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Ekeland's inverse function theorem in graded Fréchet spaces revisited for multifunctions[☆]

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Abstract

In this paper, we present some inverse function theorems and implicit function theorem for set-valued mappings between Fréchet spaces. The proof relies on Lebesgue's Dominated Convergence Theorem and on Ekeland's variational principle. An application to the existence of solutions of differential equations in Fréchet spaces with non-smooth data is given.

Keywords: Inverse function theorem, Implicit function theorem, Fréchet space, Nash-Moser theorem, graded Fréchet spaces, standard Fréchet spaces, Contingent derivative, Ekeland's variational principle, Implicit multifunction theorem.

2010 MSC: 34G20, 47J07, 49J53, 49K40, 58C15.

1. Introduction

The inverse function theorem is one of the central components of the classical and the modern variational analysis and an essential device to solving nonlinear equations. The inverse function theorem or its variants known as the implicit function theorem or the rank theorem have been established originally in Euclidean spaces and then extended to the Banach space setting. Outside this setting, for instance in Fréchet spaces, it is known that the inverse function theorem generally fails (see Lojasiewicz Jr & Zehnder [13]). This is the reason why another form of inverse function theorem, nowadays called

[☆]Dedicated to the memory of Jonathan M. Borwein.

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the Nash-Moser theorem is used as a powerful tool to prove local existence for non-linear partial differential equations in spaces of smooth functions. Some inverse theorems of Nash-Moser type have also been proved for mappings between Fréchet spaces, that are supposed to be *tame*, an additional property guaranteeing that the semi-norms satisfy some interpolation properties, see e.g. [1, 9]) or that allow the use of smoothing operators as introduced by Nash (see e.g. [9, 10, 13, 14, 20–22]). To overcome the loss of derivatives, these additional properties in Fréchet spaces allow Newton’s method on which the Nash-Moser type inverse function theorems are based to converge. Recently, Ekeland [7] (see also Ekeland & Séré [8]) produced a new result within a class of spaces much larger than the one used in the Nash-Moser literature.

Nowadays, modeling has evolved beyond equations and we know the importance and the efficacy of studying set-valued solution mappings which assign to each instance of the parameter element in a model all the corresponding solutions, if any. As it is mentioned in the book by Dontchev & Rockafeller [4], “*the central question is whether a solution mapping can be localized graphically in order to achieve single-valuedness and in that sense produce a function, the desired implicit function*”. To be more explicit, many applied problems can be modeled as differential inclusions or more generally as generalized equations, that is, inclusions governed by a set-valued mapping. For these problems which are the analogous of nonlinear equations, there is a need to use implicit multifunction theorems. During the last years a wide literature has emerged related to implicit multifunction theorems (see e.g., [3–5, 11, 12, 15–18] and the references therein). However, to our knowledge, they have been established in the framework of Banach spaces and nothing exists for Fréchet spaces. Therefore, motivated by the recent work by Ekeland, and Ekeland & Séré, it is our aim in this paper to investigate the possibility to obtain, in the context of graded Fréchet spaces, an implicit multifunction theorem for set-valued mappings.

The structure of the paper is as follows. Section 1 is devoted to recalling the notions of Fréchet, graded Fréchet, standard Fréchet spaces, and contingent derivative of multifunctions, concepts essential for the framework and assumptions on which our results are based. In Section 2, we present an inverse multifunction theorem for set-valued mappings between Fréchet spaces which is the main result of the paper. This result allows us to obtain a general version of the Ekeland inverse function for Gâteaux differentiable mappings and to establish an implicit multifunction theorem for parametrized set-valued mappings. In the final section, we present an application to the existence of solutions for differential equations in Fréchet spaces.

2. Preliminaries

We begin this section with recalling briefly some notions on Fréchet spaces, i.e. on locally convex spaces which are Hausdorff, complete and whose topology is induced by a countable family of semi-norms $(\|\cdot\|_k)_{k \in \mathbb{N}}$ with the property:

$$x \in X, \quad \|x\|_k = 0 \text{ for all } k \in \mathbb{N} \quad \Rightarrow \quad x = 0.$$

This class of spaces contains evidently Banach spaces, as well as many other locally convex spaces used in various areas of real or complex analysis. Also notice that given a Fréchet space F , we may produce the Fréchet space $C^0([a, b], F)$ of continuous paths in F equipped with the semi-norms defined by $\|f\|_k := \sup_{t \in [a, b]} \|f(t)\|_k$ and widely used in analysis.

A *graded Fréchet* space X is a Fréchet space, whose topology is generated by a fixed sequence of semi-norms $(\|\cdot\|_k)_{k \in \mathbb{N}}$, increasing in strength, so that,

$$\|x\|_k \leq \|x\|_{k+1} \quad \forall x \in X, \forall k \in \mathbb{N}.$$

This class contains the space $C^\infty([a, b])$ of infinitely differentiable real-valued functions on the interval $[a, b]$ with the grading

$$\|f\|_n = \sup_{k \leq n} \sup_{x \in [a, b]} |D^k(f(x))|,$$

as well as the space $C^\infty(\bar{\Omega}, \mathbb{R}^d)$, where $\bar{\Omega} \subset \mathbb{R}^n$ is compact, with a smooth boundary and is the closure of its interior are graded Fréchet spaces. Note that every Fréchet space can be considered as a graded Fréchet space by replacing the initial family of semi-norms by the semi-norms $\|\cdot\|_n := \sum_{k=1}^n \|\cdot\|_k$. However, as the Nash-Moser inverse function theorem highlights, the grading plays an essential role in its statement and in its proof.

Well known are the facts:

- the Cartesian product of two graded Fréchet spaces is a graded Fréchet space with the grading $\|(x, y)\|_n = \|x\|_n + \|y\|_n$;
- a closed subspace of a graded Fréchet space is a graded Fréchet space;
- a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in a Fréchet space X converges to $x \in X$, if and only if, $\|x_n - x\|_k \rightarrow 0$ for all $k \geq 0$;
- due to a classical result that (see e.g. [7]), a graded Fréchet space is a complete metric space with the metric:

$$d(x, y) = \sum_{k=1}^{\infty} \mu_k \min\{r, \|x - y\|_k\}, \quad (1)$$

where, $(\mu_k)_{k \in \mathbb{N}}$ is any sequence of non-negative numbers with unbounded support: $\sup\{k \in \mathbb{N} : \mu_k \neq 0\} = +\infty$ and $r > 0$.

Definition 1. ([7], Definition 5) *A graded Fréchet space is said to be standard if for every $x \in X$, we can find a constant $c := c(x)$ a sequence $(x_n) \subseteq X$ converging to x and a sequence of non-negative numbers (c_n) such that*

$$\|x_n\|_k \leq c\|x\|_k \quad \text{and} \quad \|x_n\|_k \leq c_n^k \quad \forall k, n \in \mathbb{N}. \quad (2)$$

The space $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ is a standard graded Fréchet space, see [7].

Let X, Y be graded Fréchet spaces. Consider a multifunction (set-valued mapping) $F : X \rightrightarrows Y$ between X and Y , that is a function between X and the subsets (possibly empty) of Y . We denote by $\text{gph } F$ and $\text{dom } F$ and $F^{-1} : Y \rightrightarrows X$, the *graph*, *domain* and *inverse* of F , respectively:

$$\text{gph } F = \{(x, y) \in X \times Y : y \in F(x)\}; \quad \text{dom } F = \{x \in X : F(x) \neq \emptyset\}$$

and

$$F^{-1}(y) := \{x \in X : y \in F(x)\}, \quad y \in Y.$$

We say that F is a *closed* multifunction if $\text{gph } F$ is a closed subset of $X \times Y$. In what follows, we will use the notion of *contingent derivative* of multifunctions. The contingent derivative of the multifunction F at a point $(\bar{x}, \bar{y}) \in \text{gph } F$ is the multifunction $DF(\bar{x}, \bar{y}) : X \rightrightarrows Y$, defined for $u \in X$ by

$$\{v \in Y : \exists (t_n)_{n \in \mathbb{N}} \downarrow 0^+, \exists (u_n, v_n)_{n \in \mathbb{N}} \rightarrow (u, v) \text{ with } (\bar{x} + t_n u_n, \bar{y} + t_n v_n) \in \text{gph } F, \forall n\}.$$

In other words, $DF(\bar{x}, \bar{y})(u) = \{v \in Y : (u, v) \in T_{\text{gph } F}(\bar{x}, \bar{y})\}$, where, $T_{\text{gph } F}(\bar{x}, \bar{y})$ stands for the *contingent cone* to $\text{gph } F$ at (\bar{x}, \bar{y}) . For more details, the reader is referred to the book by Aubin & Frankowska [2].

