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# Object digitization up to a translation

Loïc Mazo, Étienne Baudrier

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## Abstract

This paper presents a study on the set of the digitizations generated by all the translations of a planar body on a square grid. First the translation vector set is reduced to a bounded subset, then the dual introduced in [BM16] linking the translation vector to the corresponding digitization is proved to be piecewise constant. Finally, a new algorithm is proposed to compute the digitization set using the dual.

*Keywords:* digitization, plane curve, translation, dual representation

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## 1. Introduction

The digitization of a planar body depends on the digitization method and also on the object relative position with respect to the digitization grid. As a result, there is a variability in the resulting digital set and this variability may influence the digital set geometrical and topological attributes. For instance, conditions have been given to preserve the topology during the digitization step [TR02, SK05].

The focus of this paper is on the object relative position with respect to the digitization grid. This issue has been studied on some geometrical primitives, *i.e.* the straight segments and the discs. Straight segment digitizations have been discussed in function of the straight segment slope and its vertical position. The function giving the digital straight segment from these two inputs is known as the *preimage*. Several properties have been proved on the preimage and it is widely used, *e.g.* for digital straight segment recognition [DS84]. The number of oval and disc digitizations in function of their radius up to a translation was studied in [Ken48, Nag05, HZ06]. The number of digital discs including exactly  $N$  points was treated in [HZ07] and an asymptotic bound on this number was given in [HZ16]. Our study follows a previous work [BM16] which focused on function graphs digitizations.

After introducing the dual definition in Section 2, its structure is investigated in Section 3 and it is proved to be piecewise constant. Two algorithms devoted

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to the computation of any digitization and to the computation of the digitization set are presented and applied on a toy example in Section 4.

## 2. Notations and definitions

25 Let us consider a connected set  $S$  in  $\mathbb{R}^2$  whose boundary is a simple closed (Jordan) curve  $\Gamma$ . Thanks to the Jordan curve theorem, we may assume a real  $M > 0$  and a continuous map  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $S \subset D = [-M, M]^2$  and  $\Gamma$ , resp.  $S$ , is implicitly defined by  $\Gamma = \{f(x) = 0 \mid x \in \mathbb{R}^2\}$ , resp.  $S = \{x \in \mathbb{R}^2 \mid f(x) \leq 0\}$ .

30 The aim of this paper is to study the set of the digitizations of  $\Gamma$  obtained using the grids generated by the action of the group of translations on the standard grid. Equivalently, we can consider a unique grid, the standard one, and let the group of translations acts on  $S$ . This is the technical point of view that we have adopted in the present article.

35 The common methods to model the digitization of the set  $S$  are closely related to each others. In this paper, we assume a *Gauss digitization*. This method associates to the set  $S$  the digitization set  $\mathfrak{D}(S)$ . The set  $\mathfrak{D}(S)$  contains the grid points that lie inside  $S$  or, equivalently, it is a binary image defined on  $\mathbb{Z}^2$  whose 1's are the points inside  $S$ .

We write  $t_u$  for the translation of vector  $u \in \mathbb{R}^2$  and  $\mathcal{T}$  for the group of integer translations of  $\mathbb{Z}^2$ . Let  $u \in \mathbb{R}^2$ . The translate by  $-u$  of the set  $S$ ,  $t_{-u}(S)$ , is the set defined by  $f_u \leq 0$  where

$$\begin{aligned} f_u: \mathbb{R}^2 &\rightarrow \mathbb{R} \\ x &\longmapsto f(x + u) . \end{aligned}$$

Figure 1 exhibits the digitization sets for the set  $S_{\text{astro}}$  bounded by the “stretched” astroid

$$((x + u_x)/2)^{2/3} + (y + u_y)^{2/3} = 1 .$$

40

The digitization set  $\mathfrak{D}(S_u)$  is a finite subset of  $\mathbb{Z}^2$  and we are only interested in the relative positions of its elements (in other words,  $\mathbb{Z}^2$  is viewed as a geometrical subset of the Euclidean plane without any preferential origin). Thus, rather than the set  $\mathfrak{D}(S_u)$ , we will consider its equivalence class under integral translations  $[\mathfrak{D}(S_u)]$ . The choice of a canonical representative in each class could lead to surprising results. For instance, if we choose the set  $\mathfrak{D}_0(S_u)$  whose barycenter lies in  $[0, 1)^2$  as a representative of  $[\mathfrak{D}(S_u)]$ , then a small translation of the set can suddenly shift this representative if the set have long and thin horizontal, or vertical, parts that can be missed by the digitization. Therefore, we do not focus on a particular representative of the equivalence classes. We set

$$\mathcal{D}^T(f) = \{[\mathfrak{D}(S_u)] \mid u \in \mathbb{R}^2\}$$

and, for any  $u \in \mathbb{R}^2$ ,  $\varphi_S(u) = \mathfrak{D}(S_u)$ . The goal of this paper is to describe and compute  $\mathcal{D}^T(f)$ .

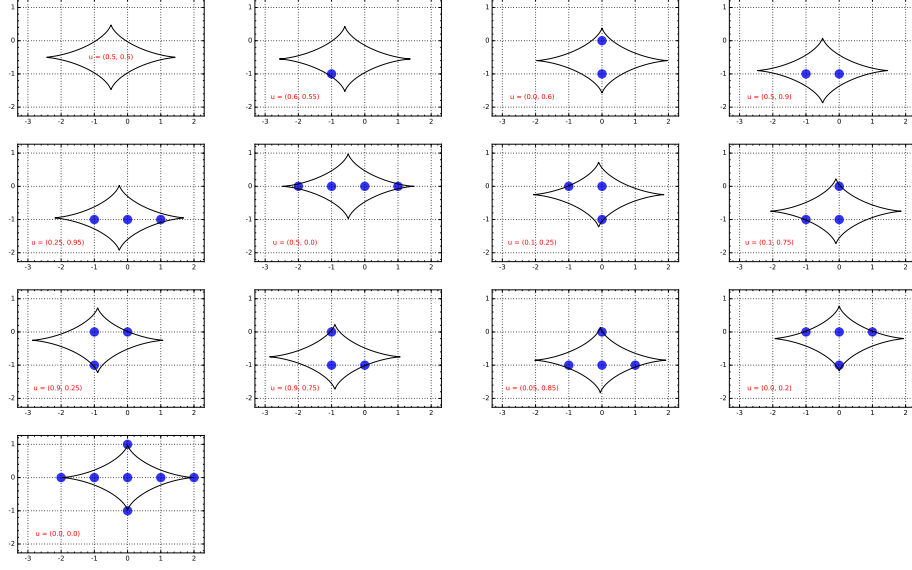


Figure 1: The thirteen digitizations of a set bounded by the stretched astroid  $((x + u_x)/2)^{2/3} + (y + u_y)^{2/3} = 1$  (the first one is the empty set).

