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# Estimation of linear operators from scattered impulse responses

Jérémie Bigot\*      Paul Escande†      Pierre Weiss‡

October 13, 2016

## Abstract

We provide a new estimator of integral operators with smooth kernels, obtained from a set of scattered and noisy impulse responses. The proposed approach relies on the formalism of smoothing in reproducing kernel Hilbert spaces and on the choice of an appropriate regularization term that takes the smoothness of the operator into account. It is numerically tractable in very large dimensions. We study the estimator's robustness to noise and analyze its approximation properties with respect to the size and the geometry of the dataset. In addition, we show minimax optimality of the proposed estimator.

*Keywords:* Integral operator, scattered approximation, estimator, convergence rate, numerical complexity, radial basis functions, Reproducing Kernel Hilbert Spaces, minimax.

*AMS classifications:* 47A58, 41A15, 41A25, 68W25, 62H12, 65T60, 94A20.

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## 1 Introduction

Let  $\Omega$  denote a subset of  $\mathbb{R}^d$  and  $H : L^2(\Omega) \rightarrow L^2(\Omega)$  denote a linear integral operator defined for all  $u \in L^2(\Omega)$  by:

$$Hu(x) = \int_{\Omega} K(x, y)u(y)dy, \tag{1.1}$$

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\*Institut de Math matiques de Bordeaux et CNRS, IMB-UMR5251, Universit  de Bordeaux, [jeremie.bigot@math.u-bordeaux.fr](mailto:jeremie.bigot@math.u-bordeaux.fr)

†D partement d'Ing nierie des Syst mes Complexes (DISC), Institut Sup rieur de l'A ronautique et de l'Espace (ISAE), Toulouse, France, [paul.escande@gmail.com](mailto:paul.escande@gmail.com)

‡Institut des Technologies Avanc es en Sciences du Vivant, ITAV-USR3505 and Institut de Math matiques de Toulouse, IMT-UMR5219, CNRS and Universit  de Toulouse, Toulouse, France, [pierre.armand.weiss@gmail.com](mailto:pierre.armand.weiss@gmail.com)

where  $K : \Omega \times \Omega \rightarrow \mathbb{R}$ , is the operator's kernel. Given a set of functions  $(u_i)_{1 \leq i \leq n}$ , the problem of operator identification consists of recovering  $H$  from the knowledge of  $f_i = Hu_i + \epsilon_i$ , where  $\epsilon_i$  is an unknown perturbation.

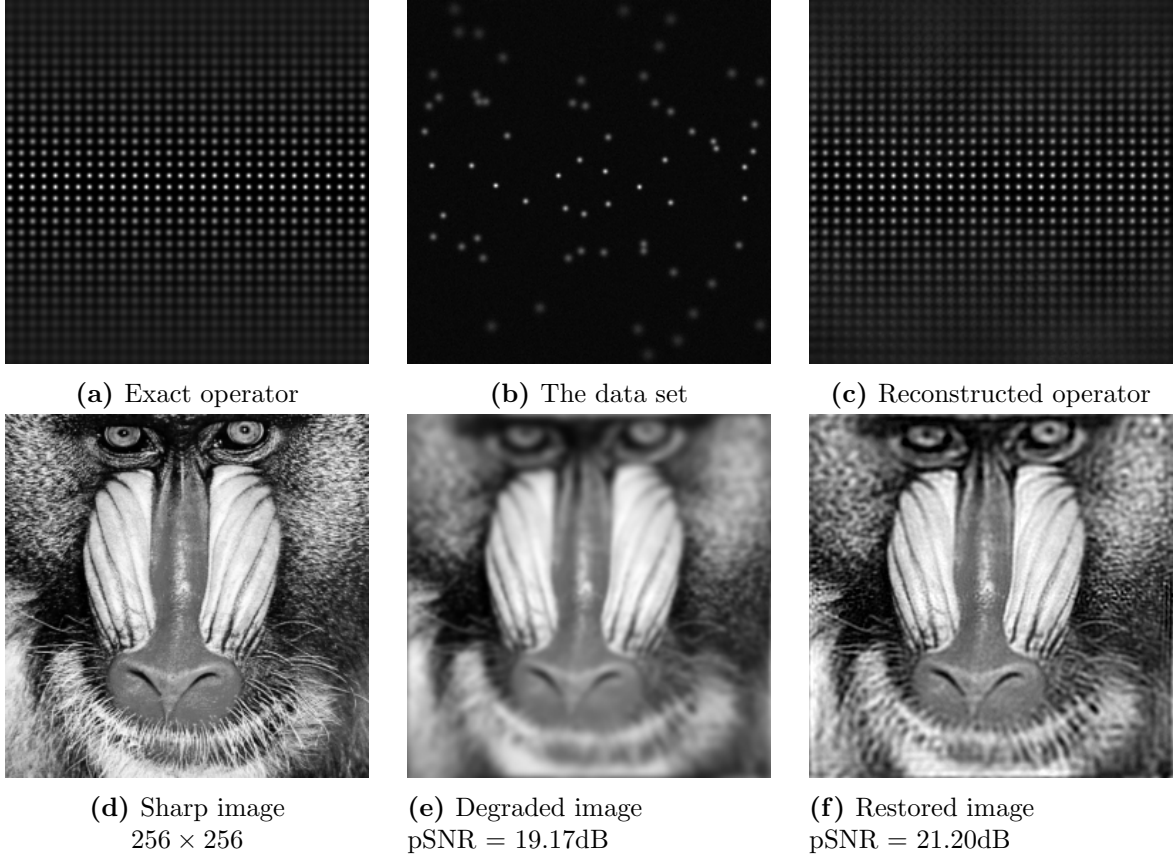
This problem arises in many fields of science and engineering such as mobile communication [13], imaging [11] and geophysics [3]. Many different reconstruction approaches have been developed, depending on the operator's regularity and the set of test functions  $(u_i)$ . Assuming that  $H$  has a bandlimited Kohn-Nirenberg symbol and that its action on a dirac comb is known, a few authors proposed extensions of Shannon's sampling theorem [13, 14, 19]. Another recent trend is to assume that  $H$  can be decomposed as a linear combination of a small number of elementary operators. When the operators are known, recovering  $H$  amounts to solving a linear system. The work [7] analyzes the conditioning of this linear system when  $H$  is a matrix applied to a random Gaussian vector. When the operator can be sparsely represented in a dictionary of elementary matrices, compressed sensing theories can be developed [20]. Finally, in astrophysics, a few authors considered interpolating the coefficients of a few known impulse responses (also called Point Spread Functions, PSF) in a well chosen basis [11, 16, 6]. This strategy corresponds to assuming that  $u_i = \delta_{y_i}$  and it is often used when the PSFs are compactly supported and have smooth variations. Notice that in this setting, each PSFs is known *independently* of the others, contrarily to the work [19].

This last approach is particularly effective in large scale imaging applications due to two useful facts. First, representing the impulse responses in a small dimensional basis allows reducing the number of parameters to identify. Second, there now exist efficient interpolation schemes based on radial basis functions. Despite its empirical success, this method still lacks of solid mathematical foundations and many practical questions remain open:

- Under what hypotheses on the operator  $H$  can this method be applied?
- What is the influence of the geometry of the set  $(y_i)_{1 \leq i \leq n}$ ?
- Is the reconstruction stable to the perturbations  $(\epsilon_i)_{1 \leq i \leq n}$ ? If not, how to make robust reconstructions, tractable in very large scale problems?
- What theoretical guarantees can be provided in this challenging setting?

