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Necessary and Sufficient Conditions on the Exponential Stability of Positive Hyperbolic Systems (full version)

Liguo Zhang and Christophe Prieur

Abstract—In this paper, a strict linear Lyapunov function is developed in order to investigate the exponential stability of a linear hyperbolic partial differential equation with positive boundary conditions. Based on the method of characteristics, some properties of the positive solutions are derived for the hyperbolic initial boundary value problems. The dissipative boundary condition in terms of linear inequalities is proven to be not only sufficient but also necessary under an extra assumption on the velocities of the hyperbolic systems. An application to control of the freeway traffic modeled by the Aw-Rascle traffic flow equations illustrates and motivates the theoretical results. The boundary control strategies are designed by integrating the on-ramp metering with the mainline speed limit. Finally, the proposed feedback laws are tested under simulation, first in the free-flow case and then in the congestion mode, which show adequate performance to stabilize the local freeway traffic.

Index Terms—Positive hyperbolic systems, Lyapunov function, distributed parameter systems, exponential stability, Aw-Rascle equations.

I. INTRODUCTION

Many physical systems having an engineering interest are represented by systems of one-dimensional hyperbolic partial differential equations (PDEs). Typical examples include the telegrapher equations for electrical circuits with transmission lines [26], the shallow water equations for open channels [11], [8], and the Aw-Rascle traffic flow model for road traffic systems [2].

The problem of analyzing exponential stability of the equilibrium for linear or quasilinear hyperbolic systems has been considered in the literature for more than 30 years. One main contribution of [18] is the trajectory-based technique via directive estimation of the solutions and their derivatives along the characteristic curves. Recent results have been extended to the case of conservation laws with perturbation source terms [23], [24], and absolutely exponential stability for switched hyperbolic systems [1], [22] under suitable dissipative boundary conditions.

On the other hand, a direct Lyapunov stability analysis technique is proposed in [7], with a strict quadratic Lyapunov

function to analyze the exponential stability of the equilibrium of conservation laws. The sufficient dissipative boundary condition is known to be strictly weaker than the one of [18] and allowing for numerically tractable methods for the design stabilizing boundary controls [9] or observers [5]. However, the approach by a quadratic Lyapunov function is not always effective to prove stability for hyperbolic systems. It has been shown in [3] that there exist stable 2×2 linear hyperbolic systems for which there does not exist any quadratic Lyapunov function.

In this paper, we consider the case in which hyperbolic systems belong to a class of positive systems whose state variables remain non-negative whenever initialized in the positive conditions. The positivity constraint on system states is motivated by the engineering physical nature of the variables involved (pressure, mass flux, density, vehicle speed and so on). The stability and other properties of positive systems often present some peculiar features [15], [16], [19], [25], which simplify the system analysis.

A necessary and sufficient condition for exponential stability of positive linear hyperbolic systems is presented by means of a novel strict Lyapunov function. To the best of our knowledge, this is the first work dealing with such a converse explicit Lyapunov result. Note that, [6, Proposition 3.7] implies that the dissipative boundary condition is a necessary and sufficient condition for stability of linear hyperbolic systems with dimension 1 to 5. Our necessary discussion relies on the assumption that velocities of hyperbolic systems have a common factor. Then, we could find a common time sample at which the dynamics of all states are projected to the boundaries; it transforms the dynamic of boundary conditions into a time-delay difference equation. The obtained conditions are written in terms of linear inequalities which can be solved numerically. The context of a continuous selection of boundary conditions in an interval matrix is also considered, supposing each entry belongs to a certain interval of the real line. Absolutely exponential stability also holds if and only if the upper bound satisfies the dissipative conditions.

The motivating application for the theoretical results presented in this paper is the development of dissipative boundary conditions with a strict linear Lyapunov function to regulate traffic states (vehicle density, speed) in a freeway link. Adopting boundary control strategies are particularly natural and necessary for freeway traffic systems, since the available control devices often depend on traffic signals for on-ramp metering and VMSs (Variable Message Signs) for regulating

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vehicle speed usually located at boundaries or cross-sections of the freeway link. Inside the freeway section, the number of vehicles is conservative. The dissipative boundary condition obtained from our theoretical contribution allows us to design the stabilizing feedback laws by integrating the on-ramp metering with the speed limiting control in the distributed control action.

The problem of freeway traffic control has been considered for a long time in the literature as reported in the survey [21] paper which involves a comprehensive bibliography.

In this paper, we go a step further since the control design is derived and analyzed directly from the linearized Aw-Rascle partial differential equations without any model approximation and discretisation. As Aw-Rascle traffic flow model, the speed dynamics might propagate upstream or downstream along the freeway for the same flow-rate of traffic, in practice, also called in the free-flow traffic or in the congestion mode. For the application, two different boundary feedback strategies are proposed in this paper by taking advantage of the traffic measurements coming from the upstream or downstream, respectively. In details, as the steady states in the free-flow mode, we detect downstream traffic states and integrate ramp metering with variable speed limit to control the upstream boundary of the freeway link, while in the congestion mode, the control design is the inverse. The theoretical contributions guarantee the exponential convergence of the system when closing the loop with the feedback control.

This paper is organized as follows. The class of positive linear hyperbolic systems is given in Section II. Wellposedness and positive properties of solutions of positive linear hyperbolic system are also discussed. In Section III, our main results on the necessary and sufficient conditions are derived for the exponential stability of positive linear hyperbolic systems. Finally, in Section IV, as a matter of illustration, an application of boundary feedback control of freeway traffic based on the linearized Aw-Rascle traffic flow equations is presented.

Notation. The set of non-negative integers and reals is \mathbb{N}_+ or \mathbb{R}_+ , respectively. $\mathbb{R}_+^{n \times n}$ or \mathbb{R}_+^n is the set of n -order square non-negative matrices or vectors. A matrix (in particular, a vector) A with entries in \mathbb{R}_+ is called a non-negative matrix (vector), and it is denoted as $A \succeq 0$. It is said to positive ($A \succ 0$), if all its entries are positive. The expression $A \succeq B$, indicate the difference $A - B$ is non-negative. The term non-positive, negative are defined analogously as \preceq and \prec . $\text{row}_i(A)$ denotes the i th row of matrix A . A real symmetric matrix A is positive definite (resp. semi-definite positive), if all its eigenvalues are positive (resp. non-negative); in that case we will use the respective notation $A > 0$ and $A \geq 0$.

The infinitum norm is denoted $|\cdot|_\infty$ and is defined by $|g|_\infty = \max_i |g_i|$ for all $g \in \mathbb{R}^n$. Given a function $g : [0, 1] \rightarrow \mathbb{R}^n$, we define its L^∞ -norm as $\|g\|_{L^\infty(0,1)} = \sup_{x \in (0,1)} |g(x)|_\infty$. We call $L^\infty(0, 1)$ the space of all measurable functions $g(x)$ for which $\|g\|_{L^\infty(0,1)} < \infty$.

II. POSITIVE HYPERBOLIC SYSTEMS: MATHEMATICAL PRELIMINARIES

Consider a class of positive linear hyperbolic systems described by the following equation:

$$\partial_t \xi(x, t) + \Lambda \partial_x \xi(x, t) = 0, \quad t \in \mathbb{R}_+, \quad x \in [0, 1], \quad (1)$$

where $\xi : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$. Assume $\Lambda \in \mathbb{R}_+^{n \times n}$ is a diagonal positive definite matrix such that $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, with $\lambda_i > 0$, for all $i \in \{1, \dots, n\}$. The boundary condition is written as

$$\xi(0, t) = G\xi(1, t), \quad t \in \mathbb{R}_+, \quad (2)$$

where matrix G belongs to $\mathbb{R}_+^{n \times n}$. We shall consider the initial condition given by

$$\xi(x, 0) = \xi^0(x), \quad x \in [0, 1], \quad (3)$$

for a given function $\xi^0 : [0, 1] \rightarrow \mathbb{R}^n$.

