambda_0^\pm \sqcup \Lambda_1^\pm$ of length T. The map $u = (a, v)$ into $\mathbb{R} \times Y$ has a *positive asymptotic* to γ at a_i if:

- $\lim_{s \to +\infty} v(\varepsilon_i(s,t)) = \gamma(Tt)$ and $\lim_{s \to +\infty} a(\varepsilon_i(s,t)) = +\infty$, given that $a_i = a_0$ is the incoming puncture, or
- $\lim_{s \to -\infty} v(\varepsilon_i(s,t)) = \gamma(T(1-t))$ and $\lim_{s \to -\infty} a(\varepsilon_i(s,t)) = +\infty$, given that $i \neq 0$.

Let γ be a Reeb chord of $\Lambda_0^\pm \sqcup \Lambda_1^\pm$ of length T. The map $u = (a, v)$ has a *negative asymptotic* to γ at a_i, $i \neq 0$, given that

- $\lim_{s \to -\infty} a(\varepsilon_i(s,t)) = -\infty$ and $\lim_{s \to -\infty} v(\varepsilon_i(s,t)) = \gamma(Tt)$.

We will never consider holomorphic curves which have a negative asymptotic at the incoming end.

Associated to a pair of cobordisms there are three types of possible targets for Lagrangian labels that will be considered here: (Σ_0, Σ_1), $(\mathbb{R} \times \Lambda_0^\pm, \mathbb{R} \times \Lambda_1^\pm)$ and $\mathbb{R} \times \Lambda_i^\pm$ for $i = 0, 1$; note that the asymptotics of the latter two are subsets of those of the first pair. We will use $\underline{L}$ to denote any of those labels. Given $x_0, \dots, x_d$ in $A(\Sigma_0, \Sigma_1)$ and $r \in \mathcal{R}^{d+1}$, we denote by

$$\mathcal{M}^r_{\underline{L}}(x_0; x_1, \dots, x_d; J)$$

the space of J-holomorphic maps from S_r to X with asymptotics to x_i at a_i modulo reparametrisations of S_r. In other words, x_0 is the asymptotic of the incoming puncture. We denote by

$$\mathcal{M}_{\underline{L}}(x_0; x_1, \dots, x_d; J)$$

the union of the $\mathcal{M}^r_{\underline{L}}(x_0; x_1, \dots, x_d; J)$ over all $r \in \mathcal{R}^{d+1}$. Note that once the asymptotics for the moduli space is fixed, the actual Lagrangian label is uniquely determined and, hence, we do not need to specify it.

In the case when both $\underline{L}$ and J are invariant under translations of the symplectisation coordinate, there is an induced $\mathbb{R}$-action on $\mathcal{M}_{\underline{L}}(x_0; x_1, \dots, x_d; J)$. We

use

$$\widetilde{\mathcal{M}}_{\underline{L}}(x_0; x_1, \dots, x_d; J)$$

to denote the quotient of the moduli space by this action.

In the following subsections we describe the moduli spaces with simple Lagrangian labels that appear in the definition of the Cthulhu differential in Section 6.

3.2.2. *Strips and half-planes.* As already seen, the pseudoholomorphic discs considered here will have a number of different types of asymptotics. However, it will be useful to make a distinction between the following two types of discs considered.

- A pseudoholomorphic disc with no jumps is called a *(punctured) half-plane*; while
- A pseudoholomorphic disc having precisely two jumps will be called a *(punctured) strip*. The puncture corresponding to the unique incoming end will be called the *output* while the puncture corresponding to the unique outgoing end at which a jump occurs will be called the *input*.

Remark 3.1. *The fact that outgoing ends are inputs and incoming ends are outputs might seem confusing. The incoming/outgoing dichotomy comes from the notion of incoming and outgoing edges in a rooted tree (see Remark 2.5) and refers to particular coordinates of the domain (the strip like ends). This follows the convention of* [64]*. The dichotomy input/output refers to what will belong to the domain/codomain of the differential as defined in Section 6.*

The above two types of pseudoholomorphic discs (i.e. halfplanes and strips) will play radically different roles in our theory. Roughly speaking, punctures halfplanes appear when defining Chekanov-Eliashberg DGA as well as DGA morphisms, while punctured strips appear when defining the Floer homology, the bilinearised Legendrian contact cohomology, and the Cthulhu homology.

3.2.3. *LCH moduli spaces.* The LCH moduli spaces are the moduli spaces appearing in the definition of linearised contact homology. We will consider two types of LCH moduli spaces, depending on whether the involved Reeb chords are mixed or not. We fix two cylindrical almost complex structures $J^{\pm} \in \mathcal{J}^{\mathrm{cyl}}(Y)$ on $X = \mathbb{R} \times Y$ and a path $\{J_t\}$ of almost complex structures in $\mathcal{J}^{\mathrm{adm}}_{J^+,J^-}(X)$. We denote $J_{(d,\underline{L})}$ the corresponding universal choice of domain dependent almost complex structures.

Denote $\Sigma = \Sigma_i$ and $\Lambda^{\pm} = \Lambda^{\pm}_i$ for $i = 0, 1$. Let $\gamma^+, \delta^+_i, \dots, \delta^+_d$ be Reeb chords of Λ^+ and $\gamma_-, \delta^-_1, \dots, \delta^-_d$ Reeb chords of Λ^-. Throughout the paper we will write a d-tuple of pure Reeb chords as a word; this notation is reminiscent of the multiplicative structure of the Chekanov-Eliashberg algebra. Therefore, we set $\boldsymbol{\delta}^{\pm} = \delta^{\pm}_1 \dots \delta^{\pm}_d$.

We will consider three types of pure LCH moduli spaces:

$$\widetilde{M}_{\mathbb{R}\times\Lambda^+}(\gamma^+; \boldsymbol{\delta}^+; J^+), \quad \widetilde{M}_{\mathbb{R}\times\Lambda^-}(\gamma^-; \boldsymbol{\delta}^-; J^-), \quad \text{and} \quad \mathcal{M}_{\Sigma}(\gamma^+; \boldsymbol{\delta}^-; J_{(d,\underline{L})}).$$

The first two moduli spaces will be called pure *cylindrical* LCH moduli spaces and are used to define the Legendrian contact homology differential of $\Lambda^{\pm}$. The third moduli space will be called pure *cobordism* LCH moduli space and is used in [36] to define maps between Legendrian contact homology algebras. Since the Lagrangian labels are constant for these moduli spaces, the almost complex structures $J_{(d,\underline{L})}$) are actually domain *independent*. However, their value depends on whether $\Sigma = \Sigma_0$

or $\Sigma = \Sigma_1$. Finally, recall that in the cylindrical moduli spaces, we take a quotient by the $\mathbb{R}$-action, while such an operation is not possible (nor desirable) for the cobordism moduli space.

Now we describe mixed LCH moduli spaces. Let $\boldsymbol{\delta}^{\pm} := \delta_1^{\pm} \dots \delta_{i-1}^{\pm}$ be Reeb chords of $\Lambda_1^{\pm}$ and $\boldsymbol{\zeta}^{\pm} = \zeta_{i+1}^{\pm} \dots \zeta_d^{\pm}$ be Reeb chords of $\Lambda_0^{\pm}$. We will consider three types of mixed LCH moduli spaces:

$$\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda_0^{\pm},\mathbb{R}\times\Lambda_1^{\pm}}(\gamma^+;\boldsymbol{\delta}^{\pm},\gamma^-,\boldsymbol{\zeta}^{\pm};J^{\pm}) \quad \text{and} \quad \mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-;J_{(d,\underline{L})}),$$

where $\gamma^{\pm} \in \mathcal{R}(\Lambda_1^+,\Lambda_0^+)$ for $\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda_0^+,\mathbb{R}\times\Lambda_1^+}(\gamma^+;\boldsymbol{\delta}^+,\gamma^-,\boldsymbol{\zeta}^+;J^+)$, $\gamma^{\pm} \in \mathcal{R}(\Lambda_1^-,\Lambda_0^-)$ for $\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda_0^-,\mathbb{R}\times\Lambda_1^-}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-;J^-)$, and $\gamma^+ \in \mathcal{R}(\Lambda_1^+,\Lambda_0^+)$, $\gamma^- \in \mathcal{R}(\Lambda_1^-,\Lambda_0^-)$ for $\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-;J_{(d,\underline{L})})$.

The first two moduli spaces will be called mixed *cylindrical* LCH moduli spaces and are used to define the bilinearised Legendrian contact homology differential of $(\Lambda_0^{\pm},\Lambda_1^{\pm})$ (see Section 5.3 or [12]). The third moduli space will be called mixed *cobordism* LCH moduli space and is used to define maps between bilinearised Legendrian contact homology groups. An illustration of a curve in the mixed cobordism LCH moduli spaces is shown in Figure 3.

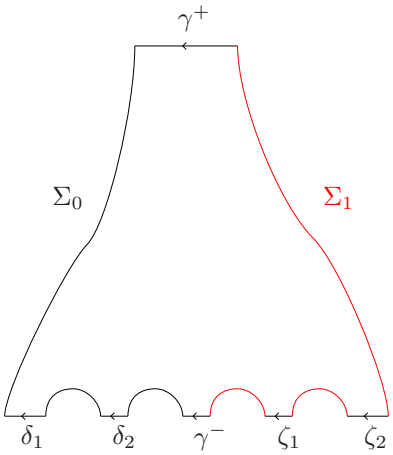

FIGURE 3. A mixed cobordism LCH curve.

3.2.4. *Floer moduli space.* Let $p, q \in \Sigma_0 \cap \Sigma_1$ be intersection points, $\boldsymbol{\delta}^- = \delta_1^- \dots \delta_{i-1}^-$ a word of Reeb chords on Λ_0^-, and $\boldsymbol{\zeta}^- = \zeta_{i+1}^-, \dots \zeta_d^-$ a word of Reeb chords on Λ_1^-.

J-holomorphic curves in the moduli space

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(p;\boldsymbol{\delta}^-,q,\boldsymbol{\zeta}^-;J_{(d,\underline{L})})$$

will be called *Floer strips*. (Note that this is an abuse of terminology as the actual domain is not a strip unless $\boldsymbol{\delta}^-$ and $\boldsymbol{\zeta}^-$ are empty.) See Figure 4.

Furthermore, the punctured disc is here required to makes a jump from Σ_1 to Σ_0 at the puncture asymptotic to p (this is the incoming puncture), while it makes a jump from Σ_0 to Σ_1 at the puncture asymptotic to q. This follows from our convention for Lagrangian labels; see Remark 2.11.

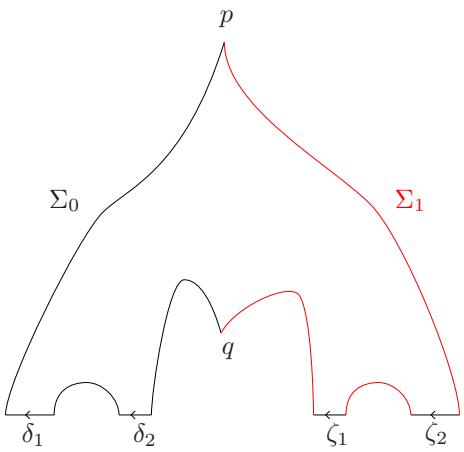

FIGURE 4. A Floer strip.

3.2.5. *LCH to Floer moduli space.* Let $\gamma^- \in \mathcal{R}(\Lambda_1^-, \Lambda_0^-)$ be a mixed chord, $p \in \Sigma_0 \cap \Sigma_1$ an intersection point, $\boldsymbol{\delta}^- = \delta_1^- \dots \delta_{i-1}^-$ a word of Reeb chords on Λ_0^-, and $\boldsymbol{\zeta}^- = \zeta_{i+1}^- \dots \zeta_d^-$ a word of Reeb chords on Λ_1^-. Curves in the moduli space

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(p; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-; J_{(d,\underline{L})})$$

will be called *holomorphic Cthulhus*. See Figure 5.

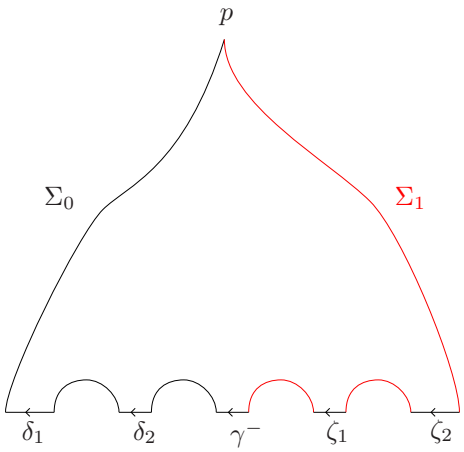

FIGURE 5. A holomorphic Cthulhu.

3.2.6. *Floer to LCH moduli space.* Let $\gamma^+ \in \mathcal{R}(\Lambda_1^+, \Lambda_0^+)$ be a mixed chord, $p \in \Sigma_0 \cap \Sigma_1$ an intersection point, $\boldsymbol{\delta}^- = \delta_1^- \dots \delta_{i-1}^-$ Reeb chords of Λ_0^- and $\boldsymbol{\zeta}^- = \zeta_{i+1}^- \dots \zeta_d^-$ Reeb chords of Λ_1^-. Curves in the moduli space

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+; \boldsymbol{\delta}^-, p, \boldsymbol{\zeta}^-; J_{(d,\underline{L})})$$

will be called *J-holomorphic cultists*. See Figure 6.

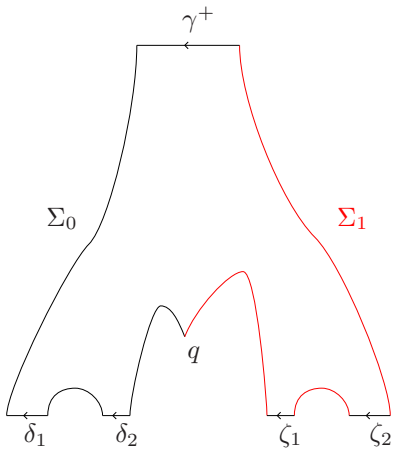

FIGURE 6. Curve contributing to the second cultist map.

3.2.7. *Bananas moduli space.* Let $\gamma_{1,0} \in \mathcal{R}(\Lambda_1^+, \Lambda_0^+)$ and $\gamma_{0,1} \in \mathcal{R}(\Lambda_0^+, \Lambda_1^+)$ be mixed Reeb chords (and note that they go in opposite directions), let $\boldsymbol{\delta}^- = \delta_1^- \dots \delta_{i-1}^-$ be Reeb chords of Λ_0^- and $\boldsymbol{\zeta}^- = \zeta_{i+1}^- \dots \zeta_d^-$ Reeb chords of Λ_1^-. Curves in the moduli space

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma_{1,0}; \boldsymbol{\delta}^-, \gamma_{0,1}, \boldsymbol{\zeta}^-; J_{(d,\underline{L})})$$

will be called *J-holomorphic bananas*. See Figure 7.

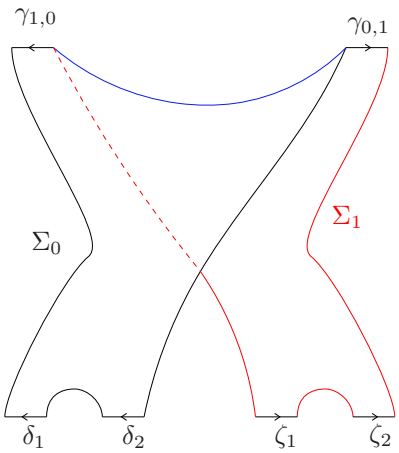

FIGURE 7. A J-holomorphic banana.

When $\Sigma_i = \mathbb{R} \times \Lambda_i$, we again use $\widetilde{\mathcal{M}}(\gamma_{1,0}; \boldsymbol{\delta}, \gamma_{0,1}, \boldsymbol{\zeta}^-)$ to denote the quotient of the moduli space by the natural $\mathbb{R}$-action.

Note that, by our our definition of an exact Lagrangian cobordism, there are no non-constant pseudoholomorphic curves with boundary on $\Sigma_0 \cup \Sigma_1$ with all punctures having negative asymptotics to Reeb chords; see Section 3.3.2 for more details.

3.3. **Energy and compactness.** In this section, we recall the notion of the Hofer energy for holomorphic curves in the symplectisation of a contact manifold as introduced in [52] and [13]. We also give estimates for this energy in terms of the asymptotics of the curves appearing in the moduli spaces of Section 3.2. The goal is to formulate the compactness theorem for pseudoholomorphic curves in the present setting.

3.3.1. *The Hofer energy.* Assume that we are given two exact Lagrangian cobordisms Σ_0 and Σ_1 in the symplectisation $(X = \mathbb{R} \times Y, d(e^t\alpha))$ of (Y, α). Let $f_i \colon \Sigma_i \to \mathbb{R}$ be primitives of $e^t\alpha|_{\Sigma_i}$ which are constant at the cylindrical ends. (By a slight abuse of notation, we will denote by $\cdot|_{\Sigma_i}$ the pull-back of $e^t\alpha$ under the inclusion of Σ_i.) Without loss of generality we will assume that both constants are 0 on the negative ends, while the constants on the positive end of Σ_i will be denoted by $\mathfrak{c}_i$, $i = 0, 1$. Here we rely on Definition 2.1 of an exact cobordism.

Take any $T > 0$ and $T > \epsilon > 0$ for which

$$\Sigma_i \cap ((-\infty, -T + \varepsilon) \times Y) = (-\infty, -T + \varepsilon) \times \Lambda_i^-$$
$$\Sigma_i \cap ((T - \varepsilon, +\infty) \times Y) = (T - \varepsilon, +\infty) \times \Lambda_i^+,$$

for $i = 0, 1$. Now, we let $\phi : \mathbb{R} \to [e^{-T}, e^T]$ be a smooth function satisfying:

- $\phi(\pm t) = e^{\pm T}$ for $t > T$;
- $\phi(t) = e^t$ for $t \in [-T + \epsilon, T - \epsilon]$;
- $\phi'(t) \geq 0$.

In the case when both Σ_0 and Σ_1 are trivial cylinders over Legendrian submanifolds, we will also allow the case $T = \epsilon = 0$, and $\phi \equiv 1$.

By construction we have $\phi\alpha|_{\Sigma_i} = e^t\alpha|_{\Sigma_i}$ for the reason that $\alpha|_{\Sigma_i} = 0$ in the subset where ϕ is not equal to e^t. A primitive of $e^t\alpha|_{\Sigma_i}$ (which exists by exactness) is hence also a primitive of $\phi\alpha|_{\Sigma_i}$.

Let $\mathcal{C}^-$ be the set of compactly supported smooth functions

$$w_- \colon (-\infty, -T + \epsilon) \to [0, +\infty)$$

satisfying $\displaystyle\int_{-\infty}^{-T+\epsilon} w_-(s)ds = e^{-T}$, and let $\mathcal{C}^+$ be the set of compactly supported smooth functions

$$w_+ \colon (T - \epsilon, +\infty) \to [0, +\infty)$$

satisfying $\displaystyle\int_{T-\epsilon}^{+\infty} w_+(s)ds = e^T$.

We are now ready to introduce different versions of energies for pseudoholomorphic curves in the symplectisation, all which are standard. We refer to [13] for the absolute case, and [30] as well as [1] for the relative case.

Definition 3.2. Let S be a punctured disc and $u = (a, v) \colon S \to \mathbb{R} \times Y$ be a smooth map.

- The $d(\phi\alpha)$-*energy* of u is given by

$$E_{d(\phi\alpha)}(u) = \int_S u^*(d(\phi\alpha)).$$

- The α-*energy of* u is given by

$$E_\alpha(u) = \sup_{(w_-, w_+) \in \mathcal{C}^- \times \mathcal{C}^+} \left(\int_S (w_- \circ a) da \wedge v^*\alpha + \int_S (w_+ \circ a) da \wedge v^*\alpha \right).$$

- The *total energy*, or the *Hofer energy*, of u is given by

$$E(u) = E_\alpha(u) + E_{d(\phi\alpha)}(u).$$

In the case when u is a proper map for which $E(u) < \infty$, we say that u is a *finite energy pseudoholomorphic disc*.

Non-constant holomorphic curves have positive total energy, as stated in the following simple lemma. We leave the proof to the reader.

Lemma 3.3. *If u is non-constant punctured pseudoholomorphic disc with boundary on a pair of exact Lagrangian cobordisms, and if the almost complex structure is cylindrical outside of $[-T+\epsilon, T-\epsilon]\times Y$, then $E(u) > 0$, $E_\alpha(u) \geq 0$, and $E_{d(\phi\alpha)}(u) \geq 0$. Moreover, $E_{d(\phi\alpha)}(u) = 0$ implies that u is contained inside a trivial cylinder over a Reeb orbit.*

The techniques in [53] can be to the applied to the current setting with non-empty boundary, similarly to what was done in [1], in order to show that

Proposition 3.4. *Assume that we endow $\mathbb{R}\times P\times\mathbb{R}$ with an admissible complex structure in the sense of Section 3.1.4. A proper punctured pseudoholomorphic disc inside the symplectisation having boundary on a Lagrangian cobordism $\Sigma_0\cup\Sigma_1$ is of finite energy if and only if all of its punctures are contained in the boundary, and such that the disc is exponentially converging to either trivial strips over Reeb chords on $\Lambda_0^\pm\cup\Lambda_1^\pm$ or intersection points $\Sigma_0\cap\Sigma_1$ at each of its boundary punctures.*

3.3.2. *Action and energy.* Consider a pair of exact Lagrangian cobordisms Σ_i, $i = 0, 1$, from Λ_i^- to Λ_i^+. Here we define the action of intersection points and Reeb chords and relate it to the different energies of holomorphic discs having boundary on this pair of cobordisms.

For a Reeb chord c we define

$$\ell(c) := \int_c \alpha.$$

Recall that we are given a choice of $T \geq 0$ in the construction of the $E_{d(\phi\alpha)}$-energy above, where equality only is possible under the assumption that both cobordisms are trivial cylinders. The action of a mixed Reeb chord γ on $\Lambda_1^\pm\cup\Lambda_0^\pm$ is defined by

$$\begin{aligned}\mathfrak{a}(\gamma) &:= e^T\ell(\alpha) + (\mathfrak{c}_i - \mathfrak{c}_j) && \text{if } \gamma \text{ is a chord from } \Lambda_i^+ \text{ to } \Lambda_j^+, \text{ and}\\ \mathfrak{a}(\gamma) &:= e^{-T}\ell(\alpha) && \text{if } \gamma \text{ is a chord on } \Lambda_0^-\cup\Lambda_1^-.\end{aligned}$$

In particular, we observe that the action of a pure Reeb chord γ on $\Lambda_i^\pm$ is defined by

$$\mathfrak{a}(\gamma) := e^{\pm T}\ell(\gamma).$$

The action of an intersection point $p \in \Sigma_0\cap\Sigma_1$ is defined by

$$\mathfrak{a}(p) := f_1(p) - f_0(p).$$

Applications of Stoke's theorem gives the following proposition (see [24] for details), where we heavily rely on the fact that each cobordism Σ_i, $i = 0, 1$, is exact.

Proposition 3.5. *Let $\gamma^\pm \in \mathcal{R}(\Lambda_1^\pm, \Lambda_0^\pm)$ be mixed Reeb chords, $\boldsymbol{\delta}^- = \delta_1^-\dots\delta_{i-1}^-$ and $\boldsymbol{\zeta}^- = \zeta_{i+1}\dots\zeta_d$ words of pure Reeb chords on Λ_1^- and Λ_0^-, respectively, and*

$p, q \in \Sigma_0 \cap \Sigma_1$ *intersection points. We denote* $\mathfrak{a}(\boldsymbol{\delta}^-) := \sum_{k=1}^{i-1} \mathfrak{a}(\delta_k^-)$ *and* $\mathfrak{a}(\boldsymbol{\zeta}^-) := \sum_{k=i+1}^{d} \mathfrak{a}(\zeta_k^-)$.

- *If* $u \in \mathcal{M}_{\underline{L}}(\gamma^+; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-)$, *then*

$$E_{d(\phi\alpha)}(u) = \mathfrak{a}(\gamma^+) - \mathfrak{a}(\gamma^-) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{10}$$
$$E_\alpha(u) \leq 2\mathfrak{a}(\gamma^+)$$

- *If* $u \in \mathcal{M}_{\underline{L}}(\gamma^+; \boldsymbol{\delta}^-, p, \boldsymbol{\zeta}^-)$, *then*

$$E_{d(\phi\alpha)}(u) = \mathfrak{a}(\gamma^+) - \mathfrak{a}(p) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{11}$$
$$E_\alpha(u) \leq 2\mathfrak{a}(\gamma^+)$$

- *If* $u \in \mathcal{M}_{\underline{L}}(p; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-)$, *then*

$$E_{d(\phi\alpha)}(u) = \mathfrak{a}(p) - \mathfrak{a}(\gamma^-) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{12}$$
$$E_\alpha(u) \leq \mathfrak{a}(p)$$

- *If* $u \in \mathcal{M}_{\underline{L}}(p; \boldsymbol{\delta}^-, q, \boldsymbol{\zeta}^-)$, *then*

$$E_{d(\phi\alpha)}(u) = \mathfrak{a}(p) - \mathfrak{a}(q) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{13}$$
$$E_\alpha(u) \leq \mathfrak{a}(p)$$

- *If* $u \in \mathcal{M}_{\underline{L}}(\gamma_{1,0}; \boldsymbol{\delta}^-, \gamma_{0,1}, \boldsymbol{\zeta}^-)$, *then*

$$E_{d(\phi\alpha)}(u) = \big(\mathfrak{a}(\gamma_{1,0}) + \mathfrak{a}(\gamma_{0,1})\big) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{14}$$
$$E_\alpha(u) \leq 2\mathfrak{a}(\gamma_{1,0}) + 2\mathfrak{a}(\gamma_{0,1})$$

For the above inequalities involving the α*-energy we must assume that the almost complex structure is cylindrical outside of* $[-T + \epsilon, T - \epsilon] \times Y$.

3.3.3. *Holomorphic buildings with boundary on Lagrangian cobordisms.* The moduli space of punctured pseudoholomorphic discs with boundary on a Lagrangian cobordism can be compactified by the space of pseudoholomorphic buildings, which we now proceed to define.

Definition 3.6. Let S be a (stable or semi-stable) nodal disc with normalisation $(\widehat{S}, \mathfrak{m} \cup \mathfrak{n}, \iota)$ as described in Section 2.3.5. By a *level function* for S we mean a locally constant map

$$f : \widehat{S} \to \{-k^-, -k^- + 1, \ldots, k^+ - 1, k^+\}$$

for some $k^-, k^+ \in \mathbb{Z}_{\geq 0}$ such that for any node $n \in \mathfrak{n}$ we have $|f(n) - f(\iota(n))| \leq 1$, while $f(n) = f(\iota(n))$ is allowed only if $f(n) = 0$.

If S is a nodal disc with level function f, a Lagrangian label $\underline{L}$ on (S, f) will be a collection of Lagrangian labels $\underline{L}_i$ on each connected component S_i of the normalisation such that:

- $\underline{L}_i$ takes values in $\{\mathbb{R} \times \Lambda_0^+, \mathbb{R} \times \Lambda_1^+\}$ if $f(S_i) > 0$, in $\{\Sigma_0, \Sigma_1\}$ if $f(S_i) = 0$ and in $\{\mathbb{R} \times \Lambda_0^-, \mathbb{R} \times \Lambda_1^-\}$ if $f(S_i) < 0$; and
- At a node $n \in \mathfrak{n}$ the asymptotic corresponding to n agrees with the asymptotic corresponding to $\iota(n)$.

In particular, performing boundary connected sums at all nodes of S, one can canonically assign a Lagrangian label on the resulting nodal disc taking values in $\{\Sigma_0, \Sigma_1\}$. A Lagrangian label on (S, f) is *simple* if, after performing boundary connected sums at all nodes, the resulting Lagrangian label is simple.

Let Σ_0, Σ_1 be two admissible exact Lagrangian cobordisms, and $\{J_t\}$ a path in $\mathcal{J}^{\mathrm{adm}}_{J^+,J^-}$ inducing domain dependent almost complex structures $J_{(d,\underline{L})}$ for any $d \geq 0$ and any Lagrangian label $\underline{L}$ for elements in $\mathcal{R}^{d+1}$.

Definition 3.7. Let S be a nodal disc with Lagrangian label $\underline{L}$ and level function f mapping onto $\{-k^-, -k^- + 1, \dots, k^+ - 1, k^+\}$ for some integers $k^\pm \geq 0$. A holomorphic building of height $k^-|1|k^+$ with domain S is given by a family $\{u_i \mid i = 1, \dots, k\}$ of punctured pseudo-holomorphic discs, where the connected component S_i of the normalisation of S corresponds to u_i, such that the following conditions moreover are satisfied:

- All discs satisfying $f(S_i) = l$ for some fixed l are said to live in level $l \in \mathbb{Z}$. We require that each level different from zero contains at least one component which is not a trivial strip over a Reeb chord.
- If $f(S_i) > 0$, then $u_i \in \mathcal{M}^{r_i}_{\mathbb{R}\times\Lambda_0^+,\mathbb{R}\times\Lambda_1^+}(y_0; y_1, \dots, y_{d_i}; J^+)$. These discs are said to live in a *top level*.
- If $f(S_i) < 0$, then $u_i \in \mathcal{M}^{r_i}_{\mathbb{R}\times\Lambda_0^-,\mathbb{R}\times\Lambda_1^-}(y_0; y_1, \dots, y_{d_i}; J^-)$. These discs are said to live in a *bottom level*.
- If $f(S_i) = 0$, then $u_i \in \mathcal{M}^{r_i}_{\Sigma_0,\Sigma_1}(y_0; y_1, \dots, y_{d_i}; J_{(d_i,\underline{L}_i)})$. These discs are said to live in the *middle level*.
- For any node n whose asymptotic corresponds to a Reeb chord, we require that $|f(n) - f(\iota(n))| = 1$;
- For any node n such that $0 \leq f(n) < f(\iota(n))$, let y_n and $y_{\iota(n)}$ be the corresponding asymptotics. Then $y_n = y_{\iota(n)}$ is a Reeb chord on $\mathbb{R} \times (\Lambda_0^+ \sqcup \Lambda_1^+)$, such that y_n and $y_{\iota(n)}$ is a positive and negative asymptotic, respectively.
- For any node n such that $f(n) < f(\iota(n)) \leq 0$ let y_n and $y_{\iota(n)}$ be the corresponding asymptotics. Then $y_n = y_{\iota(n)}$ is a Reeb chord on $\mathbb{R} \times (\Lambda_0^- \sqcup \Lambda_1^-)$, such that y_n and $y_{\iota(n)}$ is a positive and negative asymptotic, respectively.
- For any node n such that $f(n) = f(\iota(n))$ (and thus is equal to 0) the corresponding asymptotics y_n and $y_{\iota(n)}$ correspond to a given intersection point in $\Sigma_0 \cap \Sigma_1$.
- The positive (negative) punctures asymptotic to Reeb chords which are not nodes correspond precisely to the positive (negative) punctures in the k_+:th (k_-:th) level.

Furthermore, we identify two buildings consisting of the same nodal domain whenever the images of the discs in level $l = 0$ coincide, while the images in level $l \neq 0$ differ by a translation of the symplectisation coordinate by a number that only depends on the level l.

By

$$\mathcal{M}^{k^-|1|k^+}_{\Sigma_0,\Sigma_1}(x_0; x_1, \dots, x_d)$$

we denote the buildings having a single incoming puncture x_0 which is not a node (necessarily contained in level k^+), and whose outgoing punctures which are not nodes correspond to $x_1, \dots, x_d$ (respecting the order of the punctures induced by the boundary orientation after a boundary connected sum). Observe that, by definition,

we have

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(x_0;x_1,\dots,x_d) \subset \mathcal{M}^{0|1|0}_{\Sigma_0\cup\Sigma_1}(x_0;x_1,\dots,x_d).$$

The latter subspace consists of honest one-component pseudoholomorphic curves, and they will sometimes be referred to as *unbroken solutions*, while the pseudoholomorphic buildings not being of this form will be referred to as *broken solutions*.

The definition of a building is analogous in the case when the boundary condition is a *single* Lagrangian cobordism Σ, and we use

$$\mathcal{M}^{k^-|1|k^+}_{\Sigma}(x_0;x_1,\dots,x_d)$$

to denote the space of such buildings.

Remark 3.8. *Observe that the dichotomy between punctured pseudo-holomorphic half-planes and strips (see Section 3.2.2) also applies to the above pseudo-holomorphic buildings. We will call a pseudoholomorphic building a* broken punctured pseudo-holomorphic half-plane (resp. strip) *in the case when taking a boundary connected sum at each node produces a topological punctured half-plane (resp. strip).*

The space of buildings has a topology as defined in [13] and [2], where it also is shown that a sequence of buildings with fixed outgoing punctures has a subsequence converging to another building of the same type (but with possibly additional levels). Observe that, here we also rely on the compactness results for Lagrangian intersection Floer homology [46]. We do not reproduce the definition of the topology used in order to formulate this convergence, and we refer the reader to the relevant sections of the papers above. The following result is the main compactness result that we will need, obtained by combining the above mentioned compactness results in the SFT and Floer settings.

Theorem 3.9. *For an admissible almost complex structure, the disjoint unions of all buildings*

$$\bigsqcup_{k^+,k^-\geq 0} \mathcal{M}^{k^-|1|k^+}_{\Sigma_0,\Sigma_1}(x_0;x_1,\dots,x_d)$$
$$\bigsqcup_{k^+,k^-\geq 0} \mathcal{M}^{k^-|1|k^+}_{\Sigma}(x_0;x_1,\dots,x_d)$$

are compact and, in particular, these unions are finite.

The crucial property needed in order to apply the compactness theorems is that the above buildings consist of components having an a priori upper bound on the sum of their total energies. Indeed, this upper bound can be expressed solely in terms of the action $\mathfrak{a}(x_0)$ of the asymptotic x_0. This can be readily seen to follow from Proposition 3.5, using the fact that there is only a finite number of Reeb chords and intersection points below a fixed action. (In fact, in the setting considered here it is even the case that the totality of Reeb chords and intersection points for a given pair (Σ_0,Σ_1) is finite.)

4. Fredholm theory and transversality

4.1. **Linear Cauchy-Riemann operators.**

4.1.1. *The Grassmannian of Lagrangian planes.* Let (V,ω) be a symplectic vector space of dimension $2n+2$. We denote by $\mathrm{Gr}(V,\omega)$ the space of Lagrangian subspaces of (V,ω). The choice of a symplectic basis of V leads to an identification $\mathrm{Gr}(V,\omega)\cong U(n+1)/O(n+1)$. Given a path $\lambda\colon [0,1]\to \mathrm{Gr}(V,\omega)$, for $|s-r|$ small enough there is a family of linear isomorphisms $\phi_{r,s}\colon \lambda(s)\to\lambda(r)$ such that $\phi_{s,s}=\mathrm{Id}$. If $\overline{\lambda}\in\mathrm{Gr}(V,\omega)$, the *crossing form* $q_{\lambda,\overline{\lambda},s}$ is the quadratic form on $\lambda(s)\cap\overline{\lambda}$ defined by

$$q_{\lambda,\overline{\lambda},s}(v) := -\left.\frac{d}{dr}\right|_{r=s}\omega(\phi_{r,s}(v),v). \tag{15}$$

Example 4.1. Let $\eta\in\mathrm{Gr}(V,\omega)$ be a Lagrangian plane and $f\colon[0,1]\to\mathbb{R}$ a smooth function. We fix a complex structure on V compatible with ω and define the path $\lambda\colon[0,1]\to\mathrm{Gr}(V,\omega)$ as $\lambda(t):=e^{if(t)}\eta$. Then, at any point $s\in[0,1]$, the crossing form $q_{\lambda,\lambda(s),s}$ is

$$q_{\lambda,\lambda(s),s}(v)=f'(s)\|v\|^2.$$

We denote by $\mathcal{P}^-(\mathrm{Gr}(V,\omega))$ the set of paths $\lambda\colon[0,1]\to\mathrm{Gr}(V,\omega)$ such that $\lambda(0)\pitchfork\lambda(1)$ and the crossing form $q_{\lambda,\lambda(1),1}$ is negative definite. For a generic path $\lambda\in\mathcal{P}^-(\mathrm{Gr}(V,\omega))$ the spaces $\lambda(s)\cap\lambda(1)$ has positive dimension only at at finite set of s, and the crossing forms on the positive dimensional intersections is non-degenerate. (We can also assume, generically, that $\lambda(s)\cap\lambda(1)$ has dimension at most one for $s\neq 1$.) If λ is a generic path, we define

$$I(\lambda):=\sum_{s<1}\mathrm{ind}(q_{\lambda,\lambda(1),s}), \tag{16}$$

where $\mathrm{ind}(q)$ denotes the number of negative eigenvalues of a non-degenerate quadratic form q.

Let $\det\colon U(n+1)\to S^1$ denote the complex determinant. The map

$$A\mapsto\det(A)^2$$

descends to a well-defined map $\alpha\colon\mathrm{Gr}(V,\omega)\to S^1$ which induces an isomorphism $\alpha_*\colon\pi_1(\mathrm{Gr}(W,\omega))\to\pi_1(S^1)$. We regard α_* as a cohomology class $\mu\in H^1(\mathrm{Gr}(V,\omega);\mathbb{Z})$ called the *Maslov class*.

We denote by $\mathrm{Gr}^{\#}(V,\omega)\to\mathrm{Gr}(V,\omega)$ the universal covering of $\mathrm{Gr}(V,\omega)$. The elements of $\mathrm{Gr}^{\#}(V,\omega)$ are called *graded Lagrangian planes*. The map $\alpha\colon\mathrm{Gr}(V,\omega)\to S^1$ lifts, in a non-unique way, to a map $\alpha^{\#}\colon\mathrm{Gr}^{\#}(V,\omega)\to\mathbb{R}$ such that $\alpha=e^{2\pi i\alpha^{\#}}$. Different choices for $\alpha^{\#}$ differ by an integer constant. We can think of a graded Lagrangian plane $\lambda^{\#}$ as a pair $(\lambda,\alpha^{\#}(\lambda^{\#}))$.

Similarly, we denote by $\mathcal{P}^-(\mathrm{Gr}^{\#}(V,\omega))$ the set of paths $\lambda^{\#}\colon[0,1]\to\mathrm{Gr}^{\#}(V,\omega)$ that project to paths $\lambda\in\mathcal{P}^-(\mathrm{Gr}(V,\omega))$. Given $\lambda_0^{\#},\lambda_1^{\#}\in\mathrm{Gr}^{\#}(V,\omega)$ whose projections λ_0 and λ_1 intersect transversely, we chose a path $\lambda^{\#}\in\mathcal{P}^-(\mathrm{Gr}^{\#}(V,\omega))$ such that $\lambda^{\#}(0)=\lambda_0^{\#}$ and $\lambda^{\#}(1)=\lambda_1^{\#}$ and define the *absolute index*

$$i(\lambda_0^{\#},\lambda_1^{\#}):=I(\lambda).$$

Since $\lambda_0^{\#}$ and $\lambda_1^{\#}$ determine the homotopy class of λ and I is invariant under homotopies relative to the boundary, the absolute index is well defined.