The objective of this work is to address the above mentioned questions. We design a robust algorithm applicable in large scale applications. It yields a finite dimensional operator estimator of  $H$  allowing for fast matrix-vector products, which are essential for further processing. The theoretical convergence rate of the estimator as the number of observations increases is studied thoroughly. An illustration of the problem and the output of the proposed algorithm is provided in Figure 1.

The outline of this paper is as follows. We first specify the problem setting precisely in Section 2. We then describe the main outcomes of our study in Section 3. We provide a detailed explanation of the numerical algorithm in Section 4. Finally, the proofs of the main results are given in Section 5.



**Figure 1:** Top: Reconstruction of a 2D spatially varying blur operator. (a): The exact operator applied to a 2D dirac comb. (b): 64 impulse responses corrupted by additive white Gaussian noise. (c): reconstructed operator. Bottom: a deconvolution experiment. (d): original image. (e): blurry and noisy image. (f): deblurred image using the operator reconstructed in Figure (c). The operator's impulse responses are Gaussians with covariance matrices  $\Sigma(y_1, y_2) = \text{diag}(\sigma^2(y_1, y_2), \sigma^2(y_1, y_2))$  where  $\sigma(y_1, y_2) = 1 + 2 \max(1 - y_1, y_1)$  for  $(y_1, y_2) \in [0, 1]^2$ .

## 2 Problem setting

In this section, we precisely describe the problem setting. We assume that  $\Omega \subset \mathbb{R}^d$  is a bounded, open, and connected set, with Lipschitz continuous boundary. The value of a function  $f$  at  $x$  is denoted  $f(x)$ , while the  $i$ -th value of a vector  $v \in \mathbb{R}^N$  is denoted  $v[i]$ . The  $(i, j)$ -th element of a matrix  $A$  is denoted  $A[i, j]$ . The Sobolev space  $H^s(\Omega)$  is defined for  $s \in \mathbb{N}$  by

$$H^s(\Omega) = \left\{ u \in L^2(\Omega), \partial^\alpha u \in L^2(\Omega), \text{ for all multi-index } \alpha \in \mathbb{N}^d \text{ s.t. } |\alpha| = \sum_{i=1}^d \alpha[i] \leq s \right\}. \quad (2.1)$$

The space  $H^s(\Omega)$  can be endowed with a norm  $\|u\|_{H^s(\Omega)} = \left( \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}$  and the semi-norm  $|u|_{H^s(\Omega)} = \left( \sum_{|\alpha|=s} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}$ . In addition, we will use the equivalent Beppo-Levi semi-norm defined by  $|u|_{BL^s(\Omega)}^2 = \sum_{|\alpha|=s} \frac{s!}{\alpha_1! \alpha_2! \dots \alpha_d!} \|\partial^\alpha u\|_{L^2(\Omega)}^2$  and the Beppo-Levi semi-inner product defined by

$$\langle f, g \rangle_{BL^s(\Omega)} = \sum_{|\alpha|=s} \frac{s!}{\alpha_1! \alpha_2! \dots \alpha_d!} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2(\Omega)}. \quad (2.2)$$

### 2.1 Space varying impulse response regularity

An integral operator can be represented in many different ways. A key representation in this paper is the Space Varying Impulse Response (SVIR)  $S : \Omega \times \Omega \rightarrow \mathbb{R}$  defined by:

$$S(x, y) = K(x + y, y). \quad (2.3)$$

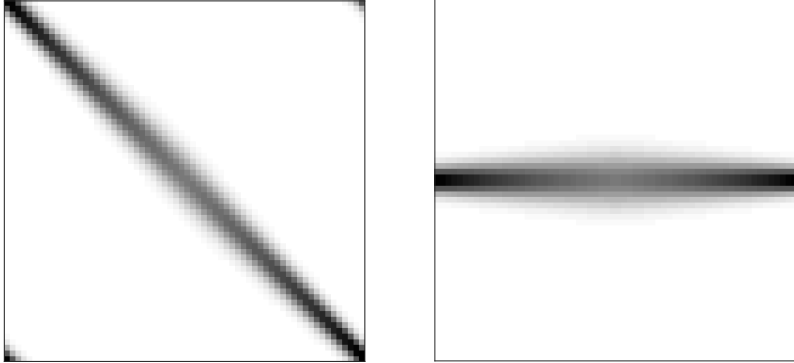
The impulse response or Point Spread Function (PSF) at location  $y \in \Omega$  is defined by  $S(\cdot, y)$ . The SVIR encodes the impulse response variations in the  $y$  direction, instead of the  $(x - y)$  direction for the kernel representation, see Figure 2 for a 1D example. It is convenient, since in many applications, the smoothness of  $S$  in the  $x$  and  $y$  directions is driven by completely different physical phenomena. For instance, in astrophysics, the regularity of  $S(\cdot, y)$  depends on the optical system, while the regularity of  $S(x, \cdot)$  may depend on exterior factors such as a weak gravitational lensing [6]. This property will be expressed through specific regularity assumptions of  $S$  defined hereafter.

Let  $(\phi_k)_{k \in \mathbb{N}}$  denote a Hilbert basis of  $L^2(\Omega)$  and  $\mathcal{E}^r(\Omega)$  denote the following Hilbert space.

**Definition 2.1.** The space  $\mathcal{E}^r(\Omega)$  is defined, for all  $r \in \mathbb{R}$  and  $r > \frac{d}{2}$ , as the set of functions  $u \in L^2(\Omega)$  such that:

$$\|u\|_{\mathcal{E}^r(\Omega)}^2 = \sum_{k \in \mathbb{N}} w[k] |\langle u, \phi_k \rangle|^2 < +\infty, \quad (2.4)$$

where  $w : \mathbb{N} \rightarrow \mathbb{R}_+^*$  is a weight function satisfying  $w[k] \gtrsim (1 + k^2)^{r/d}$ .



**Figure 2:** Two representations of an integral operator  $H$  acting on  $L^2([0,1])$ . The kernel is defined by  $K(x, y) = \frac{1}{\sqrt{2\pi\sigma(y)}} \exp\left(-\frac{1}{2\sigma(y)^2}|x-y|^2\right)$ , where  $\sigma(y) = 0.05(1 + 2\min(y, 1-y))$ . Left: kernel representation (see equation (1.1)). Right: SVIR representation (see equation (2.3)).

This definition is introduced in reference to the Sobolev spaces  $H^m(\Omega)$  on the torus  $\Omega = \mathbb{T}^d$ , which can be defined - alternatively to equation (2.1) - by:

$$H^m(\Omega) = \left\{ u \in L^2(\Omega), \sum_{k \in \mathbb{N}} (1+k^2)^{m/d} |\langle u, \phi_k \rangle|^2 < +\infty \right\}, \quad (2.5)$$

where the basis  $(\phi_k)$  is either the Fourier basis or a wavelet basis with at least  $m+1$  vanishing moments (see e.g. [15, Chapter 9]). Definition 2.1 encompasses many other spaces. For instance, it allows choosing a basis  $(\phi_k)_{k \in \mathbb{N}}$  that is best adapted to the impulse responses at hand, by using principal component analysis, as was proposed in a few applied papers [12, 4].

**Assumption 2.1** (Impulse response regularity). *In all the paper, we will assume the following regularity condition.*

$$\sup_{y \in \Omega} \|S(\cdot, y)\|_{\mathcal{E}^r(\Omega)} < +\infty. \quad (2.6)$$

When  $(\phi_k)_{k \in \mathbb{N}}$  is a Fourier or a wavelet basis, condition (2.6) simplifies to  $S(\cdot, y) \in H^r(\Omega)$  for all  $y \in \Omega$ .

## 2.2 Smooth variations of the impulse responses

In addition to the impulse responses regularity Assumption 2.1, we need to state a regularity condition for the impulse responses variations. In order to use fine approximation results based on radial basis functions [2], we will use the following regularity condition.

