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# A variational model for context-driven effects in perception and cognition

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## Abstract

Starting with a computational analysis of brightness matching, we develop a novel variational framework able to model perceptual context-driven effects that may be extended to non-physical judgments as well. The most important feature of the variational framework is the description of these phenomena as a suitable balance between contrast and dispersion. The optimal balance is defined through the simultaneous minimization of functionals characterized by two terms in opposition to each other. When the minimum is reached, the equilibrium between contrast and dispersion is attained. To show the flexibility of the proposed framework, we discuss several examples of such functionals in the field of color perception and cognition which show adherence between theoretical predictions and empirical results. With regard to social cognition theories, the simultaneous occurrence of contrast and dispersion conflicts with sequential models, thus supporting the idea of a concurrent presence of both effects in each judgment. The variational framework can serve as a view from above on perceptual and cognitive phenomena that may help in deriving new constraints for disambiguating alternative theories.

*Keywords:*

Context effects, contrast, assimilation, variational models, psychophysics.

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## 1. Introduction and state of the art

Context-driven effects are one of the most frequent observations in psychology. We can define a context-driven effect as an over- or under-estimation of a stimulus embedded in a given context compared to the same evaluation task performed in isolated conditions. Within the field of visual perception, a typical example is the contrast effect observed in the brightness matching

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6 experiment performed by Wallach (1948). Using Rudd and Zemach's reinterpretation of Wal-  
7 lach's experiment (Rudd & Zemach, 2004), we developed a general variational model where  
8 context-driven effects can be described. Formulating a problem in terms of variational prin-  
9 ciples is a common strategy in practically every scientific discipline. The main purpose is to  
10 obtain a broader view of the problem, thereby enabling us to derive a higher level explanation  
11 of the phenomenon and to detect its underlying functional constraints. Furthermore, by using  
12 this approach it is possible to highlight new constraints that may help to disambiguate alternative  
13 interpretations.

14 When the brightness matching experiment is seen as a calculus of variations problem, judg-  
15 ments of a stimulus embedded in a context can be interpreted as the result of the balancing of  
16 two opposing processes. The first (that we will call with the standard term *contrast*) tries to  
17 emphasize differences in the final percept, whereas the second (that we will call *dispersion*) tries  
18 to emphasize similarities. We attempt to present evidence that a variety of contextual effects  
19 observed in non-physical judgments (such as those of social cognition) can be analyzed using  
20 the same variational framework.

## 21 **1.1. Contextual effects for physical judgments**

22 In the domain of perception, as early as in the seventeenth century the philosopher John  
23 Locke (1690) had described the contrast effect by noticing that a hand's contact with tepid water  
24 can produce either the sensation of cold or hot if the hand had been previously placed in hot  
25 or cold water, respectively. The contrast effect has been considered since the earliest days of  
26 psychophysics (Chevreul, 1855; Wundt, 1896) and extensively investigated for judgments in the  
27 context of physical dimensions, such as the loudness of a tone (Melamed, 1971), the brightness  
28 of light (Wallach, 1948), weight (Heintz, 1950; Sherif, Taub, & Hovland, 1958), length of lines  
29 (Krantz & Campbell, 1961), and so on. In parallel, since von Bezold (1876) described a phe-  
30 nomenon "*in which a colored surface appears lighter when overlaid by thin white lines or small*  
31 *white dots and appears darker if the lines or dots are black,*" assimilation effects (after the name  
32 given to them by Evans, 1948) were also studied (Blakeslee & McCourt, 2004; De Weert &  
33 Spillman, 1995; Festinger, Coren, & Rivers, 1970; Helson, 1963; Kingdom & Moulden, 1991),  
34 especially in visual phenomena such as the white effect (White, 1979), Bressan's dungeon illu-  
35 sion (Bressan, 2001) or the *cube* illusion of Agostini and Galmonte (2002). Within the domain  
36 of brightness perception, assimilation is considered the opposite of contrast. However, given the  
37 variety of models, theoretical approaches, and empirical results, such a notion is rather contro-  
38 versial and it has been employed as a convenient catch-all in which to place anti-contrast effects  
39 (Gilchrist, 2006). In particular, when referring to brightness judgments, the term 'assimilation'  
40 has a special and different meaning from the concept described in this paper. So, as we wrote  
41 before, we will employ the term dispersion instead to indicate a positive correlation between the  
42 judgment and the context. The use of this generic term will also be helpful because we are going  
43 to refer the corresponding context effects in the social cognition and cognitive psychology fields  
44 where the word assimilation may have different meanings.

45 The debate over the variables (and the underlying mechanisms) that determine different con-  
46 text effects is still open for discussion. Indeed, there is no consensus on how surface lightness  
47 is processed by the brain (Gilchrist, 2015) and thus under which conditions different context ef-  
48 fects are observed. Within this debate, Agostini and Galmonte (2002) proposed that perceptual  
49 belongingness may determine the kind of context effect. According to it, gestalt laws (proximity,  
50 similarity, good continuation, common fate, closure, and pragnanz) (Koffka, 1935; Wertheimer,  
51 1923) can explain the tendency of the visual system to aggregate discrete stimuli within larger  
52 wholes and thus determine if a stimulus phenomenologically belongs to a larger object or not.  
53 The basic idea is that if two elements belong to different perceptual groups, their colors are con-  
54 trasted with the color of the group to which they belong (Agostini & Proffitt, 1993; Agostini &  
55 Galmonte, 2000). On the contrary, when an element is intentionally organized into one or an-  
56 other of two groups, its color is assimilated to the color of the group to which it belongs (so, in  
57 our terminology, a dispersion effect will be observed). For the sake of simplicity, in this paper we  
58 will assume Agostini and Galmonte's (2002) perspective because it is compatible with the pro-  
59 posed formal framework and it addresses an important open question in the literature. However,  
60 our analysis does not depend on this assumption and it can be compatible with other theoretical  
61 proposals.

## 62 **1.2. Context effects for non-physical judgments**

63 The results obtained for basic perceptual judgments have suggested investigating the influ-  
64 ence of the context for non-physical judgments as well. Within social cognition, the word con-  
65 trast is employed to define the case of a judgment negatively correlated with the contextual  
66 information, whereas the term assimilation refers to a positive correlation between the judgment  
67 and the contextual information. Those effects have been observed for non-physical judgments  
68 related to moral evaluations (Parducci, 1968; Pepitone & DiNubile, 1976), pleasantness of music  
69 (Parker, Bascom, Rabinovitz, & Zellner, 2008), friendliness of a person (Stapel, Koomen, & van  
70 der Pligt, 1997), attractiveness (Kenrick & Gutierrez, 1980), prices of objects (Matthews & Stew-  
71 art, 2009b), and a wide variety of social judgments and evaluations (Biernat, 2005; Moskowitz,  
72 2005). For example, when investigating the contrast effect in moral judgment, Parducci (1968)  
73 asked respondents to rate the seriousness of a number of acts, such as *poisoning a neighbor's*  
74 *dog*, alongside trivial acts, such as *keeping a dime you find in a telephone booth*, and very se-  
75 rious bad acts, such as *murdering your mother without justification or provocation*. He found  
76 that in the first case, the sentence *poisoning a neighbor's dog* was judged as more serious when  
77 compared to the second case. Different terms have been employed to indicate a judgment bi-  
78 ased towards the context. Within cognitive psychology, Tversky and Kahneman (1974) used  
79 the word anchoring to indicate the bias of a numeric judgment towards a previously considered  
80 standard. In line with the observations made previously about physical judgments, given such  
81 terminological ambiguities, we employ the word dispersion to mean such kind of bias.

82 Several studies have identified many factors that can induce dispersion or contrast in judg-  
83 ment related to person perception (Higgins & Lurie, 1983) and self-evaluation (Festinger, 1954).  
84 For example, broad contextual categories (such as traits) are likely to produce dispersion whereas

85 in the case of a context represented by narrow categories (such as exemplars), contrast effect will  
86 be observed (Stapel, Koomen, & van der Pligt, 1996). Other factors include processing goals  
87 (memorization vs. impression formation where the first induces assimilation and the second  
88 contrast, Moskowitz & Roman, 1992), distinctness of the context (in the case of high distinct-  
89 ness that is more likely to observe contrast, see Wedell, Parducci, & Geiselman, 1987), temporal  
90 distance between events (distant contextual events are more likely to induce contrast, see Strack,  
91 Schwarz, & Gschneidinger, 1985), and many others (Biernat, 2005).

92 Given the factors that can induce cognition-related context-driven effects, several theoretical  
93 models have been developed to furnish parsimonious and effective predictions about how and  
94 when dispersion or contrast occurs in a given situation. Among them, we can cite the set-reset  
95 model (Martin & Achee, 1992), the inclusion-exclusion model (Schwarz & Bless, 1992), the  
96 flexible correction model (Petty & Wegener, 1993), the interpretation-comparison model (Stapel  
97 & Koomen, 1998) and the selective accessibility model (Mussweiler, 2003). Those models differ  
98 in terms of the assumed degree of effort involved in the effects (automatic or controlled), if the  
99 two effects are simultaneous or sequential, the specific variables involved in the processes, and  
100 the assumed default process (either contrast or dispersion). However, the majority of them agree  
101 with the notion that the factors that make the context less distinct from the stimulus (in other  
102 terms, factors suggesting an inclusion of the stimulus in the context) induce a dispersion effect,  
103 whereas factors that make the context distinct from the stimulus (so, suggesting an exclusion  
104 from the context) induce a contrast effect. Such interpretation is coherent with Agostini and Gal-  
105 monte's (2002)'s perspective about the factors determining context effect in physical judgments.

### 106 **1.3. Rationale for this article**

107 In this paper, we develop a general, formal framework for context-driven effects in both  
108 perception and cognition. In the psychophysical literature about contrast, it is possible to find  
109 attempts that provide mathematical formalizations of this effect. The first, and still one of the  
110 most famous, is provided by Fechner's formalization (Fechner, 1860) of Weber's findings about  
111 the differential threshold of sensation by sense organs (Weber, 1846). Even more remarkably, it  
112 is possible to find attempts to build mathematical models that try not only to formalize measure-  
113 ments, but also to predict new phenomena. We will present a pertinent example of such a model  
114 in section 2 related to visual induction (i.e., perception of light stimuli in a non-isolated context).  
115 Recently, this model has been embedded in a variational setting (Provenzi, 2013).

116 The use of a variational principle is not yet widespread in mathematical psychology, even  
117 though a few examples can be found (Ehm & Wackermann, 2012; Noventa & Vidotto, 2012).  
118 In other disciplines (e.g., physics and signal processing), variational principles have been exten-  
119 sively employed. One of the main reasons for this success is that variational calculus allows  
120 the reinterpretation of a model in terms of the hidden basic mechanisms underlying the model  
121 itself. Crucially, these basic mechanisms are also shared with other models and this allows for  
122 a demonstration of similarities and differences that are often very difficult to discover without a  
123 variational approach.

124 We stress that the main purpose of variational calculus is not to provide new predictions  
125 compared to already existing models, but to offer a more general and profound view about the  
126 phenomena described by these models to possibly build bridges between different theories. Once  
127 the variational interpretation of a model has been established, it is possible to determine general  
128 constraints about the phenomena and it is also possible to slightly modify the analytical expres-  
129 sion of the variational model in order to generate new equations to be tested.

130 To illustrate our approach, we are going to show that the variational reinterpretation of  
131 Rudd-Zemach's model can also be applied to the comprehension of mechanisms underlying  
132 non-isolated judgments of non-physical stimuli. As stressed above, the core contribution of this  
133 variational framework is the possibility of describing context-driven judgments as a suitable bal-  
134 ance between two opposing mechanisms: contrast and dispersion. On the one side, the contrast  
135 mechanism tends to maximize as much as possible the difference between the stimulus and its  
136 surround, while on the other side, simultaneously, the dispersion mechanism tends to absorb  
137 this difference, integrating the stimulus into its surround. Finally, we will strengthen our hy-  
138 pothesis with a quantitative discussion of previous studies about context effects for non-physical  
139 judgments.

140 The paper is organized as follows: In section 2 we introduce Rudd-Zemach's analysis of a  
141 context-driven perceptual match experiment; after a brief summary of variational principles in  
142 section 3, we reformulate Rudd-Zemach's model in section 4 and interpret brightness matching  
143 as an optimal balance between dispersion control and contrast enhancement. This concept is  
144 further generalized in section 5, in which a fully general variational framework for context-driven  
145 perceptual and cognitive phenomena is provided and discussed with examples. The simple, yet  
146 already significant, variational interpretation of linear equations in the linear and logarithmic  
147 domain is discussed in great detail. Section 6 is devoted to a thorough discussion of the proposed  
148 model and several case studies. The paper ends with conclusions and an appendix showing the  
149 proof of the variational result of section 4.

## 150 **2. A significant example of modeling of context-driven phys-** 151 **ical judgments: Achromatic induction**

152 In this section, we will discuss in detail a successful model for achromatic induction. In the  
153 next section, we will provide a variational interpretation of this model and this will serve as a  
154 concrete example for the formulation, in section 5, of a more general variational model suitable  
155 for non-physical judgments as well.

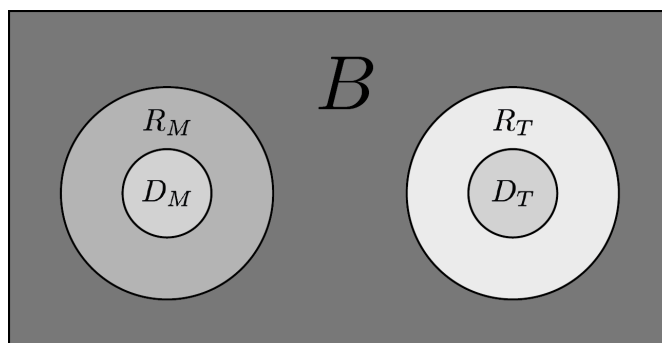
156 Human perception of a color patch is not determined by its reflectance properties alone, but  
157 also by those of the surrounding patches. This phenomenon is called *chromatic induction*, to  
158 stress the fact that color perception is induced (and altered) by the surround. This is also true  
159 for the so-called achromatic colors (i.e., perceived shades of gray), in which case one talks about  
160 *achromatic induction*. Following a common nomenclature, we will call *lightness* the *perceived*  
161 *reflectance* of a non self-luminous patch, while *brightness* will refer to the *perceived luminance*

162 emitted by a source of light. Thus, the terms lightness and brightness already incorporate the  
 163 potential effect of induction in their definition. The most elementary example of induction is the  
 164 *simultaneous contrast* phenomenon, depicted in Fig.1: The two inner squares have exactly the  
 165 same luminance, however we perceive them very differently because we are strongly influenced  
 166 by their distinct surrounds.



**Figure 1:** Achromatic simultaneous contrast: The inner gray squares have the same luminance but the left one, surrounded by a dark gray patch, looks lighter compared to the right one that is surrounded by a light gray patch.

167 Induction can be measured through psychophysical experiments. The first quantitative mea-  
 168 sure of achromatic induction was performed by Wallach (1948). In his classical experiment  
 169 (Fig.2), Wallach considered two disks,  $D_T$  and  $D_M$  for *Target* and *Match*, surrounded by two  
 170 rings  $R_T$  and  $R_M$ , embedded in a uniform background  $B$ . Let us denote with  $L_{D_M}$ ,  $L_{R_M}$ ,  $L_{D_T}$ ,  $L_{R_T}$ ,  
 171 and  $L_B$  the luminance values of  $D_M$ ,  $R_M$ ,  $D_T$ ,  $R_T$ , and  $B$ , respectively. He showed this configu-  
 172 ration to a set of observers adapted to the light conditions of a dimly illuminated room, keeping  
 173  $L_{D_T}$  and  $L_{R_M}$  fixed, using  $L_{R_T}$  as an independent variable that he could fix in every experiment,  
 174 and  $L_{D_M}$  as a dependent variable that the observers could adjust in order to achieve a perceptual  
 175 match between the two disks  $T$  and  $M$ . The stimuli presented to the observers did not have  
 176 chromatic components.



**Figure 2:** Wallach's classical experiment. Over a uniform background  $B$ , there are two inner disks,  $D_T$  and  $D_M$  ( $T$  and  $D$  for *Target* and *Match*, respectively), surrounded by two external rings  $R_T$  and  $R_M$ .