Now we want to extend the absolute index to a map

$$i\colon\mathrm{Gr}^{\#}(V,\omega)\times\mathrm{Gr}^{\#}(V,\omega)\to\mathbb{Z}.$$

Let $\lambda_0^\#$ and $\lambda_1^\#$ be graded Lagrangian planes. If $\lambda_0 \cap \lambda_1 = \mathbf{0}$, then $i(\lambda_0^\#, \lambda_1^\#)$ is defined as above. Otherwise, let $(\lambda_0 \cap \lambda_1)^\omega$ be the symplectic orthogonal to $\lambda_0 \cap \lambda_1$ and define $\widetilde{V} := (\lambda_0 \cap \lambda_1)^\omega / (\lambda_0 \cap \lambda_1)$. The symplectic form ω induces, by linear symplectic reduction, a symplectic form $\widetilde{\omega}$ on $\widetilde{V}$. If $\lambda \subset (\lambda_0 \cap \lambda_1)^\omega$ is a Lagrangian subspace of (V, ω), then $\widetilde{\lambda} = \lambda/(\lambda_0 \cap \lambda_1)$ is a Lagrangian subspace of $(\widetilde{V}, \widetilde{\omega})$. This construction gives an embedding $\mathrm{Gr}(\widetilde{V}, \widetilde{\omega}) \hookrightarrow \mathrm{Gr}(V, \omega)$ which lifts to an embedding $\mathrm{Gr}^\#(\widetilde{V}, \widetilde{\omega}) \hookrightarrow \mathrm{Gr}^\#(V, \omega)$. Let $\widetilde{\lambda}_0^\#$ and $\widetilde{\lambda}_1^\#$ be the preimages of $\lambda_0^\#$ and $\lambda_1^\#$ in $\mathrm{Gr}^\#(\widetilde{V}, \widetilde{\omega})$. Since $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1$ intersect transversely in $\widetilde{V}$, we define

$$i(\lambda_0^\#, \lambda_1^\#) := i(\widetilde{\lambda}_0^\#, \widetilde{\lambda}_1^\#). \tag{17}$$

Next we study how the index changes for small perturbation of the Lagrangian planes. From now on we fix a complex structure on V which is compatible with ω. For a Lagrangian plane $\lambda^\#$ and a real number $\delta \neq 0$, we define $e^{i\delta}\lambda^\#$ as the end point of the path $[0,1] \to \mathrm{Gr}^\#(V, \omega)$ which starts at $\lambda^\#$ and lifts the path $t \mapsto e^{i\delta t}\lambda$. In particular $\alpha^\#(e^{i\delta}\lambda^\#) = \alpha^\#(\lambda^\#) + \frac{(n+1)\delta}{\pi}$. Given two graded Lagrangian planes $\lambda_0^\#$ and $\lambda_1^\#$, we say that $\delta \in \mathbb{R}^*$ is "small" if $\lambda_0 \pitchfork e^{i\delta t}\lambda_1$ for all $t \in (0,1]$.

Lemma 4.2. *Let $\lambda_0^\#$ and $\lambda_1^\#$ be graded Lagrangian planes and $\delta \neq 0$ a small real number.*

(1) If $\delta > 0$, then $i(\lambda_0^\#, e^{i\delta}\lambda_1^\#) = i(\lambda_0^\#, \lambda_1^\#) + \dim(\lambda_0 \cap \lambda_1)$, and
(2) If $\delta < 0$, then $i(\lambda_0^\#, e^{i\delta}\lambda_1^\#) = i(\lambda_0^\#, \lambda_1^\#)$.

Proof. There is an equivalent definition of the absolute index which is less intrinsic, but more practical for this computation. Let $\eta = \lambda_0 \cap \lambda_1$, and choose a symplectic subspace $(V', \omega') \subset (V, \omega)$ containing η as a Lagrangian subspace. Let (V'', ω'') be the symplectic orthogonal of (V', ω'); then the subspaces $\zeta_i = \lambda_i \cap V''$, for $i = 0, 1$, are Lagrangian in V''. Let $\zeta \in \mathcal{P}^-(\mathrm{Gr}(V'', \omega'')$ be a path from ζ_0 to ζ_1 such that $\lambda = \eta \oplus \zeta$ lifts to a path in $\mathrm{Gr}^\#(V, \omega)$ from $\lambda_0^\#$ to $\lambda_1^\#$. Here η is regarded as a constant path. The path ζ exists because (V'', ω'') is symplectically isomorphic to $(\widetilde{V}, \widetilde{\omega})$. Then we have

$$i(\lambda_0^\#, \lambda_1^\#) = I(\zeta). \tag{18}$$

Choose a complex structure on V which is compatible with ω and preserves the direct sum decomposition $V = V' \oplus V''$, and define $\zeta^\delta(t) := e^{i\delta t}\zeta(t)$ and $\eta^\delta(t) := e^{i\delta t}\eta$. Since δ is small, $\zeta^\delta \in \mathcal{P}^-(\mathrm{Gr}(V'', \omega''))$ and $I(\zeta^\delta) = I(\zeta) = i(\lambda_0^\#, \lambda_1^\#)$.

If $\delta < 0$, then by Example 4.1 the crossing form $q_{\eta^\delta, \eta^\delta(1), 1}$ is negative definite, so $\eta^\delta \in \mathcal{P}^-(\mathrm{Gr}(V', \omega'))$ and $I(\eta^\delta) = 0$. Then $\lambda^\delta = \eta^\delta \oplus \zeta^\delta \in \mathcal{P}^-(\mathrm{Gr}(V, \omega))$, and therefore $i(\lambda_0^\#, e^{i\delta t}\lambda_1^\#) = I(\lambda^\delta) = i(\lambda_0^\#, \lambda_1^\#)$.

If $\delta > 0$, then by Example 4.1 the crossing form $q_{\eta^\delta, \eta^\delta(1), 1}$ is positive definite, so we have to modify η^δ to obtain a path in $\mathcal{P}^-(\mathrm{Gr}(V', \omega'))$. One possible modification consists in replacing the path η^δ with the path $\widetilde{\eta}^\delta$ defined as $\widetilde{\eta}^\delta(t) := e^{i\delta(-2t^2+3t)}\eta$. If δ is small enough, $\widetilde{\eta}^\delta(t) = e^{i\delta}\eta$ only for $t = 1$ and $t = \frac{1}{2}$. Moreover, by Example 4.1, the crossing form $q_{\widetilde{\eta}^\delta, \widetilde{\eta}^\delta(1), 1/2}$ is positive definite and the crossing form $q_{\widetilde{\eta}^\delta, \widetilde{\eta}^\delta(1), 1}$ is negative definite. Then $\widetilde{\eta}^\delta \in \mathcal{P}^-(\mathrm{Gr}(V', \omega')$ and $I(\widetilde{\eta}^\delta) = \dim \eta$. Therefore $i(\lambda_0^\#, e^{i\delta t}\lambda_1^\#) = I(\lambda^\delta) = i(\lambda_0^\#, \lambda_1^\#) + \dim(\lambda_0 \cap \lambda_1)$. □

4.1.2. *Cauchy-Riemann operators over punctured discs.* Let $r \in \mathcal{R}^{d+1}$. We assume that the punctured disc Σ_r is equipped with strip-like ends ε_i as described in Section 3.1 such that ε_0 is an incoming end and ε_i, for $i = 1, \dots, d$, are outgoing ends. For $i = 0, \dots, d$ we choose orientation preserving parametrisations $l_i : \mathbb{R} \to \partial\Sigma_r$ of the connected components $\partial_i\Sigma_r$ of $\partial\Sigma_r$, which we assume to be cyclically ordered according to the natural boundary orientation.

Given $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_d)$, where each λ_i is a map $\lambda_i \colon \mathbb{R} \to \mathrm{Gr}(\mathbb{C}^n, \omega_0)$ which is constant outside a compact set, for every $p > 2$ we define the Banach space

$$W^{1,p}(\Sigma_r, \boldsymbol{\lambda}) := \{\zeta \in W^{1,p}(\Sigma_r, \mathbb{C}^n) \mid \forall i \ \zeta(l_i(s)) \in \lambda_i(s)\}.$$

We denote by $D_{r,\boldsymbol{\lambda}} \colon W^{1,p}(\Sigma_r, \boldsymbol{\lambda}) \to L^p(\Sigma_r, \mathbb{C}^n)$ the standard linear Cauchy-Riemann operator. Here we have identified $T^{0,1}\Sigma \otimes_{\mathbb{C}} \mathbb{C}^n$ with $\mathbb{C}^n$.

We denote $\lambda_i^+ = \lim\limits_{s \to +\infty} \lambda_i(s)$ and $\lambda_i^- = \lim\limits_{s \to -\infty} \lambda_i(s)$. It is well known (see [47, Proposition 4.1] for the case of strips; the general case is similar) that $D_{r,\boldsymbol{\lambda}}$ is a Fredholm operator if $\lambda_i^+ \pitchfork \lambda_{i+1}^-$ for all $i = 0, \dots, d$ (with the convention that $\lambda_{d+1} = \lambda_0$). If this is the case, the index of $D_{r,\boldsymbol{\lambda}}$ can be computed as follows.

For each path $\lambda_i \colon \mathbb{R} \to \mathrm{Gr}(\mathbb{C}^n, \omega_0)$ we make the choice of a continuous lift $\lambda_i^{\#} \colon \mathbb{R} \to \mathrm{Gr}(\mathbb{C}^n, \omega_0)^{\#}$. Then it follows from [64, Proposition 11.13] that

$$\mathrm{ind}(D_{r,\boldsymbol{\lambda}}) = i((\lambda_0^-)^{\#}, (\lambda_d^+)^{\#}) - \sum_{i=0}^{d-1} i((\lambda_i^+)^{\#}, (\lambda_{i+1}^-)^{\#}). \tag{19}$$

Note that the above expression is independent of the choice of lifts made.

We will need to consider also more general boundary conditions which do not intersect transversely on the strip-like ends. It is well known that the Cauchy-Riemann operator with these boundary conditions is not Fredholm, unless we use weighted Sobolev spaces, which we are now going to describe briefly.

We define $\chi : \Sigma_r \to \mathbb{R}$ to be a smooth function which is equal to 0 outside the strip-like ends (i.e. on $\Sigma_r \setminus (\nu_0 \cup \ldots \cup \nu_d)$), is equal to 1 on the outgoing strip-like end ε_0 (where $s \geq 1$), and is equal to -1 on the incoming strip-like ends ε_i, for $i = 1, \dots, d$ (where $s \leq -1$). Given $\delta > 0$, we introduce the Banach spaces

$$\begin{aligned} W^{1,p}_{\delta}(\Sigma_r, \boldsymbol{\lambda}) &:= \{\zeta : e^{\delta\chi(z)s}\zeta \in W^{1,p}(\Sigma_r, \boldsymbol{\lambda})\} \quad \text{and} \\ L^p_{\delta}(\Sigma_r) &:= \{\zeta : e^{\delta\chi(z)s}\zeta \in L^p(\Sigma_r, \mathbb{C}^n)\}; \end{aligned}$$

note that the function $\chi(z)s$ is well defined as $\chi(z) = 0$ outside the neighbourhoods where s is a well defined coordinate. Moreover the function $e^{\delta\chi(z)s}$ goes to $+\infty$ as $s \to \pm\infty$, and this forces the functions ζ in the weighted Sobolev spaces to go to zero sufficiently fast in the strip-like ends. We will denote by $D^{\delta}_{r,\boldsymbol{\lambda}}$ the Cauchy-Riemann operator from $W^{1,p}_{\delta}(\Sigma_r, \boldsymbol{\lambda})$ to $L^p_{\delta}(\Sigma_r)$.

We define the Lagrangian label $\boldsymbol{\lambda}^{\delta} = (\lambda_0^{\delta}, \dots, \lambda_d^{\delta})$ by the isomorphism

$$W^{1,p}_{\delta}(\Sigma_r, \boldsymbol{\lambda}) \to W^{1,p}(\Sigma_r, \boldsymbol{\lambda}^{\delta}), \quad \zeta \mapsto e^{\delta\chi(z)(s+it)}\zeta.$$

In order to understand the Fredholm properties of $D^\delta_{r,\boldsymbol{\lambda}}$, we introduce the operator $\widetilde{D}_{r,\boldsymbol{\lambda}^\delta}$ defined by the following commutative diagram:

$$\begin{array}{ccc} W^{1,p}_\delta(\Sigma_r,\boldsymbol{\lambda}) & \xrightarrow{D^\delta_{r,\boldsymbol{\lambda}}} & L^p_\delta(\Sigma_r,\mathbb{C}^n) \\ \downarrow{\scriptstyle (e^{\delta\chi(z)(s+it)}).} & & \downarrow{\scriptstyle (e^{\delta\chi(z)(s+it)}).} \\ W^{1,p}(\Sigma_r,\boldsymbol{\lambda}^\delta) & \xrightarrow{\widetilde{D}_{r,\boldsymbol{\lambda}^\delta}} & L^p(\Sigma_r,\mathbb{C}^n). \end{array}$$

In other words,

$$\widetilde{D}_{r,\boldsymbol{\lambda}^\delta}(\zeta) = e^{\delta\chi(z)(s+it)} D^\delta_{r,\boldsymbol{\lambda}}(e^{-\delta\chi(z)(s+it)}\zeta) = D_{r,\boldsymbol{\lambda}^\delta}(\zeta) - \delta\overline{\partial}(\chi(z)(s+it))\zeta$$

for $\zeta \in W^{1,p}(\Sigma_r,\boldsymbol{\lambda}^\delta)$, and thus $\widetilde{D}_{r,\boldsymbol{\lambda}^\delta}$ is a compact deformation of $D_{r,\boldsymbol{\lambda}^\delta}$ because χ is constant outside a compact set. For a generic choice of δ, the asymptotic Lagrangian labels associated to $\boldsymbol{\lambda}^\delta$ are transverse, and therefore $\widetilde{D}^\delta_{r,\boldsymbol{\lambda}}$ is Fredholm. Standard Fredholm theory implies that $D^\delta_{r,\boldsymbol{\lambda}}$ also is Fredholm and that

$$\mathrm{Ind}(D^\delta_{r,\boldsymbol{\lambda}}) = \mathrm{Ind}(\widetilde{D}_{r,\boldsymbol{\lambda}^\delta}) = \mathrm{Ind}(D_{r,\boldsymbol{\lambda}^\delta}).$$

Choose lifts $(\lambda_0^\delta)^\#,\dots,(\lambda_d^\delta)^\#$; then by Equation (19) the index of $D^\delta_{r,\boldsymbol{\lambda}}$ is

$$\mathrm{ind}(D^\delta_{r,\boldsymbol{\lambda}}) = i((\lambda_0^{\delta,-})^\#,(\lambda_d^{\delta,+})^\#) - \sum_{i=0}^{d-1} i((\lambda_i^{\delta,+})^\#,(\lambda_{i+1}^{\delta,-})^\#). \tag{20}$$

In order to make Equation (20) more explicit, we relate the absolute indices of the asymptotic Lagrangian labels before and after the perturbation.

At the incoming puncture

$$(\lambda_0^{\delta,-})^\# = e^{i\delta}(\lambda_0^-)^\#, \quad \text{and} \quad (\lambda_d^{\delta,+})^\# = (\lambda_d^+)^\#,$$

while at the outgoing punctures , for $i = 0,\dots,d-1$,

$$(\lambda_i^{\delta,+})^\# = e^{-i\delta}(\lambda_i^+)^\#, \quad \text{and} \quad (\lambda_{i+1}^{\delta,-})^\# = (\lambda_{i+1}^-)^\#.$$

Since $i(e^{i\delta}(\lambda_0^-)^\#,(\lambda_d^+)^\#) = i((\lambda_0^-)^\#,e^{-i\delta}(\lambda_d^+)^\#)$ and $i(e^{-i\delta}(\lambda_i^+)^\#,(\lambda_{i+1}^-)^\#) = i((\lambda_i^+)^\#,e^{i\delta}(\lambda_{i+1}^-)^\#)$, Example 4.1 implies that the index of $D^\delta_{r,\boldsymbol{\lambda}}$ is

$$\mathrm{Ind}(D^\delta_{r,\boldsymbol{\lambda}}) = i((\lambda_0^-)^\#,(\lambda_d^+)^\#) - \sum_{i=0}^{d-1}(i((\lambda_i^+)^\#,(\lambda_{i+1}^-)^\#) + \dim(\lambda_i^+\cap\lambda_{i+1}^-)). \tag{21}$$

4.2. **Grading.** We are now ready to define the gradings of the generators that will appear in the Cthulhu complex described Section 6. For intersection points, the grading follows Seidel [64], while for Reeb chords we recover the definition in [32] by Ekholm-Etnyre-Sullivan.

The Maslov potential of a Lagrangian cobordism. We assume that $2c_1(P) = 0$, which is equivalent to saying that $\Lambda^n_{\mathbb{C}}(TX)^{\otimes 2} \simeq X\times\mathbb{C}$. Let $\widetilde{\nu}$ be such a trivialisation. Note that since Σ is Lagrangian for any basis $(v_1,\dots,v_{n+1})$ of T_pL, the value

$$\nu(v_1,\dots,v_{n+1}) := \widetilde{\nu}(v_1,\dots,v_{n+1},v_1,\dots,v_{n+1})$$

is non-zero and, moreover,

$$\frac{\nu(v_1,\dots,v_{n+1})}{\|\nu(v_1,\dots,v_{n+1})\|} \in S^1$$

does not depend on the choice of the basis. This defines a function $\alpha : L \to S^1$ whose homotopy class $[\alpha] \in [L, S^1] \simeq H^1(L)$ is mapped to the Maslov class $\mu \in H^2(M, L)$ through the connecting homomorphisms $\rho : H^1(L) \to H^2(M, L)$. We now make the assumption that the Maslov class vanishes, from which it follows that there exists a (not uniquely determined) function $\alpha^\# : L \to \mathbb{R}$ satisfying $\alpha = e^{2i\pi\alpha^\#}$. Note that, for any $p \in \Sigma$, the pair $(T_p\Sigma, \alpha^\#(p))$ can be identified with an element of $\mathrm{Gr}^\#(T_pX, \omega_p)$.

Remark 4.3. *In the case when the Maslov does not vanish one defines the potential to take values in the cover of S^1 associated to* $\mathrm{im}[\alpha] \subset H^1(S^1) = \mathbb{Z}$*; this leads to a cyclic grading defined modulo* $\mathrm{im}[\alpha]$.

Let us now assume that we have fixed choices of functions $\alpha_0^\#$ and $\alpha_1^\#$ as above for both of the cobordisms Σ_0 and Σ_1; these functions are called *Maslov potentials* for the cobordisms. Using the same notation as in Section for $p \in \Sigma_i$, we let $\alpha_i^\#(p)$ denote $(T_p\Sigma_i)^\#$.

Grading of intersection points. Let p be an intersection point between Σ_0 and Σ_1. Given a choice of ordering (Σ_0, Σ_1), we define the grading of p by

$$\mathrm{gr}(p) = i(\alpha_0^\#(p), \alpha_1^\#(p)).$$

Grading of Reeb chords. Let γ be a Reeb chord of the link $\Lambda^\pm := \Lambda_0^\pm \cup \Lambda_1^\pm$ with starting point $p^- \in \Lambda_{i^-}^\pm$, endpoint $p^+ \in \Lambda_{i^+}^\pm$, and length ℓ. The Maslov potential on $\Sigma_0 \cup \Sigma_1$ restricts to a Maslov potential on the cylindrical ends $\mathbb{R} \times \Lambda^\pm$. Let ϕ_t be the flow of the Reeb vector field $\frac{\partial}{\partial z}$. Observe that a Hamiltonian isotopy acts on the space of graded Lagrangian planes by the homotopy lifting property. Let $\alpha_{\gamma,t}^\#$ be the lift of the path of Lagrangian planes $d\phi_t(T_{(t_0,p^-)}(\mathbb{R} \times \Lambda^\pm))$ starting at $\alpha_i(t_0, p^-)^\#$ for any $t_0 \in \mathbb{R}$. We define

$$\mathrm{gr}(\gamma) = i(\alpha_{i^+}(t_0, p^+)^\#, \alpha_{\gamma,\ell}^\#) - 1.$$

Remark 4.4. *This grading coincides with the grading of the Reeb chord generators of the Chekanov-Eliashberg algebra used in* [34]*, which are defined in terms of the Conley-Zehnder index.*

Note that the above Lagrangian subspaces do not intersect transversely. Here we must use the definition that utilises the symplectic reduction as in Section 4.1.1. In this case, this symplectic reduction can be seen to geometrically correspond to taking the canonical projection $\mathbb{R} \times P \times \mathbb{R} \to P$.

4.2.1. *Virtual dimension of the moduli spaces.* We now turn our attention to the general settings and prove the virtual dimension formulae for the moduli spaces of holomorphic discs $\mathcal{M}_{\underline{L}}(x_0, x_1, \ldots, x_{j^+ + j^- + l})$ for a Lagrangian label $\underline{L}$ taking values in $\{\Sigma_0, \Sigma_1\}$. Let p and $\{q_k\}_{k=1}^l$ be intersection points of $\Sigma_0 \cap \Sigma_1$. In addition, we let $\{\gamma_0^+\}$ and $\{\gamma_k^+\}_{k=1}^{j^+}$ be Reeb chords of $\Lambda_0^+ \cup \Lambda_1^+$, and $\{\gamma_k^-\}_{k=1}^{j^-}$ be Reeb chords of $\Lambda_0^- \cup \Lambda_1^-$. We require that the Lagrangian label satisfies the property that:

- The asymptotic x_0 at the incoming puncture is either an intersection point p_0, or a positive puncture asymptotic to the Reeb chord γ_0^+. In either case, this puncture is required to be a jump from Σ_1 to Σ_0; and
- The asymptotics of the other punctures $x_1, \ldots, x_{j^+ + j^- + l}$ correspond bijectively to $\{\gamma_k^+\}_{k=1}^{j^+} \cup \{\gamma_k^-\}_{k=1}^{j^-} \cup \{q_k\}_{k=1}^l$. Here we make no requirements

regarding the jumps made at the different punctures. (In the case of a Reeb chord, such a jump is uniquely determined. The jumps at intersection points are determined by the order of the asymptotics $(x_1, \dots, x_{j^+ + j^- + l})$ since the jump at x_0 is fixed.) For an intersection point, we define the function $(\sigma_{q_k}(0), \sigma_{q_k}(1)) := (0,1)$ in case the jump is from Σ_0 to Σ_1, while we define $(\sigma_{q_k}(0), \sigma_{q_k}(1)) := (1,0)$ in case the jump is from Σ_1 to Σ_0. Note that here we do not insist on the Lagrangian label being simple.

For a punctured pseudoholomorphic disc $u : S_r \to X$ inside the above moduli space $\mathcal{M}_{\underline{L}}(x_0, x_1, \dots, x_{j^+ + j^- + l})$ we let F_u be the linearisation of Equation (9). The following result provides a computation of the Fredholm index of F_u. Recall that this Fredholm index is equal to the dimension of the moduli space of such discs modulo reparametrisation in the case when this moduli space is transversely cut out.

Theorem 4.5. *Let (Σ_0, Σ_1) be an ordered pair of Lagrangian cobordisms of dimension $n+1$, and let u be a punctured disc with boundary on this pair of Lagrangian submanifolds as described above. The Fredholm index of F_u given by*

$$\begin{aligned} \operatorname{Ind}(F_u) = {} & \operatorname{gr}(p_0) + \sum_{k=1}^{j^+} \operatorname{gr}(\gamma_k^+) - \sum_{\sigma_{q_k}(0)=1} (n+1-\operatorname{gr}(q_k)) + \\ & - \sum_{\sigma_{q_k}(0)=0} \operatorname{gr}(q_k) - \sum_{k=1}^{j^-} \operatorname{gr}(\gamma_k^-) + \\ & + (2-n)j^+ + l - 2 \quad \textit{if} \quad x_0 = p_0, \end{aligned} \tag{22}$$

$$\begin{aligned} \operatorname{Ind}(F_u) = {} & \operatorname{gr}(\gamma_0^+) + \sum_{k=1}^{j^+} \operatorname{gr}(\gamma_k^+) - \sum_{\sigma_{q_k}(0)=1} (n+1-\operatorname{gr}(q_k)) + \\ & - \sum_{\sigma_{q_k}(0)=0} \operatorname{gr}(q_k) - \sum_{k=1}^{j^-} \operatorname{gr}(\gamma_k^-) + \\ & + (2-n)j^+ + l \quad \textit{if} \quad x_0 = \gamma_0^+, \end{aligned} \tag{23}$$

where the gradings of intersection points are defined using the order (Σ_0, Σ_1) as in Section 4.2.

Proof. The operator F_u decomposes as $D_u \oplus K_u \oplus \tau_m$ where D_u acts on $W^{1,p}_\delta$, $K_u \colon \mathbb{R}^j \to 0$ with $j = j_+ + j_-$ (resp. $j = j_+ + j_- + 1$) in the case when $x_0 = p_0$ (resp. $x_0 = \gamma_0^+$), and $\tau_{j^+ + j^- + l + 1}$ is the operator associated to the deformation of the Teichmüller space at r. The latter operator has index equal to the dimension of $\mathcal{R}^{j^+ + j^- + l + 1}$, minus the dimension of the automorphisms group of the curve S_r.

The index of K_u is $j^+ + j^-$ if $x_0 = p_0$ and $j^+ + j^- + 1$ if $x_0 = \gamma_0^+$. Moreover, $\operatorname{Ind} \tau_m = j^+ + j^- + l - 2$. For $k \in \{1 \dots j^\pm\}$ we denote by $p_k^\pm$ the end of the chord

$\gamma_k^{\pm}$. From Equation (21)we get that

$$\mathrm{Ind}(D_u) = i((T_{p_0}\Sigma_0)^{\#}, (T_{p_0}\Sigma_1)^{\#}) - \sum_{k=1}^{j^+} \big(i(\alpha^{\#}_{\gamma_k^+,\ell_k}, T_{(0,p_k^+)}(\mathbb{R}\times\Lambda^+)^{\#}) + 1\big)$$

$$- \sum_k i(T_{q_k}\Sigma_{\sigma_{q_k}(0)})^{\#}, (T_{q_k}\Sigma_{\sigma_{q_k}(1)})^{\#}) - \sum_{k=1}^{j^-} \big(i(T_{(0,p_k^-)}(\mathbb{R}\times\Lambda^-)^{\#}), \alpha^{\#}_{\gamma_k^-,\ell_k}) + 1\big) \quad \text{or}$$

$$\mathrm{Ind}(D_u) = i(T_{(0,p_0^+)}(\mathbb{R}\times\Lambda^+)^{\#}, \alpha^{\#}_{\gamma_0^+,\ell_0}) - \sum_{k=1}^{j^+} \big(i(\alpha^{\#}_{\gamma_k^+,\ell_k}, T_{(0,p_k^+)}(\mathbb{R}\times\Lambda^+)^{\#}) + 1\big)$$

$$- \sum_k i(T_{q_k}\Sigma_{\sigma_{q_k}(0)})^{\#}, (T_{q_k}\Sigma_{\sigma_{q_k}(1)})^{\#}) - \sum_{k=1}^{j^-} \big(i(T_{(0,p_k^-)}(\mathbb{R}\times\Lambda^-)^{\#}, \alpha^{\#}_{\gamma_k^-,\ell_k}) + 1\big).$$

Depending on whether x_0 is p_0 or γ_0^+.

Note that for $q \in \Sigma_0 \cap \Sigma_1$, we have

$$i((T_q\Sigma_0)^{\#}, (T_q\Sigma_1)^{\#}) = n + 1 - i((T_q\Sigma_1)^{\#}, (T_q\Sigma_0)^{\#})$$

whereas that for a Reeb chord γ with end point p^+ we have that

$$i(\alpha^{\#}_{\gamma,\ell}, T_{(0,p^+)}(\mathbb{R}\times\Lambda^{\pm})^{\#}) = n - i(T_{(0,p^+)}(\mathbb{R}\times\Lambda^{\pm})^{\#}, \alpha^{\#}_{\gamma,\ell}).$$

Thus we obtain

$$\mathrm{Ind}(D_u) = \mathrm{gr}(p_0) - \sum_{i=1}^{j^+}(n - \mathrm{gr}(\gamma_k^+)) - \sum_{\sigma(q_k)(0)=0} \mathrm{gr}(q_k)$$

$$- \sum_{\sigma_{q_k}(0)=1} (n + 1 - \mathrm{gr}(q_k)) - \sum_{1}^{j^-}(\mathrm{gr}(\gamma_k^-) + 2) \quad \text{or}$$

$$\mathrm{Ind}(D_u) = \mathrm{gr}(\gamma_0^+) + 1 - \sum_{i=1}^{j^+}(n - \mathrm{gr}(\gamma_k^+))$$

$$- \sum_{\sigma(q_k)(0)=0} \mathrm{gr}(q_k) - \sum_{\sigma_{q_k}(0)=1} (n + 1 - \mathrm{gr}(q_k)) - \sum_{1}^{j^-}(\mathrm{gr}(\gamma_k^-) + 2) \quad \text{or}$$

Since $\mathrm{Ind}(F_u) = \mathrm{Ind}(D_u) + \mathrm{Ind}(K_u) + \mathrm{Ind}(\tau_m)$ the result follows. □

We apply Equation (22) to compute the virtual dimension of the moduli spaces defined in Section 3.2. For $(n+1)$-dimensional Lagrangian cobordisms, we obtain

$$\begin{aligned}
(24)\qquad \dim(\widetilde{\mathcal{M}}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)) &= \dim(\mathcal{M}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)) - 1\\
&= \mathrm{gr}(\gamma^+) - \mathrm{gr}(\gamma^-) - \mathrm{gr}(\boldsymbol{\delta}) - \mathrm{gr}(\boldsymbol{\zeta}) - 1,\\
(25)\qquad \dim(\mathcal{M}(\gamma^+;\boldsymbol{\delta}^-,q,\boldsymbol{\zeta}^-)) &= \mathrm{gr}(\gamma^+) - \mathrm{gr}(q) - \mathrm{gr}(\boldsymbol{\delta}) - \mathrm{gr}(\boldsymbol{\zeta}) + 1,\\
(26)\qquad \dim(\mathcal{M}(p;\boldsymbol{\delta}^-,q,\boldsymbol{\zeta}^-)) &= \mathrm{gr}(p) - \mathrm{gr}(q) - \mathrm{gr}(\boldsymbol{\delta}) - \mathrm{gr}(\boldsymbol{\zeta}) - 1,\\
(27)\qquad \dim(\mathcal{M}(p;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)) &= \mathrm{gr}(p) - \mathrm{gr}(\gamma^-) - \mathrm{gr}(\boldsymbol{\delta}) - \mathrm{gr}(\boldsymbol{\zeta}) - 2,\\
(28)\qquad \dim(\widetilde{\mathcal{M}}(\gamma_{2,1};\boldsymbol{\delta}^-,\gamma_{1,2},\boldsymbol{\zeta}^-)) &= \dim(\mathcal{M}(\gamma_{2,1};\boldsymbol{\delta}^-,\gamma_{1,2},\boldsymbol{\zeta}^-)) - 1\\
&= \mathrm{gr}(\gamma_{2,1}) + \mathrm{gr}(\gamma_{1,2}) - \mathrm{gr}(\boldsymbol{\delta}) - \mathrm{gr}(\boldsymbol{\zeta}) - n + 1.
\end{aligned}$$

4.3. **Transversality.** Here we have gather various results from the literature that are needed in order to achieve transversality for the relevant moduli spaces of punctured pseudoholomorphic discs inside $\mathbb{R}\times P\times\mathbb{R}$. Recall that the transversality of a moduli space in particular implies that it is a smooth manifold of dimension equal to the Fredholm index of the corresponding linearised problem (see Section 4.1.2). As usual, an almost complex structure for which the relevant moduli spaces are transversely cut out is called *regular*.

4.3.1. *Punctured discs and strips with a cylindrical boundary condition.* Consider the moduli spaces for a cylindrical boundary condition, and where all punctures are asymptotic to Reeb chords of which precisely one is positive with asymptotic γ^+. In other words, we are interested in moduli spaces of the form

$$\mathcal{M}_{\mathbb{R}\times\Lambda}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-),$$
$$\mathcal{M}_{\mathbb{R}\times\Lambda_0,\mathbb{R}\times\Lambda_1}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-),\ \ \gamma^+\in\mathcal{R}(\Lambda_1,\Lambda_0),$$

as described in Section 3.2.3. Note that solutions in the latter moduli space also are solutions (of a special kind) inside the former moduli space for $\Lambda=\Lambda_0\cup\Lambda_1$. We also consider moduli spaces of the form

$$\mathcal{M}_{\mathbb{R}\times\Lambda_0,\mathbb{R}\times\Lambda_1}(\gamma_{1,0};\boldsymbol{\delta}^-,\gamma_{0,1},\boldsymbol{\zeta}^-),$$

as considered in Section 3.2.7. Recall that here all punctures have negative asymptotics to Reeb chords except precisely *two*, where these two punctures moreover have positive asymptotics to two *distinct* Reeb chords $\gamma_{1,0}\in\mathcal{R}(\Lambda_1,\Lambda_0)$, $\gamma_{0,1}\in\mathcal{R}(\Lambda_0,\Lambda_1)$, and where $\boldsymbol{\delta}^-$ and $\boldsymbol{\zeta}^-$ denote words of chords on Λ_0 and Λ_1, respectively.

The result [24, Proposition 3.13] can be applied to all of these moduli spaces, showing that they are transversely cut-out for a Baire first category subset of almost complex structures $\mathcal{J}^{\mathrm{cyl}}(P\times\mathbb{R})$. This result can be seen as an adaptation of Dragnev's result in [28] to the case of a cylindrical boundary condition. Note that, even if the aforementioned result assumes that there is precisely one positive puncture, in this case its proof can still be applied, since the two positive punctures of the solution have different asymptotics.

For technical reasons it will be necessary to achieve transversality also for the subset $\mathcal{J}^{\mathrm{cyl}}_{\pi}(P\times\mathbb{R})\subset\mathcal{J}^{\mathrm{cyl}}(P\times\mathbb{R})$ of cylindrical lifts of almost complex structures (see Section 3.1.3). In this case, [25, Lemma 8.2] shows that a $\widetilde{J}_P$-holomorphic disc $\widetilde{u}$ in $\mathbb{R}\times P\times\mathbb{R}$ is transversely cut out if and only if its J_P-holomorphic projection $u:=\pi\circ u$ is transversely cut out. That the latter discs are transversely cut out for a Baire first category subset of admissible almost complex structures was shown in [34, Proposition 2.3] under the following additional assumption:

(A) Fix a chord-generic Legendrian submanifold $\Lambda\subset P\times\mathbb{R}$. The pair (Λ,J_P) said to be *admissible* if the following is satisfied. The almost complex structure J_P is integrable in some neighbourhood of each double point of the Lagrangian projection $\Pi_{\mathrm{Lag}}(\Lambda)$, where the two sheets of this Lagrangian immersion moreover are real-analytic.

Note that this condition imposes no restriction on the Legendrian submanifold. Namely, every transverse Lagrangian intersection is symplectomorphic to the intersection of the real and imaginary parts in $(\mathbb{C}^n,\omega_0)$ by a version of Weinstein's Lagrangian neighbourhood theorem.

Combining the above results, we obtain:

Proposition 4.6. *The above moduli spaces are transversely cut out for a Baire first category subset of the cylindrical almost complex structures $\mathcal{J}^{\mathrm{cyl}}(P\times\mathbb{R})$, as well as for a Baire first category subset of the admissible cylindrical lifts $\mathcal{J}^{\mathrm{cyl}}_{\pi}(P\times\mathbb{R})$ of almost complex structures on J_P satisfying condition (A).*

4.3.2. *Discs with boundary on a general embedded Lagrangian cobordism.* We are here interested in the case when the boundary condition is a general embedded Lagrangian cobordism Σ from Λ^- to Λ^+, and when all asymptotics are Reeb chords. In other words, we consider moduli spaces of the form

$$\mathcal{M}_{\Sigma}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-)$$

as described in Section 3.2.3.

Proposition 4.7. *Assume that we are given cylindrical almost complex structures $J^{\pm}\in\mathcal{J}^{\mathrm{cyl}}(P\times\mathbb{R})$ which are regular for $\mathcal{M}_{\mathbb{R}\times\Lambda^{\pm}}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-)$, as well as a number $T>0$ for which $\Sigma\setminus([-T,T]\times P\times\mathbb{R})$ is cylindrical. The above moduli spaces of the form*

$$\mathcal{M}_{\Sigma}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-)$$

are transversely cut out for a Baire first category subset of $\mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T}(\mathbb{R}\times P\times\mathbb{R})$.

Proof. From [17, Theorem 2.8] it follows that the discs in these moduli spaces are simple, and a standard transversality argument [56, Chapter 3] can thus be applied. To that end, observe that it suffices to find an injective point of the disc contained inside $(-T,T)\times P\times\mathbb{R}$, where the sought perturbation of the almost complex structure will be supported. (In the case when the disc is disjoint from this subset, it is in fact already transversely cut out by the assumptions of the proposition.) □

4.3.3. *Strips with at least one asymptotic to an intersection point.* Let Σ_i, $i=0,1$, be two transversely intersecting exact Lagrangian cobordisms from Λ_i^- to Λ_i^+. We now consider moduli spaces of strips being of the form

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(x_0;\boldsymbol{\delta}^-,x_1,\boldsymbol{\zeta}^-),$$

where $\boldsymbol{\delta}^-$ and $\boldsymbol{\zeta}^-$ denote words of Reeb chords on Λ_1^- and Λ_0^-, respectively, and where at least one of x_0 and x_1 is an intersection point of $\Sigma_0\cap\Sigma_1$. See Sections 3.2.4, 3.2.5, and 3.2.6. Recall that exactness implies that $x_0\neq x_1$ holds in either of the cases. It is here important to note that $\Sigma_0\cup\Sigma_1$ is not an embedded boundary condition.

One option for achieving transversality for these moduli spaces is via the technique in [34], while imposing a condition analogous to (A) near the intersection points. In other words, one has to consider almost complex structures which are integrable in a neighbourhood of $\Sigma_0\cap\Sigma_1$, which moreover make the Lagrangian submanifolds real-analytic inside the same neighbourhood.

Alternatively, transversality is also possible to achieve when using so-called "time-dependent" almost complex structures as described in Section 3.1.5. This technique goes back to Floer [46]. Recall that the strips in the above moduli space are maps of the form

$$u\colon(\mathbb{R}\times[0,1],\mathbb{R}\times\{0\},\mathbb{R}\times\{1\})\to(\mathbb{R}\times P\times\mathbb{R},\Sigma_0,\Sigma_1),$$

where the domain is endowed with the conformal structure coming from the standard holomorphic coordinate $s+it$ on $\mathbb{R}\times[0,1]\subset\mathbb{R}^2=\mathbb{C}$ (this parametrisation is determined up to a translation of the s coordinate).

