# Family of convergent numerical schemes for the incompressible Navier-Stokes equations

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**Abstract**

This paper presents the common mathematical features which are leading to convergence properties for a family of numerical schemes applied to the discretisation of the steady and transient incompressible Navier-Stokes equations with homogeneous Dirichlet's boundary conditions. This family includes the Taylor-Hood scheme, the MAC scheme, the Crouzeix-Raviart scheme generalised into the Hybrid Mixed Mimetic scheme, which can be combined with a variety of discretisations for the nonlinear convection term, each of them being more efficient than the others in particular situations. We provide tools for analyzing all the combined methods, and proving their convergence to a weak solution of the problem.

*Keywords:* incompressible Navier-Stokes equations, Gradient Discretisation Method, convergence analysis

## 1. Introduction

For the approximation of the incompressible Navier-Stokes equation, most of the numerical schemes are nonlinear extensions of schemes applying to the linear incompressible Stokes equations. Three of them are of particular importance.

1. The Taylor-Hood scheme [20] is the prototype of conforming finite element methods on general simplicial grids. The convergence of this method (among more general conforming or nonconforming methods) with the skew symmetric approximation of the convection term (detailed in the Appendix) is proved for example in [21].
2. The Marker-And-Cell (MAC) scheme, introduced in [14], is one of the most popular methods in the engineering framework [19, 22] for the approximation of the Navier-Stokes equations on structured Cartesian grids. Its convergence properties, partly provided by the pioneering works [17, 18], are detailed in [11].
3. The Crouzeix-Raviart scheme [3] is a non conforming scheme on general simplicial grids, whose advantage is to provide fluxes and exact mass balances in the simplices. This scheme can be extended to general polyhedral meshes, using the Hybrid Mixed Mimetic (HMM) methods that include in particular the Mimetic Finite Difference schemes [1].

It is proved in [5] that all of them are gradient discretisation methods for which general properties of discrete spaces and operators are allowing common convergence and error estimates properties, in the case of the steady and transient incompressible Stokes equations. Turning to the Navier-Stokes problem,

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the variety of schemes is much wider, since there are many possibilities, in the framework of a given scheme for the Stokes problem, to discretise the nonlinear convection term (see Section 4 for examples of different approximations of the nonlinear term for the Crouzeix-Raviart scheme). Our purpose is to extend to the Navier-Stokes problem the unification work done in [5]. We show in this paper that again in this case, it is possible to exhibit common mathematical properties to all the schemes quote above, combined with a variety of discretisations for the nonlinear convection term.
This paper is organised as follows. We focus in Section 2 on the steady Navier-Stokes problem. In Section 2.1, we provide the continuous equations, whose weak form is directly inspiring the general discrete formalism. We then turn in Section 2.2 to the discrete setting. In Section 2.3, we provide the mathematical analysis of the convergence of the scheme. Section 3 is devoted to the transient case. We present the continuous transient Navier-Stokes equations in Section 3.1. Section 3.2 is devoted to the adaptation of the discrete tools, given in the steady case, to the transient equations. In Section 3.3, we develop the mathematical analysis of the convergence for the gradient scheme in the transient case. In Section 4, we illustrate the variety of possible discrete nonlinear convection terms within the framework of the Crouzeix-Raviart scheme, and we show on numerical examples that none of them is always the most precise, which confirms the interest of developing tools simultaneously applying to the various schemes. Finally, some short conclusions are proposed in Section 5. We show in appendix that the properties, required on the discretisation of the nonlinear convection term for providing convergence properties, are satisfied in the case of the skew symmetric discrete convection term, whose advantage is to be generic, in the sense that it can be considered for a wide range of discretisation methods used for the Stokes problem.

## 2. Steady Navier-Stokes problem

### *2.1. The continuous equations*

Let us first recall the strong sense of the Navier-Stokes problem:

$$\left\{\begin{aligned} \eta\overline{u} - \nu\Delta\overline{u} + (\overline{u}\cdot\nabla)\overline{u} + \nabla\overline{p} &= f - \mathrm{div}(G) && \text{in } \Omega\\ \mathrm{div}\overline{u} &= 0 && \text{in } \Omega\\ \overline{u} &= 0 && \text{on } \partial\Omega, \end{aligned}\right. \tag{1}$$

where $\overline{u}$ represents the velocity field and $\overline{p}$ is the pressure, under the following assumptions

$$\begin{aligned} &\Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d\ (d\in\{2,3\}),\\ &f\in \boldsymbol{L}^2(\Omega) \text{ and } G\in\boldsymbol{L}^2(\Omega)^d,\\ &\eta\geq 0,\ \nu>0. \end{aligned} \tag{2}$$

In this paper, if $F$ is a vector space, we denote by $\boldsymbol{F}$ the space $F^d$. Thus, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$ and $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega)^d$. $L_0^2(\Omega)$ is the space of functions in $L^2(\Omega)$ with a zero mean value over $\Omega$. Finally, $\boldsymbol{H}_{\mathrm{div}}(\Omega)$ is the space of fields $v\in\boldsymbol{L}^2(\Omega)$ such that $\mathrm{div}(v)\in L^2(\Omega)$. We then give the definition of a weak solution to the steady Navier-Stokes problem (1).

**Definition 2.1 (Weak solution to the steady Navier-Stokes problem).** *Under Hypotheses (2),* $(\overline{u},\overline{p})$ *is a weak solution to (1) if*

$$\left\{\begin{aligned} &\overline{u}\in\boldsymbol{H}_0^1(\Omega),\ \overline{p}\in L_0^2(\Omega),\\ &\eta\int_\Omega \overline{u}\cdot\overline{v}\,\mathrm{d}x + \nu\int_\Omega \nabla\overline{u}:\nabla\overline{v}\,\mathrm{d}x + b(\overline{u},\overline{v})\\ &\qquad -\int_\Omega \overline{p}\,\mathrm{div}\overline{v}\,\mathrm{d}x = \int_\Omega (f\cdot\overline{v} + G:\nabla\overline{v})\,\mathrm{d}x,\ \forall\overline{v}\in\boldsymbol{H}_0^1(\Omega),\\ &\int_\Omega q\,\mathrm{div}\overline{u}\,\mathrm{d}x = 0,\ \forall q\in L_0^2(\Omega), \end{aligned}\right. \tag{3}$$

*where, for all $\xi = (\xi_{i,j})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}$ and $\chi = (\chi_{i,j})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}$, $\xi : \chi = \sum_{i,j=1}^{d} \xi_{i,j}\chi_{i,j}$ is the doubly contracted product on $\mathbb{R}^{d\times d}$, and*

$$b(u,w) = \widetilde{b}(u,u,w) \text{ with } \widetilde{b}(u,v,w) = \sum_{i,j=1}^{d} \int_\Omega u_i(x)\, \partial_i v_j(x)\, w_j(x)\, \mathrm{d}x,\ \forall u,v,w \in \boldsymbol{H}_0^1(\Omega). \tag{4}$$

We recall that $\widetilde{b}$ is a trilinear continuous form on $\boldsymbol{H}_0^1(\Omega)^3$ and that the so-called "skew symmetry property" holds [21, Ch.II, Lemma 1.2 and 1.3]

$$\widetilde{b}(u,v,w) = -\widetilde{b}(u,w,v),\ \forall u \in E(\Omega),\ \forall v,w \in \boldsymbol{H}_0^1(\Omega),$$

where

$$E(\Omega) = \{v \in \boldsymbol{H}_0^1(\Omega); \mathrm{div} v = 0 \text{ a.e. in } \Omega\}, \tag{5}$$

leading to

$$b(u,v) = \frac{1}{2}(\widetilde{b}(u,u,v) - \widetilde{b}(u,v,u)),\ \forall u \in E(\Omega),\ \forall v \in \boldsymbol{H}_0^1(\Omega). \tag{6}$$

The existence of a weak solution $(\overline{u},\overline{p})$ to Problem (1) in the sense of Definition 2.1 follows from [21, Ch.II, Theorem 1.2]. In the general framework of this paper, this solution is not unique ([21, Ch.II, Theorem 1.2] gives the uniqueness of the weak solution $(\overline{u},\overline{p})$ under the so-called "small data condition" on $\nu$, $f$ and $G$ that is not assumed here). An explicit counter-example to this uniqueness is given in [21] in [21] in the case of non-homogeneous boundary conditions in a semi-infinite domain.

*2.2. The Gradient Discretisation Method*

*2.2.1. The numerical scheme*

Let us first define the discrete spaces, operators and properties, upon which we prove in Lemma 2.12 that the numerical scheme (10) has at least one solution.

**Definition 2.2 (Gradient discretisation for the steady Navier-Stokes problem).** *A gradient discretisation $D$ for the incompressible steady Navier-Stokes problem, with homogeneous Dirichlet boundary conditions, is defined by $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \Theta_D, \mathrm{div}_D, b_D)$, where:*

1. *$X_{D,0}$ is a finite-dimensional vector space on $\mathbb{R}$ (we denote $X_{D,0}^* = X_{D,0} \setminus \{0\}$).*
2. *$Y_D$ is a finite-dimensional vector space on $\mathbb{R}$.*
3. *The linear mapping $\Pi_D : X_{D,0} \longrightarrow \boldsymbol{L}^2(\Omega)$ is the reconstruction of the approximate velocity field.*
4. *The linear mapping $\nabla_D : X_{D,0} \longrightarrow \boldsymbol{L}^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\|\cdot\|_D := \|\nabla_D \cdot\|_{\boldsymbol{L}^2(\Omega)^d}$ is a norm on $X_{D,0}$.*
5. *The linear mapping $\mathrm{div}_D : X_{D,0} \longrightarrow L^2(\Omega)$ is the discrete divergence operator.*
6. *The linear mapping $\Theta_D : Y_D \longrightarrow L^2(\Omega)$ is the reconstruction of the approximate pressure, and must be chosen such that $\|\Theta_D \cdot\|_{L^2(\Omega)}$ is a norm on $Y_D$. We then set $Y_{D,0} = \{q \in Y_D, \int_\Omega \Theta_D q \mathrm{d}x = 0\}$ (we denote $Y_{D,0}^* = Y_{D,0} \setminus \{0\}$). We assume that the quantity $\beta_D$ defined by*

$$\beta_D = \min_{q \in Y_{D,0}^*} \max_{v \in X_{D,0}^*} \frac{\int_\Omega \Theta_D q(x)\ \mathrm{div}_D v(x) \mathrm{d}x}{\|v\|_D \|\Theta_D q\|_{L^2(\Omega)}}, \tag{7}$$

   *which is non negative by definition, is different from 0 (which means $\beta_D > 0$).*
7. *The mapping $b_D : X_{D,0}^2 \longrightarrow \mathbb{R}$ is the discrete convection term. It must be chosen such that*
   - *$b_D$ is continuous,*
   - *for all $u \in X_{D,0}$, $b_D(u,u) \geq 0$,*

- *the value $B_D$ defined by*

$$B_D = \sup_{u,v \in X_{D,0}^*} \frac{|b_D(u,v)|}{\|u\|_D^2 \|v\|_D}, \tag{8}$$

*is such that $B_D < +\infty$,*

- *$b_D(u,v)$ is linear with respect to $v$.*

*We then define $C_D$ by*

$$C_D = \max_{v \in X_{D,0}^*} \frac{\|\Pi_D v\|_{\boldsymbol{L}^2(\Omega)}}{\|v\|_D} + \max_{v \in X_{D,0}^*} \frac{\|\mathrm{div}_D v\|_{L^2(\Omega)}}{\|v\|_D} + \frac{1}{\beta_D}. \tag{9}$$

The above gradient discretisation leads to the following gradient scheme for the steady Navier-Stokes problem, based on a discretisation of the weak formulation (3), in which the continuous spaces and operators are replaced with discrete ones (in (3), we wrote the property "div$\overline{u} = 0$" using test functions to make clearer this parallel between the weak formulation and the gradient scheme). If $D$ is a gradient discretisation in the sense of Definition 2.2, the scheme is given by:

$$\left\{ \begin{array}{l} u \in X_{D,0}\,,\; p \in Y_{D,0}, \\ \eta \displaystyle\int_\Omega \Pi_D u \cdot \Pi_D v + \nu \int_\Omega \nabla_D u : \nabla_D v \;\mathrm{d}x + b_D(u,v) \\ \qquad\qquad - \displaystyle\int_\Omega \Theta_D p \;\mathrm{div}_D v \mathrm{d}x = \int_\Omega (f \cdot \Pi_D v + G : \nabla_D v)\,\mathrm{d}x, \; \forall v \in X_{D,0}, \\ \displaystyle\int_\Omega \Theta_D q \,\mathrm{div}_D u \mathrm{d}x = 0, \;\; \forall q \in Y_{D,0}. \end{array} \right. \tag{10}$$

**Remark 2.3 (Approximation of the Stokes problem).** *In the case $b = 0$, the choice $b_D = 0$ done in [5] allows to define a convenient scheme.*

*2.2.2. Required properties on discrete spaces and operators*

We now turn to the properties of a sequence of gradient discretisations, which are sufficient for leading to the convergence of the corresponding sequence of discrete solutions to (10), as stated by Theorem 2.14. The coercivity of a sequence of gradient discretisations ensures that a discrete Poincaré inequality, a control of the discrete divergence and a discrete Ladyzenskaja-Babuška-Brezzi (LBB) condition can be established, all being uniform along the sequence of discretisations.

**Definition 2.4 (Coercivity).** *Let $D$ be a discretisation in the sense of Definition 2.2. A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be coercive if there exist $C_S \geq 0$ such that $C_{D_m} \leq C_S$, for all $m \in \mathbb{N}$, where $C_D$ is defined by (9).*

The consistency of a sequence of gradient discretisations states that any element of the continuous spaces containing the velocity and the pressure can be interpolated as precisely as desired.

