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A nonlinear consistent penalty method for positivity preservation in the finite element approximation of the transport equation

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Abstract

We devise and analyze a new stabilized finite element method to solve the first-order transport (or advection-reaction) equation. The method combines the usual Galerkin/Least-Squares approach to achieve stability with a nonlinear consistent penalty term inspired by recent discretizations of contact problems to weakly enforce a positivity condition on the discrete solution. We prove the existence and the uniqueness of the discrete solution. Then we establish quasi-optimal error estimates for smooth solutions bounding the usual error terms in the Galerkin/Least-Squares error analysis together with the violation of the maximum principle by the discrete solution. Numerical examples are presented to illustrate the performances of the method.

Key words: stabilized finite element method, consistent penalty, positivity preserving, transport equation, discrete maximum principle

1 Introduction

The design of robust and accurate finite element methods for first-order transport (or advection-reaction) equations or for advection-dominated advection-diffusion equations remains an active field of research. Indeed, the task of designing a numerical scheme that is of higher order than one in the zone where the exact solution is smooth, but preserves the monotonicity properties of the exact solution on the discrete level, is nontrivial. Since it is known that such a scheme necessarily must be nonlinear even for linear equations, one typical strategy adopted when working with stabilized finite element methods is to add an additional nonlinear shock-capturing term, designed to make the

method satisfy a discrete maximum principle; see, e.g., [1,2]. These methods, however, often result in ill-conditioned nonlinear equations and include parameters that may be difficult to tune and that depend on the mesh geometry. Another approach is the so-called flux-corrected finite element method [3,4]. In this scheme, the system matrix is manipulated so that it becomes a so called M-matrix, the inverse of which has positive coefficients which yields a maximum principle. Such a scheme is monotonicity preserving, but of first order. In order to improve the accuracy, anti-diffusive mechanisms, or flux-limiter techniques, have been proposed so as to reduce the amount of dissipation in the smooth regions by blending a low- and a high-order approximation [4,5].

In this paper, we consider a method that follows a completely different approach. The starting observation is that, if the problem satisfies a maximum principle of the form

$$u \geq 0, \tag{1}$$

then this constraint can be added to the problem without any perturbation. On the discrete level however, this the condition (1) is not necessarily satisfied, unless enforced by the numerical method, which is the purpose of all the methods discussed above. One may argue that one can solve the problem with any method under the constraint (1). This was proposed in [6]. The resulting method, in the form of a variational inequality, is unwieldy, with the need of Lagrange multipliers to impose the constraint and associated stability and solver issues. In the present work, we instead draw on recent advances in the field of contact problems [7,8], where the variational inequality instead is discretized on the form of a nonlinear consistent penalty method. Note however that in the present context the formulation cannot be associated with an augmented Lagrangian method, due to the lack of symmetry of the formulation.

Our method combines the well-known Galerkin/Least Squares (GaLS) discretization of the transport equation (see [9,10]) with a nonlinear switch inspired from [8] and that changes the equations in the zones where (1) is violated to a least-squares penalty on this inequality (more precisely, the negative part of the discrete solution) together with a least-squares penalty on the residual. Our method is not meant to enforce strictly a discrete maximum principle, but to blend asymptotically the satisfaction of the GaLS approximation of the PDE with the satisfaction of the discrete maximum principle, in the same spirit of the above-mentioned methods for contact problems. We first prove the existence and uniqueness of the discrete solution. Then, our main result is Theorem 3.1 where we establish a quasi-optimal error estimate bounding at the same time the error measured in the usual GaLS norm (combining the L^2 -norm on the solution, its boundary values, and the advective derivative weighted by the local mesh size) and the violation of the positivity condition (1) measured by the weighted L^2 -norm of the negative part of the discrete solution. These convergence results hold for all polynomial orders

$k \geq 1$. Another salient feature of our method is its flexibility in incorporating a priori lower and upper bounds on the discrete solution by simply adding the corresponding consistent penalty term to the discrete formulation. Finally, we report some numerical experiments illustrating that accurate solutions with mild and asymptotically vanishing violations of the discrete maximum principle can be obtained at moderate computational costs.