We consider sufficiently regular one-parameter families $\{J_t\}_{t\in[0,1]}$ of almost complex structures $J_t \in \mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(\mathbb{R}\times P\times\mathbb{R})$, and define the moduli spaces for the corresponding time-dependent version of the Cauchy-Riemann operation on a strip as in Section 3.1.5. The somewhere injectivity result in e.g. [59, Theorem 5.1] or [9, Section 8.6] also holds in the current setting, showing that:

Proposition 4.8. *Assume that we are given cylindrical almost complex structures $J^\pm \in \mathcal{J}^{\mathrm{cyl}}(P\times\mathbb{R})$ which are regular for $\mathcal{M}_{\mathbb{R}\times\Lambda^\pm}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-)$, as well as a number $T>0$ for which $\Sigma\setminus([-T,T]\times P\times\mathbb{R})$ is cylindrical. The above moduli spaces of the form*

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(x_0;\boldsymbol{\delta}^-,x_1,\boldsymbol{\zeta}^-)$$

are transversely cut out for a Baire first category subset of the time-dependent almost complex structures $J_t\colon I\to\mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T}(\mathbb{R}\times P\times\mathbb{R})$.

4.3.4. *Strips with only Reeb chord asymptotics.* We now consider moduli spaces of strips being of the form

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)$$

as above, but where $\gamma^\pm$ both are Reeb chords; see Sections 3.2.3 and 3.2.7. Observe that this strip has a puncture with a positive asymptotic to a Reeb chord γ^+ from Λ_1^+ to Λ_0^+, while γ^- is either a Reeb chord from Λ_1^- to Λ_0^-, which hence is a negative asymptotic, or a Reeb chord from Λ_0^+ to Λ_1^+, which hence is a positive asymptotic. In either case, we have $\gamma^+\neq\gamma^-$.

Proposition 4.9. *Assume that we are given cylindrical almost complex structures $J^\pm\in\mathcal{J}^{\mathrm{cyl}}(P\times\mathbb{R})$ which are regular for every moduli space of the form*

$$\mathcal{M}_{\mathbb{R}\times\Lambda_0^\pm\cup\mathbb{R}\times\Lambda_1^\pm}(\gamma^+;\gamma_1^-,\dots,\gamma_d^-),$$
$$\mathcal{M}_{\mathbb{R}\times\Lambda_0^\pm,\mathbb{R}\times\Lambda_1^\pm}(\gamma_1^+;\gamma_1^-,\dots,\gamma_{d_1}^-,\gamma_2^+,\gamma_{d_1+1}^-,\dots,\gamma_d^-),$$

as well as a number $T>0$ for which $\Sigma\setminus([-T,T]\times P\times\mathbb{R})$ is cylindrical. Any almost complex structure $J\in\mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T}(\mathbb{R}\times P\times\mathbb{R})$ can be perturbed to an almost complex structure $J'\in\mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T'}(\mathbb{R}\times P\times\mathbb{R})$ which is regular for the moduli spaces of the form

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-),$$

under the assumption that $T'\gg T$ was chosen sufficiently large.

Proof. The argument is done by induction on the energy of the discs in the moduli spaces. To that end, the following feature of the SFT compactness theorem is crucial:

- In our setting, the space of solutions of $d(\phi\alpha)$-energy at most $E>0$ has also a uniform bound on the total energy. Hence, the space $\mathcal{M}^E$ of these solutions can be compactified to $\overline{\mathcal{M}}^E$ by adding broken configurations consisting of pseudoholomorphic buildings;
- The boundary strata of the compactified moduli space $\overline{\mathcal{M}}^E$ consists of buildings whose components $u_1,\dots,u_k$ have $d(\phi\alpha)$-energies whose sum is at most E; and
- The complement of any open neighbourhood of the boundary strata in $\overline{\mathcal{M}}^E$ is a compact subspace of $\mathcal{M}^E$ in the appropriate Whitney topology.

We also note that, in our setting, the possible $d(\phi\alpha)$-energies attained by a non-trivial pseudoholomorphic curve having at most one output puncture is a *finite set of numbers* $E_k > E_{k-1} > \ldots > E_1 > 0$, as follows from the finite possibilities of asymptotics in the current setting.

The base case of the induction is as follows. The J-holomorphic discs in $\mathcal{M}^{E_1}$ of lowest $d(\phi\alpha)$-energy cannot break by the above, and must thus be compact as a space of maps. Using the asymptotic properties of finite energy discs, all solutions must have injective points in some subset of the form $[T'', T''+1] \times P \times \mathbb{R}$, for $T'' \gg T$ sufficiently big. After a perturbation of J supported inside $[T'', T''+1] \times P \times \mathbb{R}$, we may thus assume that the moduli space $\mathcal{M}^{E_1} = \overline{\mathcal{M}}^{E_1}$ of lowest energy is transversely cut out.

By induction we assume that all moduli spaces of energy strictly less than $E_i > 0$, $i > 1$, are transversely cut out. Pseudoholomorphic gluing as in Theorem 4.11 can be upgraded to the following statement: Gluing of transversely cut out solutions can always be performed and, a priori, produces transversely cut out solutions. Since the boundary strata of $\overline{\mathcal{M}}^{E_i}$ consists of transversely cut out components by the induction hypothesis together with the assumptions of the proposition, the moduli space $\mathcal{M}^{E_i}$ thus also consists of transversely cut out solutions in some neighbourhood of its boundary strata.

We are thus left to achieve transversality for a subset of solutions contained in the complement of some neighbourhood of the boundary of $\overline{\mathcal{M}}^{E_i}$. By the above, these solutions form a compact space of maps. An argument as in the base case can again be used to show the following. After a suitable perturbation of the almost complex structure J in a region containing the positive end (where these maps all have injective points by the asymptotic properties), the full moduli space $\mathcal{M}^{E_i}$ can be assumed to be transversely cut out. □

We also give the following alternative approach to transversality, using time-dependent almost complex structures.

Proposition 4.10. *Assume that we are given a pair $J^{\pm} \in \mathcal{J}^{\mathrm{cyl}}(P \times \mathbb{R})$ of cylindrical almost complex structures which are regular for all moduli spaces of the form*

$$\mathcal{M}_{\mathbb{R}\times\Lambda_0^{\pm}\cup\mathbb{R}\times\Lambda_1^{\pm}}(\gamma^+;\gamma_1^-,\ldots,\gamma_d^-),$$
$$\mathcal{M}_{\mathbb{R}\times\Lambda_0^{\pm},\mathbb{R}\times\Lambda_1^{\pm}}(\gamma_1^+;\gamma_1^-,\ldots,\gamma_{d_1}^-,\gamma_2^+,\gamma_{d_1+1}^-,\ldots,\gamma_d^-),$$

as well as a number $T > 0$ for which $\Sigma \setminus ([-T,T] \times P \times \mathbb{R})$ is cylindrical. The above moduli spaces of the form

$$\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)$$

are transversely cut out for a Baire first category subset of the time-dependent almost complex structures $J_t \colon I \to \mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T}(\mathbb{R} \times P \times \mathbb{R})$.

Proof. The somewhere injectivity result in [9, Section 8.6] can be extended to the case when all asymptotics of the strip are Reeb chords. Observe that the strips that do not pass through $[-T,T] \times P \times \mathbb{R}$ are transversely cut out by assumption. Consequently, it suffices to turn on the time-dependence in that region. □

4.4. **Gluing.** In this section we explain how transversely cut-out buildings as in Section 3.3.3 can be glued to give a family of holomorphic disc which converges to the original building. In other words, the moduli space of buildings, which can be

compactified by Theorem 3.9, will under transversality assumptions be a compact manifold whose boundary strata consists of broken solutions.

We assume from now on that all almost complex structures are regular (see Section 4.3). To a nodal curve S with normalisation $\widehat{S}$, we associate a tree $T_{\widehat{S}}$ as in Section 2.3.3 whose vertices correspond to connected components of $\widehat{S}$, interior edges connecting two vertices correspond to nodes, and whose leaves correspond to ends.

The following theorem is a standard consequence of the techniques used in [32, Section 8].

Theorem 4.11. *Assume that the almost complex structure is regular and admissible, and let $\$$ denote either "Σ" or "Σ_0, Σ_1". Consider a holomorphic building in $\mathcal{M}_{\$}^{k^-|1|k^+}(x_0; x_1, \cdots, x_d)$ associated a nodal curve S, a tree T_S, and consisting of the pseudoholomorphic discs $\{u_i\}$. For $\epsilon_0 > 0$ sufficiently small, let $\epsilon_0 > \rho_1, \dots, \rho_\nu > 0$ be numbers associated to each pairs of nodes asymptotic to an intersection point, and let $\epsilon_0 > \rho_{\nu+1}, \dots, \rho_{\nu+k} > 0$, $k = k_+ + k_- \geq 0$, be numbers associated to each non-empty level of the building except the 0:th level. Then, there exists a uniquely determined punctured pseudoholomorphic disc*

$$u_{\boldsymbol{\rho}} \in \mathcal{M}_{\$}(x_0; x_1, \cdots, x_d), \;\; \boldsymbol{\rho} = (\rho_1, \dots, \rho_{\nu+k}),$$

the so-called glued solution*, satisfying the property that $u_{\boldsymbol{\rho}_i}$, $i \to \infty$, converges to the original building whenever $\boldsymbol{\rho}_i = (\rho_{1,i}, \dots, \rho_{\nu+k,i})$ satisfies $\lim_{i\to\infty} \rho_{j,i} = 0$, $j = 1, \dots, k+\nu$.*

Remark 4.12. *The expected dimension of a glued solution in $\mathcal{M}_{\$}(x_0; x_1, \cdots, x_d)$ produced by the above theorem is given by the sum*

$$\nu + \sum_i \mathrm{Ind}(u_i),$$

in terms of the expected dimensions of all involved components $u_1, \dots, u_m$ in the original building, where ν is the total number of the involved pairs of nodes asymptotic to intersection points.

Together with the compactness result in Theorem 3.9 we obtain the following crucial result.

Corollary 4.13. *Assume that the almost complex structure is regular and admissible, and let $\$$ denote either "Σ" or "Σ_0, Σ_1". The compactification $\overline{\mathcal{M}}_{\$}(x_0; x_1, \dots, x_d)$ of a moduli space $\mathcal{M}_{\$}(x_0; x_1, \dots, x_d)$ of punctured pseudoholomorphic discs being of index one is a transversely cut out one-dimensional manifold with boundary. Moreover, its boundary points are in bijective correspondence with the broken solutions inside $\mathcal{M}_{\$}^{k^-|1|k^+}(x_0; x_1, \cdots, x_d)$ whose components $u_1, \dots, u_m$ satisfy*

$$\nu + \sum_i \mathrm{Ind}(u_i) = 1,$$

and where ν denotes the total number of pairs of nodes asymptotic to intersection points.

A geometric analysis of the a priori possibilities of building leads to the following structure of the boundary of the relevant moduli spaces of dimension one.

Let a, b be asymptotics and $\boldsymbol{\delta}$ and $\boldsymbol{\zeta}$ be sets of Reeb chords of Λ_0^- and Λ_1^- respectively such that for regular J the moduli space $\mathcal{M}(a;\boldsymbol{\delta},b,\boldsymbol{\zeta})$ is a 1-dimensional manifold. By Corollary 4.13 it follows that the compactified moduli space $\overline{\mathcal{M}}(a;\boldsymbol{\delta},b,\boldsymbol{\zeta})$ is a compact 1-dimensional manifold whose boundary consists of broken solutions.

For words w, w' and w'' in a free group such that w' is a subword of w (denoted $w' \subset w$) we denote by $w_{w'}$ the word obtain removing w' from w and by $w_{w'}(w'')$ the word obtain replacing w' by w''.

We proceed to explicitly describe the boundary of $\overline{\mathcal{M}}(a;\boldsymbol{\delta},b,\boldsymbol{\zeta})$ for different incoming and outgoing ends. Note that, by the energy estimates of Section 3.3.2 (which use the exactness assumptions), there are no components all whose punctures have negative asymptotics. Also, since every possible component of a broken configuration has non-negative index by the regularity assumptions, we have a restriction on the number of breakings involved in these compactification. In the following lists all unions are assumed to be over 0-dimensional moduli spaces.

For $a = p \in \Sigma_0 \cap \Sigma_1$ and $b = \gamma^- \in \mathcal{R}(\Lambda_1^-, \Lambda_0^-)$ we conclude that (see Figure 8)

$$\begin{aligned}(29)\qquad \partial\overline{\mathcal{M}}(p;\boldsymbol{\delta},\gamma^-,\boldsymbol{\zeta}) = \\ \bigcup_{c,\boldsymbol{\delta}'\boldsymbol{\delta}''=\boldsymbol{\delta},\boldsymbol{\zeta}'\boldsymbol{\zeta}''=\boldsymbol{\zeta}} \mathcal{M}(p;\boldsymbol{\delta}',c,\boldsymbol{\zeta}'')\times\mathcal{M}(c;\boldsymbol{\delta}'',\gamma^-,\boldsymbol{\zeta}') \\ \bigcup_{\boldsymbol{\delta}'\subset\boldsymbol{\delta},\delta_0} \mathcal{M}(p;\boldsymbol{\delta}_{\boldsymbol{\delta}'}(\delta_0),\gamma^-,\boldsymbol{\zeta})\times\mathcal{M}(\delta_0;\boldsymbol{\delta}') \\ \bigcup_{\boldsymbol{\zeta}'\subset\boldsymbol{\zeta},\zeta_0} \mathcal{M}(p;\boldsymbol{\delta},\gamma^-,\boldsymbol{\zeta}_{\boldsymbol{\zeta}'}(\zeta_0))\times\mathcal{M}(\zeta_0;\boldsymbol{\zeta}')\end{aligned}$$

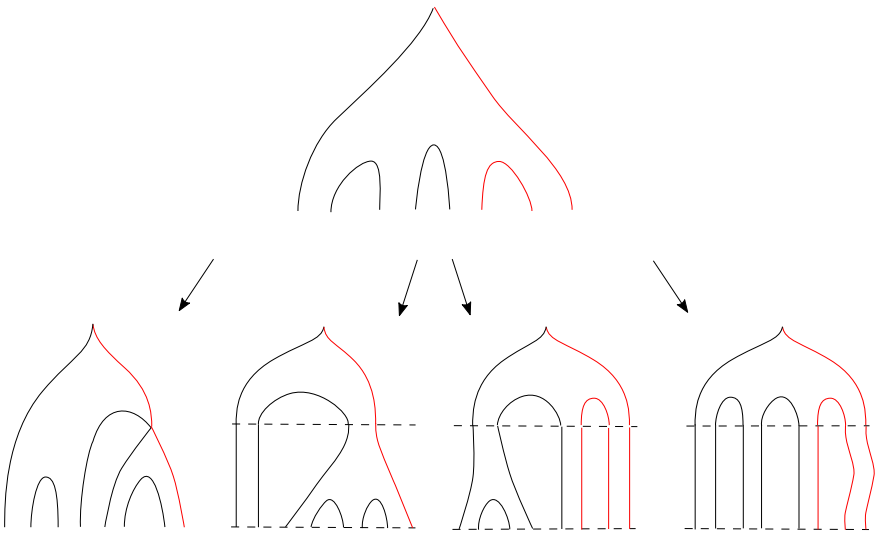

FIGURE 8. A schematic view of the boundary of $\overline{\mathcal{M}}(p;\boldsymbol{\delta},\gamma^-,\boldsymbol{\zeta})$.

Observe that the latter two types of boundary points in the above union, i.e. involving pseudoholomorphic half-planes, always can appear in the boundary of a 1-dimensional moduli space. We call them ∂-breakings and denote them by

$$\overline{\mathcal{M}}^{\partial}(a;\boldsymbol{\delta},b,\boldsymbol{\zeta}).$$

The upshot of the constructions of the Cthulhu complex will be that, when counting the boundary points of the moduli space weighted by an augmentation, the boundary points corresponding to ∂-breakings will give a total contribution by 0; see (II) below for more details.

For $a = p, b = q \in \Sigma_0 \cap \Sigma_1$ we conclude that

$$\begin{aligned}(30)\qquad \partial\overline{\mathcal{M}}(p;\boldsymbol{\delta},q,\boldsymbol{\zeta}) = \\ \bigcup_{r,\boldsymbol{\delta}'\boldsymbol{\delta}''=\boldsymbol{\delta},\boldsymbol{\zeta}'\boldsymbol{\zeta}''=\boldsymbol{\zeta}} \mathcal{M}(p;\boldsymbol{\delta}'',r,\boldsymbol{\zeta}')\times\mathcal{M}(r;\boldsymbol{\delta}',q,\boldsymbol{\zeta}'') \\ \bigcup \mathcal{M}(p;\boldsymbol{\delta}',\gamma_{10},\boldsymbol{\zeta}''')\times\mathcal{M}(\gamma_{10};\boldsymbol{\delta}'',\gamma_{01},\boldsymbol{\zeta}'')\times\mathcal{M}(\gamma_{01};\boldsymbol{\delta}''',q,\boldsymbol{\zeta}') \\ \bigcup \overline{\mathcal{M}}^{\partial}(p;\boldsymbol{\delta},q,\boldsymbol{\zeta}).\end{aligned}$$

Where the second union is over:

- $\gamma_{01} \in \mathcal{R}(\Lambda_0^-,\Lambda_1^-), \gamma_{10} \in \mathcal{R}(\Lambda_1^-,\Lambda_0^-)$; and
- $\boldsymbol{\delta}'\boldsymbol{\delta}''\boldsymbol{\delta}''' = \boldsymbol{\delta}, \boldsymbol{\zeta}'\boldsymbol{\zeta}''\boldsymbol{\zeta}''' = \boldsymbol{\zeta}$.

The second type of breaking here is depicted in Figure 9.

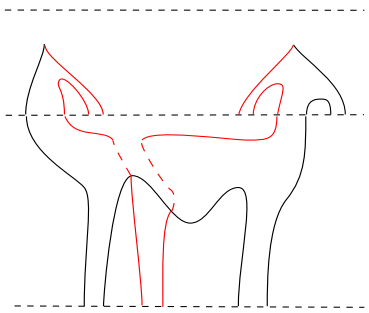

FIGURE 9. A boundary point in $\overline{\mathcal{M}}(p;\boldsymbol{\delta},q,\boldsymbol{\zeta})$

For $a = \gamma^+ \in \mathcal{R}(\Lambda_1^+,\Lambda_0^+)$ and $b = \gamma^- \in \mathcal{R}(\Lambda_1^-,\Lambda_0^-)$, we have

$$\begin{aligned}(31)\qquad \partial\overline{\mathcal{M}}(\gamma^+;\boldsymbol{\delta},\gamma^-,\boldsymbol{\zeta}) = \\ \bigcup \mathcal{M}(\gamma^+;\delta_1^+\cdots\delta_l^+,\gamma_0,\zeta_1^+\cdots\zeta_m^+)\times\big(\mathcal{M}(\delta_1^+;\boldsymbol{\delta}_1^-)\sqcup\cdots\sqcup\mathcal{M}(\delta_m^+;\boldsymbol{\delta}_m^-) \\ \sqcup\mathcal{M}(\gamma_0;\boldsymbol{\delta}_{l+1}^-,\gamma^-,\boldsymbol{\zeta}_0^-)\sqcup\mathcal{M}(\zeta_1^+;\boldsymbol{\zeta}_1^-)\sqcup\cdots\sqcup\mathcal{M}(\zeta_m^+;\boldsymbol{\zeta}_m^-)\big) \\ \bigcup \mathcal{M}(\gamma^+;\boldsymbol{\delta}',c,\boldsymbol{\zeta}'')\times\mathcal{M}(c;\boldsymbol{\delta}'',\gamma^-,\boldsymbol{\zeta}') \\ \bigcup \overline{\mathcal{M}}^{\partial}(\gamma^+;\boldsymbol{\delta},\gamma^-,\boldsymbol{\zeta})\end{aligned}$$

with the first union being over

- $\gamma^+ \in \mathcal{R}(\Lambda_1^+,\Lambda_0^+)$;
- $\delta_1^+,\dots,\delta_l^+ \in \mathcal{R}(\Lambda_0^+)$;
- $\zeta_1^+,\dots,\zeta_m^+ \in \mathcal{R}(\Lambda_1^+)$; and
- $\boldsymbol{\delta}_1^-\boldsymbol{\delta}_2^-\cdots\boldsymbol{\delta}_{l+1}^- = \boldsymbol{\delta}, \boldsymbol{\zeta}_0^-\boldsymbol{\zeta}_1^-\cdots\boldsymbol{\zeta}_m^- = \boldsymbol{\zeta}$,

and the second being over

- $c \in \Sigma_0 \cap \Sigma_1 \cup \mathcal{R}(\Lambda_1^-,\Lambda_0^-)$; and
- $\boldsymbol{\delta}'\boldsymbol{\delta}'' = \boldsymbol{\delta}, \boldsymbol{\zeta}'\boldsymbol{\zeta}'' = \boldsymbol{\zeta}$.

The first type of breaking is depicted in Figure 10.

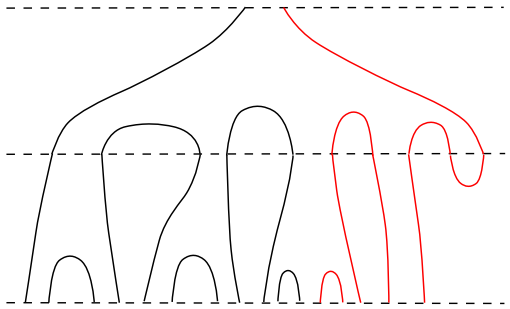

FIGURE 10. A boundary point in $\overline{\mathcal{M}}(\gamma^+;\boldsymbol{\delta},\gamma^-,\boldsymbol{\zeta})$

Finally, for $a=\gamma^+\in\mathcal{R}(\Lambda_1^+,\Lambda_0^+)$ and $b=p\in\Sigma_0\cap\Sigma_1$, we have

$$
\begin{aligned}
(32)\qquad & \partial\overline{\mathcal{M}}(\gamma^+;\boldsymbol{\delta},p,\boldsymbol{\zeta}) = \\
& \bigcup \mathcal{M}(\gamma^+;\delta_1^+\cdots\delta_l^+,\gamma_0,\zeta_1^+\cdots\zeta_m^+)\times\big(\mathcal{M}(\delta_1^+;\boldsymbol{\delta}_1^-)\sqcup\cdots\sqcup\mathcal{M}(\delta_m^+;\boldsymbol{\delta}_m^-) \\
& \sqcup\mathcal{M}(\gamma_0;\boldsymbol{\delta}_{l+1}^-,p,\boldsymbol{\zeta}_0^-)\sqcup\mathcal{M}(\zeta_1^+,\boldsymbol{\zeta}_1^-)\sqcup\cdots\sqcup\mathcal{M}(\zeta_m^+,\boldsymbol{\zeta}_m^-)\big) \\
& \bigcup_{q\in\Sigma_0\cap\Sigma_1}\mathcal{M}(\gamma^+;\boldsymbol{\delta}',q,\boldsymbol{\zeta}'')\times\mathcal{M}(q;\boldsymbol{\delta}'',p,\boldsymbol{\zeta}') \\
& \bigcup\mathcal{M}(\gamma^+;\boldsymbol{\delta}',\gamma_{10},\boldsymbol{\zeta}''')\times\mathcal{M}(\gamma_{10};\boldsymbol{\delta}'',\gamma_{01},\boldsymbol{\zeta}'')\times\mathcal{M}(\gamma_{01},\boldsymbol{\delta}''',p,\boldsymbol{\zeta}') \\
& \bigcup\overline{\mathcal{M}}^{\partial}(\gamma^+;\boldsymbol{\delta},p,\boldsymbol{\zeta})
\end{aligned}
$$

The union being over the same sets as in (31) and (30).

5. Preliminaries of Legendrian contact homology

Here we give a quick review of the theory of *Legendrian contact homology*, and we direct the reader to [34] and [30] for more details. Legendrian contact homology is a Legendrian isotopy invariant associated to a Legendrian submanifold which was defined by Chekanov for Legendrian knots in the standard $\mathbb{R}^3$ [20] and then extended to Legendrian submanifolds of contactisations of Liouville manifolds by Ekholm, Etnyre and Sullivan [34].

5.1. **The Chekanov-Eliashberg DGA.** Let (Y,ξ) be the contactisation of a Liouville manifold (P,λ) and $\Lambda\subset(Y,\xi)$ a chord-generic Legendrian submanifold. Fix a unital commutative ring R which, most often, we will be $R=\mathbb{Z}$ or $R=\mathbb{Z}/2\mathbb{Z}$. However, recall that we have to assume Λ to be spin and fix a spin structure in order to use R different from a unital algebra over a field of characteristic 2. The Legendrian contact homology differential graded algebra (DGA) of Λ, also called the *Chekanov-Eliashberg algebra* and denoted by $(\mathcal{A}(\Lambda),\partial)$, is a unital tensor R-algebra freely generated by the Reeb chords of Λ.

The grading of a Reeb chord generator $a\in\mathcal{R}(\Lambda)$ is given by

$$|\gamma| := \mathrm{gr}(\gamma)\in\mathbb{Z}/\mathbb{Z}_{\mu_\Lambda(H_2(P\times\mathbb{R},\Lambda))}$$

as defined in Section 4.2. Here $\mu_\Lambda\colon H_2(P\times\mathbb{R},\Lambda)\to\mathbb{Z}$ is the Maslov class of Λ associated to the contact planes. We define the degree of the unit to be zero.

Its differential ∂ is of degree -1 and counts certain punctured holomorphic discs in the symplectisation of Y which have boundary on $\mathbb{R}\times\Lambda$ and are asymptotic to Reeb chords. More precisely, ∂ is defined on a generator $\delta_0\in\mathcal{R}(\Lambda)$ as

$$\partial(\delta_0) := \sum_{\dim \mathcal{M}_{\mathbb{R}\times\Lambda}(\delta_0;\boldsymbol{\delta})=1} \#\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda}(\delta_0;\boldsymbol{\delta})\boldsymbol{\delta},$$

where $\boldsymbol{\delta}=\delta_1\dots\delta_{m(\boldsymbol{\delta})}$ is a formal product of Reeb chords and $\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda}(\delta_0;\boldsymbol{\delta})$ is the moduli space defined in Section 3.2.3. Here (and from now on) the symbol '#' indicates the signed count of 0-dimensional moduli spaces and we use the convention that it gives 0 for moduli spaces of dimension different from zero. Then we extend ∂ to all of $\mathcal{A}(\Lambda)$ by R-linearity together with the Leibniz rule

$$\partial(\delta\boldsymbol{\delta'}) = \partial(\delta)\boldsymbol{\delta'} + (-1)^{|\delta|}\delta\partial(\boldsymbol{\delta'}).$$

Remark 5.1. *By the results in* [25]*, we may equivalently define ∂ by counting holomorphic polygons in P having boundary on the Lagrangian projection of Λ. For us, however, the above perspective will turn out to be more useful, since it fits better with the SFT framework.*

The fact that $\partial^2=0$ and that the homology is independent of the choice of almost complex structure and invariant under Legendrian isotopy was shown in [34].

5.2. **The DGA morphism induced by an exact Lagrangian cobordism.** Given an exact Lagrangian cobordism $\Sigma\subset\mathbb{R}\times P\times\mathbb{R}$ from Λ^- to Λ^+, there is an induced unital DGA morphism

$$\Phi_\Sigma\colon (\mathcal{A}(\Lambda^+),\partial_{\Lambda^+})\to(\mathcal{A}(\Lambda^-),\partial_{\Lambda^-}),$$

constructed in [30] and [36], following the general philosophy of SFT. This DGA morphism is defined on a generator δ^+ of $\mathcal{A}(\Lambda^+)$ by the J-holomorphic disc count

$$\Phi_\Sigma(\delta^+) := \sum_{\dim \mathcal{M}_\Sigma(\delta^+;\boldsymbol{\delta}^-)=0} \#\mathcal{M}_\Sigma(\delta^+;\boldsymbol{\delta}^-)\boldsymbol{\delta}^-,$$

and then extended as a unital algebra map. Here $\boldsymbol{\delta}^-=\delta_1^-\dots\delta^-_{m(\boldsymbol{\delta}^-)}$ is an element of $\mathcal{A}(\Lambda^-)$, and the moduli space $\mathcal{M}_\Sigma(a;\mathbf{b})$ is defined in Section 3.2.3. Observe that, in order to define the above map with coefficients in $\mathbb{Z}$, we must fix a spin structure on each end $\Lambda^\pm$ which can be extended to a spin structure on Σ.

Example 5.2. For the trivial (cylindrical) cobordism $\mathbb{R}\times\Lambda$, we get $\Phi_{\mathbb{R}\times\Lambda}=\mathrm{id}_{\mathcal{A}(\Lambda)}$.

The fact that Φ_Σ is a chain map was shown in [30]. In fact, based upon the abstract perturbations in the same paper, it follows that the DGA-homotopy class (as defined in [36, Lemma 3.13]) of Φ_Σ is independent of the choices of almost complex structure and Hamiltonian isotopy class of Σ.

Remark 5.3. *The invariance properties of our Floer theory does not rely on the above abstract perturbation argument in the cases under consideration here, i.e. that of a symplectisation of a contactisation. The reason is that, in these cases, our invariance statement boils down to showing that the complex* $\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1)$ *is* acyclic.

5.3. **Augmentations and bilinearised Legendrian contact homology.** An *(R-valued) augmentation* is a unital DGA morphism $\varepsilon : \mathcal{A}(\Lambda) \to R$, where R is regarded as a DGA with trivial differential. In general, the Chekanov-Eliashberg algebra need not admit any augmentations. However, in the case when the Legendrian submanifold $\Lambda \subset Y$ has an exact Lagrangian filling $\Sigma \subset \mathbb{R} \times Y$, the above unital DGA morphism

$$\varepsilon_\Sigma := \Phi_\Sigma$$

is indeed an augmentation (also see [31]). In fact, one can think of the trivial DGA R as the Chekanov-Eliashberg DGA of the empty set.

On the other hand, there are plenty of examples of Legendrian submanifolds which do not admit any exact Lagrangian filling in the symplectisation, but whose Chekanov-Eliashberg algebra still admits augmentations. For instance, this is the case for the Legendrian twist knots constructed in [45]; see the discussion in [63, Section 10.1].

Augmentations are important for the following reason. Loosely speaking, they should be seen as "obstruction cocycles" for the Chekanov-Eliashberg algebra. Given an augmentation, Chekanov defined linearised Legendrian contact homology in [20], which is a Legendrian isotopy invariant in the form of a chain complex spanned by the Reeb chords on Λ. Linearised Legendrian contact homology is computationally tractable at the expense of discarding non-linear (and non-commutative) information.

Recently, Bourgeois and the first author have introduced a generalisation of linearised LCH which is called bilinearised Legendrian contact homology [12]. It is constructed using *a pair* of augmentations ε_0 and ε_1 of $(\mathcal{A}(\Lambda), \partial)$. We proceed to give a brief description of this complex.

The bilinearised Legendrian contact homology complex is the free module

$$LCC_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda) := R\langle \mathcal{R}(\Lambda) \rangle$$

spanned by the Reeb chords with the above grading, whose differential is of degree -1 and defined by

$$\partial^{\varepsilon_0,\varepsilon_1}(\delta) = \sum_{\dim \mathcal{M}_{\mathbb{R}\times\Lambda}(\delta;\boldsymbol{\delta})=1} \sum_{i=1}^{m_{\boldsymbol{\delta}}} \#\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda}(\delta;\boldsymbol{\delta})\varepsilon_0(\delta_1 \dots \delta_{i-1})\varepsilon_1(\delta_{i+1} \dots \delta_{m_{\boldsymbol{\delta}}})\delta_i.$$

The corresponding homology groups will be denoted by

$$LCH_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda).$$

The set of augmentations of the Chekanov-Eliashberg algebra $(\mathcal{A}(\Lambda), \partial)$ is not a Legendrian isotopy invariant of Λ. However, as shown in [12], the invariance proof of [20] can be generalised to show that the isomorphism classes

$$\{LCH_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda)\}/\sim$$

of the graded modules, where the union is over all pairs of augmentations, is a Legendrian isotopy invariant of Λ. Observe that, when $\varepsilon_0 = \varepsilon_1$, we simply recover Chekanov's linearised LCH.

We will also be interested in the dual complex of $(LCC_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda), \partial^{\varepsilon_0,\varepsilon_1})$, the so-called bilinearised Legendrian cohomology complex, where the differential $\partial^{\varepsilon_0,\varepsilon_1}$ is of degree 1. This complex will be denoted by

$$(LCC^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda), d_{\varepsilon_0,\varepsilon_1}),$$

while we write

$$LCH^{\bullet}_{\varepsilon_0,\varepsilon_1}(\Lambda)$$

for the corresponding cohomology group.

The bilinearised LCH is a stronger invariant compared to linearised LCH, due to the fact that it remembers some of the "non-commutativity" of $\mathcal{A}(\Lambda)$. In [12] it was shown that these homology groups are the morphisms spaces of an A_∞-category $\mathcal{A}\mathrm{ug}_-(\Lambda)$ called the *augmentation category*, whose objects are the augmentations of the Chekanov-Eliashberg algebra of Λ.

Given an exact Lagrangian cobordisms Σ from Λ^- to Λ^+, and two augmentations ε_0^- ε_1^- of Λ^-, the DGA map Φ_Σ described in the previous section induces a linear chain map from $LCC^{\bullet}_{\varepsilon_0^-\,\varepsilon_1^-}(\Lambda^-)$ to $LCC^{\bullet}_{\varepsilon_0^+\varepsilon_1^+}(\Lambda^+)$ via the formula

$$\Phi_\Sigma^{\varepsilon_0^-\varepsilon_1^-}(\gamma^-) = \sum_{\gamma^+,\boldsymbol{\delta}^-,\boldsymbol{\zeta}^-} \#\mathcal{M}_\Sigma(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)\varepsilon_0^-(\boldsymbol{\delta}^-)\varepsilon_1^-(\boldsymbol{\zeta}^-)\gamma^+,$$

where $\varepsilon_i^+ = \varepsilon_i^- \circ \Phi_\Sigma$ for $i = 0, 1$.

6. The Cthulhu complex

Let Σ_0 and Σ_1 be two exact Lagrangian cobordisms inside the symplectisation $(\mathbb{R}\times P\times\mathbb{R}, d(e^t\alpha))$ of a contactisation. We assume that:

- $\Sigma_0 \pitchfork \Sigma_1$ (in particular this implies that $\Lambda_0^\pm \cap \Lambda_1^\pm = \emptyset$),
- The links $\Lambda_0^\pm \sqcup \Lambda_1^\pm$ are chord-generic.

The *Cthulhu complex* of the pair (Σ_0, Σ_1) is the complex whose underlying graded R-module

$$\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1) := C^\bullet(\Lambda_0^+,\Lambda_1^+)[2] \oplus C^\bullet(\Sigma_0,\Sigma_1) \oplus C^\bullet(\Lambda_0^-,\Lambda_1^-)[1]$$

for a unital ring R. Here $C^\bullet(\Lambda_0^+,\Lambda_1^+)$ is the free graded module spanned by the Reeb chords from $\Lambda_1^\pm$ to $\Lambda_0^\pm$ and $C^\bullet(\Sigma_0,\Sigma_1)$ is the free graded module spanned by the intersection points $\Sigma_0\cap\Sigma_1$. The gradings are taken as described in Section 4.2, depending on the choice of a Maslov potential.

6.1. **The Cthulhu differential.** Fix two augmentations ε_0^- and ε_1^- of the Chekanov-Eliashberg algebras of Λ_0^- and Λ_1^-, respectively, both of which are defined using a cylindrical almost complex structure J^-. We will define the *Cthulhu differential* $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}$, which is a differential of degree 1 on the above graded module. With respect to the above decomposition, this differential takes the form

$$\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-} = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & d_{-0} & d_{--} \end{pmatrix}. \tag{33}$$

Loosely speaking, every non-zero entry in this matrix is given by a count of rigid *punctured pseudoholomorphic strips* of appropriate type, as described in Section 3.2, where the counts are "weighted by" the above augmentations.

First, however, we need to fix the choice of an admissible almost complex structure $J \in \mathcal{J}^{\mathrm{adm}}_{J^-,J^+,T}(\mathbb{R}\times P\times\mathbb{R})$, i.e. a compatible almost complex structure on $\mathbb{R}\times P\times\mathbb{R}$ satisfying the assumptions in Section 3.1.4. In particular, J coincides with the cylindrical almost complex structures J^- and J^+ in subsets of the form $(-\infty,-T]\times P\times\mathbb{R}$ and $[T,+\infty)\times P\times\mathbb{R}$, respectively. In order for these moduli

spaces to be transversely cut out, we moreover assume that the almost complex structure was generically chosen as in Section 4.3.

Below we give a careful description of each term in the above matrix, where the degrees mentioned are the degrees as maps between the above summands *without* the shifts in grading as appearing in the above definition of $\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1)$.

In the cases when we want to emphasise to which pair of Lagrangian cobordisms the differential belongs, we use the superscript "Σ_0,Σ_1", i.e. $\mathfrak{d}^{\Sigma_0,\Sigma_1}_{\varepsilon_0^+,\varepsilon_1^-}$, $d^{\Sigma_0,\Sigma_1}_{+,-}$,..., etc.

6.1.1. *The bilinearised LCH differential.* We define $\varepsilon_i^+ := \varepsilon_i^- \circ \Phi_{\Sigma_i,J}$, $i=0,1$, for the pull-backs of the above augmentations to augmentations of the Chekanov-Eliashberg algebras of Λ_0^+ and Λ_1^+, respectively.

The term $d_{\pm\pm}$ is the bilinearised Legendrian cohomology differential for $(\Lambda_0^\pm,\Lambda_1^\pm)$ induced by the pair $(\varepsilon_0^\pm,\varepsilon_1^\pm)$ of augmentations as defined in [12] and described in Section 5.3. In other words, it is given by

$$(34)\quad d_{\pm\pm}(\gamma_2^\pm) := d_{\varepsilon_0^\pm,\varepsilon_1^\pm}(\gamma_2^\pm) = \\ = \sum_{\gamma_1^\pm}\sum_{\boldsymbol{\delta}^\pm,\boldsymbol{\zeta}^\pm} \#\widetilde{\mathcal{M}}_{\mathbb{R}\times\Lambda_0^\pm,\mathbb{R}\times\Lambda_1^\pm}(\gamma_1^\pm;\boldsymbol{\delta}^\pm,\gamma_2^\pm,\boldsymbol{\zeta}^\pm;J^\pm)\cdot\varepsilon_0^\pm(\boldsymbol{\delta}^\pm)\varepsilon_1^\pm(\boldsymbol{\zeta}^\pm)\cdot\gamma_1^\pm.$$

It follows can be computed using Equation (24) that this term is of degree 1.