**Definition 2.5 (Consistency).** *Let $D$ be a gradient discretisation in the sense of Definition 2.2, and let us define the interpolation operators $I_D : \boldsymbol{H}_0^1(\Omega) \longrightarrow X_{D,0}$ and $\widetilde{I}_D : L_0^2(\Omega) \longrightarrow Y_{D,0}$ by*

$$\begin{aligned} I_D\varphi &= \operatorname*{argmin}_{v \in X_{D,0}} \left( \|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D v - \mathrm{div}\varphi\|_{L^2(\Omega)} \right), \\ \widetilde{I}_D\psi &= \operatorname*{argmin}_{z \in Y_{D,0}} \|\Theta_D z - \psi\|_{L^2(\Omega)}. \end{aligned} \tag{11}$$

*Let $S_D : \boldsymbol{H}_0^1(\Omega) \to [0,+\infty)$, and $\widetilde{S}_D : L_0^2(\Omega) \to [0,+\infty)$ be defined by*

$$\forall \varphi \in \boldsymbol{H}_0^1(\Omega)\,, \quad S_D(\varphi) = \|\Pi_D I_D\varphi - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D I_D\varphi - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D I_D\varphi - \mathrm{div}\varphi\|_{L^2(\Omega)},$$

*and*

$$\forall \psi \in L^2_0(\Omega)\,, \quad \widetilde{S}_D(\psi) = \|\Theta_D \widetilde{I}_D \psi - \psi\|_{L^2(\Omega)}.$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be consistent if, for all $\varphi \in \boldsymbol{H}^1_0(\Omega)$, $S_{D_m}(\varphi)$ tends to $0$ as $m \to \infty$ and, for all $\psi \in L^2_0(\Omega)$, $\widetilde{S}_{D_m}(\psi)$ tends to $0$ as $m \to \infty$.*

The limit conformity of a sequence of gradient discretisations states that the discrete gradient and divergence of bounded sequences whose reconstruction converges, converge to the continuous gradient and divergence of the limit (this property is immediately satisfied by conforming approximations).

**Definition 2.6 (Limit-conformity).** *Let $D$ be a gradient discretisation in the sense of Definition 2.2 and let $W_D : \boldsymbol{H}_{\rm div}(\Omega) \longrightarrow [0, +\infty)$ be defined by*

$$\forall \varphi \in \boldsymbol{H}_{\rm div}(\Omega)\,, \quad W_D(\varphi) = \max_{v \in X^*_{D,0}} \frac{\int_\Omega (\nabla_D v : \varphi + \Pi_D v \cdot {\rm div}\varphi)\,{\rm d}x}{\|v\|_D},$$

*and let $\widetilde{W}_D : L^2(\Omega) \longrightarrow [0, +\infty)$ be defined by*

$$\forall \psi \in L^2(\Omega)\,, \quad \widetilde{W}_D(\psi) = \max_{v \in X^*_{D,0}} \frac{\int_\Omega \psi \left(\sum_{i=1}^d \nabla_D^{(i,i)} v - {\rm div}_D v\right){\rm d}x}{\|v\|_D},$$

*where we denote by $\nabla_D^{(i,j)} v \in L^2(\Omega)$ the $j$-component of $\nabla_D^{(i)} v \in \boldsymbol{L}^2(\Omega)$ (recall that $\nabla_D v \in \boldsymbol{L}^2(\Omega)^d$ can be written $\nabla_D v = (\nabla_D^{(1)} v, \ldots, \nabla_D^{(d)} v)^t$).*
*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be limit-conforming if, for all $\varphi \in \boldsymbol{H}_{\rm div}(\Omega)$, $W_{D_m}(\varphi)$ tends to $0$ and for all $\psi \in L^2(\Omega)$, $\widetilde{W}_D(\psi)$ tends to $0$ as $m \to \infty$.*

**Remark 2.7 (Equivalent definitions for the limit-conformity property).** *In [5], the limit-conformity is defined by $\overline{W}_D : Z(\Omega) \longrightarrow [0, +\infty)$, with $Z(\Omega) = \{(\varphi, \psi) \in \boldsymbol{L}^2(\Omega)^d \times L^2(\Omega), {\rm div}\varphi - \nabla\psi \in \boldsymbol{L}^2(\Omega)\}$, with*

$$\forall (\varphi, \psi) \in Z(\Omega)\,, \quad \overline{W}_D(\varphi, \psi) = \max_{v \in X^*_{D,0}} \frac{\int_\Omega (\nabla_D v : \varphi + \Pi_D v \cdot ({\rm div}\varphi - \nabla\psi) - \psi {\rm div}_D v)\,{\rm d}x}{\|v\|_D}.$$

*Noticing that, for $(\varphi, \psi) \in Z(\Omega)$, we have $\widetilde{\varphi} := \varphi - \psi {\rm I}_d \in \boldsymbol{H}_{\rm div}(\Omega)$, and writing*

$$\nabla_D v : \varphi + \Pi_D v \cdot ({\rm div}\varphi - \nabla\psi) - \psi {\rm div}_D v = \nabla_D v : \widetilde{\varphi} + \Pi_D v \cdot {\rm div}\widetilde{\varphi} + \psi \left(\sum_{i=1}^d \nabla_D^{(i,i)} v - {\rm div}_D v\right),$$

*we get $\overline{W}_D(\varphi, \psi) \le W_D(\widetilde{\varphi}) + \widetilde{W}_D(\psi)$. Reciprocally, for any $\varphi \in \boldsymbol{H}_{\rm div}(\Omega)$, $(\varphi, 0) \in Z(\Omega)$ $W_D(\varphi) = \overline{W}_D(\varphi, 0)$, and for any $\psi \in L^2(\Omega)$, $(\psi {\rm I}_d, \psi) \in Z(\Omega)$ and $\widetilde{W}_D(\psi) = \overline{W}_D(\psi {\rm I}_d, \psi)$. So the above definition of limit-conformity is equivalent to the one in [5].*

The compactness of a sequence of gradient discretisations states that any bounded sequence is relatively compact in the sense that the reconstruction converges up to a subsequence.

**Definition 2.8 (Compactness).** *Let $D$ be a gradient discretisation in the sense of Definition 2.2. A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be compact if, for all sequence $(u_m)_{m\in\mathbb{N}} \in X_{D_m,0}$ such that $\|u_m\|_{D_m}$ is bounded, the sequence $(\Pi_{D_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in $\boldsymbol{L}^2(\Omega)$.*

It is shown in [5] that four schemes (the Taylor-Hood scheme, the MAC scheme, the Crouzeix-Raviart scheme and the HMM extension of the Crouzeix-Raviart scheme) can be cast as gradient schemes for the Stokes problem, such that, for regular sequences of discretisations, the corresponding discrete objects $(X_{D,0}, \Pi_D, \nabla_D, Y_D, \Theta_D, \mathrm{div}_D)$ satisfy the coercivity property in the sense of Definition 2.4, the consistency property in the sense of Definition 2.5 and the limit-conformity property in the sense of Definition 2.6 (see Remark 2.7 for a discussion on different formulations). The proof that the compactness property holds in these four cases results from the following arguments. Since the Taylor-Hood scheme is conforming, the compactness property is a consequence of Rellich's theorem. The compactness for the MAC scheme is proved by [7, Lemma 9.3 p.770], for the Crouzeix-Raviart scheme, it is proved in [10, Theorem 3.3], and for the HMM extension of the Crouzeix-Raviart scheme, it is stated by [6, Lemma 5.6].
Finally, the convection limit-conformity of a sequence of gradient discretisations states that the limit of the discrete convection term computed on converging sequences under some sense is the continuous convection term applied to the limit.

**Definition 2.9 (Convection limit-conformity).** *A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations in the sense of Definition 2.2 is said to be convection limit-conforming if the sequence $(B_{D_m})_{m\in\mathbb{N}}$ is bounded (see (8)), and, for all sequence $(u_m, v_m) \in X^2_{D_m,0}$ such that $(\|u_m\|_{D_m})_{m\in\mathbb{N}}$ and $(\|v_m\|_{D_m})_{m\in\mathbb{N}}$ are bounded, and such that there exists $(\overline{u}, \overline{v}) \in E(\Omega) \times \boldsymbol{H}^1_0(\Omega)$ such that*

- $\Pi_{D_m} u_m \to \overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$,
- $\nabla_{D_m} u_m \rightharpoonup \nabla\overline{u}$ *weakly in* $\boldsymbol{L}^2(\Omega)^d$,
- $\Pi_{D_m} v_m \to \overline{v}$ *in* $\boldsymbol{L}^2(\Omega)$,
- $\nabla_{D_m} v_m \rightharpoonup \nabla\overline{v}$ *weakly in* $\boldsymbol{L}^2(\Omega)^d$,

*then*

$$\lim_{m\to\infty} b_{D_m}(u_m, v_m) = b(\overline{u}, \overline{v})$$

*where $b$ is defined in (4) and $b_D$ is introduced in Definition 2.2.*

We show in Appendix Appendix A that this convection limit conformity holds when using the classical skew symmetric approximation for the trilinear term, based on the discrete tools for the Stokes problem arising from the schemes studied in [5], all of them satisfying the property of $p$-coercivity (see Definition Appendix A.1): this is a consequence of standard Sobolev inequalities for the Taylor-Hood scheme, of [7, Lemma 9.5 p.790] for the MAC scheme, of [12, Lemma 9.3] for the Crouzeix-Raviart scheme and of [8, Lemma 5.2] for the HMM scheme.

### *2.3. Convergence analysis*

#### *2.3.1. Estimates and existence of a discrete solution*

Let us first establish some estimates on solutions of Scheme (10).

**Lemma 2.10 (Estimate on the discrete velocity).** *Under Hypotheses (2), let $D$ be a gradient discretisation in the sense of Definition 2.2. If $(u_D, p_D)$ is a solution to Scheme (10), then there exists $C_1 > 0$ only depending on $\Omega$, $d$, $f$, $G$ and increasingly depending on $C_D$ (defined by (9)) such that*

$$\eta\|\Pi_D u_D\|^2_{L^2(\Omega)} + \nu\|u_D\|^2_D \le C_1 \tag{12}$$

**Proof** Putting $v = u_D$ and $q = p_D$ in (10), we get

$$\eta\|\Pi_D u_D\|^2_{L^2(\Omega)} + \nu\|u_D\|^2_D + \underbrace{b_D(u_D, u_D)}_{\ge 0} - \underbrace{\int_\Omega \Theta_D p_D \,\mathrm{div}_D u_D \,\mathrm{d}x}_{=0} = \int_\Omega \left(f \cdot \Pi_D u_D + G : \nabla_D u_D\right) \,\mathrm{d}x.$$

Now using the Cauchy-Schwarz inequality in the previous equation, we obtain

$$\eta\|\Pi_D u_D\|^2_{L^2(\Omega)} + \nu\|u_D\|^2_D \leq \|f\|_{\boldsymbol{L}^2(\Omega)}\|\Pi_D u_D\|_{L^2(\Omega)} + \|G\|_{\boldsymbol{L}^2(\Omega)^d}\|u_D\|_D.$$

Using $\|\Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} \leq C_D\|u\|_D$, we then conclude (12). □

**Lemma 2.11 (Estimate on the discrete pressure).** *Under Hypotheses (2), let $D$ be a gradient discretisation in the sense of Definition 2.2. If $(u_D, p_D)$ is a solution to Scheme (10), then there exists $C_2 > 0$ only depending on $\Omega$, $d$, $f$, $G$, $\eta$, $\nu$ and increasingly depending on $B_D + C_D$ (see* (8)*-*(9)*) such that*

$$\|\Theta_D p_D\|_{L^2(\Omega)} \leq C_2. \tag{13}$$

**Proof** Letting $q = p$ in Definition (7) of $\beta_D$ leads to the existence of $v \in X_{D,0}$ such that $\|v\|_D = 1$ and $\beta_D\|\Theta_D p\|_{L^2(\Omega)} \leq \int_\Omega \Theta_D p \operatorname{div}_D v \, \mathrm{d}x$. Selecting $v$ as test function in Scheme (10), we get

$$\beta_D\|\Theta_D p\|_{L^2(\Omega)} \\ \leq \left|\eta\int_\Omega \Pi_D u_D \cdot \Pi_D v \, \mathrm{d}x + \nu\int_\Omega \nabla_D u_D : \nabla_D v \, \mathrm{d}x + b_D(u_D, v) - \int_\Omega (f \cdot \Pi_D v + G : \nabla_D v) \, \mathrm{d}x\right|.$$

Using the Cauchy-Schwarz inequality, we deduce:

$$\beta_D\|\Theta_D p_D\|_{L^2(\Omega)} \leq \eta C_D\|\Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} + \nu\|u_D\|_D + |b_D(u_D, v)| + C_D\|f\|_{\boldsymbol{L}^2(\Omega)} + \|G\|_{\boldsymbol{L}^2(\Omega)^d}.$$

Thanks to (8) in Definition 2.2, we get

$$|b_D(u_D, v)| \leq B_D\|u_D\|^2_D.$$

Estimate (12) allows to conclude (13). □

**Lemma 2.12 (Existence of a discrete solution).** *Under Hypotheses (2), let $D$ be an admissible discretisation in the sense of Definition 2.2. Then there exists at least one solution $(u_D, p_D)$ to Scheme (10).*

**Proof** We follow the proof of [9, Theorem 4.3] based on a topological degree argument. Let $N$ (resp. $M$) be the dimension of $X_{D,0}$ (resp. $Y_{D,0}$) and let $(v^{(i)})_{i=1,...,N}$ (respectively $(q^{(j)})_{j=1,...,M}$) be a basis of $X_{D,0}$ (respectively $Y_{D,0}$). Let $F : \mathbb{R}^N \times \mathbb{R}^M \times [0,1] \longrightarrow \mathbb{R}^N \times \mathbb{R}^M$ be the mapping such that, for any $(u = \sum_{i=1}^N u_i v^{(i)}, p = \sum_{j=1}^M p_j q^{(j)}, \lambda) \in X_{D,0} \times Y_{D,0} \times [0,1]$, $F(u,p,\lambda) = (F_i(u,p,\lambda))_{i=1,...,N+M}$ with: for all $i = 1, ..., N$

$$F_i(u,p,\lambda) = \eta\int_\Omega \Pi_D u \cdot \Pi_D v^{(i)} \, \mathrm{d}x + \nu\int_\Omega \nabla_D u : \nabla_D v^{(i)} \, \mathrm{d}x + \lambda b_D(u, v^{(i)}) \\ - \int_\Omega \Theta_D p \operatorname{div}_D v^{(i)} \, \mathrm{d}x - \lambda\int_\Omega \left(f \cdot \Pi_D v^{(i)} + G : \nabla_D v^{(i)}\right) \mathrm{d}x,$$

and for all $j = 1, ..., M$

$$F_{j+N}(u,p,\lambda) = \int_\Omega \Theta_D q^{(j)} \operatorname{div}_D u \, \mathrm{d}x.$$

Thanks to the hypotheses on $b_D$, the mapping $F$ is continuous, and for a given $(u,p)$ such that $F_i(u,p,\lambda) = 0$ for all $i = 1, ..., N+M$, estimates of Lemmas 2.10 and 2.11 hold independently of $\lambda \in [0,1]$, replacing $(b_D, f, G)$ in Scheme (10) by $(\lambda b_D, \lambda f, \lambda G)$. Since $F(u,p,0)$ is a linear function of $(u,p)$, we deduce from the invariance of the Brouwer topological degree by homotopy that there exists at least one solution $(u_D, p_D)$ to the equation $F(u_D, p_D, 1) = 0$, which is exactly Scheme (10). □

*2.3.2. Convergence result*

The following lemma, used a few times in the course of the convergence proof of the gradient scheme, provides a regularity result on the limit of bounded sequences of discrete solutions.