When $F : X \rightarrow Y$ is a single-valued mapping, we use the notation $DF(\bar{x})$ for $DF(\bar{x}, \bar{y})$. Note that if $F : X \rightarrow Y$ is Gâteaux differentiable at $\bar{x} \in X$, then $DF(\bar{x})$ coincides with the Gâteaux derivative of F at \bar{x} .

3. Inverse and Implicit multifunction theorems

Throughout, we consider a closed multifunction $F : X \rightrightarrows Y$ between graded Fréchet spaces X and Y induced respectively by countable families of semi-norms for which we use the same notation $(\|\cdot\|_k)_{k \in \mathbb{N}}$. For each $k \in \mathbb{N}$, $x \in X$ and a subset $S \in X$, denote by $d_k(x, S) := \inf_{z \in S} \|z - x\|_k$, which is referred to as the *semi-distance* from the point x to the set S with respect to the semi-norm $\|\cdot\|_k$ in X . For given $r \in (0, +\infty]$, $k_0 \in \mathbb{N}$, we denote respectively by $B_X(\bar{x}, k_0, r) = \{x \in X : \|x - \bar{x}\|_{k_0} < r\}$ and $B_X[\bar{x}, k_0, r] = \{x \in X : \|x - \bar{x}\|_{k_0} \leq r\}$, which are also referred to as the open and closed balls in X centered at $\bar{x} \in X$, with radius r with respect to the semi-norm $\|\cdot\|_{k_0}$. Note that since $\|\cdot\|_{k_0}$ is a semi-norm, for any $z \in X$ with $\|z - \bar{x}\|_{k_0} = 0$, $B_X(z, k_0, r)$, $B_X[z, k_0, r]$ coincide with $B_X(\bar{x}, k_0, r)$, $B_X[\bar{x}, k_0, r]$, respectively.

Theorem 2. *Let $F : X \rightrightarrows Y$ be a closed multifunction between graded Fréchet spaces and let $(\bar{x}, \bar{y}) \in \text{gph } F$ be given. Assume furthermore that Y is standard. Suppose also that there are integers k_0, d_1, d_2 , real numbers $r \in (0, +\infty]$, $C \geq 0$ and non-decreasing sequences of non-negative reals $(\nu_k)_{k \in \mathbb{N}}$, $(\nu'_k)_{k \in \mathbb{N}}$, $(m_k)_{k \in \mathbb{N}}$, and $(a_k)_{k \in \mathbb{N}}$ with $m_k \geq 1$, $a_k \geq 1$ such that the following conditions are satisfied:*

- (i) *For all $(x, y) \in \text{gph } F$ with $x \in B_X(\bar{x}, k_0, r)$, $y \in B_Y(\bar{y}, k_0 + d_1 + d_2, 2r/a_{k_0+d_1})$, for every $(u, v) \in \text{gph } DF(x, y)$, there exist $c_2(u, v) > 0$ and sequences $t_n \downarrow 0$, $u_n \rightarrow u$*

and $v_n \rightarrow v$ with $(x + t_n u_n, z + t_n v_n) \in \text{gph } F$, such that for all $n \in \mathbb{N}$, all $k \in \mathbb{N}$,

$$\|v_n\|_k \leq c_2(u, v)(m_k \|u\|_{k+d_1} + \|x - \bar{x}\|_k / a_{k-d_2} + \|y - \bar{y}\|_k + \nu_k)$$

and

$$\|u_n\|_k \leq c_2(u, v)(m_k \|u\|_{k+d_1} + \|x - \bar{x}\|_k + \nu_k);$$

(ii) For all $(x, y) \in \text{gph } F$ with $x \in B_X(\bar{x}, k_0, r)$, $y \in B_Y(\bar{y}, k_0 + d_1 + d_2, 2r/a_{k_0+d_1})$, for every $v \in Y$, there exists $u \in DF(x, y)^{-1}(v)$ such that

$$\|u\|_k \leq C \left(\frac{\|x - \bar{x}\|_{k-d_1-d_2}}{m_k a_{k-d_1-d_2}} + \nu'_k \right) \|v\|_{d_1+d_2} + a_k \|v\|_{k+d_2}, \quad \forall k \in \mathbb{N}.$$

By convention we set $\|\cdot\|_k = \|\cdot\|_0$ and $a_k = 1$ for $k < 0$.

Let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of non-negative reals with unbounded support such that

$$\begin{aligned} \sum_{k=0}^{\infty} \beta_k \nu_k < +\infty \quad \sum_{k=0}^{\infty} \beta_k m_k \nu'_{k+d_1} < +\infty \\ \text{and} \quad \sum_{k=0}^{\infty} \beta_k m_k a_{k+d_1} n^k < +\infty. \end{aligned} \quad (3)$$

Then, for every $y \in Y$ with $C\gamma < 1$, where

$$\gamma := \frac{\sum_{k=0}^{\infty} (\beta_k \|y - \bar{y}\|_k + \beta_{k+d_1+d_2} \nu'_{k+d_1} / a_{k+d_1})}{\sum_{k=d_1+d_2}^{\infty} \beta_k}, \quad (4)$$

and

$$\sum_{k=0}^{\infty} \beta_k \|y - \bar{y}\|_k \left(1 - \sqrt{C\gamma}\right)^{-2} < \frac{r \beta_{k_0+d_1+d_2}}{a_{k_0+d_1}}, \quad (5)$$

there exists $x \in B_X(\bar{x}, k_0, r)$ such that $y \in F(x)$.

Proof. The proof is based on the Ekeland variational principle [6]. However, the function and space to which the Ekeland variational principle is applied are different from [7].

By translation if necessary, we can assume without loss of generality that $\bar{x} = 0$ and $\bar{y} = 0$. Let $(\alpha_k)_{k \in \mathbb{N}}$ be the sequence defined by

$$\alpha_k = \frac{\beta_{k+d_1+d_2}}{a_{k+d_1}}, \quad k \in \mathbb{N}.$$

Consider the distances on X and Y defined respectively by

$$d(x_1, x_2) := \sum_{k=0}^{\infty} \alpha_k \min\{r, \|x_1 - x_2\|_k\}, \quad x_1, x_2 \in X,$$

$$d(y_1, y_2) := \sum_{k=0}^{\infty} \beta_k \min\{r, \|y_1 - y_2\|_k\}, \quad y_1, y_2 \in Y.$$

For $\varepsilon > 0$, we define the distance $d_\varepsilon(\cdot, \cdot)$ on $X \times Y$ by

$$d_\varepsilon((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \varepsilon d(y_1, y_2), \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Equipped with these distances, the spaces X, Y and therefore $X \times Y$ are complete metric spaces. Let $y_0 \in Y$ be such that (4) and (5) are satisfied. Setting $\eta = \sqrt{C\gamma} - C\gamma$, where γ is defined by (4), consider the extended-real-valued function $f : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x, y) = \eta \sum_{k=0}^{\infty} \alpha_k \|x\|_{k+d_1} + \sum_{k=0}^{\infty} \beta_k \|y - y_0\|_k + \delta_{\text{gph } F}(x, y), \quad (x, y) \in X \times Y, \quad (6)$$

where $\delta_{\text{gph } F}$ stands for the *indicator function* of $\text{gph } F$, that is,

$$\delta_{\text{gph } F}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \text{gph } F, \\ +\infty & \text{otherwise.} \end{cases}$$

Claim 1. *The function f is lower semicontinuous and bounded from below on $X \times Y$.*

One has

$$0 \leq \inf_{(x, y) \in X \times Y} f(x, y) \leq f(0, 0) = \sum_{k=0}^{\infty} \beta_k \|y_0\|_k < +\infty. \quad (7)$$

Take a sequence $((x_n, y_n))_{n \in \mathbb{N}}$ converging to (x, y) in $X \times Y$. Then, $\|x_n - x\|_k \rightarrow 0$ and $\|y_n - y\|_k \rightarrow 0$ for every $k \in \mathbb{N}$. Two cases may happen:

1.- If $(x, y) \notin \text{gph } F$, then by the closedness of the graph of F , $(x_n, y_n) \notin \text{gph } F$ when n is sufficiently large. Hence,

$$\lim_n f(x_n, y_n) = +\infty = f(x, y).$$

2.- Suppose now that $(x, y) \in \text{gph } F$. Thanks to the Fatou lemma we have,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} f(x_n, y_n) &\geq \eta \liminf_{n \rightarrow +\infty} \left(\sum_{k=0}^{\infty} \alpha_k \|x_n\|_k + \sum_{k=0}^{\infty} \beta_k \|y_n - y_0\|_k \right) \\ &\geq \eta \sum_{k=0}^{\infty} \alpha_k \lim_{n \rightarrow +\infty} \|x_n\|_k + \sum_{k=0}^{\infty} \lim_{n \rightarrow +\infty} \beta_k \|y_n - y_0\|_k \\ &= \eta \sum_{k=0}^{\infty} \alpha_k \|x\|_k + \sum_{k=0}^{\infty} \beta_k \|y - y_0\|_k = f(x, y), \end{aligned}$$

establishing the claim. □

As

$$\frac{C\gamma + \eta}{\eta(1 - C\gamma - \eta)} = (1 - \sqrt{C\gamma})^{-2},$$

in view of assumption (5), take $\bar{r} > 0$ and $\varepsilon > 0$ such that

$$\bar{r} < r \quad \text{and} \quad \frac{C\gamma + \eta + \eta\varepsilon}{\eta(1 - C\gamma - \eta)} \sum_{k=0}^{\infty} \beta_k \|y_0\|_k < \frac{\bar{r}\beta_{k_0+d_1+d_2}}{a_{k_0+d_1}}. \quad (8)$$

Set

$$\kappa := \frac{\sum_{k=0}^{\infty} \beta_k \|y_0\|_k}{\bar{r}\alpha_{k_0}} = \frac{f(0,0)}{\bar{r}\alpha_{k_0}} < \frac{\eta(1 - C\gamma - \eta)}{C\gamma + \eta + \eta\varepsilon}. \quad (9)$$

Applying Ekeland's variational principle to the function f on $X \times Y$ endowed with the distance d_ε , we may find $(x_0, z_0) \in X \times Y$ such that

$$f(x_0, z_0) \leq f(0, 0), \quad (10)$$

$$d_\varepsilon((x_0, z_0), (0, 0)) \leq \bar{r}\alpha_{k_0}, \quad (11)$$

and

$$f(x, y) + \kappa(d(x, x_0) + \varepsilon d(y, z_0)) \geq f(x_0, z_0) \quad \forall (x, y) \in X \times Y. \quad (12)$$

Obviously, $(x_0, z_0) \in \text{gph } F$. By relations (8) and (11),

$$\alpha_{k_0} \min\{r, \|x_0\|_{k_0}\} \leq d_\varepsilon((x_0, z_0), (0, 0)) \leq \bar{r}\alpha_{k_0} < r\alpha_{k_0}.$$

Consequently, $\|x_0\|_{k_0} < r$. Since according to (5),

$$\beta_{k_0+d_1+d_2} \|y_0\|_{k_0+d_1+d_2} \leq f(0, 0) < \frac{r\beta_{k_0+d_1+d_2}}{a_{k_0+d_1}},$$

it follows $\|y_0\|_{k_0+d_1+d_2} < r/a_{k_0+d_1}$. Furthermore, as

$$\beta_{k_0+d_1+d_2} \|z_0 - y_0\|_{k_0+d_1+d_2} \leq f(x_0, z_0) \leq f(0, 0) < \frac{r\beta_{k_0+d_1+d_2}}{a_{k_0+d_1}},$$

one obtains

$$\|z_0\|_{k_0+d_1+d_2} \leq \|y_0\|_{k_0+d_1+d_2} + \|z_0 - y_0\|_{k_0+d_1+d_2} \leq 2r/a_{k_0+d_1}.$$

From (12), for all $(x, y) \in \text{gph } F$ one has

$$\begin{aligned} & \sum_{k=0}^{\infty} \beta_k (\|z_0 - y_0\|_k - \|y - y_0\|_k) \\ & \leq \eta \sum_{k=0}^{\infty} \alpha_k (\|x\|_k - \|x_0\|_k) \\ & + \kappa \left(\sum_{k=0}^{\infty} \alpha_k \min\{r, \|x - x_0\|_k\} + \varepsilon \sum_{k=0}^{\infty} \beta_k \min\{r, \|y - z_0\|_k\} \right). \end{aligned} \quad (13)$$

It suffices to prove that $z_0 = y_0$. Assume to the contrary that $z_0 \neq y_0$. Setting $v = y_0 - z_0$, and using the assumption that Y is standard, there exists a sequence $(v_n)_{n \in \mathbb{N}}$ converging to v such that

$$\|v_n\|_k \leq c_0(v)\|v\|_k, \quad \|v_n\| \leq c_1(v_n)^k \quad \forall n, k \in \mathbb{N}. \quad (14)$$

From condition (ii), for every n , there exists $u_n \in DF(x_0, z_0)^{-1}(v_n)$ such that

$$\|u_n\|_k \leq C \left(\frac{\|x_0\|_{k-d_1-d_2}}{m_k a_{k-d_1-d_2}} + \nu'_k \right) \|v_n\|_{d_1+d_2} + a_k \|v_n\|_{k+d_2}, \quad \forall k \in \mathbb{N}. \quad (15)$$

From condition (i), for every $n \in \mathbb{N}$, there exist a real $c_2(u_n, v_n) > 0$, sequences $t_{n,j} \downarrow 0$, $u_{n,j} \rightarrow u_n$, $v_{n,j} \rightarrow v_n$ as $j \rightarrow \infty$ such that

$$(x_0 + t_{n,j}u_{n,j}, z_0 + t_{n,j}v_{n,j}) \in \text{gph } F, \quad \forall j$$

$$\|v_{n,j}\|_k \leq c_2(u_n, v_n)(m_k \|u_{n,j}\|_{k+d_1} + \|x_0\|_k/a_{k-d_2} + \|z_0\|_k + \nu_k), \quad \forall j, k, \quad (16)$$

$$\|u_{n,j}\|_k \leq c_2(u_n, v_n)(m_k \|u_{n,j}\|_{k+d_1} + \|x_0\|_k + \nu_k), \quad \forall j, k. \quad (17)$$

Plugging $x := x_0 + t_{n,j}u_{n,j}$ and $y := z_0 + t_{n,j}v_{n,j}$ into relation (3), one obtains

$$\begin{aligned} \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - t_{n,j}v_{n,j}\|_k) &\leq \eta \sum_{k=0}^{\infty} \alpha_k (\|x_0 + t_{n,j}u_{n,j}\|_{k+d_1} - \|x_0\|_{k+d_1}) \\ &+ \kappa \left(\sum_{k=0}^{\infty} \alpha_k \min\{r, t_{n,j}\|u_{n,j}\|_k\} + \varepsilon \sum_{k=0}^{\infty} \beta_k \min\{r, t_{n,j}\|v_{n,j}\|_k\} \right) \quad \forall n, j. \end{aligned} \quad (18)$$

We can assume $t_{n,j} \in (0, 1)$, for all n, j . Then, one has

$$\|v - t_{n,j}v_{n,j}\|_k = \|t_{n,j}(v - v_{n,j}) + (1 - t_{n,j})v\|_k \leq t_{n,j}\|v - v_{n,j}\|_k + (1 - t_{n,j})\|v\|_k.$$

It follows that

$$\frac{\|v\|_k - \|v - t_{n,j}v_{n,j}\|_k}{t_{n,j}} \geq \|v\|_k - \|v - v_{n,j}\|_k \quad \forall n, j, k.$$

Combining this inequality with (3), one derives

$$\begin{aligned} \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_{n,j}\|_k) &\leq \eta \sum_{k=0}^{\infty} \alpha_k \|u_{n,j}\|_{k+d_1} + \\ &+ \kappa \left(\sum_{k=0}^{\infty} \alpha_k \min\left\{\frac{r}{t_{n,j}}, \|u_{n,j}\|_k\right\} + \varepsilon \sum_{k=0}^{\infty} \beta_k \min\left\{\frac{r}{t_{n,j}}, \|v_{n,j}\|_k\right\} \right) \quad \forall n, j. \end{aligned} \quad (19)$$