6.1.2. *The Floer differential.* The differential d_{00} can be seen as a modification of the differential in Lagrangian Floer homology as introduced by Floer in [46], where the version defined here has found its inspiration in [31]. For an intersection point q, it is defined by the count

$$(35)\qquad d_{00}(q) := \sum_p \sum_{\boldsymbol{\delta}^-,\boldsymbol{\zeta}^-} \#\mathcal{M}_{\Sigma_0,\Sigma_1}(p;\boldsymbol{\delta}^-,q,\boldsymbol{\zeta}^-;J)\cdot\varepsilon_0^-(\boldsymbol{\delta}^-)\varepsilon_1^-(\boldsymbol{\zeta}^-)\cdot p.$$

From Equation (26) we deduce that this map is of degree 1.

6.1.3. *The Cultist maps.* The maps d_{+0} and d_{0-} are defined using the moduli spaces described in Section 3.2.6 and in Section 3.2.5, respectively. A version of the map d_{+0} appears in [31] in the case when the negative ends of the cobordisms are empty. More precisely, we define

$$(36)\qquad d_{0-}(\gamma^-) := \sum_p \sum_{\boldsymbol{\delta}^-,\boldsymbol{\zeta}^-} \#\mathcal{M}_{\Sigma_0,\Sigma_1}(p;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)\cdot\varepsilon_0^-(\boldsymbol{\delta}^-)\varepsilon_1^-(\boldsymbol{\zeta}^-)\cdot p,$$

$$(37)\qquad d_{+0}(q) := \sum_{\gamma^+} \sum_{\boldsymbol{\delta}^-,\boldsymbol{\zeta}^-} \#\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,q,\boldsymbol{\zeta}^-)\cdot\varepsilon_0^-(\boldsymbol{\delta}^-)\varepsilon_1^-(\boldsymbol{\zeta}^-)\cdot\gamma^+.$$

Equations (25) (resp. (27)) show that d_{0-} (resp. d_{+0}) is of degree 2 (resp. -1).

6.1.4. *The LCH map.* The map d_{+-} is defined analogously to the bilinearised map in LCH induced by an exact Lagrangian cobordism. It is given as follows:

$$(38)\qquad d_{+-}(\gamma^-) := \sum_{\gamma^+} \sum_{\boldsymbol{\delta}^-,\boldsymbol{\zeta}^-} \#\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma^+;\boldsymbol{\delta}^-,\gamma^-,\boldsymbol{\zeta}^-)\cdot\varepsilon_0^-(\boldsymbol{\delta}^-)\varepsilon_1^-(\boldsymbol{\zeta}^-)\cdot\gamma^+.$$

As follows from Equation (24), this map is of degree 0.

6.1.5. *The Nessie map.* Let $C_\bullet(\Lambda_1^\pm, \Lambda_0^\pm)$ be the dual of $C^\bullet(\Lambda_1^\pm, \Lambda_0^\pm)$ with $\delta_{\pm\pm}$ the induced adjoint of the differential $d_{\pm\pm}^{\Sigma_1,\Sigma_0}$. These differentials are entries in the Cthulhu differential $\mathfrak{d}_{\varepsilon_1^-,\varepsilon_0^-}^{\Sigma_1,\Sigma_0}$. Further, let $CF_\bullet(\Sigma_0,\Sigma_1)$ be the dual of $CF^\bullet(\Sigma_0,\Sigma_1)$.

Observe that, since all the above spaces are endowed with a canonical basis, we are free to identify any such space with its dual. Moreover, with a fixed choice of Maslov potentials for the two cobordisms, there is also a canonical identification

$$CF^\bullet(\Sigma_0,\Sigma_1) = CF_\bullet(\Sigma_0,\Sigma_1) = CF_{n+1-\bullet}(\Sigma_1,\Sigma_0),$$

when reversing the ordering of the pair of cobordisms.

The count of "banana" pseudoholomorphic strips gives rise to a map

$$b\colon C_{n-1-\bullet}(\Lambda_1^\pm,\Lambda_0^\pm) \to C^\bullet(\Lambda_0^\pm,\Lambda_1^\pm),$$
$$\gamma_{01} \mapsto \sum_{\gamma_{10}} \sum_{\boldsymbol{\delta},\boldsymbol{\zeta}} \#\mathcal{M}_{\Sigma_0,\Sigma_1}(\gamma_{10};\boldsymbol{\delta},\gamma_{01},\boldsymbol{\zeta}) \cdot \varepsilon_0(\boldsymbol{\delta})\varepsilon_1(\boldsymbol{\zeta}) \cdot \gamma_{10},$$

where the degree of the map follows from Equation (28).

Using $\delta_{-0} : CF_\bullet(\Sigma_1,\Sigma_0) \to C_{\bullet-2}(\Lambda_1^-,\Lambda_0^-)$ to denote the adjoint of the map $d_{0-}^{\Sigma_1,\Sigma_0}$ in the entry of $\mathfrak{d}_{\varepsilon_1,\varepsilon_0}^{\Sigma_1,\Sigma_0}$, we are finally ready to define

$$d_{-0} := b \circ \delta_{-0}\colon CF^\bullet(\Sigma_0,\Sigma_1) = CF_{n+1-\bullet}(\Sigma_1,\Sigma_0) \to C^\bullet(\Lambda_0^-,\Lambda_1^-),$$

which is of degree 0.

The moduli spaces above were previously considered in the Floer theory involving concave ends due to Akaho in [4]. Also, see [5] where Morse homology in the presence of a non-empty boundary was considered by the same author.

6.2. **The proof of $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}{}^2 = 0$.** We are now ready to present and prove the following central result.

Theorem 6.1. *Let $\Sigma_i \subset \mathbb{R}\times P\times\mathbb{R}$, $i=0,1$, be a pair of exact Lagrangian cobordisms from Λ_i^- to Λ_i^+ as above. Given a generic admissible almost complex structure $J \in \mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(\mathbb{R}\times P\times\mathbb{R})$ in the sense of Section 3.1.4, and augmentations ε_i^- of the Chekanov-Eliashberg algebras of Λ_i^- defined using J^-, then*

- *$\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}$ is well-defined, and*
- *$\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}{}^2 = 0$,*

under the assumption that the entries above have been defined using J. In the case when Σ_i, $i=0,1$, both are spin, the above counts can moreover be defined by signed counts with coefficients in $\mathbb{Z}$ given the choice of a spin structure on each cobordism. In general, the count can always be performed in $\mathbb{Z}_2$.

In order to prove $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}{}^2 = 0$, we need to study the boundary points of one-dimensional moduli spaces of pseudoholomorphic strips. As usual in Floer theory, this composition is defined by counting broken pseudoholomorphic strips that correspond to the boundary points of these moduli spaces. Recall that, in our setting, every strip is punctured, and that all counts are *weighted* by the augmentations chosen. For this reason, we start by prescribing two important points that need to be taken into account when performing these counts:

(I) Recall that the punctured pseudoholomorphic strips used in the definition of d_{++} live in the top level and are allowed to have negative punctures. When adjoining the rigid punctured strips in the definition of d_{++} and d_{+*}, where

$* = 0, -$, i.e. the glued configurations corresponding to the compositions $d_{++}d_{+*}$, we do not necessarily obtain a broken pseudoholomorphic strip. In order to obtain a broken strip, we will adjoin the pseudoholomorphic half-planes that appear in the count defining the right-hand side of

$$\varepsilon_i^+ = \varepsilon_i^- \circ \Phi_{\Sigma_i}, \ \ i = 0, 1,$$

to the middle level. From the latter equality, it also follows that the composition $d_{++}d_{+*}$ indeed is obtained by counting buildings of precisely this form.

(II) Not all broken punctured pseudoholomorphic strips correspond to two glued pseudoholomorphic *strips*. Namely, as shown in Section 4.4, there are so-called ∂-breakings that consist of a punctured strip together with a punctured half-plane of index 1 having boundary on $\mathbb{R} \times \Lambda_i^{\pm}$, $i = 0, 1$. Recall the fact that the counts of the latter half-planes define the differential $\partial_{\pm}$ of the Chekanov-Eliashberg algebras of $\Lambda_0^{\pm} \cup \Lambda_1^{\pm}$, and that the equality

$$\varepsilon_i^{\pm} \circ \partial_{\pm} = 0, \ \ i = 0, 1,$$

holds by definition. It thus follows that the counts of the totality of the broken strips of this kind must vanish in the case when this count is weighted by the augmentations.

Proof. The fact that the map is well-defined follows form the compactness result Theorem 3.9 together with the transversality results in Section 4.3. Namely, all moduli spaces of index 0 having fixed asymptotics are compact (since there is a uniform upper bound on its total energy) 0-dimensional manifolds.

We now make a term-by-term argument for the matrix

$$\mathfrak{d}_{\varepsilon_0^-, \varepsilon_1^-}{}^2 = \begin{pmatrix} d_{++}{}^2 & d_{++}d_{+0} + d_{+0}d_{00} + d_{+-}d_{-0} & d_{++}d_{+-} + d_{+0}d_{0-} + d_{+-}d_{--} \\ 0 & d_{00}d_{00} + d_{0-}d_{-0} & d_{00}d_{0-} + d_{0-}d_{--} \\ 0 & d_{-0}d_{00} + d_{--}d_{-0} & d_{-0}d_{0-} + d_{--}d_{--} \end{pmatrix}$$

in order to show that all entries vanish.

- $d_{++}^2 = 0$. The term d_{++} is the standard bilinearised Legendrian contact cohomology differential [12] restricted to mixed chords from Λ_1^+ to Λ_0^+. More precisely, the subspace generated by these chords form a subcomplex of the linearised cohomology complex of the link $\Lambda_0^+ \sqcup \Lambda_1^+$, under the assumption that we use an augmentation of the link which vanishes on mixed chords, while it takes the value $\varepsilon_i^+(\gamma_i)$ on a chord $\gamma_i \in \mathcal{R}(\Lambda_i^+)$. This term thus vanishes, as mentioned in Section 5.3.
- $d_{++}d_{+0} + d_{+0}d_{00} + d_{+-}d_{-0} = 0$. We must study the boundary of a moduli space $\mathcal{M}_{\Sigma_0, \Sigma_1}(\gamma; \boldsymbol{\delta}, p, \boldsymbol{\zeta})$ being of dimension 1. The possibilities for the breakings involved are schematically depicted in Figure 11.

  We claim that, when counting these boundary points weighted by the augmentations ε_i^-, $i = 0, 1$, we get the contribution

  $$\langle (d_{++}d_{+0} + d_{+0}d_{00} + d_{+-}d_{-0})(p), \gamma \rangle.$$

  Indeed, inspecting the boundary components of different types as described in Equation (32) we obtain the terms $\langle d_{++}d_{+0}(p), \gamma \rangle$ (here we use $\varepsilon_i^+ =$

$\varepsilon_i^- \circ \Phi_{\Sigma_i}$; see (I)), $\langle d_{+0}d_{00}(p), \gamma\rangle$, and $\langle d_{+-}d_{-0})(p), \gamma\rangle$, together with the ∂-breakings which can be seen to contribute to zero (here we use $\varepsilon_i^{\pm} \circ \partial^{\pm} = 0$; see (II)).

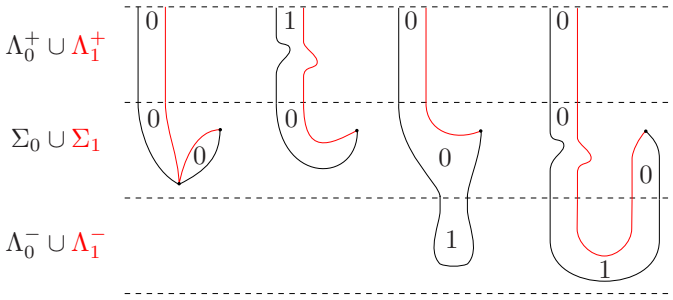

FIGURE 11. Breakings involved in $d_{++}d_{+0}+d_{+0}d_{00}+d_{+-}d_{-0} = 0$. The number on each component denotes its Fredholm index.

- $d_{++}d_{+-}+d_{+0}d_{0-}+d_{+-}d_{--} = 0$. The argument is similar to the argument above, but where Equation (31) has been used. The possibilities for the breakings involved are schematically depicted in Figure 12 (also, see Figure 10 for some of the breakings with explicit negative ends).

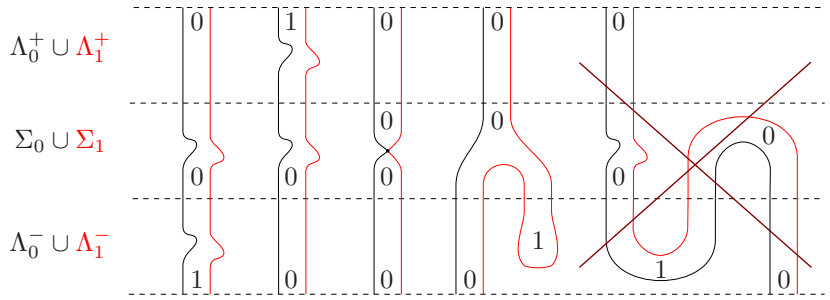

FIGURE 12. Breakings involved in $d_{++}d_{+-}+d_{+0}d_{0-}+d_{+-}d_{--} = 0$. The number on each component denotes its Fredholm index.

- $d_{00}d_{00}+d_{0-}d_{-0} = 0$. Again this follows as above, but while using Equation (30). The possibilities for the breakings involved are schematically depicted in Figure 13 (also, see Figure 9).
- $d_{00}d_{0-} + d_{0-}d_{--} = 0$. This follows similarly as above, but while using Equation (29). The possibilities for the breakings involved are schematically depicted in Figure 14.
- $d_{-0}d_{00} + d_{--}d_{-0} = 0$. Analysing the breakings of holomorphic bananas, we get that the map b satisfies $b \circ \delta_{--} = d_{--} \circ b$ (see Figure 15), where δ_{--} again denotes the adjoint of $d_{--}^{\Sigma_1,\Sigma_0}$. Hence, we get that

$$d_{-0}d_{00} + d_{--}d_{-0} = b\delta_{-0}d_{00} + d_{--}b\delta_{-0} = b\big(\delta_{-0}d_{00} + \delta_{--}\delta_{-0}\big),$$

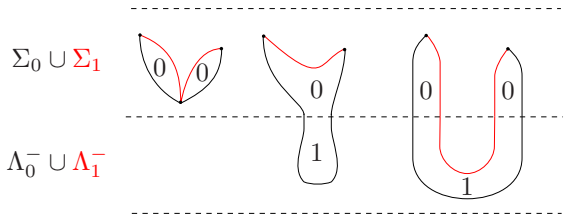

FIGURE 13. Breakings involved in $d_{00}d_{00} + d_{0-}d_{-0} = 0$. The number on each component denotes its Fredholm index.

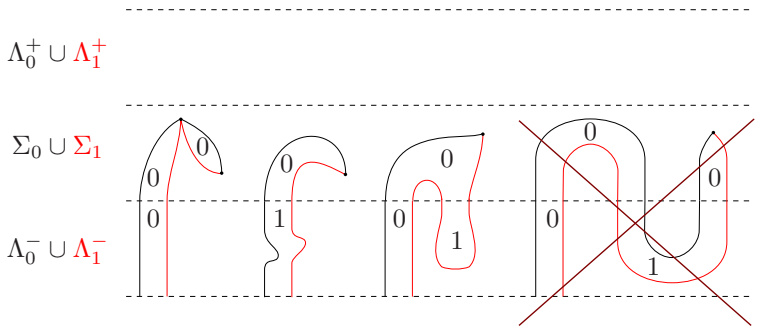

FIGURE 14. Breakings involved in $d_{00}d_{0-} + d_{0-}d_{--} = 0$. The number on each component denotes its Fredholm index.

where δ_{-0} is the adjoint of $d_{0-}^{\Sigma_1,\Sigma_0}$. Since $\partial_{00} := d_{00}$ is the adjoint of $d_{00}^{\Sigma_1,\Sigma_0}$, the factor $\delta_{-0}d_{00} + \delta_{--}\delta_{-0}$ above is actually the adjoint of $d_{00}^{\Sigma_1,\Sigma_0}d_{0-}^{\Sigma_1,\Sigma_0} + d_{0-}^{\Sigma_1,\Sigma_0}d_{--}^{\Sigma_1,\Sigma_0}$. Since the latter term vanishes by the previous case, the claim now follows. See Figure 9.

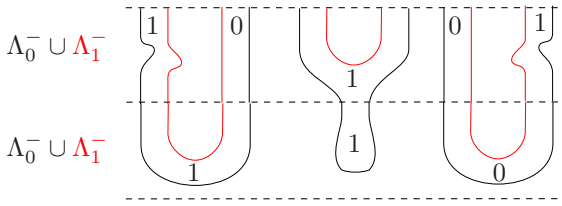

FIGURE 15. Breakings involved in $b \circ \delta_{--} = d_{--} \circ b$.

- $d_{-0}d_{0-} + d_{--}d_{--} = 0$. For action reasons we must have $d_{-0}d_{0-} = 0$. Finally, $d_{--}d_{--} = 0$ holds since d_{--} is the bilinearised Legendrian contact cohomology differential, i.e. for the same reason to why $d_{++}d_{++} = 0$.

□

7. The transfer and co-transfer map for concatenations of cobordisms

Recall that two exact Lagrangian cobordisms having a common end can be concatenated; see Section 2.2. In this section we will provide formulas which relate the Floer homologies of the different pieces of such a concatenation. This will be done by introducing a relative version of Viterbo's transfer map, originally defined in [69] for symplectic (co)homology. (Recall that Viterbo's transfer map concerns concatenations of *symplectic cobordisms*.) For the Hamiltonian formulation of wrapped Floer homology, the transfer map was constructed and treated in [3].

In the following we will consider the exact Lagrangian cobordisms $V_0, V_1, W_0, W_1 \subset \mathbb{R} \times P \times \mathbb{R}$ inside the symplectisation of a contactisation. We assume that V_i is an exact Lagrangian cobordism from Λ_i^- to Λ_i, and that W_i is an exact Lagrangian cobordism from Λ_i to Λ_i^+, $i = 0, 1$. It follows that we can form the concatenations

$$V_i \odot W_i \subset \mathbb{R} \times P \times \mathbb{R}, \ \ i = 0, 1,$$

being exact Lagrangian cobordisms from Λ_i^- to Λ_i^+.

Under the further assumption that the negative ends of V_i, $i = 0, 1$, are empty, a transfer map

$$\Phi_{W_0,W_1} \colon \ \mathrm{Cth}_\bullet(V_0, V_1) \to \mathrm{Cth}_\bullet(V_0 \odot W_0, V_1 \odot W_1)$$

was constructed in [31, Section 4.2.2]; recall that the analytic set-up of the latter article is the same as the one used here. Our construction of the transfer map will be a straight-forward generalisation of this construction to the case when the negative ends are non-empty.

We will also construct a map that we call a *co-transfer map*

$$\Phi^{V_0,V_1} \colon \ \mathrm{Cth}_\bullet(V_0 \odot W_0, V_1 \odot W_1) \to \mathrm{Cth}_\bullet(W_0, W_1).$$

This map should be thought of as a quotient projection associated to a transfer map.

7.1. Concatenations and stretching of the neck.

Recall that the Hamiltonian isotopy class of a concatenation is unique. However, for us it will be necessary to concatenate the cobordism together with an almost complex structure, i.e. keeping track of conformal data as well, thus breaking this symmetry. In order to pin-point the almost complex structure obtained it will be useful to introduce a parameter keeping track of how the concatenation was performed.

We start with the following hypotheses. Assume that we are given almost complex structures J_a and J_b on $\mathbb{R} \times P \times \mathbb{R}$ which are cylindrical in the subsets $\{t \geq -1\}$ and $\{t \leq 1\}$, respectively, and which moreover agree in the subset $\{-1 \leq t \leq 1\}$. We also assume that V and W are cylindrical in the subsets $\{t \geq -1\}$ and $\{t \leq 1\}$, respectively, where they coincide. For each $N \geq 0$ we define

$$\begin{aligned} V \odot_N W &:= (V \cap \{t \leq 0\}) \cup (\tau_N(W) \cap \{t \geq 0\}), \\ (J_a \odot_N J_b)(t, p, z) &:= \begin{cases} J_a(t, p, z) & t \leq 0 \\ J_b(t - N, p, z) & t \geq 0, \end{cases} \end{aligned}$$

where we recall that τ_T is the translation of the t-coordinate by $T \in \mathbb{R}$. We also write $J_a \odot J_b := J_a \odot_0 J_b$.

The Hamiltonian isotopy class of $V \odot_N W \subset \mathbb{R} \times P \times \mathbb{R}$ is independent of $N \geq 0$. In the case when J_a and J_b are cylindrical outside of a compact subset we have

produced a family of boundary-value problems of $J_a \odot_N J_b$-holomorphic curves in $\mathbb{R} \times P \times \mathbb{R}$ having boundary on $V \odot_N W$, $N \geq 0$. It can be seen that this family of boundary-value problems in fact is conformally equivalent to the family which "stretches the neck" along the contact-type hypersurface $\{0\} \times P \times \mathbb{R} \subset \mathbb{R} \times P \times \mathbb{R}$ with boundary condition $V \odot W$; see [13, Section 3.4] as well as [38, Section 1.3] for more details. This fact will be important below.

There is a compactness theorem analogous to Theorem 3.9 in the case of a neck-stretching sequence of almost complex structure; see [13, Section 10] for the precise formulation. The key fact is that a sequence of $J_a \odot_N J_b$-holomorphic discs, with $N \to +\infty$, has a subsequence converging to a building consisting of *several* levels whose components satisfy non-cylindrical boundary conditions. In the case under consideration, the limit buildings consist of:

- An upper level containing punctured J_b-holomorphic discs with boundary on $W_0 \cup W_1$; and
- A lower level containing punctured J_a-holomorphic discs with boundary on $V_0 \cup V_1$.

A priori there can also be intermediate levels consisting of pseudoholomorphic discs for a cylindrical almost complex structure satisfying a cylindrical boundary condition. Since we are only interested in rigid configurations, and since the latter solutions will have positive dimension (unless they are trivial strips), they can be omitted from our breaking analysis (given the assumption that transversality is achieved for every level).

The gluing result Theorem 4.11 also generalises to this setting (see [30, Lemma 3.14]), giving a bijection between buildings of the above type where all components are of Fredholm index zero, and punctured $J_a \odot_N J_b$-holomorphic discs for each $N \gg 0$ sufficiently large. Figure 16 schematically depicts two such buildings.

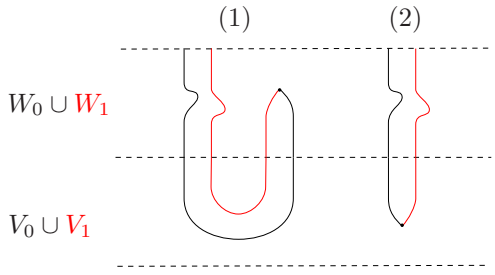

FIGURE 16. A holomorphic buildings appearing after stretching the neck along Λ

7.2. **The complex after a neck stretching procedure.** The first goal is to find a description of the complex

$$(\mathrm{Cth}_\bullet(V_0 \odot_N W_0, V_1 \odot_N W_1), \mathfrak{d}^{V \odot W}_{\varepsilon_0^- \varepsilon_1^-})$$

when the almost complex structure is given by $J_a \odot_N J_b$ for $N \gg 0$ sufficiently large. Here we assume that the almost complex structures J_a and J_b satisfy the properties described in Section 7.1, i.e. so that their concatenations can be taken.

We first consider the complex defined using J_a

$$(\mathrm{Cth}_\bullet(V_0,V_1) = C_{\bullet-2}(\Lambda_0,\Lambda_1)\oplus C_\bullet(V_0,V_1)\oplus C_{\bullet-1}(\Lambda_0^-,\Lambda_1^-), \mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-}),$$
$$\mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-} = \begin{pmatrix} -d_{++}^{V_0,V_1} & d_{+0}^{V_0,V_1} & d_{+-}^{V_0,V_1} \\ 0 & d_{00}^{V_0,V_1} & d_{0-}^{V_0,V_1} \\ 0 & d_{-0}^{V_0,V_1} & d_{--}^{V_0,V_1} \end{pmatrix}.$$

Consider the entries $d_{+0}^{V_1,V_0}$, $d_{+-}^{V_1,V_0}$, and $d_{\pm\pm}^{V_0,V_1}$ in the differential of the complex $\mathrm{Cth}_\bullet(V_1,V_0)$, where again the almost complex structure J_a has been used. We will need their adjoints

$$\begin{aligned}
\delta_{0+}^{V_0,V_1} &:= (d_{+0}^{V_1,V_0})^*\colon C_\bullet(\Lambda_1,\Lambda_0)\to C_{\bullet-2}(V_0,V_1),\\
\delta_{-+}^{V_0,V_1} &:= (d_{+-}^{V_1,V_0})^*\colon C_\bullet(\Lambda_1,\Lambda_0)\to C_{\bullet-1}(\Lambda_1^-,\Lambda_0^-),\\
\delta_{++}^{V_0,V_1} &:= (d_{++}^{V_1,V_0})^*\colon C_\bullet(\Lambda_1,\Lambda_0)\to C_{\bullet-1}(\Lambda_1,\Lambda_0),\\
\delta_{--}^{V_0,V_1} &:= (d_{--}^{V_1,V_0})^*\colon C_\bullet(\Lambda_1^-,\Lambda_0^-)\to C_{\bullet-1}(\Lambda_1^-,\Lambda_0^-),\\
\delta_{-+}^{V_0,V_1} &:= (d_{+-}^{V_1,V_0})^*\colon C_\bullet(\Lambda_1,\Lambda_0)\to C_{\bullet-1}(\Lambda_1^-,\Lambda_0^-),
\end{aligned}$$

where the canonical basis of Reeb chords and double points has been used in order to identify the modules and their duals. Observe that, exploiting the same notation, we also get $\delta_{00}^{V_0,V_1} = d_{00}^{V_0,V_1}$.

We write

$$\varepsilon_i' := \varepsilon_i\circ\Phi_{V_i,J_a},\ \ i=0,1,$$

for the pull-backs of the augmentations under the DGA morphisms induced by the respective cobordisms. These augmentations now give rise to a complex

$$(\mathrm{Cth}_\bullet(W_0,W_1) = C_{\bullet-2}(\Lambda_0^+,\Lambda_1^+)\oplus C_\bullet(W_0,W_1)\oplus C_{\bullet-1}(\Lambda_0,\Lambda_1), \mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'}),$$
$$\mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'} = \begin{pmatrix} -d_{++}^{W_0,W_1} & d_{+0}^{W_0,W_1} & d_{+-}^{W_0,W_1} \\ 0 & d_{00}^{W_0,W_1} & d_{0-}^{W_0,W_1} \\ 0 & d_{-0}^{W_0,W_1} & d_{--}^{W_0,W_1} \end{pmatrix},$$

where the moduli spaces are defined using the almost complex structure J_b. Note that $d_{--}^{W_0,W_1} = d_{++}^{V_0,V_1}$.

In addition we will also need the map

$$b^{V_0,V_1}\colon C_\bullet(\Lambda_1,\Lambda_0)\to C^{n-2-\bullet}(\Lambda_0,\Lambda_1)$$

which is defined similarly to

$$b^{\Lambda_0,\Lambda_1}\colon C_\bullet(\Lambda_1,\Lambda_0)\to C^{n-1-\bullet}(\Lambda_0,\Lambda_1)$$

as defined in Section 6.1.5, but which instead counts *rigid* "bananas" having boundary on $V_0\cup V_1$, and two punctures with positive asymptotics to Reeb chords.

Recall that the compactness theorem for a neck-stretching sequence together with pseudoholomorphic gluing shows the following. For $N\gg 0$ sufficiently large, the rigid $J_a\odot_N J_b$-holomorphic curves in $\mathbb{R}\times P\times\mathbb{R}$ having boundary on $(V_0\odot_N W_0)\cup(V_1\odot_N W_1)$ are in bijective correspondence with pseudoholomorphic buildings of the form described in Section 7.1, in which every involved component is rigid.

Analysing the possible such pseudoholomorphic buildings, we obtain the following. When $N\gg 0$ is sufficiently large, the differential of the complex for the

concatenated cobordisms, defined using the almost complex structure $J_a \odot_N J_b$ as in the previous paragraph, is given by

$$\begin{aligned}&(\mathrm{Cth}_\bullet(V_0 \odot_N W_0, V_1 \odot_N W_1)\\ &= C_{\bullet-2}(\Lambda_0^+, \Lambda_1^+) \oplus C_\bullet(W_0, W_1) \oplus C_\bullet(V_0, V_1) \oplus C_{\bullet-1}(\Lambda_0^-, \Lambda_1^-), \mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),\end{aligned}$$

$$\mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-} = \begin{pmatrix} -d^{W_0,W_1}_{++} & d^{W_0,W_1}_{+0} + d^{W_0,W_1}_{+-} b^{V_0,V_1} \delta^{W_0,W_1}_{-0} & d^{W_0,W_1}_{+-} d^{V_0,V_1}_{+0} & d^{W_0,W_1}_{+-} d^{V_0,V_1}_{+-} \\ 0 & d^{W_0,W_1}_{00} + d^{W_0,W_1}_{0-} b^{V_0,V_1} \delta^{W_0,W_1}_{-0} & d^{W_0,W_1}_{0-} d^{V_0,V_1}_{+0} & d^{W_0,W_1}_{0-} d^{V_0,V_1}_{+-} \\ 0 & d^{V_0,V_1}_{0+} \delta^{W_0,W_1}_{-0} & d^{V_0,V_1}_{00} & d^{V_0,V_1}_{0-} \\ 0 & b^{\Lambda_0^-,\Lambda_1^-} \delta^{V_0,V_1}_{-+} \delta^{W_0,W_1}_{-0} & b^{\Lambda_0^-,\Lambda_1^-} \delta^{V_0,V_1}_{-0} & d^{V_0,V_1}_{--} \end{pmatrix},$$

in terms of pseudoholomorphic strips on $V_0 \cup V_1$ and $W_0 \cup W_1$ for each $N \gg 0$ sufficiently large. (For instance the term $d^{W_0,W_1}_{+-} b^{V_0,V_1} \delta^{W_0,W_1}_{-0}$ corresponds to the breaking (1) in Figure 16 and the term $d^{W_0,W_1}_{+-} d^{V_0,V_1}_{+0}$ corresponds to the breaking (2) in the same figure.)

We have here relied on the exactness assumptions in Definition 2.1 and action considerations from Section 3.3.2 in order to rule out certain configurations.

7.3. **Definition of the transfer and co-transfer maps.** The transfer and co-transfer maps on the chain level are defined for a very "stretched" almost complex structure on a concatenated cobordism (i.e. when the parameter $N \gg 0$ in Section 7.1 is sufficiently large), so that the complexes take the form as described in Section 7.2 above.

Definition 7.1. The *transfer map* is defined by

$$\Phi_{W_0,W_1} \colon \mathrm{Cth}_\bullet(V_0, V_1) \to \mathrm{Cth}_\bullet(V_0 \odot_N W_0, V_1 \odot_N W_1),$$

$$\Phi_{W_0,W_1} = \begin{pmatrix} d^{W_0,W_1}_{+-} & 0 & 0 \\ d^{W_0,W_1}_{0-} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix},$$

while the *co-transfer map* is defined by

$$\Phi^{V_0,V_1} \colon \mathrm{Cth}_\bullet(V_0 \odot_N W_0, V_1 \odot_N W_1) \to \mathrm{Cth}_\bullet(W_0, W_1),$$

$$\Phi^{V_0,V_1} = \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & b^{V_0,V_1} \delta^{W_0,W_1}_{-0} & d^{V_0,V_1}_{+0} & d^{V_0,V_1}_{+-} \end{pmatrix}.$$

Lemma 7.2. *Under the assumption that $\mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-}$, $\mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'}$ and $\mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}$ are computed using J_a, J_b and $J_a \odot_N J_b$ (for $N \gg 0$ as before) respectively, the transfer and co-transfer maps defined above are chain maps.*

Proof. We start by showing the claim for the transfer map; i.e. we have to establish the equality

$$\Phi_{W_0,W_1} \circ \mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-} = \mathfrak{d}^{V\odot W}_{\varepsilon_0'\varepsilon_1'} \circ \Phi_{W_0,W_1}.$$

The matrices on the left and right hand sides become

$$\begin{pmatrix} -d^{W_0,W_1}_{+-} d^{V_0,V_1}_{++} & d^{W_0,W_1}_{+-} d^{V_0,V_1}_{+0} & d^{W_0,W_1}_{+-} d^{V_0,V_1}_{+-} \\ -d^{W_0,W_1}_{0-} d^{V_0,V_1}_{++} & d^{W_0,W_1}_{0-} d^{V_0,V_1}_{+0} & d^{W_0,W_1}_{0-} d^{V_0,V_1}_{+-} \\ 0 & d^{V_0,V_1}_{00} & d^{V_0,V_1}_{0-} \\ 0 & d^{V_0,V_1}_{-0} & d^{V_0,V_1}_{--} \end{pmatrix}$$

and

$$\begin{pmatrix} -d_{++}^{W_0,W_1}d_{+-}^{W_0,W_1} + d_{+0}^{W_0,W_1}d_{0-}^{W_0,W_1} + A_1 & d_{+-}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{+-}^{W_0,W_1}d_{+-}^{V_0,V_1} \\ d_{00}^{W_0,W_1}d_{0-}^{W_0,W_1} + A_2 & d_{0-}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{0-}^{W_0,W_1}d_{+-}^{V_0,V_1} \\ A_3 & d_{00}^{V_0,V_1} & d_{0-}^{V_0,V_1} \\ A_4 & d_{-0}^{V_0,V_1} & d_{--}^{V_0,V_1} \end{pmatrix},$$

respectively, where

$$\begin{aligned} A_1 &:= d_{+-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}, \\ A_2 &:= d_{0-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}, \\ A_3 &:= d_{0+}^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}, \\ A_4 &:= b^{\Lambda_0^-,\Lambda_1^-}\delta_{-+}^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}. \end{aligned}$$

As in Section 6.2

$$\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1} = 0$$

by action reasons, and hence $A_i = 0$, $i = 1,2,3,4$. What now remains is showing the equalities

$$\begin{aligned} -d_{++}^{W_0,W_1}d_{+-}^{W_0,W_1} + d_{+-}^{W_0,W_1}d_{++}^{V_0,V_1} + d_{+0}^{W_0,W_1}d_{0-}^{W_0,W_1} &= 0, \\ d_{00}^{W_0,W_1}d_{0-}^{W_0,W_1} + d_{0-}^{W_0,W_1}d_{++}^{V_0,V_1} &= 0. \end{aligned}$$

Recall that $d_{++}^{V_0,V_1} = d_{--}^{W_0,W_1}$ and, hence, both these equalities hold since the left-hand sides are defined by the signed counts of the number of boundary components of certain one-dimensional moduli spaces of pseudoholomorphic strips with boundary on (W_0, W_1). The first one comes from the boundary of $\mathcal{M}_{W_0,W_1}(\gamma^+; \boldsymbol{\delta}, \gamma^-, \boldsymbol{\zeta})$ for $\gamma^\pm$ a chord from $\Lambda_1^\pm$ to $\Lambda_0^\pm$. The second comes from $\mathcal{M}(q; \boldsymbol{\delta}, \gamma^-, \boldsymbol{\zeta})$ for $q \in W_0 \cap W_1$. Also, see Figures 12 and 14.

We now continue with the co-transfer map; i.e. we have to establish the equality

$$\Phi^{V_0,V_1} \circ \mathfrak{d}^{V \odot W}_{\varepsilon_0^-\varepsilon_1^-} = \mathfrak{d}^{W}_{\varepsilon_0'\varepsilon_1'} \circ \Phi^{V_0,V_1}.$$

The matrices on the left and right hand side become

$$\begin{pmatrix} -d_{++}^{W_0,W_1} & d_{+0}^{W_0,W_1} + d_{+-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & d_{+-}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{+-}^{W_0,W_1}d_{+-}^{V_0,V_1} \\ 0 & d_{00}^{W_0,W_1} + d_{0-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & d_{0-}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{0-}^{W_0,W_1}d_{+-}^{V_0,V_1} \\ 0 & A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= b^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1}, \\ A_2 &= b^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}d_{+0}^{V_0,V_1}, \\ A_3 &= b^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{0-}^{W_0,W_1}d_{+-}^{V_0,V_1}, \\ B_1 &= b^{V_0,V_1}\delta_{-0}^{W_0,W_1}d_{00}^{W_0,W_1} + d_{+-}^{V_0,V_1}b^{\Lambda_0^-,\Lambda_1^-}\delta_{-+}^{V_0,V_1}\delta_{-0}^{W_0,W_1} + d_{+0}^{V_0,V_1}\delta_{0+}^{V_0,V_1}\delta_{-0}^{W_0,W_1}, \\ B_2 &= d_{+0}^{V_0,V_1}d_{00}^{V_0,V_1} + d_{+-}^{V_0,V_1}d_{-0}^{V_0,V_1}, \\ B_3 &= d_{+0}^{V_0,V_1}d_{0-}^{V_0,V_1} + d_{+-}^{V_0,V_1}d_{--}^{V_0,V_1}, \end{aligned}$$

and

$$\begin{pmatrix} -d_{++}^{W_0,W_1} & d_{+0}^{W_0,W_1} + d_{+-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & d_{+-}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{+-}^{W_0,W_1}d_{+-}^{V_0,V_1} \\ 0 & d_{00}^{W_0,W_1} + d_{0-}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & d_{0-}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{0-}^{W_0,W_1}d_{+-}^{V_0,V_1} \\ 0 & d_{-0}^{W_0,W_1} + d_{--}^{W_0,W_1}b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & d_{--}^{W_0,W_1}d_{+0}^{V_0,V_1} & d_{--}^{W_0,W_1}d_{+-}^{V_0,V_1} \end{pmatrix},$$

respectively.

Again, we have $A_i = 0$, $i = 1, 2, 3$, by action reasons. The equalities remaining to be shown are

$$\begin{aligned} d_{+0}^{V_0,V_1} d_{00}^{V_0,V_1} + d_{+-}^{V_0,V_1} d_{-0}^{V_0,V_1} - d_{++}^{V_0,V_1} d_{+0}^{V_0,V_1} &= 0 \\ d_{+0}^{V_0,V_1} d_{0-}^{V_0,V_1} + d_{+-}^{V_0,V_1} d_{--}^{V_0,V_1} - d_{++}^{V_0,V_1} d_{+-}^{V_0,V_1} &= 0 \end{aligned}$$

(recall again that $d_{--}^{W_0W_1} = d_{++}^{V_0V_1}$) together with the equality

$$\begin{aligned} &(b^{\Lambda_0,\Lambda_1} + d_{++}^{V_0,V_1} b^{V_0,V_1})\delta_{-0}^{W_0,W_1} = \\ &= \quad b^{V_0,V_1}\delta_{-0}^{W_0,W_1} d_{00}^{W_0,W_1} + (d_{+-}^{V_0,V_1} b^{\Lambda_0^-,\Lambda_1^-}\delta_{-+}^{V_0,V_1} + d_{+0}^{V_0,V_1} d_{0+}^{V_0,V_1})\delta_{-0}^{W_0,W_1}. \end{aligned}$$

(Recall that $d_{-0}^{W_0,W_1} = b^{\Lambda_0,\Lambda_1}\delta_{-0}^{W_0,W_1}$.)

