Low-energy spectrum of Toeplitz operators with a miniwell

Alix Deleporte

Low-energy spectrum of Toeplitz operators with a miniwell

Alix Deleporte*

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France

October 19, 2016

Abstract

In the semiclassical limit, it is well-known that the first eigenvector of a Toeplitz operator concentrates on the minimal set of the symbol. In this paper, we give a more precise criterion for concentration in the case where the minimal set of the symbol is a submanifold, in the spirit of the "miniwell condition" of Helffer-Sjöstrand.

Contents

1 Introduction **2**
- 1.1 Motivations . . . 2
- 1.2 Outline . . . 3

2 Toeplitz quantization **3**
- 2.1 The Szegő projector . . . 3
- 2.2 Toeplitz operators . . . 6
  - 2.2.1 Toeplitz operators on $\mathbb{C}^n$ . . . 6
  - 2.2.2 Toeplitz operators on compact manifolds . . . 6
- 2.3 Main result . . . 7

3 Upper bound on the first eigenvalue **8**

4 Localization at the miniwell **8**

*deleporte@math.unistra.fr

5 Acknowledgements **13**

1 Introduction

1.1 Motivations

A few decades ago, a mathematical foundation was given for a common heuristic in the physics literature. The problem was the study, as $h \to 0$, of the lowest energy eigenfunction of a Schrödinger operator $-h^2\Delta + V$ on a Riemannian manifold, in the case where $V \geq 0$ and $\{V = 0\}$ is a submanifold. It is well-known that this eigenfunction is $O(h^\infty)$ outside every neighbourhood of $\{V = 0\}$. Helffer and Sjöstrand [5] proposed a more precise criterion for localization, based on the Hessian matrix of V on the submanifold. If this matrix is "minimal" at only one point (the miniwell condition), then, as $h \to 0$, the lowest energy eigenfunction is $O(h^{-\infty})$ outside any fixed neighbourhood of this point. An example of this is the Schrödinger operator on $L^2(\mathbb{R}^2)$ with potential:

$$V(x_1, x_2) = (x_1^2 + x_2^2)(1 + (x_1 + 1)^2),$$

which vanishes on the unit circle but which is "smaller" near $(-1, 0)$ than near any other point of the unit circle. In this case, the main result of [5] is that an eigenvector of $-h^2\Delta + V$ with minimal eigenvalue is, for h small, located near $(-1, 0)$.

This result validates, in the setting of Schrödinger operators, the physical effect of *semiclassical order from disorder*[4]: not all points in classical phase space where the energy is minimal are equivalent for quantum systems. However, the main physical application of semiclassical order from disorder lies in the setting of frustrated spin systems, where the classical symplectic manifold is a product of spheres. The mathematical setting here strongly differs from Schrödinger operators.

We propose to study Toeplitz operators, of which spin systems are a particular case. As for pseudo-differential operators, to a real function (or symbol) on a symplectic manifold we associate an auto-adjoint operator on a Hilbert space, depending on a small parameter. For this we need an additional geometric structure on the manifold. For pseudo-differential operators the symplectic manifold is supposed to be of the form T^*X and the Hilbert space is $L^2(X)$. For Toeplitz operators we suppose that the manifold has a Kähler structure, and the Hilbert space is a set of holomorphic sections in a convenient bundle.

In a previous paper [3], we developed a set of techniques in order to study the first eigenvalues of a Toeplitz operator, under the hypothesis

that the minimal set of the symbol is a finite set of non-degenerate critical points. In this article, we show that these techniques can be used to show an result analogous to the concentration on the miniwell.

1.2 Outline

In section 2, we recall the necessary material on Toeplitz operators, which allow us to state the main result in precise terms.

that was developped in our previous paper [3]. In particular, one can build approximate eigenfunctions for Toeplitz operators with non-degenerate minimal points by pulling back by normal coordinates the eigenfunctions of a model quadratic Toeplitz operator. We also recall a positivity lemma, which holds for sequence of functions sufficiently close to a point.

In section 3 we use a technique developed in a previous paper [3] to give an upper bound for the first eigenvalue in the miniwell case.