The linear hyperbolic system (1)-(3) is called positive if the trajectories of the system starting from any non-negative initial conditions remains forever non-negative. **As example ξ could be related to the gas flow in pipelines as considered in [14]. It could also coupled wave equations for dynamics of stimulated Raman scattering. See in also [12] where the positivity of the solutions is assumed.**

The existence of the solution of the hyperbolic system (1)-(3) in the set $C^0([0, \infty), H^1(0, 1)) \cap C^1([0, \infty), L^2(0, 1))$ holds for initial condition ξ^0 in $H^1(0, 1)$ **satisfying the zero-order compatibility condition (see e.g. [6] or [4, Theorem A.1]): $\xi^0(0) = G\xi^0(1)$.**

In the context of positive hyperbolic systems, we are able to state the following

Proposition 1: *Assume the matrix $G \in \mathbb{R}_+^{n \times n}$ in the boundary condition (2) is non-negative, i.e., $G \succeq 0$. Then, for any function $\xi^0 : [0, 1] \rightarrow \mathbb{R}^n$ in $H^1(0, 1)$ satisfying the zero-order compatibility condition, there exists a unique solution $\xi : [0, 1] \times [0, \infty)$ to the Cauchy problem (1)-(3).*

Moreover, the solution satisfies

- 1) *if $\xi^0(0) \in \mathbb{R}_+^n$ for all $x \in [0, 1]$, then $\xi(x, t) \succeq 0$, for all $(x, t) \in [0, 1] \times \mathbb{R}_+$;*
- 2) *if for all $i = 1, \dots, n$, $\text{row}_i(G) \neq 0$, and $\xi^0(x) \succ 0$ for all $x \in [0, 1]$, then $\xi(x, t) \succ 0$, for all $(x, t) \in [0, 1] \times \mathbb{R}_+$.*

Roughly speaking, besides the existence and uniqueness result of the Cauchy problem (1)-(3), the previous result states the positivity of hyperbolic systems, and that positivity of solutions is preserved for a special kind of boundary condition matrices.

Proof: The well-posedness follows from a classical application of Lumer-Philipp's theorem, as done e.g. in [4, Theorem A.1].

For system (1), we can calculate the propagating period of each state component $\xi_i(x, t)$ from the boundary $x = 0$ to $x = 1$ as

$$\tau_i = \frac{1}{\lambda_i}, \quad i = 1, \dots, n. \quad (4)$$

Denote $\underline{\tau}$ be the minimum time of all τ_i , i.e., $\underline{\tau} = \min_{i=1, \dots, n} \tau_i$, and for $p \in \mathbb{N}_+$, let $\Delta_p \subset \mathbb{R}_+$ be defined by

$$\Delta_p = [p\underline{\tau}, (p+1)\underline{\tau}]. \quad (5)$$

We now proceed by induction over the time interval Δ_p to prove Item 1 of Proposition 1. The first step consists to prove $\xi(x, t) \succeq 0$, for all $t \in \Delta_0$, $x \in [0, 1]$.

For an initial state $\xi^0 \in \mathbb{R}_+^n$ in $H^1(0,1)$, by the method of characteristics, the state component to the Cauchy problem (1)-(3) on $[0,1] \times \Delta_0$ is given as, for all $i = 1, \dots, n$

$$\xi_i(x, t) = \begin{cases} \sum_{j=1}^n g_{ij} \xi_j^0 \left[1 - \lambda_j \left(t - \frac{x}{\lambda_i} \right) \right], & \text{for } \lambda_i t > x, \\ \xi_i^0(x - \lambda_i t), & \text{for } \lambda_i t \leq x, \end{cases} \quad (6)$$

where g_{ij} is the entry (i, j) of the boundary condition matrix G .

Since $\xi^0(x) \in \mathbb{R}_+^n$, for all $x \in [0, 1]$, and $g_{ij} \geq 0$, it follows from (6) that $\xi(x, t) \geq 0$, for all $t \in \Delta_0$. This concludes the initial step of the induction.

Suppose for $p \geq 0$, $\xi(x, t) \geq 0$, for all $t \in \Delta_p$. Taking $\xi(\cdot, pT)$ as the initial condition of the system and applying the same argument as above, we get that the solution is uniquely defined, and $\xi(x, t) \geq 0$ for all $t \in \Delta_{p+1}$.

Therefore, the solution satisfies $\xi(x, t) \geq 0$, $t \in \mathbb{R}_+$. It concludes the proof of Item 1 in Proposition 1.

Let us prove Item 2, using again an induction argument. First from (6), noting that all the terms in the sum of the first line of (6) are non-negative and there exist at least one term g_{ij} of the sum in (6) is positive, we have $\xi_i(x, t) > 0$, for all $(x, t) \in [0, 1] \times \Delta_0$. The induction is shown in a similar way taking $\xi(\cdot, pT)$ as an initial condition.

This concludes the proof of Proposition 1. \blacksquare

Remark 1: From Proposition 1, we deduce the fact that the linear hyperbolic system (1)-(3) is positive or not depends only on whether boundary condition matrix G is non-negative. \circ

As the boundary condition matrix G including zero rows, i.e., $\text{row}_i(G) = 0$, for some row indices $i \in \{1, \dots, n\}$, the corresponding state $\xi_i(x, t)$ becomes zeros in a finite time. Then the rest state components might subsequently becoming zeros under the structure of matrix G . In the following, we give an estimation of each state component for system (1)-(3), whether it holds positive all the time or when it becomes zero.

Before stating the general results on the solutions of positive linear hyperbolic systems (1)-(3), let us construct a sequence of the index set according to the structure of matrix G .

Let $\Omega_0 = \{1, \dots, n\}$, and $\mathcal{I}_1 = \{i : \sum_{j \in \Omega_0} g_{ij} = 0\}$ which includes all indices of the zero rows of matrix G , i.e., $\text{row}_i(G) = 0$, $i = 1, \dots, n$.

We denote the following index sets iteratively, for all $k \in \mathbb{N}_+$, $k \geq 2$,

$$\Omega_{k-1} = \{1, \dots, n\} \setminus \bigcup_{l=1}^{k-1} \mathcal{I}_l, \quad (7)$$

and

$$\mathcal{I}_k = \left\{ i : i \in \Omega_{k-1}, \text{ and } \sum_{j \in \Omega_{k-1}} g_{ij} = 0 \right\}. \quad (8)$$

Then, \mathcal{I}_k is the largest subset in Ω_{k-1} such that $\sum_{j \in \Omega_{k-1}} g_{ij} = 0$ for all $i \in \mathcal{I}_k$, and for all indices $i \notin \mathcal{I}_k$ it holds $\sum_{j \in \Omega_{k-1}} g_{ij} \neq 0$.

Since G is a n -dimension matrix, the construction process (7)-(8) could be finished in a finite number of iterative steps, that is the index sequence \mathcal{I}_k , $k = 1, \dots$, is a finite set, and includes at most n subsets. Let K be such that the last subset \mathcal{I}_K is empty, $\mathcal{I}_K = \emptyset$, and other sets $\mathcal{I}_1, \dots, \mathcal{I}_{K-1}$ are non-empty.

Item 2 in Proposition 1 indicates that the positive linear hyperbolic systems (1)-(3) might experience a positive solution as the boundary condition matrix G does not include non-zero rows. Given a submatrix G_{Ω_k} (the subscripts instruct the rows and columns of G that make up the square submatrix), $k = 0, 1, \dots, K-1$, we prove that the state components ξ_i whose index i belongs to the subset Ω_k , remains positive as the submatrix G_{Ω_k} does not have non-zero rows. To be more specific the following proposition holds.

Proposition 2: Assume that the matrix G includes zero rows, i.e., $\text{row}_i(G) = 0$, for some $i \in \{1, \dots, n\}$, and the sequence of index sets \mathcal{I}_k , $k = 1, \dots, K$, is defined as (8), $\mathcal{I}_K = \emptyset$. Then, for any positive initial condition $\xi^0(x) \succ 0$, $x \in [0, 1]$, the solution to the positive linear hyperbolic system (1)-(3) satisfies

$$1) \quad \forall i \in \mathcal{I}_{K_0}, \quad K_0 = 1, \dots, K-1, \quad \xi_i(x, t) = 0, \quad (9)$$

for all $(x, t) \in [0, 1] \times \left(\sum_{k=1}^{K_0} \max_{i \in \mathcal{I}_k}(\tau_i), +\infty \right)$, where τ_i is defined by (4);

$$2) \quad \forall j \in \Omega_{K-1}, \quad \xi_j(x, t) > 0, \quad (10)$$

for all $(x, t) \in [0, 1] \times \mathbb{R}_+$.

Proof: 1) Let us prove Item 1 successively for $K_0 = 1, 2, \dots, K-1$.

Since matrix G includes some zero rows, that is $\mathcal{I}_1 \neq \emptyset$, then for all indices $i \in \mathcal{I}_1$, and $t \in \mathbb{R}_+$, at the left boundary we have $\xi_i(0, t) = 0$. Hence, for all $i \in \mathcal{I}_1$ and $x \in [0, 1]$,

$$\xi_i(x, t) = 0, \quad t \in (\max_{i \in \mathcal{I}_1}(\tau_i), +\infty). \quad (11)$$

This is (9) for $K_0 = 1$.