**Assumption 2.2** (Smooth variations). *Throughout the paper, we assume that*

$$\partial_y^\beta S(x, \cdot) \in L^2(\Omega), \quad \forall x \in \Omega \text{ and for all multi-index } \beta \text{ s.t. } |\beta| \leq s \text{ with } s > d/2. \quad (2.7)$$

The condition  $s > d/2$  ensures existence of a *continuous* representant of  $S(x, \cdot)$  for all  $x$ , by Sobolev embedding theorems [21, Thm.2, p.124].

In the particular case  $\mathcal{E}^r(\Omega) = H^r(\Omega)$ , the two assumptions 2.2 and 2.1 imply that  $S$  belongs to the *mixed-Sobolev space*  $H^{r,s}(\Omega \times \Omega)$  consisting of functions with  $\partial_x^\alpha S \in L^2(\Omega \times \Omega)$  for all multi-index  $|\alpha| \leq r$  and  $\partial_y^\beta S \in L^2(\Omega \times \Omega)$  for all multi-index  $|\beta| \leq s$ .

### 2.3 Sampling model

The main purpose of this paper is the reconstruction of the SVIR of an operator from the observation of a few impulse responses  $S(\cdot, y_i)$  at scattered (but known) locations  $(y_i)_{1 \leq i \leq n}$  in  $\Omega$ . In applications, the PSFs  $S(\cdot, y_i)$  can only be observed through a projection onto an  $N$  dimensional linear subspace  $V_N$ . In this paper, we assume that the linear subspace  $V_N$  reads

$$V_N = \text{span}(\phi_k, 1 \leq k \leq N). \quad (2.8)$$

In addition, the data is often corrupted by noise and we therefore observe a set of  $N$  dimensional vectors  $(F_i^\epsilon)_{1 \leq i \leq n}$  defined for all  $k \in \{1, \dots, N\}$  by

$$F_i^\epsilon[k] = \langle S(\cdot, y_i), \phi_k \rangle + \epsilon_i[k], \quad 1 \leq i \leq n, \quad (2.9)$$

where  $\epsilon_i$  is a random vector with independent and identically distributed (iid) components with zero mean and variance  $\sigma^2$ .

The assumption that  $V_N$  is defined using basis  $(\phi_k)$  simplifies the analysis, since the representation and observation bases coincide. It would be interesting for applications to consider cases where  $V_N$  is defined using another Hilbert basis, but we would then need to use the theory of generalized sampling, which is significantly more involved (see e.g. [1]). We therefore leave this question aside in this paper.

Finally, we will show that the approximation efficiency of our method depends on the geometry of the set of data locations, and - in particular - on the fill and separation distances defined below.

**Definition 2.2** (Fill distance). The fill distance of  $Y = \{y_1, \dots, y_n\} \subset \Omega$  is defined as:

$$h_{Y,\Omega} = \sup_{y \in \Omega} \min_{1 \leq j \leq n} \|y - y_j\|_2. \quad (2.10)$$

This is the distance for which any  $y \in \Omega$  is at most at a distance  $h_{Y,\Omega}$  of  $Y$ . It can also be interpreted as the radius of the largest ball which is completely contained in  $\Omega$  without intersecting  $Y$ .

**Definition 2.3** (Separation distance). The separation distance of  $Y = \{y_1, \dots, y_n\} \subset \Omega$  is defined as:

$$q_{Y,\Omega} = \frac{1}{2} \min_{i \neq j} \|y_i - y_j\|_2. \quad (2.11)$$

This quantity gives the maximal radius  $r > 0$  such that all balls  $\{y \in \mathbb{R}^d : \|y - y_j\|_2\}$  are disjoint.

Finally, the following condition will be shown to play a key role in our analysis.

**Definition 2.4** (Quasi-uniformity condition). A set of data locations  $Y = \{y_1, \dots, y_n\} \subset \Omega$  is said to be quasi-uniform with respect to a constant  $B > 0$  if

$$q_{Y,\Omega} \leq h_{Y,\Omega} \leq Bq_{Y,\Omega}. \quad (2.12)$$

### 3 Main results

#### 3.1 Construction of an estimator

Let  $F : \Omega \rightarrow \mathbb{R}^N$  denote the vector-valued function representing the impulse responses coefficients (IRC) in basis  $(\phi_k)_{k \in \mathbb{N}}$ :

$$F(y)[k] = \langle S(\cdot, y), \phi_k \rangle. \quad (3.1)$$

Based on the observation model (2.9), a natural approach to estimate the SVIR, consists in constructing an estimate  $\hat{F} : \Omega \rightarrow \mathbb{R}^N$  of  $F$ . The estimated SVIR is then defined as

$$\hat{S}(x, y) = \sum_{k=1}^N \hat{F}(y)[k] \phi_k(x), \text{ for } (x, y) \in \Omega \times \Omega. \quad (3.2)$$

The two assumptions 2.1 and 2.2, motivate the introduction of the following space.

**Definition 3.1** (Space  $\mathcal{H}$  of IRC). The space  $\mathcal{H}(\Omega)$  of admissible IRC is defined as the set of vector-valued functions  $G : \Omega \rightarrow \mathbb{R}^N$  such that

$$\|G\|_{\mathcal{H}(\Omega)}^2 = \alpha \int_{y \in \Omega} \sum_{k=1}^N w[k] |G(y)[k]|^2 dy + (1 - \alpha) \sum_{k=1}^N \|G(\cdot)[k]\|_{BL^s(\Omega)}^2 < +\infty, \quad (3.3)$$

where  $\alpha \in [0, 1)$  allows to balance the smoothness in each direction.

Quite obviously, we can state the following result.

**Lemma 3.1.** *The SVIRs satisfying assumptions 2.1 and 2.2 have an IRC belonging to  $\mathcal{H}(\Omega)$ .*

To construct an estimator of  $F$ , we propose to define  $\hat{F}_\mu$  as the minimizer of the following optimization problem:

$$\hat{F}_\mu = \arg \min_{F \in \mathcal{H}(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^n \|F_i^\varepsilon - F(y_i)\|_{\mathbb{R}^N}^2 + \mu \|F\|_{\mathcal{H}(\mathbb{R}^d)}^2, \quad (3.4)$$

where  $\mu > 0$  is a regularization parameter. Notice that the optimization is performed on  $\mathcal{H}(\mathbb{R}^d)$  and not  $\mathcal{H}(\Omega)$ , for technical reasons related to the use of radial basis functions.

*Remark 3.1.* The proposed formulation can be interpreted with the formalism of regression and smoothing in vector-valued Reproducing Kernel Hilbert Spaces (RKHS) [17, 18]. The space  $\mathcal{H}(\mathbb{R}^d)$  can be shown to be a vector-valued Reproducing Kernel Hilbert Space (RKHS). The formalism of vector-valued RKHS has been developed for the purpose of multi-task learning, and its application to operator estimation appears to be novel.



### 3.2 Mixed-Sobolev space interpretation

In the specific case where  $(\phi_k)_{k \in \mathbb{N}}$  is a wavelet or a Fourier basis and  $N = +\infty$ , the proposed methodology can be interpreted in terms of SVIR instead of IRC.

**Lemma 3.2.** *The cost function in problem 3.4 is equivalent to*

$$\frac{1}{n} \sum_{i=1}^n \left\| F_i^e - (\langle S(\cdot, y_i), \phi_k \rangle)_{1 \leq k \leq N} \right\|_2^2 + \mu \left( \alpha \int_{\mathbb{R}^d} |S(\cdot, y)|_{H^r(\mathbb{R}^d)}^2 dy + (1 - \alpha) \int_{\mathbb{R}^d} |S(x, \cdot)|_{BL^s(\mathbb{R}^d)}^2 dx \right). \quad (3.5)$$

In this formulation, the data fidelity term allows finding a TVIR that is close to the observed data, the first regularization term allows smoothing the additive noise on the acquired PSFs and the second interpolates the missing data.