In section 4 we give a lower bound for the first eigenvalue, which allow us to conclude.

2 Toeplitz quantization

2.1 The Szegő projector

Let M be a Kähler manifold of dimension n, with symplectic form ω. If the Chern class of ω is integer, there exists a hermitian holomorphic line bundle (L, h) over M, with curvature ω [9].

Let (L^*, h^*) be the dual line bundle of L, with dual metric. Let D be the unit ball of L^*, that is:

$$\{D = (m, v) \in L^*, \|v\|_{h^*} < 1\}.$$

The boundary of D is denoted by X. It admits an S^1 action

$$\begin{array}{rccc} r_\theta : & X & \mapsto & X \\ & (m, v) & \mapsto & (m, e^{i\theta} v). \end{array}$$

We are interested in the equivariant Hardy spaces on X, defined as follows:

Definition 2.1.

- The *Hardy space* $H(X)$ is the closure in $L^2(X)$ of

$$\{f|_X,\ f \in C^\infty(D \cup X),\ f \text{ holomorphic in } D\}.$$

- The Szegő projector S is the orthogonal projection from $L^2(X)$ onto $H(X)$.
- Let $N \in \mathbb{N}$. The *equivariant Hardy space* $H_N(X)$ is defined by:

$$H_N(X) = \{f \in H(X),\, \forall (x,\theta) \in X \times \mathbb{S}^1,\, f(r_\theta x) = e^{iN\theta} f(x)\}.$$

- Let $N \in \mathbb{N}$. The equivariant Szegő projector S_N is the orthogonal projection from $L^2(X)$ onto $H_N(X)$.

Throughout this paper, we will work with the sequence of spaces $H_N(X)$. If M is compact, then the spaces $H_N(X)$ are finite-dimensional spaces of smooth functions. Another important example is the case $M = \mathbb{C}^n$, with standard Kähler form, where the equivariant Hardy spaces are explicit:

Proposition 2.2. *If $M = \mathbb{C}^n$ with standard Kähler form, then*

$$H_N(X) \simeq B_N := L^2(\mathbb{C}^n) \cap \{z \mapsto e^{-\frac{N}{2}|z|^2} f(z),\, f \text{ is an entire function}\}.$$

The space B_N is a closed subspace of $L^2(\mathbb{C}^n)$. The orthogonal projector Π_N from $L^2(\mathbb{C}^n)$ to B_N admits as Schwartz kernel the function

$$\Pi_N : z, w \mapsto \left(\frac{N}{\pi}\right)^n \exp\left(-\frac{1}{2}N|z-w|^2 + iN\Im(z \cdot \overline{w})\right).$$

Observe that the sequence of kernels Π_N is rapidly decreasing outside the diagonal set. A very important fact is that this property holds also in the case of a compact Kähler manifold:

Proposition 2.3 ([2],prop 4.1)**.** *Let M be a compact Kähler manifold, and $(S_N)_{N\geq 1}$ be the sequence of Szegő projectors of definition 2.1. Let $\delta \in [0, 1/2)$. For every $k \geq 0$ there exists C such that, for every $N \in \mathbb{N}$, for every $x, y \in X$ such that $\mathrm{dist}(\pi(x), \pi(y)) \geq N^{-\delta}$, one has*

$$|S_N(x,y)| \leq CN^{-k}.$$

This roughly means that, though the operators S_N are non-local, their "interaction range" decreases with N.

In the spirit of the previous proposition, we define what it means for a sequence of functions in $H_N(X)$ to be localized.

Definition 2.4. Let $u = (u_N)_{N\in\mathbb{N}}$ be a sequence of unit elements of $L^2(X)$. Let $\mathrm{d}Vol$ denote the Liouville volume form on M. For every N, the probability measure $|u_N|^2 \mathrm{d}Vol \otimes \mathrm{d}\theta$ is well-defined on X, and we call μ_N the pull-back of this measure on M.