For the index $i \in \mathcal{I}_2$, it holds $\sum_{j \in \Omega_1} g_{ij} = 0$. Since $g_{ij} \geq 0$, it follows that $g_{ij} = 0$, for all $i \in \mathcal{I}_2$, and $j \in \Omega_1$. Then, due to (2),

$$\begin{aligned} \xi_i(0, t) &= \sum_{j \in \mathcal{I}_1} g_{ij} \xi_j(1, t) + \sum_{j \in \Omega_1} g_{ij} \xi_j(1, t) \\ &= \sum_{j \in \mathcal{I}_1} g_{ij} \xi_j(1, t). \end{aligned} \quad (12)$$

From (11), for the indices $j \in \mathcal{I}_1$, at the right boundary, we have $\xi_j(1, t) = 0$ when $t > \max_{i \in \mathcal{I}_1}(\tau_i)$. Substituting into (12), it yields $\xi_i(0, t) = 0$, for all $t \in (\max_{i \in \mathcal{I}_1}(\tau_i), +\infty)$.

Thus, for all $i \in \mathcal{I}_2$ and $x \in [0, 1]$,

$$\xi_i(x, t) = 0, \quad t \in \left(\sum_{k=1}^2 \max_{i \in \mathcal{I}_k}(\tau_i), +\infty \right). \quad (13)$$

This is (9) for $K_0 = 2$.

Repeating above procedures, we prove by the induction that, for each index set \mathcal{I}_{K_0} , $K_0 = 1, \dots, K-1$, the corresponding state components $\xi_i(x, t)$, $x \in [0, 1]$, with index $i \in \mathcal{I}_{K_0}$,

become zero in a finite time. It concludes the proof of Item 1 in Proposition 2.

2) We now proceed by induction over the index set Ω_k , $k = 1, \dots, K-1$ to prove that for all $j \in \Omega_k$, $x \in [0, 1]$ and $t \in [0, \sum_{k=0}^k \min_{j \in \Omega_k}(\tau_j)]$, it holds $\xi_j(x, t) > 0$.

For the index $i \in \Omega_1$, and under the boundary condition (2), we have

$$\xi_i(0, t) = \sum_{j \in \Omega_0} g_{ij} \xi_j(1, t). \quad (14)$$

On the other hand, given a positive initial condition $\xi^0(x) > 0$, $x \in [0, 1]$, it holds $\xi_j(1, t) > 0$, for all $j \in \Omega_0$ and $t \in [0, \min_{j \in \Omega_0}(\tau_j)]$. Substituting into (14), and because $\sum_{j \in \Omega_1} g_{ij} \neq 0$, for all $i \in \Omega_1$, one gets

$$\xi_i(0, t) > 0, \quad t \in [0, \min_{j \in \Omega_0}(\tau_j)]. \quad (15)$$

Thus, for $i \in \Omega_1$, $x \in [0, 1]$, the state component $\xi_i(x, t)$ satisfies

$$\xi_i(x, t) > 0, \quad t \in [0, \min_{j \in \Omega_0}(\tau_j) + \min_{j \in \Omega_1}(\tau_j)]. \quad (16)$$

This is the first step of the induction.

At the second step, for all $i \in \Omega_2$, combining

$$\xi_i(0, t) = \sum_{j \in \mathcal{I}_1} g_{ij} \xi_j(1, t) + \sum_{j \in \Omega_1} g_{ij} \xi_j(1, t), \quad (17)$$

$\sum_{j \in \Omega_1} g_{ij} \neq 0$ and (16) for $x = 1$, we get, for all j in Ω_1 ,

$$\xi_j(1, t) > 0, \quad t \in [0, \min_{j \in \Omega_0}(\tau_j) + \min_{j \in \Omega_1}(\tau_j)]. \quad (18)$$

Because again $\sum_{j \in \Omega_1} g_{ij} \neq 0$, for all $i \in \Omega_2$, one gets with (17)

$$\xi_i(0, t) > 0, \quad t \in [0, \sum_{k=0}^1 \min_{j \in \Omega_k}(\tau_j)]. \quad (19)$$

Hence, with (2), for all $i \in \Omega_2$, $x \in [0, 1]$,

$$\xi_i(x, t) > 0, \quad t \in [0, \sum_{k=0}^2 \min_{j \in \Omega_k}(\tau_j)]. \quad (20)$$

This is the second step of the induction from the first step (the other steps are proven similarly). Therefore, by induction, we prove that, for the index set Ω_{K-1} , for all $i \in \Omega_{K-1}$, $x \in [0, 1]$, the state components satisfy

$$\xi_i(x, t) > 0, \quad t \in [0, \sum_{k=0}^{K-1} \min_{j \in \Omega_k}(\tau_j)]. \quad (21)$$

Moreover, due to (2), we have

$$\begin{aligned} \xi_i(0, t) &= \left[\sum_{j \in \mathcal{I}_1} + \dots + \sum_{j \in \mathcal{I}_{K-1}} \right] g_{ij} \xi_j(1, t) \\ &+ \sum_{j \in \Omega_{K-1}} g_{ij} \xi_j(1, t). \end{aligned} \quad (22)$$

Since $\mathcal{I}_K = \emptyset$, then, for all $i \in \Omega_{K-1}$, $\sum_{j \in \Omega_{K-1}} g_{ij} \neq 0$. Taking

$$\tau' = \sum_{k=1}^{K-1} \min_{j \in \Omega_k}(\tau_j), \quad (23)$$

as the initial time, and using (21), we see the solution ξ as a strictly positive initial condition $\xi_i(x, \tau') > 0$, for all $x \in [0, 1]$.

Following the proof of Item 2 of Proposition 1, for all $i \in \Omega_{K-1}$, we have $\xi_i(x, t) > 0$, for all $t \geq \tau'$, $x \in [0, 1]$.

This concludes the proof of Proposition 2. \blacksquare

Example 1: Let us consider the following non-negative boundary condition matrix

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \{4\}$, $\mathcal{I}_3 = \{2\}$, and $\mathcal{I}_4 = \emptyset$. Thus $\{1, 2, 3, 4\} - \bigcup_{k=1}^3 \mathcal{I}_k = \{3\}$. For any initial condition $\xi^0(x) \succ 0$, the third state component satisfies $\xi_3(x, t) \succ 0$ for all $(x, t) \in [0, 1] \times \mathbb{R}_+$. For any $x \in [0, 1]$, other state components become zeros in finite time. More precisely, $\xi_1(x, t) = 0$ when $t \in [\tau_1, \infty)$, $\xi_4(x, t) = 0$ when $t \in [\tau_1 + \tau_4, \infty)$, and $\xi_2(x, t) = 0$ when $t \in [\tau_1 + \tau_4 + \tau_2, \infty)$. \star

Remark 2: We can consider more general positive linear hyperbolic systems with both negative and positive velocities for Λ , such as with $\lambda_i < 0$ for $i \in \{1, \dots, m\}$, and $\lambda_i > 0$ for $i \in \{m+1, \dots, n\}$. By defining the state description $\xi = (\xi^-, \xi^+)^T$, where $\xi^- \in \mathbb{R}^m$ and $\xi^+ \in \mathbb{R}^{n-m}$, and the change of variable $w(x, t) = [\xi^-(1-x, t), \xi^+(x, t)]^T$, we can obtain a new hyperbolic system in the same form as (1)-(2). \circ

III. NECESSARY AND SUFFICIENT CONDITIONS ON THE EXPONENTIAL STABILITY

Let us start this section by defining the notion of exponential stability under consideration in this paper.

Definition 1: The linear hyperbolic system (1)-(2) is said to be exponentially stable in L^∞ -norm, if there exist $\nu > 0$ and $C > 0$ such that, for every initial condition $\xi(\cdot, 0) \in H^1(0, 1)$ satisfying the zero-order compatibility condition, the solution ξ to the Cauchy problem (1)-(3) satisfies

$$\|\xi(\cdot, t)\|_{L^\infty(0,1)} \leq C e^{-\nu t} \|\xi(\cdot, 0)\|_{L^\infty(0,1)}, \quad (24)$$

for all $t \in \mathbb{R}_+$.

A. Conditions for the existence of a linear Lyapunov function

Let us first deal with a sufficient condition for the existence of a linear Lyapunov function yielding an exponential stability of positive linear hyperbolic systems. This sufficient condition is written in terms of a dissipative boundary condition and is also necessary under an additional assumption, as written in the following first main result.

