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# A REMARKABLE 20-CROSSING TANGLE

SHALOM ELIAHOU AND JEAN FROMENTIN

ABSTRACT. For any positive integer  $r$ , we exhibit a nontrivial knot  $K_r$  with  $(20 \cdot 2^{r-1} + 1)$  crossings whose Jones polynomial  $V(K_r)$  is equal to 1 modulo  $2^r$ . Our construction rests on a certain 20-crossing tangle  $T_{20}$  which is undetectable by the Kauffman bracket polynomial pair mod 2.

## 1. INTRODUCTION

In [6], M.B. Thistlethwaite gave two 2-component links and one 3-component link which are nontrivial and yet have the same Jones polynomial as the corresponding unlink  $U^2$  and  $U^3$ , respectively. These were the first known examples of nontrivial links undetectable by the Jones polynomial. Shortly thereafter, it was shown in [2] that, for any integer  $k \geq 2$ , there exist infinitely many nontrivial  $k$ -component links whose Jones polynomial is equal to that of the  $k$ -component unlink  $U^k$ . Yet the corresponding problem for 1-component links, i.e. for knots, is widely open: *does there exist a nontrivial knot  $K$  whose Jones polynomial is equal to that of the unknot  $U^1$ , namely to 1?*

We shall consider here the following weaker problem, consisting in looking for nontrivial knots  $K$  whose Jones polynomial is *congruent modulo some integer* to that of the unknot  $U^1$ .

**Problem 1.1.** *Given any integer  $m \geq 2$ , does there exist a nontrivial knot  $K$  whose Jones polynomial  $V(K)$  satisfies  $V(K) \equiv 1 \pmod{m}$ ?*

Naturally, for any two Laurent polynomials  $f, g$  in  $\mathbb{Z}[t, t^{-1}]$ , the notation  $f \equiv g \pmod{m}$  means that there exists an element  $h \in \mathbb{Z}[t, t^{-1}]$  such that  $f - g = m \cdot h$ . This is equivalent to require that, for each  $i \in \mathbb{Z}$ , the coefficients  $\alpha_i$  and  $\beta_i$  of  $t^i$  in  $f$  and  $g$ , respectively, are congruent modulo  $m$  as integers.

A result of M.B. Thistlethwaite [5] states that, for an alternating knot  $K$  with  $n$  crossings, the span of  $V(K)$  is exactly  $n$  and the coefficients of the terms of maximal and minimal degree in  $V(K)$  are both  $\pm 1$ . In particular, for any  $m \geq 2$ , *there is no alternating knot with trivial Jones polynomial modulo  $m$ .*

Using the *Mathematica* package **KnotTheory** of the KnotAtlas project [1], it is easy to find knots which are solutions of Problem 1.1 for the moduli  $m = 2, 3$  and 4. For  $m = 5$  there is no solution of Problem 1.1 among the knots up to 16 crossings. The following table gives the number of solutions of Problem 1.1 for the moduli 2

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to 5 up to 16 crossings, respectively:

$m$	$\leq 11$	12	13	14	15	16
2	0	4	9	35	140	582
3	0	1	0	1	2	26
4	0	0	0	0	1	0
5	0	0	0	0	0	0

In this note, we shall solve Problem 1.1 for all moduli  $m$  which are powers of 2. That is, given any integer  $r \geq 1$ , we shall construct infinitely many knots whose Jones polynomial is trivial mod  $m = 2^r$ . Our construction rests on a certain 20-crossing tangle  $T_{20}$  whose Kauffman bracket polynomial pair is trivial mod 2.

The paper is structured as follows. Section 2 is devoted to basic tangle operations. In Section 3 we describe our tangle  $T_{20}$  and we construct a family of knots  $K_r$  from a tangle  $M_r$  which is composed of  $2^r$  copies of the tangle  $T_{20}$ . In Section 4, we compute the Kauffman bracket pair of the tangle  $M_r$  modulo  $2^r$ . In Section 5, we prove that the knots  $K_r$  for  $r \geq 1$  are distinct and that the Jones polynomial of  $K_r$  is trivial modulo  $2^r$ . The paper concludes with a few related open questions.