Let moreover $Z \subset M$. We say that the sequence u localizes on Z when, for every $\delta \in [0, 1/2)$, one has

$$\mu_N(\{m \in M, \text{ dist}(m, Z) \geq N^{-\delta}\}) = O(N^{-\infty}).$$

A corollary of this definition is that, if a sequence $(u_N)_{N\in\mathbb{N}}$ concentrates on a set Z, then so does the sequence $(S_N u_N)_{N\in\mathbb{N}}$.

To complete the proposition 2.3, we have to describe how S_N acts on functions localized on a point. For this we need a convenient choice of coordinates.

Let $P_0 \in M$. The real tangent space $T_{P_0}M$ carries a Euclidian structure and an almost complex structure coming from the Kähler structure on M. We then can (non-uniquely) identify $\mathbb{C}^n$ with $T_{P_0}M$.

Definition 2.5. Let U be a neighbourhood of 0 in $\mathbb{C}^n$ and V be a neighbourhood of a point P_0 in M.

A smooth diffeomorphism $\rho : U \times \mathbb{S}^1 \to \pi^{-1}(V)$ is said to be a *normal map* or map of *normal coordinates* under the following conditions:

- $\forall (z, v) \in U \times \mathbb{S}^1$, $\forall \theta \in \mathbb{R}$, $\rho(z, ve^{i\theta}) = r_\theta \rho(z, v)$;
- Identifying $\mathbb{C}^n$ with $T_{P_0}M$ as previously, one has:

$$\forall (z, v) \in U \times \mathbb{S}^1, \ \pi(\rho(z, v)) = \exp(z).$$

Remark 2.6. The choice a normal map around a point P_0 reflects the choice of an identification of $\mathbb{C}^n$ with $T_{P_0}(M)$ and a point over P_0 in X. Hence, if ρ_1 and ρ_2 are two normal maps around the same point P_0, then $\rho_1^{-1} \circ \rho_2 \in U(n) \times SO(2)$.

We can pull-back by a normal map the projector Π_N on the Bargmann spaces by the following formula:

$$\rho^* \Pi_N(\rho(z, \theta), \rho(w, \phi)) := e^{iN(\theta-\phi)} \Pi_N(z, w).$$

By convention, $\rho^*\Pi_N$ is zero outside $\pi^{-1}(V)^2$.

Proposition 2.7. *Let $P_0 \in M$, and ρ a normal map around P_0. For every $\epsilon > 0$ there exists $\delta \in (0, 1/2)$ and $C > 0$ such that for every $N \in \mathbb{N}$, for every $u \in L^2(X)$, if the support of u lies inside $\rho(B(0, N^{-\delta}) \times \mathbb{S}^1)$, then*

$$\|(S_N - \rho^*\Pi_N)u\|_{L^2} < CN^{-\frac{1}{2}+\epsilon}.$$

In a sense, the proposition 2.7 states that the kernel S_N asymptotically looks like Π_N. This proposition was proven in [3], as a consequence of previously known results on the asymptotical behaviour of the Schwartz kernel of S_N near the diagonal set [8, 2, 1].

2.2 Toeplitz operators

Definition 2.8. Let M be a Kähler manifold, with equivariant Szegő projectors S_N.

Let $f \in C^\infty(M)$ be a smooth function on M.

The Toeplitz operator $T_N(f) : H_N(X) \to H_N(X)$ associated with the symbol f is defined as

$$T_N(f) = S_N f S_N.$$

2.2.1 Toeplitz operators on $\mathbb{C}^n$

We will use the special notation T_N^{flat} to denote Toeplitz operators on $\mathbb{C}^n$. We also release the condition that the symbol is bounded. This defines Toeplitz operators as unbounded operators on B_N.

If q is a quadratic form on $\mathbb{R}^{2n}$ identified with $\mathbb{C}^n$, then $T_N^{flat}(q)$ is essentially self-adjoint. This operator is related to the Weyl quantization $Op_W^h(q)$ with semi-classical parameter $h = N^{-1}$. In fact, $T_{h^{-1}}^{flat}(q)$ is conjugated, via a Bargmann transform, to $Op_W^h(q) + \dfrac{h}{2}\mathrm{tr}(q)$.