Theorem 1: Consider the positive linear hyperbolic system (1)-(3) with matrix $G \in \mathbb{R}_+^{n \times n}$. Let $\theta \in \mathbb{R}_+^n$ be a positive vector, i.e. $\theta \succ 0$, $\mu > 0$ be a constant such that

$$[G^T - e^{-\mu} I_n] \theta \preceq 0. \quad (25)$$

- The system (1)-(3) is exponentially stable if condition (25) holds, and moreover a Lyapunov function is given by, for all ξ in $L^1(0, 1)$,

$$V(\xi) = \int_0^1 \theta^T \Lambda^{-1} \xi e^{-\mu x} dx. \quad (26)$$

- Conversely, if the system is exponential stable and the inverse of velocities, λ_i^{-1} , are commensurable, then there exist $\theta > 0$ and $\mu > 0$ such that (25) holds.

Remark 3: Before proving this theorem, let us note that the Lyapunov function candidate V defined in (26) is inspired by [7] among other references where the same weight $x \mapsto \exp(-\mu x)$ is used combined with a L^2 integral norm (see also [10]). It is also inspired by [20] where the Lyapunov function is linear which implies a prior requirement that all state variables are non-negative. This linear Lyapunov function candidate is not appropriate for general linear hyperbolic systems. \square

Proof: Sufficiency. Using an integration by part, the time derivative of the Lyapunov function (26) along the solutions to (1) is computed as $\dot{V} = -\int_0^1 \theta^\top \partial_x \xi e^{-\mu x} dx = -[\theta^\top \xi e^{-\mu x}]_0^1 - \mu \int_0^1 \theta^\top \xi e^{-\mu x} dx$. Now under the boundary condition (2), it holds $\dot{V} = \xi^\top(1, t)[G^\top - e^{-\mu} I_n] \theta - \mu \int_0^1 \theta^\top \xi e^{-\mu x} dx$.

Moreover, using G in \mathbb{R}_+^n and Proposition 1, we have $\xi(1, t) \in \mathbb{R}_+^n$ for all $t \geq 0$ and thus $\dot{V} \leq -\mu \lambda V(\xi)$ where $\lambda = \min\{\lambda_i, i = 1, \dots, n\}$. By remarking that there exist $C_1 > 0$ and C_2 (depending on θ , Λ , and μ) such that $C_1 \|\xi\|_{L^\infty(0,1)} \leq V(\xi) \leq C_2 \|\xi\|_{L^\infty(0,1)}$, we may deduce that the solutions of the positive linear hyperbolic system (1) exponentially converge to 0 in L^∞ -norm.

This complete the proof of sufficiency.

Necessity. Since the inverse of the velocities, λ_i^{-1} , are commensurable, there exists a sufficiently small time-interval $\Delta t \in \mathbb{R}_+$, such that the time periods $\tau_j = d_j \Delta t$, for suitable integers $d_j \in \mathbb{N}_+$, $j = 1, \dots, n$.

Then, we could find a common time sample at which the dynamics of each state component $\xi_i(x, t)$ project to its left boundary. Roughly speaking the necessary condition is later derived by considering the time evolution of $\xi_i(0, t)$, for all $t \geq 0$.

Let $\bar{\tau}$ be the maximum time of all τ_i , i.e., $\bar{\tau} = \max_{i=1, \dots, n} \tau_i$. Using that state ξ_i is constant along the characteristic curves, the values of every state component at the right boundary at time t , are equivalent to those at time $t - \tau_i$, that is,

$$\xi_i(1, t) = \xi_i(0, t - \tau_i), \quad (27)$$

for all $t \geq \tau_i$, $i = 1, \dots, n$.

Combined with the boundary condition (2), for $t \geq \bar{\tau}$, the dynamics of system (1) at the left boundary $x = 0$ can be represented as

$$\xi(0, t) = \sum_{j=1}^n G[j] \xi(0, t - \tau_j), \quad (28)$$

where $G[j] = [0, \dots, \text{col}_j(G), \dots, 0]$ is a n -dimension square matrix, and $\text{col}_j(G)$ is the j -th column of the matrix G . We find $\sum_{j=1}^n G[j] = G$. Using the notation $l = \frac{\bar{\tau}}{\Delta t}$, we note that $d_j \leq l$ for all $j = 1, \dots, n$, and thus we get the following time-delay equation from (28),

$$\xi(0, s\Delta t) = \sum_{j=1}^n G[j] \xi(0, s\Delta t - d_j \Delta t), \quad (29)$$

for $s = l, l+1, \dots$

Simply denote $y_s = \xi(0, s\Delta t)$. By reorganizing the terms in (29), we get the following time-delay system, for $s = l, l+1, \dots$,

$$y_s = \sum_{d=1}^l \mathcal{G}_d y_{s-d}, \quad (30)$$

where

$$\mathcal{G}_d = \begin{cases} \sum_{d_j=d} G[j] & \text{there exists } j \text{ such that } d = d_j \\ 0 & \text{otherwise} \end{cases}$$

and initial values $y_{l-d} = \xi(0, (l-d)\Delta t)$, for all $d = 1, \dots, l$, is uniquely defined with Proposition 1.

Now, we consider the discrete-time instant

$$h = \frac{1}{\Delta t} \sum_{i=1}^n \tau_i. \quad (31)$$

Using Proposition 2, we know, as $s \geq h+1$, the state components of y_s is non-negative.

From (30), for all $s = h+2, \dots, h+p$, ($p \in \mathbb{N}_+$), we have, respectively

$$y_{h+2} = \sum_{d=1}^l \mathcal{G}_d y_{h+2-d}, \dots, y_{h+p} = \sum_{d=1}^l \mathcal{G}_d y_{h+p-d}.$$

Summing the above equalities one obtains

$$\begin{aligned} \sum_{s=h+2}^{h+p} y_s &= \sum_{s=h+2}^{h+p} \sum_{d=1}^l \mathcal{G}_d y_{s-d} = \sum_{d=1}^l \mathcal{G}_d \sum_{s=h+2}^{h+p} y_{s-d} \\ &= \sum_{d=1}^l \mathcal{G}_d \left[\sum_{s=h+2}^{h+p-d} y_s + \sum_{s=h+2-d}^{h+1} y_s \right]. \end{aligned}$$

Since the system (1) is exponentially stable, then $y_{h+p} \rightarrow 0$ as $p \rightarrow \infty$ and $\sum_{s=h+2}^{\infty} y_k < \infty$. Hence,

$$\begin{aligned} -\sum_{d=1}^l \mathcal{G}_d \sum_{s=h+2-d}^{h+1} y_s &= \left[\sum_{d=1}^l \mathcal{G}_d - I_n \right] \sum_{s=h+2}^{\infty} y_s \\ &= [G - I_n] \sum_{s=h+2}^{\infty} y_s. \end{aligned} \quad (32)$$

Moreover, due to Item 1 of Proposition 1, all the terms in the left-hand side of equality (32) are non-positive, we have

$$\begin{aligned} [G - I_n] \sum_{s=h+2}^{\infty} y_s &\preceq -\sum_{d=1}^l \mathcal{G}_d y_{h+1} \\ &\preceq -G y_{h+1}. \end{aligned} \quad (33)$$

To conclude the necessity part of the proof, and show that inequality (25) holds for suitable $\theta > 0$ and $\mu > 0$, two cases about matrix G may occur:

Case 1: $\text{row}_i(G) \neq 0$, for all $i = 1, \dots, n$.

According to Item 2 in Proposition 1, the solution of the Cauchy problem (1)-(3) with an positive initial condition $\xi^0(x) > 0$ satisfies $\xi(x, t) > 0$ for all $(x, t) \in [0, 1] \times \mathbb{R}_+$.

In particular $y_s > 0$, for all $s \geq h+1$. Thus $-G y_{h+1} < 0$. With (33), it holds

$$[G - I_n] b < 0, \quad (34)$$

by letting $b = \sum_{s=h+2}^{\infty} y_s \succ 0$.

Moreover, there exists $\mu > 0$ such that

$$[G - e^{-\mu} I_n] b \preceq 0. \quad (35)$$

Because $-[G - e^{-\mu} I_n]$ is a M-matrix and $b \succ 0$ satisfying (35), there exists a positive vector $\theta \in \mathbb{R}_+^n$, $\theta \succ 0$, such that $[G^\top - e^{-\mu} I_n] \theta \preceq 0$.

Case 2: There exists at least one row of indices $i \in \{1, \dots, n\}$, such that $\text{row}_i(G) = 0$.

Assume the sequence of index set \mathcal{I}_k as (8) is constructed, $k = 1, \dots, K-1$, and with an initial condition to (1)-(3) starting from $\xi^0(x) \succ 0$ for all $x \in [0, 1]$.

Since $h = \frac{1}{\Delta t} \sum_{i=1}^n \tau_i$, we have

$$h \geq \frac{1}{\Delta t} \sum_{k=1}^{K-1} \max_{i \in \mathcal{I}_k} (\tau_i). \quad (36)$$

Therefore according to Proposition 2, for all $s \geq h+1$, it follows, $(y_s)_{\mathcal{I}_k} = 0$, for all $k = 1, \dots, K-1$, and $(y_s)_{\Omega_{K-1}} \succ 0$.