### 3.3 Numerical complexity

Thanks to the results in [18], computing  $\hat{F}_\mu$  amounts to solving a *finite-dimensional* system of linear equations. However, for an arbitrary orthonormal basis  $(\phi_\lambda)_{\lambda \in \Lambda}$ , and without any further assumptions on the kernel of the RKHS  $\mathcal{H}(\mathbb{R}^d)$ , evaluating  $\hat{F}_\mu$  leads to the resolution of a *full*  $nN \times nN$  linear system, which is untractable for large  $N$  and  $n$ .

With the specific choice of norm introduced in Definition 3.1, the problem simplifies to the resolution of  $N$  systems of equations of size  $n \times n$ . This yields the following proposition:

**Proposition 3.1.** *The solution of (3.4) can be computed in no more than  $O(Nn^3)$  operations.*

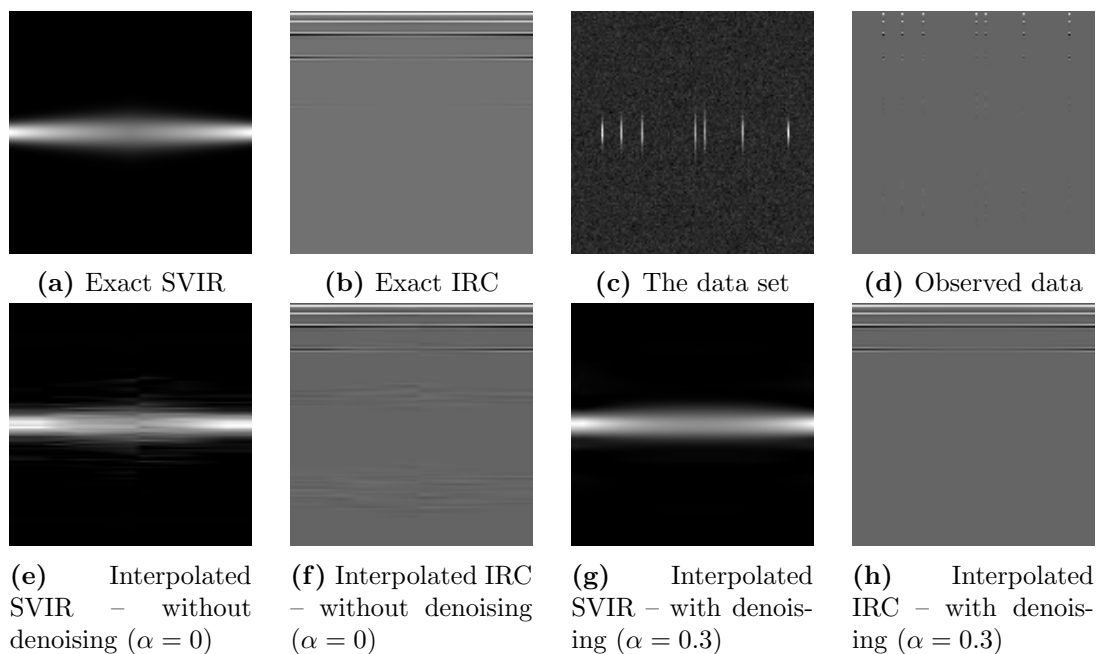
In addition, if the weight function  $w$  is piecewise constant, some  $n \times n$  matrices are identical, allowing to compute an LU factorization once for all and using it to solve many systems. In the specific case where  $(\phi_k)_{k \in \mathbb{N}}$  is a wavelet basis, it is natural to set function  $w$  as a constant over each wavelet subband [15, Thm. 9.4]. This yields the following result.

**Proposition 3.2.** *If  $w$  is set as constant over each subband of a wavelet basis, the solution of (3.4) can be computed in no more than  $O\left(\frac{\log(N)}{d}n^3 + Nn^2\right)$  operations.*

Finally for well chosen bases  $(\phi_k)_{k \in \mathbb{N}}$  - including wavelets - the impulse responses can be well approximated using a small number  $N$  of atoms, making the method tractable even in very large scale applications.

The proposed ideas are illustrated on Figure 3. As can be seen on Figure 3 (e), computing the IRC in a wavelet basis allows expressing most of the information contained in the SVIR on a few lines only. Given the noisy dataset, the proposed algorithm simultaneously interpolates along lines and denoises along rows to obtain the results in Figure 3 (c) and (f).

To conclude this paragraph, let us mention that the representation of an operator of type (3.2) can be used to evaluate matrix-vector products rapidly. We refer the interested reader to [10] for more details.



**Figure 3:** Illustration of the methodology on a 1D estimation problem. (a) Exact SVIR: Gaussian PSFs with standard deviation  $\sigma(t) = 0.05(1 + 2 \min(t, 1 - t))$  for  $t \in [0, 1]$ . (b) Exact Impulse Response Coefficients (IRC) in a wavelet basis. (c) 7 PSFs are extracted and corrupted by an additive iid Gaussian noise of standard deviation  $5 \cdot 10^{-3}$ . (d) We observe only a few noisy impulse response coefficients. (e) and (g) show the estimation result. Notice how the regularization in the vertical direction allows improving the estimator: the result is very similar to (a). (f) and (h) similar experiment and conclusions on the IRC.

### 3.4 Convergence rates

Before stating the main results, let us introduce the following definition.

**Definition 3.2.** Let  $A_1, A_2$  be positive constants. Set  $r > \frac{d}{2}$  and  $s > \frac{d}{2}$ . The ball  $\mathcal{E}^{r,s}(\Omega, A_1, A_2)$  is defined as the set of linear integral operators  $H$  with SVIR  $S$  belonging to  $L^2(\Omega \times \Omega)$ , satisfying

$$\sup_{y \in \Omega} \|S(\cdot, y)\|_{\mathcal{E}^r(\Omega)}^2 \leq A_1 \quad \text{and} \quad \sup_{x \in \Omega} \|S(x, \cdot)\|_{H^s(\Omega)}^2 \leq A_2.$$

The convergence of the proposed estimator with respect to the number  $n$  of observations is captured by the following theorem.

**Theorem 3.1.** *Assume that  $S$  satisfies Assumptions 2.1 and 2.2 and that it is sampled using model (2.9) under the quasi-uniformity condition given in Definition 2.4. Then the estimating operator  $\hat{H}$  with SVIR  $\hat{S}$  defined in equation (3.2) satisfies the following inequality*

$$\mathbb{E} \left( \|H - \hat{H}\|_{HS}^2 \right) \lesssim N^{-\frac{2r}{d}} + (N\sigma^2 n^{-1})^{\frac{2s}{2s+d}}, \quad (3.6)$$

for  $\mu \propto (N\sigma^2 n^{-1})^{\frac{2s}{2s+d}}$ . This inequality holds uniformly on the ball  $\mathcal{E}^{r,s}(\Omega, A_1, A_2)$ .

In applications where the user can choose the number of observations  $N$  (e.g. if it is sufficiently large), the upper-bound (3.6) can be optimized with respect to  $N$ .