Definition 2.9. Let q be a non-negative quadratic form on $\mathbb{R}^{2n}$, identified with $\mathbb{C}^n$.

We define $\mu(q) := \inf\left(\mathrm{Sp}(T_1^{flat}(q))\right)$.

Remark 2.10. The function μ is invariant under the $U(n)$ symmetry, and continuous on the set of semi-definite quadratic forms [6].

2.2.2 Toeplitz operators on compact manifolds

When the base manifold M is compact and f is real-valued, for fixed N the operator $T_N(f)$ is a symmetric operator on a finite-dimensional space. In this setting, we will speak freely about eigenvalues and eigenvectors of Toeplitz operators.

It turns out that the definition 2.8 is not robust enough for the set of all Toeplitz operators to be an algebra. One finds instead that the composition of two Toeplitz operators can be written, in the general case, as a formal series of Toeplitz operators [7], that is:

$$T_N(f)T_N(g) = T_N(fg) + N^{-1}T_N(C_1(f,g)) + N^{-2}T_N(C_2(f,g)) + \dots.$$

This calls for a construction of Toeplitz operators associated with formal series, which are defined modulo the $O(N^{-\infty})$ sequences of operators. In this paper we only need to use the definition 2.8 and we will not compose

two Toeplitz operators. However, the properties of the C^*-algebra of formal series of Toeplitz operators lead to the following property, which appears in previous work [3], and which is an important first step towards the study of the low-energy spectrum.

Proposition 2.11. *Let M be a compact Kähler manifold and h a real nonnegative smooth function on M. Suppose that h vanishes exactly at order 2 on $\{h = 0\}$.*

Let $u = (u_N)_{N\in\mathbb{N}}$ be a sequence of unit elements of $L^2(X)$ such that, for every N, one has

$$T_N(h)u_N = \lambda_N u_N,$$

with $\lambda_N = O(N^{-1})$.

Then the sequence u concentrates on $\{h = 0\}$.

On a minimal point of h, one can pull-back the definition 2.9 by normal coordinates:

Definition 2.12. Let $h \in \mathbb{C}^\infty(M, \mathbb{R}^+)$. Let $P \in M$ such that $P(h) = 0$. Let ρ be a normal map around P; the function $h \circ \rho$ is well-defined and non-negative on a neighbourhood of 0 in $\mathbb{C}^n$, and the image of 0 is 0. Hence the 2-jet of $h \circ \rho$ is a quadratic form q.

We define $\mu(P)$ as $\mu(q)$.

Remark 2.13. A different choice of normal coordinates corresponds to a $U(n)$ change of variables for q, under which μ is invariant. Hence $\mu(P)$ does not depend on the choice of normal coordinates.

The function $P \mapsto \mu(P)$ is continuous as it is a composition of two continuous functions.

2.3 Main result

In a previous paper, the author studied the case where $\{h = 0\}$ consists in a finite set of non-degenerate critical points of h. The main result was that the sequence of first eigenvectors concentrates only on the points where μ is minimal. A similar results holds when $\{h = 0\}$ is a submanifold.

Theorem A. *Let $h \in \mathbb{C}^\infty(M, \mathbb{R}^+)$ be such that $\{h = 0\}$ is a submanifold of M. Suppose that h vanishes exactly at order 2 on $\{h = 0\}$. Let $\mu_{\min} = \min_{h(P)=0}(\mu(P))$.*

Let $v = (v_N)_{N\in\mathbb{N}}$ be a sequence of unit vectors of $L^2(X)$ such that, for each N, v_N is an eigenvector of $T_N(h)$ with minimal eigenvalue.

Let $U \subset M$ open, and suppose that

$$\mu_{\min} < \inf(\mu(P), P \in U, h(P) = 0).$$

Then, as $N \to +\infty$, one has

$$\|v_N 1_{x \in \pi^* U}\|_{L^2(X)} = O(N^{-\infty}).$$

3 Upper bound on the first eigenvalue

Proposition 3.1. *Let $h \geq 0$ be a smooth function on M. Suppose $\{h = 0\}$ is a non-empty submanifold of M. Let $\mu_{\min} = \min_{h(P)=0}(\mu(P))$.*

Then for every $\epsilon > 0$ there exists N_0 such that, for every $N \geq N_0$, one has

$$\min \mathrm{Sp}(T_N(h)) \geq N^{-1}(\mu_{\min} + \epsilon).$$

Proof. Let $P \in M$ be such that $h(P) = 0$, and ρ a normal map around P. Let q denote the 2-jet of h at P, read from the map ρ. Then $q \geq 0$.

