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Multisymplectic variational integrators and space/time symplecticity

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Multisymplectic variational integrators are structure-preserving numerical schemes especially designed for PDEs derived from covariant spacetime Hamilton principles. The goal of this paper is to study the properties of the temporal and spatial discrete evolution maps obtained from a multisymplectic numerical scheme. Our study focuses on a $(1+1)$-dimensional spacetime discretized by triangles, but our approach carries over naturally to more general cases. In the case of Lie group symmetries, we explore the links between the discrete Noether theorems associated to the multisymplectic spacetime discretization and to the temporal and spatial discrete evolution maps, and emphasize the role of boundary conditions. We also consider in detail the case of multisymplectic integrators on Lie groups. Our results are illustrated with the numerical example of a geometrically exact beam model.

Keywords: Multisymplectic structure; discrete mechanics; variational integrator; Lie group symmetry; discrete momentum map; discrete global Noether theorem.

1. Introduction

Multisymplectic variational integrators are structure-preserving numerical schemes designed for solving PDEs arising from covariant Euler–Lagrange equations. These

schemes are derived from a discrete version of the covariant Hamilton principle of field theory and preserve, at the discrete level, the associated multisymplectic geometry.

Multisymplectic variational integrators can be seen as the spacetime generalization of the well-known variational integrators for classical mechanics (see [22]). Recall that the discrete Lagrangian flow, obtained through a classical variational integrator, preserves a symplectic form. From this property, it follows, by backward error analysis, that the energy is approximately preserved. For multisymplectic integrators, however, the situation is much more involved, the analogue of the symplectic property being given by a discrete version of the multisymplectic formula derived in [19], that will be recalled in the paper. This formula is the spacetime analogue of the symplectic property of the discrete flow associated to variational integrators in time. The continuous multisymplectic form formula is a property of the solution of the covariant Euler–Lagrange equations in field theory, see [11] to which we refer for the multisymplectic geometry of classical field theory. In particular, several important concepts, such as covariant momentum maps associated to symmetries and the covariant Noether theorem, are naturally formulated in terms of multisymplectic forms.

In the continuous case, the articles [21, 20, 8, 26, 6] are examples of papers in which multisymplectic geometry has been further developed and applied in the context of continuum mechanics. We refer to [3, 2, 7, 9] for the development and the use of the techniques of reduction by symmetries for covariant field theory.

The discrete multisymplectic geometry of field theory was first developed in [19]. In that paper, the discrete Cartan forms, the discrete covariant momentum map, and the discrete covariant Noether theorems are introduced and the discrete multisymplectic form formula is established. This work was further developed in [16] to treat more general spacetime discretizations. This allowed the development of asynchronous variational integrators which permit the selection of independent time steps in each element, while exactly preserving the discrete Noether conservation and the multisymplectic structure, and offering the possibility of imposing discrete energy conservation.

This paper is the first in a series devoted to the formulation of multisymplectic algorithms for the displacement in spacetime of a continuous medium. Many outstanding problems are not treated here and are the subject of future work. The main goal of the present paper is to study and exploit the *symplectic* properties satisfied by the solutions of a multisymplectic variational integrator. Note, however, that the solution of a discrete multisymplectic scheme is a discrete spacetime section and not a discrete curve. Assuming that the spacetime is $(1+1)$-dimensional, the discrete spacetime section can be organized either as a discrete *time-evolutionary flow* or a discrete *space-evolutionary flow*. More precisely, given a discrete field that is a solution of the multisymplectic scheme, we can first construct from it a vector of discrete positions of all spacetime nodes at a given time, and then consider the sequence of these vectors indexed by the discrete time. Conversely, the discrete

field can be organized in a space-evolutionary fashion, by first forming a vector of the discrete positions of all spacetime nodes at a given space index, and then considering the sequence of these vectors.

In order to study the symplectic character of these time-evolutionary and space-evolutionary discrete flows, we construct, from the discrete covariant Lagrangian, two discrete Lagrangians (L_d and N_d) associated to the temporal and spatial evolution, respectively. From this point of view, it follows naturally that a multisymplectic integrator gives rise to a variational integrator in time and a variational integrator in space. This also allows us to relate the discrete multisymplectic forms with the discrete symplectic forms associated to the time and space evolutionary descriptions. The type of boundary conditions imposed on the discrete spacetime domain plays a crucial role and we shall consider several types of boundary conditions. In the unrealistic situation of a spacetime without boundary, such a study would be essentially trivial.

Let us recall that, in the continuous setting, when the configuration field is not prescribed at the boundary, the variational principle yields natural boundary conditions, such as zero-traction boundary conditions. These conditions will be discretized in a structure-preserving way, by means of the discrete covariant variational principle.

A main property of variational integrators (in both the multisymplectic and the symplectic cases) is that they permit a consistent definition of discrete momentum maps and a discrete version of Noether's conservation theorem in the presence of Lie group symmetries. Our goal in this direction is to study and relate the discrete covariant Noether theorem associated to the discrete multisymplectic formulation with the discrete Noether theorems associated to the time-evolutionary and space-evolutionary discrete flows built from the discrete field. Here again, this study highly depends on the type of boundary conditions involved.

Another important goal of the paper is the derivation of multisymplectic variational integrators on Lie groups. These schemes adapt to the covariant spacetime situation the variational integrators on Lie groups developed in [1, 15] which are based on the Lie group methods of [14]. This approach involves the choice of a retraction map to consistently encode, in the Lie algebra, the discrete displacement made on the Lie group.

The theory developed here, especially the space-evolutionary point of view, is illustrated with the case of a geometrically exact beam. Geometrically exact models, developed in [24, 25], are formulated as SE(3)-valued covariant field theories in [6, §6, §7]. Using this covariant formulation, we derive a multisymplectic variational integrator for geometrically exact models. As explained later, this approach further develops the variational Lie group integrators presented in [5] for geometrically exact beams. The numerical tests presented in this paper serve only as a simple illustration of our methods. Considerably more realistic situations are presented in [4], a paper dedicated to the implementation of several simulations using the theory developed in this paper.

Plan of the paper. We begin by reviewing below some basic facts on discrete Lagrangian mechanics, following [22]. In Sec. 2, we give a quick account of the geometry of the covariant Euler–Lagrange (CEL) equations and the multisymplectic form formula. We also consider the special case when the fields are Lie group valued and present the trivialized CEL equations. In Sec. 3, we first present the main facts about multisymplectic variational integrators with spacetime discretized using triangles. In particular, we write the discrete multisymplectic form formula, the discrete covariant momentum maps, and the discrete covariant Noether theorem. We also derive the discrete zero traction and zero momentum boundary conditions via the discrete covariant variational principle. Then we describe systematically the symplectic properties of the time-evolutionary and space-evolutionary discrete flows built from the discrete field solution of the discrete covariant Euler–Lagrange (DCEL) equations. Several cases of boundary conditions are considered. We also study the link between the covariant discrete Noether theorem and the discrete Noether theorems associated to the time-evolutionary and space-evolutionary discrete flows. We will see that, while the covariant discrete Noether theorem holds independently of the imposed boundary conditions, this is not the case for the discrete Noether theorems associated to the time or space discrete evolutions. Section 4 is devoted to the particular situation when the configuration field takes values in a Lie group. In this case, it is possible to trivialize the DCEL. This is a serious advantage in the discrete setting, since it allows us to make use of the vector space structure of the Lie algebra via the use of a time difference map. Finally, in Sec. 5, we illustrate the properties of the multisymplectic variational integrator with the example of the geometrically exact beam model. The symplectic property of the space-evolutionary discrete flow is exploited to reconstruct the trajectory of the beam, knowing the evolution of the position and of the strain of one of its extremities.

Review of discrete Lagrangian dynamics. Let Q be the configuration manifold of a mechanical system. Suppose that the dynamics of this system is described by the Euler–Lagrange (EL) equations associated to a Lagrangian $\mathsf{L}\colon TQ \to \mathbb{R}$ defined on the tangent bundle TQ of the configuration manifold Q. Recall that these equations characterize the critical curves of the action functional associated to L, namely

$$\frac{d}{dt}\frac{\partial \mathsf{L}}{\partial \dot{q}} - \frac{\partial \mathsf{L}}{\partial q} = 0 \Leftrightarrow \delta \int_0^T \mathsf{L}(t, q(t), \dot{q}(t))\,\mathrm{d}t = 0,$$

for variations $\delta q(t) \in T_{q(t)}Q$ of the curve $q(t)$ vanishing at the endpoints, i.e. $\delta q(0) = \delta q(T) = 0$. The Legendre transform associated to L is the locally trivial fiber (*not* vector) bundle morphism $\mathbb{F}\mathsf{L}\colon TQ \to T^*Q$ covering the identity that associates to a velocity its corresponding conjugate momentum, where T^*Q denotes the cotangent bundle of Q. In canonical tangent and cotangent bundle charts induced by an atlas on Q, it has the expression $(q, \dot{q}) \mapsto (q, \frac{\partial \mathsf{L}}{\partial \dot{q}})$.

We recall the discrete version of this approach, following [22]. Suppose that a time step h has been fixed, denote by $\{t^j = jh \mid j = 0, \ldots, N\}$ the sequence of times discretizing $[0, T]$, and by $q_d : \{t^j\}_{j=0}^N \to Q$, $q^j := q_d(t^j)$ the corresponding discrete curve. Let $\mathsf{L}_d : Q \times Q \to \mathbb{R}, \mathsf{L}_d = \mathsf{L}_d(q^j, q^{j+1})$, be a discrete Lagrangian which we think of as approximating the action integral of L along the curve segment between q^j and q^{j+1}, that is, we have

$$\mathsf{L}_d(q^j, q^{j+1}) \approx \int_{t^j}^{t^{j+1}} \mathsf{L}(q(t), \dot{q}(t))\, \mathrm{d}t = 0,$$

where $q(t^j) = q^j$ and $q(t^{j+1}) = q^{j+1}$. The *discrete Euler–Lagrange* (*DEL*) *equations* are obtained by applying the discrete Hamilton principle to the discrete action

$$\mathfrak{S}_d(q_d) := \sum_{j=0}^{N-1} \mathsf{L}_d(q^j, q^{j+1})$$

for variations δq^j vanishing at the endpoints. We have the formula

$$\begin{aligned}\delta \mathfrak{S}_d(q_d) = &\sum_{j=1}^{N-1} (D_1 \mathsf{L}_d(q^j, q^{j+1}) + D_2 \mathsf{L}_d(q^{j-1}, q^j)) \delta q^j \\ &+ \Theta^+_{\mathsf{L}_d}(q^{N-1}, q^N)(\delta q^{N-1}, \delta q^N) - \Theta^-_{\mathsf{L}_d}(q^0, q^1)(\delta q^0, \delta q^1),\end{aligned}$$

where

$$\Theta^-_{\mathsf{L}_d}(q^0, q^1) := -D_1 \mathsf{L}_d(q^0, q^1) \mathbf{d}q^0 \quad \text{and} \quad \Theta^+_{\mathsf{L}_d}(q^0, q^1) := D_2 \mathsf{L}_d(q^0, q^1) \mathbf{d}q^1$$

are the discrete Lagrangian one-forms and D_1, D_2 denote the first and second partial derivatives of a function on the manifold $Q \times Q$. The DEL equations are thus given by

$$D_1 \mathsf{L}_d(q^j, q^{j+1}) + D_2 \mathsf{L}_d(q^{j-1}, q^j) = 0, \quad \text{for all } j = 1, \ldots, N-1.$$

The *discrete Legendre transforms* associated to L_d are the two maps $\mathbb{F}^+\mathsf{L}_d, \mathbb{F}^-\mathsf{L}_d : Q \times Q \to T^*Q$ defined by

$$\begin{aligned}\mathbb{F}^+\mathsf{L}_d(q^j, q^{j+1}) &:= D_2 \mathsf{L}_d(q^j, q^{j+1}) \in T^*_{q^{j+1}} Q, \\ \mathbb{F}^-\mathsf{L}_d(q^j, q^{j+1}) &:= -D_1 \mathsf{L}_d(q^j, q^{j+1}) \in T^*_{q^j} Q.\end{aligned} \tag{1.1}$$

Note that the DEL equations can be equivalently written as

$$\mathbb{F}^+\mathsf{L}_d(q^{j-1}, q^j) = \mathbb{F}^-\mathsf{L}_d(q^j, q^{j+1}), \quad \text{for } j = 1, \ldots, N-1 \tag{1.2}$$

and that we have $\Theta^\pm_{\mathsf{L}_d} = (\mathbb{F}^\pm \mathsf{L}_d)^* \Theta_{\mathrm{can}}$, where Θ_{can} is the canonical one-form on T^*Q, defined by $\Theta_{\mathrm{can}}(\alpha_q)(U_{\alpha_q}) := \alpha_q(T_{\alpha_q}\rho(U_{\alpha_q}))$, where $\alpha_q \in T^*_q Q, U_{\alpha_q} \in T_{\alpha_q}(T^*Q)$, and $\rho : T^*Q \to Q$ is the cotangent bundle projection.

Approximate energy conservation. The main feature of the numerical scheme $(q^{j-1}, q^j) \mapsto (q^j, q^{j+1})$, obtained by solving the DEL equations, is that the

associated scheme $(q^j, p^j) \mapsto (q^{j+1}, p^{j+1})$ induced on the phase space T^*Q through the discrete Legendre transform, defines a *symplectic integrator*. Here we assumed that the discrete Lagrangian L_d is regular, that is, both discrete Legendre transforms $\mathbb{F}^+\mathsf{L}_d, \mathbb{F}^-\mathsf{L}_d : Q \times Q \to T^*Q$ are local diffeomorphisms (for nearby q^j and q^{j+1}). The symplectic character of the integrator is obtained by showing that the scheme $(q^{j-1}, q^j) \mapsto (q^j, q^{j+1})$ preserves the discrete symplectic two-forms $\Omega^{\pm}_{\mathsf{L}_d} := (\mathbb{F}^{\pm}\mathsf{L}_d)^*\Omega_{\mathrm{can}}$, so that $(q^j, p^j) \mapsto (q^{j+1}, p^{j+1})$ preserves Ω_{can} and is therefore symplectic; see [22, 17]. Here $\Omega_{\mathrm{can}} := -\mathbf{d}\Theta_{\mathrm{can}}$ is the canonical symplectic two-form on T^*Q; in standard cotangent bundle coordinates it has the expression $\Omega_{\mathrm{can}} = \mathbf{d}q \wedge \mathbf{d}p$.

It is known (see [12]) that, given a Hamiltonian H, a symplectic integrator for H corresponds to solving a modified Hamiltonian system for a Hamiltonian $\bar{H}$ which is close to H. This explains why energy is approximately conserved for variational integrators and typically oscillates about the true energy value. We refer to [12] for a detailed account and a full treatment of backward error analysis for symplectic integrators.

2. Covariant Lagrangian Formulation

In this section, we recall basic facts about the geometry of covariant field theory such as the covariant Euler–Lagrange (CEL) equations, the Cartan forms, covariant momentum maps, and the multisymplectic form formula, following [11, 19]. We also consider the case when the fields are Lie group valued and present the trivialized CEL equations. Although the corresponding discrete multisymplectic integrators will be considered only on trivial fiber bundles $Y := X \times M \to X$, we often start with the general theory written on arbitrary fiber bundles, because it yields the correct guide to write geometrically consistent formulas, both at the continuous and discrete levels.

2.1. *Preliminaries on covariant Lagrangian formulation*

In continuum mechanics, the configuration space Q is usually a space of maps $\varphi : B \to M$ (such as embeddings) defined on a manifold B, the reference configuration space, with values in the space M of allowed deformations. Therefore, Q is an infinite-dimensional manifold whose tangent space at a point φ is $\{\dot{\varphi} : B \to TM \mid \dot{\varphi}(s) \in T_{\varphi(s)}M\}$. For example, for the geometrically exact beam, treated in Sec. 5, we have $\varphi : B = [0, L] \to M = \mathrm{SE}(3)$, where $\mathrm{SE}(3)$ is the special Euclidean group consisting of orientation-preserving rotations and translations; thus, $Q = \mathcal{F}([0, L], \mathrm{SE}(3))$ is the space of all such maps.

In many situations, the Lagrangian $\mathsf{L} : TQ \to \mathbb{R}$ of the system can be written in terms of a Lagrangian density $\mathcal{L}$ as follows

$$\mathsf{L}(\varphi, \dot{\varphi}) = \int_B \mathcal{L}(s, \varphi(s), \dot{\varphi}(s), \nabla\varphi(s)), \tag{2.1}$$

where $\nabla\varphi(s)$ denotes the derivative of φ with respect to the variable s. In other words, Lagrangians of the form (2.1) depend not only on the configuration φ but also on their first spatial derivatives.

If the Lagrangian L is defined in terms of a Lagrangian density $\mathcal{L}$ as in (2.1), one can alternatively formulate the dynamics and all its properties in terms of the Lagrangian density $\mathcal{L}$ instead of the Lagrangian L. This is the covariant, or field theoretic, point of view. In this description, instead of studying the motion $\mathbb{R} \ni t \mapsto \varphi(t) \in Q := \mathcal{F}(B, M)$, one investigates the spacetime-dependent map $\mathbb{R} \times B \ni (t, s) \mapsto \varphi(t, s) \in M$.

Abstractly, the maps φ have to be interpreted as sections of the trivial fiber bundle $\pi : Y := X \times M \to X, \pi(t, s, m) = (t, s)$, where $X := \mathbb{R} \times B$ is the base and M is the fiber. The Lagrangian density is defined on the first jet bundle $J^1Y = J^1(X \times M) \to X \times M$ of the fiber bundle $Y = X \times M \to X$ and takes values in the space $\Lambda^{n+1}X$ of $(n + 1)$-forms on $X = \mathbb{R} \times B$, where $n = \dim B$. The fiber of the first jet bundle at $y \in Y_x := \{x\} \times M$ is

$$J^1_yY = \{\gamma_y \in L(T_xX, T_yY) \,|\, T_y\pi \circ \gamma_y = \mathrm{id}_{T_xX}\}, \tag{2.2}$$

where T_xX is the tangent space of X at $x \in X, T_yY$ is the tangent space of Y at $y \in Y, T_y\pi : T_yY \to T_xX$ is the tangent map (derivative) at y of the bundle projection $\pi : Y \ni (x, m) \mapsto x \in X$, and $L(T_xX, T_yY)$ denotes the vector space of linear maps from T_xX to T_yY. We note that $\dim(J^1Y) = (n + 1) + m + (n + 1)m$, with $\dim M = m$. The first jet extension of a section φ is $j^1_x\varphi := T_x\varphi \in J^1_{\varphi(x)}Y$, so that the action functional on the interval $[0, T]$ associated to $\mathcal{L}$ can be simply written as $\int_0^T \int_B \mathcal{L}(j^1\varphi)$.

Note that a Lagrangian density defined on J^1Y may depend explicitly on time. In (2.1), however, we assumed that there is no such dependence, since this is the situation encountered in most examples in continuum mechanics. An explicit time dependence in $\mathcal{L}$, as given in (2.1), would induce an explicit time dependence in the Lagrangian L.

Since the fiber bundle $\pi : X \times M \to X$ is trivial, the first jet bundle can be identified with the bundle $L(TX, TM) \to X \times M$ over $X \times M$, whose fiber at $(x, m) \in X \times M$ is given by linear maps $L(T_xX, T_mM)$. The first jet extension $j^1\varphi(x)$ at $x = (t, s)$ is identified with the linear map $(t, s, \varphi(t, s), \dot\varphi(t, s), \nabla\varphi(t, s)) : T_{(t,s)}(\mathbb{R} \times B) \to T_{\varphi(t,s)}M$ in the following manner: if $\lambda \in \mathbb{R}, u_s \in T_sB$, so that $(\lambda, u_s) \in T_{(t,s)}X = T_{(t,s)}(\mathbb{R} \times B)$, then

$$(t, s, \varphi(t, s), \dot\varphi(t, s), \nabla\varphi(t, s)) \cdot (\lambda, u_s) := \lambda\dot\varphi(t, s) + \nabla\varphi(t, s)(u_s) \in T_{\varphi(t,s)}M.$$

Notation. We denote by $s^i, i = 1, \ldots, n = \dim B$, the coordinates on B, by $(x^0, x^1, \ldots, x^n) := (t, s^1, \ldots, s^n)$ the coordinates on $X = \mathbb{R} \times B$, and by $y^A, A = 1, \ldots, m = \dim M$, the coordinates on M. We use the notation $\mathrm{d}^{n+1}x := \mathrm{d}t \wedge \mathrm{d}^n s = \mathrm{d}t \wedge \mathrm{d}s^1 \wedge \cdots \wedge \mathrm{d}s^n$. We write locally the Lagrangian density as $\mathcal{L}(t, s^i, y^A, v^A_0, v^A_i) = L(t, s^i, y^A, v^A_0, v^A_i)\, \mathrm{d}t \wedge \mathrm{d}^n s$.

Covariant Euler–Lagrange equations. In this context, Hamilton's principle states that

$$\delta \int_0^T \int_B \mathcal{L}(j^1\varphi(t,s)) = \delta \int_0^T \int_B L(t,s,\varphi(t,s),\dot{\varphi}(t,s),\nabla\varphi(t,s))\,\mathrm{d}t \wedge \mathrm{d}^n s = 0,$$

for variations $\delta\varphi$ with $\delta\varphi|_{\partial([0,T]\times B)} = 0$, where $\partial([0,T]\times B) = (\{0\}\times B)\cup(\{T\}\times B)\cup([0,T]\times\partial B)$.