177 If the luminance of the surrounding rings failed to influence the perception of the achromatic  
 178 color of the disks, then the match between the two disks would simply be the *photometric* one,  
 179 i.e.,  $L_{D_M} = L_{D_T}$ ; instead, Wallach found that a fairly good match among the achromatic color  
 180 of the two disks was obtained when the ratios between the disk and the ring luminances were

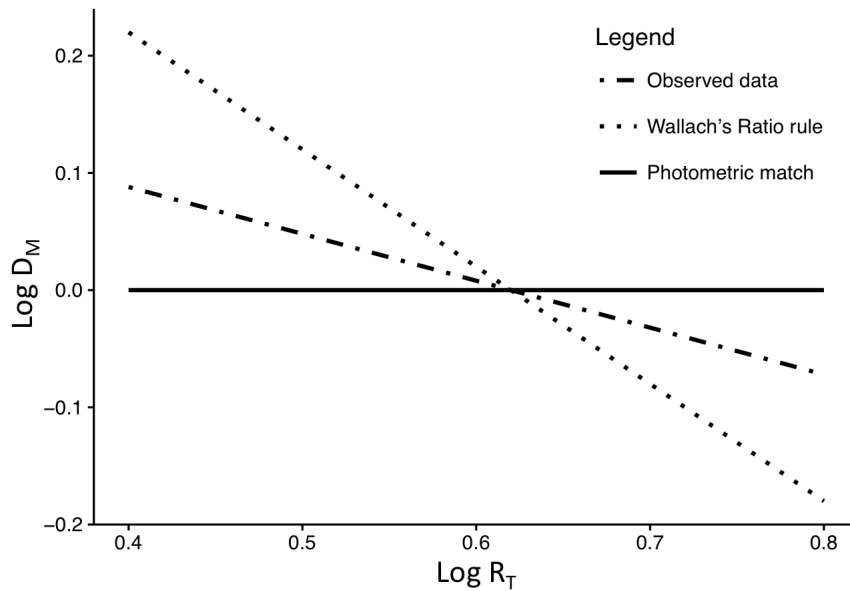
181 identical on the two sides of the display, i.e.,

$$\frac{L_{D_M}}{L_{R_M}} = \frac{L_{D_T}}{L_{R_T}}, \quad (1)$$

182 a formula called *Wallach's Ratio rule*. By taking the logarithms at both sides and solving for  $L_{D_M}$   
 183 we find:

$$\log L_{D_M} = \log L_{D_T} + \log L_{R_M} - \log L_{R_T}, \quad (2)$$

184 thus, according to Wallach's Ratio rule, the plot of the perceptual match in the plane of coordi-  
 185 nates  $(x, y) = (\log L_{R_T}, \log L_{D_M})$  should be a straight line with slope -1, against the slope 0 that a  
 186 photometric match would measure. More recent measures using the classical Wallach's experi-  
 187 ment have shown that this slope is actually between -1 and 0, as can be seen in Fig.3 (adapted  
 188 from Rudd & Zemach, 2004).



**Figure 3:** Quantitative measures of Wallach's achromatic color induction experiment for four observers performed in Rudd & Zemach (2004). The best-fit regression line slopes and associated 95% confidence limits observed by Rudd & Zemach (2004) for the four subjects of the experiment are the following:  $-0.639 \pm 0.033$ ,  $-0.791 \pm 0.034$ ,  $-0.723 \pm 0.047$ , and  $-0.657 \pm 0.042$ .

189 To account for these new psychophysical data, Rudd and Zemach (2004) have proposed a  
 190 more sophisticated model than Wallach's. They repeated Wallach's experiment adding a non-  
 191 black background  $B$ . As in Wallach's experiment,  $L_{D_T}$  and  $L_{R_M}$  are fixed and the observer's task  
 192 is to adjust  $L_{D_M}$  to achieve an achromatic color match to the test disk as a function of  $L_{R_T}$ .  $L_{R_T}$   
 193 is varied from trial to trial by sampling from a set of six luminance values spaced equally in  
 194 RGB units from 2.54 to 6.31  $\text{cd/m}^2$  (notice that Rudd and Zemach used the base 10 for their  
 195 logarithmic values, so that the logarithmic range goes from 0.405 to 0.800  $\text{cd/m}^2$ ).

196 Rudd and Zemach also pointed out some similarities between their model of achromatic  
 197 induction and the Retinex theory of color perception of Land and McCann (1971) without the



198 so-called threshold and reset mechanisms (see Provenzi, De Carli, Rizzi, & Marini, 2005, for  
 199 more details). Rudd and Zemach’s model can be described as follows. Let  $L_i$  and  $L_j$  be the  
 200 luminance of two points  $i$  and  $j$  in an image, the ratio  $\frac{L_i}{L_j}$  can be decomposed as a sequential  
 201 multiplication of the local luminance ratios at borders encountered along a path connecting  $j$  and  
 202  $i$ , for instance:

$$\frac{L_i}{L_j} = \prod_{k=i}^{j-1} \frac{L_k}{L_{k+1}}, \quad (3)$$

203 by taking the logarithm at both sides we get

$$\log \frac{L_i}{L_j} = \sum_{k=i}^{j-1} \log \frac{L_k}{L_{k+1}}. \quad (4)$$

204 Rudd and Zemach introduced *induction strength weights*  $w_k$  in order to take into account the  
 205 *locality of vision*, i.e., the fact that patches that lie in a nearby surround have a stronger influence  
 206 on the induced perception than those that are far away. This point is somewhat delicate and it  
 207 will be discussed in the next section. Notice that in the configuration shown in Fig.2, we have  
 208  $i = D_M, i+1 = j-1 = R_M, j = B$  on the left part of the visual field, and  $i = D_T, i+1 = j-1 = R_T,$   
 209  $j = B$  on the right part. The *logarithmic brightness of  $i$* , which we denote with  $\log \Phi(i)$ , that can  
 210 be inferred by Rudd and Zemach’s model, is the following:

$$\log \Phi(i) \equiv \sum_{k=i}^{j-1} w_{k-i+1} \log \frac{L_k}{L_{k+1}} + \mu, \quad (5)$$

211 where  $\mu \in \mathbb{R}$  is an arbitrary constant that will be eliminated by the matching procedure and that  
 212 we introduced to underline the fact that brightness perception is relative to a context and not  
 213 absolute. It can be seen that, if the luminances  $L_k$  and  $L_{k+1}$  are equal, then their ratio does not  
 214 give any contribution to  $\Phi(i)$ . A meaningful contribution to  $\Phi(i)$  is given only by the luminances  
 215 of points lying at the border of an edge. So,  $\Phi(i)$  represents the summed influence of all the edges  
 216 present within the spatial surround of the target point, suitably weighted. The weights index is  
 217  $k - i + 1$ , which means that small values of the index refer to patches close to  $i$ , and vice versa.  
 218 With this convention, and invoking the fact that induction strength decreases with the distance,  
 219 as proven by Wallach (1963), we have that  $w_1 > w_2 > \dots$ , i.e.,  $\frac{w_2}{w_1} < 1$ , and so on.

220 Rudd and Zemach called their model of achromatic induction ‘Weighted Log Luminance  
 221 Ratio’, or WLLR for short. WLLR predicts that the brightness match between  $L_{D_M}$  and  $L_{D_T}$  is  
 222 attained when  $\Phi(D_M) = \Phi(D_T)$ , i.e.,

$$w_1 \log \frac{L_{D_M}^{\text{Match}}}{L_{R_M}} + w_2 \log \frac{L_{R_M}}{L_B} + \mu = w_1 \log \frac{L_{D_T}}{L_{R_T}} + w_2 \log \frac{L_{R_T}}{L_B} + \mu, \quad (6)$$

223 solving this equation w.r.t.  $\log L_{D_M}$  we have

$$\log L_{D_M}^{\text{Match}} = \log L_{D_T} + \left(1 - \frac{w_2}{w_1}\right) \log L_{R_M} - \left(1 - \frac{w_2}{w_1}\right) \log L_{R_T}, \quad (7)$$

224  $L_{D_M}^{\text{Match}}$  is the luminance value of  $D_M$  selected by the observer to match  $L_{D_T}$ . If we set  $u =$   
 225  $\log L_{D_M}^{\text{Match}}$ ,  $\alpha = \log L_{D_T} + \left(1 - \frac{w_2}{w_1}\right) \log L_{R_M}$ ,  $\beta = -\left(1 - \frac{w_2}{w_1}\right)$  and  $v = \log L_{R_T}$ , then the WLLR model  
 226 predicts the following *linear behavior in the logarithmic domain*:

$$u = \alpha + \beta v, \quad (8)$$

227 with a slope  $\beta = -\left(1 - \frac{w_2}{w_1}\right) \in (-1, 0)$ , which is coherent with Rudd and Zemach's empirical  
 228 observations. In fact, the estimations of the ratio  $\frac{w_2}{w_1}$  from their interpolated data for the four  
 229 observers are: 0.361, 0.209, 0.277, and 0.343. A key assumption of *edge integration models*, like  
 230 WLLR, is that the total achromatic color induction produced by a complex surround is *the sum*  
 231 *of the individual induction effects* produced by the luminance borders comprising that surround.  
 232 Rudd and Zemach performed experiments to directly test this assumption by predicting and  
 233 then measuring the magnitude of the total induction effect produced by combining three circular  
 234 edges located at different distances from the test disk. This was done after first measuring the  
 235 magnitudes of the induction effects produced by the individual edges. Results were in accordance  
 236 with the predictions of the model.

237 In section 4, we will re-interpret the WLLR model in terms of variational principles; this will  
 238 give us the possibility of introducing in a clearer way the more general variational framework  
 239 of section 5. For the sake of clarity, and to introduce the usual nomenclature and notation of  
 240 variational calculus, we summarize the basic information about variational principles in the next  
 241 section.

### 242 3. Overview of variational principles

243 Calculus of variations is a generalization of ordinary calculus in  $\mathbb{R}^n$ . In the latter, we deal  
 244 with functions  $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $n, m$  integers  $\geq 1$ , while in variational calculus we work with  
 245 functions acting on functional spaces. More precisely, a functional space is a vector space whose  
 246 elements are functions having some specified features. To provide a concrete example, let us  
 247 consider two very well-known and useful functional spaces:

- 248 •  $C^n(D)$ ,  $D \subseteq \mathbb{R}^n$ ,  $D$  open, is the space of functions  $f : D \rightarrow \mathbb{R}$  which are  $n$ -times differen-  
 249 tiable, with continuous derivatives on the whole  $D$ , the case  $n = 0$  corresponds simply to  
 250 continuous functions on  $D$ ;
- 251 •  $L^2(\mathbb{R})$  is the space of square-integrable functions on  $\mathbb{R}$ , i.e.,  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\int_{\mathbb{R}} f(x)^2 dx <$   
 252  $+\infty$ . These functions are also said to be finite-energy functions.

253 Given an abstract functional space  $\mathcal{F}$  over the field  $\mathbb{K}$  (in general  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ), the linear  
 254 operations in  $\mathcal{F}$  are defined point-wise, i.e., given  $f, g \in \mathcal{F}$  and  $\alpha, \beta \in \mathbb{K}$ , the function  $h \in \mathcal{F}$   
 255 defined by the linear combination  $h = \alpha f + \beta g \in \mathcal{F}$  acts as follows on the arguments  $x$  of  $f$  and  
 256  $g$ :  $h(x) = \alpha f(x) + \beta g(x)$ . A functional  $\varphi$  acting on the abstract functional space  $\mathcal{F}$  is a linear form

257 over  $\mathcal{F}$ , i.e., a linear function from  $\mathcal{F}$  to the field  $\mathbb{K}$ :

$$\begin{aligned} \varphi : \mathcal{F} &\longrightarrow \mathbb{K} \\ f &\longmapsto \varphi(f). \end{aligned}$$

258 Let us also recall that, in ordinary calculus, a great deal of effort is dedicated to finding the  
259 extrema of functions. To fix ideas, let us consider a function  $f \in C^n(D)$ ,  $\vec{x} \in D$ , then:

- 260 • we call  $\underline{\vec{x}} = \operatorname{argmin}_D f$ , if  $f(\underline{\vec{x}}) = \min_{\vec{x} \in D} f(\vec{x})$ ;
- 261 • we call  $\overline{\vec{x}} = \operatorname{argmax}_D f$ , if  $f(\overline{\vec{x}}) = \max_{\vec{x} \in D} f(\vec{x})$ .

262 We recall that, given a function  $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f \in C^n(D)$  and any unit vector  $\vec{v} \in D$ , the  
263 *directional derivative* of  $f$  along  $\vec{v}$ , calculated in  $\vec{x}$ , is defined by:

$$D_{\vec{v}}f(\vec{x}) = \lim_{\varepsilon \rightarrow 0} \frac{f(\vec{x} + \varepsilon\vec{v}) - f(\vec{x})}{\varepsilon}, \quad (9)$$

264 the partial derivatives  $\partial_i f(x)$ ,  $i = 1, \dots, n$ , of  $f$  are simply the  $n$  directional derivatives computed  
265 by choosing  $\vec{v} = \vec{e}_i$ , the  $i$ -th unit vectors of the canonical basis of  $\mathbb{R}^n$ . Finally, the gradient  $\vec{\nabla}f(x)$   
266 is the  $n$ -dimensional vector whose components are the partial derivatives of  $f$  in  $x$ .

267 By virtue of Fermat's interior extrema theorem, the gradient of  $f$  (and its directional deriva-  
268 tives in every direction) must be null when computed in the argmin or argmax of  $f$ . This neces-  
269 sary condition also becomes sufficient when  $D$  and the function  $f$  are convex. The computation,  
270 either analytical or approximated, of the extrema of a function  $f \in C^n(D)$  belongs to a field called  
271 *optimization in  $\mathbb{R}^n$* . Contrary to ordinary calculus, in variational calculus *the argmin and argmax*  
272 *of a functional are functions*, more precisely, for an arbitrary functional  $E : \mathcal{F} \rightarrow \mathbb{K}$ :

- 273 • we call  $\underline{f} = \operatorname{argmin}_{\mathcal{F}} E$ , if  $E(\underline{f}) = \min_{f \in \mathcal{F}} E(f)$ ;
- 274 • we call  $\overline{f} = \operatorname{argmax}_{\mathcal{F}} E$ , if  $E(\overline{f}) = \max_{f \in \mathcal{F}} E(f)$ .

275 The possibility of progressing from an *extrema of functions*, represented by points of  $\mathbb{R}^n$ , to  
276 *extrema of functionals*, represented by functions allows us to examine, in variational calculus,  
277 much more general problems than in ordinary calculus. Of course, this comes at the expense of a  
278 greater mathematical difficulty. The computation (in some cases analytical but, most of the time,  
279 approximated) of the extrema of a functional  $E$  is called *variational* (or *functional*) *optimization*  
280 and it has been the subject of research by many mathematicians, physicists, and engineers in the  
281 past two centuries (Boyd & Vandenberghe, 2004). Later in this section, we will provide some  
282 justification for the name 'optimization.'

283 A basic tool in variational optimization is the concept of the *first variation* of a functional,  
284 which is a direct generalization of the directional derivative of a function. More precisely, given

285 a functional  $E : \mathcal{F} \rightarrow \mathbb{R}$  and any function  $g \in \mathcal{F}$ , called *perturbation*, the first variation (or  
 286 Gâteaux derivative) of  $E$  along  $g$ , calculated in  $f$ , is defined by

$$\delta E(f, g) = \lim_{\varepsilon \rightarrow 0} \frac{E(f + \varepsilon g) - E(f)}{\varepsilon}, \quad (10)$$

287 here, the perturbation  $g$  plays the role of the vector  $\vec{v}$  in the definition of directional derivative.  
 288 The generalization of Fermat's interior extrema theorem to variational calculus states that the  
 289 first variation of a functional computed in any extreme (argmin or argmax) must be null for  
 290 every perturbation. Moreover, this necessary condition also becomes sufficient under suitable  
 291 convexity hypotheses or when it is associated with some properties of the second variation, i.e.,  
 292 the first variation of the first variation interpreted as a functional.

293 The equations

$$\delta E(f, g) = 0 \quad \forall g \in \mathcal{F}, \quad (11)$$

294 are called Euler-Lagrange equations. For a better comprehension of the variational framework  
 295 that we will develop in sections 4 and 5, it is worthwhile to complete this overview by showing  
 296 explicit examples of functionals given by one or two terms and by explaining why the search  
 297 for their extrema is called optimization. Let us start with the problem of finding the extremal  
 298 function  $y = f(x)$  whose graph gives the shortest curve that connects two points  $(x_1, y_1)$  and  
 299  $(x_2, y_2)$  in  $\mathbb{R}^2$ . To find the variational principle associated with this problem, let us recall that the  
 300 arc length  $A$  of the curve represented by the graph of a differentiable function  $y = f(x)$  between  
 301 the points  $(x_1, y_1 \equiv f(x_1))$  and  $(x_2, y_2 \equiv f(x_2))$  is given by the integral  $\int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2} dx$ . The  
 302 solution to our problem will therefore be the argmin of the functional

$$\begin{aligned} A : C([x_1, x_2]) &\longrightarrow \mathbb{R} \\ f &\longmapsto A(f) = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2} dx. \end{aligned}$$

303 It can be easily proven that, as expected, the argmin of  $A(f)$  is given by the straight line function  
 304  $\underline{f}(x) = mx + b$ , where  $m = (y_2 - y_1)/(x_2 - x_1)$  and  $b = (x_2 y_1 - x_1 y_2)/(x_2 - x_1)$ .