The next obvious proposition will allow us to reduce the space of the translation vectors that has to be considered in our study.

**45 Proposition 2.1.** *Let  $\sim$  be the equivalence relation defined on  $\mathbb{R}^2$  by  $u \sim v \iff u - v \in \mathbb{Z}^2$ . Then the map  $u \mapsto [\mathfrak{D}(S_u)]$  is invariant under  $\sim$ .*

*Proof.* Let  $u, v \in \mathbb{R}^2$  s.t.  $w = u - v \in \mathbb{Z}^2$ .

$$\begin{aligned} \mathfrak{D}(S_v) &= t_v(S) \cap \mathbb{Z}^2 \\ &= t_{u+w}(S) \cap t_w(\mathbb{Z}^2) \\ &= t_w(t_u(S) \cap \mathbb{Z}^2) \\ &= t_w(\mathfrak{D}(S_u)) . \end{aligned}$$

Thus,

$$\mathfrak{D}(S_v) \in [\mathfrak{D}(S_u)] .$$

□

From now, for any  $u \in \mathbb{R}^2$ , we write  $[u]$ , resp.  $\langle u \rangle$ , for the vectors whose coordinates are respectively the integer parts and the fractional parts of the coordinates of  $u$ . Hence,  $[u] \in \mathbb{Z}^2$ ,  $\langle u \rangle \in [0, 1)^2$  and  $u = [u] + \langle u \rangle$ . Let  $\mathbb{T}$  be the torus  $\mathbb{R}^2 / \sim$ . By abuse of notation, for any equivalence class  $\mathbf{t} \in \mathbb{T}$ , we also write  $\langle \mathbf{t} \rangle$  for  $\langle u \rangle$  where  $u \in \mathbf{t}$ . The vector  $\langle \mathbf{t} \rangle$  is the canonical representative of the class  $\mathbf{t}$ .

As a consequence of Prop. 2.1, the projection theorem on equivalence relations allows us to define the dual of the set of digitizations.

**Definition 1** (Dual by translation). *The dual of  $\mathcal{D}^T(f)$  is defined on the torus  $\mathbb{T}$  as the unique function*

$$\begin{aligned}\tilde{\varphi}_S: \mathbb{T} &\rightarrow \mathcal{D}^T \\ \mathbf{t} &\mapsto \varphi_S(u) \ ,\end{aligned}$$

where  $u \in \mathbf{t}$ .

We have the following commutative diagram:

$$\begin{array}{ccc} u \in \mathbb{R}^2 & \xrightarrow{\quad} & \mathfrak{D}(S_u) \in \mathcal{P}(\mathbb{Z}^2) \\ \downarrow [\cdot] & \searrow \varphi_S & \downarrow [\cdot] \\ [u] \in \mathbb{T} & \xrightarrow{\tilde{\varphi}_S} & [\mathfrak{D}(S_u)] \in \mathcal{P}(\mathbb{Z}^2)/\mathcal{T} \end{array}$$

An example of dual is shown in Fig. 2 (deployed torus) and Fig. 3. It is the dual of the set  $S_{\text{astro}}$ . Each point  $u \in [0, 1]^2$  in Fig. 2 corresponds to a translation  $t_u$  and the color of this point corresponds to the digitization  $\mathfrak{D}(S_u)$ .  
 60 All the points having the same color in the dual correspond to translations giving the same digitization.

**Remark 2.1.** *B. Nagy represents in [Nag05] regions of the translation vector set  $[0, 1]^2$  corresponding to distinct digitizations for the special case of the disc with radius 2. The dual of the disc with radius 2 is shown on Fig. 4(a) (actually, rather a gradient image than the dual itself). One can see that Nagy's representation (Fig. 4(b)) is a sketch (using straight lines) of the dual first octant (in red).*  
 65

As seen in the example  $S_{\text{astro}}$  (Fig. 2), the cardinality of the digitizations is far from constant. Therefore, we define the *cardinal map*, that we denote  $\#_S$ , as follows:

$$\begin{aligned}\#_S: \mathbb{T} &\longrightarrow \mathbb{N} \\ \mathbf{t} &\longmapsto \mathbf{t} \cap S = \text{card } \mathfrak{D}(S_{\mathbf{t}}) \ .\end{aligned}$$

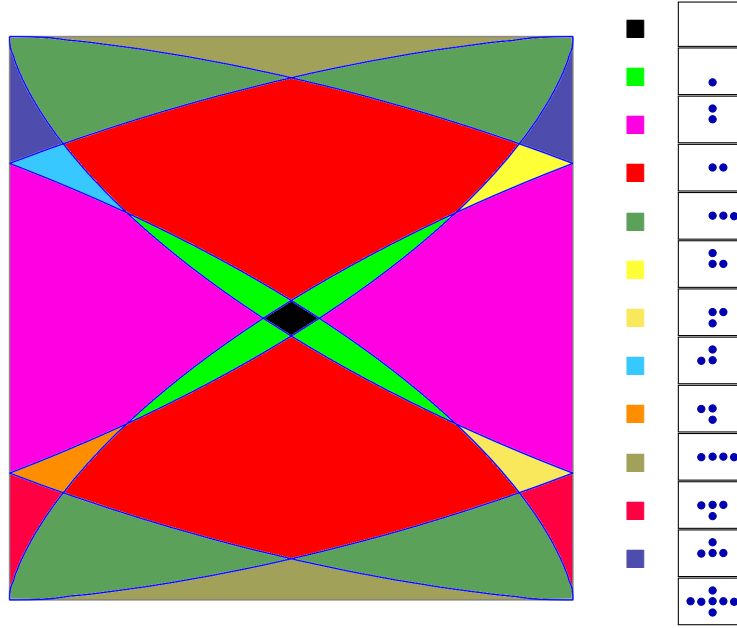


Figure 2: A deployed representation of the dual of  $\mathcal{S}_{\text{astro}}$ . Note that the 7-points digitization region contains only one point, so it is not perceptible.

