On a Data Assimilation Method coupling Kalman Filtering, MCRE Concept and PGD Model Reduction for Real-Time Updating of Structural Mechanics Model

2016 SIAM Conference on Uncertainty Quantification

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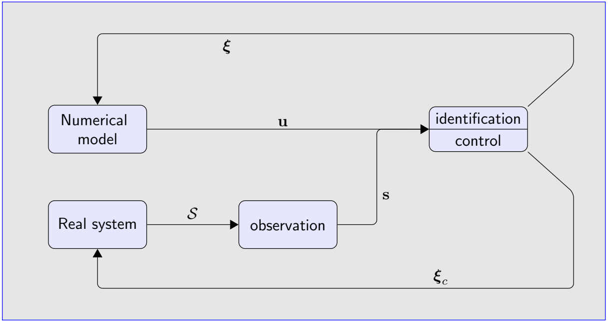
DDDAS Paradigm

DDDAS [1] paradigm : a continuous exchange between

the physical system
and
its numerical model



1. Darema, *Dynamica Data Driven Applications Systems : A New Paradigm for Application Simulations and Measurements*, 2003

In this work

Objectives :

Identification process

- for time dependent systems/parameters
- fast resolution
- robust even if highly corrupted data

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Identification process

- for time dependent systems/parameters
- fast resolution
- robust even if highly corrupted data

Tools :

Kalman filter for evolution aspect

modified Constitutive Relation Error for robustness

offline/online process based on
Proper Generalized Decomposition

Basics on Kalman Filtering

Proposed Approach

Numerical Results

Conclusion

○ Introduction
● Basics on Kalman Filtering
○ Proposed Approach
○ Numerical Results
○ Conclusion

Dynamical system :

$$\begin{cases} \mathbf{u}^{(k+1)} & = \mathcal{M}^{(k)}\mathbf{u}^{(k)} + \mathbf{e_u}^{(k)} \\ \mathbf{s}^{(k)} & = \mathcal{H}^{(k)}\mathbf{u}^{(k)} + \mathbf{e_s}^{(k)} \end{cases}$$

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Bayes theorem :

$$\boxed{\pi\left(\mathbf{u}^{(k)}\middle|\mathbf{s}^{(k)}\right) = \frac{\pi\left(\mathbf{s}^{(k)}\middle|\mathbf{u}^{(k)}\right)\pi\left(\mathbf{u}^{(k)}\middle|\mathbf{s}^{(0:k-1)}\right)}{\pi\left(\mathbf{s}^{(k)}\middle|\mathbf{s}^{(0:k-1)}\right)}}$$

under the following hypothesis :

- State $\mathbf{u}^{(k)}$ is a Markov process,
- Observations $\mathbf{s}^{(k)}$ are statistically independent of state history

Principle

Kalman filter[2] is a bayesian filter combined with *Maximum a Posteriori* method in the case of Gaussian probability density functions.

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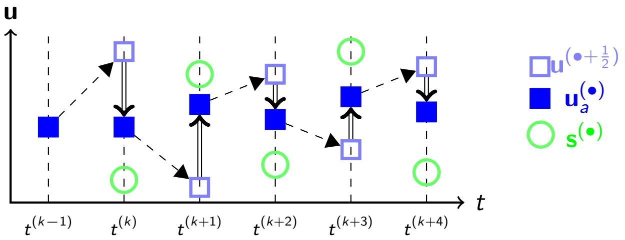
Two main steps :



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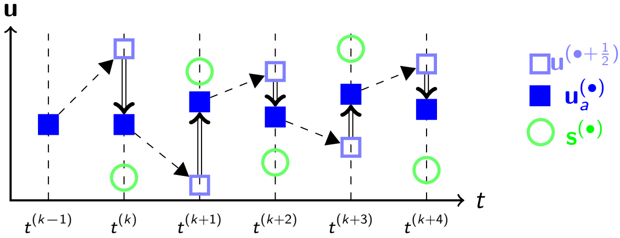
(a) Prediction step where is realized *a priori* estimation $\mathbf{u}^{(k+\frac{1}{2})}$ of state system



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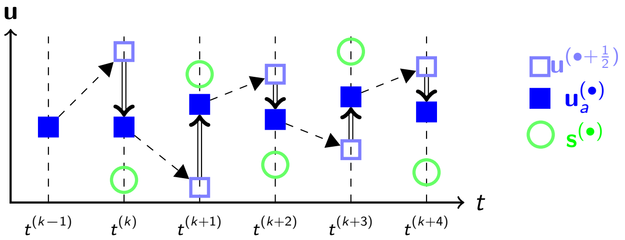
Two main steps :