305 In this case, it is very easy to understand why the argmin of  $A(f)$  represents the optimal  
 306 solution to the problem of finding the shortest curve, because of the *direct interpretation* of  $A(f)$   
 307 as the arc length functional. It is more difficult to understand what optimality means when the  
 308 functional associated with a problem is given by two or more terms. To help understand this, let  
 309 us consider the problem of determining, with a variational principle, the trajectory of a particle  
 310 moving into space between time  $t_0$  and time  $t_1$  in a conservative physical system<sup>1</sup>. The Italian-  
 311 French mathematician and physicist Lagrange solved this problem by considering the following  
 312 functional, that nowadays we call *Lagrangian* in his honor:

$$L(\vec{q}) = \int_{t_0}^{t_1} \mathcal{L}(\vec{q}(t)) dt \quad (12)$$

---

<sup>1</sup>A physical system is called conservative if the forces acting on it can be expressed as minus the gradient of a potential energy function  $V$ .

313 where  $\mathcal{L}(\vec{q}) = T(\vec{q}) - V(\vec{q})$ ,  $\vec{q} : [t_0, t_1] \rightarrow \mathbb{R}^3$  being the time-dependent position function of  
314 a moving particle in  $\mathbb{R}^3$  and  $T, V$  being the kinetic and the potential energy functions of the  
315 physical system, respectively.

316 Lagrange's outstanding achievement is that the argmin of the functional in (12) is the function  
317  $\vec{q}(t)$  which solves Newton's equation of motion  $\vec{F}(t) = m\vec{q}(t)$ ,  $\forall t \in [t_0, t_1]$ , where  $\vec{F}$  is the result of  
318 forces acting on the particle and  $\vec{q}(t)$  is the second temporal derivative of  $\vec{q}$  (i.e., its acceleration).  
319 Since it is well-known that the trajectory of a particle in a (non-relativistic) physical system sat-  
320 isfies Newton's second law of dynamics, Lagrange proved that this is equivalent to searching for  
321 the argmin of the functional (12). In other words, the minimization of functional (12) gives the  
322 optimal result because it coincides with the solution of Newton's second law of dynamics. Since  
323  $\mathcal{L}(\vec{q}(t))$ , the integrand function of (12) is given by two terms with opposite signs, i.e., the kinetic  
324 energy  $T(\vec{q}(t)) = \frac{1}{2}m\|\vec{q}(t)\|^2$  and the potential energy  $V(\vec{q}(t))$ , we can interpret Lagrange's result  
325 by saying that in every instant  $t \in [t_0, t_1]$ , the particle moves along a trajectory which minimizes  
326 the difference between the energy that the particle actually has due to its motion (i.e.,  $T$ ) and  
327 the energy that the particle could potentially attain (i.e.,  $V$ ). This interpretation is commonly  
328 summarized by saying that the trajectory of motion of a particle is given by the *optimal balance*  
329 between its kinetic and potential energy in every instant. The balance between them is always  
330 present, but it is optimal to describe the trajectory of motion only for the argmin of the functional  
331 (12). Of course, when we deal with another problem, not necessarily related to motion of parti-  
332 cles or length of curves, optimality will refer, more generally, to the match between the argmin  
333 of the functional with an empirical or theoretical law describing a phenomenon or a property.  
334 Consistent with the nomenclature just recalled, also in those cases we will say that the argmin is  
335 characterized by the optimal balance between the functional terms.

336 An illuminating example in this sense is the very deep variational interpretation of the his-  
337 togram equalization of digital images provided in Sapiro and Caselles (1997) that we will discuss  
338 in section 5.1. We will see that a digital image with an equalized histogram<sup>2</sup> can be interpreted  
339 as the argmin of a functional characterized by the difference between a functional term that de-  
340 scribes adjustment to the middle gray-level of the image and another that gives a global measure  
341 of contrast carried by the image. Coherently with the considerations above, we will say that  
342 histogram equalization is given by the optimal balance between control of the dispersion around  
343 the middle gray and contrast intensification.

344 The possibility of arriving at these highly non-intuitive interpretations of known phenomena  
345 and also to predict new ones is what makes the use of variational calculus so prevalent among  
346 many different disciplines as a sort of unifying principle. This is one of the main reasons why,  
347 in this paper, we analyze the possibility of using variational principles as a bridge between the  
348 description of perceptual and non-physical judgments.

349 *Remark 1:* Optimization of functionals is conventionally associated with the search for their

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<sup>2</sup>We recall that the intensity of a digital image is quantized and bounded in the set of values  $\{0, 1, \dots, 255\}$ . A digital image is said to be equalized if each intensity level has the same occurrence probability in the image.

350 minima. Of course, a minimization problem can be transformed into a maximization one simply  
 351 by changing the sign of the functional under analysis.

352 *Remark 2:* When parameters are involved in the definition of a functional, the interpretation  
 353 of optimality can be a little more difficult. In fact, functional minimization in this case generates  
 354 a whole family of optimal solutions that depend on the selection of the parameters appearing  
 355 in the equations. In these cases, a suitable tuning procedure that may vary from case to case,  
 356 must be used to set the parameters once and for all, thus providing the (only) optimal solution to  
 357 the problem under consideration. We will return to this subtle matter in section 5.1 during the  
 358 variational analysis of histogram equalization.

## 359 4. Variational interpretation of brightness matching

360 In this section, we will discuss an alternative version of the Rudd-Zemach model which  
 361 leads to analogous predictions and that has the advantage of being understandable in terms of  
 362 variational principles. As underlined in section 2, the choice made by Rudd and Zemach to pass  
 363 from eq. (4) to eq. (5) during the description of their WLLR model is questionable. The reason is  
 364 that the weights of spatially local induction  $w_1, w_2, \dots$  should decrease with the distance between  
 365 two point of the visual scene; however, in the decomposition of the chain of ratios in eq. (3),  
 366 the points corresponding to the indexes  $k - 1$  and  $k$  and those corresponding to  $k$  and  $k + 1$  have  
 367 exactly the same distance. Instead, if we keep fixed the target point  $i$  and consider points at an  
 368 increasing distance from it, then it is perfectly correct to consider weights of decreasing strength.

369 These considerations have been thoroughly analyzed in previous works on the interpretation  
 370 of Retinex (Bertalmío, Caselles, & Provenzi, 2009; Provenzi et al., 2005, 2007; Provenzi, Gatta,  
 371 Fierro, & Rizzi, 2008). Starting from this more coherent Retinex interpretation, we propose the  
 372 following alternative definition of logarithmic brightness of  $i$ , denoted with  $\log \Psi(i)$ :

$$\log \Psi(i) \equiv \sum_{k=i}^{j-1} w_{k-i+1} \log \frac{L_i}{L_{k+1}} + \mu, \quad (13)$$

373 where, as for  $\log \Psi(i)$ ,  $\mu \in \mathbb{R}$  is an arbitrary constant. If we compare the formulae of  $\log \Phi(i)$  and  
 374  $\log \Psi(i)$ , we see that in the latter the numerator of each ratio is  $L_i$ , this means that the contribution  
 375 to  $\Psi(i)$  is given by the logarithmic ratios between  $L(i)$  and the luminance of all the other patches  
 376 in the visual field, weighted by the distance between  $i$  and the patches.

377 It is easy to see that this model also predicts a linear relationship between  $\log L_{D_M}^{\text{Match}}$  and  
 378  $\log L_{R_T}$  with slopes in  $(-1, 0)$  for the visual match in Wallach's experiment. In fact, developing  
 379 eq. (13) for both sides of the visual field and matching the brightness, we get:

$$w_1 \log \frac{L_{D_M}^{\text{Match}}}{L_{R_M}} + w_2 \log \frac{L_{D_M}^{\text{Match}}}{L_B} + \mu = w_1 \log \frac{L_{D_T}}{L_{R_T}} + w_2 \log \frac{L_{D_T}}{L_B} + \mu, \quad (14)$$

380 solving this equation w.r.t.  $\log L_{D_M}^{\text{Match}}$  we have

$$\log L_{D_M}^{\text{Match}} = \log L_{D_T} + \frac{w_1}{w_1 + w_2} \log L_{R_M} - \frac{w_1}{w_1 + w_2} \log L_{R_T}. \quad (15)$$

381 We stress that the absolute value of the slope  $w_1/(w_1 + w_2) = \beta_1$  is naturally bounded between  
 382 -1 and 0, so it also accounts for Rudd-Zemach's observations, even without the hypothesis that  
 383  $w_2 < w_1$ . The weight  $w_2$  can be expressed in terms of the measured value of  $\beta_1$  as follows:

$$w_2 = \frac{1 - \beta_1}{\beta_1} w_1. \quad (16)$$

384 If we now add two other rings in both the match and target bipartite field, and again use the  
 385 outermost ring in the target field as an independent variable, then, by direct computation, it can  
 386 be proven that expression of the induction weight  $w_3$  corresponding to this new, and more distant,  
 387 ring predicted by our model is the following:

$$w_3 = \frac{1 - \beta_2}{\beta_2} w_2 - w_1, \quad (17)$$

388 where  $\beta_2 \in (0, 1)$  is the absolute value of the measured slope of the linear relationship in the  
 389 logarithmic domain between the  $L_{D_M}^{\text{Match}}$  and the logarithmic luminance of the new outermost  
 390 target ring. By iterating the process we find the following formula for the  $n$ -th induction weight  
 391 (corresponding to the configuration given by one disk and  $n - 1$  rings):

$$\begin{cases} w_2 = \frac{1 - \beta_1}{\beta_1} w_1 \\ w_n = \frac{1 - \beta_{n-1}}{\beta_{n-1}} w_{n-1} - \sum_{k=1}^{n-2} w_k \quad n \geq 3, \end{cases} \quad (18)$$

392  $\beta_{n-1} \in (0, 1)$  is the absolute value of the measured slope of the linear relationship in the logarithmic  
 393 domain between the  $L_{D_M}^{\text{Match}}$  and the logarithmic luminance of the outermost  $(n - 1)$ -th target  
 394 ring.

395 It is natural to search for a generalization of Rudd-Zemach's model that is valid for arbitrary  
 396 spatial configurations and not just for the special one discussed in their experiments. To do that,  
 397 we will distinguish between a discrete and a continuous context. In a discrete context, we will  
 398 denote the discrete visual field with the lattice  $\Omega \subset \mathbb{Z}^2$ , the coordinates of two arbitrary points  
 399 in  $\Omega$  as  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$ , and the corresponding luminance values as  $L(x)$  and  $L(y)$ ,  
 400 respectively. The equivalent of the logarithmic brightness of  $x$  defined in eq. (13) in this case is:

$$\log \Psi(x) = \sum_{y \in \Omega} w(\|x - y\|) \log \frac{L(x)}{L(y)} + \mu, \quad \text{Discrete context,} \quad (19)$$

401  $w(\|x - y\|)$  being a weight function which decreases with the distance  $\|x - y\|$  and  $\mu$  is an arbitrary  
 402 constant. In a continuous context,  $\Omega$  is a subset of  $\mathbb{R}^2$ , and of course the discrete sum must be  
 403 replaced by an integral:

$$\log \Psi(x) = \int_{\Omega} w(\|x - y\|) \log \frac{L(x)}{L(y)} dy + \mu, \quad \text{Continuous context.} \quad (20)$$

404 The advantage of this formulation, over that of Rudd and Zemach, is that it is possible to provide  
 405 a variational interpretation of formulae (19), (20), as the following proposition states.

406 **Proposition 4.1.** *The achromatic logarithmic brightness  $\Psi(x)$  of an arbitrary point  $x \in \Omega$  is the*  
 407 *argmin of the functional*

$$E_w(\log L) = \frac{1}{2} \sum_{x \in \Omega} \left( \log \frac{L(x)}{\mu} \right)^2 - \frac{1}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} w(\|x - y\|) \left( \log \frac{L(x)}{L(y)} \right)^2, \quad (21)$$

408 *in the discrete scenario, and of the functional*

$$E_w(\log L) = \frac{1}{2} \int_{\Omega} \left( \log \frac{L(x)}{\mu} \right)^2 dx - \frac{1}{4} \iint_{\Omega^2} w(\|x - y\|) \left( \log \frac{L(x)}{L(y)} \right)^2 dx dy. \quad (22)$$

409 *in the continuous scenario.*

410 The proof of this proposition is provided in the Appendix. Here, we are more interested in  
 411 its interpretation. To this aim it is convenient to write the terms appearing in the functional as  
 412 follows:

- 413 •  $D(\log L) \equiv \frac{1}{2} \sum_{x \in \Omega} \left( \log \frac{L(x)}{\mu} \right)^2$ : *discrete quadratic dispersion term,*
- 414 •  $C_w(\log L) \equiv \frac{1}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} w(\|x - y\|) \left( \log \frac{L(x)}{L(y)} \right)^2 dx dy$ : *discrete local quadratic contrast term,*

415 *in the discrete domain, and*

- 416 •  $D(\log L) \equiv \frac{1}{2} \int_{\Omega} \left( \log \frac{L(x)}{\mu} \right)^2 dx$ : *continuous quadratic dispersion term,*
- 417 •  $C_w(\log L) \equiv \frac{1}{4} \iint_{\Omega^2} w(\|x - y\|) \left( \log \frac{L(x)}{L(y)} \right)^2 dx dy$ : *continuous local quadratic contrast term,*

418 *in the continuous domain.*

419 As we specified in the introduction, in order to avoid confusion, we chose to use the word  
 420 dispersion instead of assimilation because of the different meanings that the latter term can have  
 421 in perception research (in particular brightness perception) and in the cognition field. From  
 422 a general point of view, in our work the term ‘dispersion’ is employed to mean a tendency  
 423 that simply weakens contrast (as it happens in the Rudd-Zemach induction model) and makes  
 424 the target appear more like the other members of its contextual set. The key observation is  
 425 that the minimization of  $E_w(\log L) = D(\log L) - C_w(\log L)$  is reached through the *simultaneous*  
 426 *minimization of  $D(\log L)$  and maximization of  $C_w(\log L)$*  (because of the minus sign appearing in  
 427 front of the contrast term).  $D(\log L)$  is minimized for values of  $L(x)$  close to  $\mu$ , and this holds for



428 all  $x \in \Omega$ . The numerical value of  $\mu$  is not important; what really matters is that  $\mu$  is a constant,  
 429 thus we infer that minimizing  $D(\log L)$  corresponds to inducing a uniform stimulus, which has  
 430 the same value in every point  $x$ . This is the reason why we called  $D(\log L)$  the dispersion term:  
 431 If only this term existed, then all the stimuli coming from different points of the scene would  
 432 be blended into a unique stimulus, uniform across the visual field. Of course, this is not what  
 433 happens when we look at a scene, so there must be an opposing mechanism to dispersion, which  
 434 is defined precisely by the action of maximizing  $C_w(\log L)$ . In fact,  $C_w(\log L)$  is maximized  
 435 when the terms of the integral, i.e.,  $(\log \frac{L(x)}{L(y)})^2 = (\log L(x) - \log L(y))^2$  are intensified, but that  
 436 means that the differences  $\log L(x) - \log L(y)$  must be amplified as much as possible, which, in  
 437 turn, corresponds to maximizing the contrast of the image content. The presence of the weight  
 438 function  $w(\|x-y\|)$  guarantees that contrast amplification respects the locality of visual perception  
 439 in the sense discussed above, which is a context-driven effect. All these considerations explain  
 440 why we named  $C_w(\log L)$  local contrast.

441 To summarize, the logarithmic brightness values  $\Psi(x)$  can be interpreted as being *the optimal*  
 442 *balance between two opposite mechanisms*: one that tends to adjust all stimuli to a constant,  
 443 uniform value, and the other that tends to do the opposite, i.e., to amplify as much as possible the  
 444 differences among all stimuli in a local, or context-driven, way. The results just obtained have  
 445 many similarities with the theory of perceptually-inspired color correction (Bertalmío, Caselles,  
 446 Provenzi, & Rizzi, 2007; Bertalmío, Caselles, & Provenzi, 2009; Palma-Amestoy, Provenzi,  
 447 Bertalmío, & Caselles, 2009; Provenzi & Caselles 2014). The functionals considered in these  
 448 papers balance dispersion and local contrast enhancement to perform perceptual color correction,  
 449 i.e., to modify the intensities of pixels in digital images to approach the sensation produced by  
 450 the real-world scene.