2 Model problem and GaLS discretization

Let Ω be an open, bounded, Lipschitz set in \mathbb{R}^d , $d \in \{2, 3\}$, let $\beta \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ be a given advection velocity, and let $\sigma \in L^\infty(\Omega; \mathbb{R})$ be a given reaction coefficient. We assume that β and σ satisfy the following (classical) positivity assumption: There exists $\sigma_0 > 0$ such that

$$0 < \sigma_0 \leq \sigma - \frac{1}{2} \nabla \cdot \beta, \quad \text{a.e. in } \Omega. \quad (2)$$

We split the boundary $\partial\Omega$ of Ω as $\partial\Omega = \partial\Omega^- \cup \partial\Omega^0 \cup \partial\Omega^+$ with $\partial\Omega^- = \{x \in \partial\Omega \mid (\beta \cdot n)(x) < 0\}$ (inflow boundary), $\partial\Omega^0 = \{x \in \partial\Omega \mid (\beta \cdot n)(x) = 0\}$ (characteristic boundary), and $\partial\Omega^+ = \{x \in \partial\Omega \mid (\beta \cdot n)(x) > 0\}$ (outflow boundary). We consider the following model problem: Find $u : \Omega \rightarrow \mathbb{R}$ such that

$$A(u) := \beta \cdot \nabla u + \sigma u = f \quad \text{in } \Omega, \quad (3a)$$

$$u = g \quad \text{on } \partial\Omega. \quad (3b)$$

We assume that $f \in L^2(\Omega; \mathbb{R})$ and $g \in L^2_{|\beta \cdot n|}(\partial\Omega^-; \mathbb{R})$ and look for a weak solution in the graph space $V := \{v \in L^2(\Omega; \mathbb{R}) \mid \beta \cdot \nabla v \in L^2(\Omega; \mathbb{R})\}$. Assuming that $\text{dist}(\partial\Omega^-, \partial\Omega^+) > 0$, one can show [11] that functions in the graph space admit a trace in the weighted space $L^2_{|\beta \cdot n|}(\partial\Omega; \mathbb{R})$ and that there exists one and only one weak solution in the graph space V to the model problem (3). In particular, we observe that the following positivity condition holds:

$$\sigma_0 \|v\|_\Omega^2 + \frac{1}{2} \| |\beta \cdot n|^{\frac{1}{2}} v \|_{\partial\Omega}^2 \leq (\beta \cdot \nabla v + \sigma v, v)_\Omega - \langle (\beta \cdot n)v, v \rangle_{\partial\Omega^-}, \quad \forall v \in V. \quad (4)$$

Several stabilized H^1 -conforming finite element methods are available in the literature to discretize (3). We focus on the Galerkin/Least-Squares method (GaLS). Let \mathcal{T}_h be a mesh from a shape-regular mesh sequence. We assume for simplicity that Ω is a polytope (polygon/polyhedron) so that \mathcal{T}_h can cover Ω exactly. Let $k \geq 1$ be the polynomial degree and consider the H^1 -conforming

finite element space

$$V_h^k := \{v_h \in C^0(\bar{\Omega}; \mathbb{R}) \mid v_h|_T \in \mathbb{P}_k(T; \mathbb{R}), \forall T \in \mathcal{T}_h\}, \quad (5)$$

where $\mathbb{P}_k(T; \mathbb{R})$ denotes the space composed of \mathbb{R} -valued functions that are the restriction to T of polynomials of degree at most k . We consider the following discrete problem: Find $u_h \in V_h^k$ such that

$$a_h^\tau(u_h, w_h) = \ell_h^\tau(w_h), \quad \forall w_h \in V_h^k, \quad (6)$$

with the following bilinear and linear forms:

$$a_h^\tau(v_h, w_h) := (A(v_h), w_h + \tau A(w_h))_\Omega - \langle (\beta \cdot n)v_h, w_h \rangle_{\partial\Omega^-}, \quad (7a)$$

$$\ell_h^\tau(w_h) := (f, w_h + \tau A(w_h))_\Omega - \langle (\beta \cdot n)g, w_h \rangle_{\partial\Omega^-}. \quad (7b)$$