(a) Prediction step where is realized *a priori* estimation $\mathbf{u}^{(k+\frac{1}{2})}$ of state system

(b) Assimilation step where is realized *a posteriori* estimation $\mathbf{u}_a$ using observations data



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Inverse problems formulation

Kalman filter is a very well-known method to solve inverse problems [3]

3. Kaipio and Somersalo, *Statistical and Computational Inverse Problems*, 2006

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Principle : Introduce model parameters vector $\boldsymbol{\xi} \in \mathbb{R}^{n_p}$

no *a priori* knowledge $\rightarrow$ stationarity hypothesis :

$$\frac{\partial \boldsymbol{\xi}}{\partial t} \simeq 0 \;\Rightarrow\; \boldsymbol{\xi}^{(k+1)} = \boldsymbol{\xi}^{(k)} + \mathbf{e}_{\boldsymbol{\xi}}^{(k)}$$

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Two formulations

Joint Kalman Filter

$$\begin{cases} \bar{\mathbf{u}}^{(k+1)} = \bar{\mathcal{M}}^{(k)} \bar{\mathbf{u}}^{(k)} + \bar{\mathbf{e}}_M^{(k)} \\ \mathbf{s}^{(k)} = \bar{\mathcal{H}}^{(k)} \bar{\mathbf{u}}^{(k)} + \mathbf{e}_s^{(k)} \end{cases}$$

Dual Kalman filter

$$\begin{cases} \boldsymbol{\xi}^{(k+1)} = \boldsymbol{\xi}^{(k)} + \mathbf{e}_{\xi}^{(k)} \\ \mathbf{s}^{(k)} = \mathcal{H}^{(k)} \mathbf{u}^{(k)}(\boldsymbol{\xi}^{(k)}) + \mathbf{e}_s^{(k)} \end{cases}$$

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$$\begin{bmatrix} \mathbf{u}^{(k)} \\ \boldsymbol{\xi}^{(k)} \end{bmatrix}$$

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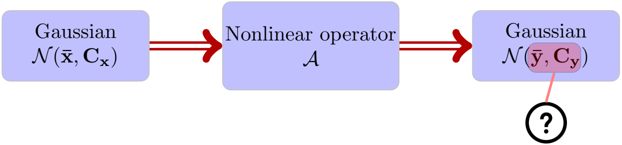
Resolution schemes : UKF vs EKF

The problem :



Two main approaches in Kalman filtering context

Resolution schemes : UKF vs EKF

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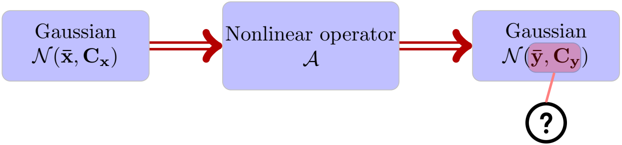

Two main approaches in Kalman filtering context

- First order linearization,

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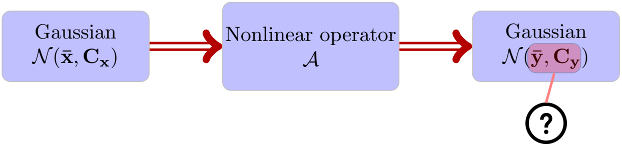

Two main approaches in Kalman filtering context

- First order linearization,

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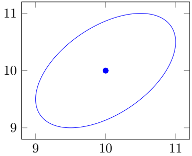
- Deterministic Monte-Carlo like method, Unscented Transform,

  Unscented Kalman filter[5]

4. Sorenson and Stubberud, *Non-linear Filtering by Approximation of the a posteriori Density*, 1968
5. Julier and Uhlmann, *A new extension of the kalman filter to nonlinear systems*, 1997

First Order Linearization

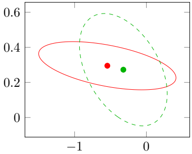
Prior



Linearization

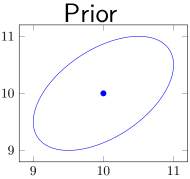
$$\mathbf{A} = \nabla_{\mathbf{x}} \mathcal{A}$$
$$\bar{\mathbf{y}} = \mathcal{A}(\bar{\mathbf{x}})$$
$$\mathbf{C_y} = \mathbf{A C_x A}^T$$