## 451 5. A general variational framework for context-driven effects

452 In this section, we generalize the concepts introduced in the previous section. The varia-  
 453 tional interpretation of brightness matching refers to the simultaneous contrast phenomenon as  
 454 initially investigated by Wallach. Until now, we have taken into account the influence of con-  
 455 text on psychological judgments along the physical dimension of luminance and for matching  
 456 experiments. However, we think that the same analysis may also hold for judgments along non-  
 457 physical dimensions, such as beauty or morality, and not necessarily for matching experiments.  
 458 The aim of this section is to provide an abstract framework that can be adapted to specific ex-  
 459 periments through an appropriate specification of parameters. To guarantee a sufficient degree of  
 460 abstraction and versatility of the setting, we introduce the following nomenclature:

- 461 •  $(\Omega, \|\cdot\|)$ : normed space describing the context. The norm  $\|\cdot\| : \Omega \rightarrow \mathbb{R}^+$  is needed to  
 462 measure distances between elements of  $\Omega$ , and thus to handle local phenomena. Typically  
 463  $\Omega$  is a subset of  $\mathbb{Z}^n$  in the discrete case, and of  $\mathbb{R}^n$ ,  $n \geq 1$ , for continuous contexts, and  $\|\cdot\|$   
 464 is the discrete or Euclidean norm, respectively, but nothing prevents  $\Omega$  and  $\|\cdot\|$  from being  
 465 a more complicated space and norm. We denote by  $x, y, \dots$  the elements of  $\Omega$  and with

- 466  $|\Omega|$  its cardinality, area, volume, or hyper-volume, depending on its discrete or continuous  
467 structure and mathematical dimension, respectively;
- 468 •  $\mathcal{S} : \Omega \rightarrow \mathbb{R}$ : stimulus function, such that  $\mathcal{S}(x)$  represents the stimulus coming from the  
469 element  $x \in \Omega$ ;
  - 470 •  $\mathcal{F}(\Omega)$ : functional space where  $\mathcal{S}$  belongs. It can be, e.g.,  $C(\Omega)$ , the space of scalar-valued  
471 continuous functions on  $\Omega$ ,  $L^2(\Omega)$ , the square integrable (or finite energy) of scalar-valued  
472 functions on  $\Omega$ , and so on, depending on the problem under analysis;
  - 473 •  $w : \Omega \times \Omega \rightarrow \mathbb{R}^+$ : induction weight function,  $w(x, y)$  and  $w(y, x)$  represent the strength of  
474 the influence of  $x$  on  $y$  and of  $y$  on  $x$ . If these strengths are equal, then we also require  
475  $w$  to be a symmetrical function, i.e.,  $w(x, y) = w(y, x)$  for all  $x, y \in \Omega$ .  $w$  is typically a  
476 decreasing function of the distance  $\|x - y\|$ , to keep into account the fact that closer stimuli  
477 (with respect to the norm  $\| \cdot \|$ ) influence each other more strongly than distant ones.

478 Table 1 provides the identification of the objects of the variational framework just defined in  
479 the concrete case of two psychological experiments concerning judgments along physical and  
480 non-physical dimensions.

	<b>Physical judgment (Brightness matching)</b>	<b>Non-physical judgment (Seriousness of an action)</b>
$x$	Pixel position on the screen	Time at which a question is formulated
$\Omega$	Screen where stimuli are presented	Real line $\mathbb{R}$ of instants of time
$\  \cdot \ $	Euclidean norm between pixels	Absolute value of the difference between two temporal moments
$\mathcal{S}(x)$	Luminous intensity of the stimulus in $x$	Question about seriousness of an action asked at the time $x$
$\mathcal{F}(\Omega)$	Space of all possible luminous stimuli	Space of all possible questions about moral seriousness

Table 1: Identification of the objects defined in the variational framework in the case of a psychological experiment along a physical (left) and a non-physical (right) dimension.

481 As discussed in the previous section, the general functional associated with a context-driven  
482 effect must be characterized by the balance between two opposing mechanisms: dispersion and  
483 contrast. In the case studied in the previous section, the dispersion is *global*, i.e., the weight  
484 function  $w$  is constant for every couple of points  $x, y \in \Omega$  but, for the sake of a more general  
485 framework, we will also allow the dispersion term to be *local*, i.e., to have a weight function  
486  $w$  which decreases with the distance between  $x$  and  $y$ . As shown in previous works (Palma-  
487 Amestoy et al., 2009), one must also add an *attachment-to-original-data constraint* that consists  
488 of an adjustment to the original stimuli values. This helps to guarantee that the context-driven

489 perceptual or cognitive values of the stimuli do not excessively depart from the original ones, and  
 490 it also helps the convergence of iterative numerical schemes for the approximation of the argmin  
 491 of the functional. To better understand the exigence of an attachment-to-original-data constraint,  
 492 we can think about the case of color perception of a surface under different light conditions,  
 493 e.g., neon and tungsten lamp. While the human visual system is able to partially discount the  
 494 noticeable spectral difference between these two illuminants, we still perceive a dominant bluish  
 495 color under neon light and a reddish color under the tungsten light, which is symptomatic of an  
 496 attachment to the original stimuli.

497 We thus introduce in our framework the three following functionals:

- 498 •  $B : \mathcal{F}(\Omega) \rightarrow \mathbb{R}$ : *attachment-to-original-stimuli functional*. The effect of its minimization  
 499 is to preserve the original values of the stimuli;
- 500 •  $C_{w_c} : \mathcal{F}(\Omega) \rightarrow \mathbb{R}$ : *contrast amplification functional*. The effect of its minimization is to  
 501 enhance the differences between any two stimuli  $\mathcal{S}(x)$ ,  $\mathcal{S}(y)$ , with  $x \neq y$ . The presence of  
 502 the weight function  $w_c$  allows contrast amplification to be context-dependent;
- 503 •  $D_{w_d} : \mathcal{F}(\Omega) \rightarrow \mathbb{R}$ : *dispersion functional*. The effect of its minimization is to adjust the  
 504 values of the stimuli in the direction of the average value of the context, which can be  
 505 global  $\mu$  (and in this case  $w_d$  is constant), or local  $\mu(x)$  (and in this case the weight function  
 506  $w_d$  is not constant);
- 507 • Moreover, we *require the minimization of these functionals to give rise to dimensionally*  
 508 *coherent equations*, where ‘dimension’ here means the unity of measurement associated  
 509 with stimuli.

510 The simultaneous presence of the three effects discussed above can be represented by the follow-  
 511 ing functional, given by a linear combination of the three previous ones:

$$E_{w_c, w_d, b, c, d}(\mathcal{S}) = bB(\mathcal{S}) + cC_{w_c}(\mathcal{S}) + dD_{w_d}(\mathcal{S}) \quad b, c, d \in \mathbb{R}, \quad (23)$$

512  $b$  is the attachment-to-original-data strength,  $c$  is the contrast strength, and  $d$  is the dispersion  
 513 strength. It is important to stress that the argmin of  $E_{w_c, w_d, b, c, d}(\mathcal{S})$  can directly represent the per-  
 514 ceived stimulus but it can also be used for brightness matching experiments, as discussed in  
 515 section 4. In this last case, once the stimulus to be matched is identified, the Euler-Lagrange  
 516 equations corresponding to the argmin must be matched and solved with respect to the perceived  
 517 quantity that one wants to measure. We also stress that the simultaneous presence of disper-  
 518 sion and contrast is essential: without the dispersion term, the contrast process would increase  
 519 without boundaries and vice-versa, if only the dispersion term were allowed to work without the  
 520 compensation of the contrast one, then the stimulus would be absorbed in the context and the  
 521 final percept would become trivial.

522 In the perception and social cognition literature there is no consensus on circumstances that  
 523 lead to either one of the two effects (Gilchrist, 2015). Many variables can affect the resulting

524 percept, as described previously in section 1. By changing parameters and the form of the func-  
525 tional, our approach is able to incorporate different assumptions about the variables involved in a  
526 particular experiment. Among the various theoretical ideas within perception, gestalt laws may  
527 provide (at least) some clues about how and when an observer will experience dispersion or con-  
528 trast (Agostini & Galmonte, 2002; Koffka, 1935; Wertheimer, 1923). According to Agostini and  
529 Galmonte (2002), when two items belong to different perceptual groups, a contrast effect will  
530 be observed. On the contrary, when an item is organized into one or the other of two perceptual  
531 groups, a dispersion effect will be observed. Such a mechanism is (at least partially) in line with  
532 social cognition studies about variables that affect context effects.

533 There are reasons to believe that the equilibrium between dispersion and contrast is also  
534 involved in non-physical judgments. One example is given by the judgment of attractiveness:  
535 an observer is exposed to a face that is considered average-looking in isolated conditions, then  
536 he/she has to judge the face again in the presence of either an attractive or an unattractive face  
537 that plays the role of the context effect. Both dispersion and contrast effects have been reported  
538 (Kenrick & Gutierrez, 1980). The same happens for a single attractive or unattractive face acting  
539 as context. For example, in the appearance of a face there can be some attractive traits (e.g.,  
540 the eyes, the form of the face and so on), some average-looking traits (e.g., the lips), and some  
541 unattractive characteristics (e.g., a scar, or other imperfections). The dispersion term indicates  
542 the tendency to create a holistic and uniform perception of face attractiveness, reducing the differ-  
543 ences that may be present. At the same time, contrast maximization implies that the differences  
544 in attractiveness of the different features of a face are enhanced. Social cognition literature about  
545 contrast and dispersion effects (Biernat, 2005; Stapel & Koomen, 1998) underlines the role of the  
546 context as the backdrop against which the stimuli is experienced. More specifically, the impor-  
547 tance of reducing the within-stimulus and the within-context differences is often stressed in order  
548 to make the comparison. Stereotypes (Biernat & Kobrynowicz, 1997; Stapel & Koomen, 1998)  
549 as well as social standards (Higgins, 1987, 1990) are unidimensional and uniform representa-  
550 tions of attitudes and traits of a given person or a group. Famous effects such as the halo effect  
551 (Nisbett & Wilson, 1977), the focusing effect (Schkade & Kahneman, 1998), and anchoring bi-  
552 ases (Tversky & Kahneman, 1974) are based on the tendency to rely very strongly on a single  
553 piece of information at the expense of other elements. Again, such effects rely on the idea that  
554 the within-differences of the single piece of information used as a criterion are minimized. Such  
555 considerations testify to the importance of the dispersion term in our variational model in order  
556 to perceive stimuli as uniform as possible to make a comparison; concurrently, this comparison  
557 is enhanced by the contrast effect.

558 To summarize, the proposed setting provides a view from above that helps put into evidence  
559 and formalize the two conflicting actions of dispersion and contrast in each judgment (either  
560 psychophysical measures in physical domains or non-physical judgments) whose balance deter-  
561 mines the final percept. In the following subsections, we will provide significant examples of  
562 functionals of the kind just described.

## 5.1. Examples of functionals $E_{w_c, w_d, b, c, d}(\mathcal{S})$ already present in the literature

We start the description of a functional of the type (23) already present in the literature with an important example of a functional that mixes dispersion and contrast. Even though it does not represent a context-driven effect, we have chosen to begin with it because it is the starting point for the development of the context-driven functionals that we will discuss afterwards.

The so-called Caselles-Sapiro functional, proposed by Sapiro and Caselles (1997), provides a highly non-intuitive variational interpretation of histogram equalization. Whenever we have the histogram of a variable, its equalization forces the histogram to be flat, so that all the realizations of that variable have the same probability of occurrence. The very remarkable finding of these authors was that, using the notations introduced above, and considering normalized stimuli  $\mathcal{S}(x) \in [0, 1]$ , the argmin of the following functional

$$E_{\text{Hist. Eq.}} = 2 \sum_{x \in \Omega} \left( \mathcal{S}(x) - \frac{1}{2} \right)^2 - \frac{1}{|\Omega|} \sum_{x \in \Omega} \sum_{y \in \Omega} |\mathcal{S}(x) - \mathcal{S}(y)| \quad (24)$$

in the discrete setting, and

$$E_{\text{Hist. Eq.}} = 2 \int_{\Omega} \left( \mathcal{S}(x) - \frac{1}{2} \right)^2 dx - \frac{1}{|\Omega|} \iint_{\Omega^2} |\mathcal{S}(x) - \mathcal{S}(y)| dx dy \quad (25)$$

in the continuous setting, has an equalized histogram. To understand this result, notice that the first term of  $E_{\text{Hist. Eq.}}$  can be interpreted as a dispersion functional with respect to the average value between the extremes 0 and 1 taken by  $\mathcal{S}(x)$ , while the second term can be interpreted as a (non context-dependent) contrast functional. In fact, the first term is clearly minimized by a constant stimulus equal to 1/2, while the second term, due to the presence of the minus sign in front of it, is minimized when the absolute value of the differences between two stimuli are maximized. In light of this result, an equalized stimulus is characterized by the equilibrium between being uniform and having as much diversity as possible, which is a highly non-trivial result.<sup>3</sup>

This profound finding has been used in previous works (Bertalmío et al., 2007, 2009; Palma-Amestoy et al., 2009) to model context-driven effects in color vision and it has been applied to the enhancement of color digital images. In this case,  $\mathcal{S}$  is the intensity function of an image in each separated chromatic channel RGB,  $I : \Omega \rightarrow [0, 1]$ , where  $\Omega$  is the spatial support of the image and  $[0, 1]$  is the normalized dynamic range of pixel intensities. The first functional inspired by the results of Caselles and Sapiro appeared in Bertalmío et al. (2007) and can be written as follows:

$$E_1(I) = d \sum_{x \in \Omega} \left( I(x) - \frac{1}{2} \right)^2 + \sum_{x \in \Omega} (I(x) - I_0(x))^2 - c \sum_{x \in \Omega} \sum_{y \in \Omega} w(x, y) S(I(x) - I(y)), \quad (26)$$

<sup>3</sup>We stress that in Sapiro and Caselles (1997) the authors show that the factor 2 in front of the dispersion term is the only choice of the weight parameter that gives rise to histogram equalization. When we deal with functionals relative to perceptual or cognitive experiments, the parameters setting must be performed by a suitable tuning procedure that may vary from case to case in order to find the optimal solution to our problem.

591 in the discrete setting, and

$$E_1(I) = d \int_{\Omega} \left( I(x) - \frac{1}{2} \right)^2 dx + \int_{\Omega} (I(x) - I_0(x))^2 dx - c \iint_{\Omega^2} w(x, y) S(I(x) - I(y)) dx dy. \quad (27)$$

592 in the continuous setting, where the function  $S$  an antiderivative for a sigmoid. It can be seen  
 593 that all the functional terms mentioned above appear in  $E_1(I)$ : The first term is the dispersion to  
 594 the average intensity level between 0 and 1, the second is the adjustment to the original intensity  
 595 values  $I_0(x)$ , and the third is a nonlinear and context-driven contrast term. The nonlinearity is  
 596 due to the presence of  $S$ , which is introduced to mimic some peculiar properties of the human  
 597 visual system (see Bertalmío et al., 2007 for more detail).

598 One of the most important characteristics of human vision is its robustness with respect to  
 599 changes of illumination<sup>4</sup> (Fairchild, 2005; Land & McCann, 1971); this property is not reflected  
 600 by the analytical expression of the functional  $E_1(I)$ , which is not invariant with respect to trans-  
 601 formations such as  $I \mapsto \lambda I$ . To remedy this problem, in Palma-Amestoy et al. (2009), another  
 602 kind of functional was considered, namely:

$$E_2(I) = d \sum_{x \in \Omega} \left( \mu \log \frac{\mu}{I(x)} - (\mu - I(x)) \right) + \sum_{x \in \Omega} \left( I_0(x) \log \frac{I_0(x)}{I(x)} - (I_0(x) - I(x)) \right) \\ + c \sum_{x \in \Omega} \sum_{y \in \Omega} w(x, y) \varphi \left( \frac{\min(I(x), I(y))}{\max(I(x), I(y))} \right) \quad (28)$$

603 in the discrete setting, and

$$E_2(I) = d \int_{\Omega} \left( \mu \log \frac{\mu}{I(x)} - (\mu - I(x)) \right) dx + \int_{\Omega} \left( I_0(x) \log \frac{I_0(x)}{I(x)} - (I_0(x) - I(x)) \right) dx \\ + c \iint_{\Omega^2} w(x, y) \varphi \left( \frac{\min(I(x), I(y))}{\max(I(x), I(y))} \right) dx dy \quad (29)$$

604 in the continuous setting, where the functional parameter  $\varphi$  is a strictly increasing positive func-  
 605 tion.

---

<sup>4</sup>Taking into account the cognitive domain, it is possible to draw a parallel between such robustness to changes of illumination with the well-known hedonic treadmill theory (Brickman & Campbell, 1971; Eysenck, 1990). This theory is based on the observed tendency of a quick restoration to a relatively stable level of happiness after the experience of major positive or negative life events (Brickman & Campbell, 1971; Diener, Lucas, & Scollon, 2006; Fujita & Diener, 2005; Lykken & Tellegen, 1996; Silver, 1982; Wildeman, Turney, & Schnittker 2014). For example, interviewing lottery winners and paraplegics in order to assess their happiness levels due to the positive (winning the lottery) or negative (becoming paralyzed) event, it was observed that after a few years, both groups returned to their baseline level of happiness (Brickman & Campbell, 1971). As a matter of fact, the hedonic treadmill metaphor explicitly refers to adaptation processes in sensory domains. Although much evidence supports this theory, there is an ongoing debate about how to measure happiness, the role of individual differences, and the differential impact of particular events on happiness (Frederick, 2007).