The cut-off parameter τ is piecewise constant on \mathcal{T}_h and is of the order of the local mesh size h_T for all $T \in \mathcal{T}_h$; more precisely, a suitable choice is $\tau|_T = \min(\sigma_0^{-1}, \beta_T^{-1}h_T)$ with $\beta_T = \|\beta\|_{L^\infty(T; \mathbb{R}^d)}$. By construction, the discrete bilinear form a_h^τ is coercive with respect to the norm:

$$a_h^\tau(v, v) \geq \|v\|^2 := \sigma_0 \|v\|_\Omega^2 + \| |\beta \cdot n|^{\frac{1}{2}} v \|_{\partial\Omega}^2 + \|\tau^{\frac{1}{2}} A(v)\|_\Omega^2, \quad \forall v \in V. \quad (8)$$

Moreover, exact consistency holds, and the following quasi-optimal error estimates can be established [9,10]: There exists C , uniform, such that

$$\|u - u_h\| \leq C \inf_{v_h \in V_h^k} (\|\tau^{-\frac{1}{2}}(u - v_h)\|_\Omega + \| |\beta \cdot n|^{\frac{1}{2}}(u - v_h) \|_{\partial\Omega} + \|\tau^{\frac{1}{2}} A(u - v_h)\|_\Omega), \quad (9)$$

and if $u \in H^{k+1}(\Omega)$, $\|u - u_h\| \leq C(\sum_{T \in \mathcal{T}_h} \phi_T h_T^{2k+1} |u|_{H^{k+1}(T; \mathbb{R})}^2)^{\frac{1}{2}}$ with $\phi_T = \max(\beta_T, \sigma_0 h_T)$.

3 The consistent penalty method

The model problem (3) has a maximum principle; for instance, if $f \geq 0$ and $g \geq 0$, then $u \geq 0$ in Ω . Unfortunately, this property rarely carries over to finite element discretizations. Our goal is to modify the GaLS finite element approximation (6) by using a consistent penalty method.

Let $\gamma > 0$ be a penalty parameter. For any function $v \in V$, let us define the function $\Omega \rightarrow \mathbb{R}$ such that

$$\xi^\gamma(v) := [v - \gamma(A(v) - f)]_-, \quad (10)$$

where $x_- = \frac{1}{2}(x - |x|)$ denotes the negative part of the real number x . Note that $\xi^\gamma(u) = 0$ in Ω for the weak solution u since $A(u) = f$ and $u_- = 0$. Let

us consider the following discrete problem: Find $u_h \in V_h^k$ such that

$$a_h^{\tau\gamma}(u_h; w_h) = \ell_h^\tau(w_h), \quad \forall w_h \in V_h^k, \quad (11)$$

with

$$a_h^{\tau\gamma}(v_h; w_h) := a_h^\tau(v_h, w_h) + (\gamma^{-1}\xi^\gamma(v_h), w_h)_\Omega. \quad (12)$$

Since $\xi^\gamma(u)$ vanishes identically in Ω , exact consistency still holds for (11). The discrete problem (11) remains meaningful and exactly consistent if the penalty parameter γ is replaced by a function taking uniformly positive values in Ω . We will take γ to be piecewise constant on the mesh \mathcal{T}_h since the error analysis below will reveal that quasi-optimal error estimates are obtained by taking γ to be locally of the order of h_T (on quasi-uniform mesh sequences, a constant function γ can be considered).

3.1 Rationale of the consistent penalty method

Before embarking on the analysis of the method, let us briefly discuss the design principle behind the approach. First, we observe that if $[u_h - \gamma(A(u_h) - f)]_- = 0$, then the formulation coincides with the standard GaLS discretization. Assume now that $[u_h - \gamma(A(u_h) - f)]_- \neq 0$ everywhere in the macroelement $\Omega_i := \text{supp}(\varphi_i)$ where φ_i is an interior nodal (or hat) basis function. Then, since $x_- = x$ if $x_- \neq 0$, we observe that the standard Galerkin part is eliminated by the second term in the penalty term, so that (11) with $w_h = \varphi_i$ becomes

$$(\gamma^{-1}u_h, \varphi_i)_{\Omega_i} + (\tau(A(u_h) - f), A(\varphi_i))_{\Omega_i} = 0. \quad (13)$$

This shows that the nonlinear penalty term changes the discrete equation locally to the sum of two least-squares contributions, one on the violation of positivity by the discrete solution and one on the PDE residual. By choosing γ small, one can expect that the violation of the maximum principle is reduced. This is indeed one of the main conclusions of the error analysis below, where we additionally prove that quasi-optimal error estimates of the form (9) also hold for the consistent penalty method.