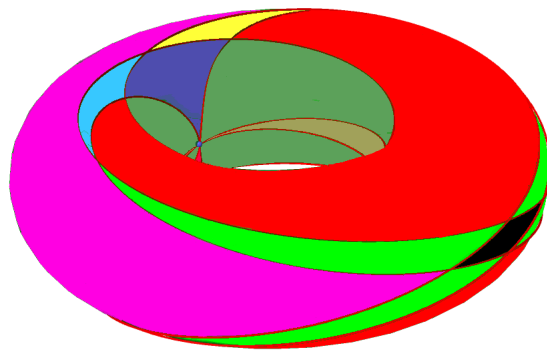


Figure 3: The dual of  $\mathcal{S}_{\text{astro}}$ .

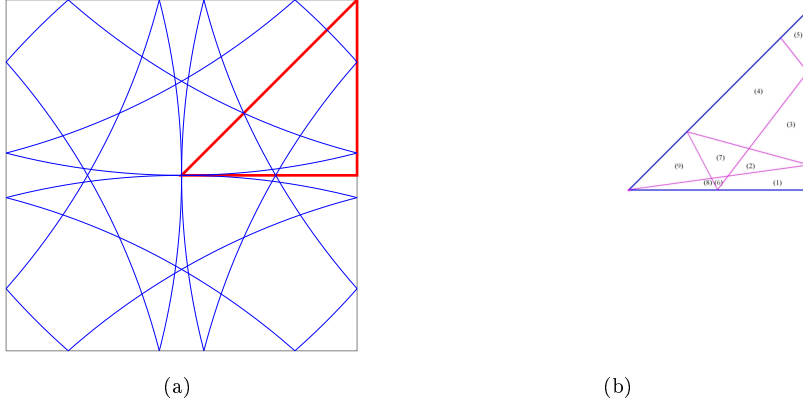


Figure 4: (a) Dual gradient image of a disc with radius 2. (b) Fig. 4 in [Nag05].

### 3. Properties

The main property of the dual is that the plot of the curve  $\Gamma$  on the torus  $\mathbb{T}$  delineates regions on which the dual function  $\tilde{\varphi}_S$  is constant. Before proving this property, we give some complementary notations related to the dual.

We define the *grid boundary*  $\mathcal{B}$  as the set of grid points that lie in the (morphological) dilation of the boundary  $\Gamma$  of  $S$  by the unit square  $(-1, 0]^2$ :

$$\mathcal{B} = (\Gamma \oplus (-1, 0]^2) \cap \mathbb{Z}^2 ,$$

where  $\oplus$  denotes the Minkowski sum (see Fig. 5).

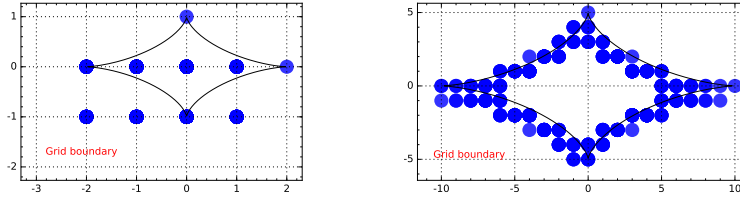


Figure 5: The grid boundary of  $S_{\text{astro}}$  at two resolutions ( $\times 1$  and  $\times 5$ ).

It is plain that  $\mathcal{B}$  contains all the points of  $\mathbb{Z}^2$  whose value can change when we shift the set  $S$  by a translation  $t_u$ ,  $u \in [0, 1]^2$  and that  $S \setminus \mathcal{B}$  is the “set core”: the points that are in any digitization. In our set instance  $S_{\text{astro}}$ , the core is empty, which results in an empty digitization (see Fig 1).

Let  $\tilde{\Gamma}$  be the “plot of  $\Gamma$  on  $\mathbb{T}$ ” :

$$\tilde{\Gamma} = \{[u] \mid u \in \Gamma\} .$$

For any  $p \in \mathbb{Z}^2$ , we define the function

$$\begin{aligned}\tilde{f}_p: \mathbb{T} &\rightarrow \mathbb{R} \\ \mathbf{t} &\mapsto f_p(\langle \mathbf{t} \rangle) = f(\langle \mathbf{t} \rangle + p) .\end{aligned}$$

We also define the restriction  $\Gamma_p$  of the curve  $\Gamma$  to the square  $\{p\} \oplus [0, 1)^2$  and the corresponding restriction  $\tilde{\Gamma}_p$  on  $\mathbb{T}$  by the implicit equations  $f_p = 0$  and  $\tilde{f}_p = 0$ . Then,

$$\tilde{\Gamma} = \bigcup_{p \in \mathcal{B}} \tilde{\Gamma}_p = \bigcup_{p \in \mathbb{Z}^2} \tilde{\Gamma}_p .$$

Indeed, let  $\mathbf{t} \in \tilde{\Gamma}$ . By definition of  $\tilde{\Gamma}$ , there exists  $u \in \mathbf{t}$  s.t.  $u \in \Gamma$ . Then,  $\lfloor u \rfloor \in \mathcal{B}$ . Moreover,

$$\begin{aligned}u \in \Gamma &\iff f(u) = 0 \\ &\iff f_{\lfloor u \rfloor}(\langle u \rangle) = 0 \\ &\iff \tilde{f}_{\lfloor u \rfloor}(\mathbf{t}) = 0 \\ &\iff \mathbf{t} \in \tilde{\Gamma}_{\lfloor u \rfloor} .\end{aligned}$$

Thereby, we have  $\tilde{\Gamma} = \bigcup_{p \in \mathcal{B}} \tilde{\Gamma}_p$  and by definition of  $\mathcal{B}$ , the set  $\tilde{\Gamma}_p$  is empty whenever  $p \notin \mathcal{B}$ .

so In other words, using the canonical plane representation of the torus  $\mathbb{T}$  as the square  $[0, 1)^2$ , the plot of  $\Gamma$  on the torus,  $\tilde{\Gamma}$ , is the superposition of the plots of the implicit functions  $f_p = 0$  on  $[0, 1)^2$ , that is, the superposition of the plots of  $\Gamma$  on the squares  $\{p\} \oplus [0, 1)^2$ ,  $p \in \mathcal{B}$ .