The first two equalities again follow from the fact that the left-hand sides correspond to counts of number of boundary component of appropriate one-dimensional moduli spaces of pseudoholomorphic discs ($\mathcal{M}(\gamma^+; \boldsymbol{\delta}, p, \boldsymbol{\zeta})$ for the first one, the second follows by an analysis similar to the one for the transfer map). Also, see Figures 14 and 12.

The third equality finally follows from

$$\begin{aligned} &\delta_{-0}^{W_0,W_1} d_{00}^{W_0,W_1} - \delta_{--}^{W_0,W_1}\delta_{-0}^{W_0,W_1} = 0 \\ &b^{\Lambda_0,\Lambda_1} + d_{++}^{V_0,V_1} b^{V_0,V_1} = b^{V_0,V_1}\delta_{--}^{W_0,W_1} + d_{+-}^{V_0,V_1} b^{\Lambda_0^-,\Lambda_1^-}\delta_{-+}^{V_0,V_1} + d_{+0}^{V_0,V_1}\delta_{0+}^{V_0,V_1}, \end{aligned}$$

both which can be shown by arguments similar to the above (more precisely, studying $\mathcal{M}(q; \boldsymbol{\delta}, \gamma^1, \boldsymbol{\zeta})$ for the first, and degeneration of bananas on (V_0, V_1) for the second). Also, see Figures 11 and 22. □

Remark 7.3. *In the special case when there are no Reeb chords from Λ_1 to Λ_0 (recall that these are the Legendrian submanifolds along which the concatenations are performed), the corresponding transfer and co-transfer maps take the following particularly simple form. Since $C(\Lambda_0, \Lambda_1) = 0$, the transfer map Φ_{W_0,W_1} is simply the inclusion of a subcomplex, while the co-transfer map Φ^{V_0,V_1} becomes the corresponding quotient projection. In fact, as will be shown below in Section 8.1, this situation can always be achieved after the application of a Hamiltonian isotopy that "wraps" the positive and negative ends of V_1 and W_1, respectively.*

The following lemma is standard. It follows from the fact that, in the cylindrical situation, regular 0-dimensional moduli space are trivial, together with a stretching-the-neck argument.

Lemma 7.4. *The (co)transfer map satisfies the following properties*

- *In the case when $W_i = \mathbb{R} \times \Lambda_i$, $i = 0, 1$, and J_b is a cylindrical almost complex structure we have*
$$\Phi_{W_0,W_1} = \Phi_{\mathbb{R}\times\Lambda_0,\mathbb{R}\times\Lambda_1} = \mathrm{id}.$$
- *In the case when $V_i = \mathbb{R} \times \Lambda_i$, $i = 0, 1$, and J_a is a cylindrical almost complex structure we have*
$$\Phi^{V_0,V_1} = \Phi^{\mathbb{R}\times\Lambda_0,\mathbb{R}\times\Lambda_1} = \mathrm{id}.$$
- *In the case when $W_i = U_i \odot_M U_i'$, $i = 0, 1$, and $J_b = J_c \odot_M J_d$ we have*
$$\Phi_{W_0,W_1} = \Phi_{U_0',U_1'} \circ \Phi_{U_0,U_1}$$
in the case when $M \gg 0$ is sufficiently large

- *In the case when* $V_i = U_i \odot_M U_i'$, $i = 0, 1$, *and* $J_a = J_c \odot_M J_d$ *we have*
$$\Phi^{V_0,V_1} = \Phi^{U_0',U_1'} \circ \Phi^{U_0,U_1}$$
in the case when $M \gg 0$ *is sufficiently large.*

In the case when $W_1 = \mathbb{R} \times \Lambda$, we write
$$\Phi_{W_0} := \Phi_{W_0,W_1} = \Phi_{W_0,\mathbb{R}\times\Lambda}$$
and, similarly, when $V_1 = \mathbb{R} \times \Lambda \subset \mathbb{R} \times Y$, we write
$$\Phi^{V_0} := \Phi^{V_0,V_1} = \Phi^{V_0,\mathbb{R}\times\Lambda}.$$

7.4. **An auxiliary complex.** Here we try to shed some light on the algebraic relationship between the transfer and the co-transfer map. We assume that all differentials are computed using almost complex structures such that Lemma 7.2 holds.

Starting with the above complex
$$(C_\bullet := \mathrm{Cth}_\bullet(V_0 \odot_N W_0, V_1 \odot_N W_1), \mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),$$
we construct the complex
$$(\widetilde{C}_\bullet := C_\bullet \oplus C_\bullet(\Lambda_0,\Lambda_1) \oplus C_{\bullet-1}(\Lambda_0,\Lambda_1), \widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),$$
$$\widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-} := \begin{pmatrix} \mathfrak{d}^{V\odot W} & 0 & 0 \\ 0 & d_{--}^{W_0,W_1} & 0 \\ 0 & \mathrm{id} & d_{++}^{V_0,V_1} \end{pmatrix}.$$
(Again, $d_{--}^{W_0W_1} = d_{++}^{V_0V_1}$.) This complex is clearly homotopic to $(C_\bullet, \mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-})$, since the canonical inclusion and projection maps
$$\iota\colon (C_\bullet, \mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}) \to (\widetilde{C}_\bullet, \widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),$$
$$\pi\colon (\widetilde{C}_\bullet, \widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}) \to (C_\bullet, \mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),$$
are homotopy inverses of each other.

We have a canonical identification
$$\widetilde{C}_\bullet = \mathrm{Cth}_\bullet(V_0, V_1) \oplus \mathrm{Cth}_\bullet(W_0, W_1)$$
on the level of modules. We let
$$\iota_{V_0,V_1}\colon \mathrm{Cth}_\bullet(V_0, V_1) \hookrightarrow \widetilde{C}_\bullet,$$
$$\iota_{W_0,W_1}\colon \mathrm{Cth}_\bullet(W_0, W_1) \hookrightarrow \widetilde{C}_\bullet,$$
be the canonical inclusions of the respective summands.

Even though ι_{V_0,V_1} is not a chain-map in general, after a change of coordinates it will be apparent that $(\widetilde{C}_\bullet, \widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-})$ in fact is the mapping-cone of a chain map
$$\delta_{VW}\colon \mathrm{Cth}_\bullet(W_0, W_1) \to \mathrm{Cth}_\bullet(V_0, V_1).$$
To that end, we start by considering the map
$$\Psi\colon C_\bullet \oplus C_\bullet(\Lambda_0,\Lambda_1) \oplus C_{\bullet-1}(\Lambda_0,\Lambda_1) \to C_\bullet \oplus C_\bullet(\Lambda_0,\Lambda_1) \oplus C_{\bullet-1}(\Lambda_0,\Lambda_1),$$
$$\Psi = \begin{pmatrix} \mathrm{id} & 0 & d_{+-}^{W_0,W_1} + d_{0-}^{W_0,W_1} \\ -(d_{+-}^{V_0,V_1} + d_{+0}^{V_0,V_1} + b^{V_0,V_1}\delta_{-0}^{W_0,W_1}) & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

The property that

$$\delta_{-0}^{W_0,W_1} d_{0-}^{W_0,W_1} = 0,$$

which holds by action considerations, implies the equality

$$(d_{+-}^{V_0,V_1} + d_{+0}^{V_0,V_1} + b^{V_0,V_1}\delta_{-0}^{W_0,W_1})(d_{+-}^{W_0,W_1} + d_{0-}^{W_0,W_1}) = 0.$$

From this it follows that Ψ in fact is an isomorphism of modules, with inverse given by

$$\Psi^{-1} = \begin{pmatrix} \mathrm{id} & 0 & -(d_{+-}^{W_0,W_1} + d_{0-}^{W_0,W_1}) \\ d_{+-}^{V_0,V_1} + d_{+0}^{V_0,V_1} + b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Using Ψ we define the complex

$$\Psi\colon (\widetilde{C}_\bullet, \overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}) \to (\widetilde{C}_\bullet, \widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),$$
$$\overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-} := \Psi^{-1} \circ \widetilde{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-} \circ \Psi,$$

obtained by applying a coordinate change to the original complex.

The upshot is that the differential of $(\widetilde{C}_\bullet, \overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-})$ has the following particularly nice and apparent cone structure. The proof follows from a breaking analysis similar to the one in the proof of Lemma 7.2.

Lemma 7.5. *The complex $(\widetilde{C}_\bullet, \overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-})$ is equal to the mapping cone*

$$(\widetilde{C}_\bullet = \mathrm{Cth}_\bullet(W_0, W_1) \oplus \mathrm{Cth}_\bullet(V_0, V_1), \overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}),$$
$$\overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-} = \begin{pmatrix} \mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'} & \delta_{VW} \\ 0 & \mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-} \end{pmatrix},$$

where

$$C_{\bullet-1}(\Lambda_0^+, \Lambda_1^+) \oplus C_\bullet(W_0, W_1) \oplus C_\bullet(\Lambda_0, \Lambda_1)$$
$$\downarrow \delta_{VW}$$
$$C_{\bullet-1}(\Lambda_0, \Lambda_1) \oplus C_\bullet(V_0, V_1) \oplus C_\bullet(\Lambda_0^-, \Lambda_1^-),$$

is of the form

$$\delta_{VW} = \begin{pmatrix} 0 & -b^{V_0,V_1}\delta_{-0}^{W_0,W_1} & \mathrm{id}_{C(\Lambda_0,\Lambda_1)} \\ 0 & d_{0+}^{V_0,V_1}\delta_{-0}^{W_0,W_1} & 0 \\ 0 & b^{\Lambda_0^-,\Lambda_1^-}\delta_{-+}^{V_0,V_1}\delta_{-0}^{W_0,W_1} & 0 \end{pmatrix}.$$

In particular, the canonical inclusion

$$\iota_{V_0,V_1}\colon (\mathrm{Cth}_\bullet(V_0, V_1), \mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-}) \hookrightarrow (\widetilde{C}_\bullet, \overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-})$$

as well as the canonical projection

$$\pi_{W_0,W_1}\colon (\widetilde{C}_\bullet, \overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}) \to (\mathrm{Cth}_\bullet(W_0, W_1), \mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'}),$$

are both chain maps. Furthermore, the transfer and co-transfer map can be expressed as

$$\begin{aligned} \Phi_{W_0,W_1} &= \pi \circ \Psi \circ \iota_{V_0,V_1}, \\ \Phi^{V_0,V_1} &= \pi_{W_0,W_1} \circ \Psi^{-1} \circ \iota, \end{aligned}$$

where all factors are chain maps.

Proof. The claim that

$$\iota_{V_0,V_1}\colon (\mathrm{Cth}_\bullet(V_0,V_1),\mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-}) \hookrightarrow (\widetilde{C}_\bullet,\overline{\mathfrak{d}}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-})$$

is a chain map can be seen by considering the relations

$$\begin{cases}\mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'}(d^{W_0,W_1}_{+-}+d^{W_0,W_1}_{0-})+(d^{W_0,W_1}_{+-}+d^{W_0,W_1}_{0-})d^{\Lambda_0,\Lambda_1}=0,\\ d^{\Lambda_0,\Lambda_1}(d^{V_0,V_1}_{+-}+d^{V_0,V_1}_{+0})=(d^{V_0,V_1}_{+-}+d^{V_0,V_1}_{+0})\mathfrak{d}^V_{\varepsilon_0^-,\varepsilon_1^-},\end{cases}$$

which follow from the fact that $(\mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'})^2=0$ and $(\mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-})^2=0$, respectively, together with the expressions of $\mathfrak{d}^{V\odot W}_{\varepsilon_0^-\varepsilon_1^-}$, $\mathfrak{d}^W_{\varepsilon_0'\varepsilon_1'}$ and $\mathfrak{d}^V_{\varepsilon_0^-\varepsilon_1^-}$ as given in Section 7.2.

To see that the corresponding quotient complex is as claimed, one must use the identity

$$(-d^{\Lambda_0,\Lambda_1}+d^{V_0,V_1}_{+-}+d^{V_0,V_1}_{+0})\delta_{VW}+b^{V_0,V_1}\delta^{W_0,W_1}_{-0})d^{W_0,W_1}_{00}=b^{\Lambda_0,\Lambda_1}\delta^{W_0,W_1}_{-0},$$

which in turn follows from

$$\begin{cases}-d^{\Lambda_0,\Lambda_1}b^{V_0,V_1}+b^{V_0,V_1}\delta^{V_0V_1}_{++}+d^{V_0,V_1}_{+-}b^{\Lambda_0^-,\Lambda_1^-}\delta^{V_0,V_1}_{-+}+\delta^{V_0,V_1}_{+0}d^{V_0,V_1}_{0+}=b^{\Lambda_0,\Lambda_1},\\ \delta^{W_0,W_1}_{-0}d^{W_0,W_1}_{00}=\delta^{W_0,W_1}_{--}\delta^{W_0,W_1}_{-0}.\end{cases}$$

The latter identities can be seen by analysing the possible boundaries of the appropriate one-dimensional moduli spaces. □

8. Proof of the acyclicity (the invariance)

In this section we establish the invariance result for our Floer theory. In fact, in our context, the invariance is simply the fact the complex $\mathrm{Cth}(\Sigma_0,\Sigma_1)$ is acyclic (actually null-homotopic). The naive reason for this is that the symplectisation of a contactisation $(\mathbb{R}\times P\times\mathbb{R}, d(e^t(dz+\theta)))$ is "subcritical": it is symplectomorphic to $(P\times\mathbb{R}^2, d\theta\oplus\omega_0)$. More precisely, the main feature that will be used is that one can use the Reeb flow (which is Hamiltonian) in order to displace an exact Lagrangian cobordism from any other given cobordism.

In order to circumvent certain technical difficulties, we will here restrict our attention to almost complex structures on the symplectisation $\mathbb{R}\times P\times\mathbb{R}$ that are admissible, as defined in Section 3.1.4, and moreover coinciding with cylindrical lifts of almost complex structures in P outside of a compact subset; see Section 3.1.3.

8.1. **Wrapping the ends.** Let Σ_i, $i=0,1$, be exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, $i=0,1$. We assume that Σ_i, $i=0,1$, both are cylindrical in the subset $\{|t|\geq T\}$ for some $T>0$.

Fix a smooth non-decreasing cut-off function $\rho\colon\mathbb{R}\to[0,1]$ satisfying $\rho(t)=0$ for $t\leq 1$ and $\rho(t)=1$ for $t\geq 2$. Consider the smooth real-valued function

$$h_N\colon\mathbb{R}\times P\times\mathbb{R}\to\mathbb{R},$$
$$(t,p,z)\mapsto e^t\rho_N(t),$$

where $\rho_N\colon\mathbb{R}\to[0,1]$ is defined uniquely by the relations $\rho_N(-t)=\rho_N(t)$, and $\rho_N(t)=\rho(t-(T+N))$ for all $t\geq 0$. We also consider the smooth real-valued functions

$$h_{N,\pm}\colon\mathbb{R}\times P\times\mathbb{R}\to\mathbb{R},$$
$$(t,p,z)\mapsto e^t\rho(\pm t-(T+N)),$$

and set $h_\pm := h_{0,\pm}$. The above functions are then used to construct the Hamiltonian isotopies

$$\begin{aligned}\phi^s_{h_N\partial_z}(t,p,z) &= (t,p,z+s\rho_N(t)),\\ \phi^s_{h_{N,\pm}\partial_z}(t,p,z) &= (t,p,z+s\rho(\pm t-(T+N))),\end{aligned}$$

which should be thought of as "wrapping" the ends of $\mathbb{R}\times P\times\mathbb{R}$. For instance, after applying the isotopy $\phi^{-S}_{h_N\partial_z}$ to Σ_0 for $S\gg 0$ sufficiently large, we get additional double points $\phi^{-S}_{h_N\partial_z}(\Sigma_0)\cap\Sigma_1$, all which correspond to the Reeb chords going from the ends of Σ_1 to the corresponding end of Σ_0. More precisely, the following is true:

Lemma 8.1. *When*

$$S\geq S_0 := 2\max_{c\in\mathcal{R}(\Lambda_1^-,\Lambda_0^-)\cup\mathcal{R}(\Lambda_1^+,\Lambda_0^+)}\ell(c)$$

there is a canonical bijective correspondence

$$w_\pm\colon \phi^{-S}_{h_{N,\pm}\partial_z}(\mathbb{R}\times\Lambda_0^\pm)\cap(\mathbb{R}\times\Lambda_1^\pm)\to\mathcal{R}(\Lambda_1^\pm,\Lambda_0^\pm)$$

induced by the Lagrangian projection, i.e. by identifying elements on both sides with a double point in $\Pi_{\mathrm{Lag}}(\Lambda_0^\pm\cup\Lambda_1^\pm)\subset P$. *On the level of gradings this bijection moreover satisfies*

$$|w_-(p)|=|p| \text{ and } |w_+(p)|=|p|-1.$$

In particular, there is a canonical identification

$$\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1)=\mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1)$$

on the level of graded modules. *After taking $N\gg 0$ sufficiently large, we may assume that the action of a generator in*

$$C(\Lambda_0^-,\Lambda_1^-)\subset\mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1),$$

is arbitrarily small, the action of a generator in

$$C(\Lambda_0^+,\Lambda_1^+)\subset\mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1)$$

is arbitrarily large, while the action of a generator in

$$C(\Sigma_0,\Sigma_1)\subset\mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1)$$

coincides with its original action.

The first goal will be showing that the identification given by the above lemma, in fact may be assumed to hold on the level of *complexes* as well.

Proposition 8.2. *For each $N\gg 0$ sufficiently large and $S\geq S_0$ as defined above, there is a canonical identification of complexes*

$$(\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-\varepsilon_1^-})=(\mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1),\mathfrak{d}_{\varepsilon_0^-\varepsilon_1^-}),$$

under the assumption that $J\in\mathcal{J}^{\mathrm{adm}}_{J^-,J^+}(\mathbb{R}\times P\times\mathbb{R})$ is a regular admissible almost complex structure and where, moreover, $J^\pm=\widetilde{J}_P$ both are the cylindrical lift of an almost complex structure on $(P,d\theta)$.

Remark 8.3. • *Recall that J is equal to $\widetilde{J}_P$ outside of a compact set by assumption, where the latter is a cylindrical almost complex structure which is invariant under the Reeb flow. Since the negative ends of Σ_0 and $\phi^{-S}_{h_N\partial_z}(\Sigma_0)$ differ by the time-S Reeb flow, it follows that these Legendrian submanifolds*

have canonically isomorphic Chekanov-Eliashberg algebras when defined by $\widetilde{J}_P$ and, in particular, we can identify their augmentations.

- *For a general choice of almost complex structure, there should again exist an analogous isomorphism, albeit non-canonical.*

Proof. Consider the transfer and co-transfer maps

$$\Phi_{\phi^{-S}_{h_+\partial_z}(\mathbb{R}\times\Lambda_0^+)}\colon\ \mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1)\to\mathrm{Cth}_\bullet(\phi^{-S}_{h_{N,+}\partial_z}(\Sigma_0),\Sigma_1),$$
$$\Phi^{\phi^{-S}_{h_-\partial_z}(\mathbb{R}\times\Lambda_0^-)}\colon\ \mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1)\to\mathrm{Cth}_\bullet(\phi^{-S}_{h_{N,+}\partial_z}(\Sigma_0),\Sigma_1),$$

defined by counting $\widetilde{J}_P$-holomorphic strips having boundary on $\phi^{-S}_{h_{N,+}\partial_z}(\mathbb{R}\times\Lambda_0^+)\cup(\mathbb{R}\times\Lambda_1^+)$ and $\phi^{-S}_{h_{N,-}\partial_z}(\mathbb{R}\times\Lambda_0^-)\cup(\mathbb{R}\times\Lambda_1^-)$, respectively. In order to identify the domains and codomains of the above maps we have used the fact that,

$$J\odot_N\widetilde{J}_P=\widetilde{J}_P\odot_N J=J,\ \ N\geq 0,$$

which holds by the assumptions made on J, as well as the facts that

$$\begin{aligned}\phi^{-S}_{h_{N,+}\partial_z}(\Sigma_0) &= \Sigma_0\odot_N\phi^{-S}_{h_+\partial_z}(\mathbb{R}\times\Lambda_0^+),\\ \phi^{-S}_{h_N\partial_z}(\Sigma_0) &= \phi^{-S}_{h_-\partial_z}(\mathbb{R}\times\Lambda_0^-)\odot_N\phi^{-S}_{h_{N,+}\partial_z}(\Sigma_0),\end{aligned}$$

hold for every $N\geq 0$ by construction.

Recall that the transfer and co-transfer maps are chain maps by Lemma 7.2, assuming that $N\gg 0$ has been chosen sufficiently large. The proposition will follow from the claim that $\Phi_{\phi^{-S}_{h_{N,+}\partial_z}(\mathbb{R}\times\Lambda_0^+)}$ and $\Phi^{\phi^{-S}_{h_{N,-}\partial_z}(\mathbb{R}\times\Lambda_0^-)}$ both are isomorphisms that, moreover, induce the respective canonical identifications of graded modules described in Lemma 8.1. The latter facts follows by the explicit disc count performed in the proof of [17, Theorem 2.15]; also, see [25, Proposition 5.11] for a similar argument. Roughly speaking, it is there shown that every $\widetilde{J}_P$-holomorphic disc of index zero in the definition of the above (co)transfer map is a transversely cut-out strip having one positive puncture and one negative puncture, and whose image under the canonical projection to P is constant. Conversely, there is an explicitly defined such strip for every double point in P corresponding to a Reeb chord. In particular, under an appropriate choice of basis, the matrices of both maps $\Phi_{\phi^{-S}_{h_{N,+}\partial_z}(\mathbb{R}\times\Lambda_0^+)}$ and $\Phi^{\phi^{-S}_{h_{N,-}\partial_z}(\mathbb{R}\times\Lambda_0^-)}$ are equal to the identity matrices. □

8.2. **Invariance under compactly supported Hamiltonian isotopies.** The following is the core of the invariance result that we need in order to deduce the acyclicity of the Cthulhu complex. (As we need to compare the complexes computed with two different almost structures, here we denote by $\mathfrak{d}_{\varepsilon_0^-\varepsilon_1^-}(J)$ the differential on $\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1)$ computed using the almost complex structure J.)

Proposition 8.4. *Let (Σ_0^s,Σ_1), $s\in[0,1]$, be a compactly supported one-parameter family of pairs of exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, $i=0,1$. Also, consider a one-parameter family $\{J_s\}_{s\in[0,1]}$ of admissible almost complex structures which agree outside of a compact set. There is an induced homotopy equivalence*

$$\Psi_{\{(\Sigma_0^s,J_s)\}}\colon\ (\mathrm{Cth}_\bullet(\Sigma_0^0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-\varepsilon_1^-}(J_0))\to(\mathrm{Cth}_\bullet(\Sigma_0^1,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-\varepsilon_1^-}(J_1)).$$

Furthermore, the restriction of $\Psi_{\{(\Sigma_0^s, J_s)\}}$ *to the subcomplex*

$$(C_\bullet(\Lambda_0^+, \Lambda_1^+), -d_{++}) \subset \mathrm{Cth}_\bullet(\Sigma_0, \Sigma_1))$$

induces an isomorphism

$$\Psi_{\{(\Sigma_0^s, J_s)\}}|_{C(\Lambda_0^+, \Lambda_1^+)} \colon (C(\Lambda_0^+, \Lambda_1^+), -d_{++}) \to (C(\Lambda_0^+, \Lambda_1^+), -d_{++}).$$

Remark 8.5.
- *Under the additional assumption that* $\Lambda_i^- = \emptyset$, $i = 0, 1$, *this result was established in* [31, Section 4.2.1].
- *Our proof of the above proposition does not rely on abstract perturbations. This is possible since we can use Proposition 8.2 in order to replace Reeb chords with intersection points, thereby breaking some of the symmetry. An alternative proof, more closely adapted to the SFT formalism, would be simply to generalise* [31, Section 4.2.1] *to the current setting. The latter approach depends on the abstract perturbation scheme outlined in* [30, Appendix B].

Proof. After an application of Proposition 8.2, we may assume that there are no Reeb chords from $\Lambda_1^\pm$ to $\Lambda_0^\pm$ and, hence, that the involved complexes are generated by intersection points only. More precisely, the aforementioned proposition provides such natural identifications

$$\begin{aligned}(\mathrm{Cth}_\bullet(\Sigma_0^0, \Sigma_1), \mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-}(J_0)) &= (\mathrm{Cth}_\bullet(\phi^{-S}_{h_N \partial_z}(\Sigma_0^0), \Sigma_1), \mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-}(J_0)),\\ (\mathrm{Cth}_\bullet(\Sigma_0^1, \Sigma_1), \mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-}(J_1)) &= (\mathrm{Cth}_\bullet(\phi^{-S}_{h_N \partial_z}(\Sigma_0^1), \Sigma_1), \mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-}(J_1)),\end{aligned}$$

of complexes for $S, N \gg 0$ sufficiently large. Obviously $\phi^{-S}_{h_N \partial_z}(\Sigma_0^s)$ still is a compactly supported family of exact Lagrangian cobordisms. This family is moreover fixed in a neighbourhood of the intersection points corresponding to the generators of $C(\Lambda_0^\pm, \Lambda_1^\pm)$, assuming that $N \gg 0$ is chosen sufficiently large.

To sum up, we have reduced the invariance problem to the case when there are no Reeb chords from $\Lambda_1^\pm$ to $\Lambda_0^\pm$. In this case the involved complexes thus have generators corresponding to the intersection points of $\Sigma_0^s \cap \Sigma_1$, $s = 0, 1$, and the one-parameter family Σ_0^s is a compactly supported family of exact Lagrangian cobordisms.

In this case, the bifurcation analysis conducted in [32], [34] can be generalised to produce the sought homotopy equivalence. We proceed to provide a sketch of this argument, highlighting the points where some additional care must be taken to adapt it to the current setting.

Recall that the differential is defined via a count of J_s-holomorphic strips having boundary on $\Sigma_0^s \cup \Sigma_1$, two punctures asymptotic to intersection points, and possibly additional negative punctures asymptotic to Reeb chords on either Λ_0^- or Λ_1^-. For a generic one-parameter family $\Sigma_0^s \cap \Sigma_1$ and J_s $s \in [0, 1]$, we can assume that for all but a finite number of instances $s \in [0, 1]$,

- the intersection $\Sigma_0^s \cap \Sigma_1$ is transverse, and
- the above spaces of J_s-holomorphic strips are transversely cut out.

Moreover, in the case when the above two conditions do not hold for some $s_0 \in [0, 1]$ we may assume that precisely one of the following cases occur.

Birth/death: All intersection points of $\Sigma_0^{s_0} \cap \Sigma_1$ are transverse *except a single intersection point*. The latter intersection point moreover arises in the family as a standard birth/death-intersection point, which was described in [32, Section 3]

(note that the Lagrangian cobordisms considered are of dimension at least *two* by assumption). Moreover, all J_{s_0}-holomorphic strips as above whose punctures are asymptotic to transverse intersection points are transversely cut out.

Handleslide: The intersection $\Sigma_0^{s_0} \cap \Sigma_1$ is transverse, and all J_s-holomorphic strips with boundary on $\Sigma_0^{s_0} \cup \Sigma_1$ are regular *except a single solution*. Moreover, the latter strip has index -1, and is generic when considered in the one-parameter family of boundary value problems.

Observe that, it suffices to consider each case above separately in order to deduce the sought invariance. Namely, for a generic family in which none of the above cases occur, all involved complexes are naturally isomorphic.

The case of a birth/death moment is treated analogously as in the closed case in [46], a careful account appearing also in [67]. Note that the difference of action between the newly created intersection points is small near the birth point and, thus, the involved holomorphic curves cannot have negative asymptotics to Reeb chords.

In the case of a handle slide we construct an isomorphism of complexes by the following formula. Assume that the unique strip u of index -1 has the intersection points x, y as incoming and outgoing ends, respectively, and that it has additional negative punctures asymptotic to the words $\mathbf{a}$, $\mathbf{b}$ of Reeb chords on Λ_0^- and Λ_1^-, respectively. On the intersection point generators we define the isomorphism

$$p \mapsto p + K(p),$$
$$K(p) = \begin{cases} \varepsilon_0(\mathbf{a})\varepsilon_1(\mathbf{b})x, & p = y, \\ 0, & p \neq y, \end{cases}$$

of graded modules.

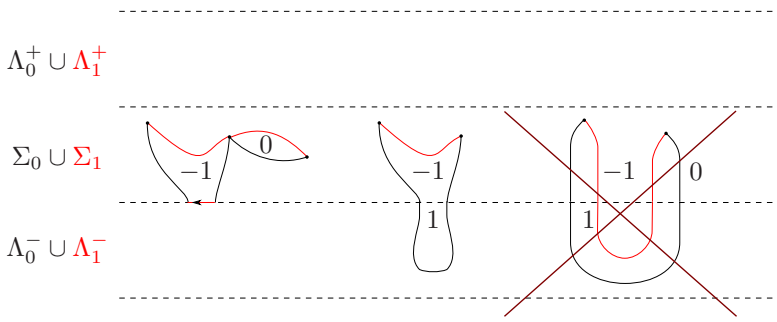

FIGURE 17. The a priori possibilities for the boundary points of a one-parameter family of rigid strips with both incoming and outgoing ends asymptotic to intersection points. The number on each component denotes its Fredholm index.

We must show that this isomorphism in fact is a chain map. Again, this is shown by gluing of pseudoholomorphic curves. The only difference between this case and the case when $\mathbf{a} = \mathbf{b} = \emptyset$ comes from the gluing of ∂-breakings. More precisely, the difference between the differential $d_{s_0-\epsilon}$ before the handle slide and $d_{s_0+\epsilon}$ come from glued rigid configuration of index 0 curve to u such configuration are of two

types (see Figure 17): The first type of gluing does not involve pure negative Reeb chords, and correspond to the term K in the definition of the isomorphism. The second type of gluing involves a pure Reeb chord, and corresponds to that of a ∂-breaking. The totality of the glued solutions of this kind thus contributes to 0 when counted utilising the augmentation (see the proof of Theorem 6.1). Finally, note that breaking involving an index -1 curve with a negative asymptotic to a mixed chord cannot occur; such a breaking would involve a holomorphic banana and, one of its two ends would necessarily be a chord from Λ_1^- to Λ_0^- (which does not exist by assumption).

Finally, in order to deduce the last claim of the proposition, it will be necessary to use the following additional property of the identifications of complexes described in the proof of Proposition 8.2. Consider the subset $C \subset \phi^{-S}_{h_N \partial_z}(\Sigma_0^s) \cap \Sigma_1$ of intersection points corresponding to the Reeb chords from Λ_1^+ to Λ_0^+; these intersection points are contained in a subset of the form $(N, +\infty) \times P \times \mathbb{R}$, which may be assumed to be fixed in the one-parameter family $\phi^{-S}_{h_N \partial_z}(\Sigma_0^s)$ of cobordisms. Even though these intersection points are fixed, their actions will in general vary with $s \in [0,1]$. We use M to denote the minimum of the action of a intersection point in $C \subset \phi^{-S}_{h_N \partial_z}(\Sigma_0^s) \cap \Sigma_1$ taken over all $s \in [0,1]$. For $N \gg 0$ sufficiently large, Lemma 8.1 shows that any intersection point $(\phi^{-S}_{h_N \partial_z}(\Sigma_0^s) \cap \Sigma_1) \setminus C$ has action strictly less than M. In conclusion, we must have $K(C(\Lambda_0^+, \Lambda_1^+)) \subset C(\Lambda_0^+, \Lambda_1^+)$ for the map K as defined above, thus implying the claim. □

8.3. **The proof of the acyclicity.** We are finally ready to prove the main result of this section. Recall that we are in the special case of a symplectisation of a contactisation. In this setting, the idea is to show that a complex having no Reeb chord generators is acyclic by alluding to Proposition 8.4, together with an explicitly defined compactly supported Hamiltonian displacement. Finally, Proposition 8.2 shows that we can replace every complex with a complex induced by a pair of Lagrangian cobordisms having no Reeb chord generators (which is obtained by wrapping the cylindrical ends of one of the Lagrangian cobordisms).

Theorem 8.6. *Assume that we are given a choice of regular admissible almost complex structure coinciding with a cylindrical lifts outside some subset of the form $[-T,T] \times Y$. For any pair $\Sigma_0, \Sigma_1 \subset \mathbb{R} \times P \times \mathbb{R}$ of exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, and choices ε_i^- of augmentations of the Chekanov-Eliashberg algebra of Λ_i^-, $i = 0, 1$, the complex*

$$(\mathrm{Cth}_\bullet(\Sigma_0, \Sigma_1), \mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-})$$

is homotopic to the trivial complex,

Proof. We use the fact that, for $S \gg 0$ sufficiently large, there is an exact Lagrangian cobordism $\Sigma_0' \subset \mathbb{R} \times (P \times \mathbb{R})$ satisfying the properties that, first, it is isotopic to $\phi^{-S}_{h_N \partial_z}(\Sigma_0)$ by a compactly supported Hamiltonian isotopy and, second, the complex

$$(\mathrm{Cth}(\Sigma_0', \Sigma_1), \mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-}) = 0$$

has no generators. To that end, observe that the complex $\mathrm{Cth}(\phi^{-S}_{h_N \partial_z}(\Sigma_0), \Sigma_1)$ has no Reeb chord generators for $S \gg 0$ sufficiently large by Lemma 8.1. Moreover, the compactly supported Hamiltonian isotopy $\phi^{-s}_{e^t} \circ \phi^{-(S-s)}_{h_N \partial_z}(\Sigma_0)$ can be seen to remove all intersection points, whenever $s \gg 0$ is sufficiently large.

The invariance result for a compactly supported Hamiltonian isotopy provided by Proposition 8.4 thus implies that the complex $\mathrm{Cth}_\bullet(\phi^{-S}_{h_N\partial_z}(\Sigma_0),\Sigma_1)$ is null-homotopic. The fact that the same is true for the complex $\mathrm{Cth}_\bullet(\Sigma_0,\Sigma_1)$ is now an immediate consequence of Proposition 8.2, for $N \gg 0$ that have been chosen sufficiently large. □

9. Proof of the main theorem

Now we proceed to prove the main result of the paper. In Section 10, this will then be used to deduce several long exact sequences.

We denote the quotient complex $C(\Sigma_0,\Sigma_1) \oplus C(\Lambda_0^-,\Lambda_1^-)$ of $\mathrm{Cth}(\Sigma_0,\Sigma_1)$ by $CF_{-\infty}(\Sigma_0,\Sigma_1)$ with differential given by $d_\infty = \begin{pmatrix} d_{00} & d_{0-} \\ d_{-0} & d_{--} \end{pmatrix}$, its homology is denoted by $HF_{-\infty}(\Sigma_0,\Sigma_1)$. The complex $\mathrm{Cth}(\Sigma_0,\Sigma_1)$ is the cone of the chain map $d_{+0}+d_{+-} : CF_\infty(\Sigma_0,\Sigma_1) \to C(\Lambda_0^+,\Lambda_1^+)$.

The acyclicity of the complex $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-\varepsilon_1^-})$ implies that this map is a quasi-isomorphism, and hence that

$$LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+,\Lambda_1^+) \simeq HF^{k+1}_{-\infty}(\Sigma_0,\Sigma_1). \tag{39}$$

Let f_i, $i=0,1$, be primitives of $e^t\alpha|_{\Lambda_i}$ as in Section 3.3. A point $p \in \Sigma_0 \cap \Sigma_1$ is *positive* (resp. *negative*) if $f_1(p)-f_0(p)$ is positive (resp. negative). This leads to a decomposition $CF(\Sigma_0,\Sigma_1) = CF_+(\Sigma_0,\Sigma_1) \oplus CF_-(\Sigma_0,\Sigma_1)$.

Proposition 9.1. *With respect to the decomposition*

$$CF_{-\infty}(\Sigma_0,\Sigma_1) = CF_+(\Sigma_0,\Sigma_1) \oplus C(\Lambda_0^-,\Lambda_1^-) \oplus CF_-(\Sigma_0,\Sigma_1),$$

the differential takes the form

$$d_{-\infty} = \begin{pmatrix} d_{0^+0^+} & d_{0^+-} & d_{0^+0^-} \\ 0 & d_{--} & d_{-0^-} \\ 0 & 0 & d_{0^-0^-} \end{pmatrix} \tag{40}$$

of an upper-triangular matrix.

Proof. This follows from the energy estimates of Section 3.3. □

This decomposition allows us to prove Theorem 1.3.

Proof of Theorem 1.3. From Equations (33) and (40), the decomposition

$$\mathrm{Cth}(\Sigma_0,\Sigma_1) = C(\Lambda_0^+,\Lambda_1^+) \oplus CF_+(\Sigma_0,\Sigma_1) \oplus C(\Lambda_0^-,\Lambda_1^-) \oplus CF_-(\Sigma_0,\Sigma_1)$$

induces a filtration for the differential $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}$. The first page thus takes the prescribed form by construction. The fact that $(\mathrm{Cth}(\Sigma_0,\Sigma_1),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-})$ is acyclic implies that the associated spectral sequence collapses on the fourth page. □

Note that if either all intersection points have positive actions or they all have negative actions, then the differential described in Formula (40) is a mapping cone from or to $C(\Lambda_0^-,\Lambda_1^-)$. This motivates the following definition for a pair (Σ_0,Σ_1) of cobordisms.

Definition 9.2. We say that (Σ_0,Σ_1) is *directed* if $CF_+(\Sigma_0,\Sigma_1)=0$, while it is called *V-shaped* if $CF_-(\Sigma_0,\Sigma_1)=0$.

We are now able to prove Corollary 1.5.