3.2 Well-posedness and convergence

Let us first establish that $a_h^{\tau\gamma}$ has reasonable monotonicity properties.

Lemma 3.1 (Monotonicity) *There exists $\alpha > 0$, uniform, such that the*

following holds for all $u_1, u_2 \in V$,

$$\alpha(\|u_1 - u_2\|^2 + \|\gamma^{-\frac{1}{2}}(\xi^\gamma(u_1) - \xi^\gamma(u_2))\|_\Omega^2) \leq a_h^{\tau\gamma}(u_1; u_1 - u_2) - a_h^{\tau\gamma}(u_2; u_1 - u_2), \quad (14a)$$

$$\alpha(\|u_1\|^2 + \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_1)\|_\Omega^2) \leq a_h^{\tau\gamma}(u_1; u_1) + \gamma\|f\|_\Omega^2. \quad (14b)$$

Proof. We observe that

$$\begin{aligned} a_h^{\tau\gamma}(u_1; u_1 - u_2) - a_h^{\tau\gamma}(u_2; u_1 - u_2) \\ = a_h^\tau(u_1 - u_2, u_1 - u_2) + (\gamma^{-1}(\xi^\gamma(u_1) - \xi^\gamma(u_2)), u_1 - u_2)_\Omega \\ \geq \|u_1 - u_2\|^2 + (\gamma^{-1}(\xi^\gamma(u_1) - \xi^\gamma(u_2)), u_1 - u_2)_\Omega, \end{aligned}$$

where we have used (8). Moreover, we have

$$\begin{aligned} (\gamma^{-1}(\xi^\gamma(u_1) - \xi^\gamma(u_2)), u_1 - u_2)_\Omega \\ = (\gamma^{-1}(\xi^\gamma(u_1) - \xi^\gamma(u_2)), u_1 - \gamma(A(u_1) - f) - (u_2 - \gamma(A(u_2) - f)))_\Omega \\ + (\xi^\gamma(u_1) - \xi^\gamma(u_2), A(u_1 - u_2))_\Omega \\ \geq \|\gamma^{-\frac{1}{2}}(\xi^\gamma(u_1) - \xi^\gamma(u_2))\|_\Omega^2 + (\xi^\gamma(u_1) - \xi^\gamma(u_2), A(u_1 - u_2))_\Omega, \end{aligned}$$

where we have used the fact that

$$|x_- - y_-|^2 \leq (x_- - y_-)(x - y), \quad \forall x, y \in \mathbb{R}. \quad (15)$$

Using a Young's inequality and the fact that $\gamma \leq \tau$, we infer that

$$((\xi^\gamma(u_1) - \xi^\gamma(u_2)), A(u_1 - u_2))_\Omega \geq -\frac{1}{2}\|\gamma^{-\frac{1}{2}}(\xi^\gamma(u_1) - \xi^\gamma(u_2))\|_\Omega^2 - \frac{1}{2}\|\tau^{\frac{1}{2}}A(u_1 - u_2)\|_\Omega^2.$$

Putting everything together shows that (14a) holds with $\alpha = \frac{1}{2}$. Proceeding similarly, we infer that

$$\begin{aligned} a_h^{\tau\gamma}(u_1; u_1) &= a_h^\tau(u_1, u_1) + (\gamma^{-1}\xi^\gamma(u_1), u_1) \\ &\geq \|u_1\|^2 + (\gamma^{-1}\xi^\gamma(u_1), u_1) \\ &\geq \|u_1\|^2 + \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_1)\|_\Omega^2 + (\xi^\gamma(u_1), A(u_1) - f)_\Omega, \end{aligned}$$

and we conclude that (14b) also holds with $\alpha = \frac{1}{2}$.

We can now prove that the discrete problem (11) is well-posed.