This principle yields the CEL equations, locally given by

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\varphi}^A} + \partial_i \frac{\partial L}{\partial \varphi^A_{,i}} - \frac{\partial L}{\partial \varphi^A} = 0. \tag{2.3}$$

For completeness, we present the derivation of these equations for $B \subset \mathbb{R}^n$ an open subset with compact closure and smooth boundary. We have

$$\begin{aligned}\delta \int_0^T \int_B \mathcal{L}(j^1\varphi(t,s)) &= \int_0^T \int_B \left(\frac{\partial L}{\partial \varphi}\cdot\delta\varphi + \frac{\partial L}{\partial \dot{\varphi}}\cdot\delta\dot{\varphi} + \frac{\partial L}{\partial \nabla\varphi}\cdot\delta\nabla\varphi\right)\mathrm{d}t\wedge\mathrm{d}^n s \\ &= \int_0^T \int_B \left(\left[\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\varphi}} - \operatorname{div}\frac{\partial L}{\partial \nabla\varphi}\right]\delta\varphi\right)\mathrm{d}t\wedge\mathrm{d}^n s \\ &\quad + \int_B \left[\frac{\partial L}{\partial \dot{\varphi}}\cdot\delta\varphi\right]_0^T \mathrm{d}^n s + \int_0^T \int_{\partial B} \frac{\partial L}{\partial \varphi^A_{,i}}\mathbf{n}^i \delta\varphi^A\,\mathrm{d}^{n-1}a\wedge\mathrm{d}t,\end{aligned} \tag{2.4}$$

where $\mathbf{n}$ is the outward pointing unit normal to the boundary ∂B and $\mathrm{d}^{n-1}a$ is the volume form induced on ∂B. Since $\delta\varphi|_{\partial([0,T]\times B)} = 0$ and $\partial([0,T]\times B) = ([0,T]\times\partial B)\cup(\{0\}\times B)\cup(\{T\}\times B)$, the boundary terms vanish, thus yielding the CEL equations.

Remark 2.1 (Boundary conditions). In the above situation, it is assumed that the configuration φ is known at $t = 0, T$ and is prescribed at the boundary for all times, which corresponds to *pure displacement boundary conditions*. If the configuration at the boundary is not prescribed, then Hamilton's principle yields the boundary condition

$$\frac{\partial \mathcal{L}}{\partial \varphi^A_{,i}}\mathbf{n}^i = 0, \quad \text{for all } A = 1,\dots,m, \tag{2.5}$$

known as *zero-traction boundary condition*. Note that the treatment of nonzero traction $\frac{\partial \mathcal{L}}{\partial \varphi^A_{,i}}\mathbf{n}^i = \boldsymbol{\tau}_A$ requires the addition of a term in the Lagrangian; see, e.g., [18].

Other conditions could be used in the variational principle, such as the assumption of pure displacement boundary conditions but without the assumption that the configuration is known at $t = 0, T$. In this case, the variational principle would yield the conditions

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^A}(0,s) = 0 = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^A}(T,s), \quad \text{for all } s \in B. \tag{2.6}$$

known as *zero momentum boundary conditions*.

Covariant Euler–Lagrange operator. For future use, we recall here an intrinsic way of writing the CEL equations. Let VY be the vertical vector subbundle of TY whose fibers are defined by

$$V_yY := \{v_y \in T_yY \mid T_y\pi(v_y) = 0\}. \tag{2.7}$$

Let V^*Y be its dual vector bundle. In the case of a trivial bundle $Y = X \times M$, the fiber $V_yY, y = (x, m)$, is identified with the tangent space T_mM.

There is a unique bundle morphism $\mathscr{E}\mathscr{L}(\mathfrak{L}) : J^1Y \to V^*Y \otimes \Lambda^{n+1}X$ covering the identity on Y, called the *covariant Euler–Lagrange operator*, such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_X \mathfrak{L}(j^1\varphi_\varepsilon(x)) = \int_X \mathscr{E}\mathscr{L}(\mathfrak{L})(j^1\varphi(x)) \cdot \delta\varphi(x), \tag{2.8}$$

for all variations φ_ε of φ, among sections of $\pi : X \times M \to X$ satisfying $\delta\varphi|_{\partial X} = 0$, where $\delta\varphi := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\varphi_\varepsilon$.

The CEL operator recovers locally the expression of the CEL equations (2.3).

2.2. *Multisymplectic forms and covariant momentum maps*

The goal of this subsection is to provide a quick review of the multisymplectic form formula. This formula is of central importance, since it generalizes to the covariant case the symplectic property of the flow of the EL equations in classical mechanics. This formula has a discrete analogue that characterizes multisymplectic integrators (see [19]). The formula is more easily formulated by staying on an arbitrary fiber bundle and using the geometry of jet bundles rather than focusing on the case of trivial bundles.

Dual jet bundles. On the Hamiltonian side, the covariant analogue of the phase space of classical mechanics is given by the dual jet bundle $J^1Y^\star \to Y$. Abstractly, the fiber of the dual jet bundle at $y \in Y_x$ consists of affine maps from J^1Y_y to $\Lambda_x^{n+1}X$, i.e.

$$J^1Y_y^\star := \mathrm{Aff}(J^1Y_y, \Lambda_x^{n+1}X).$$

The momentum bundle is, by definition, the vector bundle $\Pi \to Y$, whose fiber at y is $\Pi_y := L(L(T_xX, V_yY), \Lambda_x^{n+1}X)$. There is a line bundle $\mu : J^1Y^\star \to \Pi$ locally given by $\mu(x^\mu, y^A, p^\mu_A, p) = (x^\mu, y^A, p^\mu_A)$.

In our case, since the bundle $Y \to X$ is trivial, the dual jet bundle can be identified with the vector bundle $T^*M \otimes TX \otimes \Lambda^{n+1}X \times_{X\times M} \Lambda^{n+1}X$ over $X \times M$. Coordinates on the dual jet bundle are denoted $(t, s^i, y^A, p^0_A, p^i_A, p)$ and correspond to the affine map

$$(v_0^A, v_i^A) \mapsto (p + p^0_A v_0^A + p^i_A v_i^A)\,\mathrm{d}t \wedge \mathrm{d}^n s.$$

Similarly, the momentum bundle can be identified with the vector bundle $T^*M \otimes TX \otimes \Lambda^{n+1}X$ over $X \times M$.

The Legendre transforms. Given a Lagrangian density $\mathcal{L}: J^1Y \to \Lambda^{n+1}X$, the associated covariant Legendre transform is the fiber-preserving map $\mathbb{F}\mathcal{L}: J^1Y \to J^1Y^\star$, given locally by

$$p^0_A = \frac{\partial L}{\partial v^A_0}, \quad p^i_A = \frac{\partial L}{\partial v^A_i}, \quad p = L - \frac{\partial L}{\partial v^A_0} v^A_0 - \frac{\partial L}{\partial v^A_i} v^A_i.$$

In the case of a trivial bundle $Y = X \times M, X = \mathbb{R} \times B$, and in terms of a given field $\varphi: X \to M$, we can write

$$\mathbb{F}\mathcal{L}(\dot\varphi, \nabla\varphi) = \left(\frac{\partial \mathcal{L}}{\partial \dot\varphi}, \frac{\partial \mathcal{L}}{\partial \nabla\varphi}, \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot\varphi} \cdot \dot\varphi - \frac{\partial \mathcal{L}}{\partial \nabla\varphi} : \nabla\varphi \right),$$

where $\cdot$ and $:$ denote contractions (on one, respectively, two indices). Note that since $\dim(J^1Y^\star) = (n+1) + m + (n+1)m + 1 \neq \dim(J^1Y) = \dim(\Pi)$, the Legendre transform can never be a diffeomorphism. Therefore, the Legendre transform is sometimes defined as the map $\widehat{\mathbb{F}\mathcal{L}} := \mu \circ \mathbb{F}\mathcal{L}: J^1Y \to \Pi$.

Cartan forms. The dual jet bundle $J^1Y^\star$ is naturally endowed with a canonical $(n+1)$-form Θ. By pulling back this $(n+1)$-form with the Legendre transform, we obtain the Cartan $(n+1)$-form $\Theta_{\mathcal{L}} := \mathbb{F}\mathcal{L}^*\Theta$ on J^1Y, locally given by

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial L}{\partial v^A_0} \, \mathrm{d}y^A \wedge \mathrm{d}^n x_0 + \frac{\partial L}{\partial v^A_i} \, \mathrm{d}y^A \wedge \mathrm{d}^n x_i \\
&\quad + \left(L - \frac{\partial L}{\partial v^A_0} v^A_0 - \frac{\partial L}{\partial v^A_i} v^A_i \right) \mathrm{d}^{n+1} x \\
&= \frac{\partial L}{\partial v^A_0} \, \mathrm{d}y^A \wedge \mathrm{d}^n s + \frac{\partial L}{\partial v^A_i} \, \mathrm{d}y^A \wedge \mathrm{d}t \wedge \mathrm{d}^{n-1} s_i \\
&\quad + \left(L - \frac{\partial L}{\partial v^A_0} v^A_0 - \frac{\partial L}{\partial v^A_i} v^A_i \right) \mathrm{d}t \wedge \mathrm{d}^n s,
\end{aligned}$$

where, as usual, $\mathrm{d}^{n+1}x = \mathrm{d}t \wedge \mathrm{d}^n s$, and

$$\mathrm{d}^{n-1} s_i := \mathbf{i}_{\frac{\partial}{\partial x^i}} \mathrm{d}^n s, \quad \mathrm{d}^n x_0 := \mathbf{i}_{\frac{\partial}{\partial t}} \, \mathrm{d}^{n+1} x = \mathrm{d}^n s,$$

$$\mathrm{d}^n x_i := \mathbf{i}_{\frac{\partial}{\partial x^i}} \, \mathrm{d}^{n+1} x = \mathrm{d}t \wedge \, \mathrm{d}^{n-1} s_i.$$

The Cartan form allows to write the Lagrangian density evaluated on a first jet extension $j^1\varphi$ as

$$\mathcal{L}(j^1\varphi) = (j^1\varphi)^* \Theta_{\mathcal{L}}.$$

The Cartan form naturally appears in the covariant Hamilton principle when the variations are not necessarily vanishing at the boundary. Writing $\delta\varphi(x) = V(\varphi(x))$, where V is a vertical vector field on $\pi: Y \to X$, we have $\delta j^1\varphi(x) = j^1V(j^1\varphi(x))$, where the vertical vector field j^1V on $J^1Y \to X$ is the first jet extension of V.

With these abstract notations, (2.4) can be written as

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_X \mathcal{L}(j^1\varphi_\varepsilon(x)) = \int_X \mathscr{E}\mathscr{L}(\mathcal{L})(j^1\varphi(x))\cdot V(\varphi(x)) + \int_{\partial X}(j^1\varphi)^*(\mathbf{i}_{j^1V}\Theta_{\mathcal{L}}). \tag{2.9}$$

Finally, the CEL operator can be rewritten in terms of the $(n+2)$-form $\Omega_{\mathcal{L}} = -\mathbf{d}\Theta_{\mathcal{L}}$ as

$$\mathscr{E}\mathscr{L}(\mathcal{L})(j^1\varphi)\cdot V\circ\varphi = -(j^1\varphi)^*(\mathbf{i}_{j^1V}\Omega_{\mathcal{L}}). \tag{2.10}$$

Multisymplectic form formula. It is well known in classical Lagrangian mechanics that the flow F_t of the EL equations is symplectic relative to the symplectic form $\Omega_{\mathsf{L}} = \mathbb{F}\mathsf{L}^*\Omega_{\mathrm{can}}$ on TQ, that is, we have

$$F_t^*\Omega_{\mathsf{L}} = \Omega_{\mathsf{L}},$$

where we supposed that L is regular. In order to generalize this fact to the case of field theory, this property has to be reformulated. We follow [19]. Consider the action functional $S(q(\cdot)) = \int_0^T \mathsf{L}(q(t),\dot q(t))dt$. We have the formula

$$\mathbf{d}S(q(\cdot))\cdot\delta q(\cdot) = \int_0^T\left(\frac{\partial\mathsf{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathsf{L}}{\partial\dot q}\right)\delta q\,\mathrm{d}t + \Theta_{\mathsf{L}}(\dot q(t))\cdot\delta\dot q(t)\bigg|_0^T, \tag{2.11}$$

where $\Theta_{\mathsf{L}} := \mathbb{F}\mathsf{L}^*\Theta_{\mathrm{can}}$.

Consider now the function S_t defined on the space of solutions $\mathcal{C}_{\mathsf{L}}$ of the EL equations, which can be identified with initial conditions $v_q \in TQ$, defined by

$$S_t(v_q) := \int_0^t \mathsf{L}(q(s),\dot q(s))\,\mathrm{d}s,$$

where $(q(s),\dot q(s)) = F_s(v_q)$. In this case, (2.11) becomes

$$\mathbf{d}S_t(v_q)\cdot\delta v_q = \Theta_{\mathsf{L}}(F_t(v_q))\cdot\delta(F_t(v_q)) - \Theta_{\mathsf{L}}(v_q)\cdot\delta v_q = (F_t^*\Theta_{\mathsf{L}} - \Theta_{\mathsf{L}})(v_q)\cdot\delta v_q.$$

From the formula $\mathbf{d}S_t = F_t^*\Theta_{\mathsf{L}} - \Theta_{\mathsf{L}}$, we can deduce the symplecticity of the flow since $0 = \mathbf{dd}S_t = -F_t^*\Omega_{\mathsf{L}} + \Omega_{\mathsf{L}}$. This identity also shows that the symplecticity of the flow is equivalent to $\mathbf{dd}S_t = 0$.

Going back to (2.11), we observe that on the space $\mathcal{C}$ of curves defined on $[0,T]$, the formula can be rewritten as

$$\mathbf{d}S(q(\cdot))\cdot V = \alpha_1(q(\cdot))\cdot V + \alpha_2(q(\cdot))\cdot V,$$

where V is an arbitrary variation of the curve $q(\cdot)$, and the one-forms α_1 and α_2 on $\mathcal{C}$ are defined by

$$\alpha_1(q(\cdot))\cdot\delta q(\cdot) := \int_0^T\left(\frac{\partial\mathsf{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathsf{L}}{\partial\dot q}\right)\delta q(t)\,\mathrm{d}t,$$

$$\alpha_2(q(\cdot))\cdot\delta q(\cdot) = \Theta_{\mathsf{L}}(\dot q(t))\cdot\delta\dot q(t)|_0^T.$$

From the above formula, one deduces $0 = \mathbf{d}\mathbf{d}S = \mathbf{d}\alpha_1 + \mathbf{d}\alpha_2$. Given a solution $q(t)$ of the EL equations, a first variation at $q(t)$ is a vector field V on Q such that $t \mapsto (F_\varepsilon^V \circ q)(t)$ is also a solution curve, where F_ε^V is the flow of V. One can associate the vectors $V_{q(\cdot)} := V \circ q(\cdot)$ at $q(\cdot) \in \mathcal{C}$, also called first variations, and deduce the formula

$$\mathbf{d}\alpha_2(V_{q(\cdot)}, W_{q(\cdot)}) = 0. \tag{2.12}$$

It is this formulation of symplecticity that is generalized to the case of field theories.

In the case of field theories, (2.9) can be written as

$$\mathbf{d}S(\varphi) \cdot \delta\varphi = \alpha_1(\varphi) \cdot \delta\varphi + \alpha_2(\varphi) \cdot \delta\varphi, \tag{2.13}$$

where α_1 and α_2 are the one-forms on sections defined by

$$\alpha_1(\varphi) \cdot \delta\varphi := -\int_X (j^1\varphi)^*(\mathbf{i}_{j^1V}\Omega_{\mathcal{L}}) \quad \text{and} \quad \alpha_2(\varphi) \cdot \delta\varphi := \int_{\partial X} (j^1\varphi)^*(\mathbf{i}_{j^1V}\Theta_{\mathcal{L}}),$$

respectively. A first variation at a given solution φ of the CEL equations, is a vertical vector field $W \in \mathfrak{X}(Y)$ whose flow F_ε^W is such that $F_\varepsilon \circ \varphi$ is still a solution of the CEL equations, that is, by (2.10), $j^1(F_\varepsilon^W \circ \varphi)^*\mathbf{i}_{j^1V}\Omega_{\mathcal{L}} = 0$, for all vertical vector fields $V \in \mathfrak{X}^V(Y) := \{U \in \mathfrak{X}(Y) \,|\, U(y) \in \{0\} \times T_mM, \forall\, y = (x,m) \in X \times M = Y\}$. Taking the ε-derivative, we conclude that W satisfies the equation $(j^1\varphi)^*\pounds_{j^1W}\mathbf{i}_{j^1V}\Omega_{\mathcal{L}} = 0$, for all $V \in \mathfrak{X}^V(Y)$. From this and (2.9) and (2.10), it can be shown that if φ is a solution of the CEL equations, then, for all first variations V, W at φ, we have

$$\mathbf{d}\alpha_2(\varphi)(V \circ \varphi, W \circ \varphi) = 0 \quad \text{or, equivalently,} \quad \int_{\partial X} (j^1\varphi)^*\mathbf{i}_{j^1V}\mathbf{i}_{j^1W}\Omega_{\mathcal{L}} = 0, \tag{2.14}$$

as shown (in a slightly more general situation) in [19]. This formula is the analogue of (2.12) for the case of field theories and is referred to as the *multisymplectic form formula*.

Covariant momentum map and Noether theorem. Let G be a Lie group acting on Y by π-bundle automorphisms $\eta_Y : Y \to Y$, i.e. $\pi(\eta_Y(y)) = \eta_X(x)$, where $\eta_X : X \to X$ denotes the diffeomorphism on X induced by η_Y. A Lagrangian density $\mathcal{L}$ is said to be G-equivariant if

$$\mathcal{L}(j^1\eta_Y(\gamma)) = (\eta_X^{-1})^*\mathcal{L}(\gamma), \quad \text{for all } \gamma \in J^1Y, \;\; \eta_Y \in G,$$

where $j^1\eta_Y : J^1Y \to J^1Y$ is the lifted diffeomorphism to J^1Y. The Lagrangian momentum map $J^{\mathcal{L}} : J^1Y \to \mathfrak{g}^* \otimes \Lambda^n J^1Y$ associated to this action and to $\mathcal{L}$ is defined by

$$J^{\mathcal{L}}(\xi) = \mathbf{i}_{\xi_{J^1Y}}\Theta_{\mathcal{L}}, \tag{2.15}$$

where $\xi \in \mathfrak{g}$ and ξ_{J^1Y} is the infinitesimal generator associated to the lifted action of G on J^1Y. The Noether theorem, recalled below (see [11]), is proved by using formula (2.9) together with the G-equivariance of $\mathcal{L}$.

Theorem 2.2. *Let $\mathcal{L}: J^1Y \to \Lambda^{n+1}X$ be a G-equivariant Lagrangian density and let $J^{\mathcal{L}}: J^1Y \to \mathfrak{g}^* \otimes \Lambda^n J^1Y$ be the associated Lagrangian momentum map. If the section $\varphi: X \to Y$ is a solution of the CEL equations, then, for any subset $U \subset X$ with smooth boundary, we have*

$$\int_{\partial U} (j^1\varphi)^* J^{\mathcal{L}}(\xi) = 0, \quad \textit{for all } \xi \in \mathfrak{g}.$$

The associated local conservation law is

$$\mathbf{d}[(j^1\varphi)^* J^{\mathcal{L}}(\xi)] = 0, \quad \textit{for all } \xi \in \mathfrak{g}.$$

2.3. *Covariant Euler–Lagrange equations on Lie groups*

In this section we suppose that the fiber is a Lie group $M = G$ and use the notation $\varphi(t,s) = g(t,s) \in G$. We rewrite the CEL in a trivialized formulation, since it is this form of the CEL equations that will be discretized on Lie groups.

Trivialization on Lie groups. We can rewrite the CEL equations in a trivialized form by using the vector bundle isomorphism

$$J^1(X \times G) = L(TX, TG) \xrightarrow{\sim} L(TX, \mathfrak{g}) \times G = (T^*X \otimes \mathfrak{g}) \times G, \tag{2.16}$$

over $X \times G$, induced by the (left) trivialization $TG \ni v_g \xleftrightarrow{\sim} (g, g^{-1}v_g) \in G \times \mathfrak{g}$ of the tangent bundle TG of G. Coordinates on the trivialized jet bundle are denoted $(t, s^i, g^A, \xi_0^A, \eta_i^A)$ and the above vector bundle isomorphism reads $(t, s^i, g^A, v_0^A, v_i^A) \mapsto (t, s^i, g^A, g^{-1}v_0^A, g^{-1}v_i^A)$. The induced trivialized Lagrangian density $\bar{\mathcal{L}}$ on $L(TX, \mathfrak{g}) \times G$ satisfies

$$\begin{aligned}\mathcal{L}(t, s, g(t,s), \dot{g}(t,s), \nabla g(t,s)) &= \bar{\mathcal{L}}(t, s, g(t,s), g(t,s)^{-1}\dot{g}(t,s), g(t,s)^{-1}\nabla g(t,s)) \\ &=: \bar{\mathcal{L}}(t, s, g(t,s), \xi(t,s), \eta(t,s)),\end{aligned}$$

where $\xi(t,s) := g(t,s)^{-1}\dot{g}(t,s), \eta(t,s) := g(t,s)^{-1}\nabla g(t,s)$.