**Corollary 3.1.** *Assume that  $S$  satisfies Assumptions 2.1 and 2.2 and that it is sampled using model (2.9) under the quasi-uniformity condition given in Definition 2.4. Then the estimator  $\hat{H}$  with SVIR  $\hat{S}$  defined in equation (3.2) satisfies the following inequality*

$$\mathbb{E} \left( \|H - \hat{H}\|_{HS}^2 \right) \lesssim (\sigma^2 n^{-1})^{\frac{2q}{2q+d}}, \quad (3.7)$$

with the relation  $1/q = 1/r + 1/s$ , for  $\mu \propto (\sigma^2 n^{-1})^{\frac{2q}{2q+d}}$  and  $N \propto (\sigma^2 n^{-1})^{-\frac{dq}{r(2q+d)}}$ . This inequality holds uniformly on the ball  $\mathcal{E}^{r,s}(\Omega, A_1, A_2)$ .

Corollary 3.1 gives some insights on the estimator behavior. In particular:

- It provides an explicit way of choosing the value of the regularization parameter  $\mu$ : it should decrease as the number of observations increases.
- If the number of observations  $n$  is small, it is unnecessary to project the impulse responses on a high dimensional basis (i.e.  $N$  large). The basic reason is that not enough information has been collected to reconstruct the fine details of the kernel.

Finally, to conclude this section on convergence rates, it is shown that, under mild assumptions on the basis  $(\phi_k)_{k \geq 1}$ , the rate of convergence  $(\sigma^2 n^{-1})^{\frac{2q}{2q+d}}$  in inequality (3.7) is optimal in the case of Gaussian noise and for the expected Hilbert-Schmidt norm

$$\mathbb{E} \left\| H - \hat{H} \right\|_{HS}^2 = \mathbb{E} \left\| \hat{S} - S \right\|_{L^2(\Omega \times \Omega)}^2.$$

Optimality of the rate of converge (3.7) has to be understood in the minimax sense as classically done in the literature on nonparametric statistics (we refer to [22] for a detailed introduction to this topic). For simplicity, this optimality result is stated in the case where the domain  $\Omega = [0, 1]^d$  is the d-dimensional hypercube.

**Theorem 3.2.** *Let  $H$  be a linear operator with SVIR belonging to  $\mathcal{E}^{r,s}(\Omega, A_1, A_2)$ . Define  $q$  by  $1/q = 1/r + 1/s$ . Suppose that  $\Omega = [0, 1]^d$  and that the weights in Definition 2.1 satisfy  $w[k] \leq c_1(1+k^2)^{r/d}$  for all  $k \in \mathbb{N}$  and some constant  $c_1 > 0$ . Assume that there exists a constant  $C_\phi$  such that for any integer  $k_1 \geq 1$*

$$\sup_{x \in \Omega} \left\{ \sum_{k=1}^{k_1} |\phi_k(x)| \right\} \leq C_\phi^{1/2} k_1^{1/2}. \quad (3.8)$$

Assume that the PSF locations  $y_1, \dots, y_n$  satisfy the quasi-uniformity condition given in Definition 2.4. Assume that the random values  $(\epsilon_i[k])_{i,k}$  in the observation model (2.9) are iid Gaussian with zero mean and variance  $\sigma^2$ .

Then, there exists a constant  $c_0 > 0$  such that

$$\inf_{\hat{H}} \sup_{H \in \mathcal{E}^{r,s}(\Omega, A_1, A_2)} \mathbb{E} \left\| \hat{H} - H \right\|_{HS}^2 \geq c_0 (\sigma^2 n^{-1})^{\frac{2q}{2q+d}}, \quad (3.9)$$

where the above infimum is taken over all possible estimators  $\hat{H}$  (linear integral operators) with SVIR  $\hat{S} \in L^2(\Omega \times \Omega)$  defined as a measurable function.

We do not know whether condition (3.8) in Theorem (3.2) is necessary or not. Compactly supported wavelet bases are a particular instance of functions satisfying this condition.

## 4 Radial basis functions implementation

The objective of this section is to provide a fast algorithm to solve problem (3.4) and to prove Propositions 3.1 and 3.2. A few tools related to radial basis functions and useful for the subsequent proofs are also introduced.

A key observation is provided below.

**Lemma 4.1.** *For  $k \in \{1, \dots, N\}$ , function  $\hat{F}(\cdot)[k]$  is the solution of the following variational problem:*

$$\min_{f \in H^s(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^n (F_i^c[k] - f(y_i))^2 + \mu \left( \alpha w[k] \|f\|_{L^2(\mathbb{R}^d)}^2 + (1 - \alpha) |f|_{BL^s(\mathbb{R}^d)}^2 \right). \quad (4.1)$$

*Proof.* It suffices to remark that problem (3.4) consists of solving  $N$  independent sub-problems.  $\square$

We now focus on the resolution of sub-problem (4.1) which completely fits the framework of radial basis function approximation.

## 4.1 Standard approximation results in RKHS

Let us begin with a few results about RKHS. Most of the results can be found in the book of Wendland [26].

**Definition 4.1** (Positive definite function). A continuous function  $\Phi : \mathbb{R}^d \rightarrow \mathbb{C}$  is called positive semi-definite if, for all  $n \in \mathbb{N}$ , all sets of pairwise distinct centers  $Y = \{y_1, \dots, y_n\} \subset \mathbb{R}^d$ , and all  $\alpha \in \mathbb{C}^n$ , the quadratic form

$$\sum_{j=1}^n \sum_{k=1}^n \alpha_j \bar{\alpha}_k \Phi(x_j - x_k) \quad (4.2)$$

is nonnegative. Function  $\Phi$  is called positive definite if the quadratic form is positive for all  $\alpha \in \mathbb{C}^n \setminus \{0\}$ .

**Definition 4.2** (Reproducing kernel). Let  $\mathcal{G}$  denote a Hilbert space of real-valued functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  endowed with a scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ . A function  $\Phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is called reproducing kernel for  $\mathcal{G}$  if

1.  $\Phi(\cdot, y) \in \mathcal{G}, \quad \forall y \in \mathbb{R}^d,$
2.  $f(y) = \langle f, \Phi(\cdot, y) \rangle_{\mathcal{G}},$  for all  $f \in \mathcal{G}$  and all  $y \in \mathbb{R}^d.$

**Theorem 4.1** (RKHS). *Suppose that  $\mathcal{G}$  is a Hilbert space of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then the following statements are equivalent:*

1. *the point evaluations functionals are continuous for all  $y \in \mathbb{R}^d$ .*
2.  *$\mathcal{G}$  has a reproducing kernel.*

*A Hilbert space satisfying the properties above is called a Reproducing Kernel Hilbert Space (RKHS).*

The Fourier transform of a function  $f \in L^1(\mathbb{R}^d)$  is defined by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{x \in \mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx, \quad (4.3)$$

and the inverse transform by

$$\mathcal{F}^{-1}[\widehat{f}](x) = \int_{\xi \in \mathbb{R}^d} \widehat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi. \quad (4.4)$$

The Fourier transform can be extended to  $L^2(\mathbb{R}^d)$  and to  $\mathcal{S}'(\mathbb{R}^d)$  the space of tempered distributions.