606 Let us discuss the meaning of the terms in  $E_2$ , starting with the last one. First of all, notice that  
607 the ratio defined by  $\min(I(x), I(y)) / \max(I(x), I(y))$  is minimized when the minimum decreases  
608 and the maximum increases, which of course corresponds to a contrast amplification. Moreover,  
609 the fraction is invariant with respect to transformations like  $I \mapsto \lambda I$ . In order to be dimensionally  
610 coherent, the quadratic dispersion and attachment can no longer be used here, thus the authors  
611 selected the so-called *entropic* dispersion and attachment terms; the first and the second appear-  
612 ing in (29), respectively. Given the statistical interpretation of entropy, minimizing the first terms  
613 amounts to minimizing the variability of intensity levels around the average  $\mu$  and around the  
614 original data  $I_0(x)$ , which is what is expected from the dispersion and attachment terms. The  
615 advantage, with respect to the quadratic terms, is that the first derivatives of the entropic func-  
616 tionals are dimensionally coherent with those coming from the contrast term. We refer again to  
617 Palma-Amestoy et al. (2009) for more details. We hope that these examples may help the reader  
618 understand how the contrast and dispersion terms can vary according to the particular features  
619 exhibited by a problem.

## 620 **5.2. Variational interpretation of linear relationships in the linear and** 621 **logarithmic scale for physical and non-physical dimensions**

622 The argmin of functionals of the type (23), in the majority of cases, is expressed via an  
623 implicit equation that cannot be solved analytically. However, for some *particularly simple,*  
624 *yet already significant,* expressions of the functional terms, the argmin can be expressed via a  
625 linear equation that can be directly compared with actual observations through a linear fit. It is  
626 particularly important to analyze these situations, because they are, by far, the most common in  
627 psychology. In particular, let us distinguish between linear behavior in the linear and logarithmic  
628 scale. A linear behavior in the linear scale is simply expressed by the law:  $u = \alpha + \beta v$ , where  $u$   
629 and  $v$  are abstract variables and  $\alpha$  and  $\beta$  are coefficients. Linear relationships like the previous one  
630 are in general associated with quadratic functionals. To avoid repeating functional expressions,  
631 we will consider only the discrete formulation, as the continuous version can be easily obtained  
632 by replacing sums with integrals.

633 Let us start by considering the following functional:

$$E_{a,b,c}(\mathcal{S}) = \frac{d}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 + \frac{b}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mathcal{S}_0(x))^2 - \frac{c}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2, \quad (30)$$

634 where  $\mathcal{S}(x)$  is an abstract stimulus,  $\mathcal{S}_0(x)$  is the actual stimulus and

$$\mu = \frac{1}{|\Omega|} \sum_{x \in \Omega} \mathcal{S}(x), \quad (31)$$

635 is the constant value of an average background in which  $\mathcal{S}_0(x)$  is embedded. A practical example  
636 of such a configuration is when  $\mathcal{S}$  is a visual stimulus and  $x \in \Omega$  is a time variable, if the  
637 intervals  $x_n, x_{n+1}$  between the  $n$ -th and the  $(n + 1)$ -th presentations of the visual stimuli are long  
638 enough, then no interaction between  $\mathcal{S}(x_n)$  and  $\mathcal{S}(x_{n+1})$  is expected. Thus, only the interaction

639 between  $\mathcal{S}(x)$ ,  $x \in \Omega$  and the background  $\mu$  is expected. Notice that in this particular case, the  
 640 difference between dispersion and contrast is represented simply by the opposite sign in front of  
 641 the corresponding terms and by the two strengths  $a \neq c$ , which must be different to avoid the  
 642 trivial functional represented only by the attachment-to-data term.

643 The basic variable in this configuration is the difference  $\mathcal{S}(x) - \mu$ , which can express either  
 644 dispersion or contrast. Thus, as we have done in section 4, if we compute the Euler-Lagrange  
 645 equations of the functional (30) and express the argmin  $\underline{\mathcal{S}}(x)$  in terms of  $\mathcal{S}(x) - \mu$  as follows:

$$\underline{\mathcal{S}}(x) = \mathcal{S}_0(x) + \frac{d}{b} (\mathcal{S}(x) - \mu) - \frac{c}{b} (\mathcal{S}(x) - \mu), \quad (32)$$

646 then we can interpret  $\underline{\mathcal{S}}(x)$  as the stimulus corresponding to the optimal balance between contrast,  
 647 dispersion, and attachment to the original stimulus  $\mathcal{S}_0(x)$ .  $\underline{\mathcal{S}}(x)$  can be rearranged as follows:

$$\underline{\mathcal{S}}(x) = \mathcal{S}_0(x) + \frac{c-d}{b} \mu - \frac{c-d}{b} \mathcal{S}(x). \quad (33)$$

648 Eq. (33) expresses a linear relationship, as much as eq. (15). However, in eq. (33) it is clear that  
 649 the slope of the linear relationship, i.e.,  $-(c-d)/b$  is negative if the contrast between  $\mathcal{S}(x)$  and  
 650 background prevails over the adjustment, i.e.,  $c > d$ , whereas the slope is positive when  $d > c$ ,  
 651 i.e., if adjustment between  $\mathcal{S}(x)$  and the background overcomes the contrast effect.

652 The great generality of variational principles allows us to easily extend the previous configura-  
 653 tion to a new one, where we allow an interaction, global or local, among the  $\mathcal{S}(x)$ s. Let us start  
 654 by adding a *global quadratic interaction*:

$$\begin{aligned} \tilde{E}_{b,c_1,c_2,d}(\mathcal{S}) &= \frac{d}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 + \frac{b}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mathcal{S}_0(x))^2 \\ &\quad - \frac{c_1}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 - \frac{c_2}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} (\mathcal{S}(x) - \mathcal{S}(y))^2, \end{aligned} \quad (34)$$

655 where the contrast strengths  $c_1, c_2$  can be equal or different. The argmin of this functional is  
 656 given by the following equation:

$$\underline{\mathcal{S}}(x) = \mathcal{S}_0(x) + \frac{d}{b} (\mathcal{S}(x) - \mu) - \frac{c_1}{b} (\mathcal{S}(x) - \mu) - \frac{c_2}{b} \sum_{y \in \Omega} (\mathcal{S}(x) - \mathcal{S}(y)) \quad (35)$$

657 which, taking into consideration eq. (31), can be rearranged in this way:

$$\underline{\mathcal{S}}(x) = \mathcal{S}_0(x) + \frac{c_1 - d + c_2 |\Omega|}{b} \mu - \frac{c_1 - d + c_2 |\Omega|}{b} \mathcal{S}(x). \quad (36)$$

658 The generalization, which takes into account a *local quadratic interaction*, is provided by these  
 659 functionals:

$$\begin{aligned} \tilde{E}_{b,c_1,c_2,d,w}(\mathcal{S}) &= \frac{d}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 + \frac{b}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mathcal{S}_0(x))^2 \\ &\quad - \frac{c_1}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 - \frac{c_2}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} w(\|x - y\|) (\mathcal{S}(x) - \mathcal{S}(y))^2. \end{aligned} \quad (37)$$



660 The argmin equation in this case is:

$$\underline{\mathcal{S}}(x) = \mathcal{S}_0(x) + \frac{d}{b} (\mathcal{S}(x) - \mu) - \frac{c_1}{b} (\mathcal{S}(x) - \mu) - \frac{c_2}{b} \sum_{y \in \Omega} w(\|x - y\|) (\mathcal{S}(x) - \mathcal{S}(y)) \quad (38)$$

661 which, again, can be re-arranged as follows:

$$\underline{\mathcal{S}}(x) = \mathcal{S}_0(x) + \frac{c_1 - d + c_2}{b} \mu + \frac{c_2}{b} \sum_{y \in \Omega} \mathcal{S}(y) - \frac{c_1 - d + c_2 \sum_{y \in \Omega} w(\|x - y\|)}{b} \mathcal{S}(x). \quad (39)$$

662 For the sake of completeness, let us observe that interaction among stimuli can also be more  
 663 complicated than the quadratic one: A common example is given by the so-called *logistic* or  
 664 *sigmoid-like* interaction, in which the basic contrast variable is represented by  $\sigma(\mathcal{S}(x) - \mathcal{S}(y))$ ,  
 665 where  $\sigma$  is a sigmoid. If  $\mathbb{S}$  is an antiderivative of  $\sigma$ , then the functional corresponding to a  
 666 sigmoid-like interaction is the following:

$$\begin{aligned} \tilde{E}_{b,c_1,c_2,d,\mathbb{S}}(\mathcal{S}) &= \frac{d}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 + \frac{b}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mathcal{S}_0(x))^2 \\ &\quad - \frac{c_1}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 - \frac{c_2}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} \mathbb{S}(\mathcal{S}(x) - \mathcal{S}(y)), \end{aligned} \quad (40)$$

667 in the case of global interaction, and

$$\begin{aligned} \tilde{E}_{b,c_1,c_2,d,w,\mathbb{S}}(\mathcal{S}) &= \frac{d}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 + \frac{b}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mathcal{S}_0(x))^2 \\ &\quad - \frac{c_1}{2} \sum_{x \in \Omega} (\mathcal{S}(x) - \mu)^2 - \frac{c_2}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} w(\|x - y\|) \mathbb{S}(\mathcal{S}(x) - \mathcal{S}(y)), \end{aligned} \quad (41)$$

668 in the case of a local interaction. The analysis of the Euler-Lagrange equations in this case is  
 669 more involved than in the previous case.

670 The second type of linear behavior that we want to model with variational principles is the  
 671 one in the logarithmic scale, expressed by the law:  $\ln u = \alpha + \beta \log v$ . This relationship expresses  
 672 a *power law in the linear scale*; in fact, the exponentiation of both sides of the previous equation  
 673 gives  $u = e^\alpha e^{\beta \log v} \equiv \bar{\alpha} e^{\log v^\beta} = \bar{\alpha} v^\beta$ . To write the functionals from (30) to (37) in the logarithmic  
 674 domain, one simply has to perform the change of variable  $\mathcal{S} \mapsto \tilde{\mathcal{S}} \equiv \log \mathcal{S}$  and solve the corre-  
 675 sponding Euler-Lagrange equations with respect to the logarithmic variables. In particular, it is  
 676 interesting to re-write the functional (37) in the logarithmic domain:

$$\begin{aligned} \tilde{E}_{b,c_1,c_2,d,w}(\tilde{\mathcal{S}}) &= \frac{d}{2} \sum_{x \in \Omega} \left( \log \frac{\mathcal{S}(x)}{\mu} \right)^2 + \frac{b}{2} \sum_{x \in \Omega} \left( \log \frac{\mathcal{S}(x)}{\mathcal{S}_0(x)} \right)^2 \\ &\quad - \frac{c_1}{2} \sum_{x \in \Omega} \left( \log \frac{\mathcal{S}(x)}{\mu} \right)^2 - \frac{c_2}{4} \sum_{x \in \Omega} \sum_{y \in \Omega} w(\|x - y\|) \left( \log \frac{\mathcal{S}(x)}{\mathcal{S}(y)} \right)^2, \end{aligned} \quad (42)$$

677 if we set  $d = c_2 \equiv 1$ ,  $b = c_1 \equiv 0$  and we use the functional for a matching experiment, then  
678 we obtain exactly the same variational framework used to interpret Rudd-Zemach's model of  
679 brightness induction.

680 We conclude by noting that the ability to interpret linear equations in the linear and logarithmic  
681 domain via variational principles is of paramount importance; in fact, as we have seen in this  
682 section, the linear coefficients can be directly interpreted as the difference between the strength  
683 of dispersion and contrast in the particular phenomenon under analysis. Moreover, possible deviations  
684 from linear behavior can easily be integrated in the variational model with the introduction  
685 of nonlinear perturbation terms in the analytical expression of the functional. Finally, let us notice  
686 that, if we deal with a nonphysical dimension, all the previous considerations still work by  
687 replacing the stimulus  $\underline{S}$  with a judgment function and the average value of the background  $\mu$   
688 with a context  $K$ . Thus, in this case, a linear relationship between the judgment function  $J$  and  
689 the context  $K$  can be written as follows:

$$J(x) = \alpha + \beta K. \quad (43)$$

690 As we have seen above, depending on the analytical expression of the functional terms, we can  
691 predict the sign and the bound the value of the slope of the linear relationship. In fact, if the  
692 empirical measurements of a direct judgment experiment fit with eq. (33) with  $\mu$  replaced by  
693  $K$ , then a measured negative slope means that contrast enhancement prevails with respect to  
694 dispersion control, while a measured positive slope means the opposite. Furthermore, if the  
695 empirical measurements of a matching judgment experiment can be described by the argmin of  
696 functionals of the type (37) or (42), with  $d = c_2 \equiv 1$ ,  $b = c_1 \equiv 0$ , then we can predict that the  
697 absolute value of the slope will be bounded between 0 and 1, as discussed in section 4. Notice  
698 that a null value for the coefficient in front of a functional term means that the term does not have  
699 an influence on the global functional, and hence on the final percept; instead, a unitary value for  
700 the coefficient means that the term is accounted for its intrinsic strength, without being weighted.  
701 The great versatility of variational principles allows, in any case, the possibility of changing these  
702 coefficients to account for different theoretical proposals.

## 703 **6. The empirical case for the contextual variational frame-** 704 **work**

705 In this section, we provide a preliminary validation of the predictions of our variational frame-  
706 work with respect to the balance between dispersion and contrast. We will particularly focus on  
707 eq. (33) and we will assume that functionals of the type (37), with  $d = c_2 \equiv 1$ ,  $b = c_1 \equiv 0$ ,  
708 can be used for a variational description of the experiment under analysis. Two issues that need  
709 to be addressed are: (i) contextual phenomena can be modeled by a linear relationship even for  
710 non-physical judgments; (ii) the slope of the linear relationship is bounded between 0 and 1 when  
711 dispersion control prevails and between -1 and 0 when contrast enhancement is stronger.

## 6.1. Constraints about the slope of the linear relationship between context and judgment

The literature on context effects related to non-physical dimensions judgments is vast (Biernat, 2005; Kenrick & Gutierrez, 1980; Matthews & Stewart, 2009b; Moskowitz, 2005; Parducci, 1968; Parker et al., 2008; Pepitone & DiNubile, 1976; Stapel et al., 1997). A review of the literature on context effects in social cognition or cognitive psychology is beyond the scope of this article. Moreover, an actual check of the predictions is not always feasible because in many papers there are not enough data to verify such predictions. Indeed, only a limited number of papers, which deal with a linear model, reported the slope of the contextual effect or, alternatively, at least two points related to the same judgments in different contexts (usually mean values measured on the same scale). In the last case, of course, the presence of an actual linear relationship is questionable. Despite these difficulties, an unsystematic and limited check of the literature tends to support the variational framework. Papers that allow us to verify whether the constraints previously discussed related to eq. (43) hold have usually measured a particular dimension on a Likert scale of a stimulus  $S$ .

*Disclaimer:* the application of statistical or mathematical models to the responses collected on a Likert scale is a common but flawed procedure in the social sciences. For the sake of completeness, in this section we will provide several quantitative examples in line with this procedure. However, we stress that the comparison between the *ordinal values* that constitute the Likert scale and the *numerical* discrete or continuous values of a mathematical model is still an open problem in the social sciences (Camparo, 2013; Carifio & Perla, 2007; Casacci & Pareto, 2015; Wegener, 1982; Young, 1975). Future works should address explicitly the problem of how a monotone transformation of the non-physical judgments could affect the comparison with the variational model predictions; among the different strategies, a possible solution can be to employ the optimal scaling approach to change the distances between the categories until a particular optimum is reached (Takane, 2005). In order to provide a validation that is free from the Likert scale issue, we will describe the study of Matthews and Stewart (2009b) which deals with the psychophysics of an object's prices (White & Vilmain, 1986). In this case, even if the participants are called to produce a non-physical judgment, the responses are not collected by means of a Likert scale.

Let us start our analysis by considering a hypothetical example: The stimulus  $S$  could be the picture of a person and the dimension could be the attractiveness of the person. Taking the mean (or median) value of the observed responses in a sample, it is possible to obtain the judgment  $J$  (in isolation or in different contexts). Contextual information  $K$  consists of stimuli belonging to the same class of  $S$ , but characterized by larger or smaller values of the same dimension (e.g., a very beautiful person or a very ugly person). So, in a non-physical judgment experiment, participants are asked to judge a stimulus  $S$  and a corresponding context  $K$ .<sup>5</sup> Usually, there are only two

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<sup>5</sup>It is important to note that, in non-physical judgments, the context  $K$  must also be judged in order to be defined. This is not the case for psychophysical experiments, where the context can be measured through a physical device.