The trivialized CEL equations are obtained by applying Hamilton's principle to $\bar{\mathcal{L}}$ and using the variations induced on $\xi = g^{-1}\dot{g}$ and $\eta = g^{-1}\nabla g$, given by

$$\delta\xi = \dot{\zeta} + [\xi, \zeta] \quad \text{and} \quad \delta\eta = \nabla\zeta + [\eta, \zeta],$$

where $\zeta := g^{-1}\delta g : X \to \mathfrak{g}$ is an arbitrary map with $\zeta|_{\partial X} = 0$. We get the trivialized CEL equations

$$\frac{\partial}{\partial t}\frac{\delta\bar{\mathcal{L}}}{\delta\xi} + \operatorname{div}\frac{\delta\bar{\mathcal{L}}}{\delta\eta} = \operatorname{ad}^*_\xi\frac{\delta\bar{\mathcal{L}}}{\delta\xi} + \operatorname{ad}^*_\eta\frac{\delta\bar{\mathcal{L}}}{\delta\eta} + g^{-1}\frac{\partial\bar{\mathcal{L}}}{\partial g}. \tag{2.17}$$

Other boundary conditions can be used in the variational principle. The analogues of (2.5) and (2.6) are, respectively,

$$\frac{\partial\bar{\mathcal{L}}}{\partial\eta_i^A}\mathbf{n}^i = 0, \quad \text{for all } A = 1, \ldots, m \quad \text{and}$$

$$\frac{\partial\bar{\mathcal{L}}}{\partial\xi^A}(0,s) = \frac{\partial\bar{\mathcal{L}}}{\partial\xi^A}(T,s) = 0, \quad \text{for all } s \in B.$$

Remark 2.3 (G-invariance). The Lagrangian $\mathcal{L}$ is G-invariant if and only if $\bar{\mathcal{L}}(g,\xi,\eta) = \bar{\mathcal{L}}(e,\xi,\eta)$ for any $g \in G$. In this case, instead of working with $\bar{\mathcal{L}}: L(TX,\mathfrak{g}) \times G \to \Lambda^{n+1}X$, it suffices to consider the reduced Lagrangian $\ell(\xi,\eta)$ associated to $\mathcal{L}$ by G-invariance, i.e. $\ell(\xi,\eta) := \bar{\mathcal{L}}(e,\xi,\eta)$. Since $\frac{\partial \bar{\mathcal{L}}}{\partial g} = 0$, the trivialized CEL equations consistently recover the covariant Euler–Poincaré equations

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} + \operatorname{div}\frac{\delta \ell}{\delta \eta} = \operatorname{ad}^*_\xi \frac{\delta \ell}{\delta \xi} + \operatorname{ad}^*_\eta \frac{\delta \ell}{\delta \eta},$$

obtained by reduction, see [3]. Note that the covariant Euler–Poincaré equations can also be derived via a classical dynamic (as opposed to covariant) Lagrangian reduction approach, see [10].

Legendre transforms. Analogously to (2.16), the dual jet bundle can be trivialized by using the vector bundle isomorphism

$$\begin{aligned} J^1(X\times G)^\star &= TX \otimes T^*G \otimes \Lambda^{n+1}X \times_{X\times G} \Lambda^{n+1}X \\ &\xrightarrow{\sim} TX \otimes \mathfrak{g}^* \otimes \Lambda^{n+1}X \times_{X\times G} \Lambda^{n+1}X, \end{aligned}$$

induced by the (left) trivialization $T^*G \ni \alpha_g \xleftrightarrow{\sim} (g, T^*_e L_g \alpha_g) \simeq G \times \mathfrak{g}^*$. Local coordinates on the trivialized dual jet bundle are denoted $(t, s^i, g, \mu^0_A, \mu^i_A, p)$ and the above vector bundle isomorphism reads $(t, s^i, g, p^0_A, p^i_A, p) \mapsto (t, s^i, g, g^{-1}p^0_A, g^{-1}p^i_A, p)$. Locally, the trivialized Legendre transform

$$\mathbb{F}\bar{\mathcal{L}}: (TX \otimes \mathfrak{g}) \times G \to (TX \otimes \mathfrak{g}^* \otimes \Lambda^{n+1}X) \times_{X\times G} \Lambda^{n+1}X$$

is the fiber bundle map over $X \times G$ given by

$$\mu^0_A = \frac{\delta \bar{L}}{\delta \xi^A}, \quad \mu^i_A = \frac{\delta \bar{L}}{\delta \eta^A_i}, \quad p = \bar{L} - \frac{\delta \bar{L}}{\delta \xi^A}\xi^A - \frac{\delta \bar{L}}{\delta \eta^A_i}\eta^A_i.$$

Given a field $g: X \to G$, and defining $\xi := g^{-1}\dot{g}, \eta := g^{-1}\nabla g \in \mathfrak{g}$, we can write

$$\mathbb{F}\bar{\mathcal{L}}(g,\xi,\eta) = \left(g, \frac{\delta \bar{\mathcal{L}}}{\delta \xi}, \frac{\delta \bar{\mathcal{L}}}{\delta \eta}, \bar{\mathcal{L}} - \frac{\delta \bar{\mathcal{L}}}{\delta \xi}\xi - \frac{\delta \bar{\mathcal{L}}}{\delta \eta}\eta\right).$$

Similarly, the trivialized version of $\widehat{\mathbb{F}\mathcal{L}}$ is $\widehat{\mathbb{F}\bar{\mathcal{L}}}(g,\xi,\eta) = (g, \frac{\delta \bar{\mathcal{L}}}{\delta \xi}, \frac{\delta \bar{\mathcal{L}}}{\delta \eta})$.

The trivialized Cartan form $\bar{\Theta}_{\bar{\mathcal{L}}} := \mathbb{F}\bar{\mathcal{L}}^*\bar{\Theta}$ is found to be

$$\begin{aligned} \bar{\Theta}_{\bar{\mathcal{L}}} &= \frac{\delta \bar{L}}{\delta \xi^A} g^{-1}\mathrm{d}g^A \wedge \mathrm{d}^n s + \frac{\delta \bar{L}}{\delta \eta^A_i} g^{-1}\mathrm{d}g^A \wedge \mathrm{d}t \wedge \mathrm{d}^{n-1}s_i \\ &\quad + \left(\bar{L} - \frac{\delta \bar{L}}{\delta \xi_i}\xi_i - \frac{\delta \bar{L}}{\delta \eta^A_i}\eta^A_i\right) \mathrm{d}t \wedge \mathrm{d}^n s. \end{aligned}$$

Remark 2.4 (G-invariance). Recall that if the Lagrangian is G-invariant, we have $\bar{\mathcal{L}}(g,\xi,\eta) = \ell(\xi,\eta)$. Therefore, the maps $\mathbb{F}\bar{\mathcal{L}}$ and $\widehat{\mathbb{F}\bar{\mathcal{L}}}$ yield the reduced Legendre transforms $\mathbb{F}\ell : T^*X \otimes \mathfrak{g} \to (TX \otimes \mathfrak{g}^* \otimes \Lambda^{n+1}X) \times_X \Lambda^{n+1}X$ and

$\widehat{\mathbb{F}\ell}: T^*X \otimes \mathfrak{g} \to TX \otimes \mathfrak{g}^* \otimes \Lambda^{n+1}X$ given by

$$\mathbb{F}\ell(\xi,\eta) = \left(\frac{\delta\ell}{\delta\xi}, \frac{\delta\ell}{\delta\eta}, \ell - \frac{\delta\ell}{\delta\xi}\xi - \frac{\delta\ell}{\delta\eta}\eta\right) \quad \text{and} \quad \widehat{\mathbb{F}\ell}(\xi,\eta) = \left(\frac{\delta\ell}{\delta\xi}, \frac{\delta\ell}{\delta\eta}\right).$$

3. Multisymplectic Variational Integrators and Space/Time Splitting

In this section we study the symplectic properties and the conservation laws of a multisymplectic integrator on a $(1+1)$-dimensional spacetime discretized by triangles. This study uses a covariant point of view as well as time-evolutionary and space-evolutionary approaches.

In Sec. 3.1, we review from [19], some basic facts on multisymplectic integrators, such as the discrete covariant Euler–Lagrange (DCEL) equations, the discrete Cartan forms, the notion of multisymplecticity, the discrete covariant Legendre transform, the discrete covariant momentum map, and the discrete covariant Noether theorem. We also consider the case with external forces and write the explicit expression of the discrete Noether quantity on arbitrary rectangular subdomains.

We consider three different classes of boundary conditions: the case where the configuration is prescribed at the space and time extremities, the case when the configuration is only prescribed at the temporal extremities, and the case where the configuration is only prescribed at the spatial extremity. In the last two cases, the associated discrete zero-traction boundary conditions are derived from the discrete covariant variational principle (Proposition 3.1).

The solution of the discrete problem can be organized in a time-evolutionary fashion, by first forming a vector of the discrete positions of all nodes at a given time, and then considering the sequence of these vectors indexed by the discrete time.

Conversely, the solution of the discrete problem can be organized in a space-evolutionary fashion, by first forming a vector of the discrete positions of all nodes at a given space index, and then considering the sequence of these vectors.

In Sec. 3.2, we study the symplectic character of these time-evolutionary and space-evolutionary discrete flows. This is done by constructing, from the discrete covariant Lagrangian, two discrete Lagrangians (L_d and N_d) associated to the temporal and spatial evolution, respectively. For this construction, it is assumed that the discrete covariant Lagrangian does not depend explicitly on the discrete time, respectively, on the discrete space. We will carry out this study for each of the three boundary conditions mentioned before. The corresponding results are given in Propositions 3.5–3.11.

In Sec. 3.3, we study the various Noether conservation theorems available when the discrete covariant Lagrangian density is invariant under the action of a Lie group G. Indeed, G-invariance of the Lagrangian density induces G-invariance of the discrete Lagrangians L_d and N_d so, besides the discrete covariant Noether theorem for $\mathfrak{L}_d$, one can ask if the discrete Noether theorems associated to the time-evolutionary

and space-evolutionary discrete flows are also satisfied. This depends on the boundary conditions considered, as stated Theorem 3.13.

3.1. *Preliminaries on multisymplectic integrators*

We present below some basic facts about multisymplectic integrators, following [19]. In view of the applications we have in mind, we assume from now on that $B = [0, L]$ and hence $X = [0, T] \times [0, L]$.

3.1.1. *Discrete covariant Euler–Lagrange equations and boundary conditions*

We consider the discretization of spacetime X given by

$$X_d = \{(j, a) \in \mathbb{Z} \times \mathbb{Z} \,|\, j = 0, \ldots, N-1, a = 0, \ldots, A-1\},$$

where N and A are the number of temporal and spatial grid points, respectively. This defines the triangles $\triangle_a^j$ by specifying their vertices as the ordered triples

$$\triangle_a^j = ((j, a), (j+1, a), (j, a+1)).$$

Denote by Δt and Δs the temporal and spatial step sizes, respectively. The discrete analogue of the bundle $Y = X \times M$ is $Y_d = X_d \times M$ and the discrete sections φ_d are defined to be maps $X_d \ni (j, a) \mapsto \varphi_d(j, a) =: \varphi_a^j \in M$. Recall that M is the space of allowed deformations; $M = \mathrm{SE}(3)$ for the example of the beam that will considered in Sec. 5. The discrete analogue of the first jet bundle is $J^1 Y_d = X_d^\triangle \times M \times M \times M \to X_d^\triangle$, where $X_d^\triangle$ denotes the set of all triangles $\triangle_a^j$ defined above. Elements of $J^1 Y_d$ are of the form $(\triangle_a^j, \varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j)$. The first jet extension of a discrete section φ_d is the map $j^1\varphi_d : X_d^\triangle \to X_d^\triangle \times M \times M \times M$ defined by

$$j^1\varphi_d(\triangle_a^j) := (\triangle_a^j, \varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j). \tag{3.1}$$

Discrete covariant Euler–Lagrange equations. A discrete Lagrangian density is a map $\mathcal{L}_d : J^1 Y_d \to \mathbb{R}$ defined such that the value $\mathcal{L}_d(\triangle_a^j, \varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j)$ is an approximation of the integral

$$\int_{\square_a^j} L(t, s, \varphi(t, s), \partial_t\varphi(t, s), \partial_s\varphi(t, s))\, \mathrm{d}t \wedge \mathrm{d}s,$$

where $\square_a^j$ is the rectangle with vertices $((j, a), (j+1, a), (j, a+1), (j+1, a+1))$ and $\varphi : X \to M$ is a smooth map interpolating the field values $(\varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j)$.

The discrete action functional associated to a discrete section φ_d is

$$\mathfrak{S}_d(\varphi_d) := \sum_{j=0}^{N-1} \sum_{a=0}^{A-1} \mathcal{L}_d(\triangle_a^j, \varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j).$$

To simplify notations, we will write $\mathcal{L}_a^j := \mathcal{L}_d(\triangle_a^j, \varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j)$. Computing the derivative of the discrete action map (relative to φ_d) gives

$$\begin{aligned}
\delta\mathfrak{S}_d(\varphi_d) &= \sum_{j=0}^{N-1}\sum_{a=0}^{A-1}[D_1\mathcal{L}_a^j\cdot\delta\varphi_a^j + D_2\mathcal{L}_a^j\cdot\delta\varphi_a^{j+1} + D_3\mathcal{L}_a^j\cdot\delta\varphi_{a+1}^j] \\
&= \sum_{j=1}^{N-1}\sum_{a=1}^{A-1}(D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j)\cdot\delta\varphi_a^j \\
&\quad + \sum_{j=1}^{N-1}[(D_1\mathcal{L}_0^j + D_2\mathcal{L}_0^{j-1})\cdot\delta\varphi_0^j + D_3\mathcal{L}_{A-1}^j\cdot\delta\varphi_A^j] \\
&\quad + \sum_{a=1}^{A-1}[(D_1\mathcal{L}_a^0 + D_3\mathcal{L}_{a-1}^0)\cdot\delta\varphi_a^0 + D_2\mathcal{L}_a^{N-1}\cdot\delta\varphi_a^N] \\
&\quad + D_1\mathcal{L}_0^0\cdot\delta\varphi_0^0 + D_2\mathcal{L}_0^{N-1}\cdot\delta\varphi_0^N + D_3\mathcal{L}_{A-1}^0\cdot\delta\varphi_A^0. \qquad (3.2)
\end{aligned}$$

(A) Discrete spacetime boundary conditions. We shall first consider the case when the discrete configuration is known at the boundary of the spacetime domain. In this case, from the discrete covariant Hamilton principle, it follows that $\delta\mathfrak{S}_d(\varphi_d) = 0$, for all variations $\delta\varphi_a^j$ vanishing at the boundary, that is, such that

$$\delta\varphi_a^j = 0, \quad \text{if } a \in \{0, A\} \text{ or if } j \in \{0, N\}. \qquad (3.3)$$

We thus get from (3.2) the discrete covariant Euler–Lagrange (DCEL) equations

$$D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \;\; a = 1, \ldots, A-1, \qquad (3.4)$$

where we recall that $\mathcal{L}_a^j := \mathcal{L}_d(\triangle_a^j, \varphi_a^j, \varphi_a^{j+1}, \varphi_{a+1}^j)$ and the values of φ_a^j at the boundary are prescribed. Note that three triangles contribute to each DCEL equation in (3.4), namely, the triangles $\triangle_a^j, \triangle_a^{j-1}$, and $\triangle_{a-1}^j$. The intersection of the three triangles is (j, a), as shown in Fig. 1.

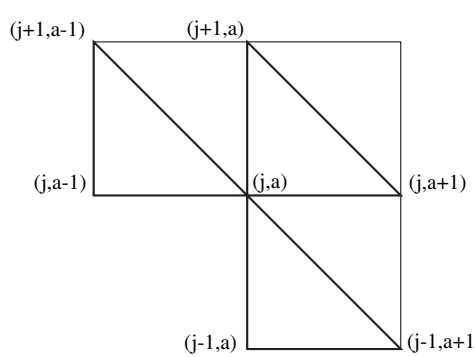

Fig. 1. The triangles $\triangle_a^j, \triangle_a^{j-1}, \triangle_{a-1}^j$.

(B) Discrete boundary conditions in time. If we assume that the discrete configuration is prescribed at $j = 0$ and $j = N$, for all $a = 0, \dots, A$, then, instead of the equalities in (3.3), only the following variations vanish:

$$\delta\varphi_a^0 = \delta\varphi_a^N = 0, \quad \text{for all } a = 0, \dots, A.$$

In this case, from (3.2), the discrete Hamilton principle yields the boundary condition

$$D_1\mathcal{L}_0^j + D_2\mathcal{L}_0^{j-1} = 0 \quad \text{and} \quad D_3\mathcal{L}_{A-1}^j = 0, \quad \text{for all } j = 1, \dots, N-1, \tag{3.5}$$

referred to as the *discrete zero-traction boundary condition.*

(C) Discrete boundary conditions in space. Conversely, if we assume that the discrete configuration is prescribed at the boundary $a = 0$ and $a = A$, for all $j = 0, \dots, N$, then the following variations vanish:

$$\delta\varphi_0^j = \delta\varphi_A^j = 0, \quad \text{for all } j = 0, \dots, N.$$

In this case, using (3.2), the discrete Hamilton principle yields the boundary condition

$$D_1\mathcal{L}_a^0 + D_3\mathcal{L}_{a-1}^0 = 0 \quad \text{and} \quad D_2\mathcal{L}_a^{N-1} = 0, \quad \text{for all } a = 1, \dots, A-1, \tag{3.6}$$

referred to as the *discrete zero momentum boundary condition.*

We summarize these facts in the following proposition.

Proposition 3.1. *Let $\mathcal{L}_d : J^1Y_d \to \mathbb{R}$ be a discrete Lagrangian density. The discrete zero-traction boundary conditions and zero momentum boundary conditions obtained via the covariant discrete Hamilton principle are, respectively, given by*

$$D_1\mathcal{L}_0^j + D_2\mathcal{L}_0^{j-1} = 0 \quad \textit{and} \quad D_3\mathcal{L}_{A-1}^j = 0, \quad \textit{for all} \quad j = 1, \dots, N-1$$

and

$$D_1\mathcal{L}_a^0 + D_3\mathcal{L}_{a-1}^0 = 0 \quad \textit{and} \quad D_2\mathcal{L}_a^{N-1} = 0, \quad \textit{for all} \quad a = 1, \dots, A-1.$$

Remark 3.2. It is important to note that the discrete boundary conditions above are obtained exactly in the same way as their continuous counterparts, namely, they arise as boundary terms in the variational principles. These boundary terms do not contribute when the configuration is prescribed at the boundary, since the corresponding variations vanish on the boundary. However, when the configuration is not prescribed at the boundary, the variational principles yield "natural" boundary conditions, given here by (3.5) and/or (3.6).

3.1.2. *Discrete Cartan forms and multisymplecticity*

The discrete Cartan forms, denoted $\Theta^1_{\mathcal{L}_d}, \Theta^2_{\mathcal{L}_d}, \Theta^3_{\mathcal{L}_d}$, are the one-forms on J^1Y_d, defined by

$$\Theta^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) := D_1\mathcal{L}_a^j d\varphi_a^j,$$

$$\Theta^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) := D_2\mathcal{L}^j_a d\varphi^{j+1}_a,$$
$$\Theta^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) := D_3\mathcal{L}^j_a d\varphi^j_{a+1}, \tag{3.7}$$

for all $j^1\varphi_d(\triangle^j_a) := (\triangle^j_a, \varphi^j_a, \varphi^{j+1}_a, \varphi^j_{a+1}) \in J^1Y_d = X^\triangle_d \times M \times M \times M$ (see (3.1) or [19]). Note that one can also interpret the Cartan forms as $X^\triangle_d$-dependent one-forms on $M \times M \times M$. Viewed this way, they satisfy the relation

$$\mathbf{d}\mathcal{L}_d = \Theta^1_{\mathcal{L}_d} + \Theta^2_{\mathcal{L}_d} + \Theta^3_{\mathcal{L}_d}. \tag{3.8}$$

The discrete Cartan 2-forms are defined as $\Omega^k_{\mathcal{L}_d} := -\mathbf{d}\Theta^k_{\mathcal{L}_d}$ and thus satisfy

$$\Omega^1_{\mathcal{L}_d} + \Omega^2_{\mathcal{L}_d} + \Omega^3_{\mathcal{L}_d} = 0.$$

The definition of the discrete Cartan forms $\Theta^k_{\mathcal{L}_d}$ is motivated by the following observation.

Given a vector field V on Y_d, we denote by V^j_a its restriction to the fiber at (j, a). Its first jet extension is the vector field j^1V on J^1Y_d defined by

$$j^1V(\triangle^j_a, \varphi^j_a, \varphi^{j+1}_a, \varphi^j_{a+1}) := (V(\varphi^j_a), V(\varphi^{j+1}_a), V(\varphi^j_{a+1})).$$

Using this notation, we can now rewrite the variations of the discrete action (3.2) in a way analogous to (2.9). Namely, given a discrete field φ_d with variations $\delta\varphi_d$, defining the vector field V on Y_d such that $\delta\varphi^j_a = V(\varphi^j_a)$, and rewriting (3.7) in the form

$$D_1\mathcal{L}^j_a\delta\varphi^j_a = [(j^1\varphi_d)^*(\mathbf{i}_{j^1V}\Theta^1_{\mathcal{L}_d})](\triangle^j_a),$$
$$D_2\mathcal{L}^j_a\delta\varphi^{j+1}_a = [(j^1\varphi_d)^*(\mathbf{i}_{j^1V}\Theta^2_{\mathcal{L}_d})](\triangle^j_a),$$
$$D_3\mathcal{L}^j_a\delta\varphi^j_{a+1} = [(j^1\varphi_d)^*(\mathbf{i}_{j^1V}\Theta^3_{\mathcal{L}_d})](\triangle^j_a),$$

equality (3.2) becomes

$$\begin{aligned}\delta\mathfrak{S}_d(\varphi_d) &= \sum_{j=1}^{N-1}\sum_{a=1}^{A-1}(D_1\mathcal{L}^j_a + D_2\mathcal{L}^{j-1}_a + D_3\mathcal{L}^j_{a-1})\cdot\delta\varphi^j_a \\ &\quad + \sum_{\{\triangle\in X^\triangle_d \,|\, \triangle\cap\partial X_d\neq\varnothing\}}\left(\sum_{k\in\{1,2,3\};\triangle^{(k)}\in\partial X_d}[(j^1\varphi_d)^*(\mathbf{i}_{j^1V}\Theta^k_{\mathcal{L}_d})](\triangle)\right) \\ &=: \alpha_1(\varphi_d)\cdot\delta\varphi_d + \alpha_2(\varphi_d)\cdot\delta\varphi_d.\end{aligned} \tag{3.9}$$

The one-forms α_1 and α_2 on the space of discrete sections are defined analogously with (2.13); see [19] for details. When evaluated on first variations V, W at a solution φ_d, the formula $0 = \mathbf{dd}\mathfrak{S}_d = \mathbf{d}\alpha_1 + \mathbf{d}\alpha_2$ yields $\mathbf{d}\alpha_2(\varphi_d)(V, W) = 0$, or equivalently

$$\sum_{\{\triangle\in X^\triangle_d \,|\, \triangle\cap\partial X_d\neq\varnothing\}}\left(\sum_{k\in\{1,2,3\};\triangle^{(k)}\in\partial X_d}[(j^1\varphi_d)^*(\mathbf{i}_{j^1V}\mathbf{i}_{j^1W}\Omega^k_{\mathcal{L}_d})](\triangle)\right) = 0. \tag{3.10}$$

This formula is referred to as the *discrete multisymplectic form formula*. It is the discrete version of (2.14) and generalizes the notion of symplecticity for integrators in mechanics to the case of integrators in field theory.