Alike, the cardinal map  $\#_S$  can be defined by means of the local functions  $\tilde{f}_p$ . For any  $p \in \mathbb{Z}^2$ , we define  $\tilde{\mathbf{1}}_p$  as the indicator function of the set  $\tilde{f}_p(\mathbf{t}) \leq 0$ . Then,

$$\#_S = \sum_{p \in \mathbb{Z}^2} \tilde{\mathbf{1}}_p = \sum_{p \in \mathcal{B} \cup S} \tilde{\mathbf{1}}_p .$$

Indeed, let  $\mathbf{t} \in \mathbb{T}$ . One has

$$\begin{aligned}\#_S(\mathbf{t}) &= \text{card } \mathbf{t} \cap S \\ &= \text{card} \{p \in \mathbb{Z}^2 \mid p + \langle \mathbf{t} \rangle \in S\} \\ &= \text{card} \{p \in \mathbb{Z}^2 \mid f_p(\langle \mathbf{t} \rangle) \leq 0\} \\ &= \text{card} \{p \in \mathbb{Z}^2 \mid \tilde{f}_p(\mathbf{t}) \leq 0\} \\ &= \sum_{p \in \mathbb{Z}^2} \tilde{\mathbf{1}}_p(\mathbf{t}) .\end{aligned}$$

Moreover, it is plain that  $p + \langle \mathbf{t} \rangle \in S$  implies  $p \in \mathcal{B} \cup S$ . It is worthy to observe that

$$\tilde{\varphi}_S(\mathbf{t}) = [\{p \in \mathbb{Z}^2 \mid \tilde{\mathbf{1}}_p(\mathbf{t}) = 1\}] .$$



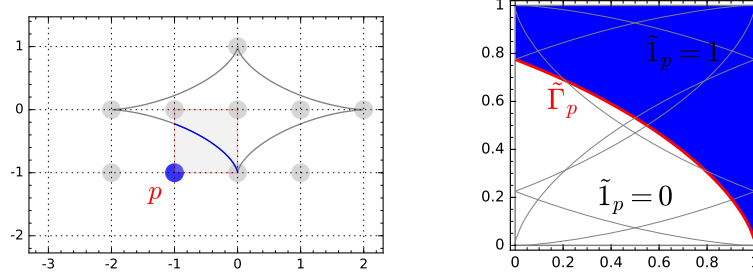


Figure 6: Left: a point  $p$  in the grid boundary and the square  $\{p\} \oplus [0, 1]^2$ . Right: the curve  $\tilde{\Gamma}_p$  (in red) and the indicator function  $\tilde{\mathbf{I}}_p$  (white: the region  $\tilde{\mathbf{I}}_p = 0$ , blue: the region  $\tilde{\mathbf{I}}_p = 1$ ). The curve  $\tilde{\mathbf{I}}_p = 0$ , which is closed at its left extremity and open at its right extremity, is included in the blue region.

We introduce other curves on the torus  $\mathbb{T}$ , the *outer equator*  $\tilde{\Gamma}_x$  which is the quotient space of the  $x$  axis of  $\mathbb{R}^2$ , and the *prime meridian*  $\tilde{\Gamma}_y$  which is the  
85 quotient space of the  $y$  axis of  $\mathbb{R}^2$ .

We now establish the main property of the dual.

**Proposition 3.1.** *Let  $S$  be a compact subset of  $\mathbb{R}^2$  whose boundary  $\Gamma$  is a Jordan curve.*

- The dual  $\tilde{\varphi}_S$  is constant on the connected components of  $\mathbb{T} \setminus \tilde{\Gamma}$ .
- 90 • For any  $p \in \mathbb{Z}^2$ , the function  $\tilde{\mathbf{I}}_p$  is constant on the connected components of  $\mathbb{T} \setminus (\tilde{\Gamma}_p \cup \tilde{\Gamma}_x \cup \tilde{\Gamma}_y)$ .

*Proof.* Let  $\mathbf{t} \in \mathbb{T} \setminus \tilde{\Gamma}$ . Then, for any  $u \in \mathbf{t}$ ,  $f(u) \neq 0$ . Since  $f$  is continuous, the sets  $f > 0$  and  $f < 0$  are open. Then, for any  $u \in \mathbf{t}$  s.t.  $[u] \in \mathcal{B}$ , there exists  $\varepsilon_u \in (0, 1/2)$  such that the open ball  $B(u, \varepsilon_u)$  does not intersect the curve  $\Gamma$ . As  $\mathcal{B}$  is finite, we can set

$$\varepsilon = \min_{u \in \mathbf{t}, [u] \in \mathcal{B}} \varepsilon_u$$

and, by definitions of  $\varepsilon$  and  $\mathcal{B}$ , no ball  $B(p + u, \varepsilon)$ ,  $(p, u) \in \mathbb{Z}^2 \times \mathbf{t}$ , intersects  $\Gamma$ . Hence, thanks to the intermediate value theorem, for any  $u \in \mathbf{t}$  and any integer point  $p \in \mathbb{Z}^2$ , the map

$$v \in B(u, \varepsilon) \mapsto \text{sign}(f_v(p))$$

is constant.