Posterior



Linearization vs Unscented Transform

First Order Linearization



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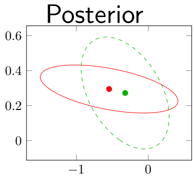
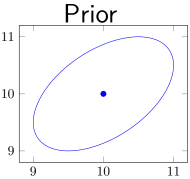
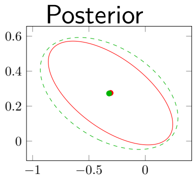


Unscented Transform



σ-points propagation

$$\{\mathbf{x}_i\}_{i=1,..,2N+1}$$
$$\{y_i\} = \mathcal{A}\left(\{\mathbf{x}_i\}\right)$$



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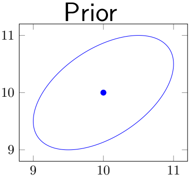

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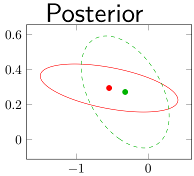

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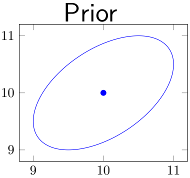

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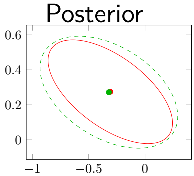

CPU For the same computational cost

Why another approach ?

Kalman Filter based methods well-adapted for evolution problems and DDDAS paradigm

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But :

methods very costly
if degrees of freedom/parameters increase

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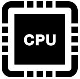

Identification quality strongly depends on measurement noise


Outline

Principle of the method

Keep the dual formulation

$$\begin{cases} \xi^{(k+1)} = \xi^{(k)} + \mathbf{e}_{\xi}^{(k)} \\ \mathbf{s}^{(k)} = \mathcal{H}^{(k)} \mathbf{u}^{(k)}(\xi^{(k)}) + \mathbf{e}_{s}^{(k)} \end{cases}$$

Classically computed using a Kalman Filter

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But use another observation operator

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But use another observation operator

Defined from the modified Constitutive Relation Error functional

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The idea[6] :

Weight the classical Constitutive Relation Error[7]
by a measurements error term

6. Ladevèze et al, *Updating of finite element models using vibration tests*, 1994

7. Ladevèze and Leguillon, *Error estimate procedure in the finite element method and application*, 1983

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mCRE functional for unsteady thermal problems :

$$\mathcal{E}_m(u,\boldsymbol{q};\boldsymbol{\xi}) = \frac{1}{2}\int_{I_t}\int_{\Omega} (\boldsymbol{q} - \mathcal{K}\nabla u)\,\mathcal{K}^{-1}\,(\boldsymbol{q} - \mathcal{K}\nabla u)\,\mathrm{d}\mathbf{x}\mathrm{d}t + \frac{\delta}{2}\int_{I_t} \|\boldsymbol{\Pi} u - \mathbf{s}\|^2 \mathrm{d}t$$

$$\mathcal{U} = \left\{ u \in H^1(\Omega) \otimes L^2(I_t) \setminus u = u^d \text{ on } \partial\Omega_u\,,\ u = u^0 \text{ at } t = t_0 \right\}$$
$$\mathcal{S}(u) = \left\{ \mathbf{q} \in [L^2(\Omega) \otimes L^2(I_t)]^d \setminus \mathbf{q}\cdot\mathbf{n} = q^d \text{ on } \partial\Omega_{\mathbf{q}}\,,\ \partial_t u + \nabla\cdot\mathbf{q} = f \right\}$$

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Solution is defined by :

$$\boldsymbol{p} = \underset{\boldsymbol{\xi} \in \mathcal{P}_{ad}}{\text{argmin}} \min_{(u,\boldsymbol{q}) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}} \mathcal{E}_m(u, \boldsymbol{q}; \boldsymbol{\xi})$$

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Admissible fields
Constrained minimization

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Parameters minimization
Gradient based methods

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Parameters minimization
Gradient based methods

Fixed point

Admissible fields
Constrained minimization

mCRE inverse problems

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Interest

(i) Robustness of the method with highly corrupted data
(ii) Strong mechanical content
(iii) Model reduction integration