If $q > 0$, it follows from the proposition 4.2 of [3] that, for every N, one can build an approximate eigenvector (with $O(N^{-1/2})$ error) for $T_N(h)$ with eigenvalue $N^{-1}\mu(P)$, hence the first eigenvalue of $T_N(h)$ is less than $N^{-1}\mu(P) + CN^{-3/2}$ for some C. Hence, for every $\epsilon > 0$, for N large enough one has

$$\min \mathrm{Sp}(T_N(h)) \geq N^{-1}(\mu(P) + \epsilon).$$

In the general case, for $\delta > 0$, let h_δ be a smooth function on M such that $h_\delta \geq h$ and such that the 2-jet of h_δ at P, read from the map ρ, is $z \mapsto q_\delta(z) := q(z) + \delta|z|^2$.

Now $q_\delta > 0$, so that, for every ϵ, there exists N_0 such that, for $N \geq N_0$, one has

$$\min \mathrm{Sp}(T_N(h_\delta)) \geq N^{-1}\left(\mu(q_\delta) + \frac{\epsilon}{2}\right).$$

On one hand, $T_N(h) \leq T_N(h_\delta)$ because the Toeplitz quantization is positive. On the other hand, μ is continuous, so for δ small enough one has $\mu(q_\delta) \leq \mu(q) + \frac{\epsilon}{2}$, which allow us to conclude. □

4 Localization at the miniwell

The following proposition was proven in previous work:

Proposition 4.1 (cf [3], prop 4.3)**.** *There exists $\delta \in [0, 1/2)$ such that, for every smooth function $h \geq 0$ on M, for every $\epsilon > 0$, there exists N_0 such that, for every $N \geq N_0$, for every $u \in L^2(X)$, if there exists $P \in M$ such that* $\operatorname{supp} u \subset B(P, N^{-\delta})$*, then*

$$\langle u, T_N(h)u \rangle \geq N^{-1}(\mu(P) - \epsilon)\|S_N u\|^2.$$

Thus, if the considered functions are sufficiently localized, then to minimize the quadratic form associated with $T_N(h)$ one has to get close to the points where μ is as small as possible.

In order to "localize" a generic function of $L^2(X)$, we wish to consider convenient open subsets of the zero set of h. They should be such that, on the overlap between two such subsets, the considered function is relatively small. A precise formulation of this lies in the following lemma.

Lemma 4.2. *Let Y be a compact Riemannian manifold. There exists a constant $C > 0$ such that, for every positive integrable function f on Y, for every $a > 0$ and $t \in (0,1)$, there exists a finite family $(U_j)_{j\in J}$ of open subsets of Y with the following properties:*

$$\forall j \in J,\ \mathrm{diam}(U_j) < a.$$

$$\forall j \in J,\ \mathrm{dist}\left(Y \setminus U_j, Y \setminus \bigcup_{i\neq j} U_i\right) \geq ta$$

$$\sum_{i\neq j} \int_{U_i \cap U_j} f \leq Ct \int_Y f.$$

Proof. Let $m \in \mathbb{N}$ be such that there exists a smooth embedding of differential manifolds from Y to $\mathbb{R}^m$, and let Φ be such an embedding. Φ may not preserve the Riemannian structure, so let c_1 be such that, for any $\xi \in TY$, one has

$$c_1 \|\Phi^* \xi\| \leq \|\xi\|.$$

We now let $L > 0$ such that any hypercube H in $\mathbb{R}^m$ of side $2/L$ is such that $\mathrm{diam}(\Phi^{-1}(H)) < a$.

At this point we make the further claim that $C = \frac{2maL}{c_1}$.