Discrete covariant Legendre transform. The discrete covariant Legendre transforms are the maps $\mathbb{F}^k\mathcal{L}_d : J^1Y_d \to T^*M$ given by

$$\begin{aligned}
\mathbb{F}^1\mathcal{L}_d(j^1\varphi_d(\triangle_a^j)) &= (\varphi_a^j, D_1\mathcal{L}_a^j),\\
\mathbb{F}^2\mathcal{L}_d(j^1\varphi_d(\triangle_a^j)) &= (\varphi_a^{j+1},\, D_2\mathcal{L}_a^j),\\
\mathbb{F}^3\mathcal{L}_d(j^1\varphi_d(\triangle_a^j)) &= (\varphi_{a+1}^j, D_3\mathcal{L}_a^j).
\end{aligned} \tag{3.11}$$

We note that the DCEL equations can be thus written in the form

$$\mathbb{F}^1\mathcal{L}_d(j^1\varphi_d(\triangle_a^j)) + \mathbb{F}^2\mathcal{L}_d(j^1\varphi_d(\triangle_a^{j-1})) + \mathbb{F}^3\mathcal{L}_d(j^1\varphi_d(\triangle_{a-1}^j)) = 0,$$

which can be regarded as a matching of momenta in $T^*_{\varphi_a^j}M$.

3.1.3. *Discrete covariant momentum maps*

We consider only vertical symmetries, that is, group actions that act trivially on the base X. Let $\Phi : G \times M \to M$ be a left action of a Lie group G on M. This action naturally induces an action on the discrete jet bundle, given by

$$\Phi_g^{J^1Y_d}(j^1\varphi_d(\triangle_a^j)) := (\triangle_a^j, \Phi_g(\varphi_a^j), \Phi_g(\varphi_a^{j+1}), \Phi_g(\varphi_{a+1}^j)), \quad g \in G,$$

whose infinitesimal generator is

$$\xi_{J^1Y_d}(j^1\varphi_d(\triangle_a^j)) := (\triangle_a^j, \xi_M(\varphi_a^j), \xi_M(\varphi_a^{j+1}), \xi_M(\varphi_{a+1}^j)).$$

We say that the discrete Lagrangian is invariant with respect to this action if $\mathcal{L}_d \circ \Phi_g^{J^1Y_d} = \mathcal{L}_d$, for all $g \in G$. As a consequence, we have the infinitesimal invariance $\mathbf{d}\mathcal{L}_d \cdot \xi_{J^1Y_d} = 0$.

The discrete momentum maps are defined by

$$J_{\mathcal{L}_d}^k : J^1Y_d \to \mathfrak{g}^*, \quad \langle J_{\mathcal{L}_d}^k, \xi\rangle := \mathbf{i}_{\xi_{J^1Y_d}}\Theta_{\mathcal{L}_d}^k, \quad \xi \in \mathfrak{g},$$

so we have the formulas

$$\begin{aligned}
\langle J_{\mathcal{L}}^1(j^1\varphi_d(\triangle_a^j)), \xi\rangle &= D_1\mathcal{L}_a^j \cdot \xi_M(\varphi_a^j),\\
\langle J_{\mathcal{L}}^2(j^1\varphi_d(\triangle_a^j)), \xi\rangle &= D_2\mathcal{L}_a^j \cdot \xi_M(\varphi_a^{j+1}),\\
\langle J_{\mathcal{L}}^3(j^1\varphi_d(\triangle_a^j)), \xi\rangle &= D_3\mathcal{L}_a^j \cdot \xi_M(\varphi_{a+1}^j).
\end{aligned} \tag{3.12}$$

We note that the infinitesimal invariance of $\mathcal{L}_d$ can be rewritten as

$$(J_{\mathcal{L}_d}^1 + J_{\mathcal{L}_d}^2 + J_{\mathcal{L}_d}^3)(j^1\varphi_d(\triangle_a^j)) = 0, \tag{3.13}$$

for all $j = 0, \ldots, N-1$ and $a = 0, \ldots, A-1$. This is the statement of the local discrete Noether theorem. To obtain the global discrete Noether theorem, one applies the formula (3.9) for variations induced by the group action. More generally, given

a restriction

$$\mathfrak{S}_d^U(\varphi_d) := \sum_{j=K}^{L}\sum_{a=B}^{C} \mathcal{L}_d(\triangle_a^j, \varphi_a^i, \varphi_a^{j+1}, \varphi_{a+1}^j)$$

of the action functional to a subdomain U of X_d induced by $U := \{(a,j) \,|\, K \leq j \leq L, B \leq a \leq C\}$, that is, U is union of triangles whose lower left vertex belongs to a given rectangular subdomain, and by applying formula (3.9), we get the following result.

Theorem 3.3 (Discrete global Noether theorem). *Suppose that the discrete Lagrangian $\mathcal{L}_d : J^1Y_d \to \mathbb{R}$ is invariant under the action of a Lie group G on M. Suppose that φ_d is a solution of the DCEL equations for $\mathcal{L}_d$. Then, for all $0 \leq B < C \leq A-1, 0 \leq K < L \leq N-1$, we have the conservation law*

$$\mathscr{J}_{B,C}^{K,L}(\varphi_d) = 0, \tag{3.14}$$

where,

$$\begin{aligned}\mathscr{J}_{B,C}^{K,L}(\varphi_d) := &\sum_{j=K+1}^{L} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_B^j) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_B^{j-1})) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_C^j))] \\ &+ \sum_{a=B+1}^{C} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^K) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^L)) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_{a-1}^K))] \\ &+ J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_B^K) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_B^L)0 + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_C^K)).\end{aligned} \tag{3.15}$$

Of course, the expression for $\mathscr{J}_{B,C}^{K,L}$ can be written in a condensed form like the one appearing in (3.9) by using the discrete Cartan forms $\Theta^k_{\mathcal{L}_d}$.

3.2. *Symplectic properties of the time and space discrete evolutions*

In this subsection we study the symplectic character of the time-evolutionary and space-evolutionary discrete flows built from a discrete solution section φ_d of the DCEL equations associated to $\mathcal{L}_d$. From Sec. 3.1.2, we already know that φ_d satisfies the discrete multisymplectic form formula (3.10), which is the multisymplectic generalization of symplecticity. However, this does not guarantee that the time-evolutionary and space-evolutionary discrete flows are symplectic. As we shall explain in detail, the conclusion depends on the type of boundary conditions considered.

We will study symplecticity in time and space by constructing, from the discrete covariant Lagrangian, two discrete Lagrangians $\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})$ and $\mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1})$ associated to the temporal and spatial evolutions, respectively. For this construction, it is assumed that the discrete covariant Lagrangian does not depend explicitly on the discrete time, respectively, on the discrete space. Knowing that the DEL

equations associated to L_d and N_d yield symplectic schemes, we will study in detail the relation between these two DEL equations and the CDEL for $\mathcal{L}_d$. The answer depends on the class of boundary conditions considered.

Of course, when $\mathcal{L}_d$ depends explicitly on discrete time, the discrete Lagrangian L_d can be defined in the same way, but it will be time dependent. The same remark applies to N_d and the dependence on discrete space.

3.2.1. *Discrete time evolution: The discrete Lagrangian* L_d

The configuration space for the discrete Lagrangian L_d is M^{A+1}. Using the notation $\boldsymbol{\varphi}^j := (\varphi_0^j, \ldots, \varphi_A^j) \in M^{A+1}$, the discrete Lagrangian $\mathsf{L}_d : M^{A+1} \times M^{A+1} \to \mathbb{R}$ is defined by

$$\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) := \sum_{a=0}^{A-1} \mathcal{L}_a^j,$$

so that the associated discrete action is

$$\mathfrak{S}_d(\varphi_d) = \sum_{j=0}^{N-1} \mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) = \sum_{j=0}^{N-1} \sum_{a=0}^{A-1} \mathcal{L}_a^j. \tag{3.16}$$

In order to analyze the relation between the discrete Hamilton principles associated to L_d and $\mathcal{L}_d$, we shall first assume that there are no boundary conditions, so that the discrete Hamilton principle for L_d yields the stationarity conditions

$$\begin{aligned} D_1\mathsf{L}_d^j + D_2\mathsf{L}_d^{j-1} = 0, \quad j = 1, \ldots, N-1, \quad \text{and} \\ D_1\mathsf{L}_d^0 = 0, \quad D_2\mathsf{L}_d^{N-1} = 0, \end{aligned} \tag{3.17}$$

since the variations do not vanish at the boundary. Computing these expressions in terms of $\mathcal{L}_d$, we get

$$\begin{aligned} \langle D_1\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \delta\boldsymbol{\varphi}^j \rangle &= \sum_{a=1}^{A-1} (D_1\mathcal{L}_a^j + D_3\mathcal{L}_{a-1}^j) \cdot \delta\varphi_a^j \\ &\quad + D_1\mathcal{L}_0^j \cdot \delta\varphi_0^j + D_3\mathcal{L}_{A-1}^j \cdot \delta\varphi_A^j, \\ \langle D_2\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \delta\boldsymbol{\varphi}^{j+1} \rangle &= \sum_{a=1}^{A-1} D_2\mathcal{L}_a^j \cdot \delta\varphi_a^{j+1} + D_2\mathcal{L}_0^j \cdot \delta\varphi_0^{j+1}. \end{aligned} \tag{3.18}$$

So the DEL equations for L_d in (3.17) yield the equations

$$\begin{aligned} D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \ \ a = 1, \ldots, A-1, \\ D_1\mathcal{L}_0^j + D_2\mathcal{L}_0^{j-1} = 0, \quad j = 1, \ldots, N-1, \\ D_3\mathcal{L}_{A-1}^j = 0, \quad j = 1, \ldots, N-1, \end{aligned} \tag{3.19}$$

and the boundary conditions in (3.17) imply the equations

$$\begin{aligned} &D_1\mathcal{L}_a^0 + D_3\mathcal{L}_{a-1}^0 = 0, \quad a = 1, \dots, A-1 \\ &D_1\mathcal{L}_0^0 = 0, \quad D_3\mathcal{L}_{A-1}^0 = 0, \quad D_2\mathcal{L}_a^{N-1} = 0, \quad a = 0, \dots, A-1. \end{aligned} \tag{3.20}$$

Of course, (3.19) and (3.20) agree with the stationarity condition (3.2) obtained from the discrete covariant Hamilton's principle when no boundary condition is imposed. Moreover, this computation shows that the DEL equations for L_d (i.e. the first equation in (3.17)) is equivalent to the DCEL equations for $\mathcal{L}_d$ together with the discrete zero-traction boundary conditions (3.5) (the second and third lines in (3.19)).

Remark 3.4 (Discrete Cartan forms). We now describe the relation between the two discrete Cartan forms $\Theta^{\pm}_{\mathsf{L}_d}$ associated to L_d and the three discrete Cartan forms $\Theta^k_{\mathcal{L}_d}, k = 1, 2, 3$, associated to $\mathcal{L}_d$. On $M^{A+1} \times M^{A+1}$ we have

$$\begin{aligned} \Theta^-_{\mathsf{L}_d}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) &= -D_1\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})d\boldsymbol{\varphi}^j \\ &= -\sum_{a=1}^{A-1}(D_1\mathcal{L}_a^j + D_3\mathcal{L}_{a-1}^j)d\varphi_a^j + D_1\mathcal{L}_0^j d\varphi_0^j + D_3\mathcal{L}_{A-1}^j d\varphi_A^j \\ &= -\sum_{a=0}^{A-1}[\Theta^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) + \Theta^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j))], \end{aligned}$$

$$\begin{aligned} \Theta^+_{\mathsf{L}_d}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) &= D_2\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})d\boldsymbol{\varphi}^{j+1} = \sum_{a=0}^{A-1} D_2\mathcal{L}_a^j d\varphi_a^{j+1} \\ &= \sum_{a=0}^{A-1}\Theta^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)). \end{aligned}$$

We abbreviate these relations as

$$\Theta^-_{\mathsf{L}_d} = -\sum_{a=0}^{A-1}[\Theta^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) + \Theta^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j))] \quad \text{and}$$

$$\Theta^+_{\mathsf{L}_d} = \sum_{a=0}^{A-1}\Theta^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)).$$

Applying identity (3.8) to $\Theta^k_{\mathcal{L}_d}$, we consistently recover the relation $\mathbf{d}\Theta^-_{\mathsf{L}_d} = \mathbf{d}\Theta^+_{\mathsf{L}_d}$:

$$\begin{aligned} \mathbf{d}\Theta^-_{\mathsf{L}_d} &= -\mathbf{d}\sum_{a=0}^{A-1}\Theta^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) - \mathbf{d}\sum_{a=0}^{A-1}\Theta^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) \\ &= \mathbf{d}\sum_{a=0}^{A-1}\Theta^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) = \mathbf{d}\Theta^+_{\mathsf{L}_d}. \end{aligned} \tag{3.21}$$

The discrete Cartan 2-form Ω_{L_d} is thus related to the 2-forms $\Omega_{\mathcal{L}_d}$ in the following manner:

$$\Omega_{\mathsf{L}_d} = \sum_{a=0}^{A-1} \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) = -\sum_{a=0}^{A-1}[\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))]. \quad (3.22)$$

Notation. Note that we maintained the index j in the expressions above, for consistency with all the notations used in the paper. The index is, however, not needed here since all the triangles are identical. One can, for example, write the expression of $\Theta^-_{\mathsf{L}_d}$ as

$$\Theta^-_{\mathsf{L}_d}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = -\sum_{a=0}^{A-1}[\Theta^1_{\mathcal{L}_d}(\varphi_a, \psi_a, \varphi_{a+1}) + \Theta^3_{\mathcal{L}_d}(\varphi_a, \psi_a, \varphi_{a+1})],$$

for $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_A)$, $\boldsymbol{\psi} = (\psi_0, \ldots, \psi_A) \in M^{A+1}$. Similarly for $\Theta^+_{\mathsf{L}_d}$ and Ω_{L_d}.

(A) Boundary conditions in time. Let us first consider the case when the configuration is prescribed at $j = 0$ and $j = N$, for all $a = 0, \ldots, A$. This means that $\boldsymbol{\varphi}^0$ and $\boldsymbol{\varphi}^N$ are prescribed, therefore Hamilton's principle only yields the first equation in (3.17), namely the DEL equations associated to L_d. So, we only get the equations in (3.19). This is in complete agreement with the results obtained above in (3.4) and (3.5) via the discrete covariant variational principle when only boundary conditions in time have been assumed. This is also in complete analogy with the continuous case, where the EL equations imply the (zero-traction) boundary conditions (2.5), given here in the discrete case by (3.5). From this discussion, we conclude that the discrete flow map

$$F_{\mathsf{L}_d} : M^{A+1} \times M^{A+1} \to M^{A+1} \times M^{A+1}, \quad F_{\mathsf{L}_d}(\boldsymbol{\varphi}^{j-1}, \boldsymbol{\varphi}^j) = (\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})$$

is symplectic relative to the symplectic form $\Omega_{\mathsf{L}_d} \in \Omega^2(M^{A+1} \times M^{A+1})$, i.e. $F^*_{\mathsf{L}_d}\Omega_{\mathsf{L}_d} = \Omega_{\mathsf{L}_d}$. In particular, we have proven the following result.

Proposition 3.5. *When boundary conditions are only imposed in time, the DEL equations for L_d are equivalent to the DCEL equations for $\mathcal{L}_d$ together with the discrete zero-traction spatial boundary conditions.*

Thus, the solution $\varphi^j_a, j = 0, \ldots, N, a = 0, \ldots, A$, of the DCEL equations with discrete zero-traction boundary conditions (3.5) *provides a symplectic-in-time discrete flow $(\boldsymbol{\varphi}^{j-1}, \boldsymbol{\varphi}^j) \mapsto (\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})$ relative to the discrete symplectic form on $M^{A+1} \times M^{A+1}$, i.e.*

$$\Omega_{\mathsf{L}_d} = \sum_{a=0}^{A-1} \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) = -\sum_{a=0}^{A-1}[\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))].$$

The equations are solved by assuming that $\boldsymbol{\varphi}^0$ and $\boldsymbol{\varphi}^1$ are known, i.e. φ^0_a and φ^1_a for all $a = 0, \ldots, A$. This corresponds to the knowledge of initial configuration and velocity.

(B) Boundary conditions in space. We now consider (for completeness and symmetry relative to the preceding case) the situation when the discrete configuration is prescribed at the boundary $a = 0$ and $a = A$, for all $j = 0, \ldots, N$, and is time independent. No boundary conditions are assumed in time. In this case, one has to incorporate these conditions in the configuration space of the discrete (dynamic) Lagrangian. Namely, we define the configuration space $M_0^{A+1} := \{\boldsymbol{\varphi} \in M^{A+1} \,|\, \varphi_0 = \bar{\varphi}_0, \varphi_A = \bar{\varphi}_A\}$ with prescribed boundary values. This is possible since the boundary conditions at $a = 0$, $a = A$ are assumed to be time independent. The discrete Lagrangian L_d is now defined as $\mathsf{L}_d : M_0^{A+1} \times M_0^{A+1} \to \mathbb{R}$. The discrete Hamilton principle yields Eq. (3.17) (both equations) but written on M_0^{A+1} instead of M^{A+1}. In this case, this leads to the following slight changes in the computations of the derivatives of L_d, namely, we have

$$\langle D_1 \mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \delta\boldsymbol{\varphi}^j \rangle = \sum_{a=1}^{A-1} (D_1 \mathcal{L}_a^j + D_3 \mathcal{L}_{a-1}^j) \cdot \delta\varphi_a^j,$$

$$\langle D_2 \mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \delta\boldsymbol{\varphi}^{j+1} \rangle = \sum_{a=1}^{A-1} D_2 \mathcal{L}_a^j \cdot \delta\varphi_a^{j+1}$$

instead of (3.18). In this case, Eq. (3.17) yield

$$D_1 \mathcal{L}_a^j + D_2 \mathcal{L}_a^{j-1} + D_3 \mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \quad a = 1, \ldots, A-1,$$

$$D_1 \mathcal{L}_a^0 + D_3 \mathcal{L}_{a-1}^0 = 0, \quad D_2 \mathcal{L}_a^{N-1} = 0, \quad a = 1, \ldots, A-1.$$

This is in agreement with Eqs. (3.4) and (3.6) obtained earlier via the covariant discrete variational principle with boundary conditions in space only.

On $M_0^{A+1} \times M_0^{A+1}$ the discrete one-forms $\Theta_{\mathsf{L}_d}^{\pm}$ are

$$\Theta_{\mathsf{L}_d}^{-}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) = -\sum_{a=1}^{A-1} [\Theta_{\mathcal{L}_d}^1(j^1\varphi_d(\triangle_a^j)) + \Theta_{\mathcal{L}_d}^3(j^1\varphi_d(\triangle_a^j))],$$

$$\Theta_{\mathsf{L}_d}^{+}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) = \sum_{a=1}^{A-1} \Theta_{\mathcal{L}_d}^2(j^1\varphi_d(\triangle_a^j)).$$

Note the slight change in the range of summation. Relations (3.21) and (3.22) hold in the same way, with the same change in the summation.

Proposition 3.6. *Boundary conditions in space can be incorporated in the configuration space of the discrete Lagrangian L_d. This yields the modified configuration space $M_0^{A+1} \subset M^{A+1}$. In this case, the DEL equations for L_d on M_0^{A+1} are equivalent to the DCEL for $\mathcal{L}_d$. The discrete covariant variational principle yields, in addition, the discrete zero momentum boundary condition in time.*

Thus, the solution $\varphi_a^j, j = 0, \ldots, N, a = 0, \ldots, A$, of the DCEL equations (3.4) *with zero momentum boundary conditions* (3.6) *provides a symplectic-in-time discrete flow $(\boldsymbol{\varphi}^{j-1}, \boldsymbol{\varphi}^j) \mapsto (\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})$ relative to the discrete symplectic form on*

$M_0^{A+1} \times M_0^{A+1}$, *i.e.*

$$\begin{aligned}\Omega_{\mathsf{L}_d} &= \sum_{a=1}^{A-1} \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) \\ &= -\sum_{a=1}^{A-1} [\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) + \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j))].\end{aligned}$$

Note that the discrete symplectic form is on $M_0^{A+1} \times M_0^{A+1}$, not on $M^{A+1} \times M^{A+1}$.

(C) Boundary conditions in space and time. Of course, one has a similar relation between the DEL equations for L_d and the DCEL equations for $\mathcal{L}_d$ in the case when both spatial and temporal boundary conditions are given. In this case, one has to choose, as before, the discrete configuration space M_0^{A+1} and to consider the DEL equations for L_d in (3.17), without the second boundary conditions. Therefore, the DEL equations for L_d read

$$D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \quad a = 1, \ldots, A-1,$$

and coincide with the DCEL equations (3.4). As before, we have the following result.

Proposition 3.7. *When boundary conditions are imposed in space and time, the DEL equations for L_d on M_0^{A+1} are equivalent to the DCEL equations for $\mathcal{L}_d$.*

Therefore, the solution $\varphi_a^j, j = 0, \ldots, N, a = 0, \ldots, A$, of the DCEL equations (3.4) *provides a symplectic-in-time discrete flow $(\boldsymbol{\varphi}^{j-1}, \boldsymbol{\varphi}^j) \mapsto (\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1})$ relative to the discrete symplectic form on $M_0^{A+1} \times M_0^{A+1}$, i.e.*

$$\begin{aligned}\Omega_{\mathsf{L}_d} &= \sum_{a=1}^{A-1} \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) \\ &= -\sum_{a=1}^{A-1} [\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) + \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j))].\end{aligned}$$

In applications, it is usually assumed that $\boldsymbol{\varphi}^0$ and $\boldsymbol{\varphi}^1$ (i.e. φ_a^0 and φ_a^1 for all $a = 0, \ldots, A$) are prescribed, corresponding to initial configuration and velocity (as opposed to $\boldsymbol{\varphi}^0$ and $\boldsymbol{\varphi}^N$). Our discussion of the prior situations carries over to this case.