The Modified Kalman Filter

$$\begin{cases} \boldsymbol{\xi}^{(k+1)} = \boldsymbol{\xi}^{(k)} + \mathbf{e}_{\boldsymbol{\xi}}^{(k)} \\ \mathbf{s}^{(k)} = \mathcal{H}_m^{(k)}\left(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)}\right) + \mathbf{e}_{\mathbf{s}}{}^{(k)} \end{cases}$$

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Two steps for $\mathcal{H}_m^{(k)}\left(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)}\right)$

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Two steps for $\mathcal{H}_m^{(k)}\left(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)}\right)$

Step 1 : admissible fields computation

$$\mathbf{u}^{(k)} = \mathcal{G}_{mCRE}(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)})$$

The Modified Kalman Filter

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Two steps for $\mathcal{H}_m^{(k)}\left(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)}\right)$

Step 1 : admissible fields computation

$$\mathbf{u}^{(k)} = \mathcal{G}_{mCRE}(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)})$$

Step 2 : projection

Typically using boolean matrix

$$\mathcal{H} := \mathbf{\Pi}$$

$$\mathcal{H}_m(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k)}) = \mathcal{H} \circ \mathcal{G}_{mCRE}(\boldsymbol{\xi}^{(k)}, \mathbf{s}^{(k-1:k)})$$

Optimization point of view

Dual Kalman filter based identification
can be seen as the minimization of

$$J(\boldsymbol{\xi}) = \sum_{k=0}^{n_t} \left(\mathbf{s}^{(k)} - \mathcal{H}^{(k)} \boldsymbol{u}^{(k)}(\boldsymbol{\xi}^{(k)})\right)^T {\mathbf{C}_\mathbf{s}^{(k)}}^{-1} \left(\mathbf{s}^{(k)} - \mathcal{H}^{(k)} \boldsymbol{u}^{(k)}(\boldsymbol{\xi}^{(k)})\right)$$

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?

Classical

$$\min_{\mathcal{U}} \left\| \mathbf{s}^{(k)} - \mathcal{H}^{(k)} \boldsymbol{u}^{(k)} \right\|_{\mathbf{C_s}^{(k)-1}}$$

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Classical

mCRE based

$$\min_{\mathcal{U}} \left\| \mathbf{s}^{(k)} - \mathcal{H}^{(k)} \boldsymbol{u}^{(k)} \right\|_{{\mathbf{C_s}^{(k)}}^{-1}}$$

$$\min_{\mathcal{U}\times\mathcal{S}} \left\| \boldsymbol{q} - \boldsymbol{\nabla u} \right\|_{\mathbf{K}^{-1}, I_t^{(k)}} + \frac{\delta}{2} \left\| \boldsymbol{\Pi u} - \mathbf{s} \right\|_{I_t^{(k)}}$$

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Observations data
strongly imposed

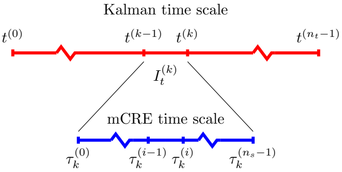
Observations data
weakly imposed

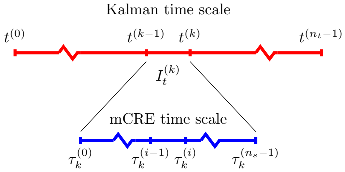
Admissible fields :

$$(u_{ad}, \boldsymbol{q}_{ad}) = \underset{(u,\boldsymbol{q}) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}}{\operatorname{argmin}} \mathcal{E}_m(u, \boldsymbol{q}; \boldsymbol{\xi}^{(k)})$$

Admissible fields :

$$(u_{ad}, \boldsymbol{q}_{ad}) = \underset{(u,\boldsymbol{q})\in\mathcal{U}_{ad}\times\mathcal{S}_{ad}}{\text{argmin}} \ \mathcal{E}_m(u, \boldsymbol{q}; \boldsymbol{\xi}^{(k)})$$





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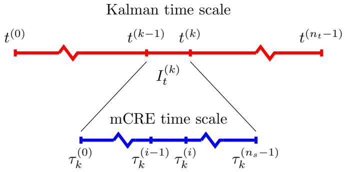
λ lagrange multiplier field **and** stationarity conditions



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λ lagrange multiplier field **and** stationarity conditions

After **FE** discretization :