Proposition 3.1 (Well-posedness) *Assume that $0 < \gamma \leq \tau$, then the discrete problem (11) admits one and only one solution.*

Proof. Uniqueness follows from (14a). To prove existence, let $N_V := \dim V_h^k$ and let $G : \mathbb{R}^{N_V} \rightarrow \mathbb{R}^{N_V}$ be the map defined by $(G(U), V)_{\mathbb{R}^{N_V}} := a_h^{\tau\gamma}(u_h; v_h) - \ell_h^\tau(v_h)$, where $U, V \in \mathbb{R}^{N_V}$ are the component vectors associated with the

functions u_h, v_h in the Lagrange basis of V_h^k . It is readily seen that G is a continuous map (observe in particular that $|x_- - y_-| \leq |x - y|$ for all $x, y \in \mathbb{R}$). Furthermore, since Cauchy–Schwarz inequalities and $\tau \leq \sigma_0^{-1}$ show that $|\ell_h^\tau(v_h)| \leq K \|v_h\|$ with $K = (2\sigma_0^{-\frac{1}{2}} \|f\|_\Omega + \|\beta \cdot n\|_{\frac{1}{2}} \|g\|_{\partial\Omega^-})$, we infer using (14b) that

$$\begin{aligned} (G(U), U)_{\mathbb{R}^{N_V}} &= a_h^{\tau\gamma}(u_h; u_h) - \ell_h^\tau(u_h) \\ &\geq \frac{1}{2}(\|u_h\|^2 + \|\gamma^{-\frac{1}{2}} \xi^\gamma(u_h)\|_\Omega^2) - \gamma \|f\|_\Omega^2 - K \|u_h\|. \end{aligned}$$

This proves that there is a real number, say K' , so that $(G(U), U)_{\mathbb{R}^{N_V}} > 0$ for all $U \in \mathbb{R}^{N_V}$ with $\|U\|_{\mathbb{R}^{N_V}} \geq K'$. Reasoning by contradiction, let us assume that $G(U) = 0$ does not have any solution in the ball with radius K' , say $B := B(0_{\mathbb{R}^{N_V}}, K')$. Let us define the map

$$\phi(U) := -\frac{G(U)}{\|G(U)\|_{\mathbb{R}^{N_V}}},$$

which maps B into itself, and ϕ is continuous by the continuity of G . By Brouwer's fixed point theorem, there exists $U \in B$ such that $U = \phi(U)$. However, this implies that

$$\|U\|_{\mathbb{R}^{N_V}}^2 = -\frac{(G(U), U)_{\mathbb{R}^{N_V}}}{\|G(U)\|_{\mathbb{R}^{N_V}}}$$

and this is impossible by the positivity property of G .

The next theorem is the main result of this paper. It shows that the GaLS finite element method with penalty has essentially the same behavior as that without penalty when approximating smooth solutions.

Theorem 3.1 (Error estimate) *Let $u \in V$ be the solution to (3) and let $u_h \in V_h^k$ be the solution to (11). Assume that $0 < \gamma \leq \tau$. Then there exists $C > 0$, uniform, such that*

$$\begin{aligned} \|u - u_h\| + \|\gamma^{-\frac{1}{2}}[u_h]_-\|_\Omega &\leq \\ C \inf_{v_h \in V_h^k} &(\|\tau^{\frac{1}{2}} A(u - v_h)\|_\Omega + \|\beta \cdot n\|_{\frac{1}{2}} \|u - v_h\|_{\partial\Omega} + \|\gamma^{-\frac{1}{2}}(u - v_h)\|_\Omega). \end{aligned} \quad (16)$$

Moreover, if $u \in H^{k+1}(\Omega)$, τ is chosen as in the GaLS method as $\tau|_T = \min(\sigma_0^{-1}, \beta_T^{-1} h_T)$, and $c\tau|_T \leq \gamma|_T$ for all $T \in \mathcal{T}_h$ with c uniformly bounded from below away from zero, then

$$\|u - u_h\| + \|\gamma^{-\frac{1}{2}}[u_h]_-\|_\Omega \leq C \left(\sum_{T \in \mathcal{T}_h} \phi_T h_T^{2k+1} \|u\|_{H^{k+1}(T; \mathbb{R})}^2 \right)^{\frac{1}{2}}. \quad (17)$$