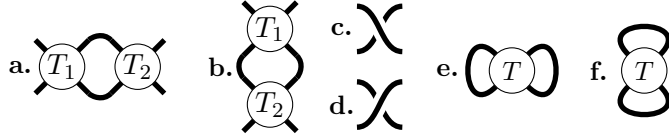


FIGURE 1. **a.** Tangle  $T_1 + T_2$ . **b.** Tangle  $T_1 * T_2$ . **c.** Tangle 1. **d.** Tangle  $-1$ . **e.** Closure  $\text{den}(T)$ . **f.** Closure  $\text{num}(T)$ .

## 2. BASIC TANGLE OPERATIONS

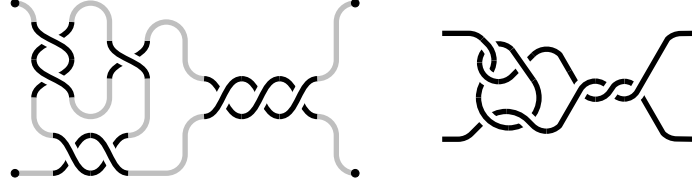
We shall use the same notation as in [2]. In particular, if  $T_1, T_2$  are two tangles with 4 endpoints, we denote by  $T_1 + T_2$  their *horizontal sum* and by  $T_1 * T_2$  their *vertical sum* (see **a** and **b** of Figure 1). The tangle 1 denotes a single crossing as in **c** of Figure 1, while the tangle  $-1$  denotes its opposite version as in **d** of Figure 1. More generally, if  $T$  is a tangle, then  $-T$  denotes the tangle obtained from  $T$  by switching the signs of all crossings in  $T$ .

As usual, for  $k \in \mathbb{N} \setminus \{0\}$  we define the tangles

$$\begin{aligned} k &= 1 + \cdots + 1, & -k &= (-1) + \cdots + (-1), \\ 1/k &= 1 * \cdots * 1, & -1/k &= (-1) * \cdots * (-1), \end{aligned}$$

with  $k$  terms in each expression. These are particular cases of *algebraic tangles*, namely tangles constructed recursively from the tangles  $\pm 1$  using horizontal and vertical sums.

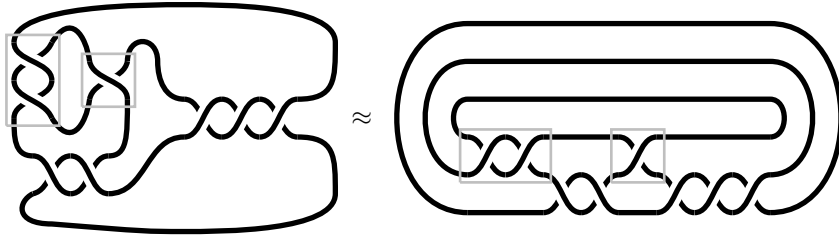
**Example 2.1.** Let  $T_{8,21}$  be the tangle  $((1/2) + 1) * 2 + (-3)$ . (See next example for the choice of this name.) Diagrams corresponding to  $T_{8,21}$  are



The parts in black correspond to the tangles  $1/2$ ,  $1$ ,  $2$  and  $-3$ , respectively, while the gray strands depict the connections between them. The four marked points represent the extremities of the global tangle. The rightmost diagram is a “smoother” representation of the tangle  $T_{8,21}$ .

If  $T$  is a tangle, we denote by  $\text{den}(T)$  and  $\text{num}(T)$  the link diagrams obtained by gluing the extremities of  $T$  as in **e** and **f** of Figure 1, respectively.

**Example 2.2.** Consider again the tangle  $T_{8,21}$  of Example 2.1. Here are  $\text{num}(T_{8,21})$  and an isotopic diagram.



The isotopy is obtained from the left diagram by rotating counterclockwise the left block in gray and clockwise the right one. Looking at Dale Rolfe's knot table [4], we remark that  $\text{num}(T_{8,21})$  is a diagram of  $K_{8,21}$  which is the 21<sup>st</sup> prime knot with 8 crossings.