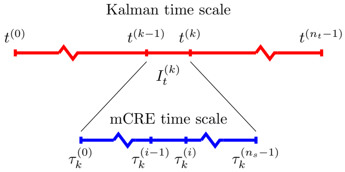
$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{K} \\ \delta \boldsymbol{\Pi}^T \boldsymbol{\Pi} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{ext} \\ \delta \boldsymbol{\Pi}^T \mathbf{s} \end{bmatrix} \quad \forall t$$

with

$$\boldsymbol{u}(\tau_k^{(0)}) = \mathbf{u}^{(k-1)} \quad \text{and} \quad \boldsymbol{\lambda}(\tau_k^{(n_s-1)}) = \mathbf{0}$$

Technical points

State estimation



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Coupled forward-backward problem in time

PGD based model reduction

Find $\mathbf{u} \in \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_D$ such that $B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v})\ \forall \mathbf{v} \in \mathcal{X}$

8. Nouy, *A priori model reduction through Proper Generalized Decomposition for solving time-dependent partial differential equations*, 2010

Find $\mathbf{u} \in \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_D$ such that $B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \ \forall \mathbf{v} \in \mathcal{X}$

Principle :

Low-rank tensor approximation

$$\mathbf{u} \simeq \mathbf{u}_m = \sum_{i=1}^{m} \boldsymbol{w}_i^1 \otimes \boldsymbol{w}_i^2 \otimes \cdots \otimes \boldsymbol{w}_i^D \ ; \ \mathbf{u}_m \in \mathcal{X}_m \subset \mathcal{X}$$

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Construction : many strategies[8] ; progressive Galerkin approach

8. Nouy, *A priori model reduction through Proper Generalized Decomposition for solving time-dependent partial differential equations*, 2010

PGD based model reduction

○ Introduction
○ Basics on Kalman Filtering
● Proposed Approach
○ Numerical Results
○ Conclusion

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Construction : many strategies [8] ; progressive Galerkin approach

$\mathbf{u}_{M-1}$ known

$$\mathbf{B_1}(\mathbf{w}^1, \mathbf{w}^\star) = L(\mathbf{w}^\star) - \mathbf{B_1}(\mathbf{u}_{M-1}, \mathbf{w}^\star)$$
$$\vdots$$
$$\mathbf{B_D}(\mathbf{w}^D, \mathbf{w}^\star) = L(\mathbf{w}^\star) - \mathbf{B_D}(\mathbf{u}_{M-1}, \mathbf{w}^\star)$$

Fixed point

Orthogonalization and update

$$\mathbf{u}_M = \mathbf{u}_{M-1} + \mathbf{w}^1 \otimes \cdots \otimes \mathbf{w}^D$$

Greedy

8. Nouy, *A priori model reduction through Proper Generalized Decomposition for solving time-dependent partial differential equations*, 2010

Two fields problem : u and λ

Two PGD decompositions simultaneously computed

PGD-mCRE

Two fields problem : u and λ

Two PGD decompositions simultaneously computed

Many parameters to consider as extra-coordinates

Two fields problem : u and λ

Two PGD decompositions simultaneously computed

Many parameters to consider as extra-coordinates

- space, time
- parameters to identify $\boldsymbol{\xi}$
- observations data
- initial condition

PGD-mCRE

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Projection into a reduced basis

$$u_0^{(k)} = \sum_{i=0}^{n_{init}} \alpha_i \psi_i(\mathbf{x})$$

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Projection into a reduced basis

$$u_0^{(k)} = \sum_{i=0}^{n_{init}} \alpha_i \psi_i(\mathbf{x})$$

$$\boldsymbol{u}_{PGD} = \sum_{i=1}^{m} \boldsymbol{\phi}_i^u \otimes \boldsymbol{\psi}_i^u \bigotimes_{j=1}^{n_p} \boldsymbol{\chi}_{j,i}^u \bigotimes_{k=1}^{n_{obs}} \boldsymbol{\theta}_{k,i}^u \bigotimes_{m=1}^{n_{obs}} \boldsymbol{\eta}_{m,i}^u \bigotimes_{q=1}^{n_{init}} \boldsymbol{\varphi}_{q,i}^u$$

$$\boldsymbol{\lambda}_{PGD} = \sum_{i=1}^{m} \boldsymbol{\phi}_i^\lambda \otimes \boldsymbol{\psi}_i^\lambda \bigotimes_{j=1}^{n_p} \boldsymbol{\chi}_{j,i}^\lambda \bigotimes_{k=1}^{n_{obs}} \boldsymbol{\theta}_{k,i}^\lambda \bigotimes_{m=1}^{n_{obs}} \boldsymbol{\eta}_{m,i}^\lambda \bigotimes_{q=1}^{n_{init}} \boldsymbol{\varphi}_{q,i}^\lambda$$