**Lemma 2.13 (Regularity of the limit of bounded sequences).** *Let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations which is coercive, limit-conforming and compact, and let for all $m \in \mathbb{N}$, $u_m \in X_{D_m,0}$ be such that the sequence $(\|u_m\|_{D_m})_{m\in\mathbb{N}}$ is bounded. Then there exists $\overline{u} \in H_0^1(\Omega)$ and a subsequence of $(D_m)_{m\in\mathbb{N}}$, again denoted by $(D_m)_{m\in\mathbb{N}}$, such that, as $m \to \infty$,*

- $\Pi_{D_m} u_m \to \overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$,
- $\nabla_{D_m} u_m \rightharpoonup \nabla\overline{u}$ *weakly in* $\boldsymbol{L}^2(\Omega)^d$,
- $\mathrm{div}_{D_m} u_m \rightharpoonup \mathrm{div}\overline{u}$ *weakly in* $L^2(\Omega)$.

*Moreover, if the sequence of gradient discretisations $(D_m)_{m\in\mathbb{N}}$ is consistent and if*

$$\forall m \in \mathbb{N},\ \forall \varphi \in Y_{D_m,0},\ \int_\Omega \mathrm{div}_{D_m} u_m \Theta_{D_m}\varphi \,\mathrm{d}x = 0, \tag{14}$$

*then* $\mathrm{div}\overline{u} = 0$.

**Proof** Using the compactness of $(D_m)_{m\in\mathbb{N}}$ gives the existence of $\overline{u} \in \boldsymbol{L}^2(\Omega)$ such that, up to a subsequence, $\Pi_{D_m} u_m \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$. From this subsequence, and using the fact that $\nabla_{D_m} u_m$ and $\mathrm{div}_{D_m} u_m$ remain bounded (we use here the coercivity of $(D_m)_{m\in\mathbb{N}}$ which provides a bound on $\|\mathrm{div}_{D_m} u_m\|_{L^2(\Omega)}$), we deduce that there exist $\zeta \in \boldsymbol{L}^2(\Omega)^d$ and $\gamma \in L^2(\Omega)$ such that, up to a subsequence indexed by $\sigma(m)$, $\nabla_{D_{\sigma(m)}} u_{\sigma(m)} \rightharpoonup \zeta$ weakly in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_{\sigma(m)}} u_{\sigma(m)} \rightharpoonup \gamma$ weakly in $L^2(\Omega)$. We extend the definition of all the previous functions by 0 outside $\Omega$. Let $\varphi \in \boldsymbol{C}^\infty(\mathbb{R}^d)^d$. From Definition 2.6 applied to the restriction of $\varphi$ to $\Omega$ and to its opposite, we have

$$|\int_{\mathbb{R}^d} (\nabla_{D_{\sigma(m)}} u_{\sigma(m)} : \varphi + \Pi_{D_{\sigma(m)}} u_{\sigma(m)} \cdot \mathrm{div}\varphi) \,\mathrm{d}x| \le W_{D_{\sigma(m)}}(\varphi|_\Omega)\|u_{\sigma(m)}\|_{D_{\sigma(m)}}.$$

Passing to the limit and using the limit-conformity of $(D_{\sigma(m)})_{m\in\mathbb{N}}$, we obtain

$$\int_{\mathbb{R}^d} (\zeta : \varphi + \overline{u} \cdot \mathrm{div}\varphi) \,\mathrm{d}x = 0.$$

The last equality shows that $\zeta = \nabla\overline{u}$ on $\mathbb{R}^d$ and therefore that $\overline{u} \in \boldsymbol{H}^1(\mathbb{R}^d)$. Since $\zeta$ vanishes outside $\Omega$, we get that $\overline{u} \in \boldsymbol{H}_0^1(\Omega)$. For any $\psi \in L^2(\Omega)$, we get that

$$|\int_\Omega \psi(\sum_{i=1}^d \nabla_{D_{\sigma(m)}}^{(i,i)} u_{\sigma(m)} - \mathrm{div}_{D_{\sigma(m)}} u_{\sigma(m)}) \,\mathrm{d}x| \le \widetilde{W}_{D_{\sigma(m)}}(\psi)\|u_{\sigma(m)}\|_{D_{\sigma(m)}}. \tag{15}$$

Passing to the limit and again using the limit-conformity of $(D_{\sigma(m)})_{m\in\mathbb{N}}$, we obtain

$$\int_\Omega \psi(\mathrm{div}\overline{u} - \gamma) \,\mathrm{d}x = 0.$$

Letting $\psi = \mathrm{div}\overline{u} - \gamma$ proves that $\gamma = \mathrm{div}\overline{u}$. The identification $\zeta = \nabla\overline{u}$ and $\gamma = \mathrm{div}\overline{u}$ prove that we can take $\sigma(m) = m$.

Let us now turn to the last part of the lemma. We then assume the consistency of the sequence of gradient discretisations and that (14) holds. Using the interpolation operator defined in Definition 2.5, we get from (14) and (15) that, for any $\psi \in L^2_0(\Omega)$,

$$|\int_\Omega \psi \ \mathrm{div}_{D_m} u_m \ \mathrm{d}x| = |\int_\Omega (\psi - \Theta_{D_m} \widetilde{I}_{D_m} \psi) \ \mathrm{div}_{D_m} u_m \ \mathrm{d}x| \leq \|\mathrm{div}_{D_m} u_m\|_{L^2(\Omega)} \|\psi - \Theta_{D_m} \widetilde{I}_{D_m} \psi\|_{L^2(\Omega)}.$$

Letting $m \to \infty$, we obtain that $\displaystyle\int_\Omega \psi \mathrm{div}\overline{u} \ \mathrm{d}x = 0$ which implies, since $\mathrm{div}\overline{u} \in L^2_0(\Omega)$, that $\mathrm{div}\overline{u} = 0$ a.e. in $\Omega$. $\square$

Our main result for the steady Navier-Stokes problem is the following theorem.

**Theorem 2.14 (Convergence of the scheme).** *Under Hypotheses (2), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 2.2, which is consistent, limit-conforming, coercive, compact and convection limit-conforming in the sense of Definitions 2.5, 2.6, 2.4, 2.8 and 2.9. Then for any $m \in \mathbb{N}$, there exists at least one solution $(u_{D_m}, p_{D_m})$ to (10) with $D = D_m$. Moreover, as $m \to \infty$, there exists a subsequence of $(D_m)_{m\in\mathbb{N}}$ again denoted $(D_m)_{m\in\mathbb{N}}$ and there exists $(\overline{u}, \overline{p})$, weak solution of the incompressible steady Navier-Stokes problem (1) in the sense of Definition 2.1, such that*

- *$\Pi_{D_m} u_{D_m}$ converges to $\overline{u}$ in $\boldsymbol{L}^2(\Omega)$,*
- *$\nabla_{D_m} u_{D_m}$ converges to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega)^d$,*
- *$\Theta_{D_m} p_{D_m}$ converges to $\overline{p}$ in $L^2(\Omega)$.*

**Remark 2.15 (Convergence for the whole sequence).** *If the solution to (1) in the sense of Definition 2.1 is unique, then the convergence holds for the whole sequence and not only for a subsequence.*

We may now prove the convergence theorem for the steady Navier-Stokes problem.

**Proof of Theorem 2.14**

**Step 1:** Extraction of a converging subsequence.

Estimate (12) allows us to apply Lemma 2.13 and to get the existence of $\overline{u} \in \boldsymbol{H}^1_0(\Omega)$ with $\mathrm{div}\,\overline{u} = 0$ and, up to a subsequence again indexed by $m$, $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} u_{D_m} \to \nabla\overline{u}$ weakly in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} u_{D_m} \to 0$ weakly in $L^2(\Omega)$. Moreover, thanks to Estimate (13), up to a subsequence of the previous one (again indexed by $m$), we get the existence of $\overline{p} \in L^2_0(\Omega)$ such that $\Theta_{D_m} p_{D_m} \to \overline{p}$ weakly in $L^2(\Omega)$.

**Step 2:** Proof that $(\overline{u}, \overline{p})$ is solution to (3).

Let $\overline{w} \in \boldsymbol{H}^1_0(\Omega)$ be given. Thanks to the consistency hypothesis, we get that $\|I_{D_m}\overline{w}\|_{D_m}$ is bounded, $\Pi_{D_m} I_{D_m}\overline{w} \to \overline{w}$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} I_{D_m}\overline{w} \to \nabla\overline{w}$ in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} I_{D_m}\overline{w} \to \mathrm{div}\overline{w}$ in $\boldsymbol{L}^2(\Omega)$. Thanks to weak/strong convergence properties, the following holds:

$$\lim_{m\to\infty} \eta \int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} I_{D_m}\overline{w} \ \mathrm{d}x = \eta \int_\Omega \overline{u} \cdot \overline{w} \ \mathrm{d}x,$$
$$\lim_{m\to\infty} \nu \int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} I_{D_m}\overline{w} \ \mathrm{d}x = \nu \int_\Omega \nabla\overline{u} : \nabla\overline{w} \ \mathrm{d}x,$$
$$\lim_{m\to\infty} \int_\Omega \Theta_{D_m} p_{D_m} \mathrm{div}_{D_m} I_{D_m}\overline{w} \ \mathrm{d}x = \int_\Omega \overline{p}\, \mathrm{div}\overline{w} \ \mathrm{d}x,$$
$$\lim_{m\to\infty} \int_\Omega \left(f \cdot \Pi_{D_m} I_{D_m}\overline{w} + G : \nabla_{D_m} I_{D_m}\overline{w}\right) \ \mathrm{d}x = \int_\Omega \left(f \cdot \overline{w} + G : \nabla\overline{w}\right) \ \mathrm{d}x,$$

and, thanks to the the convection limit-conformity of $(D_m)_{m\in\mathbb{N}}$,

$$\lim_{m\to\infty} b_{D_m}(u_{D_m}, I_{D_m}\overline{w}) = b(\overline{u}, \overline{w}).$$

Therefore, letting $v = I_{D_m}\overline{w}$ as test function in Scheme (10) and passing to the limit, we find that $(\overline{u}, \overline{p})$ is a solution to Problem (3).