### 3. THE MAIN CONSTRUCTION

We now introduce a family of tangles which includes the tangle  $T_{20}$ , the cornerstone of our solution to Problem 1.1 for moduli  $m$  which are powers of 2.

**Definition 3.1.** We define the tangles  $T_{10}$ ,  $T_{20}$  and  $M_r$  for  $r \geq 1$  as follows:

- *i*)  $T_{10} = T_{8,21} * 2 = (((1/2) + 1) * 2) + (-3) * 2$ ;
- *ii*)  $T_{20} = T_{10} + (-T_{10})$ ;
- *iii*)  $M_1 = T_{20}$  and  $M_r = M_{r-1} + M_{r-1}$  for  $r \geq 2$ .

Tangle  $M_1$  is depicted in Figure 2.

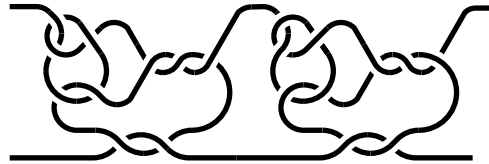


FIGURE 2. The key tangle  $M_1 = T_{20}$ .

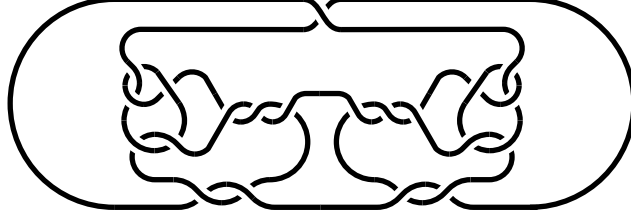
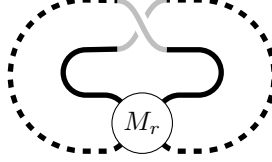


FIGURE 3. An elegant crab-like mutant version  $K'_1$  of the knot  $K_1$ .

**Definition 3.2.** For  $r \in \mathbb{N} \setminus \{0\}$ , we define  $K_r$  to be the knot represented by the diagram  $\text{den}(1 * M_r)$ .

By definition, the tangle  $M_r$  has  $2^{r-1} \times 20$  crossings. So the knot  $K_r$  has at most  $1 + 2^{r-1} \times 20$  crossings. We observe that the tangle  $M_r$  is the union of two arcs, the first one going from NW to NE and the second one from SW to SE. As illustrated by the diagram



the link diagram  $\text{den}(1 * M_r)$  has exactly one component: if we travel along the dotted arc we must meet the undotted one. The following proposition summarizes these remarks.

**Proposition 3.3.** For each  $r \geq 1$ , the link  $K_r$  is a knot with at most  $1 + 2^{r-1} \times 20$  crossings.

For all  $r \geq 1$ , a mutant knot  $K'_r$  is obtained from  $K_r$  by replacing the tangle  $-T_{10}$  in each summand  $M_1$  of  $M_r$  by its image under vertical symmetry. The knot  $K_r$  and  $K'_r$ , being mutant of each other, have the same Jones polynomial [3]. The knot  $K'_1$  is depicted on Figure 3.

#### 4. THE KAUFFMAN BRACKET PAIR OF A TANGLE

In this section we recall the definition of the Kauffman bracket pair of a tangle. This notion is very powerful to determine the Jones polynomial for a knot obtained from an algebraic tangle, which is the case for our knots  $K_r$ .

Let  $T$  be a tangle with 4 endpoints. The Kauffman bracket  $\langle T \rangle$  of  $T$  is a linear combination of two formal symbols  $\langle 0 \rangle$  and  $\langle \infty \rangle$  with coefficients in the ring of Laurent polynomials  $\Lambda = \mathbb{Z}[t, t^{-1}]$ . The bracket  $\langle T \rangle$  may be computed with the usual rules of the Kauffman bracket polynomial, namely:

- $\langle \bigcirc \rangle = 1$ ;
- $\langle \bigcirc \amalg T \rangle = \delta \langle T \rangle$  where  $\delta = -t^{-2} - t^2$ ;
- $\langle \times \rangle = t^{-1} \langle \smile \rangle + t \langle \frown \rangle$ .