Proof. Let $e = u - u_h$. Then, using (8), we infer that

$$\|e\|^2 \leq a_h^\tau(e, e) = a_h^\tau(e, u - v_h) + a_h^\tau(e, v_h - u_h).$$

Moreover, the exact consistency of the GaLS approximation and the definition of the discrete problem (11) imply that

$$\begin{aligned} a_h^\tau(e, v_h - u_h) &= \ell_h^\tau(v_h - u_h) - \ell_h^\tau(v_h - u_h) + (\gamma^{-1}\xi^\gamma(u_h), v_h - u_h)_\Omega \\ &= (\gamma^{-1}\xi^\gamma(u_h), v_h - u_h)_\Omega \\ &= (\gamma^{-1}\xi^\gamma(u_h), v_h - u + \gamma A(e))_\Omega + (\gamma^{-1}\xi^\gamma(u_h), u - u_h - \gamma A(e))_\Omega. \end{aligned}$$

Since $\xi^\gamma(u) = 0$, using the monotonicity property (15), we infer that

$$(\gamma^{-1}\xi^\gamma(u_h), u - u_h - \gamma A(e))_\Omega \leq -\|\gamma^{-\frac{1}{2}}\xi^\gamma(u_h)\|_\Omega^2.$$

As a result, we obtain

$$\|e\|^2 + \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_h)\|_\Omega^2 \leq a_h^\tau(e, u - v_h) + (\gamma^{-1}\xi^\gamma(u_h), v_h - u + \gamma A(e))_\Omega.$$

The boundedness properties of the GaLS approximation yield

$$a_h^\tau(e, u - v_h) \leq \|e\|(\|\tau^{\frac{1}{2}}A(u - v_h)\|_\Omega + \| |\beta \cdot n|^{\frac{1}{2}}(u - v_h) \|_{\partial\Omega} + \|\tau^{-\frac{1}{2}}(u - v_h)\|_\Omega).$$

Moreover, we have

$$(\gamma^{-1}\xi^\gamma(u_h), v_h - u + \gamma A(e))_\Omega \leq \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_h)\|_\Omega(\|\gamma^{-\frac{1}{2}}(u - v_h)\|_\Omega + \|\gamma^{\frac{1}{2}}A(u - v_h)\|_\Omega).$$

Collecting these two bounds and using that $\gamma \leq \tau$, we infer that $\|e\| + \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_h)\|_\Omega$ is bounded by the right-hand side of (16). To conclude that (16) holds, it thus suffices to establish that

$$\|\gamma^{-\frac{1}{2}}[u_h]_-\|_\Omega \leq \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_h)\|_\Omega + \|e\|.$$

This inequality, in turn, follows from the elementary inequality $|[x + y]_-| \leq |x_-| + |y_-|$ for arbitrary real numbers x and y , leading to

$$\|\gamma^{-\frac{1}{2}}[u_h]_-\|_\Omega \leq \|\gamma^{-\frac{1}{2}}\xi^\gamma(u_h)\|_\Omega + \|\gamma^{\frac{1}{2}}[(A(u_h) - f)]_-\|_\Omega,$$

together with the fact that $A(u) = f$ and the assumption that $\gamma \leq \tau$ so that $\|\gamma^{\frac{1}{2}}[(A(u_h) - f)]_-\|_\Omega \leq \|e\|$. Finally, (17) results from the approximation properties of finite elements and the assumptions on τ and γ .

The error estimates derived in Theorem 3.1 show that the present consistent penalty method delivers similar bounds on the error $\|u - u_h\|$ to those obtained with the usual GaLS discretization, while additionally controlling the violation of positivity by means of the measure $\|\gamma^{-\frac{1}{2}}[u_h]_-\|_\Omega$ (note that γ scales as the mesh size, so that the factor $\gamma^{-\frac{1}{2}}$ in front of $[u_h]_-$ makes the bound even

stronger.

4 Numerical example

We consider problem (3) in the domain $\Omega \subset \mathbb{R}^2$ shown in the left panel of Figure 1, with

$$\beta = (x^2 + y^2)^{-\frac{1}{2}} \begin{pmatrix} y \\ -x \end{pmatrix}, \quad \sigma = 0, \quad f = 0.$$

The advection field β rotates clockwise, and the inflow boundary corresponds to the part of $\partial\Omega$ where $x = 0$. The exact solution given by $u = \frac{1}{2}(\tanh(((x^2 + y^2)^{\frac{1}{2}} - 0.5)/\epsilon) + 1.0)$ is a consequence of inflow data imposed on the inflow boundary (see the contourlines in the right panel of Figure 1). The boundary data has a sharp layer of width ϵ at $y = 0.5$. This creates spurious under- and overshoots that are transported downstream throughout the domain. We solve the problem (3) with affine ($k = 1$) and quadratic ($k = 2$) finite elements on a sequence of quasi-uniform meshes characterized by the mesh sizes $h = 0.1/2^l$ with $l \in \{0, 1, 2, 3, 4\}$. Nodal quadrature was used for the evaluation of the nonlinear penalty term was evaluated using nodal quadrature when $k = 1$ and a quadrature exact for polynomials of order five when $k = 2$. Observe that the integrand of the nonlinear term is only H^1 in the interior of the element, and an exact quadrature requires a careful local analysis of where the nonlinearity is active.