Claim 2. For every $n \in \mathbb{N}$, one has

$$\lim_j \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_{n,j}\|_k) = \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_n\|_k)$$

and

$$\lim_{j \rightarrow \infty} \sum_{k=0}^{\infty} \beta_k \min \left\{ \frac{r}{t_{n,j}}, \|v_{n,j}\|_k \right\} = \sum_{k=0}^{\infty} \beta_k \|v_n\|_k.$$

By relations (14), (15) and (16), one has

$$\begin{aligned} \|v_{n,j}\|_k &\leq c_2(u_n, v_n) m_k \|u_n\|_{k+d_1} + c_2(u_n, v_n) \|x_0\|_k / a_{k-d_2} + c_2(u_n, v_n) \|z_0\|_k + c_2(u_n, v_n) \nu_k \\ &\leq c_2(u_n, v_n) \left[C \left(\frac{\|x_0\|_{k-d_2}}{a_{k-d_2}} + m_k \nu'_{k+d_1} \right) \|v_n\|_{d_1+d_2} + m_k a_{k+d_1} \|v_n\|_{k+d_1+d_2} \right] \\ &\quad + c_2(u_n, v_n) \|x_0\|_k / a_{k-d_2} + c_2(u_n, v_n) \|z_0\|_k \\ &\leq c_2(u_n, v_n) \left[\frac{C}{a_{k-d_2}} \|x_0\|_{k-d_2} \|v_n\|_{d_1+d_2} + m_k a_{k+d_1} c_1(v_n)^{k+d_1+d_2} \right] \\ &\quad + C c_2(u_n, v_n) \|v_n\|_{d_1+d_2} m_k \nu'_{k+d_1} + c_2(u_n, v_n) \|x_0\|_k / a_{k-d_2} + c_2(u_n, v_n) \|z_0\|_k + c_2(u_n, v_n) \nu_k. \end{aligned}$$

Thus, for every $n, j \in \mathbb{N}$,

$$\begin{aligned} &\sum_{k=0}^{\infty} \beta_k \|v_{n,j}\|_k \\ &\leq c_2(u_n, v_n) \left[C \|v_n\|_{d_1+d_2} \sum_{k=0}^{\infty} \alpha_{k-d_1-d_2} \|x_0\|_{k-d_2} + \sum_{k=0}^{\infty} \beta_k m_k a_{k+d_1} c_1(v_n)^{k+d_1+d_2} \right] \\ &\quad + \sum_{k=0}^{\infty} \alpha_{k-d_1} \|x_0\|_k + \sum_{k=0}^{\infty} \beta_k \|z_0\|_k \\ &\quad + c_2(u_n, v_n) \left(C \|v_n\|_{d_1+d_2} \sum_{k=0}^{\infty} \beta_k \nu'_{k+d_1} + \sum_{k=0}^{\infty} \beta_k \nu_k \right). \end{aligned} \tag{20}$$

By (3),

$$\begin{aligned} C \|v_n\|_{d_1+d_2} \sum_{k=0}^{\infty} \beta_k m_k \nu'_{k+d_1} + \sum_{k=0}^{\infty} \beta_k \nu_k &< +\infty; \\ \sum_{k=0}^{\infty} \beta_k m_k a_{k+d_1} c_1(v_n)^{k+d_1+d_2} &< \infty, \end{aligned}$$

and in view of relation (10),

$$\begin{aligned} \eta \sum_{k=0}^{\infty} \alpha_k \|x_0\|_{k+d_1} &\leq f(x_0, z_0) \leq f(0, 0) < \infty; \\ \sum_{k=0}^{\infty} \beta_k \|z_0\|_k &\leq \sum_{k=0}^{\infty} \beta_k \|y_0\|_k + f(0, 0) < \infty. \end{aligned}$$

Therefore, according to Lebesgue's Dominated Convergence Theorem, relation (20)

yields for every $n \in \mathbb{N}$,

$$\begin{aligned} \lim_{j \rightarrow \infty} \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_{n,j}\|_k) &= \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \lim_{j \rightarrow \infty} \|v - v_{n,j}\|_k) \\ &= \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_n\|_k), \end{aligned}$$

and

$$\lim_{j \rightarrow \infty} \sum_{k=0}^{\infty} \beta_k \min \left\{ \frac{r}{t_{n,j}}, \|v_{n,j}\|_k \right\} = \sum_{k=0}^{\infty} \beta_k \|v_n\|. \quad \square$$

Claim 3. For every $n \in \mathbb{N}$, one has

$$\lim_{j \rightarrow \infty} \sum_{k=0}^{\infty} \alpha_k \|u_{n,j}\|_{k+d_1} = \sum_{k=0}^{\infty} \alpha_k \|u_n\|_{k+d_1}, \quad (21)$$

and

$$\lim_{j \rightarrow \infty} \sum_{k=0}^{\infty} \alpha_k \min \left\{ \frac{r}{t_{n,j}}, \|u_{n,j}\|_k \right\} = \sum_{k=0}^{\infty} \alpha_k \|u_n\|_k. \quad (22)$$

From relations (14), (15) and (17), for every $k, n, j \in \mathbb{N}$, one has

$$\begin{aligned} \|u_{n,j}\|_k &\leq c_2(u_n, v_n) (m_k \|u_n\|_{k+d_1} + \|x_0\|_k + \nu_k) \\ &\leq c_2(u_n, v_n) \left(C \left(\frac{\|x_0\|_{k-d_2}}{a_{k-d_2}} + m_k \nu'_{k+d_1} \right) \|v_n\|_{d_1+d_2} + m_k a_k \|v_n\|_{k+d_2} + \|x_0\|_k + \nu_k \right) \\ &\leq c_2(u_n, v_n) \left(C \|x_0\|_{k-d_2} \|v_n\|_{d_1+d_2} + m_k a_k c_1(v_n)^{k+d_2} + \|x_0\|_k \right) \\ &\quad + c_2(u_n, v_n) \nu_k + c_2(u_n, v_n) C \|v_n\|_{d_1+d_2} m_k \nu'_{k+d_1}. \end{aligned}$$

As $\sum_{k=0}^{\infty} \beta_k m_k \nu'_{k+d_1}$, $\sum_{k=0}^{\infty} \alpha_k \|x_0\|_{k+d_1}$ and $\sum_{k=0}^{\infty} \beta_k m_{k+d_1} a_{k+d_1} c_1(v_n)^{k+d_1+d_2}$ are convergent series, we deduce (21) and (22) by Lebesgue's Dominated Convergence Theorem. \square

By virtue of Claims 2 and 3, by letting $j \rightarrow \infty$ in inequality (19), one obtains

$$\begin{aligned} \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_n\|_k) &\leq \eta \sum_{k=0}^{\infty} \alpha_k \|u_n\|_{k+d_1} + \\ &+ \kappa \left(\sum_{k=0}^{\infty} \alpha_k \|u_n\|_k + \varepsilon \sum_{k=0}^{\infty} \beta_k \|v_n\|_k \right) \quad \forall n. \end{aligned} \quad (23)$$

Next, using the first relation of (14), and the inequalities

$$\sum_{k=0}^{\infty} \beta_k \|v\|_k \leq f(x_0, z_0) \leq f(0, 0),$$

for every $n \in \mathbb{N}$, one has

$$\sum_{k=0}^{\infty} \beta_k \|v_n\|_k \leq c_0(v) \sum_{k=0}^{\infty} \beta_k \|v\|_k < \infty.$$

Applying again Lebesgue's Dominated Convergence Theorem, one obtains

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \|v - v_n\|_k) = \sum_{k=0}^{\infty} \beta_k (\|v\|_k - \lim_{n \rightarrow \infty} \|v - v_n\|_k) = \sum_{k=0}^{\infty} \beta_k \|v\|_k, \quad (24)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \beta_k \|v_n\|_k = \sum_{k=0}^{\infty} \beta_k \lim_{n \rightarrow \infty} \|v_n\|_k = \sum_{k=0}^{\infty} \beta_k \|v\|_k. \quad (25)$$

From (15), one has

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha_k \|u_n\|_k &\leq \sum_{k=0}^{\infty} \alpha_k \|u_n\|_{k+d_1} \\ &\leq \sum_{k=0}^{\infty} C \alpha_k \left(\frac{\|x_0\|_{k-d_2}}{m_{k+d_1} a_{k-d_2}} + \nu'_{k+d_1} \right) \|v_n\|_{d_1+d_2} + \sum_{k=0}^{\infty} \alpha_k a_{k+d_1} \|v_n\|_{k+d_1+d_2} \\ &\leq C \sum_{k=0}^{\infty} \alpha_k (\|x_0\|_{k+d_1} + \nu'_{k+d_1}) \|v_n\|_{d_1+d_2} + \sum_{k=0}^{\infty} \beta_{k+d_1+d_2} \|v_n\|_{k+d_1+d_2}. \end{aligned}$$