749 different context configurations,  $K_1, K_2$ , which are judged and are considered stable with respect  
750 to the introduction of a further stimulus. With  $J_1$  and  $J_2$ , we will indicate the judgment of the  
751 stimulus  $S$  in the contexts  $K_1$  and  $K_2$ , respectively. For the sake of simplicity, let us keep the  
752 same symbols  $J_1, J_2, K_1, K_2$  for their mean or median values in the Likert scale rating. If a linear  
753 relationship is found between the stimulus judgments  $J_1$  and  $J_2$  and the context judgments  $K_1$   
754 and  $K_2$ , then the slope of the linear relationship will be:

$$\beta = \frac{J_2 - J_1}{K_2 - K_1}. \quad (44)$$

755 Table 2 reports  $K$  and  $J$  values found in three studies; two about contrast effect (Leding, Hor-  
756 ton, & Wootan, 2015; Parducci, 1968), and one about dispersion effect (Tversky & Kahneman,  
757 1974), along with the slope  $\beta$  computed by using the formula (44). With regard to the contrast  
758 effect, Parducci (1968) reported the rating of the seriousness (on a 5-point Likert scale) of the  
759 same set of acts (the stimulus  $S$ ) when they were considered together with trivial or very bad  
760 actions (the context  $K$ ). The mean values of the contextual sentences were, respectively, 1.97 for  
761  $K_1$ , mild context, and 4.06 for  $K_2$ , bad context. The mean values of the stimulus judgments in  
762 the two previous contexts were 3.21 for  $J_1$  and 2.69 for  $J_2$ , respectively, thereby highlighting a  
763 contrast effect. In line with our framework, the slope  $\beta$  of equation (43) is equal to -0.25. A more  
764 recent study (Leding et al., 2015) has investigated contrast effects in judging attractiveness. There  
765 is a substantial amount of psychological literature on attractiveness ratings, as well. For example,  
766 Kenrick and Gutierrez (1980) found that exposure to attractive women caused a contrast effect  
767 in attractiveness judgments about photographs of average-looking women. Moreover, given the  
768 increasing access to the Internet and the importance of visual representations of individuals in  
769 virtual environments, Leding et al. (2015) have investigated whether exposure to avatars with  
770 different levels of attractiveness (the context  $K$ ) may influence judgments of the attractiveness  
771 of real people ( $J$ ). Judgments were made on a 5-point Likert scale (from -2 to 2). In their first  
772 study (Leding et al., 2015, experiment 1), the authors found that, when the attractiveness of the  
773 avatars was high ( $K_1 = 0.88$ ), observers rated the photo of a person as less attractive ( $J_1 = -0.87$ ).  
774 By contrast, when the attractiveness of the avatars was low ( $K_2 = -1.24$ ), the same person was  
775 judged as being more attractive ( $J_2 = -.30$ ). Furthermore, in experiment 2, they also investigated  
776 the order of presentation (the rating of the three avatars and then the evaluation of the photo and  
777 vice versa). When the avatars were presented first, the contrast effect was replicated; in the case  
778 of high attractiveness avatars ( $K_1 = 0.80$ ), observers rated the photo of the person as less attrac-  
779 tive ( $J_1 = -0.68$ ), whereas in the case of low attractiveness ( $K_2 = -1.65$ ), the photo was rated as  
780 more attractive ( $J_2 = 0.007$ ). In both cases, the slope  $\beta$  was comprised between 0 and -1 (-0.27  
781 and -0.28, respectively).

782 Switching to the dispersion effect, anchoring<sup>6</sup> provides a well-known example of such a  
783 cognitive mechanism. For example, Tversky and Kahneman (1974) described an experiment

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<sup>6</sup>Anchoring occurs when individuals use a piece of information to make subsequent judgments. Given an anchor value, subsequent judgments are biased toward such value (Tversky & Kahneman, 1974).

Study	Context $K_i$	Judgment $J_i$	$\beta$
Parducci (1968)	1.97	3.21	-.25
	4.06	2.69	
Leding et al. (2015) Exp 1	-1.24	-0.30	-.27
	0.88	-0.87	
Leding et al. (2015) Exp 2	-1.65	0.007	-.28
	0.80	-0.68	
Tversky & Kahneman (1974)	10	25	.36
	65	45	

Table 2: Contextual and stimulus values for three studies (Leding et al., 2015; Parducci, 1968; Tversky & Kahneman, 1974) and corresponding slopes. It could be noted that slopes are always constrained between -1 and 1.

784 in which participants were asked to estimate the percentage of African countries in the United  
785 Nations. Before answering, they were requested to state whether the estimate was higher than  
786 the low anchor  $K_1 = 10\%$  or lower than the high anchor  $K_2 = 65\%$ . Results indicated a strong  
787 dispersion effect; in fact, they found a value of  $J_1 = 25\%$  for the low anchor and  $J_2 = 45\%$  for the  
788 high anchor. Taking into account these points, in line with our predictions, the slope  $\beta$  is equal  
789 to .36. We also evaluated a more recent study by Mussweiler and Strack (2000) based on the  
790 same kind of task (not reported in Table 2). This study investigated the different magnitudes of  
791 the dispersion due to anchoring as a function of previous knowledge and perceived plausibility  
792 of the anchors. The slopes  $\beta$  of the reported data are always bounded between 0 and 1 (minimum  
793 0.04 - maximum 0.73).

794 All the previous cases are based on the unverified assumption of the presence of a linear  
795 relationship because the slopes are computed by using only two points. In order to overcome  
796 such a limitation, we performed an experiment aimed at evaluating (i) a contrast effect in a  
797 manner similar to that used by Parducci (1968), and (ii) a dispersion effect based on anchoring,  
798 following Tversky and Kahneman (1974). However, unlike the previous examples, we took  
799 into account several values of the context in order to check the presence of an actual linear  
800 relationship. Given the validity of such an assumption, it is possible to check if the slope of  
801 the linear relation is constrained according to the variational framework. We will also describe  
802 in more detail Matthews and Stewart’s (2009b) paper. This is a relatively recent paper that  
803 reported both dispersion and contrast effects for non-physical judgments taking into account a  
804 high number of different values for contextual information. The authors reported the slopes of  
805 such effects for the whole sample but also for each participant. Importantly, in this work contrast  
806 and dispersion effects are measured without relying on a Likert scale. For these reasons, it  
807 represents a good example of a case study for the variational framework predictions.

### 808 6.1.1. Experiment

809 In order to verify the prediction of our variational framework, we assumed a linear rela-  
810 tionship between context and judgment. The studies on context effects taken into account in  
811 the previous section evaluated such effects for only two different levels of contextual informa-

812 tion. Thus, we performed an experiment aimed at verifying the linearity assumption taking into  
813 account several contextual values. We employed two classical experimental paradigms previ-  
814 ously described: The Parducci (1968) paradigm about the moral evaluation of sentences, and  
815 the anchoring effect of Tversky and Kahneman (1974), inducing contrast and dispersion effects,  
816 respectively. Therefore, we considered two scenarios: the moral evaluation scenario in which  
817 people had to rate the moral seriousness of different actions, and the Kant's date of birth scenario  
818 in which people had to guess when Immanuel Kant was born with different anchors. In both  
819 cases, we expected to find a linear relationship. If the linearity assumption is met, according to  
820 the constraints of the proposed framework, we predict a slope between -1 and 0 for the contrast  
821 effect and a slope between 0 and 1 for the dispersion effect.

822

823 **Participants.** Three hundred and ninety subjects (285 female) participated voluntarily. Their  
824 mean age was 20.19 (sd = 3.91), with a range of 18 to 54 years. Participants were recruited  
825 in classroom settings in the University of Florence. The experimental task was part of other  
826 tasks connected to unrelated experiments. In each scenario (moral evaluation and Kant's date of  
827 birth), participants were randomly assigned to a single context condition (out of seven, including  
828 no context).

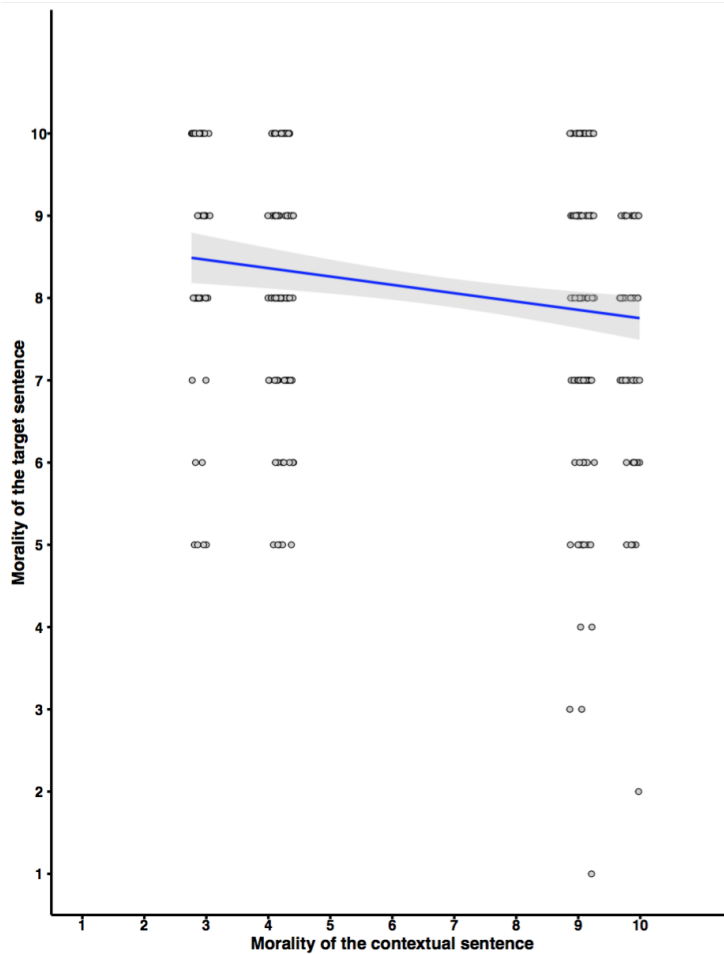
829

830 **Materials.** In the moral evaluation scenario, participants were asked to evaluate (on a Likert  
831 scale) the morality of one target sentence. Such a task could be preceded by the evaluation of  
832 a contextual sentence or it could be performed in isolation. The target sentence was *Borrowing*  
833 *a small amount of money from friends without repaying*. The six possible contextual sentences  
834 were: *Exceeding the speed limit by about 10 km/h in safe conditions when driving on a highway*;  
835 *Stealing a towel from a hotel*; *Bringing a dog on the beach where it is forbidden*; *Selling spoiled*  
836 *milk to a hospital*; *Giving false testimony at a criminal trial in exchange for a sum of money*; and  
837 *Murdering a relative*. Every sentence shown had to be evaluated on a scale from 1 (absolutely  
838 trivial) to ten (absolutely serious). In the Kant's date of birth scenario, participants were asked to  
839 estimate the year of birth of the philosopher Immanuel Kant. Such a task could be preceded by an  
840 anchor question or it could be performed in isolation. The anchor question was 'Was Immanuel  
841 Kant born before or after *anchor year*?' Anchor years were: 1574, 1624, 1674, 1774, 1824, and  
842 1874.

843

844 **Results.** With regard to the moral evaluation scenario, the mean judgment given for the target  
845 sentence in isolation was 8.05 (sd = 1.99). In Table 3 the mean judgments given to each contex-  
846 tual sentences with the corresponding judgments given to the target sentence are reported. The  
847 scatterplot (Figure 4) shows a rather weak correlation between context and judgments ( $r = -.17$ ,  
848  $p = .002$ ) with possibly six outlier points corresponding to judgments of 4 or less for the target  
849 sentence. Excluding such points, the Pearson's correlation becomes weaker but it is still statisti-  
850 cally significant ( $r = -.14$ ,  $p = .014$ ); in this case, the regression equation is  $y = 8.66 - 0.07x$ .

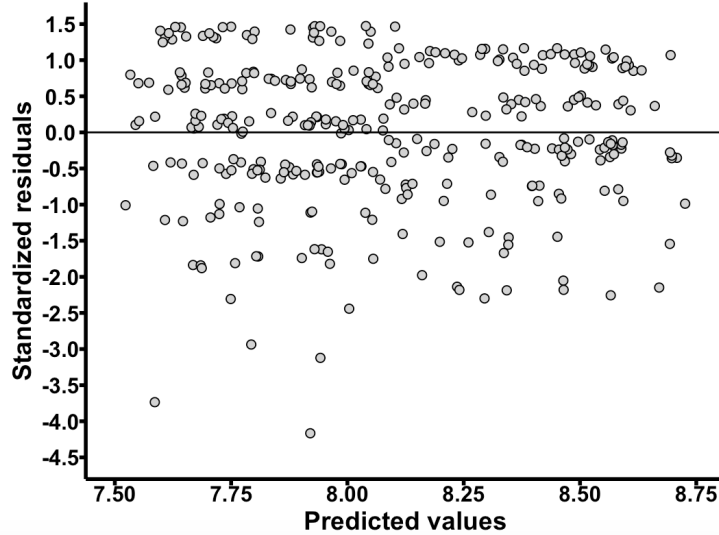
851 The residual plot for the moral evaluation scenario (Figure 5) was patternless and it showed  
852 no clear trend, suggesting a linear relationship between the context and moral judgments. By us-



**Figure 4:** Morality of the target sentence as a function of the morality of the contextual sentence. Noise was added for better visibility. Regression equation is  $y = 8.77 - 0.1x$  (blue line) where the shadowed area corresponds to the confidence interval.

853 ing an ANOVA to verify the linearity between the context and the judgments, the linear compo-  
 854 nent was statistically significant ( $\mathcal{F}_{(1,325)} = 9.67, p = .002$ ), whereas the deviation from linearity  
 855 component was not ( $\mathcal{F}_{(4,325)} = 1.02, p = .365$ ). The linear relationship between the context and  
 856 the moral judgments was confirmed by means of a Ramsey RESET test (Ramsey, 1969) against  
 857 a quadratic ( $\text{RESET}_{(1,328)} = 0.62, p = .431$ ), cubic ( $\text{RESET}_{(1,328)} = 1.11, p = .292$ ) and quartic  
 858 ( $\text{RESET}_{(1,328)} = 1.67, p = .198$ ) power effect as well as the square root ( $\text{RESET}_{(1,328)} = 0.23, p =$   
 859  $.626$ ).

860 With regard to the Kant scenario, the mean estimated year in isolation was 1768 (sd = 81).  
 861 Table 4 shows the relationship between anchor years and the mean estimated year of Kant's birth.  
 862 The scatterplot (Figure 6) shows a positive and statistically significant relationship ( $r = .46, p <$   
 863  $.001$ ) between the anchor and the estimated values. Also for the Kant scenario, the residual plot  
 864 (Figure 7) was patternless, suggesting a linear relationship between the anchor and the estimated  
 865 year of birth. Indeed, the linear component of the ANOVA was statistically significant ( $\mathcal{F}_{(1,294)}$



**Figure 5:** The residual plot shows the relation between standardized residuals and predicted values for the moral evaluation scenario. Noise was added for better visibility. Other than a few outliers, residuals are randomly distributed around the horizontal line corresponding to zero mean.

866 = 77.48,  $p < .001$ ) whereas the deviation from linearity was not ( $\mathcal{F}_{(4,294)} = 0.33$ ,  $p = .857$ ). The  
 867 Ramsey RESET test was not statistically significant for the quadratic ( $\text{RESET}_{(1,297)} = 0.82$ ,  $p =$   
 868  $.365$ ), cubic ( $\text{RESET}_{(1,297)} = 0.80$ ,  $p = .371$ ) and quartic ( $\text{RESET}_{(1,297)} = 0.78$ ,  $p = .377$ ) power  
 869 effect or as a square root function ( $\text{RESET}_{(1,297)} = 0.85$ ,  $p = .357$ ).

870 For both the moral and the Kant scenarios, the relationship between context values and judg-  
 871 ments was linear and constrained between 0 and 1 or -1 and 0 for contrast and dispersion effects,  
 872 respectively, in line with the constraints predicted by the proposed variational framework. It must  
 873 be underlined that only in the moral scenario were data collected by means of a Likert scale (and  
 874 thus with the consequent limitations associated with the use of an ordinal scale).