**Step 3:** Proof of the strong convergence of $\nabla_{D_m} u_{D_m}$.
Taking $v = u_{D_m}$ as test function in Scheme (10), passing to the supremum limit as $m \to \infty$, using the convergence of $\Pi_{D_m} u_{D_m}$ to $\overline{u}$ in $\boldsymbol{L}^2(\Omega)$ and the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega)^d$ and using $b_{D_m}(u_{D_m}, u_{D_m}) \geq 0$, we get that:

$$\eta\|\overline{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \nu \limsup_{m\to\infty} \|u_{D_m}\|^2_{D_m} \leq \int_\Omega (f \cdot \overline{u} + G : \nabla\overline{u}) \, \mathrm{d}x.$$

Now choosing $\overline{v} = \overline{u}$ as test function in Problem (3), recalling that $b(\overline{u}, \overline{u}) = 0$, we find

$$\eta\|\overline{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \nu\|\nabla\overline{u}\|^2_{\boldsymbol{L}^2(\Omega)^d} = \int_\Omega (f \cdot \overline{u} + G : \nabla\overline{u}) \, \mathrm{d}x.$$

Combining the last two equations we get

$$\limsup_{m\to\infty} \|u_{D_m}\|^2_{D_m} \leq \|\nabla\overline{u}\|^2_{\boldsymbol{L}^2(\Omega)^d}.$$

Furthermore, owing to the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega)^d$, we may write that

$$\liminf_{m\to\infty} \|u_{D_m}\|^2_{D_m} \geq \|\nabla\overline{u}\|^2_{\boldsymbol{L}^2(\Omega)^d}.$$

This implies that $\|u_{D_m}\|_{D_m} \to \|\nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d}$ and concludes the proof.
**Step 4:** Proof of the strong convergence of the approximate pressure in $L^2(\Omega)$.
We select $v_m \in X_{D_m}$ such that $\|v_m\|_{D_m} = 1$ and

$$\beta_{D_m}\|\Theta_{D_m}(\widetilde{I}_{D_m}\overline{p} - p_{D_m})\|_{L^2(\Omega)} \leq \int_\Omega \Theta_{D_m}(\widetilde{I}_{D_m}\overline{p} - p_{D_m})\mathrm{div}_{D_m} v_m \mathrm{d}x.$$

Letting $v = v_m$ in the scheme, we get

$$\eta\int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} v_m + \nu\int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} v_m + b_{D_m}(u_{D_m}, v_m) - \int_\Omega \Theta_{D_m} p_{D_m} \,\mathrm{div}_{D_m} v_m = \int_\Omega f \cdot \Pi_{D_m} v_m.$$

Combining the two above relations, we get

$$\begin{aligned}\beta\|\Theta_{D_m}(\widetilde{I}_{D_m}\overline{p} - p_{D_m})\|_{L^2(\Omega)} \leq & \int_\Omega f \cdot \Pi_{D_m} v_m + \int_\Omega \Theta_{D_m}\widetilde{I}_{D_m}\overline{p} \,\mathrm{div}_{D_m} v_m \\ & - \eta\int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} v_m - \nu\int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} v_m - b_{D_m}(u_{D_m}, v_m).\end{aligned}$$

Thanks to the triangle inequality, we deduce

$$\begin{aligned}\beta\|\overline{p} - \Theta_{D_m} p_{D_m}\|_{L^2(\Omega)} \leq \beta\|\Theta_{D_m}\widetilde{I}_{D_m}\overline{p} - \overline{p}\|_{L^2(\Omega)} + & \int_\Omega f \cdot \Pi_{D_m} v_m + \int_\Omega \Theta_{D_m}\widetilde{I}_{D_m}\overline{p} \,\mathrm{div}_{D_m} v_m \\ & - \eta\int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} v_m - \nu\int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} v_m - b_{D_m}(u_{D_m}, v_m).\end{aligned}$$

Since $\|v_m\|_{D_m} = 1$, Lemma 2.13 shows the existence of $\overline{v} \in \boldsymbol{H}^1_0(\Omega)$ and of a subsequence, again indexed by $m$, such that $\Pi_{D_m} v_m$ tends to $\overline{v}$ in $\boldsymbol{L}^2(\Omega)$ and such that $\nabla_{D_m} v_m$ weakly converges to $\nabla\overline{v}$ in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} v_m$ weakly converges to $\mathrm{div}\overline{v}$ in $L^2(\Omega)$.

Using the (already proved) strong convergence properties for the velocity and the convection limit-conformity of $(D_m)_{m\in\mathbb{N}}$, we may now pass to the limit $m \to \infty$, since all integrals involve weak/strong convergence properties. We get

$$\beta \limsup_{m\to\infty} \|\overline{p} - \Theta_{D_m} p_{D_m}\|_{L^2(\Omega)} \le \int_\Omega f \cdot \overline{v} + \int_\Omega \overline{p}\ \mathrm{div}\overline{v} - \eta \int_\Omega \overline{u} \cdot \overline{v} - \nu \int_\Omega \nabla\overline{u} : \nabla\overline{v} - b(\overline{u}, \overline{u}, \overline{v}).$$

It now suffices to use the fact that we already proved that $(\overline{u}, \overline{p})$ is a weak solution to the steady Navier-Stokes equation (3). We then get that the right hand side of the previous inequality vanishes, which shows the convergence in $L^2(\Omega)$ for this subsequence. Using a standard uniqueness argument, we deduce that the whole subsequence built at step 1 converges in this sense. □

## 3. Transient Navier-Stokes problem

### *3.1. The continuous equations*

Let us now give the strong sense of the transient problem that we consider in this paper.

$$\begin{cases} \partial_t \overline{u} - \nu \Delta \overline{u} + (\overline{u} \cdot \nabla)\overline{u} + \nabla \overline{p} & = f - \mathrm{div}(G) & \text{in } \Omega \times (0,T) \\ \mathrm{div}\overline{u} & = 0 & \text{in } \Omega \times (0,T) \\ \overline{u} & = 0 & \text{on } \partial\Omega \times (0,T) \\ \overline{u}(\cdot, 0) & = u_{\mathrm{ini}} & \text{in } \Omega, \end{cases} \tag{16}$$

under the following assumptions:

$$\begin{array}{l} \Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d \ (d \in \{2,3\}), \\ T > 0,\ \nu > 0 \\ f \in \boldsymbol{L}^2(\Omega \times (0,T)) \text{ and } G \in \boldsymbol{L}^2(\Omega \times (0,T))^d \\ u_{\mathrm{ini}} \in \boldsymbol{L}^2(\Omega). \end{array} \tag{17}$$

We now give the sense for a weak solution to Problem (16).

**Definition 3.1 (Weak solution to the transient Navier-Stokes problem).** *Under Hypotheses (17), $\overline{u}$ is a weak solution to (16) if $\overline{u} \in L^2(0,T,E(\Omega)) \cap L^\infty(0,T,\boldsymbol{L}^2(\Omega))$ and*

$$\begin{cases} -\displaystyle\int_0^T \int_\Omega \overline{u}(x,t) \cdot \partial_t \overline{v}(x,t)\ \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \overline{u}_{\mathrm{ini}}(x) \cdot \overline{v}(x,0)\ \mathrm{d}x \\ \qquad + \nu \displaystyle\int_0^T \int_\Omega \nabla\overline{u}(x,t) : \nabla\overline{v}(x,t)\ \mathrm{d}x\mathrm{d}t + \int_0^T b(\overline{u}(\cdot,t), \overline{v}(\cdot,t))\ \mathrm{d}t \\ \qquad = \displaystyle\int_0^T \int_\Omega (f(x,t) \cdot \overline{v}(x,t) + G(x,t) : \nabla\overline{v}(x,t))\ \mathrm{d}x\mathrm{d}t, \\ \forall \overline{v} \in L^2(0,T,E(\Omega)) \cap \boldsymbol{C}_c^\infty(\Omega \times (-\infty, T)), \end{cases} \tag{18}$$

*where $E(\Omega)$ is defined by (5) and $b$ is defined by (4).*

We recall that a weak solution $\overline{u}$ of (16) in the sense of Definition 3.1 satisfies $\partial_t \overline{u} \in L^{4/d}(0,T,E'(\Omega))$, and the weak sense could be equivalently defined, introducing $\int_0^T \langle \partial_t \overline{u} \cdot \overline{v} \rangle \mathrm{d}t$ instead of an integrate by parts with respect to time.

*3.2. The space-time Gradient Discretisation method*

As in the steady case, we need to define a space-time gradient discretisation, which includes an adaptation of Definition 2.2.

**Definition 3.2 (Space-time gradient discretisation).** *A space-time gradient discretisation $D$ for the transient Navier-Stokes problem, with homogeneous Dirichlet boundary conditions, is defined by a family $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \Theta_D, \mathrm{div}_D, b_D, (t^{(n)})_{n=0,\ldots,N}, J_D)$ where:*

- *$D^s = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \Theta_D, \mathrm{div}_D, b_D)$ is a gradient discretisation in the sense of Definition 2.2,*
- *$J_D : \boldsymbol{L}^2(\Omega) \longrightarrow X_{D,0}$ is an interpolation operator;*
- *$t^{(0)} = 0 < t^{(1)} < ... < t^{(N)} = T$ is the finite sequence of discrete times.*

*We define $k^{n+\frac{1}{2}} = t^{(n+1)} - t^{(n)}$ for all $n = 0, ..., N-1$ and $k_D = \max\limits_{n=0,\ldots,N-1}(k^{n+\frac{1}{2}})$ and we set*

$$\forall n = 0, \ldots, N-1\,,\ \forall v = (v^n)_{n=0,\ldots,N} \in X_{D,0}^{N+1}\,, \delta_D^{n+\frac{1}{2}} v = \frac{\Pi_D v^{(n+1)} - \Pi_D v^{(n)}}{k^{n+\frac{1}{2}}}.$$

*We extend the definition of the operators $\Pi_D$, $\nabla_D$ and $\delta_D$ to space-time functions by the following definition: if $v = (v^n)_{n=0,\ldots,N} \in X_{D,0}^{N+1}$, the functions $\Pi_D v : \Omega \times (0,T) \to \mathbb{R}^d$, $\nabla_D v : \Omega \times (0,T) \to \mathbb{R}^{d\times d}$ and $\delta_D v : \Omega \times (0,T) \to \mathbb{R}^{d\times d}$ are defined by*

$$\begin{aligned}&\forall n = 0, \ldots, N-1\,,\ \forall t \in (t^{(n)}, t^{(n+1)}]\,,\ \ \textit{for a.e. } x \in \Omega\,,\\ &\Pi_D v(x,t) = \Pi_D v^{(n+1)}(x)\,,\ \nabla_D v(x,t) = \nabla_D v^{(n+1)}(x) \textit{ and } \delta_D v(x,t) = \delta_D^{n+\frac{1}{2}} v(x).\end{aligned} \tag{19}$$

A sequence of space-time gradient discretisations $(D_m)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming and compact) if its spatial component $(D^s_m)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming and compact). We now need to adapt the definition of consistency, with respect to the time step size and the interpolation of the initial condition.

**Definition 3.3 (Space-time consistency).** *A sequence $(D_m)_{m\in\mathbb{N}}$ of space-time gradient discretisations in the sense of Definition 3.2 is said to be consistent if*

1. *$(D^s_m)_{m\in\mathbb{N}}$ is consistent in the sense of Definition 2.5,*
2. *for all $\varphi \in \boldsymbol{L}^2(\Omega)$, $\Pi_{D_m} J_{D_m} \varphi \to \varphi$ in $\boldsymbol{L}^2(\Omega)$,*
3. *$k_{D_m} \to 0$ as $m \to \infty$.*

The following definition for the space-time convection limit-conformity is given in order to include, as in the steady case, approximations for the convection term such as the skew symmetric one based on the chosen discretisation for the Stokes problem (see Appendix Appendix B). Note that we do no longer require that the sequence $(B_{D_m})_{m\in\mathbb{N}}$ remains bounded, since this is only used in the steady case for obtaining a uniform estimate on the pressure (we are not able to provide such an estimate in the transient case).

**Definition 3.4 (Space-time convection limit-conformity).** *A sequence $(D_m)_{m\in\mathbb{N}}$ of space-time gradient discretisations in the sense of Definition 3.2 is said to be space-time convection limit-conforming if for all sequence $u_m, v_m \in (X_{D_m,0}^{N_m+1})^2$ such that $(\|\nabla_{D_m} u_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d})_{m\in\mathbb{N}}$ and $(\|\nabla_{D_m} v_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d})_{m\in\mathbb{N}}$ are bounded, and such that there exists $(\overline{u}, \overline{v}) \in L^2(0,T,E(\Omega)) \times L^2(0,T,\boldsymbol{H}^1_0(\Omega))$ such that*

- *$\Pi_{D_m} u_m \to \overline{u}$ in $L^1(0,T,\boldsymbol{L}^2(\Omega))$ and $\|\Pi_{D_m} u_m\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))}$ is bounded,*
- *$\nabla_{D_m} u_m \rightharpoonup \nabla\overline{u}$ weakly in $\boldsymbol{L}^2(\Omega\times(0,T))^d$,*

- $\Pi_{D_m} v_m \to \overline{v}$ *in* $L^\infty(0,T,\boldsymbol{L}^2(\Omega))$,
- $\nabla_{D_m} v_m \to \nabla\overline{v}$ *in* $L^\infty(0,T,\boldsymbol{L}^2(\Omega)^d)$,

*then*

$$\lim_{m\to\infty}\sum_{n=0}^{N_m-1} k^{n+\frac{1}{2}} b_{D_m}(u_m^{(n+1)}, v_m^{(n+1)})\mathrm{d}t = \int_0^T b(\overline{u},\overline{v})\mathrm{d}t.$$

**Remark 3.5.** *If we replace Definition 3.4 by Definition 2.9, we cannot conclude to the convergence of the scheme in the transient case. Indeed, the compactness properties obtained in the proof of Theorem 3.10 are that there exists $\overline{u}\in L^2(0,T,E(\Omega))$ such that $\Pi_{D_m} u_{D_m}\to\overline{u}$ in $\boldsymbol{L}^2(\Omega\times(0,T))$ and $\nabla_{D_m} u_{D_m}\to\nabla\overline{u}$ weakly in $\boldsymbol{L}^2(\Omega\times(0,T))^d$. This does not imply that, for a.e. $t\in]0,T[$, $\nabla_{D_m} u_{D_m}(t)$ remains bounded in $\boldsymbol{L}^2(\Omega)^d$, nor that it converges weakly in $\boldsymbol{L}^2(\Omega)^d$ to $\nabla\overline{u}(t)$.*

Let $D$ be a space-time gradient discretisation in the sense of Definition (3.2). The implicit gradient scheme for (16) is based on the following approximation of (18):

$$\left\{\begin{array}{l}
u_D = (u_D^{(n)})_{n=0,...,N}\,,\ p_D = (p_D^{(n)})_{n=1,...,N} \text{ such that } u_D^{(0)} = J_D u_{\rm ini} \text{ and, } \forall n=0,...,N-1:\\
u_D^{(n+1)}\in X_{D,0}\,,\ p_D^{(n+1)}\in Y_{D,0}\,,\\
\displaystyle\int_\Omega \delta_D^{n+\frac{1}{2}} u_D\cdot\Pi_D v\mathrm{d}x + \nu\int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v\mathrm{d}x + b_D(u^{n+1},v) - \int_\Omega \Theta_D p_D^{(n+1)}\mathrm{div}_D v\mathrm{d}x\\
\quad\displaystyle = \frac{1}{k^{n+\frac{1}{2}}}\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f\cdot\Pi_D v\mathrm{d}x\mathrm{d}t + \frac{1}{k^{n+\frac{1}{2}}}\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega G\cdot\nabla_D v\mathrm{d}x\mathrm{d}t,\quad \forall v\in X_{D,0},\\
\displaystyle\int_\Omega \mathrm{div}_D u_D^{(n+1)}\Theta_D q\mathrm{d}x = 0,\quad \forall q\in Y_{D,0}.
\end{array}\right. \tag{20}$$

*3.3. Convergence analysis*

*3.3.1. Estimates and existence of a discrete solution*

We first establish some estimates on the discrete velocity.