Let $1 \leq k \leq m$, and let Φ_k denote the k-th component of Φ. The function Φ_k is continuous from Y to a segment of $\mathbb{R}$. Without loss of generality this segment is $[0,1]$. Let g_k denote the integral of f along the level sets of Φ_k. The function g_k is a positive integrable function on $[0,1]$. Let $t' > 0$ be the inverse of an integer, and $0 \leq \ell \leq L-1$. In the interval $[\ell/L, (\ell+1)/L]$, there exists a subinterval I, of length t'/L, such that

$$\int_I g_k \leq t' \int_{\ell/L}^{(\ell+1)/L} g_k. \tag{1}$$

Indeed, one can cut the interval $[\ell/L, (\ell+1)/L]$ into $1/t'$ intervals of size t'/L. If none of these intervals was verifying (1), then the total integral would be strictly greater than itself.

Let $x_{k,\ell}$ denote the centre of such an interval. Then, let

$$\begin{aligned} V_{k,0} &= \left[0, x_{k,0} + \frac{t'}{2L}\right) \\ V_{k,\ell} &= \left(x_{k,\ell-1} - \frac{t'}{2L}, x_{k,\ell} + \frac{t'}{2L}\right) \text{ for } 1 \leq \ell \leq L \\ V_{k,L+1} &= \left(x_{k,L} - \frac{t'}{2L}, 1\right]. \end{aligned}$$

Each open set $V_{k,l}$ has a length smaller than $2/L$. The overlap of two consecutive sets has a length t', and the sum of the integrals on the overlaps is less than $t' \int_0^1 g_k = t' \int_Y f$.

Now let ν denote a polyindex $(\nu_k)_{1 \leq k \leq m}$, with $\nu_k \leq L+1$ for every k. Define

$$U_\nu = \Phi^{-1}\left(V_{1,\nu_1} \times V_{2,\nu_2} \times \ldots \times V_{m,\nu_m}\right).$$

Then $\operatorname{diam} U_\nu \leq a$ because it is the pull-back of an open set contained in a hypercube of side $2/L$. Moreover, one has

$$\operatorname{dist}\left(Y \setminus U_\nu, Y \setminus \bigcup_{\nu' \neq \nu} U_{\nu'}\right) \geq \frac{c_1 t'}{L}.$$

To conclude, observe that

$$\sum_{\nu \neq \nu'} \int_{U_\nu \cap U_{\nu'}} f = \sum_{k=1}^{m} \sum_{\ell=0}^{L} \int_{V_{k,\ell} \cap V_{k,\ell+1}} g_k \leq m t' \int_Y f.$$

It only remains to choose t' conveniently. The fraction $t\frac{aL}{c_1}$ may not be the inverse of an integer; however the inverse of some integer lies in $[\frac{aL}{2c_1}, \frac{aL}{c_1}]$. This allow us to conclude. □

Remark 4.3. In the previous lemma, the number of elements of J is bounded by a polynomial in a that depends only on the geometry of Y.

Let now Z denote the zero set of h. Let V_0 a small tubular neighbourhood of Z. It is well-known that V_0 is diffeomorphic to a neighbourhood of the zero section in a vector bundle over Z. We let $p : V \mapsto Z$ denote the composition of such a diffeomorphism and the projection on the base point.

For $N \in \mathbb{N}$, let $\lambda_N = \min \operatorname{Sp}(T_N(h))$. Because of proposition 3.1, one has $\lambda_N = O(N^{-1})$.

Let $u = (u_N)_{N\in\mathbb{N}}$ denote a sequence of normalized elements of $L^2(X)$ such that, for every $N \in \mathbb{N}$, one has $T_N(h)u_N = \lambda_N u_N$. Then u concentrates on Z. In particular, u_N is $O(N^{-\infty})$ outside V. Hence, with

$$\begin{array}{rcl} f_N : Z & \mapsto & \mathbb{R}^+ \\ z & \mapsto & \int_{p^{-1}(z)} |u_N(x)|^2 \mathrm{d}x, \end{array}$$

one has $\|f_N\|_{L^1(Z)} = 1 - O(N^{-\infty})$.