Thus, after removing the crossings and all free loops using the rules above, we end up with a unique expression of the form  $\langle T \rangle = f(T) \langle 0 \rangle + g(T) \langle \infty \rangle$  where 0 is the tangle  $\smile$  and  $\infty$  is the tangle  $\frown$ . The notation 0 and  $\infty$  comes from the shape of the link obtained when we take the den closure of the corresponding tangles.

We define the *bracket pair*  $\text{br}(T)$  of  $T$  as

$$(1) \quad \text{br}(T) = \begin{bmatrix} f(T) \\ g(T) \end{bmatrix} \in \Lambda^2.$$

For example we compute

$$\langle -1 \rangle = \langle \text{X} \rangle = t^{-1} \langle \text{Z} \rangle + t \langle \text{I} \rangle = t^{-1} \langle 0 \rangle + t \langle \infty \rangle,$$

which implies  $\text{br}(-1) = \begin{bmatrix} t^{-1} \\ t \end{bmatrix}$ . Since by definition of  $-T$ , we have  $f(-T) =$

$f(T)|_{t \leftarrow t^{-1}}$  and  $g(-T) = g(T)|_{t \leftarrow t^{-1}}$  we obtain  $\text{br}(1) = \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$ .

Proposition 2.2 of [2] gives computation rules for the bracket pair of the horizontal and vertical sum of tangles together with the bracket of the num and den closures of tangles. Here they are.

**Proposition 4.1.** *For two tangles  $T$  and  $U$ , we have:*

- i)  $\text{br}(T+U) = \begin{bmatrix} f(T)f(U) \\ f(T)g(U) + g(T)f(U) + \delta g(T)g(U) \end{bmatrix};$
- ii)  $\text{br}(T * U) = \begin{bmatrix} \delta f(T)f(U) + f(T)g(U) + g(T)f(U) \\ g(T)g(U) \end{bmatrix};$
- iii)  $\langle \text{num}(T) \rangle = \delta f(T) + g(T)$  and  $\langle \text{den}(T) \rangle = f(T) + \delta g(T)$ .

A direct computation from the expression of  $\text{br}(1)$  gives

$$\text{br}(2) = \begin{bmatrix} t^2 \\ -t^{-4} + 1 \end{bmatrix}, \quad \text{br}(3) = \begin{bmatrix} t^3 \\ t^{-7} - t^{-3} + t \end{bmatrix} \quad \text{and} \quad \text{br}(1/2) = \begin{bmatrix} 1 - t^4 \\ t^{-2} \end{bmatrix}.$$

Using these values, we determine the bracket pair of  $T_{8,21} = (((1/2) + 1) * 2) + (-3)$ :

$$(2) \quad \text{br}(T_{8,21}) = \begin{bmatrix} -2t^{-6} + 2t^{-2} - 2t^2 + t^6 \\ -2t^{-4} + 3 - 4t^4 + 3t^8 - 2t^{12} + t^{16} \end{bmatrix}.$$

**Notation.** In the sequel, for a tangle  $T$  and an integer  $m \geq 2$ , we will denote by  $\text{br}_m(T)$  the bracket pair of  $T$  modulo  $m$ .

**4.1. The case of tangle  $T_{20}$ .** We now analyze the bracket pair of the tangle  $T_{20}$  introduced in Definition 3.1.