Note that the values at the extremities are prescribed and time independent: $\varphi_0^j = \bar{\varphi}_0, \varphi_A^j = \bar{\varphi}_A$ for all $j = 0, \ldots, N$ (or, equivalently, $\boldsymbol{\varphi}^j \in M_0^{A+1} = \{\boldsymbol{\varphi} \in M^{A+1} \,|\, \varphi_0 = \bar{\varphi}_0, \varphi_A = \bar{\varphi}_A\}$).

3.2.2. *Discrete spatial evolution*: *The discrete Lagrangian* N_d

We consider now the converse situation to the one before, that is, we regard the spatial coordinate as the dynamic variable, whereas time is considered as a parameter. Mathematically speaking, this is simply a switching between the s- and t-variables.

The configuration space for the discrete "spatial-evolution" Lagrangian N_d is thus M^{N+1}. Using the notation $\boldsymbol{\varphi}_a := (\varphi_a^0, \ldots, \varphi_a^N) \in M^{N+1}$, the discrete Lagrangian $\mathsf{N}_d : M^{N+1} \times M^{N+1} \to \mathbb{R}$ is defined by

$$\mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) := \sum_{j=0}^{N-1} \mathcal{L}_a^j,$$

so that the discrete action reads

$$\mathfrak{S}_d(\varphi_d) = \sum_{a=0}^{A-1} \mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) = \sum_{j=0}^{N-1} \sum_{a=0}^{A-1} \mathcal{L}_a^j. \tag{3.23}$$

In order to analyze the relation between the discrete Hamilton principles associated to L_d and N_d, we shall first assume that there are no boundary conditions, so that the discrete Hamilton principle for N_d yields the stationarity conditions

$$\begin{aligned} D_1\mathsf{N}_d^a + D_2\mathsf{N}_d^{a-1} = 0, \quad a = 1, \ldots, A-1, \quad \text{and} \\ D_1\mathsf{N}_d^0 = 0, \quad D_2\mathsf{N}_d^{A-1} = 0. \end{aligned} \tag{3.24}$$

We compute

$$\begin{aligned} \langle D_1\mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}), \delta\boldsymbol{\varphi}_a \rangle &= \sum_{j=1}^{N-1} (D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1}) \cdot \delta\varphi_a^j \\ &\quad + D_1\mathcal{L}_a^0 \cdot \delta\varphi_a^0 + D_2\mathcal{L}_a^{N-1} \cdot \delta\varphi_a^N, \\ \langle D_2\mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}), \delta\boldsymbol{\varphi}_{a+1} \rangle &= \sum_{j=1}^{N-1} D_3\mathcal{L}_a^j \cdot \delta\varphi_{a+1}^j + D_3\mathcal{L}_a^0 \cdot \delta\varphi_{a+1}^0. \end{aligned} \tag{3.25}$$

So, the DEL equations for N_d in (3.24), yield

$$\begin{aligned} D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \quad a = 1, \ldots, A-1, \\ D_1\mathcal{L}_a^0 + D_3\mathcal{L}_{a-1}^0 = 0, \quad a = 1, \ldots, A-1, \\ D_2\mathcal{L}_a^{N-1} = 0, \quad a = 1, \ldots, A-1. \end{aligned} \tag{3.26}$$

The boundary conditions in (3.24) imply the equations

$$\begin{aligned} &D_1\mathcal{L}_0^j + D_2\mathcal{L}_0^{j-1} = 0, \quad j = 1, \ldots, N-1, \\ &D_1\mathcal{L}_0^0 = 0, \quad D_2\mathcal{L}_0^{N-1} = 0, \quad \text{and} \quad D_3\mathcal{L}_{A-1}^j = 0, \quad j = 0, \ldots, N-1. \end{aligned} \tag{3.27}$$

So we recover exactly the stationarity conditions obtained from the discrete covariant Hamilton principle (3.2) when no boundary condition is imposed.

Of course, (3.26) and (3.27) agree with the stationarity condition (3.2) obtained from the discrete covariant Hamilton principle when no boundary condition is imposed. Moreover, this computation shows that the DEL equations for N_d (i.e. the first equation in (3.24)) is equivalent to the DCEL equations for $\mathcal{L}_d$ together

with the discrete zero momentum boundary conditions in time (3.6) (the second and third lines in (3.26)).

Remark 3.8 (Discrete Cartan forms). The discrete Cartan one-forms $\Theta^{\pm}_{\mathsf{N}_d}$ on $M^{N+1} \times M^{N+1}$ are computed to be

$$\begin{aligned}
\Theta^{-}_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) &= -D_1 \mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) d\boldsymbol{\varphi}_a \\
&= -\sum_{j=1}^{N-1} (D_1 \mathcal{L}^j_a + D_2 \mathcal{L}^{j-1}_a) d\varphi^j_a + D_1 \mathcal{L}^0_a d\varphi^0_a + D_2 \mathcal{L}^{N-1}_a d\varphi^N_a \\
&= -\sum_{j=0}^{N-1} [\Theta^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + \Theta^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))], \\
\Theta^{+}_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) &= D_2 \mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) d\boldsymbol{\varphi}_{a+1} \\
&= \sum_{j=0}^{N-1} D_3 \mathcal{L}^j_a d\varphi^j_{a+1} \\
&= \sum_{j=0}^{N-1} \Theta^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)).
\end{aligned}$$

The discrete Cartan 2-forms Ω_{N_d} are related to the 2-forms $\Omega_{\mathcal{L}_d}$ by

$$\Omega_{N_d} = \sum_{j=0}^{N-1} \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) = -\sum_{j=0}^{N-1} [\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))].$$

These formulas should be compared with those obtained in Remark 3.4.

(A) Boundary conditions in space. When $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_A$ are prescribed, we only get the first equation in (3.24). These equations are equivalent to the results obtained in (3.4) and (3.6) via the discrete covariant variational principle when only boundary conditions in space have been assumed. The discrete flow map is now given by

$$\begin{aligned}
&F_{\mathsf{N}_d} : M^{N+1} \times M^{N+1} \to M^{N+1} \times M^{N+1}, \\
&F_{N_d}(\boldsymbol{\varphi}_{a-1}, \boldsymbol{\varphi}_a) = (\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1})
\end{aligned}$$

and is symplectic relative to the discrete symplectic form Ω_{N_d}. In complete analogy with Proposition 3.5, we get the following result.

Proposition 3.9. *When boundary conditions are imposed in space, the DEL equations for N_d are equivalent to the DCEL equations for $\mathcal{L}_d$ together with the discrete zero momentum boundary condition in time.*

Thus, the solution $\varphi^j_a, j = 0, \ldots, N, a = 0, \ldots, A$, of the DCEL equations with discrete zero momentum boundary conditions (3.6) *provides a symplectic-in-space*

discrete flow $(\boldsymbol{\varphi}_{a-1}, \boldsymbol{\varphi}_a) \mapsto (\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1})$ *for the discrete symplectic form on* $M^{N+1} \times M^{N+1}$, *i.e.*

$$\Omega_{\mathsf{N}_d} = \sum_{j=0}^{N-1} \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) = -\sum_{j=0}^{N-1} [\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))].$$

(B) Boundary conditions in time. When the discrete configuration is prescribed at temporal boundaries (i.e. at $j = 0$ and $j = N$, for all $a = 0, \ldots, A$), since we are working with the discrete spatial evolution, one has to include them in the discrete configuration space M^{N+1}, that is, we define the discrete configuration space $M_0^{N+1} = \{\boldsymbol{\varphi} \in M^{N+1} \,|\, \varphi^0 := \bar{\varphi}^0, \varphi^N = \bar{\varphi}^N\}$, where $\bar{\varphi}^0$ and $\bar{\varphi}^N$ are given.

This is possible, if these boundary conditions at $t = 0$ and $t = T$ do not depend on the spatial index. The discrete Lagrangian is now defined as $\mathsf{N}_d : M_0^{N+1} \times M_0^{N+1} \to \mathbb{R}$. The discrete Hamilton principle yields both equations in (3.24) on M_0^{N+1}. Using (3.25), with the obvious modifications due to the fact that we work on $M_0^{N+1} \subset M^{N+1}$, we get the equations

$$D_1\mathcal{L}^j_a + D_2\mathcal{L}^{j-1}_a + D_3\mathcal{L}^j_{a-1} = 0, \quad j = 1, \ldots, N-1, \quad a = 1, \ldots, A-1,$$
$$D_1\mathcal{L}^j_0 + D_2\mathcal{L}^{j-1}_0 = 0, \quad D_3\mathcal{L}^j_{A-1} = 0, \quad j = 1, \ldots, N-1.$$

This is in agreement with the results obtained in (3.4) and (3.5) via the discrete covariant variational principle when only boundary conditions in time have been assumed.

The discrete one-forms $\Theta^{\pm}_{\mathsf{N}_d}$ on $M_0^{N+1} \times M_0^{N+1}$ are

$$\Theta^-_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) = -\sum_{j=1}^{N-1} [\Theta^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + \Theta^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))],$$

$$\Theta^+_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) = \sum_{j=1}^{N-1} \Theta^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)),$$

where we note the slight change in the range of summation. We get the following result.

Proposition 3.10. *Boundary conditions in time can be incorporated in the configuration space of the discrete Lagrangian* N_d. *This yields the modified configuration space* $M_0^{N+1} \subset M^{N+1}$. *In this case, the DEL equations for* N_d *on* M_0^{N+1} *are equivalent to the DCEL equations for* $\mathcal{L}_d$. *The discrete variational principles yield, in addition, the discrete zero-traction boundary conditions.*

Therefore, the solution φ^j_a, $j = 0, \ldots, N$, $a = 0, \ldots, A$, *of the DCEL equations* (3.4) *with discrete zero-traction boundary condition* (3.5) *provides a symplectic-in-space discrete flow* $(\boldsymbol{\varphi}_{a-1}, \boldsymbol{\varphi}_a) \mapsto (\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1})$ *relative to the discrete*

symplectic form on $M_0^{N+1} \times M_0^{N+1}$, *i.e.*

$$\Omega_{\mathsf{N}_d} = \sum_{j=1}^{N-1} \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) = -\sum_{j=1}^{N-1} [\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) + \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j))].$$

Note that the discrete symplectic form is on $M_0^{N+1} \times M_0^{N+1}$, not on $M^{N+1} \times M^{N+1}$.

(C) Boundary conditions in time and space. Of course, one has a similar relation between the DEL equations for N_d and the DCEL equations for $\mathcal{L}_d$ in the case when both spatial and temporal boundary conditions are assumed. In this case, one has to choose, as before, the discrete configuration space M_0^{N+1} and to consider the DEL equations for N_d in (3.24), without the second boundary conditions. In this case, the DEL equations for N_d read

$$D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \ a = 1, \ldots, A-1,$$

and coincide with the DCEL equations (3.4). As before, we have the following result.

Proposition 3.11. *When boundary conditions are imposed in space and time, the DEL equations for* N_d *on* M_0^{N+1} *are equivalent to the DCEL equations for* $\mathcal{L}_d$.

Therefore, the solution φ_a^j, $j = 0, \ldots, N$, $a = 0, \ldots, A$, *of the DCEL equations* (3.4) *provides a symplectic-in-space discrete flow* $(\boldsymbol{\varphi}_{a-1}, \boldsymbol{\varphi}_a) \mapsto (\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1})$ *relative to the discrete symplectic form on* $M_0^{N+1} \times M_0^{N+1}$, *i.e.*

$$\Omega_{\mathsf{N}_d} = \sum_{j=1}^{N-1} \Omega^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) = -\sum_{j=1}^{N-1} [\Omega^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j)) + \Omega^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^j))].$$

3.3. *Discrete momentum maps*

Suppose that the discrete covariant Lagrangian density $\mathcal{L}_d : J^1Y_d \to \mathbb{R}$ is invariant under the action of a Lie group G on M. The associated discrete classical Lagrangians $\mathsf{L}_d : M^{A+1} \times M^{A+1} \to \mathbb{R}$ and $\mathsf{N}_d : M^{N+1} \times M^{N+1} \to \mathbb{R}$ associated to the "temporal evolution" and "spatial evolution", respectively, inherit this G-invariance. Indeed, both L_d and N_d are G-invariant under the diagonal action of G on M^{A+1} and M^{N+1}, respectively.

The associated discrete momentum maps are $\mathsf{J}^{\pm}_{\mathsf{L}_d} : M^{A+1} \times M^{A+1} \to \mathfrak{g}^*$ and $\mathsf{J}^{\pm}_{\mathsf{N}_d} : M^{N+1} \times M^{N+1} \to \mathfrak{g}^*$, given by

$$\begin{aligned}
\langle \mathsf{J}^+_{\mathsf{L}_d}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \zeta \rangle &= \langle D_2\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \zeta_{M^{A+1}}(\boldsymbol{\varphi}^{j+1}) \rangle, \\
\langle \mathsf{J}^-_{\mathsf{L}_d}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \zeta \rangle &= \langle -D_1\mathsf{L}_d(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}), \zeta_{M^{A+1}}(\boldsymbol{\varphi}^j) \rangle, \\
\langle \mathsf{J}^+_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}), \zeta \rangle &= \langle D_2\mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}), \zeta_{M^{N+1}}(\boldsymbol{\varphi}_{a+1}) \rangle, \\
\langle \mathsf{J}^-_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}), \zeta \rangle &= \langle -D_1\mathsf{N}_d(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}), \zeta_{M^{N+1}}(\boldsymbol{\varphi}_a) \rangle,
\end{aligned} \tag{3.28}$$

for all $\zeta \in \mathfrak{g}$.

From the definition of L_d and N_d in terms of $\mathcal{L}_d$, we have the relations

$$\mathsf{J}^+_{\mathsf{L}_d}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) = \sum_{a=0}^{A-1} J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)),$$

$$\mathsf{J}^-_{\mathsf{L}_d}(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^{j+1}) = -\sum_{a=0}^{A-1} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))],$$

$$\mathsf{J}^+_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) = \sum_{j=0}^{N-1} J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)),$$

$$\mathsf{J}^-_{\mathsf{N}_d}(\boldsymbol{\varphi}_a, \boldsymbol{\varphi}_{a+1}) = -\sum_{j=0}^{N-1} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a)) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_a))],$$

between the various discrete momentum maps. The G-invariance of $\mathcal{L}_d$ yields (3.13), which consistently implies $\mathsf{J}^+_{\mathsf{L}_d} = \mathsf{J}^-_{\mathsf{L}_d}$ and $\mathsf{J}^+_{\mathsf{N}_d} = \mathsf{J}^-_{\mathsf{N}_d}$.

Covariant versus evolutionary Noether theorem. In the next lemma, we relate the expression $\mathscr{J}^{K,L}_{B,C}$ in (3.14) with the discrete momentum maps $\mathsf{J}^\pm_{\mathsf{L}_d}$ and $\mathsf{J}^\pm_{\mathsf{N}_d}$. This follows from a direct computation.

Lemma 3.12. *When $B = 0$ and $C = A-1$, or $K = 0$ and $L = N-1$, we have, respectively*

$$\mathscr{J}^{K,L}_{0,A-1}(\varphi_d) = \sum_{j=K+1}^{L} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_0) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^{j-1}_0)) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^j_{A-1}))]$$
$$+ \mathsf{J}^+_{\mathsf{L}_d}(\boldsymbol{\varphi}^L, \boldsymbol{\varphi}^{L+1}) - \mathsf{J}^-_{\mathsf{L}_d}(\boldsymbol{\varphi}^K, \boldsymbol{\varphi}^{K+1}) \tag{3.29}$$

$$\mathscr{J}^{0,N-1}_{B,C}(\varphi_d) = \sum_{a=B+1}^{C} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle^0_a) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle^{N-1}_a)) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle^0_{a-1}))]$$
$$+ \mathsf{J}^+_{\mathsf{N}_d}(\boldsymbol{\varphi}_C, \boldsymbol{\varphi}_{C+1}) - \mathsf{J}^-_{\mathsf{N}_d}(\boldsymbol{\varphi}_B, \boldsymbol{\varphi}_{B+1}). \tag{3.30}$$

Recall from Theorem 3.3, that if $\mathcal{L}_d$ is G-invariant and if φ_d satisfies the DCEL equations $D_1\mathcal{L}^j_a + D_2\mathcal{L}^{j-1}_a + D_3\mathcal{L}^j_{a-1} = 0, j = 1, \ldots, N-1, a = 1, \ldots, A-1$, then $\mathscr{J}^{K,L}_{B,C}(\varphi_d) = 0$ for all $0 \le B < C \le A-1, 0 \le K < L \le N-1$. As is apparent from Lemma 3.12, at this point, the discrete covariant Noether theorem $\mathscr{J}^{K,L}_{B,C}(\varphi_d) = 0$ does not imply the discrete Noether theorem for $\mathsf{J}_{\mathsf{L}_d}$ and $\mathsf{J}_{\mathsf{N}_d}$. This is due to the fact that the DEL equations for L_d (or for N_d) imply (but are not equivalent to) the DCEL equations for $\mathcal{L}_d$. To analyze this situation further, we have to take into account the boundary conditions involved.

(A) Boundary condition in time. In this case, the discrete equations are given by the DEL equations $D_1\mathsf{L}^j_d + D_2\mathsf{L}^{j-1}_d = 0, j = 1, \ldots, N-1$. They are equivalent

to the DCEL equations together with the zero-traction boundary conditions

$$D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, \quad j = 1, \ldots, N-1, \quad a = 1, \ldots, A-1,$$
$$D_1\mathcal{L}_0^j + D_2\mathcal{L}_0^{j-1} = 0, \quad D_3\mathcal{L}_{A-1}^j = 0, \quad j = 1, \ldots, N-1.$$

The first of these equations implies $\mathscr{J}_{0,A-1}^{K,L}(\varphi_d) = 0$, while the second and third equations imply that the first term of the right-hand side of (3.29) vanishes. So, we get

$$\mathscr{J}_{0,A-1}^{K,L}(\varphi_d) = \mathsf{J}_{\mathsf{L}_d}(\boldsymbol{\varphi}^L, \boldsymbol{\varphi}^{L+1}) - \mathsf{J}_{\mathsf{L}_d}(\boldsymbol{\varphi}^K, \boldsymbol{\varphi}^{K+1}) = 0,$$

where we used $\mathsf{J}_{\mathsf{L}_d}^+ = \mathsf{J}_{\mathsf{L}_d}^-$ because L_d is G-invariant. This shows that the covariant discrete Noether theorem implies the discrete Noether theorem by choosing the special case $B = 0, C = A-1$.

Recall that, when using the discrete Lagrangian N_d, we have to restrict to the space M_0^{N+1}. The equations above are equivalent to $D_1\mathsf{N}_d^a + D_2\mathsf{N}_d^{a-1} = 0, a = 1, \ldots, A-1, D_1\mathsf{N}_d^0 = 0$, and $D_2\mathsf{N}_d^{A-1} = 0$. Note that in this case, the Noether theorem for the Lagrangian N_d does not apply, since G does not act on M_0^{N+1}. We can, nevertheless, consider the expressions $\mathsf{J}_{\mathsf{N}_d}^{\pm}$. Using Lemma 3.12 and the discrete covariant Noether theorem $\mathscr{J}_{B,C}^{0,N-1}(\varphi_d) = 0$, we see explicitly how the Noether theorem fails for $\mathsf{J}_{\mathsf{N}_d}^{\pm}$, namely,

$$\begin{aligned}
&\mathsf{J}_{\mathsf{N}_d}^+(\boldsymbol{\varphi}_C, \boldsymbol{\varphi}_{C+1}) - \mathsf{J}_{\mathsf{N}_d}^-(\boldsymbol{\varphi}_B, \boldsymbol{\varphi}_{B+1}) \\
&\quad = -\sum_{a=B+1}^{C} [J_{\mathcal{L}_d}^1(j^1\varphi_d(\triangle_a^0) + J_{\mathcal{L}_d}^2(j^1\varphi_d(\triangle_a^{N-1})) + J_{\mathcal{L}_d}^3(j^1\varphi_d(\triangle_{a-1}^0))]. \qquad (3.31)
\end{aligned}$$

(B) Boundary condition in space. The same discussion holds when the configuration is prescribed at the spatial boundary and when zero momentum boundary conditions in time are used, by exchanging the role of L_d and N_d. In this case, we have

$$\mathscr{J}_{B,C}^{0,N-1}(\varphi_d) = \mathsf{J}_{\mathsf{N}_d}^+(\boldsymbol{\varphi}_C, \boldsymbol{\varphi}_{C+1}) - \mathsf{J}_{\mathsf{N}_d}^-(\boldsymbol{\varphi}_B, \boldsymbol{\varphi}_{B+1}) = 0,$$

and $\mathscr{J}_{0,A-1}^{K,L}(\varphi_d) = 0$ implies

$$\begin{aligned}
&\mathsf{J}_{\mathsf{L}_d}^+(\boldsymbol{\varphi}^L, \boldsymbol{\varphi}^{L+1}) - \mathsf{J}_{\mathsf{L}_d}^-(\boldsymbol{\varphi}^K, \boldsymbol{\varphi}^{K+1}) \\
&\quad = -\sum_{j=K+1}^{L} [J_{\mathcal{L}_d}^1(j^1\varphi_d(\triangle_0^j) + J_{\mathcal{L}_d}^2(j^1\varphi_d(\triangle_0^{j-1})) + J_{\mathcal{L}_d}^3(j^1\varphi_d(\triangle_{A-1}^j))]. \qquad (3.32)
\end{aligned}$$

(C) Boundary condition in both space and time. In this case, the equations are given by the DEL equations $D_1\mathsf{L}_d^j + D_2\mathsf{L}_d^{j-1} = 0, j = 1, \ldots, N-1$, or, equivalently, $D_1\mathsf{N}_d^a + D_2\mathsf{N}_d^{a-1} = 0, a = 1, \ldots, A-1$, defined on M_0^{A+1} and M_0^{N+1}, respectively. They are both equivalent to $D_1\mathcal{L}_a^j + D_2\mathcal{L}_a^{j-1} + D_3\mathcal{L}_{a-1}^j = 0, j = 1, \ldots, N-1, a = 1, \ldots, A-1$. In this case, Noether's theorem for the Lagrangians

L_d and N_d does not apply, since G does not act on M_0^{A+1} and M_0^{N+1}. However, the covariant Noether theorem does apply, so that $\mathscr{J}_{B,C}^{K,L}(\varphi_d) = 0$. We can also see directly how the Noether's theorems fail for L_d and N_d, namely

$$\begin{aligned}
&\mathsf{J}^+_{\mathsf{L}_d}(\boldsymbol{\varphi}^L, \boldsymbol{\varphi}^{L+1}) - \mathsf{J}^-_{\mathsf{L}_d}(\boldsymbol{\varphi}^K, \boldsymbol{\varphi}^{K+1}) \\
&\qquad = -\sum_{j=K+1}^{L} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_0^j) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_0^{j-1})) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_{A-1}^j))], \\
&\mathsf{J}^+_{\mathsf{N}_d}(\boldsymbol{\varphi}_C, \boldsymbol{\varphi}_{C+1}) - \mathsf{J}^-_{\mathsf{N}_d}(\boldsymbol{\varphi}_B, \boldsymbol{\varphi}_{B+1}) \\
&\qquad = -\sum_{a=B+1}^{C} [J^1_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^0) + J^2_{\mathcal{L}_d}(j^1\varphi_d(\triangle_a^{N-1})) + J^3_{\mathcal{L}_d}(j^1\varphi_d(\triangle_{a-1}^0))].
\end{aligned}$$

The situation can be summarized as follows.