Because again $b = \sum_{s=h+2}^{\infty} y_s$ is constant, it follows $b_{\mathcal{I}_k} = 0$, for all $k = 1, \dots, K-1$, and $b_{\Omega_{K-1}} \succ 0$.

Furthermore, under the structure of matrix G , the (i, i) entries of $[G - I_n]_{\mathcal{I}_k}$ (a submatrix of $[G - I_n]$ whose row indices belong to \mathcal{I}_k) is -1 , $i \in \mathcal{I}_k$, and the (i, j) entries of $[G - I_n]_{\mathcal{I}_k}$ is 0, for $i \in \mathcal{I}_k$, $j \in \Omega_k$, $k = 1, \dots, K-1$. Hence, we have, for all $k = 1, \dots, K-1$,

$$([G - I_n] b)_{\mathcal{I}_k} = 0. \quad (37)$$

On the other hand, from inequality (33), we have

$$([G - I_n] b)_{\Omega_{K-1}} \preceq (-G y_{h+1})_{\Omega_{K-1}} \prec 0. \quad (38)$$

Let vector \tilde{b} in \mathbb{R}^n such that $\tilde{b}_{\mathcal{I}_k} = \epsilon_k$, for all $k = 1, \dots, K-1$, and $\tilde{b}_{\Omega_{K-1}} = 0$ for the remaining entries of \tilde{b} , ϵ_k are sufficiently small positive values satisfying

$$\epsilon_k > \max_{i \in \mathcal{I}_k} \left\{ \sum_{r=1}^k \sum_{j \in \mathcal{I}_r} g_{ij} \epsilon_r \right\}. \quad (39)$$

Hence, $b + \tilde{b} \succ 0$, and, due to (37) and (38), $[G - I_n] (b + \tilde{b}) \prec 0$ holds. The remaining proof of this case is as in the first case.

It concludes the proof of Theorem 1. \blacksquare

B. Conditions for the existence of a quadratic Lyapunov function

Before commenting on the second main result, let us recall the following sufficient condition for the existence of a quadratic Lyapunov function.

Proposition 3: [9] Let $P \in \mathbb{R}^{n \times n}$ be a diagonal positive definite matrix, $\mu > 0$ be a constant, the function

$$V(\xi) = \int_0^1 \xi^\top \Lambda^{-1} P \xi e^{-2\mu x} dx, \quad (40)$$

is a quadratic Lyapunov function for system (1)-(3), that is $\dot{V} \leq -\frac{2\mu}{(\min_i \lambda_i)^{-1}} V$ along the solutions to (1)-(3), if the matrix inequality

$$e^{-2\mu} P - G^\top P G \geq 0 \quad (41)$$

holds.

The second main result, namely Theorem 2 below, proves that the sufficient condition for the existence of a linear Lyapunov function as written in item 1) of Theorem 1 is equivalent to the dissipative boundary condition (41) considered in Proposition 3. In details, Theorem 2 below gives three characterizations for the matrix inequality (41) to hold, each of them implying the existence of a quadratic Lyapunov function for system (1)-(3).

Theorem 2: Given a positive linear hyperbolic system (1)-(3), the following are equivalent:

- 1) there exist $\mu > 0$, and a vector $\theta \succ 0$, such that condition (25) holds;
- 2) there exist $\mu > 0$ and a diagonal positive definite matrix $P > 0$, such that inequality (41) holds;
- 3) $\rho(G) < 1$, where $\rho(G)$ is the spectral radius of the matrix G .

Proof: 1) \Rightarrow 2) We denote $[G^\top - e^{-\mu} I_n] \theta = v \preceq 0$, and $[G - e^{-\mu} I_n] b = u \preceq 0$, where $\theta = [\theta_1, \dots, \theta_n]^\top$, and $b = [b_1, \dots, b_n]^\top$.

Using θ and b , we construct a diagonal positive definite matrix as $P = \text{diag} \left\{ \frac{\theta_1}{b_1}, \frac{\theta_2}{b_2}, \dots, \frac{\theta_n}{b_n} \right\}$. We have $Pb = \theta$ and $Gb = u + e^{-\mu} b$. Then

$$\begin{aligned} (e^{-2\mu} P - G^\top P G) b &= e^{-2\mu} \theta - G^\top P (u + e^{-\mu} b) \\ &= e^{-\mu} (e^{-\mu} I_n \theta - G^\top \theta) - G^\top P u \\ &= -e^{-\mu} v - G^\top P u \succeq 0. \end{aligned} \quad (42)$$

Since $e^{-2\mu} P - G^\top P G$ is a symmetric M-matrix and satisfies inequality (42), it must be a semi-definite positive matrix, and thus (41) holds.

2) \Rightarrow 3) Using the congruence transformation defined by the positive definite matrix $e^{-\mu} P^{\frac{1}{2}}$ on both sides of $e^{-2\mu} P - G^\top P G = Q$, yields

$$\begin{aligned} (e^\mu P^{-\frac{1}{2}} G^\top P^{\frac{1}{2}}) (P^{\frac{1}{2}} G P^{-\frac{1}{2}} e^\mu) - I_n \\ = -(e^\mu P^{-\frac{1}{2}}) Q (P^{-\frac{1}{2}} e^\mu). \end{aligned} \quad (43)$$

Since the right-hand side of (43) is semi-definite negative, this requires that $\|P^{\frac{1}{2}} G P^{-\frac{1}{2}} e^\mu\|_2 \leq 1$. Thus, with $\mu > 0$, $\rho(G) = \rho(P^{\frac{1}{2}} G P^{-\frac{1}{2}}) \leq \|P^{\frac{1}{2}} G P^{-\frac{1}{2}}\|_2 < 1$.

3) \Rightarrow 1) follows from the Perron-Frobenius theorem.

This concludes the proof of Theorem 2. \blacksquare

Example 2: Consider a positive hyperbolic linear system (1)-(2) with the characteristic and the boundary condition matrices given, respectively, by $\Lambda = \text{diag}\{1, -1, 2\}$, and

$$G = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.8 & 0.2 & 0.1 \\ 0.4 & 0 & 0.2 \end{bmatrix}. \quad (44)$$

The spectral radius of non-negative matrix G is $\rho(G) = 0.8990 < 1$. After solving the linear inequality condition (25), one can easily construct a linear Lyapunov function (26) with $\theta = (6, 4, 3)^\top \succ 0$ and $\mu = 0.01 > 0$. Following the results of Theorem 2, a quadratic Lyapunov function (40) can be further constructed with $b = (4, 5, 5)^\top \succ 0$ and $P = \text{diag}\{1.5, 0.8, 0.6\} > 0$. \star

IV. APPLICATION TO FREEWAY TRAFFIC CONTROL

To illustrate the usefulness of the results, we will consider the stabilization problem of vehicle transportation systems in a freeway link which is modeled by the positive hyperbolic linear systems.

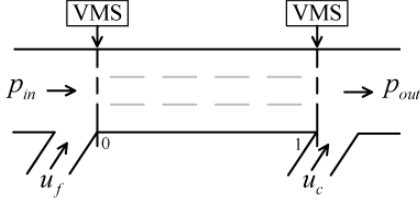


Fig. 1. A freeway link controlled by local on-ramp metering and variable speed sign.

A homogeneous freeway section between two successive on-ramps is sketched in Fig. 1, where u_f and u_c are two boundary feedback laws that will be defined in this section. We assume that the upstream and the downstream boundaries of the freeway section are provided with the on-ramp metering to regulate the flow-rate of driving-in vehicles, and with the VMSs to limit the driving speed of the mainline traffic.

The control of the freeway traffic is defined from upstream to downstream and regulates the traffic limit of the freeway section (see Section IV-C below for more details). Usually the transmission speed of information from upstream to downstream is faster than the speed of the vehicles in the freeway. Therefore, for this application, the transmission delay will be neglected.

A. Aw-Rascle traffic flow model

The traffic dynamics of the freeway link are described by a system of two laws of conservation, the so-called Aw-Rascle traffic flow model [2]. It is

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(v + p(\rho)) + v \partial_x(v + p(\rho)) = 0 \end{cases}, \quad (45)$$

where $\rho(x, t)$ is vehicle density, $v(x, t)$ is average speed, $x \in [0, 1]$, $t \geq 0$, and $p(\rho)$ means the traffic pressure term, which is supposed to increase over the vehicle density.