**Lemma 4.2.** *The bracket pair  $\text{br}_2(T_{20})$  is equal to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Moreover, the leading term of  $f(T_{20})$  is  $2t^{28}$  and that of  $g(T_{20})$  is  $2t^{26}$ .*

*Proof.* By relation (2), we obtain

$$\text{br}(T_{10}) = \text{br}(T_{8,21} * 2) = \begin{bmatrix} 2t^{-10} - 2t^{-6} + 2t^{-2} - 2t^6 + 2t^{10} - 2t^{14} + t^{18} \\ 2t^{-8} - 5t^{-4} + 7 - 7t^4 + 5t^8 - 3t^{12} + t^{16} \end{bmatrix}.$$

Replacing  $t$  by  $t^{-1}$ , we get

$$\text{br}(-T_{10}) = \begin{bmatrix} t^{-18} - 2t^{-14} + 2t^{-10} - 2t^{-6} + 2t^2 - 2t^6 + 2t^{10} \\ t^{-16} - 3t^{-12} + 5t^{-8} - 7t^{-4} + 7 - 5t^4 + 2t^8 \end{bmatrix}.$$

The formula for the computation of  $\text{br}(T_{10} + (-T_{10}))$  given in i) of Proposition 4.1 implies that the leading term of  $f(T_{20})$  is  $t^{18} \cdot 2t^{10} = 2t^{28}$  and that of  $g(T_{20})$  is

$$t^{18} \cdot 2t^8 + t^{16} \cdot 2t^{10} - t^2 \cdot t^{16} \cdot 2t^8 = 2t^{26}$$

as expected. Taking coefficient modulo 2, we obtain

$$\begin{aligned} \text{br}_2(T_{10}) &= \begin{bmatrix} t^{18} \\ t^{-4} + 1 + t^4 + t^8 + t^{12} + t^{16} \end{bmatrix}, \\ \text{br}_2(-T_{10}) &= \begin{bmatrix} t^{-18} \\ t^{-16} + t^{-12} + t^{-8} + t^{-4} + 1 + t^4 \end{bmatrix}, \end{aligned}$$

and so  $\text{br}_2(T_{20}) = \text{br}_2(T_{10} + (-T_{10})) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .  $\square$

**4.2. The general case of tangles  $M_r$ .** We now analyze the bracket pair (1) of the tangle  $M_r$  constructed in Definition 3.1. For convenience, for  $r \geq 1$ , we denote by  $\ell_r \in \mathbb{Z}[t, t^{-1}]$  the leading term of  $f(M_r)$ .

**Proposition 4.3.** *For all  $r \geq 1$ , we have  $\text{br}_{2r}(M_r) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Moreover, we have  $\ell_r = (2t^{28})^{2^{r-1}}$  while the leading term of  $g(M_r)$  is equal to  $t^{-2}\ell_r$ .*

*Proof.* By induction on  $r \geq 1$ . The case  $r = 1$  is Lemma 4.2. Assume now  $r \geq 2$ . By the induction hypothesis, we have  $f(M_{r-1}) \equiv 1 \pmod{2^{r-1}}$  and  $g(M_{r-1}) \equiv 0 \pmod{2^{r-1}}$ . Hence, there exist two Laurent polynomials  $P$  and  $Q$  in  $\mathbb{Z}[t, t^{-1}]$  such that the relations  $f(M_{r-1}) = 1 + 2^{r-1}P$  and  $g(M_{r-1}) = 2^{r-1}Q$  hold. By Proposition 4.1 and formula  $\text{br}(M_r) = \text{br}(M_{r-1} + M_{r-1})$  we obtain:

$$\begin{aligned} \text{br}(M_r) &= \begin{bmatrix} f(M_{r-1})^2 \\ 2g(M_{r-1})f(M_{r-1}) + \delta g(M_{r-1})^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2^r P + 2^{2r-2} P^2 \\ 2^r Q + 2^{2r-1} PQ + \delta 2^{2r-2} Q^2 \end{bmatrix}. \end{aligned}$$