**Lemma 3.6 (Estimates on the discrete velocity).** *Under Hypotheses (17), let $D$ be a space-time gradient discretisation in the sense of Definition 3.2. If $(u_D,p_D)$ is a solution to Scheme (20), then for all $n=0,...,N$,*

$$\nu\|\nabla_D u_D\|^2_{\boldsymbol{L}^2(\Omega\times(0,t^{(n)}))^d} + \frac{1}{2}\|\Pi_D u_D^{(n)}\|^2_{\boldsymbol{L}^2(\Omega)} \le \int_0^{t^{(n)}}\int_\Omega f\cdot\Pi_D u_D\mathrm{d}x\mathrm{d}t + \int_0^{t^{(n)}}\int_\Omega G:\nabla_D u_D\mathrm{d}x\mathrm{d}t + \frac{1}{2}\|\Pi_D J_D u_{\rm ini}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{21}$$

*Therefore, there exists $C_3>0$ only depending on $\Omega$, $d$, $\nu$, $f$, $G$ and increasingly depending on $C_D$, such that*

$$\|\Pi_D u_D\|^2_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))} + \nu\|\nabla_D u_D\|^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \le C_3 + \|\Pi_D J_D u_{\rm ini}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{22}$$

**Proof** Putting $v = k^{n+\frac{1}{2}} u_D^{(n+1)}$ and $q = p_D^{(n+1)}$ in (20), since $b_D(u_D^{(n+1)}, u_D^{(n+1)}) = 0$, we get

$$\int_\Omega\left(\Pi_D u_D^{(n+1)} - \Pi_D u_D^{(n)}\right)\cdot\Pi_D u_D^{(n+1)}\mathrm{d}x + \nu\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega |\nabla_D u_D^{(n+1)}|^2\mathrm{d}x\mathrm{d}t = \int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f\cdot\Pi_D u_D^{(n+1)}\mathrm{d}x\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega G:\nabla_D u_D^{(n+1)}\mathrm{d}x\mathrm{d}t.$$

Using the inequality $(a-b)\cdot a \geq \frac{1}{2}(|a|^2 - |b|^2)$ (valid for any $a, b \in \mathbb{R}^d$) on the first term, it comes:

$$\frac{1}{2}\int_\Omega \left[\left|\Pi_D u_D^{(n+1)}\right|^2 - \left|\Pi_D u_D^{(n)}\right|^2\right] \mathrm{d}x + \nu \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f\cdot \Pi_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t.$$

We take $n \in \{0, \ldots, N\}$ and sum the obtained equation over $0, \ldots, n-1$. This gives (21). Estimate (22) is a straightforward consequence of the definition of $C_D$ and Young's inequality applied to (21) with $m = N$. □

We can then, in a similar way to the steady case, establish the existence of at least one solution to the scheme.

**Lemma 3.7 (Existence of a discrete solution).** *Under Hypotheses (17), let $D$ be a space-time gradient discretisation in the sense of Definition 3.2. Then there exists at least one solution $(u_D, p_D)$ to Scheme (20).*

**Proof** We remark that, for a given $n = 0, ..., N-1$, the existence of $u_D^{(n+1)}$ solution to (20) is identical to the existence of a solution to Scheme (10) with $\eta = \dfrac{1}{k^{n+\frac{1}{2}}}$. Therefore, the existence of at least one solution follows from Lemma 2.12. □

**Definition 3.8 (Dual semi-norm).** *The semi-norm $|\cdot|_{*,D}$ is defined on $\boldsymbol{L}^2(\Omega)$ by*

$$|w|_{*,D} = \sup\left\{\int_\Omega w(x)\cdot \Pi_D v(x) \mathrm{d}x,\ v \in E_D \text{ such that } \|v\|_D = 1\right\},$$

*where $E_D = \{v \in X_{D,0}, \mathrm{div}_D v = 0\}$.*

**Lemma 3.9 (Estimate on the dual semi-norm of the discrete time derivative).** *Under Hypotheses (17), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisations in the sense of Definition 3.2, let $(u_D, p_D)$ be a solution to Scheme (20). Then there exists $C_4 \geq 0$ only depending on $\Omega$, $d$, $T$, $\nu$, $f$, $G$ and increasingly depending on $C_D$ such that*

$$\int_0^T |\delta_D u_D|_{*,D} \mathrm{d}t \leq C_4(1 + \|\Pi_D J_D u_{\mathrm{ini}}\|_{\boldsymbol{L}^2(\Omega)}). \tag{23}$$

**Proof** Taking any $v \in E_D$ in Scheme (20) such that $\|v\|_D = 1$ , we have, for $n = 0, \ldots, N-1$:

$$\int_\Omega \delta_D^{n+\frac{1}{2}} u_D \cdot \Pi_D v \mathrm{d}x \leq \nu \|u^{(n+1)}\|_D + \frac{1}{k^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega (f\cdot \Pi_D v + G : \nabla_D v)\mathrm{d}x\mathrm{d}t.$$

Using the Cauchy-Schwarz inequality on the last term, the definition of the coercivity and that of $|\cdot|_{\star,D}$, we get

$$|\delta_D^{n+\frac{1}{2}} u_D|_{\star,D} \leq \nu\|u^{(n+1)}\|_D + \frac{1}{k^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} (C_D\|f(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)} + \|G(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)^d})\mathrm{d}t.$$

Multiplying by $k^{n+\frac{1}{2}}$ and summing over $n$ gives the desired estimate, thanks to (22) and to the Cauchy-Schwarz inequality. □

An estimate on the preceding semi-norm, which is a norm on the space $B = \{\Pi_D v,\ v \in E_D\}$, will allow us to apply theorem Appendix C.1 given in Appendix.

*3.3.2. Convergence result*

We can now state the convergence result for the transient Navier-Stokes problem.

**Theorem 3.10 (Convergence of the scheme).** *Under hypotheses (17), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisations in the sense of Definition 3.2 which is space-time consistent, limit-conforming, coercive, compact and space-time convection limit conforming in the sense of Definitions 3.3, 2.6, 2.5, 2.8 and 3.4. For any $m \in \mathbb{N}$, let $(u_{D_m}, p_{D_m})$ be a solution to Scheme (20) with $D = D_m$. Then there exists a subsequence of $(D_m)_{m\in\mathbb{N}}$ again denoted $(D_m)_{m\in\mathbb{N}}$ such that, as $m \to \infty$, $\Pi_{D_m} u_{D_m}$ converges in $L^2(0,T,\boldsymbol{L}^2(\Omega))$ to $\overline{u}$, $\nabla_{D_m} u_{D_m}$ weakly converges to $\nabla\overline{u}$ in $L^2(0,T,\boldsymbol{L}^2(\Omega)^d)$, $\mathrm{div}_{D_m} u_{D_m}$ weakly converges to $0$ in $L^2(0,T,L^2(\Omega))$, where $\overline{u}$ is a weak solution to the incompressible transient Navier-Stokes problem in the sense of Definition 3.1.*

**Proof**

Since the space-time consistency implies that $\Pi_{D_m} J_{D_m} u_{\rm ini} \to u_{\rm ini}$ in $\boldsymbol{L}^2(\Omega)$, we get that $(\|\Pi_{D_m} J_{D_m} u_{\rm ini}\|_{\boldsymbol{L}^2(\Omega)})_{m\in\mathbb{N}}$ is bounded and therefore that the estimates given by Lemmas 3.6 and 3.9 are independent on $m \in \mathbb{N}$.

**Step 1**: Application of Theorem Appendix C.1 and consequences.

In our setting, the space $B$ of the theorem is $\boldsymbol{L}^2(\Omega)$ and $B_m = \{\Pi_{D_m} v, v \in E_{D_m}\}$. The norm $\|\cdot\|_{X_m}$ is defined by

$$\|z\|_{X_m} = \min\{\|v\|_{D_m}, v \in X_{D_m,0} \text{ such that } \Pi_{D_m} v = z\},$$

and the norm $\|\cdot\|_{Y_m}$ is the semi-norm $|\cdot|_{\star,D_m}$ defined in Definition 3.8.

The compactness property in the sense of Definition 2.8 of the sequence of discretisations $(D_m)_{m\in\mathbb{N}}$ and Estimate (22) give the existence of $\overline{u} \in \boldsymbol{L}^2(\Omega\times(0,T))$ and $\zeta \in \boldsymbol{L}^2(\Omega\times(0,T))^d$ such that, up to a subsequence (still indexed by $m$), $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $\boldsymbol{L}^2(\Omega\times(0,T))$ and $\nabla_{D_m} u_{D_m} \to \zeta$ weakly in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ and thus Assumption $(h_1)$ is satisfied.

Assumption $(h_2)$ of the theorem is a consequence of Definition 3.8. Indeed, let $(z_m)_{m\in\mathbb{N}}$ such that $v_m = \mathrm{argmin}\{\|v\|_{D_m}, v \in X_{D_m,0} \text{ such that } \Pi_{D_m} v = z_m\}$ satisfies that $\|v_m\|_{D_m}$ remains bounded, that $z_m \to v$ in $B$ and that $|z_m|_{\star,D_m} \to 0$ as $m \to \infty$. We have

$$\int_\Omega \Pi_{D_m} v_m \cdot \Pi_{D_m} v_m \,\mathrm{d}x \le |z_m|_{\star,D_m} \|v_m\|_{D_m} \to 0 \text{ as } m \to \infty,$$

which shows that $v = 0$.

From estimates (22) and (23), we get that Assumptions $(h_3)$ and $(h_4)$ are satisfied. Therefore, we deduce that there exists $\overline{u} \in L^1(0,T,\boldsymbol{L}^2(\Omega))$ and a subsequence of $(D_m)_{m\in\mathbb{N}}$, denoted in the same way, such that $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $L^1(0,T,\boldsymbol{L}^2(\Omega))$ as $m \to \infty$.

**Step 2**: Proof that $\overline{u} \in L^2(0,T,E(\Omega))$.

Let $\varphi \in C^\infty(\overline{\mathbb{R}^d})^d$ and $\xi \in C_c^\infty(0,T)$ be given. We have, for all $n = 0,\dots,N-1$, and all $t \in (t^{(n)}, t^{(n+1)})$,

$$|\int_{\mathbb{R}^d} \big(\nabla_{D_m} u_m^{(n+1)} : \varphi\,\xi(t) + \Pi_{D_m} u_m^{(n+1)} \cdot \mathrm{div}\varphi\,\xi(t)\big)\mathrm{d}x| \le W_{D_m}(\varphi|_\Omega)\,\xi(t)\,||\nabla_{D_m} u_{D_m}||_{\boldsymbol{L}^2(\Omega)^d}.$$

Integrating the above inequality over $t \in (t^{(n)}, t^{(n+1)})$, summing on $n = 0,\dots,N-1$ and using Estimate (22), allows to follow the proof of Lemma 2.13, hence leading to $\overline{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$ and $\mathrm{div}\overline{u} = 0$.

**Step 3**: Proof that $\overline{u}$ is the solution to (18).

We use the density in $L^2(0,T;E(\Omega))$ of the space of finite sums of functions under the form $\xi(t)\overline{w}(x)$ with $\xi \in C_c^\infty([0,T))$ and $\overline{w} \in E(\Omega)$. Let $\xi \in C_c^\infty([0,T))$ and $\overline{w} \in E(\Omega)$. As $(\overline{w}, 0)$ is the solution of the incompressible steady Stokes problem with $f = \eta\overline{w}$ and $G = \nabla\overline{w}$ (Problem (1) with $b = 0$, see Remark 2.15), Scheme (10) provides for a given $m \in \mathbb{N}$ an approximation $w_{D_m^s} \in X_{D_m,0}$ such that $\int_\Omega \Theta_{D_m} q \,\mathrm{div}_{D_m} w_{D_m^s} = 0$ for all $q \in Y_{D_m,0}$, $\Pi_{D_m} w_{D_m^s} \to \overline{w}$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_{D_m} w_{D_m^s} \to \nabla\overline{w}$ in $\boldsymbol{L}^2(\Omega)^d$.