Proof of Corollary 1.5. For directed cobordisms, from Equation (40) we get that the differential on $CF_{-\infty}(\Sigma_0, \Sigma_1)$ is the cone of the map $d_{-0} : CF(\Sigma_0, \Sigma_1) \to C(\Lambda_0^-, \Lambda_1^-)$. Thus we get a long exact sequence

$$\begin{array}{ccccccc} \cdots \to & HF^k_{-\infty}(\Sigma_0, \Sigma_1) & & & & & \\ & \downarrow & & & & & \\ & HF^k_-(\Sigma_0, \Sigma_1) & \to & LCH^k_{\varepsilon_0^-, \varepsilon_1^-}(\Lambda_0^-, \Lambda_1^-) & \to & HF^{k+1}_{-\infty}(\Sigma_0, \Sigma_1) & \to \cdots \end{array}$$

For the case of V-shaped cobordisms we get that the differential on $CF_{-\infty}(\Sigma_0, \Sigma_1)$ is the cone of the map $d_{0-} : C(\Lambda_0^-, \Lambda_1^-) \to CF(\Sigma_0, \Sigma_1)$ which leads to the following exact sequence

$$\begin{array}{ccccccc} \cdots \to & HF^{k+1}_{-\infty}(\Sigma_0, \Sigma_1) & \to & LCH^k_{\varepsilon_0^-, \varepsilon_1^-}(\Lambda_0^-, \Lambda_1^-) & \to & HF^{k+2}_+(\Sigma_0, \Sigma_1) & \\ & & & & & \downarrow & \\ & & & & & HF^{k+2}_{-\infty}(\Sigma_0, \Sigma_1) & \to \cdots \end{array}$$

In both cases the isomorphism $LCH^k_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \simeq HF^{k+1}_{-\infty}(\Sigma_0, \Sigma_1)$ of equation (39) concludes the proof. □

10. Long exact sequences induced by an exact Lagrangian cobordism

Here we establish the long exact sequences described in Section 1.2 associated to a Lagrangian cobordism Σ. In order to do this, we need to make use of the invariance result proven in the former section. More precisely, the sought long exact sequence will be induced by Corollary 1.5 applied to a pair (Σ_0, Σ_1) of exact Lagrangian cobordisms, where $\Sigma_0 := \Sigma$ and Σ_1 is obtained from Σ by a suitable Hamiltonian perturbation.

10.1. **Various push-offs of an exact Lagrangian cobordism.** In the following, we assume that we are given an exact Lagrangian cobordism $\Sigma \subset \mathbb{R} \times P \times \mathbb{R}$ from Λ_- to Λ_+ inside the symplectisation of a contactisation. We furthermore assume that Σ is cylindrical outside of the set $[A, B] \times P \times \mathbb{R}$ for some $A < B$. We shall write

$$\begin{aligned} \overline{\Sigma} &:= \Sigma \cap \{t \in [A, B]\}, \\ \partial_- \overline{\Sigma} &:= \Sigma \cap \{t = A\}, \\ \partial_+ \overline{\Sigma} &:= \Sigma \cap \{t = B\}, \end{aligned}$$

so that clearly $\partial \overline{\Sigma} = \partial_- \overline{\Sigma} \cup \partial_+ \overline{\Sigma}$.

We will consider different push-offs constructed via autonomous Hamiltonians $h \colon \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R}$ induced by a function $h(t)$ only depending on the symplectisation coordinate. We write

$$\Sigma^h := \phi_h^1(\Sigma)$$

and observe that the Hamiltonian flow takes the particularly simple form

$$\phi_h^s(t, p, z) = (t, p, se^{-t}h'(t) + z).$$

In particular, the one-form $e^t\alpha$ pulls back to $e^t\alpha + (h'' - h')dt$ under ϕ_h^1. So, if Σ_0 is an exact Lagrangian cobordism with a primitive $f_0 \colon \Sigma_0 \to \mathbb{R}$ of $\Sigma_0^* e^t\alpha$, then the

primitive of $(\phi_h^1(\Sigma_0))^*(e^t\alpha)$ is given by

$$\widetilde{f}_0(q) = f_0(q) + (h' - h)(a(\Sigma_0(q))), \tag{41}$$

where a is the canonical projection to $\mathbb{R}_t$, and $\Sigma_0 \colon \Sigma_0 \hookrightarrow \mathbb{R} \times P \times \mathbb{R}$ denotes the inclusion. Assuming that h coincides with λe^t near $-\infty$ for a constant $\lambda \in \mathbb{R}$ (i.e. $h' = h$), the primitive f_0 vanishes at $-\infty$ if and only if $\widetilde{f}_0$ vanishes there.

We are now ready to define the Hamiltonians $h \colon \Sigma \to \mathbb{R}$ needed for the different perturbations needed. The long exact sequences of Theorems 1.6, 1.7 and 1.8 are then derived by combining Corollary 1.5 applied to (Σ, Σ^h) with Theorems 10.3 and 10.5 proven below.

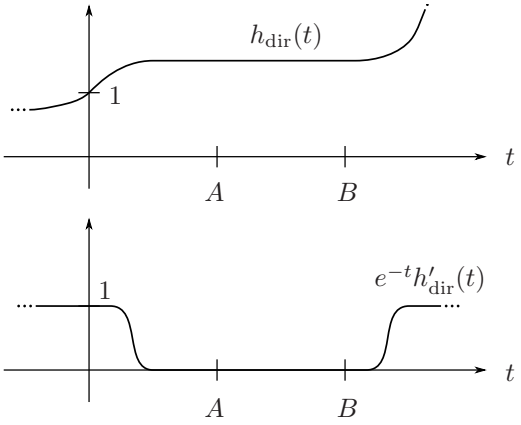

FIGURE 18. The Hamiltonian $h_{\mathrm{dir}} \colon \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R}$ applied to an exact Lagrangian cobordism Σ produces a directed pair. The corresponding Hamiltonian vector field is given by $e^{-t}h'_{\mathrm{dir}}(t)\partial_z$.

10.1.1. *The push-off inducing the long exact sequence of a pair.* Consider the Hamiltonian $h_{\mathrm{dir}} \colon \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R}$ depending only on the t-coordinate and satisfying

- $h_{\mathrm{dir}}(t) = e^t$ for $t \leq 0$;
- $h_{\mathrm{dir}}(t) = e^t - C$ for all $t \geq B + 1$, and some $C \geq 0$; and
- $(h'_{\mathrm{dir}})^{-1}(0)$ is a connected interval containing $[A, B]$ (in particular h_{dir} is constant on $[A, B]$.

See Figure 18 for a schematic picture of $h_{\mathrm{dir}}(t)$ and the corresponding Hamiltonian vector field $e^{-t}h'_{\mathrm{dir}}(t)\partial_z$.

Let $\Sigma' := \Sigma^{\epsilon h_{\mathrm{dir}}}$, for $\epsilon > 0$ sufficiently small, and take Σ_1 to be a generic sufficiently small and compactly supported Hamiltonian perturbation of Σ'. Let f_0 be the primitive of $\Sigma^*(e^t\alpha)$ which vanishes at $-\infty$. From Equation (41) and from the fact that $h > 0$ and $h' = 0$ on $[A, B]$ we get that the primitive of $(\Sigma')^*(e^t\alpha)$ is smaller than f_0 on $[A, B] \times Y$. Writing $\Sigma_0 := \Sigma$, for $\epsilon > 0$ sufficiently small, all intersections between Σ_0 and Σ_1 are contained inside $[A, B] \times Y$. Thus, if the Hamiltonian perturbation of Σ' is small enough, we get that $CF_+(\Sigma_0, \Sigma_1) = 0$ or, using the terminology of Definition 9.2, that the pair of cobordisms (Σ_0, Σ_1) is directed.

Furthermore, under the assumption that Σ_1 is a sufficiently C^1-small perturbation of Σ_0, and that the almost complex structures are chosen appropriately,

Theorem 1.6 now follows by applying Theorems 10.3 and 10.5 to the terms in the long exact sequence (2) of Corollary 1.5 (also see Remark 10.6).

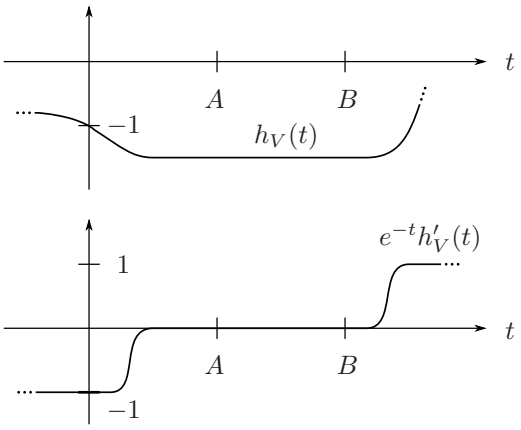

FIGURE 19. The Hamiltonian $h_V : \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R}$ applied to an exact Lagrangian cobordism Σ produces a V-shaped pair. The corresponding Hamiltonian vector field is given by $e^{-t}h'_V(t)\partial_z$.

10.1.2. *The push-off inducing the duality long exact sequence.* Consider a Hamiltonian $h_V : \mathbb{R} \times P \times \mathbb{R}$ depending only on the t-coordinate and satisfying

- $h_V(t) = -e^t$ for $t \leq A - 1$;
- $h_V(t) = e^t - C$ for all $t \geq B + 1$, and some $C > 0$; and
- $(h'_V)^{-1}(0)$; is a connected interval containing $[A, B]$.

See Figure 18 for a schematic picture of $h_V(t)$ as well as of the corresponding Hamiltonian vector field $e^{-t}h'_V(t)\partial_z$.

Let $\Sigma' := \Sigma^{\epsilon h_V}$, for $\epsilon > 0$ sufficiently small, and let Σ_1 be a generic, sufficiently C^1-small, and compactly supported Hamiltonian perturbation of Σ'. Again, from Equation (41), we deduce that the primitive of $e^t\alpha$ on Σ' is equal to $f_0 + (h'_V - h_V) \circ a \circ \Sigma$, which is this time greater than f_0 on $[A, B]$. In conclusion, $CF_-(\Sigma_0, \Sigma_1) = 0$ or, using the terminology of Definition 9.2, (Σ_0, Σ_1) is a V-shaped pair.

When Σ_1 is a sufficiently small perturbation of Σ_0, and the almost complex structures are chosen appropriately, Theorem 1.7 follows by applying Proposition 10.1 and Theorems 10.3 to the terms in the long exact sequence (3) of Corollary 1.5 (also see Remark 10.6).

10.1.3. *The push-off inducing the Mayer-Vietoris long exact sequence.* The Hamiltonian h_V considered above also gives rise to the long exact sequence in Theorem 1.8. It is simply a matter of applying Theorem 1.3 to a *different* filtration of the same complex.

Consider the filtration

$$C(\Lambda_0^+, \Lambda_1^+) > C_+(\Lambda_0^-, \Lambda_1^-) \oplus CF_+(\Sigma_0, \Sigma_1) > C_0(\Lambda_0^-, \Lambda_1^-)$$

where the decomposition

$$C(\Lambda_0^-, \Lambda_1^-) = C_+(\Lambda_0^-, \Lambda_1^-) \oplus C_0(\Lambda_0^-, \Lambda_1^-)$$

has been made so that the left summand is generated by those Reeb chords corresponding to Reeb chords on Λ^- (being of length bounded from below by the minimal length of a Reeb chord on Λ^-, i.e. of some significant length), while the right summand is generated by those Reeb chords corresponding to the critical points of a Morse function on Λ^- (being of length roughly equal to $\epsilon > 0$, which thus may be assumed to be arbitrarily small). To see this, observe that we have $\Lambda_0^- = \Lambda^-$, and that Λ_1^- is obtained as the one-jet $j^1 g$ of a negative Morse function $g\colon \Lambda^- \to (-\epsilon, 0)$ in some standard contact neighbourhood of Λ^-, where the latter is identified with a neighbourhood of the zero-section in $J^1\Lambda^-$. For simplicity, we here make the assumption that there is a unique local minimum and maximum of g.

The above filtration induces a spectral sequence whose first page is given by

$$\begin{gathered} E_1^{2,\bullet} \oplus E_1^{1,\bullet} \oplus E_1^{0,\bullet} \\ \| \\ LCH^{\bullet-2}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+) \oplus (LCH^{\bullet-1}_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \oplus HF^+_\bullet(\Sigma_0, \Sigma_1)) \oplus H_\bullet(\Lambda^-), \end{gathered}$$

where we rely on [35] together with Proposition 10.1 in order to make the identifications

$$\begin{aligned} H^i(C^\bullet_+(\Lambda_0^-, \Lambda_1^-)) &= LCH^i_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-), \\ H^i(C^\bullet_-(\Lambda_0^-, \Lambda_1^-)) &= H^{\mathrm{Morse}}_{n-1-i}(g). \end{aligned}$$

Note that the middle term on the first page of the above spectral sequence really is a direct sum. This follows from the fact that the positive intersection points may be assumed to have arbitrarily small action and hence, in particular, action smaller than the length of any Reeb chord generator of $C_+(\Lambda_0^-, \Lambda_1^-)$. Consequently, there are no pseudoholomorphic strips as in the definition of the differential with input being a Reeb chord in $C_+(\Lambda_0^-, \Lambda_1^-)$ and output being an intersection point contained in $CF_+(\Sigma_0, \Sigma_1)$, or vice versa.

The technique in the proof of [25, Theorem 6.2(ii)] shows that the Cthulhu differential restricts to the natural map

$$H_{n-1-i}(\Lambda^-) \simeq H^i(C^\bullet_-(\Lambda_0^-, \Lambda_1^-)) \to H_{n-1-i}(\Sigma)$$

in homology induced by the topological inclusion $\Lambda^- \hookrightarrow \Sigma$. To that end, we start by considering a perturbation $\widetilde{h}_V(t)$ of $h_V(t)$ being of the form as shown in Figure 20. Note that any sufficiently C^1-small perturbation $f\colon \Sigma \to \mathbb{R}$ of the restriction $\widetilde{h}_V|_\Sigma$ has the property that the map $H_\bullet(\Lambda^-) \to H_\bullet(\Sigma)$ can be realised as the inclusion of the Morse homology complex generated by the critical points near $\{t = a\}$ into the full Morse homology complex of f. Also, see Section 11.3.2 for a similar analysis. It then suffices to show that the differential is equal to the identity map

$$H_{n-1-i}(\Lambda^-) \simeq H^i(C^\bullet_-(\Lambda_0^-, \Lambda_1^-)) \to H_{n-1-i}(\Lambda^-)$$

under the appropriate natural identifications.

Finally, the statements concerning the fundamental classes is a consequence of [35, Theorem 5.5], a result which is only valid if we are working with $\varepsilon_0^- = \varepsilon_1^- = \varepsilon$. Namely, the latter result shows that the minimum of $-g$ defines a nonvanishing cycle inside $LCH^{-1}_{\varepsilon,\varepsilon}(\Lambda_0^-, \Lambda_1^-)$. Similarly, under the additional assumption that Λ^-

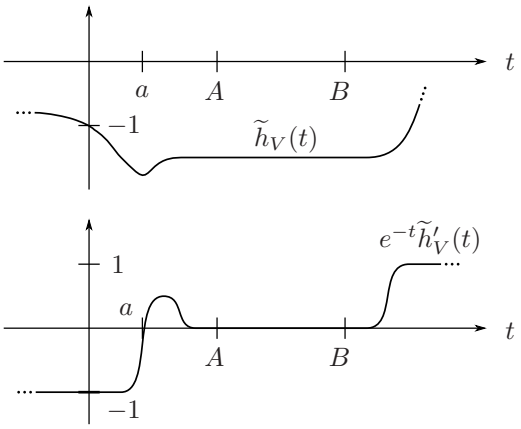

FIGURE 20. The Hamiltonian $\widetilde{h}_V : \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R}$ applied to an exact Lagrangian cobordism Σ produces a V-shaped pair. The corresponding Hamiltonian vector field is given by $e^{-t}\widetilde{h}'_V(t)\partial_z$.

is *horizontally displaceable*, the differential of the maximum of $-g$ is a non-zero class in

$$H^n(C^\bullet_+(\Lambda_0^-, \Lambda_1^-)) \simeq LCH^n_{\varepsilon,\varepsilon}(\Lambda^-)$$

called the *fundamental class in Legendrian contact cohomology* (also see Section 11.3).

With the above considerations, we have now managed to establish Theorem 1.8.

10.2. **Computing the Floer homology of a Hamiltonian push-off.** We now consider the pair (Σ_0, Σ_1), where $\Sigma_0 = \Sigma$ and Σ_1 is equal to Σ^h for a C^1-small smooth function $h \colon \mathbb{R} \to \mathbb{R}$ as constructed above.

To that end, we will assume that $\widetilde{J}_P$ is the cylindrical lift of a regular almost complex structure J_P on P, i.e. the unique cylindrical almost complex structure on $\mathbb{R} \times P \times \mathbb{R}$ for which the canonical projection to P is $(\widetilde{J}_P, J_P)$-holomorphic.

Recall that the time-s Hamiltonian flow generated by the Hamiltonian e^t on the symplectisation $\mathbb{R} \times P \times \mathbb{R}$ simply is a translation of the z-coordinate by s. Recall that this is the same as the time-s flow $\phi^s \colon P \times \mathbb{R}$ of the Reeb vector field induced by the standard contact form.

Proposition 10.1. *For the cylindrical lift $\widetilde{J}_P$ of an almost complex structure, we have a canonical isomorphism*

$$LCC^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda, \phi^\epsilon(\Lambda')) \simeq LCC^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda)$$

of complexes for any sufficiently small epsilon > 0, and generic and sufficiently C^1-small Legendrian perturbation Λ' of Λ.

Under the additional assumption that Λ is horizontally displaceable, we moreover have a quasi-isomorphism

$$LCC^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda, \phi^{-\epsilon}(\Lambda')) \sim LCC^{\varepsilon_0,\varepsilon_1}_{n-1-\bullet}(\Lambda)$$

of complexes, where n is the dimension of Λ.

Remark 10.2. *The above identifications of augmentations, despite the fact that the DGAs are associated to geometrically* different *Legendrians, can be justified as follows. For $\Lambda, \Lambda'' \subset P \times \mathbb{R}$ being sufficiently C^1-close together with a fixed choice of compatible almost complex structure J_P on P, the invariance theorem in* [34] *gives a canonical isomorphism between the Chekanov-Eliashberg algebras $(\mathcal{A}(\Lambda), \partial_\Lambda)$ and $(\mathcal{A}(\Lambda''), \partial_{\Lambda''})$ induced by the canonical bijection identifying the Reeb chords on Λ with the Reeb chords on Λ''.*

Proof. The first isomorphism is simply [17, Proposition 2.7].

The second isomorphism follows by combining this result with the isomorphism

$$LCC^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda, \phi^{-\epsilon}(\Lambda')) \simeq LCC^{\varepsilon_0,\varepsilon_1}_{n-1-\bullet}(\Lambda)$$

established in [35, Proposition 4.1]. Here we must use the assumption of horizontal displaceability. □

Consider an admissible almost complex structure J on $\mathbb{R} \times P \times \mathbb{R}$ which coincides with the cylindrical lifts $\widetilde{J}^+_P$ and $\widetilde{J}^-_P$ of almost complex structures on P in subsets of the form

$$(-\infty, -T] \times P \times \mathbb{R},$$
$$[T, +\infty) \times P \times \mathbb{R},$$

respectively. In the following we will also assume that $h \colon \mathbb{R} \times P \times \mathbb{R}$ is an autonomous Hamiltonian being of the form $\pm e^t + C$ in each of the two latter cylindrical subsets.

Theorem 10.3. *Let Σ be an $(n+1)$-dimensional exact Lagrangian cobordisms from Λ^- to Λ. Assume that there are augmentations ε_i, $i = 0, 1$, of the Chekanov-Eliashberg algebra of Λ^- defined using $\widetilde{J}^-_P$. For $\epsilon > 0$ sufficiently small, and Σ' being a sufficiently C^1-small compactly supported perturbation of Σ, we may assume that J is a regular admissible almost complex structure for which:*

(1) There is a natural isomorphism

$$LCC^\bullet_{\varepsilon_0 \circ \Phi_\Sigma, \varepsilon_1 \circ \Phi_{\phi^\epsilon_h(\Sigma')}}(\Lambda, \phi^\epsilon(\Lambda)) = LCC^\bullet_{\varepsilon_0 \circ \Phi_\Sigma, \varepsilon_1 \circ \Phi_\Sigma}(\Lambda)$$

of complexes;

(2) There is an equality

$$\Phi_\Sigma^{\varepsilon_0^-, \varepsilon_1^-} = d_{+-}$$

of chain maps, where the former is the map induced by the linearised DGA morphism Φ_Σ, as described in Section 5.3, and the latter is the corresponding component of the differential $\mathfrak{d}_{\varepsilon_0^- \varepsilon_1^-}$ on $\mathrm{Cth}_\bullet(\Sigma, \phi^\epsilon_h(\Sigma'))$.

Remark 10.4. *All Legendrian contact homology complexes above are induced by $\widetilde{J}^\pm_P$, while the DGA morphisms, as well as d_{+-}, are induced by J.*

Proof. Both results follow from Proposition 10.1 together with the bijections provided by [17, Theorem 2.15], where the latter provides the necessary identifications of pseudoholomorphic strips on the cobordism Σ and its two-copy $\Sigma \cup \phi^\epsilon_h(\Sigma')$ that are used in the definitions of the DGA morphisms induced by the respective cobordisms. To that end, an admissible almost complex structure as above must be used, i.e. which coincides with cylindrical lifts in the prescribed subsets. □

Theorem 10.5. *For an appropriately chosen Maslov potential and for any generic smooth function $f\colon \Sigma \to \mathbb{R}$ obtained as a sufficiently C^1-small and compactly supported perturbation of $h|_{\Sigma'}\colon \Sigma \to \mathbb{R}$, there is an induced Hamiltonian perturbation Σ' of Σ inducing a canonical identification of complexes*

$$CF_\bullet(\Sigma, \phi_h^\epsilon(\Sigma')) = C^{\mathrm{Morse}}_{n+1-\bullet}(f).$$

Here we require that the almost complex structure is regular, admissible, and induced by a Riemannian metric on g for which (f, g) is a Morse-Smale pair in some compact neighbourhood of the non-cylindrical part of Σ. If Σ is pin, then the Hamiltonian perturbation admits an pin structure for which the above identification holds with coefficients in $\mathbb{Z}$.

Proof. Observe that the Weinstein Lagrangian neighbourhood theorem implies that there is a symplectomorphism identifying a neighbourhood of Σ with a neighbourhood of $T^*\Sigma$, such that Σ moreover gets identified with the zero-section; see Section 11.3.2. For an appropriate such identification, we may assume that $\phi_h^\epsilon(\Sigma) = \phi^1_{\epsilon h}(\Sigma)$ is contained in this neighbourhood and that this Lagrangian is identified with the section $-\epsilon d(h|_\Sigma)$. In particular, every intersection point $\phi^1_{\epsilon h}(\Sigma) \cap \Sigma$ corresponds to a critical point of ϵh and may be assumed to have action corresponding to ϵh. After replacing ϵh with ϵf, we can choose Σ' to be the exact Lagrangian cobordism given as the graph of ϵdf, while using the above Weinstein neighbourhood.

The isomorphism of Morse and Floer complexes is standard, going back to the original work made by Floer [46]. For the current setting, the analogous computation made in [25, Theorem 6.2] is also relevant. □

Remark 10.6. *Consider the autonomous Hamiltonian h in Theorem 10.5 above, together with the induced perturbation f of $h|_\Sigma$.*

(1) If $h = h_{\mathrm{dir}}$ as defined in Section 10.1.1, then

$$H(C^{\mathrm{Morse}}_{n+1-\bullet}(f)) = H_{n+1-\bullet}(\overline{\Sigma}, \partial_- \overline{\Sigma}; R) \simeq H^\bullet(\Sigma, \partial_+ \overline{\Sigma}; R).$$

(2) If $h = h_V$ as defined in 10.1.2, then

$$H(C^{\mathrm{Morse}}_{n+1-\bullet}(f)) = H_{n+1-\bullet}(\overline{\Sigma}; R).$$

10.3. **Seidel's isomorphism.** We end this section by recalling the definition of Seidel's isomorphism. Let $h\colon \mathbb{R} \times P \times \mathbb{R}$ be an autonomous Hamiltonian coinciding with $e^t + C_+$ on the positive end, and with $-e^t$ on the negative end; we can take e.g. the Hamiltonian h_V constructed in Section 10.1.2. Combining Theorem 10.5 and Proposition 10.1 we obtain a module morphism

$$G_\Sigma^{\varepsilon_0^-, \varepsilon_1^-} : C^{\mathrm{Morse}}_{n+1-\bullet}(f) \to LCC^{\bullet-1}_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+, \phi^\epsilon(\Lambda^+); R) = LCC^{\bullet-1}_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+; R),$$

identified with the term d_{+0} in the differential of $(\mathrm{Cth}(\Sigma, \phi_f^\epsilon(\Sigma)), \mathfrak{d}_{\varepsilon_0^-, \varepsilon_1^-})$. Observe that, since the pair of cobordisms is V-shaped, we can conclude that

Lemma 10.7. *The above map $G_\Sigma^{\varepsilon_0^-, \varepsilon_1^-}$ is a chain map which, in the case when the negative end of Σ is empty, is a quasi-isomorphism.*

11. Twisted coefficients, L^2-completions, and applications

In order to deduce information about the fundamental group of an exact Lagrangian cobordism it is necessary to introduce a version of our Floer complex coefficients twisted by the fundamental group, analogous to that defined for Lagrangian Floer homology in [67] by Sullivan and in [22] by Damian. Since it is not possible to make sense of the rank of a general module with group ring coefficients, it will also be necessary to introduce a version of an L^2-completion of this complex. So-called L^2-coefficients were first considered by Atiyah in [8]. We start by describing the version of the complex with twisted coefficients, and we also introduce a version of the fundamental class in this setting. The fundamental class will be crucial for the proof of Theorems 1.16 (see Section 12.3.1). We then continue by defining the L^2-completion of this complex, for which we recall some basic properties. The proof of Theorem 1.18 will use this theory (see Section 12.3.2).

11.1. **Floer homology with twisted coefficients.** First, we introduce the algebraic setup needed in order to define the Chekanov-Eliashberg algebra, along with its linearisations, in the setting of twisted coefficients. We refer the reader to [18] by the authors for a more detailed treatment.

11.1.1. *The tensor ring of a bimodule.* First we recall a few classical and general algebraic constructions. Let A be a (not necessarily commutative) unital ring. We start with the definition of tensor product of $A-A$-bimodules. Given two $A-A$-bimodules M and N , their (balanced) tensor product $M \otimes_A N$ is the quotient of the abelian group $M \otimes_{\mathbb{Z}} N$ by the relation

$$\forall m \in M,\ n \in N,\ a \in A: \quad ma \otimes n = m \otimes an.$$

Observe that $M \otimes_A N$ has the structure of a $A-A$-bimodule, where left and right multiplications are given by the formula $a(m \otimes n)a' = (am) \otimes (na')$ with $m \in M$, $n \in N$, and $a, a' \in A$. In order to avoid confusion, we would like to emphasise that in the bimodule $M \otimes_A N$ elements of the form $m \otimes n$ and $m \otimes an$ are in general not in relation to each other.

Given a $A-A$-bimodule M, *the tensor ring of* M is the graded ring

$$T_A(M) = \bigoplus_{k=0}^{\infty} M^{\otimes_A k},$$

where $M^{\otimes_A 0} = A$, and $M^{\otimes_A k} = M \otimes_A M^{\otimes_A k-1}$. Multiplication in $T_A(M)$ is induced by the natural isomorphism from $M^{\otimes_A l} \otimes_A M^{\otimes_A m}$ to $M^{\otimes_A l+m}$. Observe that $T_A(M)$ contains a subring $M^{\otimes_A 0} = A$ as well as a $A-A$-bimodule $M^{\otimes_A 1} = M$, and that $T_A(M)$ is the universal ring satisfying these properties. (Note that, as A is not necessarily commutative, $T_A(M)$ is not an algebra according to standard terminology. This is the reason why we refer to it as the *tensor ring*).

If M is freely generated by elements $\{\gamma_1, \dots, \gamma_k\}$ as a bimodule, homogeneous elements of $T_A(M)$ are generated by elements of the form

$$a_1\gamma_{i_1} \otimes a_2\gamma_{i_2} \otimes \cdots \otimes a_j\gamma_{i_j}a_{j+1},$$

where $a_1, \dots, a_{j+1} \in A$. In most cases A will be the group ring $R[\pi]$ over a commutative ring R. In this situation, the tensor ring of M over $A = R[\pi]$ will be denoted by $T_\pi(M)$ for simplicity.

11.1.2. *The Chekanov-Eliashberg algebra with twisted coefficients.* Legendrian contact homology with twisted coefficients has previously been considered in [37], and a detailed account is currently under development in [42]. Here we consider a version of the Chekanov-Eliashberg algebra for a Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ with twisted coefficients, constructed as tensor ring over the group ring $R[\pi_1(\Lambda)]$, as defined above, where R is a commutative ring.

Fix a base point $* \in \Lambda$ and write $\pi_1(\Lambda) := \pi_1(\Lambda, *)$ for short. Let A be a unital, not necessarily commutative, ring for which:

- There is a ring homomorphism $i : R[\pi_1(\Lambda)] \to A$. This induces an $R[\pi_1(\Lambda)]-R[\pi_1(\Lambda)]$-bimodule structure on A;
- There is an augmentation homomorphism $a : A \to R$ such that $\Pi := a \circ i$ is the standard augmentation $\Pi : R[\pi_1(\Lambda)] \to R$.

By abuse of notation we will see any element a in $R[\pi_1(\Lambda)]$ as an element of A by identifying it with its image under the ring homomorphism i. For example, take any group homomorphism $\pi_1(\Lambda) \to G$, this induces a ring homomorphism $R[\pi_1(\Lambda)] \to R[G]$ and the augmentation corresponds to the standard ring homomorphism $R[G] \to R$. When $G = \{1\}$, the construction we describe below will recover the standard Chekanov-Eliashberg DGA.

For any Reeb chord γ of Λ, we fix a *capping path* ℓ^e_γ (resp. ℓ^s_γ) on Λ which connects the end point (resp. starting point) of γ to the base point $*$. (Such paths exist because we assume that Λ is connected.) Let $C(\Lambda)$ be the free $A-A$-bimodule generated by the Reeb chords of Λ. A punctured pseudoholomorphic disc $u \in \mathcal{M}^r(\gamma^+; \gamma_1^-, \gamma_2^-, \dots, \gamma_k^-)$ determines an element $\mathbf{c}_u$ of $C(\Lambda)^{\otimes_A k}$ via the following procedure. Let $\partial_0 S_r, \dots, \partial_k S_r$ be the connected components of ∂S_r ordered as in Section 2.3.1. We denote by p the canonical projection $\mathbb{R} \times P \times \mathbb{R} \to P \times \mathbb{R}$ from the symplectisation to the contact manifold.

- For $j \in \{1, \dots, k-1\}$, we denote by a_j the based loop $(\ell^e_{\gamma_{j+1}^-})^{-1} * (p \circ u|_{\partial_j}) * \ell^s_{\gamma_j^-}$;
- For $j = 0$, we denote by a_j the based loop $\ell^e_{\gamma_1^-} * (p \circ u|_{\partial_0}) * (\ell^e_{\gamma^+})^{-1}$; and
- For $j = k$, we denote by a_j the based loop $\ell^s_{\gamma^+} * (p \circ u|_{\partial_k}) * (\ell^s_{\gamma_k^-})^{-1}$.

The element $\mathbf{c}_u$ is then given by

$$\mathbf{c}_u = a_0 \gamma_1^- a_1 \otimes \gamma_2^- a_2 \otimes \cdots \otimes \gamma_k^- a_k.$$

The Chekanov-Eliashberg differential is defined on $T_A(C(\Lambda))$ by the formula

$$\partial(\gamma^+) = \sum_{\gamma_1, \dots, \gamma_k} \sum_{u \in \mathcal{M}(\gamma^+; \gamma_1^-, \gamma_2^-, \dots, \gamma_k^-)} \operatorname{sign}(u) \mathbf{c}_u$$

on generators, where the sum is taken over the rigid components of the moduli spaces. The differential is then extended as a bimodule homomorphism to $C(\Lambda)$, and ultimately to the whole tensor ring using the Leibniz rule. The DGA obtained will be denoted by $\mathcal{A}_A(\Lambda)$ (or $\mathcal{A}_G(\Lambda)$ if $A = R[G]$). If in this notation we omit the subscript A, then we just mean the standard Chekanov-Eliashberg algebra of Λ, which is the particular case $\mathcal{A}(\Lambda) := \mathcal{A}_R(\Lambda)$.

By an *augmentation* of the Chekanov-Eliashberg DGA we mean a homomorphism of $R[\pi_1(\Lambda)] - R[\pi_1(\Lambda)]$-bimodules $\varepsilon : T_A(C(\Lambda)) \to A$ being a unital ring homomorphism satisfying $\varepsilon \circ \partial = 0$.

Remark 11.1. *(1) An augmentation in this setting is still determined by its values on the Reeb chord generators.*

(2) Any homomorphism $G \to H$ of groups induces a unital DGA morphism $r\colon \mathcal{A}_G(\Lambda) \to \mathcal{A}_H(\Lambda)$. In particular, when H is the trivial group, we get a canonical DGA homomorphism $r\colon \mathcal{A}_G(\Lambda) \to \mathcal{A}(\Lambda)$. The pre-composition $\widetilde{\varepsilon} := \varepsilon \circ r$ of an augmentation ε of $\mathcal{A}(\Lambda)$ is clearly an augmentation of $\mathcal{A}_G(\Lambda)$; this augmentation will be called the lift of ε.

(3) Similarly to the definition of the differential of the DGA with twisted coefficients, an exact Lagrangian cobordism Σ from Λ^- to Λ^+ can be seen to induce a unital DGA homomorphism $\widetilde{\Phi}_\Sigma\colon \mathcal{A}_{\pi_1(\Sigma)}(\Lambda^+) \to \mathcal{A}_{\pi_1(\Sigma)}(\Lambda^-)$ with twisted coefficients. In particular, an exact Lagrangian filling induces an augmentation in the group ring of its fundamental group.

For any pair of augmentations, the linearisation procedure gives rise to a differential $d_{\varepsilon_0,\varepsilon_1} : LCC_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda; A) \to LCC_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda; A)$ on the free $A-A$-bimodule spanned by the Reeb chords in the usual way. The map $d_{\varepsilon_0,\varepsilon_1}$ is in this situation a bimodule homomorphism. We again denote the resulting homology by $LCH_\bullet^{\varepsilon_0,\varepsilon_1}(\Lambda; A)$ called the *bilinearised Legendrian contact homology with twisted coefficients*.

In the case when $\Lambda = \Lambda_0 \sqcup \Lambda_1$ and ε_i comes from an augmentation of Λ_i for $i = 0, 1$, we can again define the sub-complex $LCC^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda_0, \Lambda_1; A)$ which is the free right A-module spanned by the Reeb chords starting on Λ_1 and ending on Λ_0, as well as the corresponding cohomology groups $LCH^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda_0, \Lambda_1; A)$. The result [17, Proposition 2.7] carries over immediately to this setting, and thus the identification

$$LCH^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda, \Lambda'; A) = LCH^\bullet_{\varepsilon_0,\varepsilon_1}(\Lambda; A)$$

holds on the level of homology (again, for a suitable small push-off Λ' of Λ, together with a suitable lifted almost complex structure).

Remark 11.2. *Note that, if ε_1 takes values in R, then the left action of $\varepsilon_1(\gamma)$ on the free module is the same as the right action. In this case, the whole module structure factors through a complex defined as a free left $R[\pi_1(\Sigma_0)]$-module generated by the Reeb chords. This is relevant for the next section, where we will twist coefficients using $R[\pi_1(\Sigma_0)]$ in a way so that the augmentation ε_1 still will take values in R.*

11.1.3. *The Floer complex with twisted coefficients.* Let R be a unital commutative ring. We are now ready to define our Floer complex for a pair (Σ_0, Σ_1) of exact Lagrangian cobordisms with twisted coefficients, defined as an $R[\pi_1(\Sigma_0)] - R[\pi_1(\Sigma_0)]$ bimodule. For Lagrangian intersection Floer homology, such a construction has previously been carried out in [67] and [22]. This was subsequently generalised to the case of Wrapped Floer homology in [17, Section 4.2], i.e. for a pair of Lagrangian cobordisms having empty negative ends. Here we will define this theory for a pair of exact Lagrangian cobordisms having non-empty negative ends whose Chekanov-Eliashberg algebras admit augmentations.

As before, we let $\Sigma_i \subset \mathbb{R} \times P \times \mathbb{R}$ be exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, $i = 0, 1$, where both Σ_0 and Λ_0^- are assumed to be connected. We consider the non-free $R[\pi_1(\Lambda_0^\pm)] - R[\pi_1(\Lambda_0^\pm)]$-bimodule $R[\pi_1(\Sigma_0)]$ with structure coming from the ring homomorphism induced by the inclusion maps $\{\pm T\} \times \Lambda_0^\pm \to \Sigma_0$. In order to obtain this bimodule structure, the base point of Σ_0 need to be chosen of the form $(-T, *)$, where $*$ is the based point of Λ_0^-. We then choose, once and for all, a path connecting $(-T, *)$ and $(T, *')$, where $*'$ is the based point of Λ_0^+.

Fix augmentations ε_0^- and ε_1^- of $\mathcal{A}_{\pi_1(\Sigma_0)}(\Lambda_0^-)$ and $\mathcal{A}(\Lambda_1^-)$, respectively (which by definition take values in $R[\pi_1(\Sigma_0)]$ and R, respectively). Let $u \in \mathcal{M}^r(x;\boldsymbol{\zeta},y,\boldsymbol{\delta})$ be a pseudoholomorphic strip involved in the Cthulhu differential $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}$ as defined in Section 3.2. Order the connected components of the boundary of ∂S_r which are mapped to Σ_0 starting with the arc adjacent to the incoming puncture (as in Section 2.3.1), and denote them by $\partial_0 S_r, \dots, \partial_k S_r$. We associate to each of these arcs an element $a_j \in \pi_1(\Sigma_0)$ in the same manner as in the definition of the differential of the Chekanov-Eliashberg algebra with twisted coefficients described above. These paths, together with the above augmentations, now determine an element

$$c_u^{\varepsilon_0^-,\varepsilon_1^-} = a_1\varepsilon_0^-(\zeta_1)a_2\varepsilon_0^-(\zeta_2)\cdots a_{k-1}\varepsilon_0^-(\zeta_{k-1})a_k\varepsilon_1^-(\boldsymbol{\delta}) \in R[\pi_1(\Sigma_0)].$$

This construction allows us to define the Cthulhu differential $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}$ on the non-free $R[\pi_1(\Sigma_0)]$–$R[\pi_1(\Sigma_0)]$-bimodule

$$\mathrm{Cth}(\Sigma_0,\Sigma_1;R[\pi_1(\Sigma_0)]) := \mathrm{Cth}(\Sigma_0,\Sigma_1)\otimes_R R[\pi_1(\Sigma_0)]$$

First, when y is either a intersection point or a Reeb chord from Λ_0^- to Λ_1^-, we define

$$\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}(y) = \sum_x \sum_{u\in\mathcal{M}^r(x;\boldsymbol{\zeta},y,\boldsymbol{\delta})} \mathrm{sign}(u) c_u^{\varepsilon_0^-,\varepsilon_1^-} x,$$

where the sum is taken over the rigid components of the moduli space. The formula for y being a Reeb chord from Λ_1^+ to Λ_0^+ is similar, but involves the pull-backs $\varepsilon_0^-\circ\widetilde{\Phi}_{\Sigma_0}$ and $\varepsilon_1^-\circ\Phi_{\Sigma_1}$ of the augmentations under the DGA homomorphism induced by the cobordisms with and without twisted coefficients, respectively. The differential is then extended to all of $\mathrm{Cth}(\Sigma_0,\Sigma_1;R[\pi_1(\Sigma_0)])$ as a *right* $R[\pi_1(\Sigma_0)]$-module homomorphism.