As  $2r - 2 \geq r$  holds since  $r \geq 2$ , we find  $\text{br}_{2r}(M_r) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Let us now prove the statement about the leading terms. Since  $f(M_r)$  is equal to  $f(M_{r-1})^2$ , the induction hypothesis implies that the leading term of  $f(M_r)$  is the square of the leading term of  $f(M_{r-1})$ , i.e., the square of  $\ell_{r-1}$ . Since  $\ell_{r-1}^2$  is equal to  $\ell_r$ , we have the desired result for the leading term of  $f(M_r)$ . Denoting by  $\text{lt}(P)$  the leading term of a Laurent polynomial  $P \in \Lambda$ , we have

$$\text{lt}(g(M_r)) = \text{lt}(2g(M_{r-1})f(M_{r-1}) + \delta g(M_{r-1})^2).$$

By the induction hypothesis, we compute

$$\begin{aligned} \text{lt}(g(M_{r-1})f(M_{r-1})) &= t^{-2}\ell_{r-1} \cdot \ell_{r-1} = t^{-2}\ell_{r-1}^2 = t^{-2}\ell_r \\ \text{lt}(\delta g(M_{r-1})^2) &= -t^2 \cdot (t^{-2}\ell_{r-1})^2 = -t^{-2}\ell_{r-1}^2 = -t^{-2}\ell_r \end{aligned}$$

and so  $\text{lt}(g(M_r)) = 2(t^{-2}\ell_r) - (t^{-2}\ell_r) = t^{-2}\ell_r$ , as desired.  $\square$

## 5. ON THE JONES POLYNOMIAL OF $K_r$

The *writhe* of an oriented link diagram  $D$ , denoted by  $\text{wr}(D)$ , is the sum of the signs of the crossings of  $D$  following conventions **a** and **b** of Figure 4.

The notion of writhe can be naturally extended to an oriented tangle  $T$ . We say that a tangle  $T$  is *left-right orientable* if it can be equipped with an orientation as in **c** of Figure 4, in which case we denote by  $\text{wr}(T)$  the writhe of  $T$  with respect to that orientation.

**Lemma 5.1.** *The tangle  $M_r$  is left-right orientable and we have  $\text{wr}(M_r) = 0$ .*

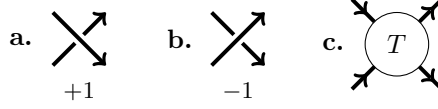


FIGURE 4. **a.** Crossing with positive writhe. **b.** Crossing with negative writhe. **c.** Left-right oriented tangle.

*Proof.* We first remark that the tangle  $T_{10}$  is left-right orientable. Since  $-T_{10}$  is obtained from  $T_{10}$  by switching the signs of the crossings, the tangle  $-T_{10}$  is also left-right orientable and  $\text{wr}(-T_{10}) = -\text{wr}(T_{10})$  holds. As the horizontal sum of tangles is compatible with the left-right orientation, for any left-right orientable tangles  $U$  and  $V$ , we have  $\text{wr}(U + V) = \text{wr}(U) + \text{wr}(V)$ . As  $M_1 = T_{10} + (-T_{10})$ , we have

$$\text{wr}(M_1) = \text{wr}(T_{10}) + \text{wr}(-T_{10}) = \text{wr}(T_{10}) - \text{wr}(T_{10}) = 0.$$

Again by the compatibility between the left-right orientation and the horizontal sum  $+$ , a straightforward induction yields  $\text{wr}(M_r) = \text{wr}(M_{r-1}) + \text{wr}(M_{r-1}) = 0 + 0 = 0$ .  $\square$

The *normalized Kauffman bracket polynomial* of a link  $L$  depicted by an oriented diagram  $D$  is

$$\chi(L) = (-t^3)^{-\text{wr}(D)} \langle D \rangle,$$

which is an invariant of the link  $L$  and so is independent of the choice of the oriented diagram  $D$  representing  $L$ . The *Jones polynomial* of a link  $L$  is then

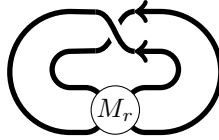
$$V(L) = \chi(L)|_{t \leftarrow t^{-1/4}}.$$

The Kauffman bracket of the unknot  $\bigcirc$  is  $\langle \text{den}(0) \rangle = 1$ , which gives  $\chi(\bigcirc) = 1$  and so  $V(\bigcirc) = 1$ .

Recall that  $K_r$  is the knot represented by the diagram  $\text{den}(1 * M_r)$  and that  $\ell_r$  is the leading term of  $f(M_r)$ .