**Theorem 4.2** ([26, Theorem 10.12]). *Suppose that  $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  is a real-valued positive definite function. Define  $\mathcal{G} = \left\{ f \in L^2(\mathbb{R}^d) \cap C(\mathbb{R}^d) : \widehat{f}/\sqrt{\widehat{\Phi}} \in L^2(\mathbb{R}^d) \right\}$  equipped with*

$$\langle f, g \rangle_{\mathcal{G}} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{f}(\xi) \overline{\widehat{g}(\xi)}}{\widehat{\Phi}(\xi)} d\xi. \quad (4.5)$$

*Then  $\mathcal{G}$  is a real Hilbert space with inner-product  $\langle \cdot, \cdot \rangle_{\mathcal{G}}$  and reproducing kernel  $\Phi(\cdot - \cdot)$ .*

**Theorem 4.3.** *Let  $\mathcal{G}$  be an RKHS with positive definite reproducing kernel  $\Phi$ . Let  $(y_1, \dots, y_n)$  denote a set of points in  $\mathbb{R}^d$  and  $z \in \mathbb{R}^n$  denote a set of altitudes. The solution of the following approximation problem*

$$\min_{u \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^n (u(y_i) - z[i])^2 + \frac{\mu}{2} \|u\|_{\mathcal{G}}^2 \quad (4.6)$$

can be written as:

$$u(x) = \sum_{i=1}^n c[i] \Phi(x - y_i), \quad (4.7)$$

where vector  $c \in \mathbb{R}^n$  is the unique solution of the following linear system of equations

$$(G + n\mu \text{Id})c = z \text{ with } G[i, j] = \Phi(y_i - y_j). \quad (4.8)$$

It is shown in [26], that the conditioning number of  $G$  depends on the ratio  $h_{Y, \Omega}/q_{Y, \Omega}$ . For numerical reasons it might therefore be useful to discard locations that are too close to each other.

## 4.2 Application to our problem

Let us now show how the above results help solving problem (4.1).

**Proposition 4.1.** *Let  $\mathcal{G}$  be the Hilbert space of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $|f|_{BL^s(\mathbb{R}^d)}^2 + \|f\|_{L^2(\Omega)}^2 < +\infty$ , equipped with the inner product:*

$$\langle f, g \rangle_{\mathcal{G}} = (1 - \alpha) \langle f, g \rangle_{BL^s(\mathbb{R}^d)} + \alpha w[k] \langle f, g \rangle_{L^2(\mathbb{R}^d)}. \quad (4.9)$$

Then  $\mathcal{G}$  is an RKHS and its scalar product reads

$$\langle f, g \rangle_{\mathcal{G}} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{f}(\xi) \overline{\widehat{g}(\xi)}}{\widehat{\Phi}(\xi)} d\xi, \quad (4.10)$$

where the reproducing kernel  $\Phi$ , is defined by:

$$\widehat{\Phi}_k(\xi) = ((1 - \alpha) \|\xi\|^{2s} + \alpha w[k])^{-1}. \quad (4.11)$$

*Proof.* The proof is a direct application of the different results stated previously.  $\square$

The Fourier transform  $\widehat{\Phi}_k$  is radial, so that  $\Phi_k$  is radial too and the resolution of (4.1) fits the formalism of radial basis functions interpolation/approximation [5].

*Remark 4.1.* For some applications, it makes sense to set  $w[k] = 0$  for some values of  $k$ . For instance, if  $(\phi_k)_{k \in \mathbb{N}}$  is a wavelet basis, then it is usually good to set  $w[k] = 0$  when  $k$  is the index of a scaling wavelet. In that case, the theory of conditionally positive definite kernels should be used instead of the one above. We do not detail this aspect since it is well described in standard textbooks [26, 5].

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**Algorithm 1**

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- 1: **Input:** Weight vector  $w \in \mathbb{R}^N$
  - 2: Regularity  $s \in \mathbb{N}$
  - 3: PSF locations  $Y = \{y_1, \dots, y_n\} \in \mathbb{R}^{d \times n}$
  - 4: Observed data  $(F_i^\epsilon)_{1 \leq i \leq n}$ , where  $F_i^\epsilon \in \mathbb{R}^N$ .
  - 5: Identify the  $m \leq N$  weights of identical values in vector  $w \in \mathbb{R}^N$ .  $\triangleright O(N \log(N))$
  - 6: **for** Each unique weight  $\omega$  **do**  $\triangleright O(mn^3)$
  - 7:     Compute matrix  $G$  from formula (4.8) with  $\Phi$  defined in (4.11).
  - 8:     Compute an LU decomposition of  $M_\omega = (G + n\mu\text{Id}) = L_\omega U_\omega$ .
  - 9: **end for**
  - 10: **for**  $k = 1$  to  $N$  **do**  $\triangleright O(Nn^2)$
  - 11:     Identify the value  $\omega$  such that  $w[k] = \omega$ .
  - 12:     Set  $z = (F_i^\epsilon[k])_{1 \leq i \leq n}$ .
  - 13:     Solve the linear system  $L_\omega U_\omega c_k = z$ .
  - 14:     Possibly reconstruct  $\hat{F}$  by (see equation (4.7))
- $$\hat{F}(x)[k] = \sum_{i=1}^n c_k[i] \Phi(x - y_i).$$
- 15: **end for**
- 

## 5 Proofs of the main results

In this section, we prove Theorem 3.1 about the convergence rate of the quadratic risk  $\mathbb{E}\|S - \hat{S}\|_{HS}^2$ .

### 5.1 Operator norm risk

To analyse the theoretical properties of a given estimator of the operator  $H$ , we introduce the quadratic risk defined as:

$$R(\hat{H}, H) = \mathbb{E} \left\| \hat{H} - H \right\|_{HS}^2, \tag{5.1}$$

where  $\hat{H}$  is the operator associated to the SVIR  $\hat{S}$  defined in (3.2). The above expectation is taken with respect to the distribution of the observations in (2.9). Notice that  $\|H\|_{HS} =$

$\|K\|_{L^2(\Omega \times \Omega)} = \|S\|_{L^2(\Omega \times \Omega)}$ . From this observation we get that:

$$\begin{aligned}
R(\hat{H}, H) &= \mathbb{E} \left\| \hat{H} - H \right\|_{HS}^2 \\
&\leq 2 \left( \|H - H_N\|_{HS}^2 + \mathbb{E} \left\| H_N - \hat{H} \right\|_{HS}^2 \right) \\
&= 2 \left( \underbrace{\|S - S_N\|_{L^2(\Omega \times \Omega)}^2}_{\epsilon_d(N)} + \mathbb{E} \underbrace{\left\| S_N - \hat{S} \right\|_{L^2(\Omega \times \Omega)}^2}_{\epsilon_e(n)} \right), \tag{5.2}
\end{aligned}$$

where  $H_N$  is the operator associated to the SVIR  $S_N$  defined by  $S_N(x, y) = \sum_{k=1}^N F(y)[k]\phi_k(x)$  and  $\hat{H}$  the estimating operator associated to the SVIR  $\hat{S}$  as in (3.2).

In equation (5.2), the risk is decomposed as the sum of two terms  $\epsilon_e(n)$  and  $\epsilon_d(N)$  (standard bias/variance decomposition in statistics). The first one  $\epsilon_d(N)$  is the error introduced by the discretization step. The second term  $\epsilon_e(N)$  is the quadratic risk between  $S_N$  and the estimator  $\hat{S}$ . In the next sections, we provide upper-bounds for  $\epsilon_d(N)$  and  $\epsilon_e(n)$ .

## 5.2 Discretization error $\epsilon_d$

The discretization error  $\epsilon_d(N)$  can be controlled using the standard approximation result below (see e.g. [15, Theorem 9.1, p. 437]).

**Theorem 5.1.** *Let  $f \in \mathcal{E}^r(\Omega)$  and let  $f_N = \sum_{k=1}^N \langle f, \phi_k \rangle \phi_k$ . Then*

$$\|f - f_N\|_2^2 \leq c \|f\|_{\mathcal{E}^r(\Omega)}^2 N^{-2r/d}, \tag{5.3}$$

where  $c$  is a universal constant.