In [27], a special pressure function $p(\rho)$ is given as

$$p(\rho) = v_f - V(\rho), \quad (46)$$

where v_f is the free (maximal) speed, and $V(\rho)$ is the speed-density fundamental diagram. Typically, with the Greenshields fundamental diagram [13],

$$V(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right), \quad (47)$$

where ρ_m is the maximal density. Thus, we have $p(\rho) = a\rho$, and $a = \frac{v_f}{\rho_m}$.

Let $w = v + a\rho$, $z = v$. The nonlinear hyperbolic equation (45) may be written in the characteristic Riemann coordinates as

$$\begin{cases} \partial_t w + z \partial_x w = 0 \\ \partial_t z + (2z - w) \partial_x z = 0 \end{cases}. \quad (48)$$

In (48), the first velocity is assumed to be positive, $z > 0$, and the second velocity $2z - w$ is assumed to be nonzero so that its sign does not change. The sign of the second velocity indicates the transfer direction of the vehicle speed from the freeway upstream section to the downstream, or inverse. It is usually used as the feature to determine the freeway traffic lies in the free-flow mode or in the congestion mode [17]. In practice, also determines how to regulate the freeway traffic by using the upstream or the downstream traffic measurements (see Section V-C below).

B. Steady-state and linearization

A steady-state of the freeway traffic is a constant traffic state (ρ^*, v^*) , which satisfies one of the following relation

$$\begin{aligned} p_{in} + r_f &= \rho^* v^*, & \text{or} \\ \rho^* v^* &= p_{out} - r_c, \end{aligned} \quad (49)$$

where p_{in} , r_f are constant flow-rates of the driving-in vehicles through the upstream mainline and the upstream on-ramp of the freeway section, and p_{out} , r_c are constant driving-out flow-rates of the downstream mainline and on-ramp.

In order to linearize the model (45), we define the deviations of the state $\rho(x, t)$, $v(x, t)$ with respect to the steady-state as, respectively

$$\begin{aligned} \hat{\rho} &= \rho - \rho^*, \\ \hat{v} &= v - v^*. \end{aligned} \quad (50)$$

The linearization of system (48) with the steady-state (\hat{w}, \hat{z}) is written as

$$\begin{cases} \partial_t \hat{w} + (z^* + \hat{z}) \partial_x \hat{w} = 0 \\ \partial_t \hat{z} + (2z^* - w^* + 2\hat{z} - \hat{w}) \partial_x \hat{z} = 0 \end{cases}, \quad (51)$$

In Riemann coordinates, the deviations are $\hat{w} = w - w^*$, and $\hat{z} = z - z^*$, with the associated steady-state given as $w^* = v^* + a\rho^*$, $z^* = v^*$.

For the traffic steady-state, we only assume that the product of vehicle density times speed, that is the flow-rate is constant. Then there exists a set of the steady-states (ρ^*, v^*) satisfying the condition (49). In fact, since

$$\lambda_2^* = 2z^* - w^* = v^* - a\rho^*, \quad (52)$$

we regard just as two categories of the traffic steady-states. While the steady-state (ρ^*, v^*) satisfies $v^* - a\rho^* > 0$, we call it the free-flow steady-state, while $v^* - a\rho^* < 0$, the congestion steady-state.

C. Boundary feedback control

Now we are going to show how Theorem 2 may be applied to analyze the local stability of the freeway traffic under the boundary feedback control. As mentioned above, the regulation strategies (using different controller and measurement) are designed depending on the steady-state of the freeway traffic lying in the free-flow or congestion traffic modes.

Two cases are discussed separately.

Case 1: $\lambda_2^* = 2z^* - w^* > 0$, the steady-state (ρ^*, v^*) lies in the free-flow mode.

In this case, the velocity information (z or v) is propagating from upstream to downstream, **and thus it is natural to control u_f and $v(0, t)$** . We assume that the vehicle density $\rho(1, t)$ and the average speed $v(1, t)$ at the downstream boundary are measured, and the control units are the upstream on-ramp metering and the driving-in speed limit of the freeway section. Precisely, we introduce the following boundary feedback law:

$$\begin{cases} u_f(t) = r_f + k_\rho^f(\rho(1, t) - \rho^*) \\ v(0, t) = v^* + k_v^f(v(1, t) - v^*) \end{cases}, \quad (53)$$

with feedback gains $k_\rho^f > 0$, and $k_v^f > 0$. **The previous controllers are proportional controllers that do not need the knowledge of the entire state, but only the measurement of the vehicle density $\rho(1, t)$ and average speed $v(1, t)$ at the downstream boundary.**

At the upstream boundary of the freeway section, *i.e.*, $x = 0$, the driving-in flow-rate conservation equation holds

$$u_f(t) + p_{in} = \rho(0, t)v(0, t). \quad (54)$$

After the linearization of the boundary condition (54) with integrating the feedback control law (53), we have the following boundary condition

$$\hat{\rho}(0, t) = \frac{k_\rho^f}{v^*} \hat{\rho}(1, t) - \rho^* \frac{k_v^f}{v^*} \hat{v}(1, t). \quad (55)$$

Since $\hat{v}(1, t) = \hat{z}(1, t)$, and $\hat{\rho}(x, t) = (\hat{w}(x, t) - \hat{v}(x, t))/a$, as $x = 0, 1$, we could rewrite the condition (55) in the Riemann coordinates for system (51) as

$$\begin{aligned} \hat{w}(0, t) &= \frac{k_\rho^f}{v^*} \hat{w}(1, t) \\ &+ \left[k_v^f - a\rho^* \frac{k_v^f}{v^*} - \frac{k_\rho^f}{v^*} \right] \hat{z}(1, t). \end{aligned} \quad (56)$$

Then the boundary condition that need to be imposed for system (51) is written as

$$\begin{bmatrix} \hat{w}(0, t) \\ \hat{z}(0, t) \end{bmatrix} = G_f \begin{bmatrix} \hat{w}(1, t) \\ \hat{z}(1, t) \end{bmatrix}, \quad (57)$$

where

$$G_f = \begin{bmatrix} \frac{k_\rho^f}{v^*} & k_v^f - a\rho^* \frac{k_v^f}{v^*} - \frac{k_\rho^f}{v^*} \\ 0 & k_v^f \end{bmatrix}. \quad (58)$$

In the boundary condition matrix G_f ,

$$k_v^f - a\rho^* \frac{k_v^f}{v^*} - \frac{k_\rho^f}{v^*} = \frac{\lambda_2^* k_v^f - k_\rho^f}{v^*}. \quad (59)$$

Let us choose $\lambda_2^* k_v^f - k_\rho^f \geq 0$, the boundary condition matrix G_f is non-negative, *i.e.*, $G_f \geq 0$. Therefore, applying Proposition 1, system (51) with the boundary condition (57) is a positive linear hyperbolic system.

Straightforward calculations show that Theorem 1 holds for the matrix G_f , if and only if, k_ρ^f, k_v^f of the feedback control law (53) can be selected as $0 < k_\rho^f < v^*$, $0 < k_v^f < 1$, and $\lambda_2^* k_v^f - k_\rho^f \geq 0$.

Case 2: $\lambda_2^* = 2z^* - w^* < 0$, the steady-state (ρ^*, v^*) lies in the congestion mode.

In this case, velocity information is propagating **from upstream to downstream and thus it is natural to control u_c and**

$v(1, t)$. Thus, conversely, the measurements are the vehicle density $\rho(0, t)$ and the average speed $v(0, t)$ at the upstream boundary, and the control units are the downstream on-ramp metering and the driving-out speed limit of the freeway section.

Then, we introduce the other boundary feedback law:

$$\begin{cases} u_c(t) = r_c - k_\rho^c(\rho(0, t) - \rho^*) \\ v(1, t) = v^* + k_v^c(v(0, t) - v^*) \end{cases}, \quad (60)$$

with feedback gains $k_\rho^c > 0$, and $k_v^c > 0$. **The previous controllers are again proportional controllers that need only the measurement of the vehicle density $\rho(0, t)$ and average speed $v(0, t)$ at the upstream boundary.**

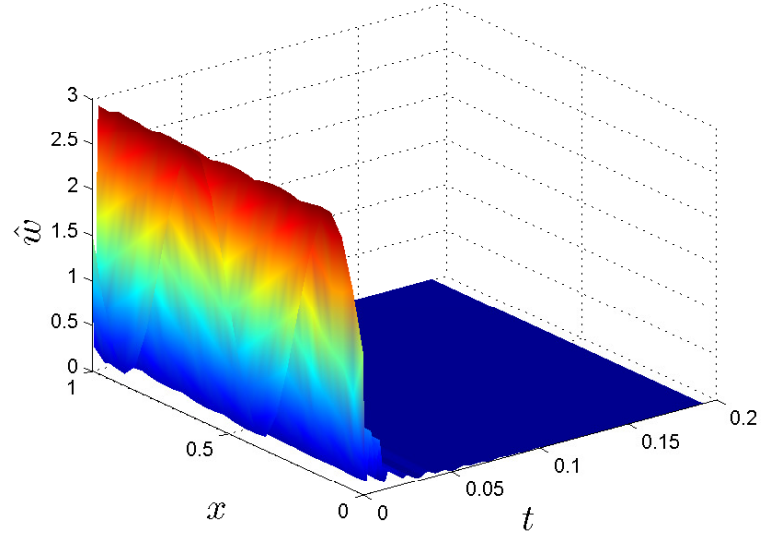


Fig. 2. The evolution of the first component \hat{w} of the positive linear hyperbolic system (51) with matrix G_f .