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$$\boldsymbol{\lambda}_{PGD} = \sum_{i=1}^{m} \boldsymbol{\phi}_i^\lambda \otimes \boldsymbol{\psi}_i^\lambda \bigotimes_{j=1}^{n_p} \boldsymbol{\chi}_{j,i}^\lambda \bigotimes_{k=1}^{n_{obs}} \boldsymbol{\theta}_{k,i}^\lambda \bigotimes_{m=1}^{n_{obs}} \boldsymbol{\eta}_{m,i}^\lambda \bigotimes_{q=1}^{n_{init}} \boldsymbol{\varphi}_{q,i}^\lambda$$

⚠ $n_p + 2 \cdot n_{obs} + n_{init} \lessapprox 20$

8. Marchand et al, *Real-time updating of structural mechanics models using Kalman filtering, modified Constitutive Relation Error and Proper Generalized Decomposition*, 2016

offline

- Reduced basis computation for initial condition projection
- PGD admissible fields computation

CPU

8. Marchand et al, *Real-time updating of structural mechanics models using Kalman filtering, modified Constitutive Relation Error and Proper Generalized Decomposition*, 2016

Synthesis

offline

- Reduced basis computation for initial condition projection
- PGD admissible fields computation

CPU

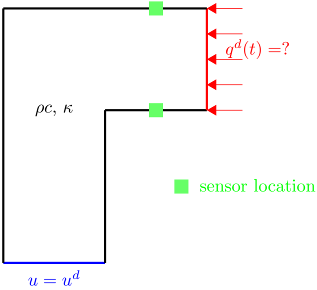
online

at each time step

- Project current initial condition in reduced basis
- Evaluate PGD parametric solution for set of σ-points
- Project state into observation space
- Kalman parameters update

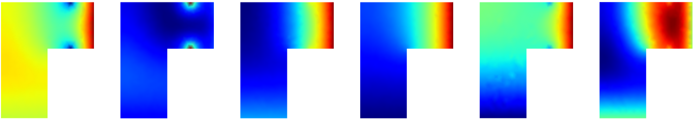
8. Marchand et al, *Real-time updating of structural mechanics models using Kalman filtering, modified Constitutive Relation Error and Proper Generalized Decomposition*, 2016

Example 1

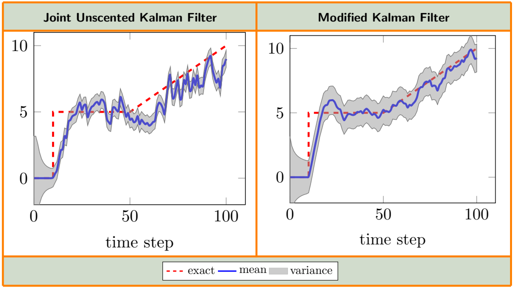


Time stepping for observation : 1000
Time stepping for identification : 100
Noise level : 20%

PGD modes

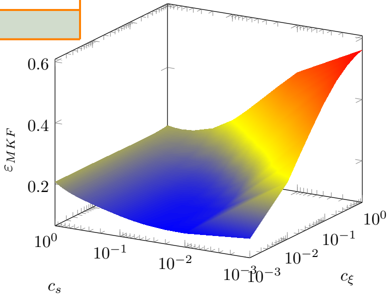


Exemple 1 : Neumann B.C. identification

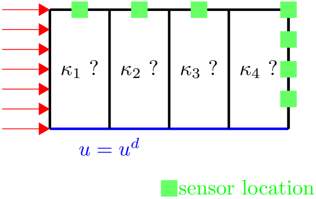


Better accuracy

Tuning parameters impact 

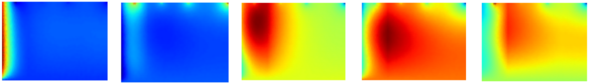


$$\varepsilon_{MKF} = \frac{\left\| \xi_{true} - \mathbf{E}\left[\xi_{MKF}\right] \right\|_{L^2(I_t)}}{\left\| \xi_{true} \right\|_{L^2(I_t)}}$$