**Corollary 5.1.** *Under assumptions 2.1 and 2.2, the discretization error satisfies:*

$$\epsilon_d(N) \lesssim N^{-2r/d}. \tag{5.4}$$

*Proof.* By assumption 2.1,  $S(\cdot, y) \in \mathcal{E}^r(\Omega)$  for all  $y \in \Omega$ . Therefore, by Theorem 5.1:

$$\|S(\cdot, y) - S_N(\cdot, y)\|_{L^2(\Omega)}^2 \leq c N^{-2r/d}. \tag{5.5}$$

Finally:

$$\begin{aligned}
\|S - S_N\|_{L^2(\Omega \times \Omega)}^2 &= \int_{y \in \Omega} \|S(\cdot, y) - S_N(\cdot, y)\|_{L^2(\Omega)}^2 dy \\
&\leq |\Omega| c \left( \sup_{y \in \Omega} \|S(\cdot, y)\|_{\mathcal{E}^r(\Omega)}^2 \right) N^{-2r/d}.
\end{aligned}$$

□



### 5.3 Estimation error $\epsilon_e$

This section provides an upper-bound on the estimation error

$$\epsilon_e(n) = \mathbb{E} \left\| S_N - \hat{S} \right\|_{L^2(\Omega \times \Omega)}^2. \quad (5.6)$$

This part is significantly harder than the rest of the paper. Let us begin with a simple remark.

**Lemma 5.1.** *The estimation error satisfies*

$$\epsilon_e(n) = \|F - \hat{F}\|_{\mathbb{R}^N \times L^2(\Omega)}^2. \quad (5.7)$$

*Proof.* Since  $(\phi_k)_{1 \leq k \leq N}$  is an orthonormal basis, Parseval's theorem gives

$$\begin{aligned} \|S_N - \hat{S}\|_{L^2(\Omega \times \Omega)}^2 &= \int_{\Omega} \int_{\Omega} \left( S_N(x, y) - \hat{S}(x, y) \right)^2 dx dy \\ &= \int_{\Omega} \int_{\Omega} \left( \sum_{k=1}^N (F(y)[k] - \hat{F}(y)[k]) \phi_k(x) \right)^2 dx dy \\ &= \int_{\Omega} \sum_{k=1}^N (F(y)[k] - \hat{F}(y)[k])^2 dy \\ &= \sum_{k=1}^N \|F(\cdot)[k] - \hat{F}(\cdot)[k]\|_{L^2(\Omega)}^2 =: \|F - \hat{F}\|_{\mathbb{R}^N \times L^2(\Omega)}^2. \end{aligned} \quad (5.8)$$

□

By Lemma 4.1 the estimator defined in (3.4) can be decomposed as  $N$  independent estimators. Lemma 5.2 below provides a convergence rate for each of them. This result is strongly related to the work in [24] on smoothing splines. Unfortunately, we cannot directly apply the results in [24] to our setting since the kernel defined in (4.11) is not that of a thin-plate smoothing spline.

**Lemma 5.2.** *Suppose that  $\Omega \subset \mathbb{R}^d$  is a bounded connected open set in  $\mathbb{R}^d$  with Lipschitz continuous boundary and that the set  $Y = \{y_1, \dots, y_n\} \subset \Omega$  of PSF locations satisfies a quasi-uniformity condition in the sense of Definition 2.4. Then, each function  $\hat{F}(\cdot)[k]$  solution of problem (4.1) satisfies:*

$$\mathbb{E} \|\hat{F}(\cdot)[k] - F(\cdot)[k]\|_{L^2(\Omega)}^2 \lesssim \mu \|F(\cdot)[k]\|_{H^s(\Omega)}^2 + n^{-1} \sigma^2 \mu^{-\frac{d}{2s}}, \quad (5.9)$$

provided that  $n\mu^{d/2s} \geq 1$ .

*Proof.* In order to prove the upper-bound (5.9), we first decompose the expected squared error  $\mathbb{E} \|\hat{F}(\cdot)[k] - F(\cdot)[k]\|_{L^2(\Omega)}^2$  into bias and variance terms:

$$\mathbb{E} \|\hat{F}(\cdot)[k] - F(\cdot)[k]\|_{L^2(\Omega)}^2 \leq 2 \left( \underbrace{\|\hat{F}^0(\cdot)[k] - F(\cdot)[k]\|_{L^2(\Omega)}^2}_{\text{Bias term}} + \underbrace{\mathbb{E} \|\hat{F}^0(\cdot)[k] - \hat{F}(\cdot)[k]\|_{L^2(\Omega)}^2}_{\text{Variance term}} \right), \quad (5.10)$$

where  $\hat{F}^0(\cdot)[k]$  is the solution of the noise-free problem

$$\hat{F}^0(\cdot)[k] = \arg \min_{f \in H^s(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^n (F(y_i)[k] - f(y_i))^2 + \mu \left( \alpha w[k] \|f\|_{L^2(\mathbb{R}^d)}^2 + (1 - \alpha) \|f\|_{BL^s(\mathbb{R}^d)}^2 \right). \quad (5.11)$$

We then treat the bias and variance terms separately.

**Control of the bias** The bias control relies on sampling inequalities in Sobolev spaces. They first appeared in [9] to control the norm of functions in Sobolev spaces with scattered zeros. They have been generalized in different ways, see e.g. [27] and [2]. In this paper, we will use the following result from [2].

**Theorem 5.2** ([2, Theorem 4.1]). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded connected open set with Lipschitz continuous boundary.*

*Let  $p, q, x \in [1, +\infty]$ . Let  $s$  be a real number such that  $s \leq d$  if  $p = 1$ ,  $s > d/p$  if  $1 < p < \infty$  or  $s \in \mathbb{N}^*$  if  $p = \infty$ . Furthermore, let  $l_0 = s - d(1/p - 1/q)_+$  and  $\gamma = \max(p, q, x)$  where  $(\cdot)_+ = \max(0, \cdot)$ .*

*Then, there exist two positive constants  $\eta_s$  (depending on  $\Omega$  and  $s$ ) and  $C$  (depending on  $\Omega$ ,  $n, s, p, q$  and  $x$ ) satisfying the following property: for any finite set  $Y \subset \bar{\Omega}$  (or  $Y \subset \Omega$  if  $p = 1$  and  $s = d$ ) such that  $h_{Y, \Omega} \leq \eta_s$ , for any  $u \in W^{s,p}(\Omega)$  and for any  $l = 0, \dots, \lceil l_0 \rceil - 1$ , we have*

$$\|u\|_{W^{l,q}(\Omega)} \leq C \left( h_{Y, \Omega}^{s-l-d(1/p-1/q)_+} \|u\|_{W^{s,p}(\Omega)} + h_{Y, \Omega}^{d/\gamma-l} \|u|_Y\|_x \right), \quad (5.12)$$

where  $\|u|_Y\|_x = \left( \sum_{i=1}^n u(y_i)^x \right)^{1/x}$ . If  $s \in \mathbb{N}^*$  this bound also holds with  $l = l_0$  when either  $p < q < \infty$  and  $l_0 \in \mathbb{N}$  or  $(p, q) = (1, \infty)$  or  $p \geq q$ .

The above theorem is the key to obtain Proposition 5.1 below.

**Proposition 5.1.** *Set  $a > 0$  and let  $\mathcal{G}(\Omega)$  be the RKHS with norm defined by  $\|\cdot\|_{\mathcal{G}(\Omega)}^2 = \|\cdot\|_{BL^s(\Omega)}^2 + a \|\cdot\|_{L^2(\Omega)}^2$ . Let  $u \in H^s(\Omega)$  denote a target function and  $Y = \{y_1, \dots, y_n\} \subset \Omega$  a data site set. Let  $f_\mu$  denote the solution of the following variational problem*

$$f_\mu = \arg \min_{f \in \mathcal{G}(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^n (u(y_i) - f(y_i))^2 + \mu \|f\|_{\mathcal{G}(\mathbb{R}^d)}^2. \quad (5.13)$$

Then

$$\|f_\mu - u\|_{L^2(\Omega)} \leq C \left( h_{Y, \Omega}^s + h_{Y, \Omega}^{d/2} \sqrt{n\mu} \right) \|u\|_{H^s(\Omega)}, \quad (5.14)$$

where  $C$  is a constant depending only on  $\Omega$  and  $s$  and  $h_{Y, \Omega}$  is the fill distance defined in 2.2.