- Since, for any integer point  $p \in \mathbb{Z}^2$  and any  $u \in \mathbf{t}$ , the map  $v \in B(u, \varepsilon) \mapsto \text{sign}(f_v(p))$  is constant,  $\varphi_S$  is constant on the open ball  $B(u, \varepsilon)$  for any  
95  $u \in \mathbf{t}$ . Therefore, the set  $B_{\mathbf{t}}$  of  $\mathbb{T}$  which is the (common) projection of the balls  $B(u, \varepsilon)$ ,  $u \in \mathbf{t}$ , on the torus  $\mathbb{T}$  is open and  $\tilde{\varphi}_S$  is constant on  $B_{\mathbf{t}}$ :  $\tilde{\varphi}_S$  is locally constant on  $\mathbb{T} \setminus \tilde{\Gamma}$ . We conclude that  $\tilde{\varphi}_S$  is constant on any connected component of  $\mathbb{T} \setminus \tilde{\Gamma}$ .

- From now, we assume that  $\mathbf{t}$  does not lie on the outer equator nor on the prime meridian, we take  $u = \langle \mathbf{t} \rangle$  and we consider some  $p \in \mathbb{Z}^2$ . Then, we can choose  $\varepsilon_{\langle \mathbf{t} \rangle}$  such that the open ball  $B(\langle \mathbf{t} \rangle, \varepsilon_{\langle \mathbf{t} \rangle})$  does not intersect the grid lines. Thereby,

$$B(\langle \mathbf{t} \rangle, \varepsilon_{\langle \mathbf{t} \rangle}) \subseteq [0, 1)^2$$

and, with  $\varepsilon' = \min(\varepsilon, \varepsilon_{\langle \mathbf{t} \rangle})$ , for any  $v \in B(\langle \mathbf{t} \rangle, \varepsilon')$ ,

$$f_v(p) = f_p(v) = \tilde{f}_p([v]) .$$

Since the map  $v \in B(\langle \mathbf{t} \rangle, \varepsilon) \mapsto \text{sign}(f_v(p))$  is constant, we derive that  $\tilde{\mathbf{I}}_p$  is constant on the open neighborhood  $[B(\langle \mathbf{t} \rangle, \varepsilon')]$  of  $\mathbf{t}$ . Thereafter,  $\tilde{\mathbf{I}}_p$  is locally constant, which achieves the proof.

□

For instance, the region boundaries of the dual of the set  $S_{\text{astro}}$  (Fig. 3) are obtained by the following SAGE program whose raw result is shown in Fig. 7.

```

105 f(x,y) = ((x/2)^2+y^2 - 1)^3 + 27*(x/2)^2*y^2
    p = polygon([(0,0), (1,0), (1,1), (0,1)], fill=false)
    for i in range(-3, 3):
        for j in range(-2, 2):
            g(x,y)= f(i+x, j+y)
110     p += implicit_plot(g,(x,0,1),(y,0,1))
    p.show()

```

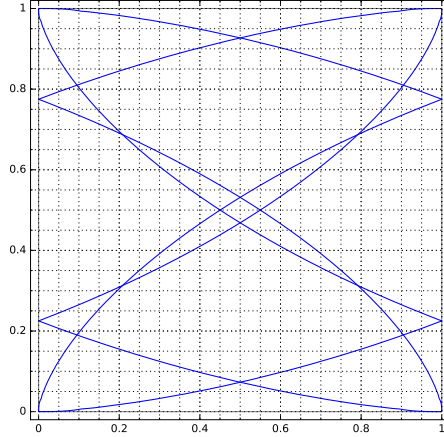


Figure 7: Region boundaries for the dual of our set instance  $S_{\text{astro}}$ .

In order to precise the relationship between neighboring flat regions of the dual, we now establish that crossing the curve  $\tilde{\Gamma}$  on the torus  $\mathbb{T}$  generally amounts to remove, or to add, a particular point in the digitization of the set.

**Lemma 3.1.** *Let  $p \in \mathcal{B}$  and  $\mathbf{t} \in \tilde{\Gamma}_p$ . If  $\mathbf{t} \notin (\bigcup_{q \neq p} \tilde{\Gamma}_q) \cup \tilde{\Gamma}_x \cup \tilde{\Gamma}_y$ , then there exists a neighborhood  $N$  of  $\mathbf{t}$  on which the functions  $\tilde{\mathbf{1}}_q$ ,  $q \neq p$ , are constant and  $\tilde{\mathbf{1}}_p$  is not constant.*

*Proof.* Let  $p \in \mathcal{B}$  and  $\mathbf{t} \in \tilde{\Gamma}_p \setminus \left( \left( \bigcup_{q \neq p} \tilde{\Gamma}_q \right) \cup \tilde{\Gamma}_x \cup \tilde{\Gamma}_y \right)$ .

Let  $v \in \{p\} \oplus (0,1)^2$  such that  $v \in \mathbf{t}$ . We write  $G$  for the grid in  $\mathbb{R}^2$  ( $G = \mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ ). In  $\mathbb{R}^2$ , let define

$$E = (\Gamma \cup G) \setminus (\{p\} \oplus (0,1)^2) .$$

Note that

$$[E] = \left( \bigcup_{q \neq p} \tilde{\Gamma}_q \right) \cup \tilde{\Gamma}_x \cup \tilde{\Gamma}_y .$$

The set  $E$  is a closed set and  $\mathcal{B}$  is finite. Then there exists  $\varepsilon > 0$  such that, for any  $u \in \mathbf{t}$ , the open ball  $B(u, \varepsilon)$  does not intersect the set  $E$ . Let  $N$  be the common projection of the balls  $B(u, \varepsilon)$  on the torus:  $N$  is an open neighborhood of  $\mathbf{t}$  that is included in

$$\mathbb{T} \setminus (\tilde{\Gamma}_x \cup \tilde{\Gamma}_y \cup \bigcup_{q \neq p} \tilde{\Gamma}_q) .$$

120 From the second part of Prop. 3.1, we derive that, for any  $r \in \mathbb{Z}^2$ ,  $r \neq p$ , the function  $\tilde{\mathbf{1}}_r$  is constant on  $N$ . Moreover, since  $\Gamma$  is a Jordan curve, the ball  $B(v, \varepsilon)$  intersects both the interior and the exterior of the set  $S$ . Then,  $\tilde{\mathbf{1}}_p$  is not constant on  $N$ .  $\square$

125 Thus, if the set  $S$  is gradually translated and the corresponding translation vector on  $\mathbb{T}$  crosses once  $\Gamma$ , the corresponding digitization will change in one point. Let see what happens when the meridian or the outer equator is crossed.