We take $v_{D_m}^{(n+1)} = \xi(t^{(n)})w_{D_m^s}$ as test function in Scheme (20), we multiply by $k^{(n+\frac{1}{2})}$ and we sum the resulting equation on $n = 0, ..., N-1$ to get $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} = T_5^{(m)}$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} k^{n+\frac{1}{2}}\xi(t^{(n)}) \int_\Omega \delta^{(n+\frac{1}{2})}u_{D_m}(x) \cdot \Pi_{D_m} w_{D_m^s}(x)\mathrm{d}x,$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} k^{n+\frac{1}{2}}\xi(t^{(n)}) \int_\Omega \nabla_{D_m} u_{D_m}^{(n+1)}(x) : \nabla_{D_m} w_{D_m^s}(x)\mathrm{d}x,$$

$$T_3^{(m)} = -\sum_{n=0}^{N-1} k^{n+\frac{1}{2}}\xi(t^{(n)}) \int_\Omega \Theta_{D_m} p_{D_m}^{(n+1)}(x)\mathrm{div}_{D_m} w_{D_m^s}(x)\mathrm{d}x,$$

$$T_4^{(m)} = \sum_{n=0}^{N-1} k^{n+\frac{1}{2}}\xi(t^{(n)}) b_{D_m}(u_{D_m}^{(n+1)}, w_{D_m^s}) = \sum_{n=0}^{N-1} k^{n+\frac{1}{2}} b_{D_m}(u_{D_m}^{(n+1)}, v_{D_m}^{(n+1)}),$$

$$T_5^{(m)} = \sum_{n=0}^{N-1} \xi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega (f(x,t) \cdot \Pi_{D_m} w_{D_m^s}(x) + G(x,t) : \nabla_D w_{D_m^s}(x))\mathrm{d}x\mathrm{d}t.$$

First, we remark that $T_3^{(m)} = 0$ since $\int_\Omega \Theta_{D_m} q \mathrm{div}_{D_m} w_{D_m^s} = 0$ for all $q \in Y_{D_m,0}$. Using discrete integration by parts and writing $\xi(t^{(n+1)}) - \xi(t^{(n)}) = \int_{t^{(n)}}^{t^{(n+1)}} \xi'(t)\mathrm{d}t$, we find

$$T_1^{(m)} = -\int_0^T \int_\Omega \xi'(t)\Pi_{D_m} u_{D_m}(x,t) \cdot \Pi_{D_m} w_{D_m^s}(x)\mathrm{d}x\mathrm{d}t - \xi(0)\int_\Omega \Pi_{D_m} u_{D_m}^{(0)}(x) \cdot \Pi_{D_m} w_{D_m^s}(x)\mathrm{d}x.$$

Recall that $u_{D_m}^{(0)} = J_{D_m} u_{\mathrm{ini}}$, so that the space–time consistency (Definition 3.3) gives $\Pi_{D_m} u_{D_m}^{(0)} \to u_{\mathrm{ini}}$ in $\boldsymbol{L}^2(\Omega)$ as $m \to \infty$. Thus, by strong convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_{D_m} w_{D_m^s}$ to $\overline{w}$,

$$T_1^{(m)} \to -\int_0^T \int_\Omega \xi'(t)\overline{u}(x,t) \cdot \overline{w}(x)\,\mathrm{d}x\mathrm{d}t - \xi(0)\int_\Omega \overline{u}_{\mathrm{ini}}(x) \cdot \overline{w}(x)\,\mathrm{d}x.$$

Using the regularity of $\xi$ and the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0,T))^d$, it easily comes

$$T_2^{(m)} \to \int_0^T \xi(t) \int_\Omega \nabla\overline{u}(x,t) : \nabla\overline{w}(x)\mathrm{d}x\mathrm{d}t$$

and

$$T_5^{(m)} \to \int_0^T \xi(t) \int_\Omega (f(x,t) \cdot \overline{w}(x) + G(x,t) : \nabla\overline{w}(x))\mathrm{d}x\mathrm{d}t.$$

For the limit of $T_4^{(m)}$, we remark that the sequences $(\Pi_{D_m} u_{D_m})_{m\in\mathbb{N}}$, $(\nabla_{D_m} u_{D_m})_{m\in\mathbb{N}}$, $(\Pi_{D_m} v_{D_m})_{m\in\mathbb{N}}$ and $(\nabla_{D_m} v_{D_m})_{m\in\mathbb{N}}$ satisfy the required conditions for applying the space-time convection limit-conformity in the sense of Definition 3.4. This gives that

$$\lim_{m\to\infty} T_4^{(m)} = \int_0^T b(\overline{u}, \xi\overline{w})\,\mathrm{d}t.$$

Finally, passing to the limit in $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} = T_5^{(m)}$ concludes the proof that $\overline{u}$ satisfies (18).
□

## 4. Numerical comparison of different approximations of the convection term for the steady problem

The aim of this section is to exhibit the large influence of the choice for the approximation of the convection term on the accuracy of the results, keeping the same scheme for the approximation of the Stokes problem. We therefore select here the Crouzeix–Raviart scheme [3] for the approximation of the velocity and of the pressure, whose advantage is to lead to accurate approximations for the Stokes problem on a large variety of simplicial grids, available on many practical cases. We first recall, as in [5], how these spaces and operators can be defined in this case. We consider a conforming simplicial mesh $\mathcal{M}$ of a bounded polyhedral domain $\Omega$, that is a finite family of simplices (triangles if $d = 2$, tetrahedra if $d = 3$) such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For $K \in \mathcal{M}$ we denote by $|K| > 0$ the measure of $K$. The set of the faces (in 3D) or edges (in 2D) is denoted by $\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$, where $\mathcal{E}_{\rm int}$ is the set of faces (edges) included in $\Omega$, and $\mathcal{E}_{\rm ext}$ is the set of faces (edges) included in $\partial\Omega$. The $(d-1)$-dimensional measure and the centre of gravity of $\sigma \in \mathcal{E}$ are respectively denoted by $|\sigma|$ and $\overline{x}_\sigma$. For all $K \in \mathcal{M}$, we denote by $\mathcal{E}_K \subset \mathcal{E}$ the set of the $d+1$ faces (edges) of $K$, and for all $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{K \in \mathcal{M} \,:\, \sigma \in \mathcal{E}_K\}$ (if $\mathcal{M}_\sigma \subset \mathcal{E}_{\rm ext}$, it contains exactly one element, otherwise $\mathcal{M}_\sigma \subset \mathcal{E}_{\rm int}$ and it contains exactly two elements). For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to $\sigma$ outward to $K$. For all $K \in \mathcal{M}$, we denote by $x_K$ the center of gravity of $K$. The Crouzeix–Raviart gradient discretisation is defined as follows.

1. The space of the discrete velocities is $X_{D,0} = \{v = (v_\sigma)_{\sigma\in\mathcal{E}} \,:\, v_\sigma \in \mathbb{R}^d\,,\ v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\rm ext}\}$.
2. The space of the discrete pressures is $Y_D = \{p = (p_K)_{K\in\mathcal{M}} \,:\, p_K \in \mathbb{R}\}$.
3. The linear mapping $\Pi_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ is the nonconforming piecewise affine reconstruction of each component of the velocity defined by
$$\forall v \in X_{D,0},\ \Pi_D v = \sum_{\sigma\in\mathcal{E}} v_\sigma \varphi_\sigma, \tag{24}$$
where $\varphi_\sigma$ is the non-conforming $\mathbb{P}^1$ basis function associated with the face $\sigma$.
4. The linear mapping $\Theta_D : Y_D \to L^2(\Omega)$ is defined by: for $p \in Y_D$ and $K \in \mathcal{M}$, $\Theta_D p = p_K$ on $K$.
5. The linear mapping $\nabla_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ is the piecewise constant "broken gradient":
$$\forall v \in X_{D,0},\ \forall K \in \mathcal{M}\,,\ (\nabla_D v)_{|K} = (\nabla(\Pi_D v))_{|K}. \tag{25}$$
6. The linear mapping $\mathrm{div}_D : X_{D,0} \to L^2(\Omega)$ is defined by
$$\forall v \in X_{D,0}\,,\ \forall K \in \mathcal{M}\,,\ (\mathrm{div}_D v)_{|K} = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma| v_\sigma \cdot \mathbf{n}_{K,\sigma} = (\mathrm{div}(\Pi_D v))_{|K}. \tag{26}$$
7. Let us now review a few possible definitions for the discrete convection term, that are compared below on numerical examples. For all the five following examples, the definition of $b_D(u, v)$ is obtained through an expression $b_D(u, v) = \widehat{B}_D(u, u, v)$, where $\widehat{B}_D$ happens to be a trilinear form. As a consequence, all these expressions necessarily meet some of the properties requested by item 7 of Definition 2.2: continuity, existence of $B_D < +\infty$ such that (8) holds, linearity with respect to $v$. The property $b_D(u, u) \geq 0$ is only satisfied by four or the following five examples.

   The first example is the case of a centred approximation $b_D(u, v)$, computed from the face velocities:
$$b_D(u, v) := b_D^{(1)}(u, v) = \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \mathcal{M}_\sigma=\{K,L\}}} |\sigma|\ u_\sigma \cdot \mathbf{n}_{K,\sigma}\ (\Pi_D u(x_L) - \Pi_D u(x_K)) \cdot \frac{\Pi_D v(x_K) + \Pi_D v(x_L)}{2}. \tag{Scheme 1}$$
   This definition implies that
$$b_D^{(1)}(u, u) = -\sum_{K\in\mathcal{M}} \frac{1}{2}|\Pi_D u(x_K)|^2 \sum_{\sigma\in\mathcal{E}_K} |\sigma| u_\sigma \cdot \mathbf{n}_{K,\sigma} = 0. \tag{27}$$

**Remark 4.1.** *It is easy to define an upstream weighting version of the above scheme, letting*

$$b_D^{(1,\mathrm{up})}(u,v) = \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \mathcal{M}_\sigma=\{K,L\}}} |\sigma|((u_\sigma\cdot\mathbf{n}_{K,\sigma})^+\Pi_D v(x_K) - (u_\sigma\cdot n_{K|L})^-\Pi_D v(x_L))\cdot(\Pi_D u(x_L) - \Pi_D u(x_K)).$$

*This upstream weighting version leads to the inequality $b_D(u,u)\geq 0$, instead of* (27).

The second example, which is mathematically analysed in details in this paper, is generic, and can be considered in any gradient discretisation framework. In this scheme, $b_D$ is defined by:

$$b_D(u,v) := b_D^{(2)}(u,v) = \frac{1}{2}\left(\widetilde{b}_D(u,u,v) - \widetilde{b}_D(u,v,u)\right), \qquad \text{(Scheme 2)}$$

where $\widetilde{b}_D : X_{D,0}^3 \longrightarrow \boldsymbol{L}^2(\Omega)$ is given, in the same way as the continuous trilinear form $b$ defined by (4), by

$$\widetilde{b}_D(u,v,w) = \sum_{i,j=1}^{d}\int_\Omega \Pi_D^{(i)}u\nabla_D^{(i,j)}v\Pi_D^{(j)}w\ \mathrm{d}x.$$

This example, which is the classical skew symmetric definition, naturally satisfies $b_D(u,u)=0$. The fact that it only involves the discrete tools defined in the framework of the Stokes problem is useful, and is standard in the case of the conforming finite element method, but also, for example, in the framework of Discontinuous Galerkin methods [4]. It is studied in Appendix Appendix A.

The third example does not meet the mathematical framework of this paper, since it does not satisfy $b_D(u,u)\geq 0$. In this scheme, $b_D$ is defined by:

$$b_D(u,v) := b_D^{(3)}(u,v) = \widetilde{b}_D(u,u,v), \qquad \text{(Scheme 3)}$$

The next final examples are inspired by Scheme 1 above. These centered approximation are based on the decomposition of any simplex in covolumes centered on the edges:

$$b_D(u,v) := b_D^{(4,5)}(u,v) = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\sum_{\tau\in\mathcal{E}_K} F_{\sigma,\tau}^{(4,5)}(u)(u_\sigma - u_\tau)\cdot\frac{v_\sigma+v_\tau}{2}. \qquad \text{(Schemes 4-5)}$$

Denoting by $S_{\sigma,\tau}$ the interface between the covolumes centered on $\sigma$ and $\tau$ (see Figure 1) and by $\mathbf{n}_{\sigma,\tau}$ the normal vector to this interface oriented from $\sigma$ to $\tau$, Scheme 4 is defined by

$$F_{\sigma,\tau}^{(4)}(u) = \int_{S_{\sigma,\tau}} \Pi_D u(x)\cdot\mathbf{n}_{\sigma,\tau}\mathrm{d}s(x). \qquad \text{(Scheme 4)}$$

Scheme 5 is algebraically defined by

$$\sum_{\tau\in\mathcal{E}_K} F_{\sigma,\tau}^{(5)}(u) + |\sigma|u_\sigma\cdot\mathbf{n}_{K,\sigma} = 0, \qquad \text{(Scheme 5)}$$

(this property is also satisfied by Scheme 4) but a simple constraint of a null sum on oriented $F_{\sigma,\tau}(u)$ is substituted to Scheme 4 [16]. In both cases of Schemes 4 and 5, $b_D(u,u)=0$ holds.

Let us now compare the expressions given by Schemes 1-5, in addition with the Crouzeix-Raviart gradient discretisation as specified above used in the Gradient Scheme (10). All these examples are run on the case where $d=2$ and $\Omega=(0,1)^2$. The triangular meshes are issued from a benchmark on anisotropic diffusion problem [15]. These triangle meshes show no symmetry which could artificially increase the convergence rate. This family of meshes is built through the same pattern, whose size is divided by 2

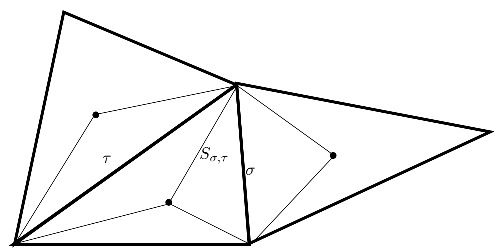


Figure 1: Co-volumes centred on faces.