Theorem 3.13. *Let $\mathcal{L}_d : J^1Y_d \to \mathbb{R}$ be a discrete covariant Lagrangian density and consider the associated discrete Lagrangians $\mathsf{L}_d : M^{A+1} \times M^{A+1} \to \mathbb{R}$ and $\mathsf{N}_d : M^{N+1} \times M^{N+1} \to \mathbb{R}$. Consider a Lie group action of G on M and the associated discrete covariant momentum maps $J^k_{\mathcal{L}_d}, k = 1, 2, 3$, and discrete momentum maps $\mathsf{J}^\pm_{\mathsf{L}_d}, \mathsf{J}^\pm_{\mathsf{N}_d}$. Suppose that the discrete covariant Lagrangian density $\mathcal{L}_d : J^1Y_d \to \mathbb{R}$ is invariant under the Lie group G. While the discrete covariant Noether theorem $\mathscr{J}_{B,C}^{K,L}(\varphi_d) = 0$ (Theorem 3.3) is always satisfied, independently of the imposed boundary conditions, the validity of the discrete Noether theorems for $\mathsf{J}_{\mathsf{L}_d}$ and $\mathsf{J}_{\mathsf{N}_d}$ depends on the boundary conditions.*

If the configuration is prescribed at the temporal extremities and zero-traction boundary conditions are used, then the discrete momentum map $\mathsf{J}_{\mathsf{L}_d}$ is conserved. Conservation of $\mathsf{J}^\pm_{\mathsf{N}_d}$ does not hold in this case, as illustrated by formula (3.31).

If the configuration is prescribed at the spatial extremities and zero momentum boundary conditions are used, then the discrete momentum map $\mathsf{J}_{\mathsf{N}_d}$ is conserved. Conservation of $\mathsf{J}^\pm_{\mathsf{L}_d}$ does not hold in this case, as illustrated by formula (3.32).

4. Multisymplectic Variational Integrators on Lie Groups

In this section, we consider the particular case when the configuration field φ takes values in a Lie group G. Completely analogous to the continuous case treated in Sec. 2.3, the discrete equations also admit a formulation that uses the trivialization of the tangent bundle of the Lie group. The resulting equations present clear advantages in the discrete setting, since one can make use of the vector space structure of the Lie algebra via the use of a time difference map.

We are thus in the particular case when the fiber $M = G$ is a Lie group and we use the notation $\varphi_a^j = g_a^j$. Recall that the discrete version of the first jet bundle $J^1(X \times G)$ is $J^1(X_d \times G) = X_d^\triangle \times G^3$, where $G^3 := G \times G \times G$. Note also that we

have the isomorphism

$$X_d^{\triangle} \times G^3 \ni (\triangle_a^j, g_a^j, g_a^{j+1}, g_{a+1}^j)$$
$$\mapsto (\triangle_a^j, g_a^j, (g_a^j)^{-1} g_a^{j+1}, (g_a^i)^{-1} g_{a+1}^j) \in X_d^{\triangle} \times G^3.$$

In order to discretize the relations $\xi = g^{-1}\dot{g}$ and $\eta = g^{-1}g'$, we shall fix a local diffeomorphism $\tau : \mathfrak{g} \to G$ in a neighborhood of the identity, such that $\tau(0) = e$. Examples for τ are provided by the exponential map or the Cayley transform (see [1, 14]). The approach will involve the right trivialized derivative $\mathrm{d}^R\tau$ of τ defined by

$$\mathrm{d}^R\tau_\xi : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{d}^R\tau_\xi(\eta) := (T_\xi\tau(\eta))\tau(\xi)^{-1}, \tag{4.1}$$

where $T_\xi\tau : \mathfrak{g} \to T_{\tau(\xi)}G$ is the tangent map (derivative) of τ. The right trivialized derivative of τ^{-1} is defined by

$$\mathrm{d}^R\tau_\xi^{-1} : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{d}^R\tau_\xi^{-1}(\eta) := T_g\tau^{-1}(\eta g),$$

where $g := \tau(\xi)$. It is readily checked that $\mathrm{d}^R\tau_\xi^{-1} = (\mathrm{d}^R\tau_\xi)^{-1}$.

Using the local diffeomorphism τ, the relations $\xi = g^{-1}\dot{g}$ and $\eta = g^{-1}g'$ are discretized as

$$\begin{aligned} \xi_a^j &:= \tau^{-1}((g_a^j)^{-1} g_a^{j+1})/\Delta t \in \mathfrak{g}, \\ \eta_a^j &:= \tau^{-1}((g_a^j)^{-1} g_{a+1}^j)/\Delta s \in \mathfrak{g}. \end{aligned} \tag{4.2}$$

From these definitions, we can define the discrete Lagrangian $\bar{\mathcal{L}}_d : X_d^{\triangle} \times G \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$\bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) := \mathcal{L}_d(\triangle_a^j, g_a^j, g_a^{j+1}, g_{a+1}^j).$$

Note that $X_d^{\triangle} \times G \times \mathfrak{g} \times \mathfrak{g}$ is thought of as the discretization of the trivialized first jet bundle $(T^*X \otimes \mathfrak{g}) \times G$, see (2.16), and the discrete Lagrangian $\bar{\mathcal{L}}_d$ is the discretization of the trivialized Lagrangian $\bar{\mathcal{L}} = \bar{\mathcal{L}}(g, \xi, \eta)$ defined in Sec. 2.3.

We have the following relations between the partial derivatives of $\mathcal{L}_d$ and $\bar{\mathcal{L}}_d$:

$$\begin{aligned} (g_a^j)^{-1} D_1\mathcal{L}_a^j &= (g_a^j)^{-1} D_g\bar{\mathcal{L}}_a^j - \frac{1}{\Delta t}\mu_a^j - \frac{1}{\Delta s}\lambda_a^j, \\ (g_a^{j+1})^{-1} D_2\mathcal{L}_a^j &= \frac{1}{\Delta t}\,\mathrm{Ad}^*_{\tau(\Delta t\xi_a^j)}\mu_a^j, \\ (g_{a+1}^j)^{-1} D_3\mathcal{L}_a^j &= \frac{1}{\Delta s}\,\mathrm{Ad}^*_{\tau(\Delta s\eta_a^j)}\lambda_a^j. \end{aligned} \tag{4.3}$$

Discrete covariant Euler–Lagrange equations on Lie groups. The discrete covariant Hamilton principle reads

$$\delta\bar{\mathfrak{S}}_d(g_d) = \delta \sum_{j=0}^{N-1} \sum_{a=0}^{A-1} \bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) = 0. \tag{4.4}$$

Using the definitions (4.2), we obtain the variations

$$\begin{aligned}\delta\xi_a^j &= \mathrm{d}^R\tau^{-1}_{\Delta t\xi_a^j}(-\zeta_a^j + \mathrm{Ad}_{\tau(\Delta t\xi_a^j)}\zeta_a^{j+1})/\Delta t,\\ \delta\eta_a^j &= \mathrm{d}^R\tau^{-1}_{\Delta s\eta_a^j}(-\zeta_a^j + \mathrm{Ad}_{\tau(\Delta s\eta_a^j)}\zeta_{a+1}^j)/\Delta s,\end{aligned} \tag{4.5}$$

where we defined $\zeta_a^j := (g_a^j)^{-1}\delta g_a^j$ and we used (4.1).

For simplicity, we will use the notation $\bar{\mathcal{L}}_a^j := \bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j)$. Defining the discrete momenta

$$\mu_a^j := (\mathrm{d}^R\tau^{-1}_{\Delta t\xi_a^j})^* D_\xi\bar{\mathcal{L}}_a^j, \quad \lambda_a^j := (\mathrm{d}^R\tau^{-1}_{\Delta s\eta_a^j})^* D_\eta\bar{\mathcal{L}}_a^j$$

and applying the covariant discrete Hamilton principle we get

$$\begin{aligned}\delta\bar{\mathfrak{S}}_d(g_d) &= \sum_{j=0}^{N-1}\sum_{a=0}^{A-1}(D_g\bar{\mathcal{L}}_a^j\cdot\delta g_a^j + D_\xi\bar{\mathcal{L}}_a^j\cdot\delta\xi_a^j + D_\eta\bar{\mathcal{L}}_a^j\cdot\delta\eta_a^j)\\
&= \sum_{j=0}^{N-1}\sum_{a=0}^{A-1}\left((g_a^j)^{-1}D_g\bar{\mathcal{L}}_a^j - \frac{1}{\Delta t}\mu_a^j - \frac{1}{\Delta s}\lambda_a^j\right)\cdot\zeta_a^j\\
&\quad + \frac{1}{\Delta t}\,\mathrm{Ad}^*_{\tau(\Delta t\xi_a^j)}\mu_a^j\cdot\zeta_a^{j+1} + \frac{1}{\Delta s}\,\mathrm{Ad}^*_{\tau(\Delta s\eta_a^j)}\lambda_a^j\cdot\zeta_{a+1}^j\\
&= \sum_{j=1}^{N-1}\sum_{a=1}^{A-1}\Bigg((g_a^j)^{-1}D_g\bar{\mathcal{L}}_a^j + \frac{1}{\Delta t}(\mathrm{Ad}^*_{\tau(\Delta t\xi_a^{j-1})}\mu_a^{j-1} - \mu_a^j)\\
&\quad + \frac{1}{\Delta s}(\mathrm{Ad}^*_{\tau(\Delta s\eta_{a-1}^j)}\lambda_{a-1}^j - \lambda_a^j)\Bigg)\cdot\zeta_a^j\\
&\quad + \sum_{j=1}^{N-1}\Bigg[\left((g_0^j)^{-1}D_g\bar{\mathcal{L}}_0^j + \frac{1}{\Delta t}(\mathrm{Ad}^*_{\tau(\Delta t\xi_0^{j-1})}\mu_0^{j-1} - \mu_0^j) - \frac{1}{\Delta s}\lambda_0^j\right)\cdot\zeta_0^j\\
&\quad + \frac{1}{\Delta s}\,\mathrm{Ad}^*_{\tau(\Delta s\eta_{A-1}^j)}\lambda_{A-1}^j\cdot\zeta_A^j\Bigg]\\
&\quad + \sum_{a=1}^{A-1}\Bigg[\left((g_a^0)^{-1}D_g\bar{\mathcal{L}}_a^0 - \frac{1}{\Delta t}\mu_a^0 + \frac{1}{\Delta s}(\mathrm{Ad}^*_{\tau(\Delta s\eta_{a-1}^0)}\lambda_{a-1}^0 - \lambda_a^0)\right)\cdot\zeta_a^0\\
&\quad + \frac{1}{\Delta t}\,\mathrm{Ad}^*_{\tau(\Delta t\xi_a^{N-1})}\mu_a^{N-1}\cdot\zeta_a^N\Bigg]\\
&\quad + \left((g_0^0)^{-1}D_g\bar{\mathcal{L}}_0^0 - \frac{1}{\Delta t}\mu_0^0 - \frac{1}{\Delta s}\lambda_0^0\right)\cdot\zeta_0^0 + \left(\frac{1}{\Delta t}\,\mathrm{Ad}^*_{\tau(\Delta t\xi_0^{N-1})}\mu_0^{N-1}\right)\cdot\zeta_0^N\\
&\quad + \left(\frac{1}{\Delta s}\,\mathrm{Ad}^*_{\tau(\Delta s\eta_{A-1}^0)}\lambda_{A-1}^0\right)\cdot\zeta_A^0.\end{aligned} \tag{4.6}$$

This can be also obtained directly from (3.2) by using (4.3).

For later use, we now list the stationarity conditions obtained if there are no boundary conditions imposed on the variations

$$\frac{1}{\Delta t}(\mu_a^j - \operatorname{Ad}^*_{\tau(\Delta t\xi_a^{j-1})}\mu_a^{j-1}) + \frac{1}{\Delta s}(\lambda_a^j - \operatorname{Ad}^*_{\tau(\Delta s\eta_{a-1}^j)}\lambda_{a-1}^j) = (g_a^j)^{-1}D_g\bar{\mathcal{L}}_a^j,$$

$$\text{for all } j = 1,\dots,N-1,\ a = 1,\dots,A-1, \tag{4.7}$$

$$\frac{1}{\Delta t}(\mu_0^j - \operatorname{Ad}^*_{\tau(\Delta t\xi_0^{j-1})}\mu_0^{j-1}) + \frac{1}{\Delta s}\lambda_0^j = (g_0^j)^{-1}D_g\bar{\mathcal{L}}_0^j, \quad \text{and}$$

$$\frac{1}{\Delta s}\operatorname{Ad}^*_{\tau(\Delta s\eta_{A-1}^j)}\lambda_{A-1}^j = 0, \quad \text{for all } j = 1,\dots,N-1, \tag{4.8}$$

$$\frac{1}{\Delta t}\mu_a^0 + \frac{1}{\Delta s}(\lambda_a^0 - \operatorname{Ad}^*_{\tau(\Delta s\eta_{a-1}^0)}\lambda_{a-1}^0) = (g_a^0)^{-1}D_g\bar{\mathcal{L}}_a^0, \quad \text{and}$$

$$\frac{1}{\Delta t}\operatorname{Ad}^*_{\tau(\Delta t\xi_a^{N-1})}\mu_a^{N-1} = 0, \quad \text{for all } a = 1,\dots,A-1, \tag{4.9}$$

$$\frac{1}{\Delta t}\mu_0^0 + \frac{1}{\Delta s}\lambda_0^0 = (g_0^0)^{-1}D_g\bar{\mathcal{L}}_0^0, \quad \frac{1}{\Delta t}\operatorname{Ad}^*_{\tau(\Delta t\xi_0^{N-1})}\mu_0^{N-1} = 0 \quad \text{and}$$

$$\frac{1}{\Delta s}\operatorname{Ad}^*_{\tau(\Delta s\eta_{A-1}^0)}\lambda_{A-1}^0 = 0. \tag{4.10}$$

Equations (4.7) will be referred to as the *Lie group DCEL equations*.

Remark 4.1 (Discrete Cartan forms). In terms of the trivialized discrete Lagrangian $\bar{\mathcal{L}}_d$, the discrete Cartan forms (3.7) are computed to be

$$\begin{aligned}
\theta^1_{\bar{\mathcal{L}}_d}(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= \left\langle (g_a^j)^{-1}D_g\bar{\mathcal{L}}_a^j - \frac{1}{\Delta t}\mu_a^j - \frac{1}{\Delta s}\lambda_a^j, (g_a^j)^{-1}dg_a^j \right\rangle, \\
\theta^2_{\bar{\mathcal{L}}_d}(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= \left\langle \frac{1}{\Delta t}\operatorname{Ad}^*_{\tau(\Delta t\xi_a^j)}\mu_a^j, (g_a^{j+1})^{-1}dg_a^{j+1} \right\rangle \\
&= \left\langle \frac{1}{\Delta t}g_a^j\mu_a^j, dg_a^j \right\rangle + \left\langle D_\xi\bar{\mathcal{L}}_a^j, d\xi_a^j \right\rangle, \qquad (4.11)\\
\theta^3_{\bar{\mathcal{L}}_d}(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= \left\langle \frac{1}{\Delta s}\operatorname{Ad}^*_{\tau(\Delta s\eta_a^j)}\lambda_a^j, (g_{a+1}^j)^{-1}dg_{a+1}^j \right\rangle \\
&= \left\langle \frac{1}{\Delta t}g_a^j\lambda_a^j, dg_a^j \right\rangle + \langle D_\eta\bar{\mathcal{L}}_a^j, d\eta_a^j \rangle.
\end{aligned}$$

We note the relation

$$\theta^k_{\bar{\mathcal{L}}_d} = (\phi_\tau)_*\Theta^k_{\mathcal{L}_d}, \quad k = 1,2,3, \tag{4.12}$$

where $\phi_\tau : X_d^\triangle \times G \times G \times G \to X_d^\triangle \times G \times \mathfrak{g} \times \mathfrak{g}$ is the local diffeomorphism defined by $\phi_\tau(\triangle_a^i, g_a^i, g_a^{i+1}, g_{a+1}^i) = (\triangle_a^i, g_a^i, \xi_a^i, \zeta_a^i)$ and $\Theta^k_{\mathcal{L}_d}$ are the discrete one-forms defined in (3.7). From the relations (4.12), $\bar{\mathcal{L}}_d \circ \phi_\tau = \mathcal{L}_d$, and the formula (3.8), we get

$$\mathbf{d}\bar{\mathcal{L}}_d = \theta^1_{\bar{\mathcal{L}}_d} + \theta^2_{\bar{\mathcal{L}}_d} + \theta^3_{\bar{\mathcal{L}}_d}. \tag{4.13}$$

Given a vector field V on $X_d^{\triangle} \times G$ and its first jet extension j^1V on the discrete jet bundle $X_d^{\triangle} \times G \times G \times G$, we define the vector field $\overline{j^1V}$ induced on $X_d^{\triangle} \times G \times \mathfrak{g} \times \mathfrak{g}$ by ϕ_τ. Similarly, given a discrete section g_d and its first jet extension $j^1 g_d : X_d^{\triangle} \to X_d^{\triangle} \times G \times G \times G$ we define $\overline{j^1 g_d} := \phi_\tau \circ j^1 g_d$. With these notations, we can write the formulas

$$\left\langle (g_a^j)^{-1} D_g \bar{\mathcal{L}}_a^j - \frac{1}{\Delta t}\mu_a^j - \frac{1}{\Delta s}\lambda_a^j, (g_a^j)^{-1}\delta g_a^j \right\rangle = [(\overline{j^1 g_d})^* \mathbf{i}_{\overline{j^1 V}} \theta^1_{\bar{\mathcal{L}}_d}](\triangle_a^j),$$

$$\left\langle \frac{1}{\Delta t} \operatorname{Ad}^*_{\tau(\Delta t \xi_a^j)} \mu_a^j, (g_a^{j+1})^{-1}\delta g_a^{j+1} \right\rangle = [(\overline{j^1 g_d})^* \mathbf{i}_{\overline{j^1 V}} \theta^2_{\bar{\mathcal{L}}_d}](\triangle_a^j),$$

$$\left\langle \frac{1}{\Delta s} \operatorname{Ad}^*_{\tau(\Delta s \eta_a^j)} \lambda_a^j, (g_{a+1}^j)^{-1}\delta g_{a+1}^j \right\rangle = [(\overline{j^1 g_d})^* \mathbf{i}_{\overline{j^1 V}} \theta^3_{\bar{\mathcal{L}}_d}](\triangle_a^j),$$

from which we deduce, as in (3.9), that (4.6) can be written as

$$\begin{aligned}
\delta \bar{\mathfrak{S}}_d(g_d) = \sum_{j=1}^{N-1} \sum_{a=1}^{A-1} \Bigg\langle & (g_a^j)^{-1} D_g \bar{\mathcal{L}}_a^j + \frac{1}{\Delta t}(\operatorname{Ad}^*_{\tau(\Delta t \xi_a^{j-1})} \mu_a^{j-1} - \mu_a^j) \\
& + \frac{1}{\Delta s}(\operatorname{Ad}^*_{\tau(\Delta s \eta_{a-1}^j)} \lambda_{a-1}^j - \lambda_a^j), (g_a^j)^{-1}\delta g_a^j \Bigg\rangle \\
& + \sum_{\{\triangle \in X_d^{\triangle} \mid \triangle \cap \partial U \neq \varnothing\}} \left(\sum_{k \in \{1,2,3\}; \triangle^{(k)} \in \partial U} [(\overline{j^1 g_d})^* (\mathbf{i}_{\overline{j^1 V}} \theta^k_{\bar{\mathcal{L}}_d})](\triangle) \right).
\end{aligned} \tag{4.14}$$

(A) Spacetime boundary conditions. When the values of the discrete configuration g_a^j are prescribed at the spacetime boundary, then the covariant Hamilton principle only yields the Lie group DCEL equations (4.7).

(B) Boundary conditions in time. When the values of the discrete configuration g_a^j are prescribed for $j = 0$ and $j = N$, then the covariant Hamilton principle yields the Lie group DCEL equations (4.7) together with the *discrete zero-traction boundary conditions* (4.8).