Let  $\mathbf{i} = (1,0)$  and  $\mathbf{j} = (0,1)$ . Let  $p \in \mathbb{Z}^2$  and  $\mathbf{t} \in \tilde{\Gamma}_y \setminus \tilde{\Gamma}_p$ . Since  $f$  is continuous, there exists a neighborhood  $N$  of  $p + \langle \mathbf{t} \rangle$  on which  $f$  does not vanish (see Fig. 8). Then, Prop. 3.1 allows us to define  $\tilde{\mathbf{1}}_{p-\mathbf{i}}(\mathbf{t}^-)$  as the value of  $\tilde{\mathbf{1}}_{p-\mathbf{i}}$  on the connected component of

$$\mathbb{T} \setminus (\tilde{\Gamma}_{p-\mathbf{i}} \cup \tilde{\Gamma}_x \cup \tilde{\Gamma}_y)$$

which contains the projection on the torus of the set

$$N \cap (\{p - \mathbf{i}\} \oplus (0,1)^2) .$$

By the intermediate value theorem, the value of  $\tilde{\mathbf{1}}_{p-\mathbf{i}}(t^-)$  is also the value of  $\tilde{\mathbf{1}}_p(t)$ , which in turn is the value of  $\tilde{\mathbf{1}}_p$  on the connected component of

$$\mathbb{T} \setminus (\tilde{\Gamma}_p \cup \tilde{\Gamma}_x \cup \tilde{\Gamma}_y)$$

which contains the projection of the set

$$N \cap (\{p\} \oplus (0,1)^2) .$$

Alike, we define  $\tilde{\mathbf{1}}_p(\mathbf{t}^-)$  when  $\mathbf{t} \in \tilde{\Gamma}_x \setminus \tilde{\Gamma}_p$  and we state

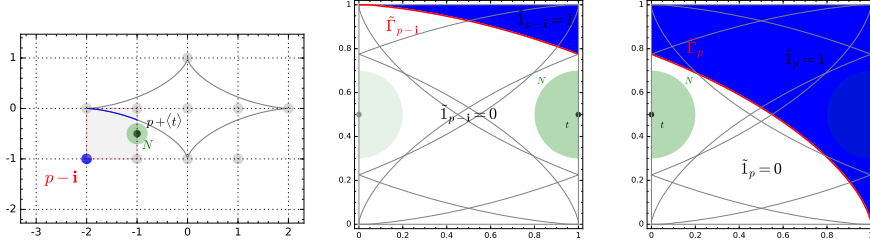


Figure 8: A point on the torus prime meridian is used to link two successive indicator functions.

**Lemma 3.2.** *Let  $p \in \mathbb{Z}^2$ .*

- For any  $\mathbf{t} \in \tilde{\Gamma}_x \setminus \tilde{\Gamma}_p$ , one has,  $\tilde{\mathbf{i}}_p(\mathbf{t}) = \tilde{\mathbf{i}}_{p-j}(\mathbf{t}^-)$ .
- 130 • For any  $\mathbf{t} \in \tilde{\Gamma}_y \setminus \tilde{\Gamma}_p$ , one has,  $\tilde{\mathbf{i}}_p(\mathbf{t}) = \tilde{\mathbf{i}}_{p-i}(\mathbf{t}^-)$ .

Lemma 3.2 expresses the fact than crossing the prime meridian, resp. the outer equator, on the torus results in a unit horizontal, resp. vertical shift, on the indicator functions (provided the cross does not occur on a boundary point).

In the next section, we use the previous results to propose two algorithms  
 135 for the generation of the digitization classes.

## 4. Algorithms

### 4.1. Pointwise determination of the dual

Starting from a set  $S$  and a point  $\mathbf{t} \in \mathbb{T}$ , it is obviously possible to translate the set  $S$  of  $\mathbf{t}$  and to compute the corresponding Gauss digitization with one of  
 140 the existing algorithms. The following algorithms show that it is possible to find any digitization by overlapping the grid squares  $p \oplus [0, 1)^2$ ,  $p \in \mathbb{Z}^2$ , containing the plot of the boundary of  $S$  provided each square is labeled by its reference point  $p$ .

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**Algorithm 1:** Computing a digitization class
 

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**Input:** The family of curves  $\tilde{\Gamma}_p$ ,  $p \in \mathbb{Z}^2$  and a point  $\mathbf{t} \in \mathbb{T}$ .

**Output:** The digitization class  $\tilde{\varphi}_S(\mathbf{t})$ .

**if**  $\mathbf{t} \in \tilde{\Gamma}$  **then**

$C \leftarrow \{p \in \mathbb{Z}^2 \mid \mathbf{t} \in \tilde{\Gamma}_p\};$

Replace  $\mathbf{t}$  by some point of a connected component of  $\mathbb{T} \setminus \tilde{\Gamma}$  whose boundary contains  $\mathbf{t}$ ;

Plot a loop  $\Delta$  with base-point  $\mathbf{t}$  on  $\mathbb{T}$  crossing once  $\tilde{\Gamma}_y$ , oriented like  $\tilde{\Gamma}_x$  which is not crossed and avoiding the extremities of the curves  $\tilde{\Gamma}_p$ ;

$A \leftarrow \{p \in \mathbb{Z}^2 \mid \Delta \text{ crosses an odd number of times } \tilde{\Gamma}_p \text{ after crossing } \tilde{\Gamma}_y\};$

$B \leftarrow \{p \in \mathbb{Z}^2 \mid \Delta \text{ crosses an odd number of times } \tilde{\Gamma}_p \text{ before crossing } \tilde{\Gamma}_y\};$