The techniques in Section 8 can be used to prove the following theorem.

Theorem 11.3. *The map* $\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-} : \mathrm{Cth}(\Sigma_0,\Sigma_1;R[\pi_1(\Sigma_0)]) \to \mathrm{Cth}(\Sigma_0,\Sigma_1;R[\pi_1(\Sigma_0)])$ *satisfies* $\mathfrak{d}^2_{\varepsilon_0^-,\varepsilon_1^-} = 0$*, i.e. it is a differential, and it gives rise to an acyclic complex, i.e.* $H(\mathrm{Cth}(\Sigma_0,\Sigma_1;R[\pi_1(\Sigma_0)]),\mathfrak{d}_{\varepsilon_0^-,\varepsilon_1^-}) = 0$.

Proof. The proof is similar to the one in Sections 6.2 and 8.3. To that end, we observe the following important feature of the above construction of the coefficient $c_u^{\varepsilon_0^-,\varepsilon_1^-}$. Let u and v be holomorphic strips whose outgoing and incoming punctures, respectively, agree so that one can pre-glue them to a strip $u*v$. It then follows that $c_{u*v}^{\varepsilon_0^-,\varepsilon_1^-} = c_v^{\varepsilon_0^-,\varepsilon_1^-} * c_u^{\varepsilon_0^-,\varepsilon_1^-}$. It now follows from the compactness theorem that that the boundary of a component of a one-dimensional moduli space of holomorphic strips either consists of two broken configurations $u*v$ and $u'*v'$, both which contribute with the same coefficient (with opposite signs)

$$c_v^{\varepsilon_0^-,\varepsilon_1^-} * c_u^{\varepsilon_0^-,\varepsilon_1^-} = -c_{v'}^{\varepsilon_0^-,\varepsilon_1^-} * c_{u'}^{\varepsilon_0^-,\varepsilon_1^-},$$

or at least one boundary point corresponds to a broken configuration involving a pure chord. However, when counted with the augmentations as above, the counts of the latter boundary points cancel for the algebraic reason that the augmentation is a chain map (see (II) in Section 6). □

In view of the above theorem, the computations in Section 10 can be carried over immediately to the case of twisted coefficients. We proceed to explicitly describe the long exact sequence analogous to (4) in Theorem 1.6. Let Σ be an exact

Lagrangian cobordism from Λ^- to Λ^+. Let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}_{\pi(\Sigma)}(\Lambda^-)$ and $\mathcal{A}(\Lambda^-)$ into $R[\pi(\Sigma)]$ and R, respectively. Further, we consider the pull-backs $\varepsilon_0^+ := \varepsilon_0^- \circ \widetilde{\Phi}_\Sigma$ and $\varepsilon_1^+ := \varepsilon_0^- \circ \Phi_\Sigma$ of these augmentations.

Remark 11.4. *It is important to note that ε_0^+ need* not *be the lift of an augmentation into R in general, even in the case when ε_0^- is.*

Writing $\widetilde{\Sigma}$ for the universal cover of Σ, and $\overline{\widetilde{\Sigma}}$ for its compactification to a manifold with boundary, there is a long exact sequence:

$$\begin{aligned}(42)\qquad \cdots \to LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+;R[\pi_1(\Sigma)]) \\ \downarrow \\ H_{n+1-k}(\overline{\widetilde{\Sigma}},\partial_-\overline{\widetilde{\Sigma}};R) \to LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-;R[\pi_1(\Sigma)]) \\ \downarrow \\ LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+;R[\pi_1(\Sigma)]) \to \cdots\end{aligned}$$

The identification of the topological term $H_{n+1-k}(\overline{\widetilde{\Sigma}},\partial_-\overline{\widetilde{\Sigma}};R)$ is proven in the same manner as before (see Theorem 10.5), while making the observation that the Morse homology of a manifold with coefficients twisted by its fundamental group computes the homology of its universal cover.

Finally, we point out that

$$LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-;R[\pi_1(\Sigma)]) = LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-)\otimes R[\pi_1(\Sigma)]$$

is satisfied in the case when Λ^- is simply connected.

11.2. **Augmentations in finite-dimensional non-commutative algebras.** The DGAs with "coefficients" in non-commutative unital algebras as treated in Section 11.1 can also be considered in a setting where the algebra is more general than a group ring, and without using twisted coefficient. Here we describe augmentations in non-commutative unital algebras, as done by the second and fourth authors in [27], which can be seen to fit into this framework. Since we will be interested in computing the ranks of involved linearised complexes, we will restrict ourselves to the case when the involved algebra is finite-dimensional over the ground field $\mathbb{F}$.

A finite-dimensional augmentation of the Chekanov-Eliashberg algebra is a unital DGA homomorphism

$$\varepsilon\colon (\mathcal{A}(\Lambda),\partial)\to(A,0),$$

where A is a *not necessarily commutative* unital algebra which is finite-dimensional over the ground field $\mathbb{F}$. Here $\mathbb{F}$ denotes the field that was used as coefficient ring for $\mathcal{A}(\Lambda)$. Recall that the existence of such a (graded) augmentation is equivalent to the existence of a finite-dimensional representation of the so-called *characteristic algebra*, which is defined as the quotient algebra $\mathcal{A}(\Lambda)/\langle\partial(\mathcal{A})\rangle$ by the two-sided ideal generated by the boundaries (see [58]).

Given two such augmentations

$$\varepsilon_i\colon (\mathcal{A}(\Lambda_i^-),\partial_i)\to(A_i,0),\ \ i=0,1,$$

we can form all our complexes as free $A_0\otimes_{\mathbb{F}}(A_1)^{\mathrm{op}}$-modules or, differently put, a free A_0-A_1 bimodule. It is important to note that the dimension over $\mathbb{F}$ still

makes sense, since the latter modules are of dimension $(\dim_{\mathbb{F}} A_0)(\dim_{\mathbb{F}} A_1)$ times the number of generators.

To construct the differentials in this setting one proceeds as in [27], which also is analogous to the construction of the complexes in Section 11.1 with twisted coefficients. In particular, the differentials are of the form

$$d((a_0 \otimes a_1)x) = \sum_y \sum_{\boldsymbol{\delta}^-, \boldsymbol{\zeta}^-} \#\mathcal{M}(y; \boldsymbol{\delta}^-, x, \boldsymbol{\zeta}^-) \cdot \varepsilon_0(\boldsymbol{\delta}^-) a_0 \otimes a_1 \varepsilon_1(\boldsymbol{\zeta}^-) \cdot y,$$

where x and y denote either intersection points or Reeb chords, and where $a_i \in A_i$, $i = 0, 1$.

Remark 11.5. *This convention tells us that the differential is defined by multiplication of $A_0 \otimes A_1^{\mathrm{op}}$ from the* left*, and is hence a morphism of* right *modules.*

The long exact sequences in homology for these bimodules now follow verbatim from the proofs in the case when the augmentation is taken into $\mathbb{F}$. It is important to notice that all the complexes above are of finite dimension over $\mathbb{F}$. More precisely,

$$\dim_{\mathbb{F}} LCC^\bullet(\Lambda_0^\pm, \Lambda_1^\pm) = |\mathcal{R}(\Lambda_1^\pm, \Lambda_0^\pm)| \cdot \dim_{\mathbb{F}} A_0 \cdot \dim_{\mathbb{F}} A_1,$$

while

$$\dim_{\mathbb{F}} H_\bullet(X, Y; A_0 \otimes_{\mathbb{F}} (A_1)^{\mathrm{op}}) = \dim_{\mathbb{F}} H_\bullet(X, Y; \mathbb{F}) \dim_{\mathbb{F}}(A_0) \dim_{\mathbb{F}}(A_1)$$

holds by the universal coefficients theorem.

Example 11.6. There are examples of Legendrian submanifolds which admit augmentations into finite-dimensional non-commutative unital algebras, but which do not admit any augmentation into any commutative unital algebra. We refer to Part (1) of Example 12.4 below for such Legendrian torus knots found by Sivek in [65], which admit augmentations into the matrix algebra $\mathrm{M}_2(\mathbb{Z}_2)$. The second and the fourth author later used these examples in order to construct plenty of Legendrian submanifolds inside contact spaces $(\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ for arbitrary $n \in \mathbb{N}$ whose Chekanov-Eliashberg algebras admit augmentations into $\mathrm{M}_2(\mathbb{Z}_2)$, but not into any commutative algebra.

11.3. **The fundamental class and twisted coefficients.** In this section we will introduce the fundamental class in the setting of twisted coefficients. We will prove that this class coincides with the fundamental class introduced in [35] in the general twisted coefficients setting. We will also prove that this class is functorial under exact Lagrangian cobordisms. In Section 12.3.1 we will use the naturality of this class to prove Theorem 1.16. In the following we let Σ be a connected exact Lagrangian cobordism from Λ^- to Λ^+, where the latter Legendrian submanifolds are connected as well.

11.3.1. *The definition of the fundamental class.* Recall the map

$$G_\Sigma^{\varepsilon_0^-, \varepsilon_1^-} : H_\bullet(\Sigma; R) \to LCH^{n-\bullet}_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+, \Lambda_1^+; R)$$

in homology constructed in Section 10.3, whose underlying chain map is defined by a count of punctured strips with boundary on $\Sigma_1 \cup \Sigma$. Here Σ_1 is an exact Lagrangian cobordism from Λ_1^- to Λ_1^+ obtained by the Hamiltonian push-off of Σ as defined in Section 10.1.2. Roughly speaking, Σ_1 is obtained by a small perturbation using the positive and negative Reeb flows at the positive and negative end of Σ, respectively.

The underlying chain map of $G_\Sigma^{\varepsilon_0^-,\varepsilon_1^-}$ lifts to the corresponding complexes with twisted coefficients. Namely, we define the chain map by the same counts of pseudoholomorphic strips, but where the count takes the homotopy class of the boundary of the strips into account in the manner described above. The lifted map on homology will be denoted by

$$\widetilde{G}_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} : H_\bullet(\Sigma; R[\pi_1(\Sigma)]) \to LCH^{n-\bullet}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+, \Lambda_1^+; R[\pi_1(\Sigma)]).$$

We will be particularly interested in the restriction to the degree 0 part of $H_0(\Sigma; R[\pi_1(\Sigma)])$

$$\widetilde{G}_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} : H_0(\Sigma; R[\pi_1(\Sigma)]) \to LCH^{n}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+, \Lambda_1^+; R[\pi_1(\Sigma)]).$$

Observe that this map is linear over $R[\pi_1(\Sigma)]$ or, put differently, it is $\pi_1(\Sigma)$-equivariant. Also, we recall that $H_\bullet(\Sigma; R[\pi_1(\Sigma)])$ of the twisted complex here computes the homology $H_\bullet(\widetilde{\Sigma}, R)$ of the universal cover $\widetilde{\Sigma} \to \Sigma$, and that

$$LCH^\bullet_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+, \Lambda_1^+; R[\pi_1(\Sigma)]) \simeq LCH^\bullet_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+; R[\pi_1(\Sigma)]). \tag{43}$$

holds by [17, Proposition 2.7]. Later we will be particularly interested in the case when Λ^+ is simply connected and when ε_i^+, $i = 0, 1$, both take values in R. In this situation the universal coefficients theorem gives us an identification

$$LCH^\bullet_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+; R[\pi_1(\Sigma)]) = LCH^\bullet_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+; R) \otimes_R R[\pi_1(\Sigma)]. \tag{44}$$

Choosing a generator $m \in H_0(\Sigma; R[\pi_1(\Sigma)])$, the fundamental class induced by Σ is defined to be the image

$$\widetilde{c}^{\varepsilon_0^-,\varepsilon_1^-}_{\Sigma,m} := \widetilde{G}_\Sigma^{\varepsilon_0^-,\varepsilon_1^-}(m) \in LCH^n_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+; R[\pi_1(\Sigma)]),$$

where we rely on the identification (43) above.

Let Λ_1^+ be obtained from Λ by a C^1-small perturbation of its image under the time-ϵ Reeb flow, where $\epsilon > 0$ is sufficiently small. We moreover assume that Λ_1^+ can be identified with the graph $j^1 f^+ \subset J^1\Lambda^+$ in a standard contact neighbourhood of Λ^+, where $f^+ \colon \Lambda^+ \to \mathbb{R}$ is a Morse function with a unique local minimum $m^+ \in \Lambda^+$. Recall the definition of the fundamental class in [35], which was defined by the following count of pseudoholomorphic strips having boundary on $\Lambda_1^+ \cup \Lambda^+$:

$$\widetilde{c}^{\varepsilon_0^+,\varepsilon_1^+}_{\Lambda^+,m^+} := \sum_{u \in \widetilde{\mathcal{M}}(\gamma,\boldsymbol{\delta},m^+,\boldsymbol{\zeta})} \operatorname{sign}(u) c_u^{\varepsilon_0^+,\varepsilon_1^+} \gamma \in LCH^n_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+; R[\pi_1(\Sigma)]).$$

Here we have again used identification (43) above. In the case when $\varepsilon_0^+ = \varepsilon_1^+$ and Λ^+ is horizontally displaceable, this class has moreover been shown to be non-vanishing; see [35, Theorem 5.5].

The following proposition shows that the two definitions of the fundamental class given above in fact coincide.

Proposition 11.7. *Assume that the natural map*

$$H_0(\Lambda^+; R[\pi_1(\Sigma)]) \to H_0(\Sigma; R[\pi_1(\Sigma)])$$

sends $m^+ \in H_0(\Lambda^+, R[\pi_1(\Sigma)])$ to $m \in H_0(\Sigma; R[\pi_1(\Sigma)])$. For appropriate choices of almost complex structures and Hamiltonian perturbations in the constructions,

there is an identification

$$\widetilde{c}^{\varepsilon_0^+,\varepsilon_1^+}_{\Lambda^+,m^+} = \widetilde{c}^{\varepsilon_0^-,\varepsilon_1^-}_{\Sigma,m} \in LCH^n_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+; R[\pi_1(\Sigma)])$$

of fundamental classes.

In Section 12.3.1 we will see that Theorem 1.16 is a direct consequence of the above proposition. Its proof is postponed until Section 11.3.3, and it relies on studying a specific push-off of Σ to be describe in the subsequent subsection.

11.3.2. *A more careful version of the push-off defined in Section 10.1.2.* In order to facilitate the proof of Proposition 11.7 we will choose a push-off Σ_1 of Σ of a very special form, where Σ_1 will be an exact Lagrangian cobordism from Λ_1^- to Λ_1^+. We will suppose that Σ is cylindrical outside of the subset $[-T,T]\times P\times\mathbb{R}$.

First we need to construct a Weinstein neighbourhood of Σ of a particular form.

Begin by fixing contact-form preserving contactomorphisms $\phi^\pm\colon U^\pm \to J^1\Lambda^\pm$ identifying a neighbourhood $U^\pm \subset P\times\mathbb{R}$ of $\Lambda^\pm$ with a neighbourhood $\phi^\pm(U^\pm)$ of $\Lambda^\pm \subset (J^1\Lambda^\pm, dz-\theta_{\Lambda^\pm})$, where $\phi^\pm(\Lambda^\pm) = 0_{\Lambda^\pm} \subset J^1\Lambda^\pm$ is the zero-section. We also get induced exact symplectomorphisms

$$(\mathrm{Id}_{\mathbb{R}}, \phi^\pm)\colon \mathbb{R}\times U^\pm \to \mathbb{R}\times\phi^\pm(U^\pm) \subset \mathbb{R}\times J^1\Lambda^\pm.$$

Pre-composing $(\mathrm{Id}_{\mathbb{R}}, (\phi^\pm)^{-1})$ with the (non-exact) symplectomorphism

$$\psi^\pm\colon (T^*(I_\pm\times\Lambda^\pm), d(p_idq_i)) \to (\mathbb{R}\times J^1(\Lambda^\pm), d(e^t(dz-y_idx_i))),$$
$$((q,\mathbf{q}),(p,\mathbf{p})) \mapsto (\log q, (\mathbf{q}, -\mathbf{p}/q, -p)),$$

where $I_+ = [e^T,+\infty)$ and $I_- = (0,e^{-T}]$, we get an induced Weinstein neighbourhood of the cylindrical ends of Σ parametrised by $(\mathrm{Id}_{\mathbb{R}}, (\phi^\pm)^{-1})\circ\psi^\pm$.

Fix a proper open embedding

$$I_-\times\Lambda^- \sqcup I_+\times\Lambda^+ \hookrightarrow \Sigma.$$

An adaptation of the proof of Weinstein's Lagrangian neighbourhood theorem (see e.g. [55]) shows that we can construct a symplectic identification

$$\Psi\colon (D^*_\delta\Sigma, d(p_idq_i)) \hookrightarrow (\mathbb{R}\times P\times\mathbb{R}, d(e^t(dz+\theta)))$$

of a co-disc bundle of some small radius $\delta>0$, for which the zero-section is identified with $\Psi(0_\Sigma) = \Sigma \subset \mathbb{R}\times P\times\mathbb{R}$, and whose restriction to

$$(D^*_\delta(I_-\times\Lambda^- \sqcup I_+\times\Lambda^+), d(p_idq_i)) \subset (D^*_\delta\Sigma, d(p_idq_i)),$$

coincides with the symplectomorphisms $(\mathrm{Id}_{\mathbb{R}}, (\phi^\pm)^{-1})\circ\psi^\pm$ constructed above. For the definition of the above disc bundle we will pick a metric on the cotangent bundle $T^*\Sigma$ which is induced by a Riemannian metric on Σ being of the form $dq\otimes dq + q^{-1}g_\pm$ on the ends $I_\pm\times\Lambda^\pm$, where $g_\pm$ denote Riemannian metrics on $\Lambda^\pm$.

We are now ready to describe the construction of the Hamiltonian push-off Σ_1 of Σ, which is done by performing the push-off in Section 10.1.2 while taking extra care.

First, we choose a C^1-small push-off Σ_1' of Σ by applying the time-ϵ flow of the Hamiltonian vector-field $e^{-t}h'_\Lambda(t)\partial_z$ defined in Section 10.1.2. Here $\epsilon>0$ is chosen smaller than the shortest Reeb chords on both ends $\Lambda^\pm$. (Recall that this flow coincides with the positive and negative Reeb flow at the positive and negative end, respectively.) Second, we perform a non-compact perturbation of the cylindrical ends of Σ_1' induced by cylindrically extending a perturbation of the Legendrian

ends (where this perturbation has been appropriately cut off). Finally, we perform a compactly supported Hamiltonian perturbation of Σ_1' yielding the sought exact Lagrangian cobordism Σ_1. These perturbations will moreover be performed so that the following properties are satisfied.

(1) Under the above identifications of standard contact neighbourhoods of $\Lambda^\pm$, we require that $\phi^\pm(\Lambda_1^\pm) = \pm j^1 f^\pm \subset J^1\Lambda^\pm$ for positive Morse functions $f^\pm\colon \Lambda^\pm \to (0, \epsilon]$. We moreover require that f^+ has a *unique* local minimum m^+. (This is possible since Λ^+ is connected by assumption.)

(2) Under the above identification of a Weinstein neighbourhood of Σ, we require that $\Psi^{-1}(\Sigma_1) = -dF \subset T^*\Sigma$ for a Morse function $F\colon \Sigma \to \mathbb{R}$. We moreover require that F has a *unique* local minimum m (note the sign!). (This is possible since Σ is connected and since F increases as $|t| \to +\infty$ along either of the cylindrical ends.)

(3) Above $I_- \times \Lambda^-$, the function F is required to restrict to a function of the form
$$F|_{I_- \times \Lambda^-} = -qf^-,$$
where q denotes the standard coordinate on $I_- = (0, e^{-T}] \subset \mathbb{R}$.

(4) Above $I_+ \times \Lambda^+ \subset \Sigma$, the function F restricts to a function of the form
$$F|_{I_+ \times \Lambda^+} = qf^+ + C(q)$$
by construction, where q is the standard coordinate on $I_+ = [e^T, +\infty) \subset \mathbb{R}$ and $C(q)$ is constant outside of a compact set. We moreover require that $C\colon \mathbb{R} \to \mathbb{R}$ satisfies:
   (a) $C(q) = -2(\max_{\Lambda^+} f^+) \cdot q < \min_{I_- \times \Lambda^-} F < 0$ holds near $q = e^T$;
   (b) $C(q) \equiv 0$ for all $q \gg 0$ sufficiently large;
   (c) $C'(q)$ is non-decreasing;
   (d) $C'(e^{T'}) = -f^+(m^+)$, i.e. $(e^{T'}, m^+) \in I_+ \times \Lambda^+$ is a critical point being a local minimum for some $T' > T$; and
   (e) $C''(e^{T'}) > 0$, i.e. this critical point is a non-degenerate local minimum.
Note that, in particular, we require that the unique local minimum of F is given by the above critical point $m = (e^{T'}, m^+) \in I_+ \times \Lambda^+ \subset \Sigma$. Also, see Figure 21.

In order to see that we indeed can find the above function F, we observe that a function defined on the subsets $I_- \times \Lambda^- \cup I_+ \times \Lambda^+ \subset \Sigma$ satisfying the above requirements (3) and (4) can be extended to a Morse function $F\colon \Sigma \to \mathbb{R}$ without introducing additional local minima, as follows by considering property (4a).

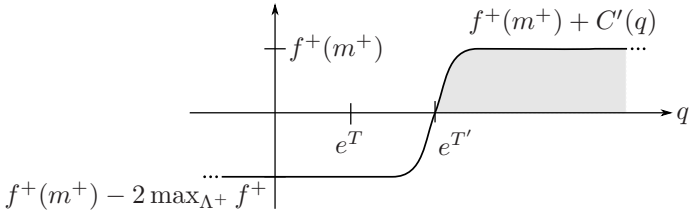

FIGURE 21. The graph of the differential $\partial_q F$ along $[e^T, +\infty) \times \{m^+\} \subset I_+ \times \Lambda^+$. Observe that $F|_{\{q\} \times \Lambda^+}$ has a non-degenerate local minimum at (q, m^+) for each $q \in I_+$.

Lemma 11.8. *Let Σ_1 be the Hamiltonian push-off of Σ described by all the previous conditions. For a suitable choice of almost complex structure J there is a* unique *and transversely cut out J-holomorphic disc having boundary on $\Sigma \cup \Sigma_1$ and a single positive puncture asymptotic to γ_{m^+}. This disc is moreover a rigid strip having precisely two punctures, where the second puncture maps to the intersection point $m = (e^{T'}, m_+) \in \Sigma \cap \Sigma_1$.*

Proof. We will choose an admissible almost complex structure J on $\mathbb{R} \times P \times \mathbb{R}$ which, on the subset $[T, +\infty) \times P \times \mathbb{R}$, will be the cylindrical lift of a compatible almost complex structure J_P on $(P, d\theta)$ as described in Section 3.1.3.

First we show that a punctured disc as in the assumption cannot pass through the hypersurface $\{t = T\}$ for a suitable choice of almost complex structure J. Namely, consider a neck-stretching limit around this hypersurface, and observe that

$$\{t = T\} \cap (\Sigma \cup \Sigma_1) = \mathbb{R} \times (\Lambda^+ \cup \phi^{-2\max_{\Lambda^+} f^+}(\Lambda_1^+))$$

holds by property (4a) above. (Here $\phi^s(t, x, z) = (t, x, z + s)$ denotes the Reeb flow as usual.) By action reasons it thus follows that no such strip can pass through this hypersurface for a sufficiently stretched almost complex structure, since otherwise we would get a component of negative energy in the SFT limit. Observe that stretching the neck in this setting can be equivalently performed by fixing the almost complex structure J, but while changing the boundary condition in a way so that $T' \gg 0$ above becomes arbitrarily large.

We may hence assume that any disc as in the assumption is contained in $\{t \geq T\}$. The canonical (J, J_P)-holomorphic projection $[T, +\infty) \times P \times \mathbb{R} \to P$ maps $\{t \geq T\} \cap (\Sigma \cup \Sigma_1)$ to the Lagrangian projection $\Pi_{\mathrm{Lag}}(\Lambda^+ \cup \Lambda_1^+) \subset (P, d\theta)$. The fact that the positive puncture of the projection of the disc in the assumption is mapped to $m^+ \in \Pi_{\mathrm{Lag}}(\Lambda^+) \cap \Pi_{\mathrm{Lag}}(\Lambda_1^+)$ implies that this disc must have a constant projection to P. In other words, the disc is contained inside the J-holomorphic plane $[T, +\infty) \times \{m^+\} \times \mathbb{R} \subset \mathbb{R} \times P \times \mathbb{R}$.

Finally, it can be checked by hand that there exists a unique J-holomorphic strip contained in the above plane $[T, +\infty) \times \{m^+\} \times \mathbb{R}$ having boundary on $\Sigma \cup \Sigma_1$; see Figure 21 for a picture. This strip is transversely cut out by the explicit calculation made in [25, Lemma 8.2]. (This argument is similar to the proof of [17, Theorem 2.15].) □

11.3.3. *The proof of Proposition 11.7.* We now proceed with the proof of Proposition 11.7.

Proof of Proposition 11.7. Let l_γ denote the coefficient of the Reeb chord generator γ of the fundamental class $c^{\varepsilon_0^+, \varepsilon_1^+}_{\Lambda^+, m^+}$ (using the canonical basis of the Reeb chord generators). Recall that this coefficient is given by the count of rigid punctured strips inside the moduli spaces of the form $\widetilde{\mathcal{M}}(\gamma; \boldsymbol{\delta}, m^+, \boldsymbol{\zeta})$, where each strip is counted with the weight $\varepsilon_0^+(\boldsymbol{\delta})\varepsilon_1^+(\boldsymbol{\zeta})$.

Consider punctured Floer strips with boundary on $\Sigma_0 \cup \Sigma_1$ having precisely two positive punctures asymptotic to Reeb chords: one being the Reeb chord m^+ from Λ_0^+ to Λ_1^+ corresponding to the minimum of the Morse function f, and one corresponding to the above Reeb chord γ from Λ_1^+ to Λ_0^+. By the non-negativity of the Fredholm index for a generic almost complex structure, together with the positivity of the energy, the compactification of this moduli space a priori consists of pseudoholomorphic buildings of the following form:

(1) Pseudoholomorphic buildings with:
   - A top level consisting of a single punctured strip with boundary on $\mathbb{R}\times\Lambda_0^+\cup\mathbb{R}\times\Lambda_1^+$ (which hence is rigid up to translation);
   - A middle level consisting of punctured half-planes of index zero having boundary on Σ_i, $i=0,1$.

(2) Pseudoholomorphic buildings with:
   - A top level consisting of one punctured strip of index one having boundary on $\mathbb{R}\times\Lambda_0^+\cup\mathbb{R}\times\Lambda_1^+$ (which hence is rigid up to translation) together with a trivial strip over a Reeb chord;
   - A middle level consisting of a single punctured strip of index zero having boundary on $\Sigma_0\cup\Sigma_1$.

(3) Pseudoholomorphic buildings with:
   - A middle level consisting of two punctured strips of index zero having boundary on $\Sigma_0\cup\Sigma_1$;
   - A bottom level consisting of a single punctured strip with boundary on $\mathbb{R}\times\Lambda_0^-\cup\mathbb{R}\times\Lambda_1^-$ which is of index one (and hence rigid up to translation).

(4) Pseudoholomorphic buildings with:
   - A middle level consisting of a single punctured strip of index zero having boundary on $\Sigma_0\cup\Sigma_1$;
   - A bottom level consisting of a single punctured half-plane of index one with boundary on $\mathbb{R}\times\Lambda_i^-$, $i=0,1$ (which hence are rigid up to translation), together with additional trivial strips over Reeb chords.

(5) A broken punctured strip having boundary on $\Sigma_0\cup\Sigma_1$.

See Figure 22 for a schematic picture of the above pseudoholomorphic buildings.

A gluing argument implies that the configurations in (1) are in bijection with the configurations contributing to the above coefficient l_γ in front of γ of the fundamental class. Furthermore, the count of the configurations in (5) gives the coefficients of $\widetilde{G}_\Sigma^{\varepsilon_0^-,\varepsilon_1^-}$ by Lemma 11.8. We proceed to infer that the signed counts of all buildings of type (2)-(4) is equal to the coefficient of γ in the expression $d_{\varepsilon_0^+,\varepsilon_1^+}\circ b^{\Sigma_1,\Sigma}(m_+)$, from which the sought equality on the level of homology now follows. (As usual, all counts above are weighted by the augmentations ε_i^-, $i=0,1$.)

(2): There are two cases: either the non-trivial strip in the top level has a positive puncture asymptotic to m^+, or it has positive puncture asymptotic to γ. The former case can be excluded by actions reasons, while the count of the latter configurations corresponds exactly to the coefficient in front of γ of the boundary $d_{\varepsilon_0^+,\varepsilon_1^+}\circ b^{\Sigma_1,\Sigma}(m_+)$.

(3): There are no buildings of this type. Namely, by Lemma 11.8, we may assume there are no punctured pseudoholomorphic strips with boundary on $\Sigma_0\cup\Sigma_1$ having positive asymptotic to the minimum m^+ and a negative asymptotic to a Reeb chord from Λ_0^- to Λ_1^-.

(4): The sum of these contributions vanishes, as follows from the fact that ε_i^-, $i=0,1$, vanishes on any boundary of the Chekanov-Eliashberg algebra of Λ_i^- (see (II) in Section 6). Recall that the latter differential is defined by a count of punctured pseudoholomorphic half-planes of index one having boundary on $\mathbb{R}\times\Lambda_i^-$.

□

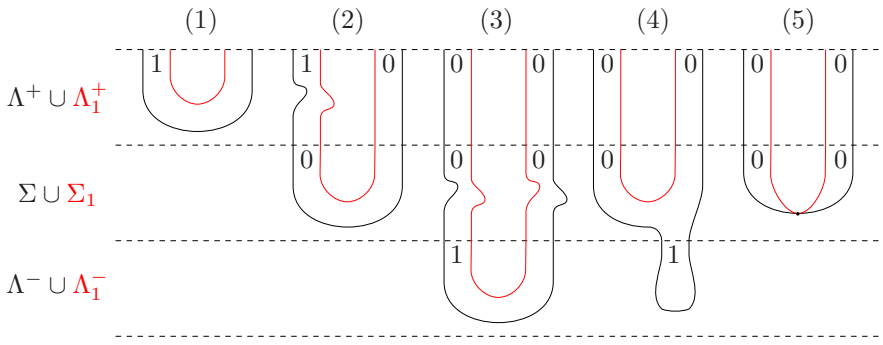

FIGURE 22. The pseudoholomorphic buildings (1)-(5) described in the proof of Proposition 11.7. The number on each component denotes its Fredholm index.

Not that the construction allows us to give a proof of the fact (already pointed out in [36]) that the fundamental class is functorial with respect to exact Lagrangian cobordisms. Indeed, stretching the neck in the slice $\{t = -T\}$ decomposes the map $G_\Sigma^{\varepsilon_0^-,\varepsilon_1^-}$ into $\Phi_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} \circ G_{\mathbb{R}\times\Lambda^-}^{\varepsilon_0^-,\varepsilon_1^-}$, where Φ_Σ is the DGA morphism induced by the cobordism. Alternatively, one can also use the long exact sequence produced by Theorem 1.8 together with Proposition 11.7 in order to deduce this. In either case, we have:

Theorem 11.9. [36, Theorem 7.7] *Let Σ be a connected exact Lagrangian cobordism from Λ^- to Λ^+, and let ε_i^-, $i = 0, 1$, be augmentations of the Chekanov-Eliashberg algebra of Λ^- which pull back to augmentations ε_i^+ under the DGA morphism Φ_Σ induced by Σ. It follows that*

$$\Phi_\Sigma^{\varepsilon_0^-,\varepsilon_1^-}(c_{\Lambda^-,m^-}^{\varepsilon_0^-,\varepsilon_1^-}) = c_{\Lambda^+,m^+}^{\varepsilon_0^+,\varepsilon_1^+},$$

i.e. the fundamental class is preserved under the bilinearised dual of the DGA morphism induced by Σ, under the additional assumption that the images of $m^\pm$ under the natural maps $H_0(\Lambda^\pm, \mathbb{F}) \to H_0(\Sigma)$ agree.

11.4. **A brief introduction to homology with L^2-coefficients.** We use the technology of L^2-Betti numbers, introduced by Atiyah in [8], as a tool to study rank properties of Legendrian contact cohomology when the coefficient ring is a group ring $\mathbb{C}[\pi]$ for a group π. In the following, π denotes the fundamental group of the cobordism. Observe that in this case π is countable. The main idea is to replace $\mathbb{C}[\pi]$, which is not a priori a Noetherian ring, with a more manageable module. Namely, we consider its L^2-completion $\ell^2(\pi)$ defined by the set of functions $f\colon \pi \to \mathbb{C}$ satisfying $\sum_{g\in\pi} |f(g)|^2 < \infty$, endowed with its natural structure of a Hilbert space.

We do not intend to give a comprehensive introduction to the subject of L^2-ranks and refer the reader to the book of Lück [54] and the introductory paper of Eckmann [29] as the main references for the results used, but we still try to give an understandable overview of the needed techniques. The main result that we will need is a version of the snake lemma for L^2-cohomology, due to Cheeger and Gromov [19] (also, see [54, Theorem 1.21]), which applies since all our complexes are finitely generated as $\ell^2(\pi)$-modules.

A *Hilbert π-module* V is a Hilbert space on which π acts by isometry. It is said to be of finite type if it can be realised as a closed subspace of $\ell^2(\pi) \otimes_{\mathbb{C}} \mathbb{C}^m$ for a certain $m \in \mathbb{N}$. Morphisms of Hilbert π-modules are bounded linear maps which are π-equivariant. Given an endomorphism $f\colon V \to V$ of a Hilbert module of finite type, we define its *von Neumann trace* by

$$\operatorname{tr}_{L^2}(f) := \sum_{i=1}^{m} \langle \overline{f}(1 \otimes e_i), 1 \otimes e_i \rangle. \tag{45}$$

Here $\overline{f} := i \circ f \circ p$, $i\colon V \to \ell^2(\pi) \otimes_{\mathbb{C}} \mathbb{C}^m$ is the inclusion, $p\colon \ell^2(\pi) \otimes_{\mathbb{C}} \mathbb{C}^m \to V$ is the orthogonal projection, and $\{e_i\}$ is the standard basis of $\mathbb{C}^m$. A simple computation shows that this trace only depends on f and not on the particular choice of the embedding.

The *von Neumann dimension* of V is $\operatorname{rk}_{L^2}(V) = \operatorname{tr}_{L^2} \mathrm{Id}$, which clearly is a non-negative number bounded from above by m, under the assumption that V can be embedded in $\ell^2(\pi) \otimes_{\mathbb{C}} \mathbb{C}^m$. Note that the von Neumann dimension can take non-integer values. The following basic properties will be crucial:

Lemma 11.10 (Theorem 1.12 in [54])**.** *(1) $V = 0$ if and only if $\operatorname{rk}_{L^2}(V) = 0$; and*

(2) If $0 \to U \to V \to W \to 0$ is weakly exact, i.e. $\overline{\operatorname{im} i} = \ker p$, then $\operatorname{rk}_{L^2}(V) = \operatorname{rk}_{L^2}(U) + \operatorname{rk}_{L^2}(W)$.

We will be interested in applying this theory to a complex $(\overline{C}_\bullet, \overline{\partial})$ which is the L^2-completion of a G-equivariant complex $(C_\bullet, \partial)$ consisting of finitely generated free $\mathbb{C}[G]$-modules. In this case, the L^2-modules are all of finite type. The corresponding L^2-homology will be denoted by $H^{(2)}_\bullet(C_\bullet, \partial)$, where we note that $H^{(2)}_i(C_\bullet, \partial)$ is defined as the quotient of the subspace of cycles by the *closure* of the subspace of boundaries. It follows that $H^{(2)}_\bullet(C_\bullet, \partial)$ again is a G-equivariant L^2-module of finite type.

Lemma 11.11. *In the above situation, we have*

$$\operatorname{rk}_{L^2} H^{(2)}_i(C_\bullet, \partial) \leq \operatorname{rk}_{\mathbb{C}[G]} C_i.$$

Furthermore, for a finite-dimensional complex $(C'_\bullet, \partial')$ over $\mathbb{C}$, we have

$$H^{(2)}_i(C'_\bullet \otimes \mathbb{C}[G], \partial' \otimes \mathrm{Id}_{\mathbb{C}[G]}) = H_i(C'_\bullet, \partial') \otimes_{\mathbb{C}} \ell^2(G),$$

and thus, in particular,

$$\operatorname{rk}_{L^2} H^{(2)}_i(C'_\bullet \otimes \mathbb{C}[G], \partial' \otimes \mathrm{Id}_{\mathbb{C}[G]}) = \dim_{\mathbb{C}} H_i(C'_\bullet, \partial').$$

Proof. The first statement follows from Lemma 11.10 together with the Hodge decomposition in [54, Lemma 1.18]. The second statement follows by a direct computation. □

For a pair of CW complexes (X, Y), $Y \subset X$ and a choice of homomorphism $\varphi\colon \pi_1(X) \to G$, there is an induced covering $(\widetilde{X}, \widetilde{Y}) \to (X, Y)$ with fibre G, monodromy described by φ, and where there is a natural free G-action on the covering. In the case when $(C_\bullet, \partial)$ is the G-equivariant cellular complex associated to such a covering, we will write the corresponding L^2-homology groups by $H^{(2)}_\bullet(X, Y; \varphi)$ or, by abuse of notation, $H^{(2)}_\bullet(X, Y; G)$.

11.5. **Estimating the first L^2-Betti number of a tower.** Suppose that $\overline{\Sigma}$ is a compact $(n+1)$-dimensional manifold with boundary $\partial\overline{\Sigma} = \partial_+\overline{\Sigma} \sqcup \partial_-\overline{\Sigma}$, such that $\partial_\pm\overline{\Sigma} \cong \Lambda$ both are simply connected. Let $\overline{\Sigma}^{\odot k}$ be the quotient of $\sqcup_{i=1}^k \overline{\Sigma}_i$, $\overline{\Sigma}_i \cong \overline{\Sigma}$, which identifies $\partial_+(\overline{\Sigma}) \subset \overline{\Sigma}_i$ with $\partial_-(\overline{\Sigma}) \subset \overline{\Sigma}_{i+1}$. We will write $\partial\overline{\Sigma}^{\odot k} = \partial_-\overline{\Sigma}^{\odot k} \sqcup \partial_+\overline{\Sigma}^{\odot k}$, where $\partial_-\overline{\Sigma}^{\odot k} = \partial_-\overline{\Sigma}_1$, $\partial_+\overline{\Sigma}^{\odot k} = \partial_+\overline{\Sigma}_k$.