We compute approximate solutions for both a smooth layer ($\epsilon = 0.1$) and a sharp layer ($\epsilon = 0.01$) using either the nonlinear method (11) or the standard GaLS method obtained by dropping the nonlinear term. We set the penalty parameter to $\gamma = 0.0001h$ for $k = 1$ and to $\gamma = 0.005h$ for $k = 2$. Our results show that these choices are sufficient to reduce undershoots to less than one percent in all cases. Strengthening the penalty in the case of quadratic approximation did not improve the positivity, but increased the stiffness of the nonlinear problem. The results for the error $e = u - u_h$ in the L^2 -norm and in the streamline derivative, the violations of the maximum principle evaluated as $e_{\min} := -\min_{x \in \Omega} (u_h(x))_-$ and $e_{\max} := \max_{x \in \Omega} u(x) - 1$, and the error on the global conservation property $\Phi(u_h) := |\int_{\partial\Omega} (\beta \cdot n) u_h \, ds| = 0$ are presented in Tables 1 to 6 where the symbol ‘-’ in the columns for e_{\min} or e_{\max} means that the discrete maximum principle was satisfied. Note that the lack of exact global conservation is due to quadrature errors. In all cases we solved the nonlinear system using fixed-point iteration to an accuracy of 10^{-6} on the increment $\|u_h^n - u_h^{n+1}\|_{\Omega}$. Such a stringent convergence criterion has been used for the sole purpose of numerical illustrations. In practice, one computationally-effective

| l | $\ e\ _{\Omega}$ | $\ \beta \cdot \nabla e\ _{\Omega}$ | e_{\min} | e_{\max} | $\Phi(u_h)$ |
|-----|------------------|-------------------------------------|------------|------------|-------------|
| 0 | 1.13e-2 | 9.13e-2 | – | 1.09e-2 | 4.59e-3 |
| 1 | 2.33e-3 | 2.65e-2 | – | 1.93e-3 | 1.49e-4 |
| 2 | 9.57e-4 | 1.94e-2 | – | 9.20e-4 | 2.72e-5 |
| 3 | 2.54e-4 | 9.92e-3 | – | 3.90e-4 | 8.59e-6 |
| 4 | 6.35e-5 | 4.64e-3 | – | 1.6e-4 | 2.17e-6 |

Table 1

Nonlinear method, $k = 1$, $\gamma = 0.0001h$, $\tau = h/2$, $\epsilon = 0.1$

| l | $\ e\ _{\Omega}$ | $\ \beta \cdot \nabla e\ _{\Omega}$ | e_{\min} | e_{\max} | $\Phi(u_h)$ |
|-----|------------------|-------------------------------------|------------|------------|-------------|
| 0 | 1.18e-2 | 9.69e-2 | 2.37e-2 | 1.09e-2 | 3.44e-4 |
| 1 | 2.33e-3 | 2.65e-2 | – | 1.93e-3 | 1.45e-4 |
| 2 | 9.57e-4 | 1.94e-2 | – | 9.20e-4 | 2.72e-5 |
| 3 | 2.54e-4 | 9.92e-3 | – | 3.90e-4 | 8.59e-6 |
| 4 | 6.35e-5 | 4.64e-3 | – | 1.60e-4 | 2.17e-6 |

Table 2

Linear method, $k = 1$, $\tau = h/2$, $\epsilon = 0.1$

possibility is to set the fixed-point convergence tolerance to $(0.01/l)^{\frac{3}{2}}$ (where l denotes the level of mesh refinement) rather than to the fixed (and very small) value of 10^{-6} . For example, for the case presented in Table 3, this modified tolerance leads to identical results as for the fixed tolerance, but using only two iterations at each mesh refinement level. In this context it is interesting to compare the behaviour of the present method with diffusion based shock-capturing terms with artificial viscosity depending on the residual. We observe that for the present method, the computational cost is reduced as the solution improves and the violation of the DMP is isolated close to layers, whereas no such reduction is observed for nonlinear diffusion, where the nonlinearity appears to have a much more global character.