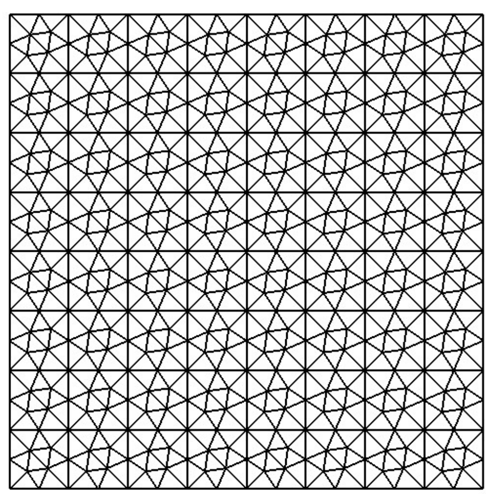

Figure 2: The 3rd mesh of [15].

| h | scheme 1 | scheme 2 | scheme 3 | scheme 4 | scheme 5 |
|---|---|---|---|---|---|
| 0.2500 | 0.565E+00 | 0.691E+00 | 0.543E+00 | 0.602E+00 | 0.585E+00 |
| 0.1250 | 0.159E+00 | 0.181E+00 | 0.155E+00 | 0.165E+00 | 0.164E+00 |
| 0.0625 | 0.412E-01 | 0.467E-01 | 0.409E-01 | 0.427E-01 | 0.426E-01 |
| 0.0312 | 0.104E-01 | 0.118E-01 | 0.104E-01 | 0.108E-01 | 0.108E-01 |

Table 1: Analytical case, $L^2(\Omega)$ error of the velocity with respect to the mesh size

for generating the next one: the third mesh is shown on Figure 2. The first test case that we consider is an analytical case, where the velocity is prescribed by $u(x,y) = (\partial_y \Phi(x,y), -\partial_x \Phi(x,y))^t$ with $\Phi(x,y) = 1000\big(x(1-x)y(1-y)\big)^2$, and the pressure is given by $p(x,y) = (x+y-1)/1000$. Table 1 shows the convergence in the $L^2(\Omega)$ norm of the velocity with respect to the mesh size. On this case, all the schemes have a similar convergence rate (around 2 for the finest meshes).
The second test case that we consider is the Poiseuille flow with a Reynolds number equal to 1000. In this case, Scheme 2 did not converge on the coarsest mesh. Again all orders are about 2, but we see on Table 2 that the finest error is more than 10 times lower than the greatest one. It is obtained in this case using Scheme 3, which is the one on which no convergence proof is available.

| h | scheme 1 | scheme 2 | scheme 3 | scheme 4 | scheme 5 |
|---|---|---|---|---|---|
| 0.2500 | 0.948E+01 | - | - | 0.512E+02 | 0.476E+02 |
| 0.1250 | 0.281E+01 | - | 0.184E+01 | 0.140E+02 | 0.138E+02 |
| 0.0625 | 0.710E+00 | 0.715E+01 | 0.361E+00 | 0.413E+01 | 0.411E+01 |
| 0.0312 | 0.179E+00 | 0.178E+01 | 0.868E-01 | 0.111E+01 | 0.111E+01 |

Table 2: Poiseuille flow, $L^2$ error of the velocity with respect to the mesh size

The final comparison test case is that of the lid driven cavity with the Reynolds number equal to 1000. In Figure 3, we present the profile of the vertical velocity $u^{(2)}$ along the horizontal line $y = .5$. We compare the above schemes with the results provided in the literature [13]. Let us observe that we could not obtain any result with the generic choice Scheme 2, for which no convergence of the Newton-Raphson iterations were observed. We observe that the most precise scheme is Scheme 5.
Let us now turn to the accuracy of Scheme 5 with respect to the mesh size. We see on Figure 4 that the numerical convergence to the reference values is quite good using mesh 5 or mesh 6. In this case we did not succeed to derive a numerical convergence order, although we tried to compute it on one hand from the reference values and on the other hand from the values obtained by mesh 6.
These three test cases show that there is no scheme being the most precise in all cases. So this enhances the interest of the generic framework of this paper.

## 5. Conclusion

This paper shows that the mathematical features leading to the convergence properties of a diversity of numerical schemes are in fact the same ones:

1. The properties of the discrete objects used in the numerical scheme for getting the existence of a stable solution can be summarized by the notion of gradient discretisation, which simultaneously includes conforming schemes like the Taylor-Hood scheme and nonconforming ones such as the Crouzeix-Raviart scheme and more surprisingly the finite volume MAC scheme.

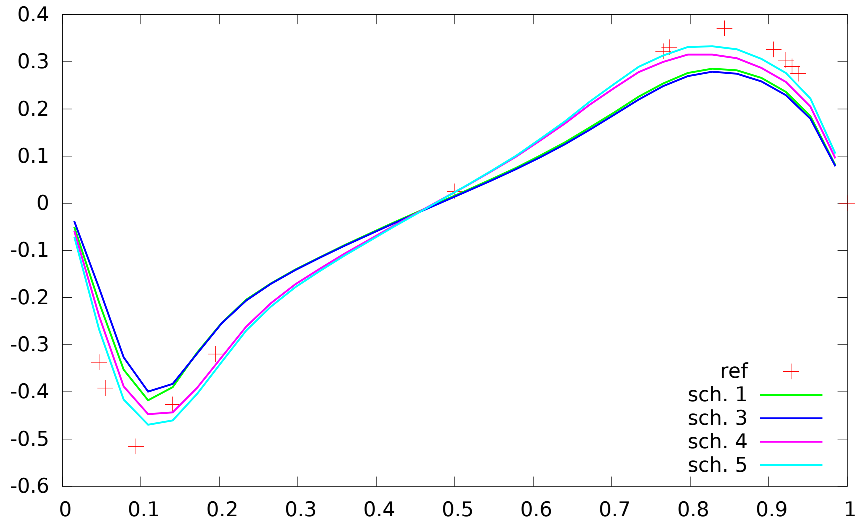


Figure 3: Lid driven cavity, vertical velocity profile along the line $y = 0.5$ using Mesh 4 and different schemes

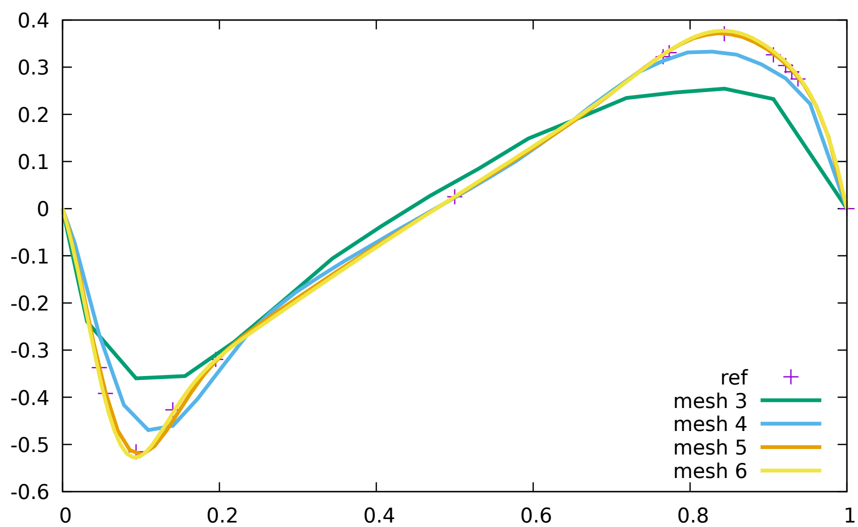


Figure 4: Lid driven cavity, vertical velocity profile along the line $y = 0.5$ using Scheme 5 and different meshes

2. The properties of these objects leading to the convergence properties (coercivity, consistency, limit-conformity, compactness and convection limit-conformity) are generic: they are sufficient for proving that the schemes are converging, and they hold for the large number of schemes based on a scheme for the Stokes equation and a scheme for the convection term.
3. Considering the same scheme for the Stokes problem, there is no scheme for the convection term which can be considered as the “best one”: the accuracy of each of them is depending on the test case, and the use of a given scheme may also depend on relevant coupled physical phenomena.

The continuation of this study will be to show how a larger number of schemes can be proved to enter into this framework.

## Appendix A. Convection limit-conformity of the skew symmetric example

In this appendix, we prove, under some general hypotheses on the sequence of gradient discretisations, the convection limit-conformity of Scheme 2 in Section 4, that is the skew symmetric convection scheme, which corresponds to the case where $b_D$ is defined by:

$$b_D(u,v) = \frac{1}{2}\left(\widetilde{b}_D(u,u,v) - \widetilde{b}_D(u,v,u)\right), \tag{A.1}$$

where $\widetilde{b}_D : X_{D,0}^3 \longrightarrow \mathbb{R}$ is given by

$$\widetilde{b}_D(u,v,w) = \sum_{i,j=1}^{d} \int_\Omega \Pi_D^{(i)} u \nabla_D^{(i,j)} v \Pi_D^{(j)} w \, \mathrm{d}x.$$

As stated in the introduction of this paper, this proof holds for the Crouzeix-Raviart scheme as well as for the other three cases studied in [5] (conforming Taylor-Hood scheme, MAC scheme, HMM scheme), which all meet the $p$-coercivity property for $p = 6$ in the sense of the following definition.

**Definition Appendix A.1 ($p$-coercivity).** *Let $D$ be a discretisation in the sense of Definition 2.2. Let $p \in [1,+\infty)$ and let $C_D^{(p)}$ be defined by*

$$C_D^{(p)} = \max_{v \in X_{D,0}^*} \frac{\|\Pi_D v\|_{\boldsymbol{L}^p(\Omega)}}{\|v\|_D} + C_D. \tag{A.2}$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be p-coercive if there exist $C_S \geq 0$ such that $C_{D_m}^{(p)} \leq C_S$ for all $m \in \mathbb{N}$.*

Note that the $p$-coercivity implies the $q$-coercivity for any $q \in [1,p]$.

**Remark Appendix A.2.** *This section does not cover the case of the classical choice for the nonlinear convection term of the MAC scheme. Note that the discrete Sobolev inequality [7, Lemma 3.5] yields p-coercivity properties for the MAC Gradient Discretisation, and therefore that [11, Lemma 3.11] completes the proof that the classical choice for the MAC scheme has the space convection limit-conformity property.*

Let us now give the result obtained for a $p$-coercive sequence of gradient discretisations.

**Lemma Appendix A.3 (Space convection limit-conformity).** *Let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 2.2 which is $p-$coercive with $p > 4$ in the sense of Definition Appendix A.1 and such that $b_{D_m}$ is defined by* (A.1)*. Then $(D_m)_{m\in\mathbb{N}}$ is convection limit-conforming in the sense of Definition 2.9.*

**Proof** First, by definition of $b_D$, we remark that $b_D$ is continuous, is such that for all $u \in X_{D,0}$, $b_D(u,u) = 0$ and such that $b_D(u,v)$ is linear with respect to $v$. Moreover, we may write

$$\widetilde{b}_D(u,u,v) \leq \|\Pi_D u\|_{\boldsymbol{L}^4(\Omega)} \|u\|_D \|\Pi_D v\|_{\boldsymbol{L}^4(\Omega)}.$$

Thanks to the p-coercivity of the discretisation, we obtain

$$\widetilde{b}_D(u,u,v) \leq (C_D^{(4)})^2 \|u\|_D^2 \|v\|_D.$$

Using the same idea for $\widetilde{b}_D(u,v,u)$, we finally get the admissibility of $b_D$ in the sense of Definition 2.2, using $B_D \le 2(C_D^{(4)})^2$ (see (8)). Therefore the sequence $(B_{D_m})_{m\in\mathbb{N}}$ is bounded. It remains to prove that, for a sequence $(u_m,v_m)_{m\in\mathbb{N}} \in X^2_{D_m,0}$ with the properties given in Definition 2.9, $b_{D_m}(u_m,v_m) \to b(\overline{u},\overline{v})$. We remark that the strong convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_{D_m}u_m$ to $\overline{u}$ and $\Pi_{D_m}v_m$ to $\overline{v}$ combined with the p-coercivity for $p>4$, gives us the convergence in $\boldsymbol{L}^4(\Omega)$ of $\Pi_{D_m}u_m \to \overline{u}$ and $\Pi_{D_m}v_m \to \overline{v}$. Thus, for the first term of the right hand-side of $b_D$, the weak convergence in $\boldsymbol{L}^2(\Omega)^d$ of $\nabla_{D_m}v_m \to \nabla\overline{v}$ suffices for passing to the limit. Using the same reasoning for the second term of the right hand-side allows us to write the following result:

$$\lim_{m\to\infty} b_{D_m}(u_m,v_m) = \frac{1}{2}(\widetilde{b}(\overline{u},\overline{u},\overline{v}) - \widetilde{b}(\overline{u},\overline{v},\overline{u})) = b(\overline{u},\overline{v}),$$

recalling Property (6) since $\overline{u}\in E(\Omega)$. $\square$

## Appendix B. Space-time convection limit-conformity of the skew symmetric example

**Lemma Appendix B.1 (Space-time convection-conformity).** *Let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisations in the sense of Definition 3.2 which is $p-$coercive in the sense of Definition Appendix A.1 with $p>4$ and such that $b_{D_m}$ is defined by* (A.1). *Then $(D_m)_{m\in\mathbb{N}}$ is space-time convection limit-conforming in the sense of Definition 3.4.*

**Proof** We consider sequences $(u_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$ satisfying the properties given in Definition 3.4. Let us check that the four items which are assumed in Lemma Appendix B.2 are satisfied. Items 1 and 3 are assumed on $(\Pi^{(i)}_{D_m}u_m)_{m\in\mathbb{N}}$ and $(\nabla^{(i,j)}_D u_m)_{m\in\mathbb{N}}$. The fact that $(\Pi^{(i)}_{D_m}u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(0,T,\boldsymbol{L}^p(\Omega))$ is a consequence of the p-coercivity assumption thanks to the fact that $\|\nabla_{D_m}u_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}$ is bounded. Similarly, using the fact that $\|\nabla_{D_m}v_m\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega)^d)}$ is bounded, we get a bound on $\|\Pi^{(j)}_{D_m}v_m\|_{L^\infty(0,T,\boldsymbol{L}^p(\Omega))}$, hence items 2 and 4 hold. Thus we can apply Lemma Appendix B.2, which leads to

$$\lim_{m\to\infty}\int_0^T \widetilde{b}_{D_m}(u_m,u_m,v_m) = \int_0^T \widetilde{b}(\overline{u},\overline{u},\overline{v}).$$

We check as well that the three items assumed in Lemma Appendix B.3 hold, which shows that

$$\lim_{m\to\infty}\int_0^T \widetilde{b}_{D_m}(u_m,v_m,u_m) = \int_0^T \widetilde{b}(\overline{u},\overline{v},\overline{u}).$$

Finally, the last two limits combined with Property (6), since $\overline{u}\in L^2(0,T,E(\Omega))$, conclude the space-time convection limit-conformity. $\square$

To complete the proof of Lemma Appendix B.1, we now state and prove two technical lemmas.