**for**  $j = -\infty$  **to**  $+\infty$  **do**

$b \leftarrow 0;$

**for**  $i = -\infty$  **to**  $+\infty$  **do**

**if**  $(i, j) \in A$  **then**  $\tilde{\mathbf{1}}_{(i,j)}(\mathbf{t}) \leftarrow 1 - b;$

**else**  $\tilde{\mathbf{1}}_{(i,j)}(\mathbf{t}) \leftarrow b;$

**if**  $(i, j) \in B$  **then**  $b \leftarrow 1 - \tilde{\mathbf{1}}_{(i,j)}(\mathbf{t});$

**else**  $b \leftarrow \tilde{\mathbf{1}}_{(i,j)}(\mathbf{t});$

**return**  $\tilde{\varphi}_S(\mathbf{t}) = [C \cup \{p \in \mathbb{Z}^2 \mid \tilde{\mathbf{1}}_p(\mathbf{t}) = 1\}];$

---

*Proof.* Recall that, for any  $\mathbf{t} \in \mathbb{T}$ ,  $\tilde{\varphi}_S(\mathbf{t})$  is a subset of  $\mathbb{Z}^2$ , that is a function from  $\mathbb{Z}^2$  to  $\{0, 1\}$  whose value in  $p \in \mathbb{Z}^2$  is  $\tilde{\mathbf{1}}_p(\mathbf{t})$ . Let  $\mathbf{t}_0$  be the intersection point of  $\Delta$  with the prime meridian. Let  $j \in \mathbb{Z}$ . The proof is made par induction on  $i$ . We set  $p = (i, j)$  and  $q = (i + 1, j)$ . When  $i$  is small enough, say  $i < -M - 1$ , it is plain that  $\tilde{\mathbf{1}}_p(t) = \tilde{\mathbf{1}}_p(t_0^-) = b = 0$ . Let us assume that for some  $i$ ,  $b = \tilde{\mathbf{1}}_p(t_0^-)$ . Then, by Lemma 3.2,  $\tilde{\mathbf{1}}_q(t_0) = b$ . If  $q \in A$ , from Lemma 3.1 we derive that  $\tilde{\mathbf{1}}_q(t) = 1 - b$ . Otherwise,  $\tilde{\mathbf{1}}_q(t) = b$ . This is the value of  $\tilde{\mathbf{1}}_q(t)$  calculated by the algorithm. Alike, if  $q \in B$ , Lemma 3.1 implies  $\tilde{\mathbf{1}}_q(t_0^-) = 1 - \tilde{\mathbf{1}}_q(t)$ . Otherwise,  $\tilde{\mathbf{1}}_q(t_0^-) = \tilde{\mathbf{1}}_q(t)$ . Then, in any case, the next value of  $b$  computed by the algorithm is equal to  $\tilde{\mathbf{1}}_q(t_0^-)$  which achieves the induction.  $\square$

In this algorithm, the value of  $b$  propagates until a point in  $A \cup B$  is encountered and this value codes for the membership of the points to the set. Then, when a point in  $A \cup B$  is reached, depending whether the point is in  $A \setminus B$ ,  $B \setminus A$  or  $A \cap B$ , the value of  $b$  is changed, or/and the membership rule is violated. Fig. 9 and Tab. 1 exemplifies Algo 1 on our set instance  $S_{\text{astro}}$ .

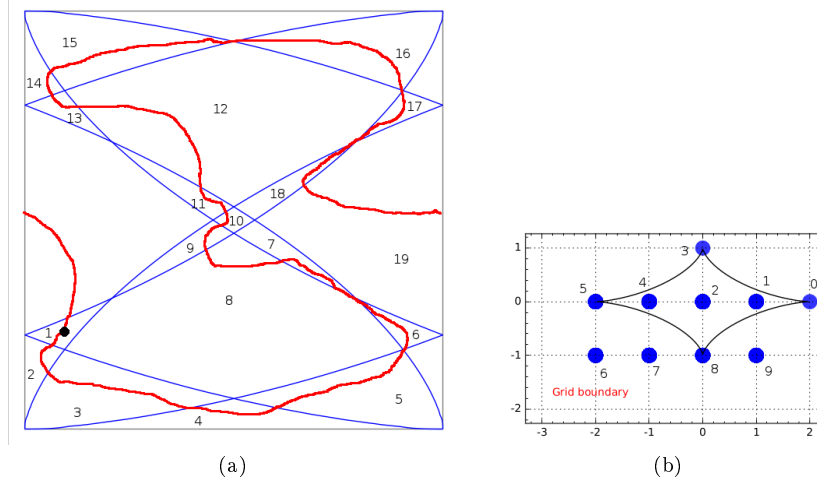


Figure 9: (a) A loop (in red) whose base-point is the black point in the region 1, runs through the connected components of  $\mathbb{T} \setminus \tilde{\Gamma}$ . The numbers on the figure label these components according to the traveling order. The ordered list of the curves  $\tilde{\Gamma}_p$  crossed by the red loop before, resp. after, crossing the prime meridian, labeled according to the dictionary shown in (b), is  $[1, 8, 5, 1, [5, 7], [2, 7], 2, 4, 2, 8, 7, 2, 9, 2, [6, 9], [4, 6], [4, 8], 4]$ , resp.  $[4]$ . Then, in Algorithm 1,  $A = \{4\}$  and  $B = \{2, 7, 8\}$ . Table 1 is the trace of the execution of Algo 1.

$j$	$i$	label	$A$	$B$	$b_{\text{in}}$	$\tilde{\mathbf{1}}_{(i,j)}$	$b_{\text{out}}$
-1	-2	6			0	0	0
	-1	7		x	0	0	1
	0	8		x	1	1	0
-1	1	9			0	0	0
	-2	5			0	0	0
	-1	4	x		0	1	1
	0	2		x	1	1	0
	1	1			0	0	0
0	2	0			0	0	0
1	0	3			0	0	0

Table 1: Values of the variables used in Algo 1 during the iteration process (if  $(i, j) \notin \mathcal{B}$ ,  $b = \tilde{\mathbf{1}}_{(i,j)} = 0$ ).

#### 4.2. Global determination of the dual

160 Thanks to Lemma 3.1, we easily derive from Algo 1 a propagation algorithm that provides any digitization encountered when performing the torus loop.