(C) Boundary conditions in space. When the values of the discrete configuration g_a^j are prescribed for $a = 0$ and $a = A$, then the covariant Hamilton principle yields the Lie group DCEL equations (4.7) together with the *discrete zero momentum boundary conditions* (4.9).

Discrete Legendre transforms. The discrete covariant Legendre transforms have been defined in (3.11). Their expressions, $\mathbb{F}^k \bar{\mathcal{L}}_d : X_d^{\triangle} \times G \times \mathfrak{g} \times \mathfrak{g} \to G \times \mathfrak{g}^*, k = 1, 2, 3,$

in terms of the discrete trivialized Lagrangian $\bar{\mathcal{L}}_d$, are given by

$$\begin{aligned}
\mathbb{F}^1\bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= \left(g_a^j, (g_a^j)^{-1}D_g\bar{\mathcal{L}}_a^j - \frac{1}{\Delta t}\mu_a^j - \frac{1}{\Delta s}\lambda_a^j\right),\\
\mathbb{F}^2\bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= \left(g_a^{j+1}, \frac{1}{\Delta t}\operatorname{Ad}^*_{\tau(\Delta t\xi_a^j)}\mu_a^j\right),\\
\mathbb{F}^3\bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= \left(g_{a+1}^j, \frac{1}{\Delta s}\operatorname{Ad}^*_{\tau(\Delta s\eta_a^j)}\lambda_a^j\right).
\end{aligned} \tag{4.15}$$

Note that the formulas (4.15) are related to the discrete Legendre transforms $\mathbb{F}^k\mathcal{L}_d, k = 1, 2, 3$, via the identity

$$\rho_L \circ \mathbb{F}^k\mathcal{L}_d = \mathbb{F}^k\bar{\mathcal{L}}_d \circ \phi_\tau,$$

where $\rho_L : T^*G \to G \times \mathfrak{g}^*$, $\rho_L(\alpha_g) := (g, g^{-1}\alpha_g)$. Also, as before, the DCEL equations (4.7) can be written as

$$\begin{aligned}
&\mathbb{F}^1\bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) + \mathbb{F}^2\bar{\mathcal{L}}_d(\triangle_a^j, g_a^{j-1}, \xi_a^{j-1}, \eta_a^{j-1})\\
&\quad + \mathbb{F}^3\bar{\mathcal{L}}_d(\triangle_a^j, g_{a-1}^j, \xi_{a-1}^j, \eta_{a-1}^j) = 0.
\end{aligned}$$

Discrete momentum maps. We now consider symmetries given by a subgroup H of the Lie group fiber G. We assume that H acts on the left by translation, i.e. $\Phi : H \times G \to G, \Phi_h(g) = hg$. The infinitesimal generator associated to $\zeta \in \mathfrak{h}$ is $\zeta_G(g) = \zeta g$. Using the formulas (3.12), adapted to this special case and written in terms of the trivialized discrete Lagrangian $\bar{\mathcal{L}}_d$, we get the discrete momentum maps $J^k_{\bar{\mathcal{L}}_d} : X_d^\triangle \times G \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}^*, k = 1, 2, 3$,

$$\begin{aligned}
J^1_{\bar{\mathcal{L}}_d}(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= i^*\operatorname{Ad}^*_{(g_a^j)^{-1}}\left((g_a^j)^{-1}D_g\bar{\mathcal{L}}_a^j - \frac{1}{\Delta t}\mu_a^j - \frac{1}{\Delta s}\lambda_a^j\right),\\
J^2_{\bar{\mathcal{L}}_d}(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= i^*\frac{1}{\Delta t}\operatorname{Ad}^*_{(g_a^j)^{-1}}\mu_a^j,\\
J^3_{\bar{\mathcal{L}}_d}(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) &= i^*\frac{1}{\Delta s}\operatorname{Ad}^*_{(g_a^j)^{-1}}\lambda_a^j,
\end{aligned} \tag{4.16}$$

where $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ denotes the dual map to the Lie algebra inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. We have the formula

$$\langle J^k_{\bar{\mathcal{L}}_d}, \xi\rangle = \mathbf{i}_{\overline{\xi}_{J^1(X_d\times G)}}\theta^k_{\bar{\mathcal{L}}_d}, \quad \xi \in \mathfrak{h} \quad \text{and} \quad J^k_{\bar{\mathcal{L}}_d} \circ \phi_\tau = J^k_{\mathcal{L}_d}, \tag{4.17}$$

where $\xi_{J^1(X_d\times G)}$ is the infinitesimal generator of the H-action induced on $J^1Y_d = X_d^\triangle \times G \times G \times G$ and $\overline{\xi}_{J^1(X_d\times G)}$ is the corresponding vector field induced on $X_d^\triangle \times G \times \mathfrak{g} \times \mathfrak{g}$.

Let us assume that $\bar{\mathcal{L}}_d$ is H-invariant, i.e. $\bar{\mathcal{L}}_d(\triangle_a^j, hg_a^j, \xi_a^j, \eta_a^j) = \bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j)$, for all $h \in H$. This implies the infinitesimal H-invariance $\langle D_g\bar{\mathcal{L}}_a^j, \zeta g_a^j\rangle = 0$, for all $\zeta \in \mathfrak{h}$, i.e. $i^*D_g\bar{\mathcal{L}}_a^j(g_a^j)^{-1} = 0$. Using formulas (4.16), this can be equivalently

written as

$$(J^1_{\bar{\mathcal{L}}_d} + J^2_{\bar{\mathcal{L}}_d} + J^3_{\bar{\mathcal{L}}_d})(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a) = 0,$$

which is the statement of the local discrete Noether theorem. We now state its global version.

Theorem 4.2. *Suppose that the discrete Lagrangian* $\bar{\mathcal{L}}_d : X^{\triangle}_d \times G \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *is invariant under the left action of the Lie group* H *on* G*. Suppose that* g_d *is a solution of the DCEL equations for* $\bar{\mathcal{L}}_d$*. Then, for all* $0 \le B < C \le A-1, 0 \le K < L \le N-1$*, we have the conservation law*

$$\bar{\mathscr{J}}^{K,L}_{B,C}(g_d) = 0, \tag{4.18}$$

where $\bar{\mathscr{J}}^{K,L}_{B,C}$ *is given by* (3.15)*, with* $J^k_{\mathcal{L}_d}(j^1\varphi_d(\triangle^K_a))$ *replaced by* $J^k_{\bar{\mathcal{L}}_d}(j^1 g_d(\triangle^K_a))$*.*

Proof. From the H-invariance of $\bar{\mathcal{L}}_d$ we have $\mathfrak{S}_d(h \cdot g_d) = \mathfrak{S}_d(g_d)$, so, the derivative of this expression with respect to h vanishes. Using this in (4.14), together with the fact that g_d is a solution of the DCEL equations, we get

$$0 = \sum_{\{\triangle \in X^{\triangle}_d \,|\, \triangle \cap \partial X_d \neq \varnothing\}} \left(\sum_{k \in \{1,2,3\}; \triangle^{(k)} \in \partial X_d} [(\overline{j^1 g_d})^* (\mathbf{i}_{\bar{\xi}_{J^1(X_d \times G)}} \theta^k_{\bar{\mathcal{L}}_d})](\triangle) \right),$$

for all $\xi \in \mathfrak{h}$. More generally, this can be done for a rectangular subdomain U as in Theorem 3.3. The global Noether theorem follows from the first formula in (4.17). $\square$

The G-invariant case and discrete covariant Euler–Poincaré equations. If the given Lagrangian density $\mathcal{L} : J^1(X \times G) \to \mathbb{R}$ is G-invariant, it induces the function $\ell = \ell(\xi, \eta) : L(TX, \mathfrak{g}) \to \mathbb{R}$, as recalled in Sec. 2.3. The CEL equations for $\mathcal{L}$ are equivalent to the covariant Euler–Poincaré equations for ℓ.

In the case of a G-invariant Lagrangian, we shall choose a discrete Lagrangian $\mathcal{L}_d$ that inherits the same invariance. Consider the left action of G on itself by left translation. This action naturally lifts to $J^1 Y_d = X^{\triangle}_d \times G \times G \times G$. Then the discrete covariant Lagrangian $\mathcal{L}_d : J^1 Y_d \to \mathbb{R}$ is G-invariant if and only if its trivialized expression $\bar{\mathcal{L}}_d(\triangle^j_a, g^j_a, \xi^j_a, \eta^j_a)$, defined through a local diffeomorphism $\tau : \mathfrak{g} \to G$, does not depend on g^j_a. We thus obtain a discrete reduced Lagrangian

$$\ell_d(\triangle^j_a, \xi^j_a, \eta^j_a) : X^{\triangle}_d \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

that approximates the reduced Lagrangian ℓ, namely,

$$\ell_d(\triangle^j_a, \xi^j_a, \eta^j_a) \simeq \int_{\square^j_a} \ell(t, s, \xi(t,s), \eta(t,s))\, \mathrm{d}s\, \mathrm{d}t.$$

The stationarity conditions are computed exactly as in (4.6) and are obtained by setting $D_g \bar{\mathcal{L}} = 0$ in (4.7)–(4.10).

For example, if there are only temporal boundary conditions, we get the equations

$$\frac{1}{\Delta t}(\mu_a^j - \mathrm{Ad}^*_{\tau(\Delta t \xi_a^{j-1})}\mu_a^{j-1}) + \frac{1}{\Delta s}(\lambda_a^j - \mathrm{Ad}^*_{\tau(\Delta s \eta_{a-1}^j)}\lambda_{a-1}^j) = 0, \tag{4.19}$$

for all $j = 1, \ldots, N-1, a = 1, \ldots, A-1$, with the natural (zero-traction-like) boundary conditions

$$\begin{aligned} &\frac{1}{\Delta t}(\mu_0^j - \mathrm{Ad}^*_{\tau(\Delta t \xi_0^{j-1})}\mu_0^{j-1}) + \frac{1}{\Delta s}\lambda_0^j = 0, \quad \text{and} \\ &\frac{1}{\Delta s}\,\mathrm{Ad}^*_{\tau(\Delta s \eta_{A-1}^j)}\lambda_{A-1}^j = 0, \end{aligned} \tag{4.20}$$

for all $j = 1, \ldots, N-1$. Equations (4.19) are called the *discrete covariant Euler–Poincaré equations.*

Time and space discrete evolutions. As in Sec. 3.2, given the discrete covariant Lagrangian $\mathcal{L}_d(\triangle_a^j, g_a^j, g_a^{j+1}, g_{a+1}^j)$, we can define the discrete Lagrangians $\mathsf{L}_d = \mathsf{L}_d(\mathbf{g}^j, \mathbf{g}^{j+1}) : G^{A+1} \times G^{A+1} \to \mathbb{R}$ and $\mathsf{N}_d = \mathsf{N}_d(\mathbf{g}_a, \mathbf{g}_{a+1}) : G^{N+1} \times G^{N+1} \to \mathbb{R}$ associated to the temporal and spatial discrete evolutions, respectively, and where $\mathbf{g}^j = (g_0^j, \ldots, g_A^j)$ and $\mathbf{g}_a = (g_a^0, \ldots, g_a^N)$. If we choose a local diffeomorphism $\tau : \mathfrak{g} \to G$ defined on an open neighborhood of the origin in $\mathfrak{g}$ onto an open neighborhood of the identity in G such that $\tau(0) = e$, then, to L_d and N_d are naturally associated the discrete Lagrangians $\bar{\mathsf{L}}_d : G^{A+1} \times \mathfrak{g}^{A+1} \to \mathbb{R}$ and $\bar{\mathsf{N}}_d : G^{N+1} \times \mathfrak{g}^{N+1} \to \mathbb{R}$ defined, respectively, by

$$\bar{\mathsf{L}}_d(\mathbf{g}^j, \boldsymbol{\xi}^j) := \mathsf{L}_d(\mathbf{g}^j, \mathbf{g}^{j+1}) \quad \text{and} \quad \bar{\mathsf{N}}_d(\mathbf{g}_a, \boldsymbol{\eta}_a) := \mathsf{N}_d(\mathbf{g}_a, \mathbf{g}_{a+1}),$$

where $\boldsymbol{\xi}^j := \frac{1}{\Delta t}\tau^{-1}((\mathbf{g}^j)^{-1}\mathbf{g}^{j+1}) \in \mathfrak{g}^{A+1}$ and $\boldsymbol{\eta}_a := \frac{1}{\Delta s}\tau^{-1}((\mathbf{g}_a)^{-1}\mathbf{g}_{a+1}) \in \mathfrak{g}^{N+1}$. These discrete Lagrangians are related to the trivialized covariant Lagrangian $\bar{\mathcal{L}}_d$ by the formulas

$$\bar{\mathsf{L}}_d(\mathbf{g}^j, \boldsymbol{\xi}^j) = \sum_{a=0}^{A-1} \bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j) \quad \text{and}$$

$$\bar{\mathsf{N}}_d(\mathbf{g}_a, \boldsymbol{\eta}_a) = \sum_{j=0}^{N-1} \bar{\mathcal{L}}_d(\triangle_a^j, g_a^j, \xi_a^j, \eta_a^j).$$

One can now carry out the study of the symplectic properties of the time and space discrete evolutions for various boundary conditions, by following the results obtained in Secs. 3.2 and 3.3. Taking advantage of the Lie group structure, all the results can be formulated in terms of the trivialized discrete Lagrangians $\bar{\mathcal{L}}_d, \bar{\mathsf{L}}_d$, and $\bar{\mathsf{N}}_d$. In presence of H-symmetries, the expression of the discrete Lagrangian momentum maps $\mathsf{J}^{\pm}_{\bar{\mathsf{L}}_d} : G^{A+1} \times \mathfrak{g}^{A+1} \to \mathfrak{h}^*$ and $\mathsf{J}^{\pm}_{\bar{\mathsf{N}}_d} : G^{N+1} \times \mathfrak{g}^{N+1} \to \mathfrak{h}^*$ are easily

obtained from the expressions of $\mathsf{J}^{\pm}_{\mathsf{L}_d} : G^{A+1} \times G^{A+1} \to \mathfrak{h}^*$ and $\mathsf{J}^{\pm}_{\mathsf{N}_d} : G^{N+1} \times G^{N+1} \to \mathfrak{h}^*$; see (3.28).

Remark 4.3 (Covariant versus dynamic reduction). Note that in the G-invariant case, the discrete covariant Euler–Poincaré equations for $\ell_d(\triangle^j_a, \xi^j_a, \eta^j_a)$ can be interpreted as a symmetry reduced version of the DEL equations for the discrete Lagrangian $\mathsf{L}_d(\mathbf{g}^j, \mathbf{g}^{j+1})$ associated to time evolution. To a discrete spacetime covariant reduction for $\mathcal{L}_d$ thus corresponds a discrete dynamic reduction for L_d. The same comment applies to N_d. It would be interesting to analyze the link between this observation and the approach carried out in [10] which relates, in the continuous case, the covariant and dynamic reductions in principal bundle field theories.

5. Numerical Example: The Three-Dimensional Geometrically Exact Beam Model

In this section, we illustrate the results obtained in this paper with the example of a geometrically exact beam (see [23–25]). The problem considered here should be regarded only as an illustration of our methods. We devoted [4] to the simulation of considerably more realistic situations.

We will take advantage of the multisymplectic character of the integrator to simulate the motion of the beam knowing the time evolution of position and the strain of one of the extremities. This unusual boundary value problem can be treated simply by our integrator for which time and space are discretized in the same way. By switching the space and time variables, this boundary problem reduces to a standard boundary problem with given position and velocity at initial time.

5.1. *Time and space evolutions of the beam*

In geometrically exact models, the instantaneous configuration of a beam is described by its line of centroids as a map $\mathbf{r} : [0, L] \to \mathbb{R}^3$ and the orientation of all its cross-sections at points $\mathbf{r}(s)$, where $s \in [0, L]$, by a moving orthonormal basis $\{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$. The attitude of this moving basis is described by a map $\Lambda : [0, L] \to \mathrm{SO}(3)$ satisfying $\mathbf{d}_I(s) = \Lambda(s)\mathbf{E}_I$, $I = 1, 2, 3$, where $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a fixed orthonormal basis (Fig. 2).

Covariant formulation. The convective covariant formulation of geometrically exact beams has been developed in [6, §6–7]. In this approach, the maps $\Lambda, \mathbf{r}$ are interpreted as spacetime-dependent fields

$$\mathbb{R} \times [0, L] \ni (t, s) \mapsto g(t, s) := (\Lambda(t, s), \mathbf{r}(t, s)) \in G = \mathrm{SE}(3),$$

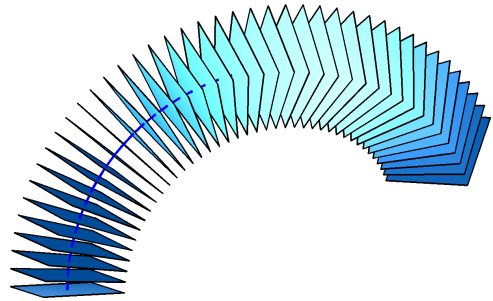

Fig. 2. The geometrically exact model of the beam is defined by the position $\mathbf{r}(t,s) \in \mathbb{R}^3$ of the line of centroids and by the orientation $\Lambda(t,s) \in \mathrm{SO}(3)$ of the cross-sections. This figure illustrates the discretized beam at a given time t^j giving all cross-sections at the spatial points s_a.

taking values in the special Euclidean group. The fiber bundle of the problem is therefore $X \times G \to X$, with $X = \mathbb{R} \times [0, L] \ni (t, s)$ and this formulation fits into the framework of Sec. 2.3. The convected variables $\xi(t,s) = g(t,s)^{-1}\partial_t g(t,s)$ and $\eta(t,s) = g(t,s)^{-1}\partial_s g(t,s)$ are here given by the convected angular and linear velocities and strains, i.e.

$$\xi = g^{-1}\partial_t g = (\Lambda^{-1}\dot{\Lambda}, \Lambda^{-1}\dot{\mathbf{r}}) = (\omega, \boldsymbol{\gamma}),$$

$$\eta = g^{-1}\partial_s g = (\Lambda^{-1}\Lambda', \Lambda^{-1}\mathbf{r}') = (\Omega, \boldsymbol{\Gamma}).$$

The Lagrangian density of geometrically exact beams reads

$$\begin{aligned} \mathfrak{L}(g,\xi,\eta) &= \frac{1}{2}\langle \mathbb{J}\xi, \xi\rangle - \frac{1}{2}\langle \mathbb{C}(\eta - \mathbf{E}_6), (\eta - \mathbf{E}_6)\rangle - \Pi(g) \\ &=: K(\xi) - \Phi(\eta) - \Pi(g), \end{aligned} \tag{5.1}$$

where $K(\xi), \Phi(\eta)$, and $\Pi(g)$ are, respectively, the kinetic energy density, the strain energy density, and the external (such as gravitational) potential energy density; in the right-hand side of (5.1), ξ, η are regarded as elements of $\mathbb{R}^6$. In (5.1), $\mathbb{J}$ is a 6×6 diagonal matrix whose diagonal elements are composed of the principal moments of inertia and the mass of the cross-section; the linear strain tensor $\mathbb{C}$ is a 6×6 diagonal matrix, whose diagonal elements depend on the cross-sectional area, the principal moments of inertia of the cross-sections, Young's modulus, and the Poisson's ratio; $\mathbf{E}_6 = (0,0,0,0,0,1) \in \mathbb{R}^6$. Both $\mathbb{J}$ and $\mathbb{C}$ are assumed to be independent of (t,s).

The covariant Hamilton principle reads

$$\delta \int_0^T \int_0^L \left(K(\xi) - \Phi(\eta) - \Pi(g)\right) \, \mathrm{d}s \, \mathrm{d}t = 0, \tag{5.2}$$

for arbitrary variations δg of g, vanishing at the boundary. It yields the trivialized CEL equations,

$$\frac{\partial}{\partial t}\frac{\partial K}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial K}{\partial \xi} = \frac{\partial}{\partial s}\frac{\partial \Phi}{\partial \eta} - \mathrm{ad}^*_\eta \frac{\partial \Phi}{\partial \eta} - g^{-1}\frac{\partial \Pi}{\partial g}; \tag{5.3}$$

see (2.17). We refer to [6] for a detailed derivation of these equations for geometrically exact models.

Time and space evolutions. Following the theory developed in Sec. 3.2, we now define the Lagrangians associated to the temporal and spatial evolutions. We first do this at the continuous level. The Lagrangian associated to the time evolution is $\bar{\mathsf{L}}(g,\xi) = \int_0^L (K(\xi) - \Phi(g^{-1}\partial_s g) - \Pi(g))\,\mathrm{d}s$ and the associated energy is

$$\begin{aligned}\mathsf{E}_{\bar{\mathsf{L}}}(g,\xi) &= \int_0^L \left\langle \frac{\partial K}{\partial \xi}, \xi \right\rangle \mathrm{d}s - \bar{\mathsf{L}}(g,\xi) \\ &= \int_0^L (K(\xi) + \Phi(g^{-1}\partial_s g) + \Pi(g))\,\mathrm{d}s. \end{aligned} \tag{5.4}$$

The Lagrangian associated to the spatial evolution is

$$\bar{\mathsf{N}}(g,\eta) = \int_0^T (K(g^{-1}\partial_t g) - \Phi(\eta) - \Pi(g))\,\mathrm{d}t.$$

One can also associate to $\bar{\mathsf{N}}$ an energy function $\mathsf{E}_{\bar{\mathsf{N}}}$ defined by the same formula, namely,

$$\begin{aligned}\mathsf{E}_{\bar{\mathsf{N}}}(g,\eta) &= -\int_0^T \left\langle \frac{\partial \Phi}{\partial \eta}, \eta \right\rangle \mathrm{d}t - \bar{\mathsf{N}}(g,\eta) \\ &= \int_0^T (-K(g^{-1}\partial_t g) - \langle \mathbb{C}(\eta - \mathbf{E}_6), \mathbf{E}_6\rangle \\ &\quad - \Phi(\eta) + \Pi(g))\,\mathrm{d}t. \end{aligned} \tag{5.5}$$

This energy function does not correspond to the physical energy.