Further, consider the covering space $\widetilde{\Sigma}^{\odot k} \to \overline{\Sigma}^{\odot k}$ obtained by gluing the boundary of the universal cover $\sqcup_{i=1}^k \widetilde{\Sigma}_i \to \sqcup_{i=1}^k \overline{\Sigma}_i$ via the identification of the induced cover

$$\widetilde{\Sigma}_i \supset \sqcup_{g\in\pi_1(\overline{\Sigma})} \partial_+(\overline{\Sigma}) \to \partial_+(\overline{\Sigma}) \subset \overline{\Sigma}_i$$

with the induced cover

$$\widetilde{\Sigma}_{i+1} \supset \sqcup_{g\in\pi_1(\overline{\Sigma})} \partial_-(\overline{\Sigma}) \to \partial_-(\overline{\Sigma}) \subset \overline{\Sigma}_{i+1}.$$

Observe that the covering $\widetilde{\Sigma}^{\odot k} \to \overline{\Sigma}^{\odot k}$ obtained is induced by a group epimorphism

$$\pi_1(\overline{\Sigma}^{\odot k}) \simeq \underbrace{\pi * \ldots * \pi}_{k} \to \pi.$$

Lemma 11.12. *In the case when π is finite, we have*

$$\dim_{\mathbb{F}}(H_1(\widetilde{\Sigma}^{\odot k}, \partial_-\widetilde{\Sigma}^{\odot k}; \mathbb{F})) = (|\pi| - 1)k, \;\; k \geq 1.$$

If $\pi := \pi_1(\overline{\Sigma})$ is infinite, the L^2-homology $H_1^{(2)}(\overline{\Sigma}^{\odot k}, \partial_-\overline{\Sigma}^{\odot k}; \pi)$ with coefficients twisted by the above covering $\widetilde{\Sigma}^{\odot k} \to \overline{\Sigma}^{\odot k}$ satisfies

$$\mathrm{rk}_{L^2}(H_1^{(2)}(\overline{\Sigma}^{\odot k}, \partial_-\overline{\Sigma}^{\odot k}; \pi)) \geq k.$$

Proof. The first statement is proven by induction on k, using a standard comparison of dimensions obtained via the Mayer-Vietoris long exact sequence

$$\begin{aligned}
\ldots \to\; & H_1(\partial_-\widetilde{\Sigma}; \mathbb{F}) \to \\
\to\; & H_1(\widetilde{\Sigma}; \mathbb{F}) \oplus H_1(\widetilde{\Sigma}^{\odot(k-1)}, \partial_-\widetilde{\Sigma}^{\odot(k-1)}; \mathbb{F}) \to \\
\to\; & H_1(\widetilde{\Sigma}^{\odot k}, \partial_-\overline{\Sigma}^{\odot k}; \mathbb{F}) \\
\to\; & H_0(\partial_-\widetilde{\Sigma}; \mathbb{F}) \to \ldots.
\end{aligned}$$

We now show the statement concerning the L^2-ranks, which follows by analogous computations. Lemma 11.11 implies that

$$H_0^{(2)}(\partial_\pm\overline{\Sigma}; \pi) = \ell^2(\pi) \text{ and } H_1^{(2)}(\partial_\pm\overline{\Sigma}; \pi) = 0,$$

since $\partial_\pm\overline{\Sigma}$ are simply connected. Observe that we also have

$$H_0^{(2)}(\overline{\Sigma}^{\odot k}; \pi) = 0, \;\; k \geq 1,$$

as follows from [54, Theorem 1.35(8)], using the fact that π is infinite. The (weak) long exact sequence of a pair [54, Theorem 1.21] immediately implies the base case

$$\mathrm{rk}_{L^2} H_1^{(2)}(\overline{\Sigma}^{\odot 1}, \partial_-\overline{\Sigma}^{\odot 1}; \pi) = \mathrm{rk}_{L^2} H_1^{(2)}(\overline{\Sigma}, \partial_-\overline{\Sigma}; \pi) \geq 1$$

as well as the vanishing

$$H_0^{(2)}(\overline{\Sigma}^{\odot k}, \partial_-\overline{\Sigma}^{\odot k}; \pi) = 0.$$

The Mayer-Vietoris long (weakly) exact sequence

$$\begin{aligned}\ldots \to H_1^{(2)}(\partial_-\overline{\Sigma};\pi) \to \\ \to \quad H_1^{(2)}(\overline{\Sigma};\pi) \oplus H_1^{(2)}(\overline{\Sigma}^{\odot(k-1)}, \partial_-\overline{\Sigma}^{\odot(k-1)};\pi) \to \\ \to \quad H_1^{(2)}(\overline{\Sigma}^{\odot k}, \partial_-\overline{\Sigma}^{\odot k};\pi) \\ \to \quad H_0^{(2)}(\partial_-\overline{\Sigma};\pi) \to 0 \to \ldots,\end{aligned}$$

together with $H_1^{(2)}(\partial_-\overline{\Sigma};\ell^2(\pi)) = 0$ and [54, Theorem 1.12(2)] gives that

$$\begin{aligned}&\mathrm{rk}_{L^2} H_1^{(2)}(\overline{\Sigma}^{\odot k}, \partial_-\overline{\Sigma}^{\odot k};\pi) \geq \\ &\geq \quad \mathrm{rk}_{L^2} H_1^{(2)}(\overline{\Sigma}^{\odot(k-1)}, \partial_-\overline{\Sigma}^{\odot(k-1)};\pi) + \mathrm{rk}_{L^2} H_0^{(2)}(\partial_-\overline{\Sigma};\pi).\end{aligned}$$

Since $\mathrm{rk}_{L^2} H_0^{(2)}(\partial_-\overline{\Sigma};\pi) = 1$, the claim now follows by induction. □

12. Applications and examples

In this section we deduce all applications mentioned in the introduction of the paper. In addition, we provide explicit examples of Lagrangian cobordisms: both examples to which our results apply, but also examples showing the importance of the different hypotheses used.

12.1. **The homology of an endocobordism.** The following proofs of Theorems 1.11 and 1.13 are similar to the proofs given in [26].

Proof of Theorem 1.11. We begin by showing the result in the case when $\mathbb{F} = \mathbb{Z}_2$.

(i): First, recall the elementary fact from algebraic topology that

$$\dim_{\mathbb{F}} H(\Sigma;\mathbb{F}) \geq \dim_{\mathbb{F}} H(\Lambda;\mathbb{F}) \tag{46}$$

is satisfied, which follows by studying the long exact sequence of the pair $(\overline{\Sigma}, \partial\overline{\Sigma})$ together with Poincaré duality (see [26, Lemma 2.1]).

We proceed to prove the opposite inequality $\dim_{\mathbb{F}} H(\Sigma;\mathbb{F}) \leq \dim_{\mathbb{F}}(\Lambda;\mathbb{F})$. The linearised Legendrian contact cohomology satisfies the bound

$$\dim_{\mathbb{F}} LCH_{\varepsilon'}(\Lambda) \leq |\mathcal{R}(\Lambda)|$$

for any ε'. Thus we can fix an augmentation ε of $\mathcal{A}(\Lambda;\mathbb{F})$ satisfying

$$\dim_{\mathbb{F}} LCH_{\varepsilon}(\Lambda;\mathbb{F}) = \max_{\varepsilon'}\{\dim_{\mathbb{F}} LCH_{\varepsilon'}(\Lambda;\mathbb{F})\}. \tag{47}$$

The exact triangle in Theorem 1.8 gives us

$$\begin{aligned}&\dim_{\mathbb{F}} LCH_{\varepsilon_+}(\Lambda;\mathbb{F}) \geq \\ &\geq \quad \dim_{\mathbb{F}} LCH_{\varepsilon}(\Lambda;\mathbb{F}) + \dim_{\mathbb{F}} H(\Sigma;\mathbb{F}) - \dim_{\mathbb{F}} H(\Lambda;\mathbb{F})\end{aligned}$$

where ε_+ is the augmentation of $\mathcal{A}(\Lambda;\mathbb{F})$ obtained as the pull-back $\varepsilon_+ := \varepsilon \circ \Phi_\Sigma$. Formula (47) implies that $\dim_{\mathbb{F}} H(\Sigma;\mathbb{F}) - \dim_{\mathbb{F}} H(\Lambda;\mathbb{F}) \leq 0$. Together with inequality (46), we obtain

$$\dim_{\mathbb{F}} H(\Sigma;\mathbb{F}) = \dim_{\mathbb{F}} H(\Lambda;\mathbb{F}) \tag{48}$$

of the dimension of the total homology.

In order to show that $\dim_{\mathbb{F}} H_i(\Sigma;\mathbb{F}) = \dim_{\mathbb{F}} H_i(\Lambda;\mathbb{F})$ for all i, we argue by contradiction, assuming that

$$d_{i_0}(\Sigma) := \dim_{\mathbb{F}} H_{i_0}(\Sigma;\mathbb{F}) - \dim_{\mathbb{F}} H_{i_0}(\Lambda;\mathbb{F}) > 0$$

for some i_0. By the Mayer-Vietoris sequence we conclude that the inequality

$$\dim_{\mathbb{F}} H_{i_0}(\Sigma \odot \Sigma; \mathbb{F}) \geq 2 \dim_{\mathbb{F}} H_{i_0}(\Sigma; \mathbb{F}) - \dim_{\mathbb{F}} H_{i_0}(\Lambda; \mathbb{F})$$

holds. In particular,

$$d_{i_0}(\Sigma \odot \Sigma) := \dim_{\mathbb{F}} H_{i_0}(\Sigma \odot \Sigma; \mathbb{F}) - \dim_{\mathbb{F}} H_{i_0}(\Lambda; \mathbb{F}) \geq 2 d_{i_0}(\Sigma),$$

which by induction leads to a contradiction with equality (48).

(ii): The argument is the same as the one in the proof of [26, Theorem 1.6 (ii)], and follows form Part (i) applied to the concatenation $\Sigma \odot \Sigma$. Namely the Mayer-Vietoris sequence for the concatenation $\Sigma \odot \Sigma$ seen as two copies of $\overline{\Sigma}$ glued along the boundary component Λ shows that

$$\dim_{\mathbb{F}} H(\Sigma \odot \Sigma; \mathbb{F}) \geq 2 \dim_{\mathbb{F}} H(\Sigma; \mathbb{F}) - \dim_{\mathbb{F}} \operatorname{im}(i_*^-, i_*^+)$$

and by the above result, we conclude that

$$\dim_{\mathbb{F}} \operatorname{im}(i_*^-, i_*^+) = \dim_{\mathbb{F}} H(\Sigma; \mathbb{F}) = \dim_{\mathbb{F}} H(\Lambda; \mathbb{F}),$$

from which the claim follows.

(iii): By contradiction, we assume that $i_*^+ \oplus i_*^- \colon H(\Lambda \sqcup \Lambda) \to H(\Sigma)$ is not a surjection. Considering a representative $V \subset H(\Sigma)$ of the cokernel of this map, which hence is of dimension $\dim_{\mathbb{F}} V > 0$, the Mayer-Vietoris long exact sequence implies that the image of $V \oplus V$ under the map

$$H(\Sigma) \oplus H(\Sigma) \to H(\Sigma \odot \Sigma)$$

has image being of dimension $2 \dim_{\mathbb{F}} V > 0$. Moreover, $V \oplus V$ can again be seen to not be contained in the image of

$$i_*^+ \oplus -i_*^- \colon H(\Lambda \sqcup \Lambda) \to H(\Sigma \odot \Sigma).$$

Namely, the above inclusion factorises through the canonical maps as

$$i_*^+ \oplus -i_*^- \colon H(\Lambda \sqcup \Lambda) \to H(\Sigma \sqcup \Sigma) \to H(\Sigma \odot \Sigma),$$

where the latter morphism is the one from the above Mayer-Vietoris long exact sequence. In conclusion, the cokernel of

$$i_*^+ \oplus i_*^- \colon H(\Lambda \sqcup \Lambda) \to H(\Sigma \odot \Sigma)$$

is of dimension at least $2 \dim_{\mathbb{F}} V$. Arguing by induction, now we arrive at the sought contradiction with Part (i) above.

The proof is now complete for $\mathbb{Z}_2$.

Under the additional assumption that Λ is spin, and admitting an augmentation in an arbitrary field $\mathbb{F}$, Corollary 12.3 of Theorem 1.14 implies that any endocobordism of Λ is spin as well. This allows us to repeat the previous argument with coefficients in the field $\mathbb{F}$. Note that Theorem 1.14 relies on Theorem 1.11, which we established above in the needed case $\mathbb{F} = \mathbb{Z}_2$. □

We now prove the following Theorem of which Theorem 1.13 is an immediate corollary. Observe that the following result also can be proved by alluding to Theorem 1.11.

Theorem 12.1. *Let Λ be a Legendrian homology sphere inside a contactisation, Σ be an exact Lagrangian cobordism from Λ to itself inside the symplectisation, and $\mathbb{F}$ a field. If $\mathcal{A}(\Lambda; \mathbb{F})$ admits an augmentation, then $H_\bullet(\Sigma, \Lambda; \mathbb{F}) = 0$, i.e. Σ is a $\mathbb{F}$-homology cylinder.*

Proof. Let $\Sigma^{\odot k}$, $k \geq 1$, be the k-fold concatenation of Σ with itself, which again is an exact Lagrangian cobordism from Λ to Λ. Since Λ is a homology sphere it is spin and, hence, $\Sigma^{\odot k}$ is spin for all $k \geq 1$ by Corollary 12.3.

We fix an augmentation ε of $\mathcal{A}(\Lambda;\mathbb{F})$ and let ε_k be the augmentation of $\mathcal{A}(\Lambda;\mathbb{F})$ obtained by the pull-back of ε under the unital DGA morphism induced by $\Sigma^{\odot k}$.

The (ungraded version of the) long exact sequence in Theorem 1.6 becomes

$$\begin{array}{ccc} LCH^\bullet_\varepsilon(\Lambda) & \longrightarrow & LCH^\bullet_{\varepsilon_k}(\Lambda) \\ & \nwarrow \quad \swarrow & \\ & H_\bullet(\overline{\Sigma}^{\odot k}, \partial_- \overline{\Sigma}^{\odot k}; \mathbb{F}) & \end{array} \tag{49}$$

Observe that

$$\dim H_i(\overline{\Sigma}^{\odot k}, \partial_- \overline{\Sigma}^{\odot k}; \mathbb{F}) = \begin{cases} 0, & i = 0, n+1, \\ k \dim H_i(\overline{\Sigma}, \partial_- \overline{\Sigma}; \mathbb{F}), & 0 < i < n+1, \end{cases}$$

as follows from the Mayer-Vietoris long exact sequence together with the assumption that Λ is a $\mathbb{F}$-homology sphere.

Since the linearised contact cohomology satisfies the bound

$$\dim_{\mathbb{F}} LCH_{\varepsilon'}(\Lambda) \leq |\mathcal{R}(\Lambda)|$$

for any ε', we get the inequality

$$k \dim H_i(\overline{\Sigma}, \partial_- \overline{\Sigma}; \mathbb{F}) = \dim H_i(\overline{\Sigma}^{\odot k}, \partial_- \overline{\Sigma}^{\odot k}; \mathbb{F}) \leq 2|\mathcal{R}(\Lambda)|, \;\; 0 < i < n+1,$$

for each k, where the exactness of the above triangle has been used to show the last inequality. In conclusion, we have established

$$\dim H_i(\overline{\Sigma}, \partial_- \overline{\Sigma}; \mathbb{F}) = 0, \;\; 0 < i < n+1,$$

which finishes the proof. □

Proof of Theorem 1.13. Since Λ is assumed to have an augmentation over $\mathbb{Z}$ it admits an augmentation over $\mathbb{Q}$ as well. And thus it follows from Theorem 12.1 that $H_\bullet(\Sigma, \Lambda; \mathbb{Q}) = 0$ and thus that $H_\bullet(\Sigma, \Lambda; \mathbb{Z})$ is torsion. The augmentation over $\mathbb{Z}$ also induces an augmentation over any finite field, and thus Theorem 12.1 implies that $H_\bullet(\Sigma, \Lambda; \mathbb{Z})$ has no p-torsion for any prime p. Thus $H_\bullet(\Sigma, \Lambda; \mathbb{Z}) = 0$. □

Remark 12.2. *Following the discussion in Section 11.2 we get that Theorem 1.13 holds under the weaker assumption that the Chekanov-Eliashberg algebra admits a non-commutative augmentation in a finite-dimensional $\mathbb{F}$-algebra. (The proof is a verbatim reproduction of the precedent.)*

12.2. **Characteristic classes of endocobordisms.** Recall that from Section 1.1.1, that Theorem 1.11 still applies in the case when the cobordism Σ is not orientable and of Maslov number one i.e. when the Cthulhu complexes involving Σ necessarily are *ungraded*. In this case, we obtain exact triangles instead of long exact sequences.

Assume now that we are given a chord-generic *orientable* Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ whose Chekanov-Eliashberg algebra admits an augmentation over $\mathbb{Z}_2$ (or, more generally, a linear m-dimensional representation over $\mathbb{Z}_2$). The dual statement of Part (iii) of Theorem 1.11 reads as follows. Let Σ be an exact Lagrangian endocobordism Σ of Λ. The map $(i_+^*, i_-^*) : H^*(\Sigma, \mathbb{Z}_2) \to H^*(\Lambda \sqcup \Lambda, \mathbb{Z}_2)$ is injective.

Theorem 1.14 is an immediate corollary of this and of the naturality of characteristic classes. In turn, it implies the following:

Corollary 12.3. *If Λ is orientable (respectively, spin), and that it admits an augmentation into a finite-dimensional algebra, then any exact endocobordism Σ of Λ is orientable (respectively, spin) as well.*

This result can be seen as a generalisation of the result of Capovilla-Searle and Traynor, see [14, Theorem 1.2]. The proof of Theorem 1.14 for Pontryagin classes follows similarly assuming that Λ is spin.

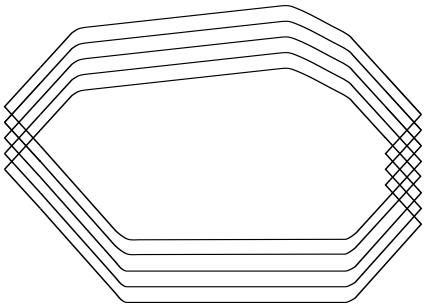

FIGURE 23. The front projection of $\Lambda_{(5,-8)}$.

Example 12.4. Recall that a Legendrian knot in the standard contact $\mathbb{R}^3$ for which the Kauffman bound on tb is not sharp does not admit an augmentation in a commutative ring [61].

(1) Consider a family of the Legendrian representatives of torus $(p,-q)$-knots $\Lambda_{(p,-q)} \subset \mathbb{R}^3$ with $q > p \geq 3$ and p odd; see Figure 23. Following Sivek [65], we observe that $\mathrm{tb}(\Lambda_{(p,-q)}) = -pq$ and, hence, from the classification result of Etnyre and Honda [43] it follows that $\Lambda_{(p,-q)}$ is tb-maximising. Recall that Sivek [65] proved that the Chekanov-Eliashberg algebra of $\Lambda_{(p,-q)}$ admits a 2-dimensional representation over $\mathbb{Z}_2$, but for which the Kauffman bound on tb is not sharp. Therefore, these Legendrian knots do not admit non-orientable exact Lagrangian endocobordisms.

(2) Consider $\Lambda_{(p,-q)}\#\Lambda$, where p is odd, $q > p \geq 3$, and let Λ be a tb-maximising Legendrian knot of $\mathbb{R}^3$ whose Chekanov-Eliashberg algebra admits an augmentation (or, more generally, m-dimensional linear representation) over $\mathbb{Z}_2$. Then, following the discussion in [27, Lemma 4.3], we see that the Kauffman bound for $\Lambda_{(p,-q)}\#\Lambda$ is not sharp and that the Chekanov-Eliashberg algebra of $\Lambda_{(p,-q)}\#\Lambda$ admits a finite-dimensional linear representation over $\mathbb{Z}_2$. In addition, from the fact that $\Lambda_{(p,-q)}$ and Λ are tb-maximising, together with [44, Corollary 3.5] (or [68, Theorem 1.1]), it follows that $\Lambda_{(p,-q)}\#\Lambda$ also is tb-maximising. This leads us to many other examples, besides $\Lambda_{(p,-q)}$, which do not admit non-orientable exact Lagrangian endocobordisms.

(3) There is also an example due to Sivek, see [65, Sections 2.2 and 3], of a tb-maximising knot with non-sharp Kauffman bound on tb, whose Chekanov-Eliashberg algebra does not admit a finite-dimensional linear representation over $\mathbb{Z}_2$.

Remark 12.5. *The above examples provide a negative answer to a question of Capovilla-Searle and Traynor, see* [14, Question 6.1].

12.3. **Restrictions on the fundamental group of an endocobordism between simply connected Legendrians.** We now prove the results concerning the fundamental groups of endocobordisms between simply connected Legendrian submanifolds.

12.3.1. *Proof of Theorem 1.16.*

Proof of Theorem 1.16. Recall the construction of the fundamental class in the setting of twisted coefficients carried out in Section 11.3. The proof will be a straightforward consequence of Proposition 11.7 therein.

From the assumptions of the theorem, the Legendrian submanifold Λ^+ has a unique augmentation. It follows that [35, Theorem 5.5] can be applied, and hence the fundamental class $\widetilde{c}^{\varepsilon_0^+,\varepsilon_1^+}_{\Lambda^+,m^+}$ is non-vanishing. By Proposition 11.7 we, moreover, conclude that this fundamental class is the image of a generator m of $H_0(\Sigma; R[\pi_1(\Sigma)])$ under the map $\widetilde{G}^{\varepsilon^-,\varepsilon^-}_{\Sigma}$. Since Λ^+ is simply connected by assumption, it follows from (44) above that this image is not torsion. In particular

$$g \cdot \widetilde{c}^{\varepsilon_0^+,\varepsilon_1^+}_{\Lambda^+,m^+} \neq \widetilde{c}^{\varepsilon_0^+,\varepsilon_1^+}_{\Lambda^+,m^+}, \ \ \forall g \in \pi_1(\Sigma).$$

Thus, m is not torsion either, and since it generates $H_0(\Sigma; R[\pi_1(\Sigma)])$ we conclude that $H_0(\Sigma; R[\pi_1(\Sigma)]) = R[\pi_1(\Sigma)]$. However, since $\widetilde{\Sigma}$ is connected, we know that $H_0(\Sigma; R[\pi_1(\Sigma)]) = H_0(\widetilde{\Sigma}) = R$. In other words, $\pi_1(\Sigma)$ is the trivial group, as sought. □

12.3.2. *Proof of Theorem 1.18.*

Proof of Theorem 1.18. Here it will be crucial to use the machinery of L^2-coefficients as described in Section 11.4.

We will let $\Sigma^{\odot k}$, $k \geq 1$, denote the k-fold concatenation of the cobordism Σ from Λ to Λ. Since Λ is spin by assumption, it follows from Corollary 12.3 that the cobordisms $\Sigma^{\odot k}$ are spin for all $k \geq 1$. We also consider the cover $p\colon \widetilde{\Sigma}^{\odot k} \to \overline{\Sigma}^{\odot k}$ as constructed in the previous section.

First, we argue that the claim follows from the fact $|\pi_1(\Sigma)| < \infty$, which will be shown below. Indeed, under these assumptions, the version of the long exact sequence in Theorem 1.6 applied to the system of local coefficients induced by the above covering (see Section 11.1), becomes

$$\begin{array}{ccc} LCH^\bullet_\varepsilon(\Lambda; \mathbb{C}[\pi_1(\Sigma)]) & \longrightarrow & LCH^\bullet_{\varepsilon_k}(\Lambda; \mathbb{C}[\pi_1(\Sigma)]) \\ & \nwarrow \quad \swarrow & \\ & H_\bullet(\widetilde{\Sigma}^{\odot k}, \partial_- \widetilde{\Sigma}^{\odot k}; \mathbb{C}) & \end{array} \tag{50}$$

Here the augmentation ε_k is the pull-back of the augmentation ε under the unital DGA morphism induced by $\Sigma^{\odot k}$ and the covering. Observe that ε_k takes values in $\mathbb{C}[\pi_1(\Sigma)]$.

For $k \gg 0$ sufficiently large, unless $|\pi_1(\Sigma)| = 1$, the equality

$$\dim_{\mathbb{C}}(H_1(\widetilde{\Sigma}^{\odot k}, \partial_- \widetilde{\Sigma}^{\odot k}; \mathbb{C})) = (|\pi_1(\Sigma)| - 1)k, \ \ k \geq 1,$$

as shown in Lemma 11.12, together with the universal bound

$$\dim_{\mathbb{C}} LCH^{\bullet}_{\varepsilon_0,\varepsilon_1}(\Lambda;\mathbb{C}[\pi_1(\Sigma)]) \leq |\pi_1(\Sigma)||\mathcal{R}(\Lambda)|$$

gives a contradiction.

It remains to show that $|\pi_1(\Sigma)|$ is finite. Assuming the contrary, we use the (weakly) long exact sequence obtained from (50) by taking the L^2-completions of the above $\mathbb{C}[\pi_1(\Sigma)]$-equivariant complexes (since the complexes are *freely and finitely generated* we can again apply Cheeger and Gromov's result in [19]), establishing the exact triangle

$$\begin{array}{ccc} LCH^{(2)\bullet}_{\varepsilon}(\Lambda) & \longrightarrow & LCH^{(2)\bullet}_{\varepsilon_k}(\Lambda) \\ & \nwarrow \quad \swarrow & \\ & H^{(2)}_{\bullet}(\overline{\Sigma}^{\odot k}, \partial_- \overline{\Sigma}^{\odot k}; \pi_1(\Sigma)) & \end{array}$$

The inequality

$$\mathrm{rk}_{L^2}(H^{(2)}_1(\overline{\Sigma}^{\odot k}, \partial_- \overline{\Sigma}^{\odot k}; \pi_1(\Sigma))) \geq k,$$

as shown in Lemma 11.12, together with the universal bound

$$\mathrm{rk}_{L^2} LCH^{(2)\bullet}_{\varepsilon'}(\Lambda) \leq |\mathcal{R}(\Lambda)|,$$

which follows by Lemma 11.11, finally gives the sought contradiction, from which it follows that $\pi_1(\Sigma)$ is finite. □

12.4. **Explicit examples of Lagrangian cobordisms.** We start by recalling a few general constructions of Legendrian submanifolds and exact Lagrangian cobordisms. Below these will be used in order to construct explicit examples of Lagrangian cobordisms.

12.4.1. *A Legendrian ambient surgery on the front-spin.* The front S^m-spinning construction described in [50] by the fourth author constructs a Legendrian embedding $\Sigma_{S^m}\Lambda \subset (\mathbb{R}^{2(m+n)+1}, \xi_{\mathrm{std}})$ of $S^m \times \Lambda$, given a Legendrian embedding $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$. In the same article, it was also shown that the same construction can be applied to an exact Lagrangian cobordism $\Sigma \subset \mathbb{R} \times \mathbb{R}^{2n+1}$ from Λ^- to Λ^+ inside the symplectisation, producing an exact Lagrangian cobordism $\Sigma_{S^m}\Sigma \subset \mathbb{R} \times \mathbb{R}^{2(n+m)+1}$ from $\Sigma_{S^m}\Lambda^-$ to $\Sigma_{S^m}\Lambda^+$ that is diffeomorphic to $S^m \times \Sigma$.

Consider a Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$. Its left-most cusp edge in the front projection for a generic representative corresponds to a cusp edge diffeomorphic to S^m in the front projection of the front spin $\Sigma_{S^m}\Lambda \subset (\mathbb{R}^{2m+3}, \xi_{\mathrm{std}})$. Moreover, this cusp edge bounds an obvious embedding of a Lagrangian $(m+1)$-disc $D \subset (\mathbb{R}^{2(m+n)+1}, \xi_{\mathrm{std}})$ whose interior is disjoint from $\Sigma_{S^m}\Lambda$, while its boundary coincides with this cusp edge; see Figure 24.

A Legendrian ambient m-surgery, as described in [24] by the second author, can be applied to the sphere $S^m \hookrightarrow \Sigma_{S^m}\Lambda$ corresponding to the cusp edge ∂D, utilising the bounding Legendrian disc D. The Legendrian submanifold $\Lambda^+ \subset (\mathbb{R}^{2(m+n)+1}, \xi_{\mathrm{std}})$ resulting from the surgery has the front projection shown in Figure 25 in the case of $m = 1 = \dim \Lambda$. Recall that there also is a corresponding elementary Lagrangian $(m+1)$-handle attachment, which is an exact Lagrangian cobordism from $\Sigma_{S^m}\Lambda$ to the Legendrian submanifold Λ^+ obtained after the surgery. Topologically, this cobordism is simply the handle attachment corresponding to the surgery.

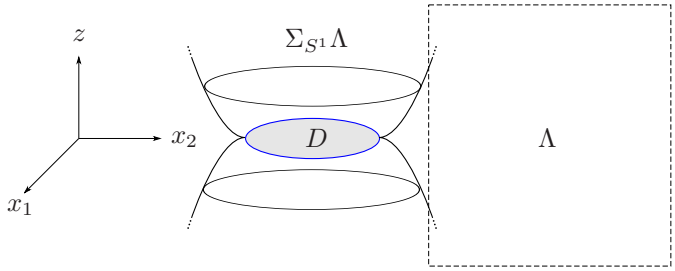

FIGURE 24. The front projection of the front spin $\Sigma_{S^1}\Lambda \subset (\mathbb{R}^5, \xi_{\text{std}})$ near the left-most cusp of $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$. The corresponding cusp-edge for the front projection of the front spin bounds an obvious embedded Legendrian disc D intersecting $\Sigma_{S^1}\Lambda$ cleanly along this cusp edge.

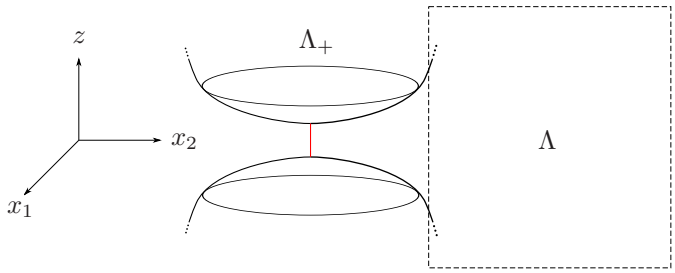

FIGURE 25. The front projection of the Legendrian submanifold $\Lambda^+ \subset (\mathbb{R}^5, \xi_{\text{std}})$ obtained after a Legendrian ambient surgery on the front spin $\Sigma_{S^1}\Lambda \subset (\mathbb{R}^5, \xi_{\text{std}})$, utilising the Legendrian disc D as shown in Figure 24.

12.4.2. *Non-simply connected exact Lagrangian fillings of Legendrian spheres (the proof of Proposition 1.22).* Using the constructions in Section 12.4.1 above, the sought examples will not be difficult to produce. We start with a Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$ which admits a non-simply connected Lagrangian filling Σ. For instance, we can take the Legendrian right handed trefoil knot and its exact Lagrangian filling by a punctured torus; see [36]. It follows that $\Sigma_{S^m}\Lambda \subset (\mathbb{R}^{2(m+1)+1}, \xi_{\text{std}})$ is a Legendrian $S^m \times S^1$ which admits an exact Lagrangian filling $\Sigma_{S^m}\Sigma$ diffeomorphic to $S^m \times \Sigma$ that is not simply connected.

The Legendrian ambient surgery along a cusp-edge in the class $S^m \times \{p\}$ for $p \in \Lambda$ corresponding to the left-most cusp edge of $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$ as described above produces a Legendrian sphere, and concatenating $\Sigma_{S^m}\Sigma$ with the corresponding elementary Lagrangian $(m+1)$-handle provides a non-simply connected filling of $\Sigma_{S^m}\Lambda$. These are the sought non-simply connected exact Lagrangian cobordisms.

Remark 12.6. *Theorem 1.18 is applicable to the constructed sphere in order to rule out non-simply connected endocobordisms. In addition, note that since the conclusion of Theorem 1.16 is not satisfied, the Reeb chords created by the surgery are essential, and the Legendrian sphere admits at least two distinct augmentations.*

12.4.3. *Non-invertible Lagrangian concordances.* Here we will prove the statement that

Proposition 12.7. *In all contact spaces* $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ *with* $n \geq 1$ *there exists a Legendrian* n*-sphere* Λ *of* $\text{tb} = -1$ *which is fillable by a Lagrangian disc, but for which there is no Lagrangian concordance to the standard Legendrian sphere* Λ_0 *of* $\text{tb} = -1$. *(Recall that the filling induces a Lagrangian concordance from* Λ_0 *to* Λ*.)*

In [16] the first author proved that the relation of Lagrangian concordance is not symmetric by establishing the above proposition in the case $n = 1$. In particular, it was shown that the Legendrian representative $\Lambda_{9_{46}} \subset (\mathbb{R}^3, \xi_{\text{std}})$ of the knot 9_{46} as depicted in Figure 26 (satisfying $\text{tb} = -1$; this is maximal for this smooth knot class), which is fillable by a Lagrangian disc, is not concordant to the standard Legendrian unknot Λ_0 of $\text{tb} = -1$.

Recall that an exact Lagrangian filling by a disc can be used to construct a concordance C from Λ_0 to $\Lambda_{9_{46}}$, which was explicitly described in the same article. One such concordance is described in Figure 27 below. Notice that along the entire concordance the leftmost cusp-edge p is fixed, and so we can assume that the cylinder C coincides with the trivial cylinder $\mathbb{R} \times l$ for a small arc $p \in l \subset \Lambda_{9_{46}}$ inside a neighbourhood of this cusp. This fact will be important below.

Using the results in the current article, the non-existence of a concordance from $\Lambda_{9_{46}}$ to Λ_0 can be reproved by applying Corollary 1.20 together with the calculations in [16]. Namely, in the latter article it is shown that, for an appropriate pair $\varepsilon_0, \varepsilon_1$ of augmentations of the Chekanov-Eliashberg algebra of Λ_0, we have

$$LCH_{-1}^{\varepsilon_0,\varepsilon_1}(\Lambda_{9_{46}}) \neq 0,$$

and no concordance going the other way can thus exist by Corollary 1.20.

The front spinning construction produces exact Lagrangian concordances $\Sigma_{S^m} C \subset \mathbb{R} \times \mathbb{R}^{3+2m}$, obtained as the front spin of C, from $\Sigma_{S^m}\Lambda_0 \subset (\mathbb{R}^{3+2m}, \xi_{\text{std}})$ to $\Sigma_{S^m}\Lambda_{9_{46}} \subset (\mathbb{R}^{3+2m}, \xi_{\text{std}})$. Here, the latter Legendrian submanifolds are the front spins of Λ_0 and $\Lambda_{9_{46}}$, respectively. In [17, Section 5] the authors proved using the Künneth formula in Floer homology that again

$$LCH_{-1}^{\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1}(\Sigma_{S^m}\Lambda_{9_{46}}) \neq 0$$

holds for a suitable pair of augmentations, which together with Corollary 1.20 implies that there is no Lagrangian concordance from $\Sigma_{S^m}\Lambda_{9_{46}}$ to $\Sigma_{S^m}\Lambda_0$.

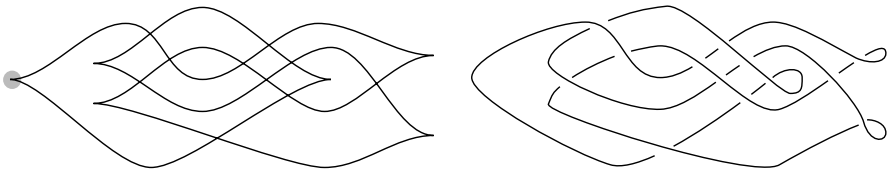

FIGURE 26. Front (left) and Lagrangian (right) projections of the maximal TB $m(9_{46})$ knot.

Recall that $\Sigma_{S^m}\Lambda_0 \simeq \Sigma_{S^m}\Lambda_{9_{46}} \simeq S^m \times S^1$, while $\Sigma_{S^m} C \simeq \mathbb{R} \times S^m \times S^1$. We will now perform an explicit modification of the above example to produce an example of Legendrian *spheres* in all dimensions which admit a concordance *from* the standard sphere, but which do not admit a concordance *to* the standard sphere; this establishes Proposition 12.7.

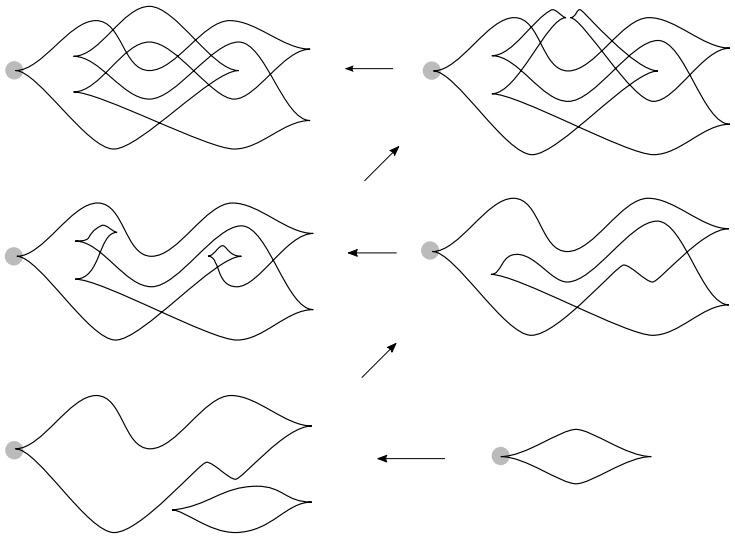

FIGURE 27. A Lagrangian concordance from Λ_0 to $\Lambda_{9_{46}}$.

Proof of Proposition 12.7. The Legendrian ambient surgery can be performed to the cusp-edge of the front projection of $\Sigma_{S^m}\Lambda_{9_{46}}$ corresponding to the left-most cusp edge $p \in \Lambda_{9_{46}}$. In this way, a Legendrian sphere $\Lambda^+ \subset (\mathbb{R}^{2(m+1)+1}, \xi_{\text{std}})$ is produced. Since the concordance C moreover may be assumed to be a trivial cylinder over a neighbourhood of $p \in \Lambda$ and, hence, so is $\Sigma_{S^m}C$, we obtain a Lagrangian concordance from Λ^- to Λ^+, where Λ^- is the Legendrian sphere obtained by performing the corresponding Legendrian ambient surgery on $\Sigma_{S^m}\Lambda_0$. In fact, the latter sphere is the standard Legendrian $(m+1)$-sphere of $\text{tb} = -1$.

Recall that the Legendrian ambient surgery also produces an exact Lagrangian handle attachment cobordism from $\Sigma_{S^m}\Lambda_{9_{46}}$ to Λ^+. Inspecting the long exact sequence induced by Theorem 1.6, we immediately conclude that there are augmentations ε_i^+, $i = 0, 1$ for the Chekanov-Eliashberg algebra of the Legendrian sphere Λ^+ satisfying

$$LCH_{-1}^{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \simeq LCH_{-1}^{\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1}(\Sigma_{S^m}\Lambda_{9_{46}}) \neq 0.$$

Once again, Corollary 1.20 shows that there is no concordance from Λ^+ to Λ^-.

□

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