At the right boundary of the freeway link, *i.e.*, $x = 1$, the driving-out flow-rate conservation equation holds

$$p_{out} - u_c(t) = \rho(1, t)v(1, t). \quad (61)$$

Combining boundary condition (61) with the feedback control law (60), we have

$$\hat{\rho}(0, t) = \frac{v^*}{k_\rho^c} \hat{\rho}(1, t) + \rho^* \frac{k_v^c}{k_\rho^c} \hat{v}(0, t). \quad (62)$$

Using Riemann coordinates, since $\hat{v}(0, t) = \hat{z}(0, t)$ and $\hat{\rho}(x, t) = (\hat{w}(x, t) - \hat{v}(x, t))/a$, as $x = 0, 1$, we have

$$\begin{aligned} \hat{w}(0, t) &= \frac{v^*}{k_\rho^c} \hat{w}(1, t) \\ &+ \left[1 + a\rho^* \frac{k_v^c}{k_\rho^c} - v^* \frac{k_v^c}{k_\rho^c} \right] \hat{z}(1, t). \end{aligned} \quad (63)$$

The corresponding boundary condition for system (51) is given as

$$\begin{bmatrix} \hat{w}(0, t) \\ \hat{z}(1, t) \end{bmatrix} = G_c \begin{bmatrix} \hat{w}(1, t) \\ \hat{z}(0, t) \end{bmatrix}, \quad (64)$$

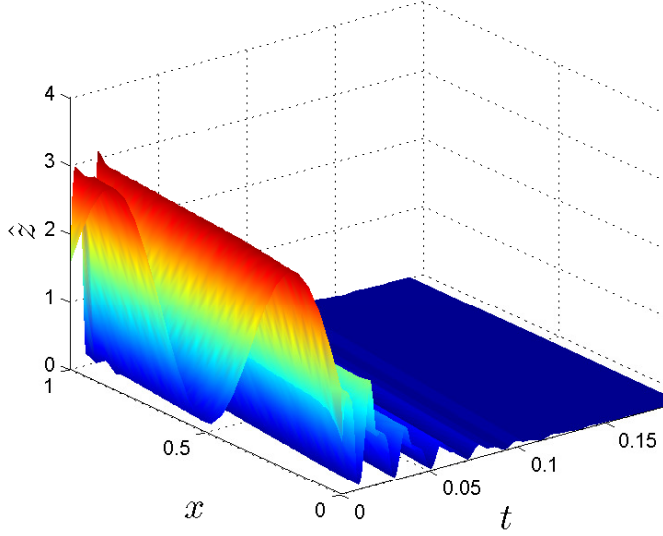


Fig. 3. The evolution of the second component \hat{z} of the positive linear hyperbolic system (51) with matrix G_f .

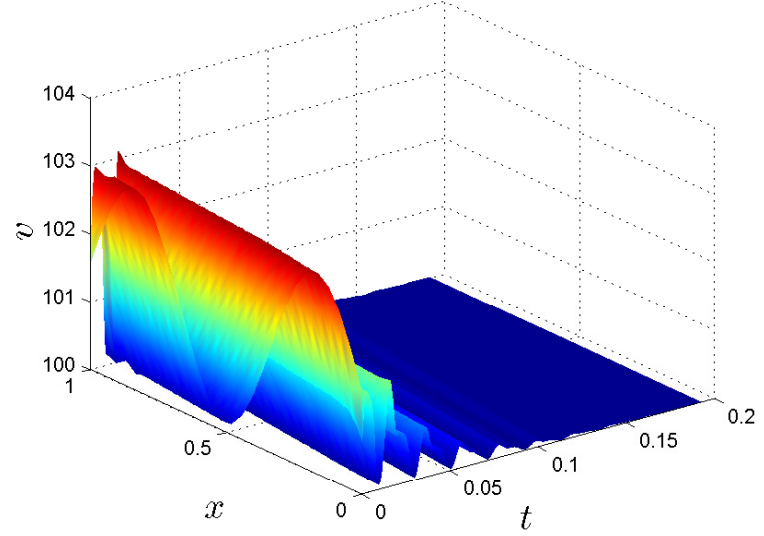


Fig. 5. The evolution of average speed v of the Aw-Rascle traffic flow equation (45) with the steady-state (ρ_1^*, v_1^*) .

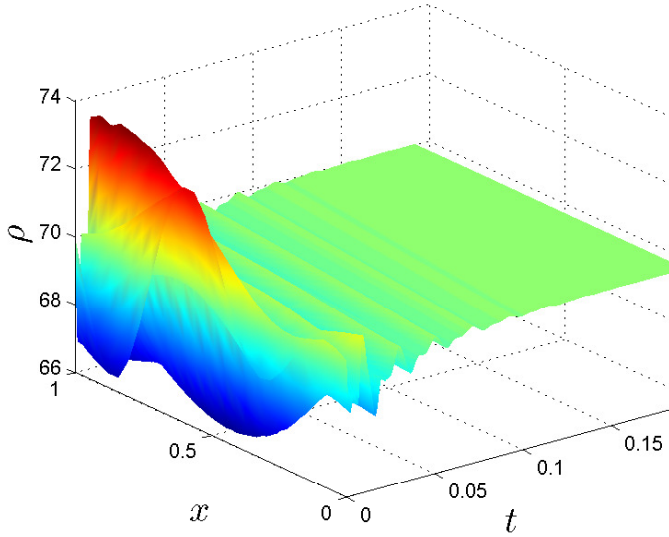


Fig. 4. The evolution of the density ρ of the Aw-Rascle traffic flow equation (45) with the steady-state (ρ_1^*, v_1^*) .

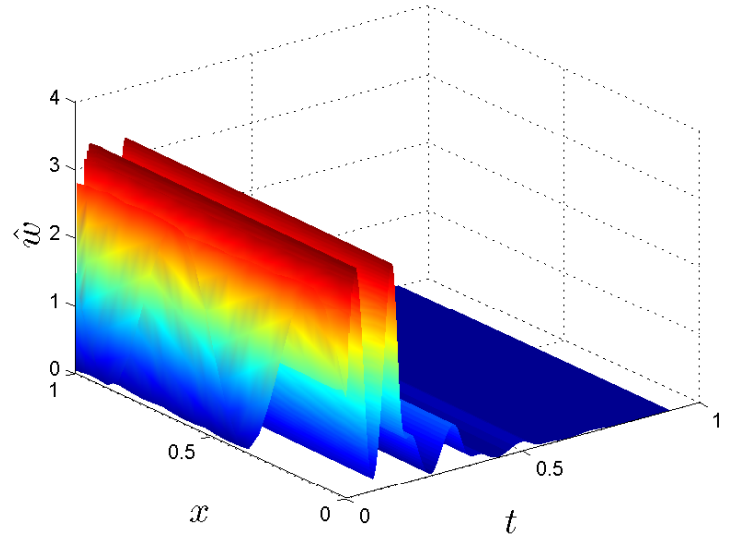


Fig. 6. The evolution of the first component \hat{w} of the positive linear hyperbolic system (51) with matrix G_c .

where

$$G_c = \begin{bmatrix} \frac{v^*}{k_\rho^c} & 1 + a\rho^* \frac{k_v^c}{k_\rho^c} - v^* \frac{k_v^c}{k_\rho^c} \\ 0 & k_v^c \end{bmatrix}. \quad (65)$$

Because

$$1 + a\rho^* \frac{k_v^c}{k_\rho^c} - v^* \frac{k_v^c}{k_\rho^c} = 1 - \lambda_2^* \frac{k_v^c}{k_\rho^c} > 0, \quad (66)$$

the boundary condition matrix G_c is non-negative, *i.e.*, $G_c \succeq 0$, and system (51) is positive all the time, if only $k_\rho^c > 0$, and $k_v^c > 0$.

Moreover, as the freeway traffic is in the congestion mode, system (51) with the boundary condition matrix G_c is local

exponentially stable if and only if the control gains k_ρ^c, k_v^c of the boundary feedback strategy (60) satisfy $k_\rho^c > v^*$, $0 < k_v^c < 1$.