As

$$\eta \sum_{k=0}^{\infty} \alpha_k \|x_0\|_{k+d_1} \leq f(x_0, z_0) \leq f(0, 0) = \sum_{k=0}^{\infty} \beta_k \|y_0\|_k,$$

one deduces that

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha_k \|u_n\|_k &\leq \sum_{k=0}^{\infty} \alpha_k \|u_n\|_{k+d_1} \\ &\leq \frac{C\gamma}{\eta} \sum_{k=d_1+d_2}^{\infty} \beta_k \|v_n\|_{d_1+d_2} + \sum_{k=0}^{\infty} \beta_{k+d_1+d_2} \|v_n\|_{k+d_1+d_2} \\ &\leq \frac{C\gamma}{\eta} \sum_{k=d_1+d_2}^{\infty} \beta_k \|v_n\|_k + \sum_{k=0}^{\infty} \beta_{k+d_1+d_2} \|v_n\|_{k+d_1+d_2} \\ &\leq \left(\frac{C\gamma}{\eta} + 1 \right) \sum_{k=0}^{\infty} \beta_k \|v_n\|_k. \end{aligned} \quad (26)$$

By virtue of this inequality, letting $n \rightarrow \infty$ in relation (23), in view of relations (24) and (25), one obtains

$$\sum_{k=0}^{\infty} \beta_k \|v\|_k \leq [(C\gamma + \eta) + \kappa(C\gamma/\eta + 1 + \varepsilon)] \sum_{k=0}^{\infty} \beta_k \|v\|_k,$$

from which it follows that

$$\kappa \geq \frac{\eta(1 - C\gamma - \eta)}{C\gamma + \eta + \eta\varepsilon}.$$

This contradicts (8), (9). The proof is completed. \square

Remark 3. We can add to the conclusion of Theorem 2, that the inverse image x of an arbitrary $y \in Y$ is controlled by the distance to the reference point \bar{x} . More precisely thanks to relation (11) we have $d(x, \bar{x}) < r\alpha_{k_0}$, where d is a metric defining X .

In the last part of this section, we consider two graded Fréchet spaces X and Y , a

topological space P and a multifunction $F : X \times P \rightrightarrows Y$. For $p \in P$, set

$$S(p) = \{x \in X : 0 \in F(x, p)\}. \quad (27)$$

The multifunction $S : P \rightrightarrows X$ is referred to as the *solution mapping* associated to F . For $p \in P$, denote by $F_p := F(\cdot, p) : X \rightrightarrows Y$. By making use Theorem 2, we derive the following implicit multifunction theorem.

Theorem 4. *Let X, Y be graded Fréchet spaces and let P be a topological space. Consider a multifunction $F : X \times P \rightrightarrows Y$ and a given point $(\bar{x}, \bar{p}) \in X \times P$ with $0 \in F(\bar{x}, \bar{p})$. Assume furthermore that Y is standard. Suppose also that there are integers k_0, d_1, d_2 , real numbers $r_0 \in (0, +\infty]$, $C > 0$ and non-decreasing sequences of non-negative reals $(\mu_k)_{k \in \mathbb{N}}$, $(\mu'_k)_{k \in \mathbb{N}}$, and $(m_k)_{k \in \mathbb{N}}$, $(a_k)_{k \in \mathbb{N}}$ with $m_k \geq 1$, $a_k \geq 1$ such that the following conditions are satisfied:*

- (i) *For each $p \in P$, $\text{gph } F_p$ is closed and the multifunction $F(\bar{x}, \cdot) : P \rightrightarrows Y$ is lower semicontinuous at \bar{p} ;*
- (ii) *For all p near \bar{p} , for all $(x, y) \in \text{gph } F_p$ with $x \in B_X(\bar{x}, k_0, r_0)$, $y \in B_Y(0, k_0 + d_1 + d_2, 2r_0/a_{k_0+d_1})$, for every $v \in DF_p(x, z)u$ with $u \in X$, there exist $c_2(u, v) > 0$ and sequences $t_n \downarrow 0$, $u_n \rightarrow u$ and $v_n \rightarrow v$ with $(x + t_n u_n, y + t_n v_n) \in \text{gph } F_p$, such that for all $n \in \mathbb{N}$, all $k \in \mathbb{N}$,*

$$\|v_n\|_k \leq c_2(u, v)(m_k \|u\|_{k+d_1} + \|x - \bar{x}\|_k / a_{k-d_2} + \|y\|_k + \mu_k)$$

and

$$\|u_n\|_k \leq c_2(u, v)(m_k \|u\|_{k+d_1} + \|x - \bar{x}\|_k + \mu_k);$$

- (ii) *For $p \in P$ near \bar{p} , for all $(x, y) \in \text{gph } F_p$ with $x \in B_X(\bar{x}, k_0, r_0)$, $y \in B_Y(0, k_0 + d_1 + d_2, 2r/a_{k_0+d_1})$, for every $v \in Y$, there exists $u \in DF_p(x, y)^{-1}(v)$ such that*

$$\|u\|_k \leq \frac{C}{m_k a_{k-d_1-d_2}} \|x - \bar{x}\|_{k-d_1-d_2} \|v\|_{d_1+d_2} + a_k \|v\|_{k+d_2}, \quad \forall k \in \mathbb{N}.$$

Then for every $\tau > a_{k_0+d_1}$, there exist $r \in (0, r_0)$ and a neighborhood W in P of \bar{p} such that

$$d_{k_0}(x, S(p)) \leq \tau d_{k_0+d_1+d_2}(0, F(x, p)), \quad (28)$$

for all $(x, p) \in B_X(\bar{x}, k_0, r) \times W$.

Proof. Pick a positive real $r \in (0, r_0)$ such as $Cr < 1/2$. Since $F(\bar{x}, \cdot)$ is lower semicontinuous, for any $\varepsilon > 0$, there exists a neighborhood W of \bar{p} in P such that $F(\bar{x}, p) \cap B_Y(0, k_0 + d_1 + d_2, \varepsilon) \neq \emptyset$, for all $p \in W$. That is,

$$d_{k_0+d_1+d_2}(0, F(\bar{x}, p)) < \varepsilon, \quad \forall p \in W.$$

Suppose that for this neighborhood W , conditions (i) and (ii) are satisfied for all $p \in W$. For a given $p \in W$, there is $\bar{y}_p \in F(\bar{x}, p)$ such as

$$\|\bar{y}_p\|_{k_0+d_1+d_2} \leq (1 + \varepsilon)d_{k_0+d_1+d_2}(0, F(\bar{x}, p)) < \varepsilon.$$

Let $\tau > a_{k_0+d_1}$ and $p \in W$ with $0 \notin F(\bar{x}, p)$ be given. Pick $\varepsilon > 0$ with $\tau\varepsilon < r/4$ and $a_{k_0+d_1} < \tau/(1 + \varepsilon)$ and $0 < r' < r$ with

$$a_{k_0+d_1}\|\bar{y}_p\| < r' < \frac{\tau}{1 + \varepsilon}\|\bar{y}_p\|_{k_0+d_1+d_2} \leq \tau d_{k_0+d_1+d_2}(0, F(\bar{x}, p)).$$

Take a sequence $(\beta_k)_{k \in \mathbb{N}}$ of non-negative reals with unbounded support such that

$$\sum_{k=0}^{\infty} \beta_k (\|\bar{y}_p\|_k + \mu_k) < +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} \beta_k m_k a_{k+d_1} n^k < +\infty; \quad (29)$$

$$\frac{C \sum_{k=0}^{\infty} \beta_k \|\bar{y}_p\|_k}{\sum_{k=d_1+d_2}^{\infty} \beta_k} < 1, \quad (30)$$

and

$$\sum_{k=0}^{\infty} \beta_k \|y_p\|_k \left(1 - \sqrt{\frac{C \sum_{k=0}^{\infty} \beta_k \|y_p\|_k}{\sum_{k=d_2}^{\infty} \beta_k}} \right)^{-2} < \frac{r' \beta_{k_0+d_1+d_2}}{a_{k_0+d_1}}. \quad (31)$$