For every $N \in \mathbb{N}$, we apply lemma 4.2 with the following data:

- $Y = Z$
- $f = f_N$
- $a = N^{-\delta}$
- $t = N^{-\alpha}$.

Here α and δ will be chosen later on. Let $(U_{j,N})_{j\in J_N}$ denote a family of open subsets obtained by lemma 4.2, and for every $c > 0$, let

$$U_{j,N}^c = \{z \in Z, \mathrm{dist}(z, Z \setminus U_{j,N}) \geq cN^{-\alpha-\delta}\}.$$

If $c < \frac{1}{2}$, then by the second property of lemma 4.2, for every N, the family $(U_{j,N}^\epsilon)_{j\in J_N}$ covers Z. For every N, let $(\chi_{j,N})_{j\in J_N}$ denote a partition of the unity on $\pi^{-1}V$, associated with the family $(\pi^{-1}p^{-1}U_{j,N}^c)_{j\in J_N}$.

Now choose $\delta < \frac{1}{2}$ from the proposition 4.1 and let $\epsilon > 0$. There exists N_0 such that, for every $N \geq N_0$, for every $j \in J_N$, one has

$$\langle u_N\chi_{j,N}, T_N(h), \chi_{j,N}u_N\rangle \geq N^{-1}(\inf(\mu(z), z \in U_{j,N}^c) + \epsilon)\|S_N\chi_{j,N}u\|^2.$$

Let now $i \neq j \in J_N$. We wish to estimate the quantity

$$\begin{aligned} &|\langle u_N\chi_{i,N}, T_N(h), u_N\chi_{j,N}\rangle| \\ &\qquad \leq \iiint_{V_{i,N}^c\times X\times V_{j,N}^c} \left|\overline{u_N(x)}S_N(x,y)h(y)S_N(y,z)u_N(z)\right| \mathrm{d}x\mathrm{d}y\mathrm{d}z. \end{aligned}$$

Here $V_{j,N}^c = \pi^{-1}p^{-1}U_{j,N}^c$. Then, by definition of $U_{j,N}^c$, one has

$$\mathrm{dist}(V_{i,N}^c, X \setminus \pi^{-1}p^{-1}U_{i,N}) = cN^{\alpha+\delta}.$$

Lemma 4.4. *Choose α such that $\alpha+\delta < \frac{1}{2}$. There exists a constant C such that, for every N, for every $i \neq j \in J_N$, for every $\delta' < \frac{1}{2}$, there holds*

$$|\langle u_N\chi_{i,N}, T_N(h), u_N\chi_{j,N}\rangle| \leq CN^{-2\delta'} \int_{U_{i,N}\cap U_{j,N}} f_N + O(N^{-\infty}).$$

Proof. Let $V^c_{j,N} = \pi^{-1}p^{-1}U^c_{j,N} \subset X$. We wish to estimate the integral

$$\iiint_{V^c_{i,N}\times X\times V^c_{j,N}} \left|\overline{u_N(x)}S_N(x,y)h(y)S_N(y,z)u_N(z)\right| \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

For this, we reduce in two steps the domain of integration.
By definition of $U^c_{j,N}$, one has

$$\mathrm{dist}(V^c_{i,N}, X \setminus \pi^{-1}p^{-1}U_{i,N}) = cN^{\alpha+\delta}.$$

If $x \in V^c_{i,N}$, then $S_N(x,y)S_N(y,z)$ is $O(N^{-\infty})$ unless $z \in \pi^{-1}p^{-1}U_{i,N}$. Hence, up to an $O(N^{-\infty})$ error, the domain of integration can be replaced with

$$\pi^{-1}p^{-1}(U_{i,N}\cap U_{j,N}) \times X \times \pi^{-1}p^{-1}(U_{i,N}\cap U_{j,N}).$$