Figure 1. Left: computational mesh. Right: contourlines of solution

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| l | $\ e\ _\Omega$ | $\ \beta \cdot \nabla e\ _\Omega$ | e_{\min} | e_{\max} | $\Phi(u_h)$ |
|-----|----------------|-----------------------------------|------------|------------|-------------|
| 0 | 7.43e-2 | 5.64e-1 | – | 8.86e-2 | 2.24e-3 |
| 1 | 5.23e-2 | 5.04e-1 | – | 1.13e-1 | 1.18e-3 |
| 2 | 2.68e-2 | 6.72e-1 | – | 6.77e-2 | 2.20e-3 |
| 3 | 1.58e-2 | 6.46e-1 | – | 5.88e-2 | 2.03e-3 |
| 4 | 5.19e-3 | 2.97e-1 | – | 2.00e-2 | 4.34e-4 |

Table 3

Nonlinear method, $k = 1$, $\gamma = 0.0001h$, $\tau = h/2$, $\epsilon = 0.01$

| l | $\ e\ _\Omega$ | $\ \beta \cdot \nabla e\ _\Omega$ | e_{\min} | e_{\max} | $\Phi(u_h)$ |
|-----|----------------|-----------------------------------|------------|------------|-------------|
| 0 | 7.43e-2 | 5.64e-1 | 9.94e-2 | 8.86e-2 | 5.68e-4 |
| 1 | 5.23e-2 | 5.04e-1 | 7.01e-2 | 1.13e-1 | 1.41e-4 |
| 2 | 2.69e-2 | 6.67e-1 | 6.41e-2 | 6.77e-2 | 1.84e-5 |
| 3 | 1.59e-2 | 6.47e-1 | 6.01e-2 | 5.88e-2 | 1.20e-5 |
| 4 | 5.31e-3 | 2.95e-1 | 1.94e-2 | 2.0e-2 | 2.01e-6 |

Table 4

Linear method, $k = 1$, $\tau = h/2$, $\epsilon = 0.01$

| l | $\ e\ _\Omega$ | $\ \beta \cdot \nabla e\ _\Omega$ | e_{\min} | e_{\max} | $\Phi(u_h)$ |
|-----|----------------|-----------------------------------|------------|------------|-------------|
| 0 | 5.60e-2 | 6.47e-1 | 4.36e-3 | 7.58e-2 | 5.82e-3 |
| 1 | 2.77e-2 | 7.17e-1 | 5.25e-3 | 6.27e-2 | 2.47e-3 |
| 2 | 1.05e-2 | 4.39e-1 | 1.81e-4 | 3.91e-2 | 1.06e-3 |
| 3 | 2.86e-3 | 1.90e-1 | 4.07e-4 | 8.81e-3 | 1.10e-4 |
| 4 | 3.96e-4 | 4.07e-2 | 2.31e-5 | – | 3.78e-9 |

Table 5

Nonlinear method, $k = 2$, $\gamma = 0.005h$, $\tau = h/2$, $\epsilon = 0.01$

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| l | $\ e\ _\Omega$ | $\ \beta \cdot \nabla e\ _\Omega$ | e_{\min} | e_{\max} | $\Phi(u_h)$ |
|-----|----------------|-----------------------------------|------------|------------|-------------|
| 0 | 5.33e-2 | 6.58e-1 | 9.26e-2 | 7.68e-2 | 3.34e-4 |
| 1 | 2.56e-2 | 7.29e-1 | 7.82e-2 | 6.22e-2 | 5.84e-5 |
| 2 | 1.05e-2 | 4.70e-1 | 4.17e-2 | 3.83e-2 | 2.07e-6 |
| 3 | 2.93e-3 | 1.92e-1 | 1.03e-2 | 8.82e-3 | 5.31e-8 |
| 4 | 3.96e-4 | 4.07e-2 | 4.76e-18 | – | 3.78e-9 |

Table 6

Linear method, $k = 2$, $\tau = h/2$, $\epsilon = 0.01$

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