It is not difficult to show the existence of such a sequence $(\beta_k)_{k \in \mathbb{N}}$. Let us apply Theorem 2 with $F(\cdot, p)$, r' , \bar{y}_p and 0, instead of F , r , \bar{y} and y , respectively. Obviously, (i) and (ii) are satisfied with $\nu_k = \mu_k + \|\bar{y}_p\|_k$, $\nu'_k = 0$. We can find $\bar{x}_p \in S(p)$ such that

$$\begin{aligned} d_k(\bar{x}, S(p)) &\leq \|\bar{x}_p - \bar{x}\|_{k_0} \\ &\leq r' < \tau d_{k_0+d_1+d_2}(0, F(\bar{x}, p)). \end{aligned} \quad (32)$$

Now let $x \in B_X(\bar{x}, k_0, r/2)$ and $p \in W$. If $d_{k_0+d_1+d_2}(0, F(x, p)) \geq r/(2\tau)$ be given, then by relation (32), there is $\bar{x}_p \in S(p)$ such that $\|\bar{x} - \bar{x}_p\|_{k_0} \leq \tau\varepsilon < r/4$. Therefore,

$$\begin{aligned} d_{k_0}(x, S(p)) &\leq \|x - x_p\|_{k_0} \\ &\leq \|x - \bar{x}\|_{k_0} + \|\bar{x} - x_p\|_{k_0} \\ &\leq r/2 + r/4 < \tau d_{k_0+d_1+d_2}(0, F(x, p)). \end{aligned} \quad (33)$$

Let us consider the case $d_{k_0+d_1+d_2}(0, F(x, p)) < r/(2\tau)$. Then there is $y_p \in F(x, p)$ with $\|y_p\|_{k_0+d_1+d_2} < r/2$. Pick a non-negative number $r' < r/2$ with

$$a_{k_0+d_1}\|y_p\|_{k_0+d_1+d_2} < r' < \tau\|y_p\|_{k_0+d_1+d_2}.$$

Since $C(\|y_p\|_{k_0+d_1+d_2} + \|x - \bar{x}\|_{k_0}) < Cr < 1/2$, we can pick a sequence $(\beta_k)_{k \in \mathbb{N}}$ of

non-negative reals with unbounded support such that

$$\begin{aligned} \sum_{k=0}^{\infty} \beta_k (\|y_p\|_k + \mu_k) < +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} \beta_k m_k \|x - \bar{x}\|_{k-d_2} < +\infty; \\ \sum_{k=0}^{\infty} \beta_k < +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} \beta_k m_k a_{k+d_1} n^k < +\infty \quad \text{and} \quad (3); \\ \frac{C \sum_{k=0}^{\infty} (\beta_k \|y_p\|_k + \beta_{k+d_1+d_2} \|x - \bar{x}\|_{k-d_2})}{\sum_{k=d_1+d_2}^{\infty} \beta_k} := s < 1; \\ \sum_{k=0}^{\infty} \beta_k \|y_p\|_k (1 - \sqrt{s})^{-2} < \frac{r' \beta_{k_0+d_1+d_2}}{a_{k_0+d_1}}. \end{aligned}$$

Then apply Theorem 2 with $F(\cdot, p)$, r' , y_p , 0 , x instead of F , r , \bar{y} , y , and \bar{x} , respectively, since conditions (i) and (ii) are verified for $\nu_k = \|x - \bar{x}\|_k + \|y_p\|_k + \mu_k$, $\nu'_k = \|x - \bar{x}\|_{k-d_1-d_2}$. We obtain the existence of $x_p \in S(p)$ verifying

$$d_k(x, S(p)) \leq \|x_p - x\|_{k_0} \leq r' < \tau d_{k_0+d_1+d_2}(0, F(\bar{x}, p)).$$

Thus (28) is shown. \square

To a given multifunction $F : X \rightrightarrows Y$, we associate the new multifunction $\Phi : X \times Y \rightrightarrows Y$ defined by

$$\Phi(x, y) = F(x) - y, \quad (x, y) \in X \times Y.$$

Applying Theorem 4 to the multifunction Φ , we derive the following result of metric regularity type in Fréchet spaces.

Corollary 5. *Let X, Y be graded Fréchet spaces and assume that Y is standard. Let $F : X \rightrightarrows Y$ be a closed multifunction and let $(\bar{x}, \bar{y}) \in \text{gph } F$ be given. Suppose also that there are integers k_0, d_1, d_2 , real numbers $r \in (0, +\infty]$, $C > 0$ and non-decreasing sequences of non-negative reals $(\nu_k)_{k \in \mathbb{N}}$, $(\nu'_k)_{k \in \mathbb{N}}$, and $(m_k)_{k \in \mathbb{N}}$, $(a_k)_{k \in \mathbb{N}}$ with $m_k \geq 1$, $a_k \geq 1$ such that the conditions (i) and (ii) of Theorem 2 are satisfied. Then for every $\tau > a_{k_0+d_1}$, there exists a neighborhood W in Y of \bar{y} such that*

$$d_{k_0}(x, F^{-1}(y)) \leq \tau d_{k_0+d_1+d_2}(y, F(x)), \quad (34)$$

for all $(x, y) \in B_X(\bar{x}, k_0, r) \times W$.

When the mapping $F : X \rightarrow Y$ is Gâteaux differentiable, Theorem 2 yields the following theorem which covers Theorem 3 in [7].

Theorem 6. *Let $F : X \rightarrow Y$ be a continuous mapping between graded Fréchet spaces X, Y with Y standard, and let $\bar{x} \in X$ be given. Suppose F is Gâteaux differentiable on X with derivative $DF(x)$, and that there are integers k_0, d_1, d_2 , real numbers $r \in (0, +\infty]$,*

$C > 0$ and non-decreasing sequences of non-negative reals $(\nu_k)_{k \in \mathbb{N}}$, and $(m_k)_{k \in \mathbb{N}}$, $(a_k)_{k \in \mathbb{N}}$ with $m_k \geq 1$, $a_k \geq 1$ such that the following conditions are satisfied:

(i) For every $u \in X$, there exists $c_2(u) > 0$ such that for all $x \in B_X(\bar{x}, k_0, r)$, all $k \in \mathbb{N}$,

$$\|DF(x)u\|_k \leq c_2(u)(m_k\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \|F(x) - F(\bar{x})\|_k + \nu_k)$$

(ii) For all $x \in B_X(\bar{x}, k_0, r)$, there exists a linear mapping $L(x) : Y \rightarrow X$ such that $DF(x)L(x) = I_Y$ (the identity mapping on Y) and for every $v \in Y$,

$$\|L(x)v\|_k \leq \frac{C}{m_k a_{k-d_1-d_2}} \|x - \bar{x}\|_{k-d_1-d_2} \|v\|_{d_1+d_2} + a_k \|v\|_{k+d_2}, \quad \forall k \in \mathbb{N}.$$

Let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of non-negative reals with unbounded support such that

$$\sum_{k=0}^{\infty} \beta_k \nu_k < +\infty, \quad (35)$$

Then, for every $y \in Y$ with

$$\frac{C \sum_{k=0}^{\infty} \beta_k \|y - F(\bar{x})\|_k}{\sum_{k=d_1+d_2}^{\infty} \beta_k} < 1, \quad (36)$$

and

$$\sum_{k=0}^{\infty} \beta_k \|y - F(\bar{x})\|_k \left(1 - \sqrt{\frac{C \sum_{k=0}^{\infty} \beta_k \|y - F(\bar{x})\|_k}{\sum_{k=d_1+d_2}^{\infty} \beta_k}} \right)^{-2} < \frac{r \beta_{k_0+d_1+d_2}}{a_{k_0+d_1}}, \quad (37)$$

there exists $x \in B_X(\bar{x}, k_0, r)$ such that $F(x) = y$.