Remark 4: Our designed boundary feedback strategies (53), (60) depend on the sign of the second velocity which keeps either positive or negative in the whole spatial domain. For the spatial-varying steady-states containing transition between free-flow mode and congestion mode, the boundary feedback stabilization is not discussed in the present note. \circ

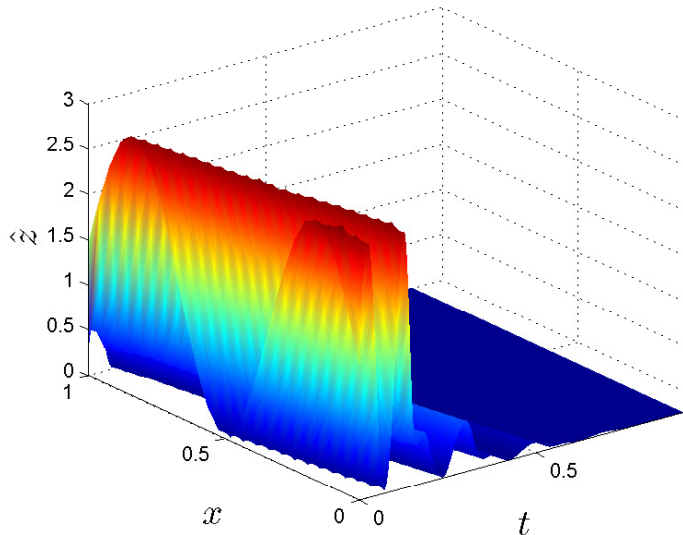


Fig. 7. The evolution of the second component \hat{z} of the positive linear hyperbolic system (51) with matrix G_e .

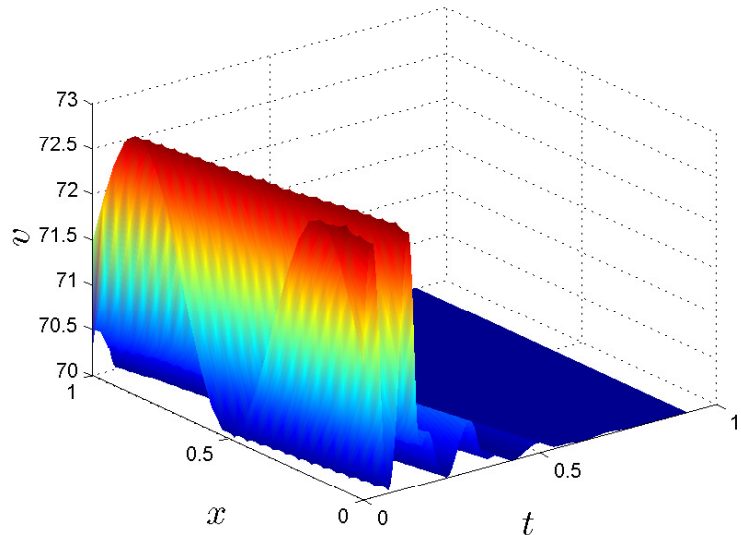


Fig. 9. The evolution of average speed v of the Aw-Rasclé traffic flow equation (45) with the steady-state (ρ_2^*, v_2^*) .

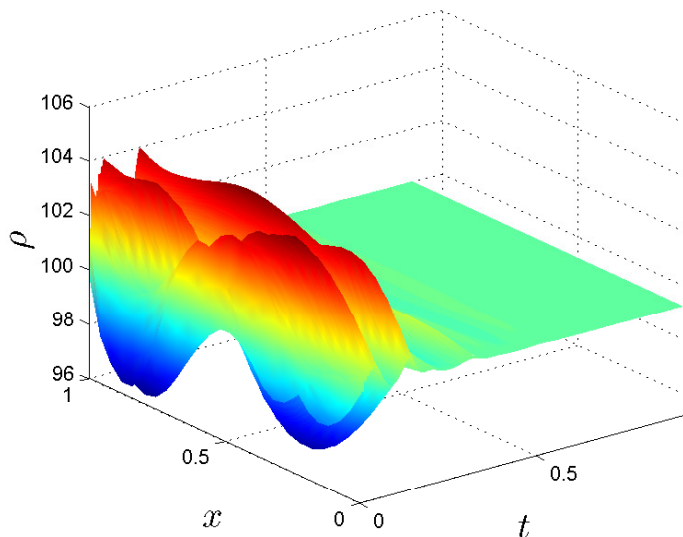


Fig. 8. The evolution of the density ρ of the Aw-Rasclé traffic flow equation (45) with the steady-state (ρ_2^*, v_2^*) .

D. Simulation

The developed boundary feedback strategies (53) and (60) are now tested with some numerical simulations in the two traffic modes presented above. To this end, we consider a local freeway section with parameters $\rho_m = 200$ veh./km, $v_f = 150$ km/hour, $a = 0.75$, $p_{in} = 6000$ veh./hour, $r_f = r_c = 1000$ veh./hour, $p_{out} = 8000$ veh./hour, and the total road length is 1 km, *i.e.*, $x \in [0, 1]$.

Firstly, for the steady-state $(\rho_1^*, v_1^*) = (70, 100)$, we have $\lambda_2^* = 47.5 > 0$. In the neighborhood of this steady-state, the freeway traffic lies in the free-flow mode. Thus we apply the boundary feedback strategy (53), and the control gains are

chosen as $k_\rho^f = 20$, and $k_v^f = 0.5$.

The associated boundary condition matrix G_f in (58) is given as

$$G_f = \begin{bmatrix} 0.2 & 0.0375 \\ 0 & 0.5 \end{bmatrix}. \quad (67)$$

The initial deviations for the Aw-Rasclé traffic flow equation (45) are chosen as

$$\begin{cases} \rho(x, 0) = \rho^* + \sqrt{2} \sin(4\pi x) + 1.45 \\ v(x, 0) = v^* + \sqrt{2} \sin(3\pi x) + 1.45 \end{cases}. \quad (68)$$

Figs. 2 and 3 show the time evolution of system (51) with the boundary feedback control (53), boundary condition matrix (58), and the initial conditions $\hat{w}(x, 0) = \hat{v}(x, 0) + a\hat{\rho}(x, 0)$, and $\hat{z}(x, 0) = \hat{v}(x, 0)$, $x \in [0, 1]$, under Riemann coordinates, satisfying the zero-order compatibility condition.

It is observed that solutions of $\hat{w}(x, t)$, $\hat{z}(x, t)$ of hyperbolic system (51) preserve non-negativity and converge to the origin as time increases, as expected from Proposition 1 and Theorem 1. Figs. 4 and 5 show the solution $\rho(x, t)$ and $v(x, t)$ to the Aw-Rasclé equation (45). The states $\rho(x, t)$ and $v(x, t)$ of Aw-Rasclé traffic flow model clearly converge to their steady-state $\rho_1^* = 70$ veh./km, and $v_1^* = 100$ km/hour, respectively, in these simulation.

In the other simulation, we assume freeway traffic lies in the congestion mode, and consider the steady-state as $(\rho_2^*, v_2^*) = (100, 70)$. In this case, the second eigenvalue $\lambda_2^* = -5 < 0$. We apply the boundary feedback strategy (60), and the control gains are chosen as $k_\rho^c = 400$, and $k_v^c = 0.5$.

Then the boundary condition matrix G_c in (65) is calculated as

$$G_c = \begin{bmatrix} 0.175 & 1.0025 \\ 0 & 0.2 \end{bmatrix}. \quad (69)$$

Using the same initial conditions (68), Figs. 6 and 7 show that the boundary feedback control (60) stabilizes the

linearized system (51). A common practice of traffic control during the congestion mode is to first regulate the downstream traffic to a constant value corresponding to the upstream measurements. Figs. 8 and 9 show such strategy is effective to stabilize the congestion freeway traffic, by simulating the Aw-Rascle equation (45).

V. CONCLUSION

In this paper, an important contribution was the issue of necessary and sufficient conditions for the exponential stability of the positive linear hyperbolic systems using a constructive linear Lyapunov function. This condition is written in terms of linear inequalities which are numerically tractable. The theoretical contribution was applied to stabilize freeway traffic represented by the Aw-Rascle traffic flow model. The control strategies combine the on-ramp metering with the speed limiting in the distributed action.

Future work shall extend above theoretical results, such as Propositions 1, 2, and Theorem 1 with space-varying coefficients in the velocity matrix and with the discontinuous selection of boundary conditions. Some effort will also be devoted to the connection between the positivity of hyperbolic systems and the positivity of the associate Riemann coordinates.

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