### 875 **6.1.2. The Matthews & Stewart (2009b) study**

876 The Matthews and Stewart (2009b) paper is focused on how the context influences the judg-  
 877 ment of non-physical dimensions. In particular, they investigated judgments of objects' prices in  
 878 a sequential decision making paradigm. Within the study of basic perceptual properties, as we  
 879 have seen before, it has been observed that psychophysical judgments are strongly influenced by  
 880 the local context: In particular, in sequential decision making tasks, the stimuli and the responses  
 881 given in previous trials may influence the response of the current trial. Matthews and Stewart  
 882 (2009b) cite Jesteadt, Luce, and Green (1977) who propose a linear regression model to account  
 883 for context effects that arise in sequential decisional making experiments, based on the following  
 884 equation:

$$J_n = \gamma + \alpha_0 P_n + \alpha_1 P_{n-1} + \beta_1 J_{n-1} + \epsilon, \quad (45)$$

885 where  $J_n$  is the current judgment at trial  $n$ , which is a function of  $P_n$ ,  $P_{n-1}$ , the objective mag-  
 886 nitude of the stimuli presented in the current trial  $n$  and in the previous trial  $n - 1$ , respectively,

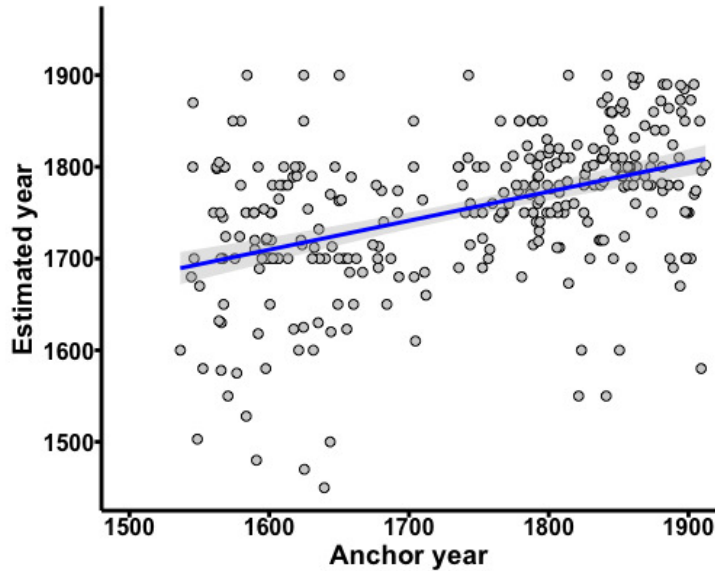


<b>Context Sentence</b>	<b>Context</b>		<b>Target Sentence</b>	
	<b>Mean</b>	<b>sd</b>	<b>Mean</b>	<b>sd</b>
Exceeding the speed limit by about 10 km/h in safe conditions when driving on a highway	2.92	1.66	8.49	1.52
Stealing a towel from a hotel	4.15	2.02	8.38	1.40
Bringing a dog on the beach where it is forbidden	4.25	2.26	8.24	1.52
Giving false testimony at a criminal trial in exchange for a sum of money	9.02	1.23	8.07	1.73
Selling spoiled milk to a hospital	9.10	1.55	7.97	1.82
Murdering a relative	9.83	0.68	7.41	1.45

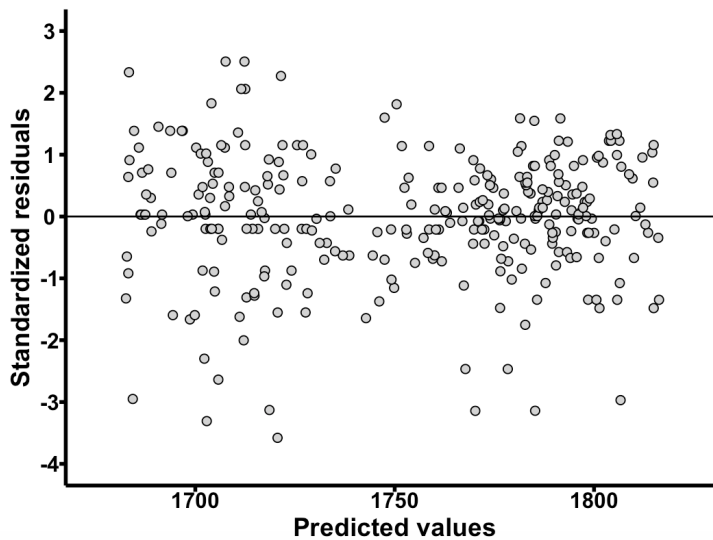
Table 3: Results of the moral evaluation experiment. There is a contrast effect between the judgments given to the contextual sentences and the corresponding judgments related to the target sentence.

<b>Anchor</b>	<b>Estimated year</b>	
	<b>Mean</b>	<b>sd</b>
1574	1704	92
1624	1713	90
1674	1720	76
1774	1763	46
1824	1784	69
1874	1802	69

Table 4: Results of the Kant's date of birth experiment. There is a dispersion effect between the years that play the role of the anchor and the estimation of the year in which Kant was born.



**Figure 6:** Estimated year of Kant's birth as a function of the anchor year. Noise was added for better visibility. Regression equation is  $y = 1164 + 0.33x$  (blue line) where the shadowed area corresponds to the confidence interval.



**Figure 7:** The residual plot shows the relation between standardized residuals and predicted values for the Kant scenario. Noise was added for better visibility. Other than a few outliers, residuals are randomly distributed around the horizontal line corresponding to zero mean.

887 and of  $J_{n-1}$ , which is the judgment relative to the trial  $n - 1$ . Apart from the error  $\epsilon$ , the other  
 888 constants appearing in the previous formula are the regression coefficients relative to the current  
 889 and previous objective stimulus ( $\alpha_0$  and  $\alpha_1$ ), and to the previous judgment ( $\beta_1$ ). The key obser-  
 890 vation is that there is a contrast effect between  $J_n$  and  $P_{n-1}$  expressed by  $\alpha_1$  and, *at the same*  
 891 *time*, there is a dispersion effect between  $J_n$  and  $J_{n-1}$  expressed by  $\beta_1$  (DeCarlo & Cross, 1990;

	<b>Experiment 1</b>	<b>Experiment 2a</b>	<b>Experiment 2b</b>
Total participants	25	28	28
Positive values	23	26	14
Significant positive values	9	12	5
Maximum and minimum of positive values	0.032, 0.475	0.027, 0.465	0.052, 0.311
Mean	0.163	0.162	0.046

Table 5: Dispersion effects in the Matthews & Stewart (2009b) study. For each experiment, the table reports the total number of participants, how many unstandardized  $\beta_1$  were positive, positive and significant, the maximum and minimum values, and the mean across the whole sample. Experiment 1 did not include particular manipulations, whereas experiment 2 provided two conditions: (a) no feedback and fixed time of presentation (b) feedback after each judgment and a fixed presentation time of each item.

	<b>Experiment 1</b>	<b>Experiment 2a</b>	<b>Experiment 2b</b>
Total participants	25	28	28
Negative values	19	17	11
Significant negative values	2	4	2
Maximum and minimum of negative values	-0.017, -0.188	-0.014, -0.278	-0.008, -0.226
Mean	-0.034	-0.049	0.058

Table 6: Contrast effects in the Matthews & Stewart (2009b) study. For each experiment, the table reports the total number of participants, how many unstandardized  $\alpha_1$  were negative, negative and significant, the maximum and minimum values, and the mean across the whole sample. Experiment 1 did not include particular manipulations, whereas experiment 2 provided two conditions: (a) no feedback and fixed time of presentation (b) feedback after each judgment and a fixed presentation time of each item.

892 Jesteadt et al., 1977; Matthews & Stewart, 2009a; Mori, 1998).

893 Matthews and Stewart (2009b) investigated such sequential effects in the domain of non-  
894 physical dimensions asking for judgments related to the prices of various items (such as chairs  
895 and shoes). In the first experiment, participants rated the price of a set of 100 chairs where  
896 each item was shown until a response was given. They then employed the regression model of  
897 Jesteadt et al. (1977) to evaluate the influence of context on the judgment of price (after a log  
898 transformation of each variable). The authors performed three experiments: A preliminary study  
899 with free presentation time and no particular manipulation, and a second experiment with two  
900 conditions. The first one was without feedback and with the presentation of each item for a fixed  
901 amount of time, whereas in the second condition, there was feedback given after each judgment  
902 and a fixed presentation time of each item. In Table 5, we report some statistics related to the  
903 estimation of the unstandardized coefficient  $\beta_1$  obtained in the three experiments. Crucially for  
904 our framework, all coefficients related to the dispersion effect were constrained between 0 and 1.  
905 With regard to the contrast effect, Table 6 summarized the findings related to the unstandardized  
906 coefficients  $\alpha_1$ . Although in experiment 2b contrast effect was not observed, in line with our  
907 predictions, no coefficient lower than -1 was observed. Therefore, the empirical cases reported  
908 testify to the overall accuracy of the variational framework predictions. Naturally, future studies  
909 are needed to further empirically validate the predictions of the variational framework proposed  
910 in this paper.

## 7. Discussion and conclusions

Starting with the variational interpretation of brightness matching, we have proposed a general variational framework for context effects that may hold both for physical and non-physical judgments. Our variational interpretation indicates that contrast and dispersion effects may occur simultaneously in every judgment, the final percept being the balance between the opposing actions of contrast (the tendency to maximize the differences among the dimensions that characterize the object to be judged) and dispersion (the tendency to minimize the differences among the dimensions).

In the first part of the paper, we showed how the variational interpretation of brightness matching represents an alternative version of Rudd and Zemach's model (Rudd & Zemach, 2004) that makes the same predictions but has the added advantage of being able to explain brightness matching in terms of a balance between contrast and dispersion, and also able to be directly generalizable to physical configurations that are much more complex than those considered in Rudd and Zemach's experiments. Within perception studies, there are many different quantitative models and theoretical approaches to context effects (Gilchrist, 2006). We have shown that our framework can describe existing theories through the specification of suitable functional terms and parameters. We expect that future works will apply the proposed variational techniques to derive new constraints and, hopefully, disambiguate different theoretical proposals.

In the second part of the paper, we advocated the plausibility of the proposed framework for non-physical judgments as well, such as those observed in social cognition or cognitive psychology. Several case studies drawn from the existing literature, along with new empirical data, are in line with the constraints set by our variational approach. In particular, we performed an experiment aimed at observing classic dispersion and contrast effects. Crucially, the relationship between context and judgments was in line with the predictions of the variational model. Although this evidence tend to support our framework, it is necessary to stress some of its limitations. First, since non-physical judgments are usually collected by means of a Likert scale, the applicability of our variational approach to ordinal data it is not straightforward (Camparo, 2013; Carifio & Perla, 2007; Wegener, 1982). Thus, this issue requires a systematic investigation in order to understand the extent to which a variational solution may be applied to Likert scale data. In any case, we deem it important to stress that the study of Matthews and Stewart (2009b) on context effects in non-physical judgments supports our proposal without using a Likert scale. Second, in order to further confirm our view, a complete meta-analysis is needed, along with a direct empirical investigation into the predictions related to the constraints of the proposed model. If our variational approach will prove its validity in non-physical judgments domains also, then this work may represent a first step toward a formal analysis of psychological mechanisms able to account for perceptual and cognitive phenomena with similar characteristics. The Kahneman and Tversky research program began as the study of the statistical intuitions of experts (Tversky & Kahneman, 1971) where intuitive judgments were considered as extensions of perception to judgments about objects that are not currently present (Kahneman & Frederick, 2005). Since then, psychological literature often draws a parallel between intuitive non-physical judgments

951 and the mechanisms of perception (Kahneman, 2011; Thaler & Sunstein, 2008). However, this  
952 parallelism is sometimes superficially stated and not fully investigated, especially from a formal  
953 point of view. Our work tries to fill this gap, representing an attempt to use a common framework  
954 to account for context-driven effects both in perception and cognition. Moreover, we hope that  
955 our proposal will stimulate the mathematical modeling of context-driven effects in fields such as  
956 social cognition where these kinds of approaches are seldom employed (Biernat, 2005). Given  
957 that our analysis emphasizes the concurrent presence of contrast and dispersion effects in every  
958 judgment, the proposed variational framework supports the theoretical accounts that hypothe-  
959 size the simultaneous occurrence of the two effects at the expense of sequential models (Biernat,  
960 2005). Future research about context-driven effects in social cognition and cognitive psychology  
961 could also use our framework to devise new experiments and disambiguate different theoretical  
962 positions.

963 In general terms, the proposed variational framework does not address the question about  
964 under which conditions we observe either a prevalence of contrast or dispersion (both within  
965 perception or cognition fields). The main aim of our approach is not to solve these kinds of open  
966 problems, rather to help understand in a more profound way the interpretation of observations in  
967 psychology. We consider this to be as important as making new predictions. In fact, being able  
968 to predict a phenomenon does not necessarily mean understanding it, as the renowned example  
969 of Ptolemaic orbital system shows: planetary orbits, as predicted by Ptolemy, were accurate, but  
970 his geocentric interpretation of planetary movement was totally wrong, showing that description  
971 and comprehension can be separated in a scientific theory. One of the powerful advantages of  
972 a variational approach is that it allows for conjugating description and comprehension simulta-  
973 neously. Consequently, we hope that this work (and especially the included tutorial) will also  
974 stimulate further application of variational methods in mathematical psychology beyond the is-  
975 sue of context-driven effects.

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## References

- 980
- 981 Agostini, T. & Galmonte, A. (2000). Contrast and assimilation: the belongingness paradox.  
982 *Review of Psychology*, 7(1-2), 3–7.
- 983 Agostini, T. & Galmonte, A. (2002). Perceptual organization overcomes the effect of local sur-  
984 rund in determining simultaneous lightness contrast. *Psychological Science*, 13(1), 88–92.
- 985 Agostini, T. & Proffitt, D. R. (1993). Perceptual organization evokes simultaneous lightness  
986 contrast. *Perception*, 22(3), 263–272.
- 987 Bertalmío, M., Caselles, V., & Provenzi, E. (2009). Issues about the retinex theory and contrast  
988 enhancement. *International Journal of Computer Vision*, 83, 101–119.
- 989 Bertalmío, M., Caselles, V., Provenzi, E., & Rizzi, A. (2007). Perceptual color correction through  
990 variational techniques. *IEEE Trans. on Image Processing*, 16, 1058–1072.
- 991 Biernat, M. (2005). *Standard and expectancies: Contrast and assimilation in judgments of self  
992 and others*. NY: Psychology Press.
- 993 Biernat, M. & Kobrynowicz, D. (1997). Gender- and race-based standards of competence: Lower  
994 minimum standards but higher ability standards for devalued groups. *Journal of Personality  
995 and Social Psychology*, 72(3), 544–557.
- 996 Blakeslee, B. & McCourt, M. E. (2004). A unified theory of brightness contrast and assimilation  
997 incorporating oriented multiscale spatial filtering and contrast normalization. *Vision Research*,  
998 44(21), 2483–2503.
- 999 Boyd, S. & Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press.
- 1000 Bressan, P. (2001). Explaining lightness illusions. *Perception*, 30, 1031–1046.
- 1001 Brickman, P. & Campbell, D. T. (1971). Hedonic relativism and planning the good society. In  
1002 M. H. Appley (Ed.), *Adaptation-level theory* (pp. 287–302). New York: Academic Press.
- 1003 Camparo, J. (2013). A geometrical approach to the ordinal data of likert scaling and attitude  
1004 measurements: The density matrix in psychology. *Journal of Mathematical Psychology*, 57(1-  
1005 2), 29–42.
- 1006 Carifio, J. & Perla, R. J. (2007). Ten common misunderstandings, misconceptions, persistent  
1007 myths and urban legends about likert scales and likert response formats and their antidotes.  
1008 *Journal of Social Sciences*, 3(3), 143–159.
- 1009 Casacci, S. & Pareto, A. (2015). Methods for quantifying ordinal variables: a comparative study.  
1010 *Quality & Quantity*, 49(5), 1859–1872.

- 1011 Chevreul, M. E. (1855/2007). *The Principles of Harmony and Contrast of Colours, and Their*  
1012 *Applications to the Arts* (2 ed.). London: Longman, Brown, Green, and Longmans.
- 1013 De Weert, C. M. M. & Spillman, L. (1995). Assimilation: Asymmetry between brightness and  
1014 darkness. *Vision Research*, 35, 1413–1420.
- 1015 DeCarlo, L. T. & Cross, D. V. (1990). Sequential effects in magnitude scaling: Models and  
1016 theory. *Journal of Experimental Psychology: General*, 119(4), 375–396.
- 1017 Diener, E., Lucas, R. E., & Scollon, C. N. (2006). Beyond the hedonic treadmill: revising the  
1018 adaptation theory of well-being. *American Psychologist*, 61(4), 305–314.
- 1019 Ehm, W. & Wackermann, J. (2012). Modeling geometric-optical illusions: A variational ap-  
1020 proach. *Journal of Mathematical Psychology*, 56(6), 404–416.
- 1021 Evans, R. (1948). *An introduction to color*. Wiley.
- 1022 Eysenck, M. W. (1990). *Happiness*. Hove, UK: Laurence Erlbaum Associates Ltd.
- 1023 Fairchild, M. (2005). *Color appearance models*. Hoboken, NJ.: Wiley.
- 1024 Fechner, G. (1860). *Elemente der Psychophysik I u. II*. Leipzig: Breitkopf und Hartel.
- 1025 Festinger, L. (1954). A theory of social comparison processes. *Human Relations*, 7, 117–140.
- 1026 Festinger, L., Coren, S., & Rivers, G. (1970). The effect of attention on brightness contrast and  
1027 assimilation. *American Journal of Psychology*, 83, 189–207.
- 1028 Frederick, S. (2007). Hedonic treadmill. In R. Baumeister & K. D. Vohs (Eds.), *Encyclopedia of*  
1029 *Social Psychology* (pp. 419–420). London: Sage.
- 1030 Fujita, F. & Diener, E. (2005). Life satisfaction set point: Stability and change. *Journal of*  
1031 *Personality and Social Psychology*, 88(1), 158–164.
- 1032 Gelfand, I. & Fomin, S. (1963). *Calculus of variations*. Prentice Hall: Englewood Cliffs, NJ.
- 1033 Gilchrist, A. (2006). *Seeing black and white*. Oxford University Press.
- 1034 Gilchrist, A. (2015). Perceptual organization in lightness. In J. Wagemans (Ed.), *Oxford hand-*  
1035 *book of perceptual organization* (pp. 391–412). Oxford: Oxford University Press.
- 1036 Heintz, R. K. (1950). The effect of remote anchoring upon the judgment of lifted weights.  
1037 *Journal of Experimental Psychology*, 40(5), 584–591.
- 1038 Helson, H. (1963). Studies of anomalous contrast and assimilation. *Journal of the Optical Society*  
1039 *of America*, 53, 179–184.