Proof. It suffices to show that conditions (i) and (ii) of Theorem 2 are satisfied. Indeed, for every $x \in B(\bar{x}, k_0, r)$, $v \in Y$, setting $u = L(x)v$, one has obviously $u \in DF^{-1}(x)(v)$. Thus condition (2) implies (ii) of Theorem 2. For given $u \in X$ and $x \in B(\bar{x}, k_0, r)$, pick $\bar{t} \in (0, 1)$ such that $\|x + tu\|_{k_0} < r$ for all $t \in [0, \bar{t}]$. For each $k \in \mathbb{N}$, define the function $f_k : [0, \bar{t}] \rightarrow \mathbb{R}$ by

$$f_k(t) = \|F(x + tu) - F(\bar{x})\|_k, \quad t \in [0, \bar{t}].$$

Obviously, f has a right derivative everywhere and

$$f'_{k+}(t) \leq \|DF(x + tu)\|_k, \quad \forall t \in [0, \bar{t}].$$

Therefore, by assumption (i), one has

$$f'_{k+}(t) \leq c_2(u)(m_k\|u\|_{k+d_1} + \|x + tu - \bar{x}\|_k/a_{k-d_2} + f(t) + \nu_k), \quad \forall t \in [0, \bar{t}],$$

and consequently,

$$f'_{k+}(t) - c_2(u)f_k(t) \leq c_2(u)((m_k + 1)\|u\|_{k+d_1} + \|x - \bar{x}\|_k + \nu_k), \quad \forall t \in [0, \bar{t}].$$

Equivalently,

$$\begin{aligned} & e^{-tc_2(u)}[f'_{k+}(t) - c_2(u)f_k(t)] \\ & \leq e^{-tc_2(u)}c_2(u)((m_k + 1)\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \nu_k), \quad \forall t \in [0, \bar{t}]. \end{aligned}$$

By integration, one obtains

$$\begin{aligned} & e^{-tc_2(u)}f_k(t) - f_k(0) \\ & \leq (1 - e^{-tc_2(u)})((m_k + 1)\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \nu_k). \end{aligned}$$

That is, for all $t \in [0, \bar{t}]$,

$$\begin{aligned} & m_k\|u\|_{k+d_1} + \|F(x + tu) - F(\bar{x})\|_k \\ & \leq e^{tc_2(u)}[(m_k + 1)\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \|F(x) - F(\bar{x})\|_k + \nu_k]. \end{aligned}$$

This together with (i) yield

$$\begin{aligned} & \|DF(x + tu)u\|_k \\ & \leq c_2(u)e^{tc_2(u)}[(m_k + 1)\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \|F(x) - F(\bar{x})\|_k + \nu_k], \quad \forall t \in [0, t_0]. \end{aligned}$$

Next, pick a sequence $(t_n)_{n \in \mathbb{N}}$ converging to 0, with $t_n \in (0, \bar{t})$, and set

$$u_n := u, \quad v_n := \frac{F(x + t_n u) - F(x)}{t_n}, \quad n \in \mathbb{N}.$$

Then, $F(x) + t_n v_n = F(x + t_n u_n)$ and $\lim_{n \rightarrow \infty} (u_n, v_n) = (u, DF(x)u)$. Setting $C_2(u) = c_2(u)e^{\bar{t}c_2(u)}$, the Mean Value inequality yields,

$$\begin{aligned} & \|v_n\|_k \\ & \leq \sup_{s \in [0, t_n]} \|DF(x + su)u\|_k \\ & \leq C_2(u)[(m_k + 1)\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \|F(x) - F(\bar{x})\|_k + \nu_k], \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

Thus condition (ii) of Theorem 2 holds. \square

Corollary 7. *Under the assumptions of Theorem 6, for every $\tau > a_{k_0+d_1}$, there exist $r > 0$ and a neighborhood W in Y of \bar{y} such that*

$$d_{k_0}(x, F^{-1}(y)) \leq \tau \|y - F(x)\|_{k_0+d_1+d_2}, \quad (38)$$

for all $(x, y) \in B_X(\bar{x}, k_0, r) \times W$.

Using the same argument as in the proof of Theorem 6, from Theorem 4, we obtain,

when $F(\cdot, p)$ is Gâteaux differentiable, the following implicit multifunction theorem for the system (27).

Corollary 8. *Let X, Y be graded Fréchet spaces and let P be a topological space. Consider a mapping $F : X \times P \rightarrow Y$ and a given point $(\bar{x}, \bar{p}) \in X \times P$ with $F(\bar{x}, \bar{p}) = 0$. Assume furthermore that Y is standard. Suppose $F(\cdot, p)$ is Gâteaux differentiable on X with derivative $DF_p(x)$ for p near \bar{p} , and that there are integers k_0, d_1, d_2 , real numbers $r_0 \in (0, +\infty]$, $C > 0$ and non-decreasing sequences of non-negative reals $(\nu_k)_{k \in \mathbb{N}}$, and $(m_k)_{k \in \mathbb{N}}$, $(a_k)_{k \in \mathbb{N}}$ with $m_k \geq 1$, $a_k \geq 1$ such that the following conditions are satisfied:*

- (i) *The mapping F is continuous at (\bar{x}, \bar{p}) ;*
- (ii) *For p near \bar{p} , for every $u \in X$, there exists $c_2(u) > 0$ such that for all $x \in B_X(\bar{x}, k_0, r_0)$, all $k \in \mathbb{N}$,*

$$\|DF_p(x)u\|_k \leq c_2(u)(m_k\|u\|_{k+d_1} + \|x - \bar{x}\|_k/a_{k-d_2} + \|F_p(x) - F_p(\bar{x})\|_k + \nu_k);$$

- (iii) *For p near \bar{p} and for all $x \in B_X(\bar{x}, k_0, r_0)$, there exists a linear mapping $L_p(x) : Y \rightarrow X$ such that $DF_p(x)L_p(x) = I_Y$ (the identity mapping on Y) and for every $v \in Y$,*

$$\|L_p(x)v\|_k \leq \frac{C}{m_k a_{k-d_1-d_2}} \|x - \bar{x}\|_{k-d_1-d_2} \|v\|_{d_1+d_2} + a_k \|v\|_{k+d_2}, \quad \forall k \in \mathbb{N}.$$

Then, for every $\tau > a_{k_0+d_1}$, there exist $r > 0$ and a neighborhood W in P of \bar{p} such that

$$d_{k_0}(x, S(p)) \leq \tau \|F(x, p)\|_{k_0+d_1+d_2}, \quad (39)$$

for all $(x, p) \in B_X(\bar{x}, k_0, r) \times W$.

4. Application: Differential equations in Fréchet spaces

In this final section, we present an application to the existence of solutions for ordinary differential equations in Fréchet spaces. Let X be a graded Fréchet space, let $U \subseteq X$ be an open set, let $t_0 \in \mathbb{R}$, $r_0 > 0$ be given, and let $f : [t_0 - r_0, t_0 + r_0] \times U \rightarrow X$ be a continuous mapping. For given $r > 0$ and $x_0 \in U$, consider the initial value problem:

$$(DEF) \quad \begin{cases} x'(t) &= f(t, x(t)), & t \in [t_0 - r, t_0 + r], \\ x(t_0) &= x_0. \end{cases}$$

When the function f is of class C^2 , Poppenberg [19] established a result on existence of solutions for equation (DEF). In the following theorem, the data function is assumed merely to be Gâteaux differentiable.

Theorem 9. *Let E be graded Fréchet space such that X and $C([-1, 1], E)$ are standard. Suppose that the function f is continuous on $[t_0 - r_0, t_0 + r_0] \times U$. Suppose also that for*

each $t \in [t_0 - r_0, t_0 + r_0]$, $f(t, \cdot) : U \rightarrow X$ is Gâteaux differentiable on U with derivative $D_x f(t, \cdot)$, and that there is a non-decreasing sequence of non-negative reals $(c_k)_{k \in \mathbb{N}}$ such that for all $(t, x) \in [t_0 - r_0, t_0 + r_0] \times U$, one has

$$\|D_x f(t, x)u\|_k \leq c_k \|u\|_k, \quad \text{for all } u \in X, k \in \mathbb{N}. \quad (40)$$

Then, there is $r \in (0, r_0]$ such that problem (DEF) has a solution $x(\cdot) \in C^1([t_0 - r, t_0 + r], E)$. If in addition, f is a C^1 -mappings on $[t_0 - r_0, t_0 + r_0] \times U$, and that, say, for a sequence $(c_k)_{k \in \mathbb{N}}$ above,

$$\|Df(t, x)\|_k \leq c_k \quad \forall k \in \mathbb{N}, \quad \forall (t, x) \in [t_0 - r_0, t_0 + r_0] \times U, \quad (41)$$

then the solution $x(\cdot)$ is unique.