Moreover, recall u_N is an eigenvector of $T_N(h)$ with eigenvalue $O(N^{-1})$. Hence, the sequence u concentrates on Z so that for any δ', up to an $O(N^{-\infty})$ error, the domain of integration can be replaced with

$$\pi^{-1}p^{-1}(U_{i,N}\cap U_{j,N})\times\{z\in X, \mathrm{dist}(\pi(X),Z)\leq N^{-\delta'}\}\times\pi^{-1}p^{-1}(U_{i,N}\cap U_{j,N}).$$

On $\{z \in X, \mathrm{dist}(\pi(X), Z) \leq N^{-\delta'}\}$, the function h is smaller than $CN^{-2\delta'}$ for some constant C. In particular, there holds

$$|\langle u_N\chi_{i,N}, T_N(h), u_N\chi_{j,N}\rangle| \leq CN^{-2\delta'}\int_{U_{i,N}\cap U_{j,N}} f_N + O(N^{-\infty}).$$

□

The number of elements of J_N grows polynomially with N. Hence, one can sum the previous inequality:

$$\sum_{i\neq j}|\langle u_N\chi_{i,N}, T_N(h), u_N\chi_{j,N}\rangle| \leq CN^{-2\delta'}\sum_{i\neq j}\int_{U_{i,N}\cap U_{j,N}} f_N + O(N^{-\infty}).$$

Now, by lemma 4.2, and the fact that $\|u_N\| = 1$, as $N \to +\infty$ there holds

$$\sum_{i\neq j}|\langle u_N\chi_{i,N}, T_N(h), u_N\chi_{j,N}\rangle| \leq CN^{-2\delta'-\alpha} + O(N^{-\infty}).$$

Choose δ' such that $2\delta' + \alpha > 1$. Then, as $N \to +\infty$, one has

$$\langle u_N, T_N(h)u_N\rangle \geq N^{-1}\sum_{j\in J_N}(\inf(\mu(z), z \in U^c_{j,N})+\epsilon)\|S_N\chi_{j,N}u_N\|_{L^2}+o(N^{-1}).$$

Let now W be an open set of Z such that $\inf_{z\in W}\mu(z) \geq \mu_{\min} + 2\epsilon$. Then

$$\sum_{U^c_{j,N}\subset W} \|S_N\chi_{j,N}u\|_{L^2} = o(1).$$

In particular, with $\chi_N = \sum_{U^c_{j,N}\subset W}\chi_{j,N}$, one has $\|S_N\chi_N u\|_{L^2} = o(1)$. Moreover, for every $W' \subset\subset W$, there is N_1 such that $\chi_N = 1$ on W' for $N \geq N_0$. Hence, for every $W'' \subset\subset W'$, on W'' there holds $u = S_N u = S_N\chi_N u + O(N^{-\infty})$.

We are able to conclude: for every $W'' \subset\subset W$, the L^2 norm of u_N on W'' is $o(1)$. This concludes the proof.

5 Acknowledgements

The author thanks N. Anantharaman, L. Charles and B. Helffer for their helpful insight and encouragement while writing this paper.

References

[1] L. Charles. *Aspects semi-classiques de la quantification géométrique*. PhD thesis, Université Paris 9, 2000.

[2] X. Dai, K. Liu, and X. Ma. On the asymptotic expansion of Bergman kernels. *J. differential Geometry*, 72:1–41, 2006.

[3] A. Deleporte. Low-energy spectrum of Toeplitz operators with a mini-well. 2016. Under publication.

[4] B. Douçot and P. Simon. A semiclassical analysis of order from disorder. *Journal of Physics A: Mathematical and General*, 31(28):5855, 1998.

[5] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique V : Étude des minipuits. *Current topics in partial differential equations*, pages 133–186, 1986.

[6] A. Melin. Lower bounds for pseudo-differential operators. *Arkiv för Matematik*, 9(1):117–140, 1971.

[7] M. Schlichenmaier. Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization. In *Conférence Moshé Flato 1999*, pages 289–306. Springer, 2000.

[8] B. Shiffman and S. Zelditch. Asymptotics of almost holomorphic sections on symplectic manifolds. *J. reine angew. Math.*, 544:181–222, 2002.

[9] N. M. J. Woodhouse. *Geometric quantization*. Oxford University Press, 1997.