**Lemma Appendix B.2.** *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences such that there exists $p>4$ with*

1. *$u_n \to u$ in $L^1(0,T;L^2(\Omega))$,*
2. *$u_n$ is bounded in $L^2(0,T;L^p(\Omega))$ and in $L^\infty(0,T;L^2(\Omega))$,*
3. *$v_n \to v$ weakly in $L^2(0,T;L^2(\Omega))$,*
4. *$w_n \to w$ in $L^\infty(0,T;L^2(\Omega))$ and is bounded in $L^\infty(0,T;L^p(\Omega))$.*

*then*

$$\lim_{n\to\infty}\int_0^T\int_\Omega u_n(x,t)v_n(x,t)w_n(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega u(x,t)v(x,t)w(x,t)\mathrm{d}x\mathrm{d}t.$$

**Proof** It suffices to remark that $u_n w_n$ tends to $uw$ in $L^2(0,T;L^2(\Omega))$. Indeed, we have

$$\int_0^T \int_\Omega (u_n(x,t)w_n(x,t) - u(x,t)w(x,t))^2 \mathrm{d}x\mathrm{d}t \leq 2(A_n + B_n)$$

with

$$A_n = \int_0^T \int_\Omega (u_n(x,t) - u(x,t))^2 w_n(x,t)^2 \mathrm{d}x\mathrm{d}t$$

and

$$B_n = \int_0^T \int_\Omega u(x,t)^2 (w_n(x,t) - w(x,t))^2 \mathrm{d}x\mathrm{d}t$$

Owing to the inequality

$$\int_0^T \int_\Omega U(x,t)^2 W(x,t)^2 \mathrm{d}x\mathrm{d}t \leq \int_0^T \left(\int_\Omega U(x,t)^4 \mathrm{d}x\right)^{1/2} \left(\int_\Omega W(x,t)^4 \mathrm{d}x\right)^{1/2} \mathrm{d}t$$

which leads to

$$\int_0^T \int_\Omega U(x,t)^2 W(x,t)^2 \mathrm{d}x\mathrm{d}t \leq \|W\|^2_{L^\infty(0,T;L^4(\Omega))} \|U\|^2_{L^2(0,T;L^4(\Omega))},$$

we get that $A_n \to 0$ and $B_n \to 0$, since by $L^p - L^q$ interpolation, $u^n \to u$ in $L^2(0,T;L^4(\Omega))$ and $w^n \to w$ in $L^\infty(0,T;L^4(\Omega))$. $\square$

**Lemma Appendix B.3.** *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences such that there exists $p > 4$ with*

1. *$u_n \to u$ and $v_n \to v$ in $L^1(0,T;L^2(\Omega))$,*
2. *$u_n$ and $v_n$ are bounded in $L^2(0,T;L^p(\Omega))$ and in $L^\infty(0,T;L^2(\Omega))$,*
3. *$w_n \to w$ in $L^\infty(0,T;L^2(\Omega))$.*

*then*

$$\lim_{n\to\infty} \int_0^T \int_\Omega u_n(x,t)v_n(x,t)w_n(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega u(x,t)v(x,t)w(x,t)\mathrm{d}x\mathrm{d}t.$$

**Proof**

$$\int_0^T \int_\Omega (u_n(x,t)v_n(x,t)w_n(x,t) - u(x,t)v(x,t)w(x,t))\mathrm{d}x\mathrm{d}t = A_n + B_n + C_n$$

with

$$A_n = \int_0^T \int_\Omega (u_n(x,t) - u(x,t))v_n(x,t)w_n(x,t)\mathrm{d}x\mathrm{d}t$$

$$B_n = \int_0^T \int_\Omega u(x,t)(v_n(x,t) - v(x,t))w_n(x,t)\mathrm{d}x\mathrm{d}t$$

and

$$C_n = \int_0^T \int_\Omega u(x,t)v(x,t)(w_n(x,t) - w(x,t))\mathrm{d}x\mathrm{d}t$$

The conclusion follows from the fact that, by $L^p - L^q$ interpolation, $u^n \to u$ and $v_n \to v$ in $L^2(0,T;L^4(\Omega))$, and $w^n \to w$ in $L^\infty(0,T;L^4(\Omega))$, and from the inequality

$$\int_0^T \int_\Omega |U(x,t)V(x,t)W(x,t)|\mathrm{d}x\mathrm{d}t \le \|W\|_{L^\infty(0,T;L^2(\Omega))}\|U\|_{L^2(0,T;L^4(\Omega))}\|V\|_{L^2(0,T;L^4(\Omega))}.$$

Indeed,

$$\int_0^T \int_\Omega |U(x,t)V(x,t)W(x)|\mathrm{d}x\mathrm{d}t \le \sup_{t\in]0,T[}(\int_\Omega W(x,t)^2\mathrm{d}x)^{1/2}(\int_\Omega U(x,t)^2V(x,t)^2\mathrm{d}x)^{1/2}\mathrm{d}t,$$

so

$$\int_0^T \int_\Omega |U(x,t)V(x,t)W(x)|\mathrm{d}x\mathrm{d}t \le \sup_{t\in]0,T[}(\int_\Omega W(x,t)^2\mathrm{d}x)^{1/2}\int_0^T(\int_\Omega U(x,t)^4\mathrm{d}x\int_\Omega V(x,t)^4\mathrm{d}x)^{1/4}\mathrm{d}t,$$

and therefore we have

$$\int_0^T \int_\Omega |U(x,t)V(x,t)W(x)|\mathrm{d}x\mathrm{d}t$$
$$\le \sup_{t\in]0,T[}(\int_\Omega W(x,t)^2\mathrm{d}x)^{1/2}\left(\int_0^T(\int_\Omega U(x,t)^4\mathrm{d}x)^{1/2}\mathrm{d}t\right)^{1/2}\left(\int_0^T(\int_\Omega V(x,t)^4\mathrm{d}x)^{1/2}\mathrm{d}t\right)^{1/2},$$

which concludes the proof. □

## Appendix C. Discrete Aubin-Simon theorem

The following theorem is a discrete version of the Aubin-Simon theorem. It is proved in [2, Theorem 7.1].

**Theorem Appendix C.1 (Discrete Aubin-Simon theorem).** *Let $T > 0$ and let $B$ be a Banach space. Let $(B_\ell)_{\ell\in\mathbb{N}}$ be a sequence of finite dimensional subspaces of $B$. For any $\ell \in \mathbb{N}$, let $N_\ell \in \mathbb{N}^*$, $t_\ell^{(0)} = 0 < t_\ell^{(1)} < \ldots < t_\ell^{(N_\ell)} = T$ and $k_\ell^{(n)} = t_\ell^{(n)} - t_\ell^{(n-1)}$, $n = 1,\ldots,N_\ell$. Let $\{u_\ell^{(n)}, n = 0,\ldots,N_\ell\} \subset B_\ell$ and let $u_\ell \in L^1(0,T;B_\ell)$ be defined, for a given real family $(\alpha_\ell^{(n)})_{n=1,\ldots,N_\ell}$, by*

$$\begin{gathered} u_\ell(\cdot,t) = (1-\alpha_\ell^{(n)})u_\ell^{(n-1)} + \alpha_\ell^{(n)}u_\ell^{(n)} \in B_\ell, \\ \text{for a.e. } t \in (t_\ell^{(n-1)}, t_\ell^{(n)}), \text{ and } n \in \{1,\ldots N_\ell\}. \end{gathered} \tag{C.1}$$

*Let $\delta_\ell u_\ell$ be the "discrete time derivative", defined by:*

$$\delta_\ell u_\ell(\cdot,t) = \delta_\ell^{(n)}u_\ell := \frac{1}{k_\ell^{(n)}}(u_\ell^{(n)} - u_\ell^{(n-1)}) \text{ for a.e. } t \in (t_\ell^{(n-1)}, t_\ell^{(n)}), n \in \{1,\ldots,N_\ell\}.$$

*Let $\|\cdot\|_{X_\ell}$ and $\|\cdot\|_{Y_\ell}$ be two norms on $B_\ell$. We denote by $X_\ell$ the space $B_\ell$ endowed with the norm $\|\cdot\|_{X_\ell}$ and by $Y_\ell$ the space $B_\ell$ endowed with the norm $\|\cdot\|_{Y_\ell}$. We assume that*

*(h1) For any sequence $(w_\ell)_{\ell\in\mathbb{N}}$ such that $w_\ell \in B_\ell$ and $(\|w_\ell\|_{X_\ell})_{\ell\in\mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_\ell \to w$ in $B$ as $\ell \to +\infty$.*

*(h2) For any sequence $(w_\ell)_{\ell\in\mathbb{N}}$ such that $w_\ell \in B_\ell$, $(\|w_\ell\|_{X_\ell})_{\ell\in\mathbb{N}}$ is bounded, there exists $w \in B$ such that $w_\ell \to w$ in $B$ and $\|w_\ell\|_{Y_\ell} \to 0$ as $\ell \to +\infty$, then $w = 0$.*

*(h3) The family $(\alpha_\ell^{(n)})_{n=1,\ldots,N_\ell,\ell\in\mathbb{N}}$ and the sequence $(\|u_\ell\|_{L^1(0,T;X_\ell)})_{\ell\in\mathbb{N}}$ are bounded.*

*(h4) The sequence $(\|\delta_\ell u_\ell\|_{L^1(0,T;Y_\ell)})_{\ell\in\mathbb{N}}$ is bounded.*

*Then there exists $u \in L^1(0,T;B)$ such that, up to a subsequence, $u_\ell \to u$ in $L^1(0,T;B)$ as $\ell \to +\infty$.*

## References


[1] F. Brezzi, K. Lipnikov, V. Simoncini, A family of mimetic finite difference methods on polygonal and polyhedral meshes, Math. Mod. Meth. Appl. S. 15 (10) (2005) 1533–1551.

[2] E. Chénier, R. Eymard, T. Gallouët, R. Herbin, An extension of the MAC scheme to locally refined meshes: convergence analysis for the full tensor time-dependent Navier–Stokes equations, Calcolo (2012) 1–39.

[3] M. Crouzeix, P.-A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 7 (R-3) (1973) 33–75.

[4] D. A. Di Pietro, A. Ern, Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations, Math. Comp. 79 (271) (2010) 1303–1330.
URL `https://dx-doi-org.fennec.u-pem.fr/10.1090/S0025-5718-10-02333-1`

[5] J. Droniou, R. Eymard, P. Feron, Gradient schemes for Stokes problem, IMA J. Numer. Anal. 36 (4) (2016) 1636–1669.
URL `http://dx.doi.org/10.1093/imanum/drv061`

[6] J. Droniou, R. Eymard, T. Gallouët, R. Herbin, Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations, Math. Models Methods Appl. Sci. (M3AS) 23 (13) (2013) 2395–2432.

[7] R. Eymard, T. Gallouët, R. Herbin, Finite volume methods, in: P. G. Ciarlet, J.-L. Lions (eds.), Techniques of Scientific Computing, Part III, Handbook of Numerical Analysis, VII, North-Holland, Amsterdam, 2000, pp. 713–1020.

[8] R. Eymard, T. Gallouët, R. Herbin, Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI: a scheme using stabilization and hybrid interfaces, IMA J. Numer. Anal. 30 (4) (2010) 1009–1043.
URL `http://dx.doi.org/10.1093/imanum/drn084`

[9] R. Eymard, R. Herbin, J.-C. Latché, Convergence analysis of a colocated finite volume scheme for the incompressible Navier–Stokes equations on general 2D or 3D meshes, SIAM Journal on Numerical Analysis 45 (1) (2007) 1–36.

[10] T. Gallouët, R. Herbin, J.-C. Latché, A convergent finite element-finite volume scheme for the compressible stokes problem. part i: The isothermal case, Mathematics of Computation 78 (267) (2009) 1333–1352.

[11] T. Gallouët, R. Herbin, J.-C. Latché, K. Mallem, Convergence of the Marker-and-Cell Scheme for the Incompressible Navier–Stokes Equations on Non-uniform Grids, Foundations of Computational Mathematics.
URL `https://doi.org/10.1007/s10208-016-9338-4`

[12] T. Gallouët, R. Herbin, D. Maltese, A. Novotny, Error estimates for a numerical approximation to the compressible barotropic Navier-Stokes equations, IMA J. Numer. Anal. 36 (2) (2016) 543–592.
URL `https://dx-doi-org.fennec.u-pem.fr/10.1093/imanum/drv028`

[13] U. Ghia, K. N. Ghia, C. Shin, High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method, Journal of computational physics 48 (3) (1982) 387–411.

[14] F. Harlow, J. Welch, Numerical calculation of time-dependent viscous incompressible flow of fluid with a free surface, Physics of Fluids 8 (1965) 2182–2189.

[15] R. Herbin, F. Hubert, Benchmark on discretization schemes for anisotropic diffusion problems on general grids for anisotropic heterogeneous diffusion problems, in: R. Eymard, J.-M. Hérard (eds.), Finite Volumes for Complex Applications V, Wiley, 2008.

[16] J.-C. Latché, Personal communication.

[17] R. Nicolaïdes, Analysis and convergence of the mac scheme i: The linear problem, SIAM J. Numer. Anal. 29 (1992) 1579–1591.

[18] R. Nicolaïdes, X. Wu, Analysis and convergence of the mac scheme ii, Navier-Stokes equations, Math. Comp. 65 (1996) 29–44.

[19] S. Patankar, Numerical heat transfer and fluid flow, vol. XIII of Series in Computational Methods in Mechanics and Thermal Sciences, Washington - New York - London: Hemisphere Publishing Corporation; New York. McGraw-Hill Book Company, 1980.

[20] C. Taylor, P. Hood, A numerical solution of the Navier–Stokes equations using the finite element technique, Internat. J. Comput. & Fluids 1 (1) (1973) 73–100.

[21] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, vol. 2 of Studies in Mathematics and its Applications, 3rd ed., North-Holland Publishing Co., Amsterdam, 1984, with an appendix by F. Thomasset.

[22] P. Wesseling, Principles of computational fluid dynamics, vol. 29 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2001.