**Proposition 5.2.** *For all  $r \geq 1$ , the Jones polynomial of  $K_r$  is equal to 1 modulo  $2^r$ . Moreover the leading term of  $\chi(K_r)$  is equal to  $\ell_r$ .*

*Proof.* Let  $r$  be an integer  $\geq 1$ . We denote by  $D_r$  the diagram  $\text{den}(1 * M_r)$ . The left-right orientation of  $M_r$  induces the following orientation on  $D_r$ :



As the writhe of  $M_r$  is 0 by Lemma 5.1, the writhe of  $D_r$  with respect to the above orientation is  $+1$ .

Let us now determine the Kauffman bracket of  $D_r$ . We have

$$\text{br}(1 * M_r) = \begin{bmatrix} \delta t f(M_r) + t g(M_r) + t^{-1} f(M_r) \\ t^{-1} g(M_r) \end{bmatrix}$$

and so

$$\begin{aligned}\langle D_r \rangle &= \delta t f(M_r) + t g(M_r) + t^{-1} f(M_r) + \delta t^{-1} g(M_r) \\ &= (\delta t + t^{-1}) f(M_r) + (t + \delta t^{-1}) g(M_r) \\ &= -t^3 f(M_r) - t^{-3} g(M_r).\end{aligned}$$

Hence the normalized Kauffman bracket of  $K_r$  is

$$\begin{aligned}\chi(K_r) &= (-t^3)^{-\text{wr}(D_r)} \cdot \langle D_r \rangle = (-t^3)^{-1} (-t^3 f(M_r) - t^{-3} g(M_r)) \\ &= f(M_r) + t^{-6} g(M_r).\end{aligned}$$

As  $f(M_r) = 1$  and  $g(M_r) = 0$  modulo  $2^r$  by Proposition 4.3, we obtain  $\chi(K_r) = 1$  modulo  $2^r$  and so  $V(K_r)$  is trivial modulo  $2^r$ . Since the leading term of  $g(M_r)$  is equal to  $t^{-2}\ell_r$  by Proposition 4.3, the leading term of  $\chi(K_r)$  is  $\ell_r$ .  $\square$

We can now state and prove our main result.

**Theorem 5.3.** *For all  $r \geq 1$ , there exist infinitely many pairwise distinct knots with trivial Jones polynomial modulo  $2^r$ .*

*Proof.* Let  $r \geq 1$  be an integer. The knots  $K_i$  with  $i \geq r$  satisfy the statement. Indeed, by Proposition 5.2, for all  $i \geq r$  the Jones polynomial of  $K_i$  is trivial modulo  $2^i$  and thus modulo  $2^r$ . Since for  $j \geq 1$ , the leading term of  $\chi(K_j)$  is

$$\ell_j = (2t^{28})^{2^{j-1}}$$

by Proposition 4.3, the map  $j \mapsto \chi(K_j)$  is injective. In particular the knots  $K_i$  for  $i \geq r$  have distinct Jones polynomials and so they are pairwise distinct.  $\square$

## 6. CONCLUDING REMARKS

We have exhibited a 20-crossing tangle  $T_{20}$  whose Kauffman bracket polynomial pair is trivial mod 2, i.e. congruent to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  mod 2. That tangle allowed us to construct, for any  $r \geq 1$ , a nontrivial knot  $K_r$  whose Jones polynomial is congruent to 1 mod  $2^r$ .

Having thus solved Problem 1.1 for  $m = 2^r$  and knowing solutions for  $m = 3$  from the Tables, what about the existence of solutions for the next moduli, such as  $m = 5, 6$  or  $7$  for instance? More ambitiously perhaps, given an integer  $m \geq 3$ , does there exist a tangle, analogous to  $T_{20}$ , whose Kauffman bracket polynomial pair would be trivial mod  $m$ ? If yes, what should be the expected minimal number of crossings as a function of  $m$ ?

Even more intriguing: does there exist a tangle whose Kauffman bracket polynomial pair is trivial *over*  $\mathbb{Z}$ ? The existence of such a tangle would probably imply the existence of a nontrivial knot with trivial Jones polynomial.

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