Of course, E_{L} is conserved along the solutions of the EL equations for L on $\mathcal{F}([0,L],\mathrm{SE}(3))$ and E_{N} is conserved along the solutions of the EL equations for N on $\mathcal{F}([0,T],\mathrm{SE}(3))$. Recall also that the EL equations for L and N both imply not only the CEL equations for $\mathcal{L}$ but also the boundary conditions: zero-traction boundary conditions in the case of the EL equations associated to L and zero momentum boundary conditions in space for the EL equations associated to N, given respectively by

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial g'}(t,0) &= \frac{\partial \mathcal{L}}{\partial g'}(t,L) = 0, \quad \forall\, t \quad \text{and} \\ \frac{\partial \mathcal{L}}{\partial \dot{g}}(0,s) &= \frac{\partial \mathcal{L}}{\partial \dot{g}}(T,s) = 0, \quad \forall\, s. \end{aligned} \tag{5.6}$$

5.2. *Space-stepping algorithm*

We present below the complete algorithm obtained via the covariant variational integrator.

Space Integrator

Given: $g_a^j, \eta_a^{j-1}, \lambda_a^{j-1}, D_g\Pi(g_a^j), \quad$ **for** $j = 0, \ldots, N,$

Compute:

$$\begin{aligned}
\xi_a^j &= \frac{1}{\Delta t}\tau^{-1}((g_a^j)^{-1}g_a^{j+1}),\\
\mu_a^j &:= (\mathrm{d}^R\tau^{-1}_{\Delta t\xi_a^j})^*\partial_\xi K(\xi_a^j),\\
\lambda_a^j &= \mathrm{Ad}^*_{\tau(\Delta s\eta_{a-1}^j)}\lambda_{a-1}^j\\
&\quad - \begin{cases} \Delta s\left(\dfrac{1}{\Delta t}\mu_a^j + (g_a^j)^{-1}D_g\Pi(g_a^j)\right), \\ \qquad \textbf{for } j = 0, \\ \Delta s\left(\dfrac{1}{\Delta t}(\mu_a^j - \mathrm{Ad}^*_{\tau(\Delta t\xi_a^{j-1})}\mu_a^{j-1}) + (g_a^j)^{-1}D_g\Pi(g_a^j)\right), \\ \qquad \textbf{for } j = 1, \ldots, N-1, \end{cases}
\end{aligned}$$

Solve the discrete Legendre transform: $\lambda_a^j = -(\mathrm{d}\tau^{-1}_{\Delta s\,\eta_a^j})^*\partial_\eta\Phi(\eta_a^j),$

for η_a^j

Update: $g_{a+1}^j = g_a^j\tau(\Delta s\,\eta_a^j).$

Note that g_a^N is obtained using the boundary condition $\mathrm{Ad}^*_{\tau(\Delta t\xi_a^{N-1})}\mu_a^{N-1} = 0$ (see (4.9)). The external load at the point (a, j) is denoted above by $(g_a^j)^{-1}D_g\Pi(g_a^j)$. The only implicit part of the algorithm is the solution of the Legendre transform which is locally invertible for small space steps. All other parts of the algorithm are explicit.

Numerical tests. We shall use the multisymplectic integrator on Lie groups obtained in Sec. 4 from the discrete covariant variational principle. We consider a geometrically exact beam of length 0.8 m, and with cross-section given by a square of side $a = 0.01$ m. We assume that there are no exterior forces and that $\Pi(g) = 0$ so that $\mathcal{L}$ is SE(3)-invariant.

We choose the spacetime $X = [0, T] \times [0, L]$, with time of simulation $T = 2$ s, and length $L = 0.8$ m. The space and time steps are $\Delta s = 0.02$ m and $\Delta t = 0.04$ s. The spacetime is discretized as in Sec. 3.1.1, namely, $X_d = \{(j, a) \in \mathbb{Z} \times \mathbb{Z} \,|\, j = 0, \ldots, N-1, a = 0, \ldots, A-1\}$, where $N-1$ and $A-1$ correspond to T and L, respectively. Recall that for all $(j, a) \in X_d$, we consider the triangles $\triangle_a^j = ((j, a), (j+1, a), (j, a+1))$ that also involve the nodes (N, a) for all $a = 0, \ldots, A-1$, and (j, A) for all $j = 0, \ldots, N-1$.

The construction of the discrete Lagrangian density $\mathcal{L}_d(\Delta_a^j, \xi_a^j, \eta_a^j)$ as well as the detailed derivation of the associated discrete scheme obtained from the formula (4.7) are described in [4].

In this example we consider the space evolution through the multisymplectic variational integrator, followed by time reconstruction.

(1) ***Space evolution*:** The problem treated here corresponds to the following situation. We assume that at the initial time $t = 0$ and at the final time $t = T$, the velocity of the beam is zero. This corresponds to zero momentum boundary conditions. The configuration of the beam at $t = 0$ and $t = T$ is unknown. We assume, however, that we know the evolution (for all $t \in [0, T]$) of one of the extremities, say $s = 0$, as well as the evolution if its strain (for all $t \in [0, T]$). The approach described in this paper, that makes use of both the temporal and spatial evolutionary descriptions at both the continuous and discrete level, is especially well designed to discretize this problem in a structure-preserving way.

Note that we do not impose zero-traction boundary conditions, given here by

$$\begin{cases} (\Gamma - \mathbf{E}_3)|_{s=0} = 0, \\ (\Gamma - \mathbf{E}_3)|_{s=L} = 0, \\ \Omega(0) = \Omega(L) = 0, \end{cases} \tag{5.7}$$

at the two extremities of the beam.

The initial conditions are given by the configurations $\mathbf{g}_0 = (g_0^0, \ldots, g_0^N)$ and the strains $\boldsymbol{\eta}_0 = (\eta_0^0, \ldots, \eta_0^N)$ at the extremity $s = s_0$. In this example, we choose the following configuration and strain:

$$g_0^0 = (\mathrm{Id}, (0, 0, 0)), \quad g_0^{j+1} = g_0^j \,\mathrm{cay}(\Delta t\, \xi_0^j), \quad \text{for all } j = 0, \ldots, N-1,$$

where $\xi_0^j = (0, -0.85, 0; 0, -0.1, 0)$, for all $j = 0, \ldots, N-1$, and

$$\eta_0^j = \frac{1}{\Delta s} \mathrm{cay}^{-1}((g_0^j)^{-1} g_1^j), \quad \text{for all } j = 0, \ldots, N-1,$$

where $g_1^0 = (\mathrm{Id}, (0, 0, \Delta s))$ and $g_1^{j+1} = g_1^j \mathrm{cay}(\Delta t\, \xi_1^j)$, with $\xi_1^j = (0.06, -0.849, -0.04; -0.03, -0.1, 0)$, for all $j = 0, \ldots, N-1$ (Fig. 3).

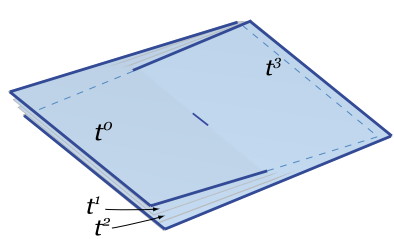

Fig. 3. Initial conditions $\mathbf{g}_0$ (enlarged), when $j \in \{0, 1, 2, 3\}$. The figure shows the discrete trajectory of a cross-section at the times t^0, t^1, t^2, t^3 at a given spatial node a.

For the problem treated here, the boundary conditions are thus given by (4.9), which are the discretization of the right-hand side condition in (5.6), i.e. discrete zero momentum boundary conditions. Note also that since $\Pi(g) = 0$, we have $D_g\bar{\mathcal{L}}_a^j = 0$. So, the discrete scheme is

$$\frac{1}{\Delta t}(\mu_a^j - \mathrm{Ad}^*_{\tau(\Delta t\xi_a^{j-1})}\mu_a^{j-1}) + \frac{1}{\Delta s}(\lambda_a^j - \mathrm{Ad}^*_{\tau(\Delta s\eta_{a-1}^j)}\lambda_{a-1}^j) = 0,$$

$$\text{for all } j = 1, \ldots, N-2, a = 1, \ldots, A-2,$$

$$-\frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\xi_a^{N-2})}\mu_a^{N-2} + \frac{1}{\Delta s}(\lambda_a^{N-1} - \mathrm{Ad}^*_{\tau(\Delta s\eta_{a-1}^{N-1})}\lambda_{a-1}^{N-1}) = 0,$$

$$\text{for all } a = 1, \ldots, A-2,$$

$$\frac{1}{\Delta t}\mu_a^0 + \frac{1}{\Delta s}(\lambda_a^0 - \mathrm{Ad}^*_{\tau(\Delta s\eta_{a-1}^0)}\lambda_{a-1}^0) = 0, \quad \text{and}$$

$$\frac{1}{\Delta t}\,\mathrm{Ad}^*_{\tau(\Delta t\xi_a^{N-1})}\mu_a^{N-1} = 0, \quad \text{for all } a = 1, \ldots, A-2,$$

where $\mu_a^j := (\mathrm{d}\tau^{-1}_{\Delta t\xi_a^j})^*\partial_\xi K(\xi_a^j)$ and $\lambda_a^j := -(\mathrm{d}\tau^{-1}_{\Delta s\eta_a^j})^*\partial_\eta\Phi(\eta_a^j)$.

This variational integrator produces the following displacement $\mathbf{g}_1, \ldots, \mathbf{g}_A$ "in space" of the trajectories $\mathbf{g}_a = (g_a^0, \ldots, g_a^N)$ "in time" of the beam sections (see Figs. 4 and 5).

Energy behavior. The above DCEL equations together with the boundary conditions are equivalent to the DEL equations for $\bar{\mathsf{N}}_d(\mathbf{g}_a, \boldsymbol{\eta}_a)$; see the discussion in Sec. 3.3. In particular, the solution of the discrete scheme defines a discrete symplectic flow in space $(\mathbf{g}_a, \boldsymbol{\eta}_a) \mapsto (\mathbf{g}_{a+1}, \boldsymbol{\eta}_{a+1})$ relative to the discrete symplectic form $\Omega_{\bar{\mathsf{N}}_d}$. As a consequence, the energy $\mathsf{E}_{\bar{\mathsf{N}}_d}$ of to the Lagrangian $\bar{\mathsf{N}}_d$ associated to the spatial evolution description is approximately conserved (Fig. 6 and 8).

Momentum map conservation. Recall that the Lagrangian density $\mathcal{L}$ is SE(3)-invariant, so the covariant Noether theorem holds. At the discrete level, since the

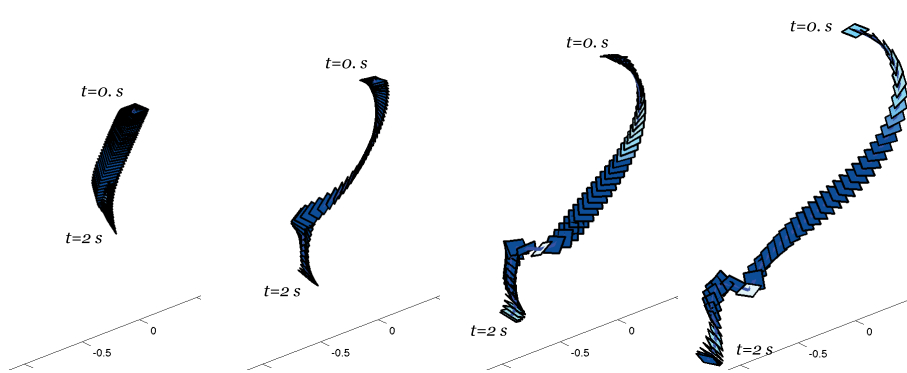

Fig. 4. Each figure represents the time evolution $\mathbf{g}_a = \{g_a^j, j = 1, \ldots, N\}$ of a given node a of the beam. The chosen nodes correspond to $s = 13\Delta s, 23\Delta s, 31\Delta s, 40\Delta s$, where $\Delta s = 0.02\,\mathrm{m}$.

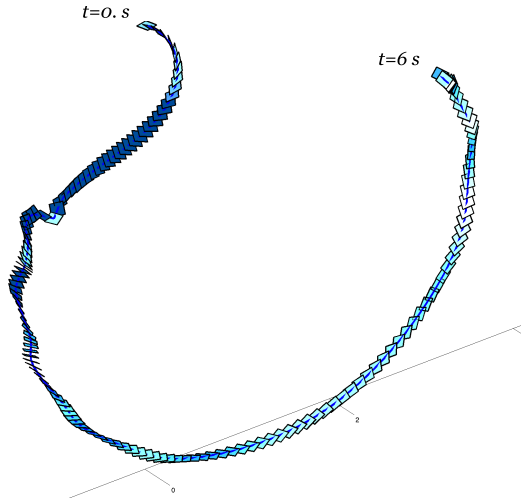

Fig. 5. This figure represents the time evolution $\mathbf{g}_a = \{g_a^j, j = 1, \ldots, N\}$, of a given node a of the beam, when $t^N = 150\Delta t$, where $\Delta t = 0.04\,\mathrm{s}$. The chosen node corresponds to $s = 40\Delta s$, where $\Delta s = 0.02\,\mathrm{m}$.

discrete Lagrangian density is also SE(3)-invariant (see [4]), we get the discrete covariant Noether theorem $\bar{\mathscr{J}}^{K,L}_{B,C}(g_d) = 0$ (according to (4.18)). Since the discrete Lagrangian $\bar{\mathsf{N}}_d$ is SE(3)-invariant, the discrete momentum maps coincide: $\mathsf{J}^+_{\bar{\mathsf{N}}_d} = \mathsf{J}^-_{\bar{\mathsf{N}}_d} = \mathsf{J}_{\bar{\mathsf{N}}_d}$, and we have

$$\mathsf{J}_{\bar{\mathsf{N}}_d}(\mathbf{g}_a, \boldsymbol{\eta}_a) = \sum_{j=0}^{N-1} \Delta t \mathrm{Ad}^*_{(g_a^j)^{-1}} \lambda_a^j,$$

according to (4.16). In view of the boundary conditions used here, from the discussion in Sec. 3.3 it follows that the discrete momentum map $\mathsf{J}_{\mathsf{N}_d}$ is conserved. This can be seen as a consequence of the covariant discrete Noether theorem $\bar{\mathscr{J}}^{0,N-1}_{B,C}(g_d) = 0$.

The discrete energy behavior and the conservation of the discrete momentum map $\mathsf{J}_{\bar{\mathsf{N}}_d} = (\mathsf{J}^1, \ldots, \mathsf{J}^6) \in \mathbb{R}^6$ are illustrated in Fig. 6.

The discrete covariant Noether theorem is also numerically satisfied on the solutions of the discrete scheme. We checked, for example, that $\bar{\mathscr{J}}^{0,N-1}_{0,A-1}(g_d) = 0$. Recall that this follows from Lemma 3.12 and the discussion after it. Indeed, we can write

$$\begin{aligned}\bar{\mathscr{J}}^{0,N-1}_{0,A-1}(g_d) = &\sum_{a=1}^{A-1} (J^1_{\hat{\mathcal{L}}_d}(j^1 g_d(\triangle_a^0)) + J^2_{\hat{\mathcal{L}}_d}(j^1 g_d(\triangle_a^{N-1})) + J^3_{\hat{\mathcal{L}}_d}(j^1 g_d(\triangle_{a-1}^0))) \\ &+ \mathsf{J}_{\bar{\mathsf{N}}_d}(\mathbf{g}_{A-1}, \boldsymbol{\eta}_{A-1}) - \mathsf{J}_{\bar{\mathsf{N}}_d}(\mathbf{g}_0, \boldsymbol{\eta}_0).\end{aligned}$$

The first line vanishes because of the boundary condition and the second line vanishes from the discrete Noether theorem.

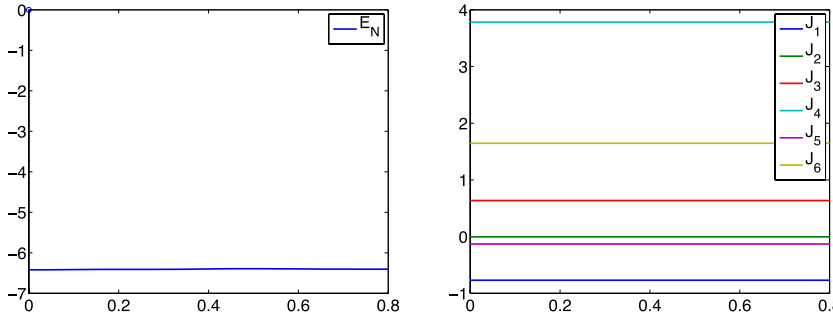

Fig. 6. The discrete energy function $\mathsf{E}_{\bar{\mathsf{N}}_d}$ (left) is approximately conserved, and the discrete momentum maps $\mathsf{J}_{\bar{\mathsf{N}}_d}$ (right) are exactly preserved.

One can also consider the discrete Lagrangian $\bar{\mathsf{L}}_d$. However, as explained in Sec. 3.2.1 (see (3.19)), the DEL equations for $\bar{\mathsf{L}}_d$ yield, in addition to the DCEL, zero-traction boundary conditions, that are not satisfied here. As we have seen, one can include boundary conditions in space by restricting $\bar{\mathsf{L}}_d$ to a subspace determined by these conditions. However, as explained in Sec. 3.2.1 case (B), these conditions have to be time independent, which is not the case in the problem considered here. So, the equations of motion cannot be written as DEL equations for $\bar{\mathsf{L}}_d$ and the energy $\mathsf{E}_{\bar{\mathsf{L}}_d}$ is not expected to be conserved. Of course, the same discussion holds for the continuous case as well. For similar reasons, the discrete momentum maps associated to $\bar{\mathsf{L}}_d$ are not conserved, but satisfy the identity (3.32).

(2) ***Reconstruction*:** The initial conditions are given by the set of configurations $\mathbf{g}_1, \ldots, \mathbf{g}_A$ obtained through space evolution (see Fig. 4). Thus we can immediately reconstruct the time advancement $\mathbf{g}^1, \ldots, \mathbf{g}^N$ of the configuration of the beam, where $\mathbf{g}^j = (g_0^j, \ldots, g_A^j)$; see Fig. 7.

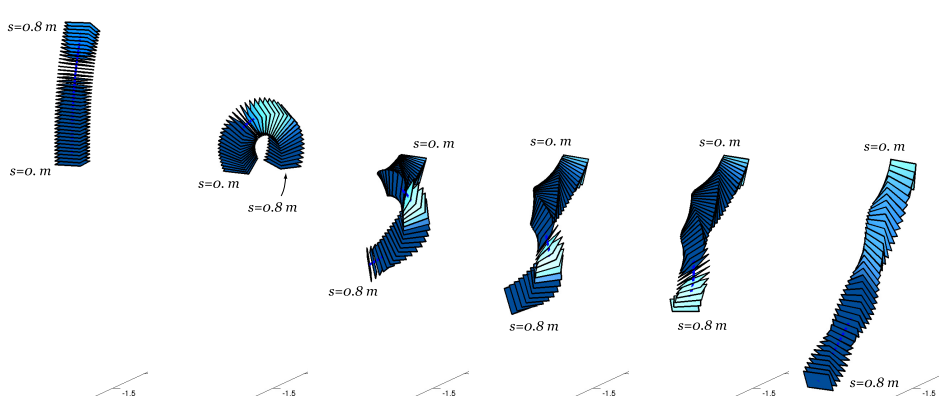

Fig. 7. From left to right: reconstruction of the trajectories in space of the sections, at times $t = 0.1\,\mathrm{s}, 0.6\,\mathrm{s}, 1.25\,\mathrm{s}, 1.4\,\mathrm{s}, 1.45\,\mathrm{s}, 1.95\,\mathrm{s}$.

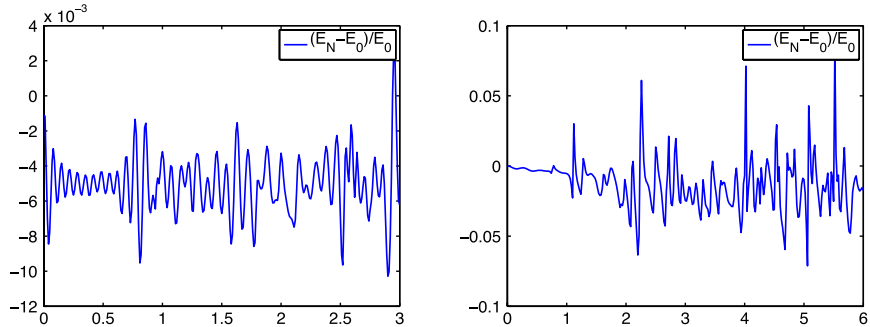

Fig. 8. Relative error $\frac{\mathsf{E}_{\bar{\mathsf{N}}_d} - \mathsf{E}^0_{\bar{\mathsf{N}}_d}}{\mathsf{E}^0_{\bar{\mathsf{N}}_d}}$: on the left, $\Delta s = 0.01\,\mathrm{m}$ and $\Delta t = 0.1105\,\mathrm{s}$ with $L = 3\,\mathrm{m}$; on the right, $\Delta s = 0.02\,\mathrm{m}$ and $\Delta t = 0.2291047\,\mathrm{s}$ with $L = 6\,\mathrm{m}$.

(3) ***Concluding remarks and future directions*:** During our numerical experiments we observed that finding the appropriate combination of space and time steps for stability is a delicate issue. It is well known that, for a time-stepping algorithm, the ratio $\Delta t/\Delta s$ has to verify the Courant condition (see, e.g., [13]). It would be important to derive an analogous estimate for our space-stepping integrator.

We now list several possible ways to achieve stability in a more systematic manner. First, we could use a rectangular mesh in order to get a second-order integrator, obtained through the generalized trapezoidal rule. Second, one could define an implicit integrator obtained through the mid-point rule, which is known to be much more stable. Finally, taking full advantage of the multisymplectic properties, we could design a variational integrator with adaptive mesh.

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