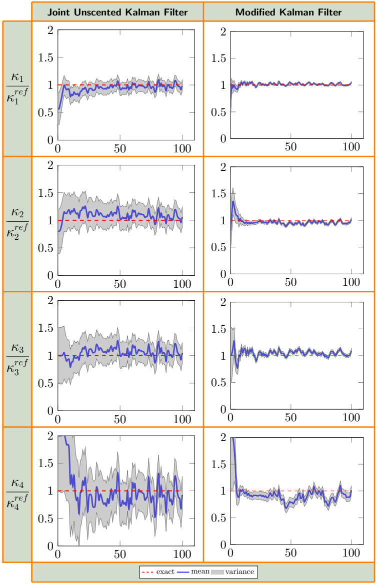


Time stepping for observation : 1000
Time stepping for identification : 100
Noise level : 10%

Space modes



Exemple 2 : conductivity identification



Better

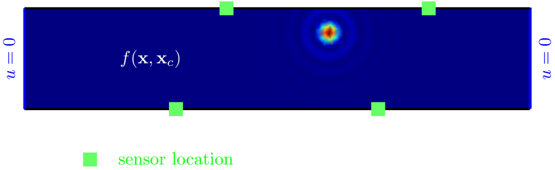
accuracy

and

robustness

Thermal source :

$$f(\mathbf{x};\mathbf{x}_c) = \mathrm{sinc}^2\left(\pi\|\mathbf{x}-\mathbf{x}_c(t)\|\right)$$

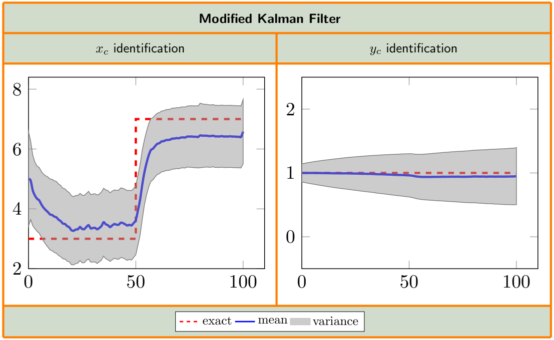


To include $\mathbf{x}_c$ as PGD's extra-coordinate

$$f(\mathbf{x};\mathbf{x}_c) \simeq \sum_{i=1}^{N} F_i(\mathbf{x}) \cdot G_i(\mathbf{x}_c)$$

Using SVD

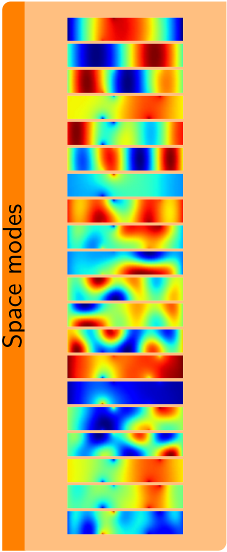
Time stepping for observation : 1000
Time stepping for identification : 100
Noise level : 10%

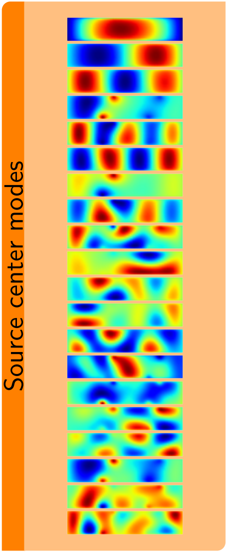


Results not compared to UKF since this problem requires to solve **5000** problems at each time step with the UKF approach

Exemple 3 : source localization

Limits of PGD here





PGD limits

Solution is relatively *singular*

involves

Initial condition should be project on many modes

$$n_{init} \gg 1$$

but

$$n_p + 2 \cdot n_{obs} + n_{init} \lessapprox 20$$

Outline

Basics on Kalman Filtering

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Conclusion

Conclusion and future works

Unscented Kalman Filter	**Modified Kalman Filter**	**modified CRE**
Implementation	**Implementation**	**Robustness**
Cost	**Robustness**	**Cost**
Robustness	**Cost**	**2∗Minimizations**

Proper Generalized Decomposition

Conclusion and future works

Unscented Kalman Filter	**Modified Kalman Filter**	**modified CRE**
Implementation	**Implementation**	**Robustness**
Cost	**Robustness**	**Cost**
Robustness	**Cost**	**2∗Minimizations**

Proper Generalized Decomposition

Extension to field identification

Number of parameters significantly increases

split state and parameters meshes
adaptive strategy [9]