- 1040 Higgins, E. T. (1987). Self-discrepancy: A theory relating self and affect. *Psychological Review*,  
1041 94(3), 319–340.
- 1042 Higgins, E. T. (1990). Personality, social psychology, and person-situation relations: Standards  
1043 and knowledge activation as a common language. In L. A. Pervin (Ed.), *Handbook of Person-*  
1044 *ality: Theory and Research* (pp. 301–338). NY: The Guilford Press.
- 1045 Higgins, E. T. & Lurie, L. (1983). Context, categorization, and memory: The 'change of stan-  
1046 dard' effect. *Cognitive Psychology*, 15, 525–547.
- 1047 Jesteadt, W., Luce, R. D., & Green, D. M. (1977). Sequential effects in judgments of loudness.  
1048 *Journal of Experimental Psychology: Human Perception and Performance*, 3(1), 92–104.
- 1049 Kahneman, D. (2011). *Thinking, fast and slow*. NY: Macmillan.
- 1050 Kahneman, D. & Frederick, S. (2005). A model of heuristic judgement. In *The cambridge*  
1051 *handbook of thinking and reasoning* (pp. 267–293). Cambridge: Cambridge University Press.
- 1052 Kenrick, D. T. & Gutierrez, S. A. (1980). Contrast effects and judgments of physical attractive-  
1053 ness: When beauty becomes a social problem. *Journal of Personality and Social Psychology*,  
1054 38(1), 131–140.
- 1055 Kingdom, F. & Moulden, B. (1991). White's effect and assimilation. *Vision Research*, 31(1),  
1056 151–159.
- 1057 Koffka, K. (1935). *Principles of Gestalt Psychology*. NY: Harcourt, Brace, and World.
- 1058 Krantz, D. L. & Campbell, D. T. (1961). Separating perceptual and linguistic effects of context  
1059 shifts upon absolute judgments. *Journal of Experimental Psychology*, 62(1), 35–42.
- 1060 Land, E. & McCann, J. (1971). Lightness and Retinex theory. *Journal of the Optical Society of*  
1061 *America*, 61(1), 1–11.
- 1062 Leding, J. K., Horton, J. C., & Wootan, S. S. (2015). The contrast effect with avatars. *Computers*  
1063 *in Human Behavior*, 44, 118–123.
- 1064 Locke, J. (1690/1979). *An essay concerning human understanding*. Oxford: Oxford University  
1065 Press.
- 1066 Lykken, D. & Tellegen, A. (1996). Happiness is a stochastic phenomenon. *Psychological Sci-*  
1067 *ence*, 7(3), 186–189.
- 1068 Martin, L. L. & Achee, J. W. (1992). Beyond accessibility: The role of processing objectives  
1069 in judgment. In L. L. Martin & A. Tesser (Eds.), *The Construction of Social Judgments* (pp.  
1070 195–216). Hillsdale, NJ: Lawrence Erlbaum Associates.



- 1071 Matthews, W. J. & Stewart, N. (2009a). The effect of interstimulus interval on sequential effects  
1072 in absolute identification. *The Quarterly Journal of Experimental Psychology*, 62(10), 2014–  
1073 2029.
- 1074 Matthews, W. J. & Stewart, N. (2009b). Psychophysics and the judgment of price: Judging  
1075 complex objects on a non-physical dimension elicits sequential effects like those in perceptual  
1076 tasks. *Judgment and Decision Making*, 4(1), 64–81.
- 1077 Melamed, M. E. (1971). Response requirements and the contrast effect in loudness judgments.  
1078 *Perception & Psychophysics*, 10(4), 309–312.
- 1079 Mori, S. (1998). Effects of stimulus information and number of stimuli on sequential dependen-  
1080 cies in absolute identification. *Canadian Journal of Experimental Psychology/Revue canadi-  
1081 enne de psychologie expérimentale*, 52(2), 72–83.
- 1082 Moskowitz, G. B. (2005). *Social Cognition: Understanding Self and Others*. Guildford Press.
- 1083 Moskowitz, G. B. & Roman, R. J. (1992). Spontaneous trait inferences as self-generated primes:  
1084 Implications for conscious social judgement. *Journal of Personality and Social Psychology*,  
1085 62, 728–738.
- 1086 Mussweiler, T. (2003). Comparison processes in social judgment: Mechanisms and conse-  
1087 quences. *Psychological Review*, 110, 472–489.
- 1088 Mussweiler, T. & Strack, F. (2000). The use of category and exemplar knowledge in the solution  
1089 of anchoring tasks. *Journal of Personality and Social Psychology*, 78(6), 1038.
- 1090 Nisbett, R. E. & Wilson, T. D. (1977). The halo effect: Evidence for unconscious alteration of  
1091 judgments. *Journal of Personality and Social Psychology*, 35(4), 250–256.
- 1092 Noventa, S. & Vidotto, G. (2012). A variational approach to behavioral and neuroelectrical laws.  
1093 *Biological Cybernetics*, 106, 339–358.
- 1094 Palma-Amestoy, R., Provenzi, E., Bertalmío, M., & Caselles, V. (2009). A perceptually inspired  
1095 variational framework for color enhancement. *IEEE Transactions on Pattern Analysis and  
1096 Machine Intelligence*, 31(3), 458–474.
- 1097 Parducci, A. (1968). The relativism of absolute judgments. *Scientific American*, 219(6), 84–90.
- 1098 Parker, S., Bascom, J., Rabinovitz, B., & Zellner, D. (2008). Positive and negative hedonic  
1099 contrast with musical stimuli. *Psychology of Aesthetics, Creativity and the Arts*, 2(3), 171–  
1100 174.
- 1101 Pepitone, A. & DiNubile, M. (1976). Contrast effects in judgments of crime severity and the  
1102 punishment of criminal violators. *Journal of Personality and Social Psychology*, 33(4), 448–  
1103 459.

- 1104 Petty, R. E. & Wegener, D. T. (1993). Flexible correction processes in social judgment: Correct-  
1105 ing for context-induced contrast. *Journal of Experimental Social Psychology*, 29, 137–165.
- 1106 Provenzi, E. (2013). Achromatic induction: A variational interpretation of Rudd-Zemach's Edge  
1107 Integration Model. In *Signal-Image Technology & Internet-Based Systems (SITIS), 2013 In-*  
1108 *ternational Conference*, (pp. 424–429). IEEE.
- 1109 Provenzi, E. & Caselles, V. (2014). A wavelet perspective on variational perceptually-inspired  
1110 color enhancement. *International Journal of Computer Vision*, 106, 153–171.
- 1111 Provenzi, E., De Carli, L., Rizzi, A., & Marini, D. (2005). Mathematical definition and analysis  
1112 of the retinex algorithm. *Journal of the Optical Society of America A*, 22(12), 2613–2621.
- 1113 Provenzi, E., Fierro, M., Rizzi, A., De Carli, L., Gadia, D., & Marini, D. (2007). Random spray  
1114 retinex: A new retinex implementation to investigate the local properties of the model. *IEEE*  
1115 *Transactions on Image Processing*, 16, 162–171.
- 1116 Provenzi, E., Gatta, C., Fierro, M., & Rizzi, A. (2008). Spatially variant white patch and gray  
1117 world method for color image enhancement driven by local contrast. *IEEE Transactions on*  
1118 *Pattern Analysis and Machine Intelligence*, 30, 1757–1770.
- 1119 Ramsey, J. B. (1969). Tests for specification errors in classical linear least squares regression  
1120 analysis. *Journal of the Royal Statistical Society Series B*, 31(2), 350–371.
- 1121 Rudd, M. & Zemach, I. (2004). Quantitative properties of achromatic color induction: An edge  
1122 integration analysis. *Vision Research*, 44, 971–981.
- 1123 Sapiro, G. & Caselles, V. (1997). Histogram modification via differential equations. *Journal of*  
1124 *Differential Equations*, 135, 238–266.
- 1125 Schkade, D. A. & Kahneman, D. (1998). Does living in California make people happy? A  
1126 focusing illusion in judgments of life satisfaction. *Psychological Science*, 9(5), 340–346.
- 1127 Schwarz, N. & Bless, H. (1992). Constructing reality and its alternatives: An inclusion/exclusion  
1128 model of assimilation and contrast effects in social judgments. In L. L. Martin & A. Tesser  
1129 (Eds.), *The Construction of Social Judgments* (pp. 217–245). Mahwah, NJ: Lawrence Erlbaum  
1130 Associates.
- 1131 Sherif, M., Taub, D., & Hovland, C. L. (1958). Assimilation and contrast effects of anchoring  
1132 stimuli on judgments. *Journal of Experimental Psychology*, 55(2), 150–155.
- 1133 Silver, R. L. (1982). *Coping with an undesirable life event: A study of early reactions to physical*  
1134 *disability*. PhD thesis, Unpublished Doctoral Dissertation, Northwestern University.

- 1135 Stapel, D. A. & Koomen, W. (1998). When stereotype activation results in (counter) stereotypical  
1136 judgments: Priming stereotype-relevant traits and exemplar. *Journal of Experimental Social*  
1137 *Psychology*, *34*, 136–163.
- 1138 Stapel, D. A., Koomen, W., & van der Pligt, J. (1996). The referents of trait inferences: The im-  
1139 pact of trait concepts versus actor-trait links on subsequent judgements. *Journal of Personality*  
1140 *and Social Psychology*, *70*, 437–450.
- 1141 Stapel, D. A., Koomen, W., & van der Pligt, J. (1997). Categories of category accessibility: The  
1142 impact of trait concept versus exemplar priming on person judgments. *Journal of Experimental*  
1143 *Social Psychology*, *33*(1), 47–76.
- 1144 Strack, F., Schwarz, N., & Gschneidinger, E. (1985). Happiness and reminiscing: The role of  
1145 time perspective, affect, and mode of thinking. *Journal of Personality and Social Psychology*,  
1146 *49*, 1460–1469.
- 1147 Takane, Y. (2005). Optimal scaling. In B. Everitt & D. Howell (Eds.), *Encyclopedia of Statistics*  
1148 *for Behavioral Sciences* (pp. 1479–1482). Chichester: Wiley.
- 1149 Thaler, R. H. & Sunstein, C. R. (2008). *Nudge: Improving Decisions about Health, Wealth, and*  
1150 *Happiness*. New Haven and London: Yale University Press.
- 1151 Tversky, A. & Kahneman, D. (1971). Belief in the law of small numbers. *Psychological Bulletin*,  
1152 *76*(2), 105–110.
- 1153 Tversky, A. & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Sci-*  
1154 *ence*, *185*(4157), 1124–1131.
- 1155 von Bezold, W. (1876/2014). *The Theory of Color in Its Relation to Art and Art-Industry*.  
1156 Charleston, SC: Nabu Press.
- 1157 Wallach, H. (1948). Brightness constancy and the nature of achromatic colors. *Journal of Ex-*  
1158 *perimental Psychology*, *38*(3), 310–324.
- 1159 Wallach, H. (1963). The perception of neutral colors. *Scientific American*, *208*, 107–116.
- 1160 Weber, E. (1846). Der tastsinn und das gemeingefuhl. In R. Wagner (Ed.), *Handwörterbuch der*  
1161 *Physiologie*. Braunschweig.
- 1162 Wedell, D. H., Parducci, A., & Geiselman, R. E. (1987). A formal analysis of rating of physical  
1163 attractiveness: Successive contrast and simultaneous assimilation. *Journal of Experimental*  
1164 *Social Psychology*, *23*, 230–249.
- 1165 Wegener, B. (Ed.). (1982). *Social attitudes and psychophysical measurement*. Hillsdale, NJ:  
1166 Lawrence Erlbaum Associates.

- 1167 Wertheimer, M. (1923). Untersuchungen zur lehre von der gestalt. *Psychologische Forschung*,  
1168 4, 301–350.
- 1169 White, J. D. & Vilmain, J. A. (1986). The psychophysics of price: A critique of the weber-  
1170 fechner approach in consumer behavior. In N. K. Malhotra (Ed.), *Proceedings of the 1986*  
1171 *Academy of Marketing Science (AMS) Annual Conference* (pp. 30–34). Springer International  
1172 Publishing.
- 1173 White, M. (1979). A new effect of the nature on perceived lightness. *Perception*, 8, 413–416.
- 1174 Wildeman, C., Turney, K., & Schnittker (2014). Hedonic consequences of punishment revisited.  
1175 *Journal of Criminal Law and Criminology*, 104, 133–164.
- 1176 Wundt, W. M. (1896). *Grundriss der Psychologie*. Leipzig: Wilhelm engelmann.
- 1177 Young, F. W. (1975). Methods for describing ordinal data with cardinal models. *Journal of*  
1178 *Mathematical Psychology*, 12(4), 416–436.

## Appendix: Proof of Proposition 4.1

We will perform the proof only in the continuous case, since the one in the discrete scenario is analogous. Let us compute the variation of the functional  $E_w(\log L) = A(\log L) - C_w(\log L)$  with respect to  $\log L$ . To do this, it is convenient to write  $\log L(x) \equiv \tilde{L}(x)$ . The variation of the first term of the functional is trivial to compute and gives:

$$\delta A(\tilde{L}, \tilde{J}) = \int_{\Omega} (\tilde{L}(x) - \tilde{\mu}) \tilde{J}(x) dx \quad (46)$$

$\tilde{J}(x)$  being a generic functional perturbation of  $\tilde{L}(x)$ . The variation of the second term instead can be written as follows

$$\begin{aligned} \delta C_w(\tilde{L}, \tilde{J}) &= \frac{1}{2} \iint_{\Omega^2} w(\|x - y\|) [\tilde{L}(x) - \tilde{L}(y)] \tilde{J}(x) dx dy \\ &\quad - \frac{1}{2} \iint_{\Omega^2} w(\|x - y\|) [\tilde{L}(y) - \tilde{L}(x)] \tilde{J}(y) dx dy. \end{aligned} \quad (47)$$

Now, interchanging the role of the ‘mute’ variables  $x$  and  $y$  in the second integral, and thanks to the symmetry of the induction weight, i.e.,  $w(\|x - y\|) = w(\|y - x\|)$ , we can write

$$\delta C_w(\tilde{L}, \tilde{J}) = \iint_{\Omega^2} w(\|x - y\|) [\tilde{L}(x) - \tilde{L}(y)] \tilde{J}(x) dx dy. \quad (48)$$

Since  $\delta E(\tilde{L}, \tilde{J}) = \delta A(\tilde{L}, \tilde{J}) - \delta C_w(\tilde{L}, \tilde{J})$ , thanks to Fubini’s theorem, we have

$$\delta E_w(\tilde{L}, \tilde{J}) = \int_{\Omega} \left( \tilde{L}(x) - \int_{\Omega} w(\|x - y\|) [\tilde{L}(x) - \tilde{L}(y)] dy \right) \tilde{J}(x) dx, \quad (49)$$

The argmin of  $E_w(\tilde{L}, \tilde{J})$  satisfies the Euler-Lagrange equation  $\delta E_w(\tilde{L}, \tilde{J}) = 0$  for all perturbations  $\tilde{J}$ . Thanks to the fundamental lemma of variational calculus (Gelfand & Fomin, 1963), this is possible if and only if  $\tilde{L}(x)$  satisfies

$$\tilde{L}(x) - \tilde{\mu} - \int_{\Omega} w(\|x - y\|) [\tilde{L}(x) - \tilde{L}(y)] dy = 0, \quad (50)$$

i.e.,

$$\tilde{L}(x) = \tilde{\mu} + \int_{\Omega} w(\|x - y\|) [\tilde{L}(x) - \tilde{L}(y)] dy. \quad (51)$$

Turning back to the original variables, we have

$$\log L(x) = \mu + \int_{\Omega} w(\|x - y\|) \log \frac{L(x)}{L(y)} dy, \quad (52)$$

so we see that the argmin of  $E_w$  satisfies eq. (13) that defines the logarithmic brightness, and thus proposition 4.1 is proven.  $\square$