Proof. Using the transformations $t = t_0 + rs$, $z(s) = x(t_0 + rs) - x_0$, $s \in [-1, 1]$, we can rewrite problem (DEF) as

$$(DEF1) \quad \begin{cases} z'(s) &= r f(t_0 + rs, z(s) + x_0), \quad s \in [-1, 1], \\ z(0) &= 0. \end{cases}$$

Denote by

$$W = \{(z, r) \in C^1([-1, 1], X) \times \mathbb{R} : r \in (-r_0, r_0), z(s) \in U \forall s \in [-1, 1]\}.$$

W is an open subset of the graded Fréchet space $C^1([-1, 1], X) \times \mathbb{R}$. Set $F : W \rightarrow C([-1, 1], X) \times X$, defined by

$$F(z, r) = (z'(s) - r f(t_0 + rs, z(s) + x_0), z(0)), \quad s \in [-1, 1], (z, r) \in W.$$

Then for each $r \in (0, r_0)$, $F(r, \cdot)$ is Gâteaux differentiable on W with derivative given by

$$D_z F(z, r)u = (u'(s) - r D_x f(t_0 + rs, z(s) + x_0)u(s), u(0)), \quad s \in [-1, 1], (r, s) \in W, u \in C^1([-1, 1], X).$$

Obviously, $(0, 0) \in W$ and $F(0, 0) = (0, 0)$, and moreover, $z \in C^1([-1, 1], X)$ is a solution of problem (DEF1) with respect to $r \in (-r_0, r_0)$ if and only if $F(z, r) = (0, 0)$. So it suffices to show that the mapping F verifies all the assumptions (i), (ii) and (iii) of Corollary 8 with $(\bar{x}, \bar{p}) := (0, 0)$, $(x, p) := (z, r)$. Assumption (i) is obvious. To verify (ii), for any $k \in \mathbb{N}$, for $(z, r) \in W$, $u \in C^1([-1, 1], X)$, making use of relation (40), one has

$$\begin{aligned} \|D_z F(z, r)u\|_k &= \|u'(\cdot) - r D_x f(t_0 + rs, z(\cdot) + x_0)u(\cdot)\|_k + \|u(0)\|_k \\ &\leq \|u'(\cdot)\|_k + r_0 c_k \|u(\cdot)\|_k + \|u(\cdot)\|_k \leq (2 + r_0 c_k) \|u\|_k. \end{aligned}$$

Thus (ii) follows. To verify (iii), for $(z, r) \in W$, $(v, v_0) \in C([-1, 1], X) \times X$, then $D_z F(z, r)u = (v, v_0)$, $u \in C^1([-1, 1], X)$ if and only if u is a solution of the linear

differential equation:

$$(LDE) \quad \begin{cases} u'(s) &= A(s)u(s) + v(s), & s \in [-1, 1], \\ u(0) &= v_0 \end{cases}$$

where, $A(s) := rD_x f(t - 0 + rs, z(s) + x_0)$ ($s \in [-1, 1]$) is a continuous linear mapping from X to itself, according to condition (40). Thanks to Proposition 3.4 in [19], Problem (LDE) has a unique solution $u \in C^1([-1, 1], X)$. That is, $D_z F(z, r)$ is invertible with $L(z, r) := D_z F(z, r)^{-1}(v, v_0) = u$, u solving (LDE), for $(v, v_0) \in C([-1, 1], X) \times X$. Let now $(v, v_0) \in C([-1, 1], X) \times X$, be given, and let $u \in C^1([-1, 1], X)$ be a solution of problem (LDE). One has

$$\|A(s)u(s)\|_k \leq r_0 c_k \|u(s)\|_k, \quad \text{for all } k \in \mathbb{N}, s \in [-1, 1]. \quad (42)$$

Therefore, for any $k \in \mathbb{N}$, by considering the corresponding integral equation of (LDE), one has

$$\|u(t)\|_k \leq \|v\|_k + \|v_0\|_k + r_0 c_k \int_0^t \|u(s)\|_k, \quad \forall t \in [0, 1].$$

Thanks to the Gronwall lemma, applied to the function $\alpha(t) := \|u(t)\|_k$, $t \in [0, 1]$, one obtains

$$\|u(t)\|_k \leq e^{r_0 c_k t} \|(v, v_0)\|_k, \quad \text{for all } t \in [0, 1].$$

Similarly, for $t \in [-1, 0]$, by setting $w(s) := u(-s)$, $s \in [0, 1]$, one also has

$$\|w(t)\|_k \leq \|v\|_k + \|v_0\|_k + r_0 c_k \int_0^t \|w(s)\|_k, \quad \forall t \in [0, 1].$$

Hence,

$$\|u(t)\|_k \leq e^{-r_0 c_k t} \|(v, v_0)\|_k, \quad \text{for all } t \in [-1, 0].$$

Thus,

$$\sup_{s \in [-1, 1]} \|u(s)\|_k \leq e^{r_0 c_k} \|(v, v_0)\|_k, \quad \forall k \in \mathbb{N}. \quad (43)$$

Furthermore, from equation (LDE), using relations (42), (43), one obtains

$$\begin{aligned} \sup_{s \in [-1, 1]} \|u'(s)\|_k &\leq r_0 c_k \sup_{s \in [-1, 1]} \|u(s)\|_k + \sup_{s \in [-1, 1]} \|v(s)\|_k \\ &\leq r_0 c_k e^{r_0 c_k} \|(v, v_0)\|_k + \|v\|_k \leq (r_0 c_k e^{r_0 c_k} + 1) \|(v, v_0)\|_k. \end{aligned}$$

Hence,

$$\|u(\cdot)\|_k \leq (e^{r_0 c_k} + r_0 c_k e^{r_0 c_k} + 1) \|(v, v_0)\|_k,$$

and (iii) follows. According to Corollary 8, there is a neighborhood $U(x_0) \subseteq U$ of x_0 and $r_1 > 0$, as well as a sequence of non-negative reals (τ_k) such that for all $k \in \mathbb{N}$,

$$d_k(x, S(r)) \leq \tau_k \|F(x, r)\|_k, \quad \text{for all } (x, r) \in U(x_0) \times [0, r_1],$$

where,

$$S(r) = \{z \in C^1([-1, 1], X) : (z, r) \in W, F(z, r) = 0\}.$$

Suppose now f is a mapping of class C^1 on $[t_0 - r_0, t_0 + r_0] \times U$. We shall show that when $r > 0$ is sufficiently small, there is a unique $z \in C^1([-1, 1], X)$ with $(z, r) \in W$ such that $F(x, r) = (0, 0)$. Indeed, since f is a C^1 -mapping on $[t_0 - r_0, t_0 + r_0] \times U$, there is a sequence of non-negative reals (ε_k) such that for all $k \in \mathbb{N}$,

$$\begin{aligned} & \|f(t_1, x_1)\|_k - f(t_2, x_2)\|_k \\ & \leq \|Df(t_0, x_0)\|_k \|(t_1 - t_2, x_1 - x_2)\|_k \\ & + \varepsilon_k \|(t_1 - t_2, x_1 - x_2)\|_k, \quad \forall (t_1, x_1), (t_2, x_2) \in [t_0 - r_0, t_0 + r_0] \times U. \end{aligned}$$

This shows that there is some sequence of non-negative (d_k) such that

$$\|f(t_1, x_1)\|_k - f(t_2, x_2)\|_k \leq d_k \|(t_1 - t_2, x_1 - x_2)\|_k \quad \text{for all } k \in \mathbb{N}, (t, x) \in [t_0 - r_0, t_0 + r_0] \times U. \quad (44)$$

Let $r \in (0, r_0)$ be given, and let z_1, z_2 in $C^1([-1, 1], X)$ be two solutions of equation (DEF1) with respect to r . Then, one has

$$\begin{cases} z_1'(s) - z_2'(s) & = r[f(t_0 + rs, z_1(s) + x_0) - f(t_0 + rs, z_2(s) + x_0)], \quad s \in [-1, 1], \\ z_1(0) - z_2(0) & = 0. \end{cases}$$

By making use similarly of the Gronwall lemma as above for this differential equation with its solution $z_1 - z_2$, one derives $z_1 = z_2$. The proof is completed. \square

Remark 10. Note that, by the same argument, the conclusion of the preceding theorem remains true if condition (40) is replaced by the following one: there is a constant $C > 0$ such that for all $k \in \mathbb{N}$, one has

$$\|D_x f(t, x)u\|_k \leq C(\|x\|_k + \|u\|_k), \quad \text{for all } u \in X, (t, x) \in [t_0 - r_0, t_0 + r_0] \times U. \quad (45)$$

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