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#### Algorithm 2: Digitization propagation

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**Input:**  $\tilde{\varphi}_S(\mathbf{t})$ ,  $L_\Gamma$ ,  $L_{CC}$ ,  
 where  $\tilde{\varphi}_S(\mathbf{t})$  is the digitization class return by Algorithm 1 (under the form of a boolean function over  $\mathcal{B}$ ),  
 $L_\Gamma$  is the ordered list of the curves crossed by the loop in Algorithm 1, before crossing the prime meridian for the indices less than  $N$  and after crossing the prime meridian for the indices greater than, or equal to  $N$ ,  
 $L_{CC}$  is the list of the connected components of  $\mathbb{T} \setminus \tilde{\Gamma}$  crossed by the loop such that  $L_\Gamma[i]$  contains the curve(s)  $\tilde{\Gamma}_p$  crossed to go from  $L_{CC}[i]$  to  $L_{CC}[i+1]$ .  
**Output:** The list  $L_{\mathfrak{D}}$  of the  $L_{CC}$  element digitization classes.

$L_{\mathfrak{D}}[0] \leftarrow \tilde{\varphi}_S(\mathbf{t});$   
**for**  $i = 1$  **to**  $N$  **do**  
 |  $L_{\mathfrak{D}}[i] \leftarrow L_{\mathfrak{D}}[i-1];$   
 |   **foreach**  $p \in L_\Gamma[i-1]$  **do**  $L_{\mathfrak{D}}[i](p) \leftarrow \neg L_{\mathfrak{D}}[i](p);$   
 $L_{\mathfrak{D}}[\text{length}(L_\Gamma)] \leftarrow \tilde{\varphi}_S(\mathbf{t});$   
**for**  $i = \text{length}(L_\Gamma) - 1$  **to**  $N$  **do**  
 |  $L_{\mathfrak{D}}[i] \leftarrow L_{\mathfrak{D}}[i+1];$   
 |   **foreach**  $p \in L_\Gamma[i]$  **do**  $L_{\mathfrak{D}}[i](p) \leftarrow \neg L_{\mathfrak{D}}[i](p);$

---

Table 2 shows the execution of Algo 2 on  $S_{\text{astro}}$ .

$i$	$L_{CC}[i]$	$L_{\Gamma}[i-1]$	0	1	2	3	4	5	6	7	8	9
0	1		0	0	1	0	1	0	0	0	1	0
1	2	1	0	1	1	0	1	0	0	0	1	0
2	3	8	0	1	1	0	1	0	0	0	0	0
3	4	5	0	1	1	0	1	1	0	0	0	0
4	5	1	0	0	1	0	1	1	0	0	0	0
5	6	5, 7	0	0	1	0	1	0	0	1	0	0
6	7	2, 7	0	0	0	0	1	0	0	0	0	0
7	8	2	0	0	1	0	1	0	0	0	0	0
8	9	4	0	0	1	0	0	0	0	0	0	0
9	10	2	0	0	0	0	0	0	0	0	0	0
10	11	8	0	0	0	0	0	0	0	0	1	0
11	12	7	0	0	0	0	0	0	0	1	1	0
12	13	2	0	0	1	0	0	0	0	1	1	0
13	14	9	0	0	1	0	0	0	0	1	1	1
14	15	2	0	0	0	0	0	0	0	1	1	1
15	16	6, 9	0	0	0	0	0	0	1	1	1	0
16	17	4, 6	0	0	0	0	1	0	0	1	1	0
17	18	4, 8	0	0	0	0	0	0	0	1	0	0
18	19	4	0	0	0	0	1	0	0	1	0	0
$i$	$L_{CC}[i]$	$L_{\Gamma}[i]$										
19	1		0	0	1	0	1	0	0	0	1	0
18	19	4	0	0	1	0	0	0	0	0	1	0

Table 2: Execution of Algorithm 2 on the output of Algorithm 1: the digitizations associated to the 19 visible regions of the dual of the set  $S_{\text{astro}}$  are computed by propagation. The numbers in Columns 2 (regions) and 3 (points in  $\mathcal{B}$  - also in the first line of Column 3) refer to the labeling of regions and points in Fig. 9. Observe that we obtain two distinct digitizations for the region 19 ( $L_{CC}[18]$ ) but they are in the same class of digitizations composed by two points vertically aligned (see Fig. 2).



## 5. Conclusion

We present in this paper a theoretical study on the digitizations obtained from a planar set under translation. The dual linking a translation vector class to a digitization class is defined. The dual is a piecewise constant function and the link between the dual constant region boundaries and the set frontier is established. Two other properties allow us to give two proved algorithms for the digitization class generation. The dual can be used as a visualization tool for the digitization variability under the action of the translation group and it allows an estimation of the weights of the different digitizations.

Future works on the digitization dual include its extension to the (non-trivial) action of the rotation group and its application in the study of the digitization combinatorial properties.

## References

- [BM16] Etienne BAUDRIER et Loïc MAZO : Curve digitization variability. *In DGC I Proceedings*, volume 9647 de *Lecture Notes in Computer Science*, pages 59–70, 2016.
- [DS84] Leo DORST et Arnold W M SMEULDERS : Discrete Representation of Straight Lines. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-6(4):450–463, jul 1984.
- [HZ06] Martin N. HUXLEY et Jovisa ZUNIC : Different Digitisations of Displaced Discs. *Foundations of Computational Mathematics*, 6(2):255–268, apr 2006.
- [HZ07] Martin N. HUXLEY et Jovisa ZUNIC : The number of N-point digital discs. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(1):159–161, 2007.
- [HZ16] M N HUXLEY et Jovisa ZUNIC : The Number of Different Digital N-Discs. *Journal of Mathematical Imaging and Vision*, 56(3):403–408, 2016.
- [Ken48] D.G. KENDALL : On the number of lattice points inside a random oval. *The Quarterly Journal of Mathematics*, 19:1–26, 1948.
- [Nag05] Benedek NAGY : An algorithm to find the number of the digitizations of discs with a fixed radius. *Electronic Notes in Discrete Mathematics*, 20:607–622, 2005.
- [SK05] Peer STELLDINGER et Ullrich KÖTHE : Towards a general sampling theory for shape preservation. *Image Vision Comput.*, 23(2):237–248, 2005.
- [TR02] Mohamed TAJINE et Christian RONSE : Topological properties of hausdorff discretization, and comparison to other discretization schemes. *Theoretical Computer Science*, 283(1):243 – 268, 2002.