*Proof.* By applying the Sobolev sampling inequality of Theorem 5.2 for  $p = q = x = 2$ ,  $l = 0$ , we get

$$\|v\|_{L^2(\Omega)} \leq C \left( h_{Y, \Omega}^s \|v\|_{H^s(\Omega)} + h_{Y, \Omega}^{d/2} \left( \sum_{i=1}^n v(y_i)^2 \right)^{1/2} \right), \quad (5.15)$$

for all  $v \in H^s$ . This inequality applied to function  $v = f_\mu - u$  yields

$$\|f_\mu - u\|_{L^2(\Omega)} \leq C \left( h_{Y,\Omega}^s |f_\mu - u|_{H^s(\Omega)} + h_{Y,\Omega}^{d/2} \left( \sum_{i=1}^n (f_\mu(y_i) - u(y_i))^2 \right)^{1/2} \right), \quad (5.16)$$

The remaining task is to bound  $|f_\mu - u|_{H^s(\Omega)}$  and  $(\sum_{i=1}^n (f_\mu(y_i) - u(y_i))^2)^{1/2}$  by  $\|u\|_{H^s(\Omega)}$ . One part of the difficulty lies in the fact that  $f_\mu$  minimizes the semi-norm over  $\mathbb{R}^n$  and that we wish a control over the domain  $\Omega$ . This motivates the introduction of an extension operator  $P : H^s(\Omega) \rightarrow H^s(\mathbb{R}^d)$  defined by

$$Pu = \arg \min_{\substack{f \in H^s(\mathbb{R}^d) \\ f|_\Omega = u}} \|f\|_{H^s(\mathbb{R}^d)}^2. \quad (5.17)$$

By [9, Lemma 3.1], this operator is continuous, i.e. there exists a constant  $K > 0$  (depending on  $\Omega$  and  $s$ ) such that for all  $u \in H^s(\Omega)$ ,  $\|Pu\|_{H^s(\mathbb{R}^d)} \leq K\|u\|_{H^s(\Omega)}$ . Now, let  $f_0 : \Omega \rightarrow \mathbb{R}$  denote the solution of

$$f_0 = \arg \min_{\substack{f \in \mathcal{G}(\mathbb{R}^d) \\ f(y_j) = u(y_j)}} \|f\|_{\mathcal{G}(\mathbb{R}^d)}^2, \quad (5.18)$$

By strict convexity of the squared norm  $\|\cdot\|_{\mathcal{G}(\mathbb{R}^d)}^2$ , function  $f_0$  is uniquely determined. Moreover, it satisfies  $\|f_0\|_{\mathcal{G}(\Omega)} \leq \|f_0\|_{\mathcal{G}(\mathbb{R}^d)} \leq \|Pu\|_{\mathcal{G}(\mathbb{R}^d)} \leq \|Pu\|_{H^s(\mathbb{R}^d)} \leq K\|u\|_{H^s(\Omega)}$ . Now, let us define two functionals  $f \mapsto E(f) = \frac{1}{n} \sum_{i=1}^n (u(y_i) - f(y_i))^2$  and  $f \mapsto J(f) = \|f\|_{\mathcal{G}(\mathbb{R}^d)}^2$ . Since  $f_\mu$  is the minimizer of (5.13), it satisfies

$$E(f_\mu) + \mu J(f_\mu) \leq E(f_0) + \mu J(f_0). \quad (5.19)$$

In addition  $E(f_0) = 0$  and  $J(f_0) \leq \|Pu\|_{\mathcal{G}(\mathbb{R}^d)}^2 \leq \|Pu\|_{H^s(\mathbb{R}^d)}^2 \leq K^2\|u\|_{H^s(\Omega)}^2$ . Hence,

$$E(f_\mu) + \mu J(f_\mu) \leq K\mu\|u\|_{H^s(\Omega)}^2 \quad (5.20)$$

which yields

$$E(f_\mu) = \frac{1}{n} \sum_{i=1}^n (u(y_i) - f_\mu(y_i))^2 \leq K^2\mu\|u\|_{H^s(\Omega)}^2. \quad (5.21)$$

To finish, the triangle inequality yields  $|f_\mu - u|_{H^s(\Omega)} \leq |f_\mu|_{H^s(\Omega)} + |u|_{H^s(\Omega)}$ . Then,  $|u|_{H^s(\Omega)} \leq \|u\|_{H^s(\Omega)}$  and bound (5.20) leads to:

$$|f_\mu|_{H^s(\Omega)} \leq \|f_\mu\|_{\mathcal{G}(\mathbb{R}^d)} \leq \|f_0\|_{\mathcal{G}(\mathbb{R}^d)} \leq K\|u\|_{H^s(\Omega)}. \quad (5.22)$$

Therefore

$$|f_\mu - u|_{H^s(\Omega)} \leq (K + 1)\|u\|_{H^s(\Omega)}. \quad (5.23)$$

Replacing bounds (5.23) and (5.21) in the sampling inequality (5.16) completes the proof of Proposition 5.1.  $\square$

Applying Proposition 5.1 to  $\hat{F}^0(\cdot)[k]$ , we get

$$\|\hat{F}^0(\cdot)[k] - F(\cdot)[k]\|_{L^2(\Omega)}^2 \leq C \left( h_{Y,\Omega}^s + \sqrt{\mu n} h_{Y,\Omega}^{d/2} \right)^2 \|F(\cdot)[k]\|_{H^s(\Omega)}^2. \quad (5.24)$$

The trick is now to use the quasi-uniformity condition given in Definition 2.4 to control  $h_{Y,\Omega}^s$  and  $\sqrt{\mu n} h_{Y,\Omega}^{d/2}$ . This is achieved using the following proposition.

**Proposition 5.2** ([26, Proposition 14.1] or [24]). *Let  $Y = \{y_1, \dots, y_n\} \subset \Omega$  be a quasi-uniform set with respect to  $B$ . Then, there exist constants  $c > 0$  and  $C > 0$  depending only on  $d$ ,  $\Omega$  and  $B$  such that,*

$$cn^{-1} \leq h_{Y,\Omega}^d \leq Cn^{-1}. \quad (5.25)$$

Condition  $n\mu^{d/2s} \geq 1$  combined with the right-hand-side of (5.25) yields  $h_{Y,\Omega}^d \leq C\mu^{d/2s}$ , so that  $h_{Y,\Omega}^s \lesssim \sqrt{\mu}$ . Similarly, the right-hand-side of (5.25) yields  $\sqrt{\mu n} h_{Y,\Omega}^{d/2} \lesssim \sqrt{\mu}$ . Hence

$$\|\hat{F}^0(\cdot)[k] - F(\cdot)[k]\|_{L^2(\Omega)}^2 \lesssim \mu \|F(\cdot)[k]\|_{H^s(\Omega)}^2. \quad (5.26)$$

**Control of the variance** The variance term is treated following arguments similar to those in [24]. However, the change of kernel needs additional treatments. First of all, note that due to the linearity of the estimators (4.1), we have  $\hat{F}_\mu^0(\cdot)[k] - \hat{F}_\mu(\cdot)[k] = f_k^\eta$  with  $\eta \in \mathbb{R}^n$  defined as  $\eta[i] = \epsilon_i[k]$  and

$$f_k^\eta = \arg \min_{f \in H^s(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^n (f(y_i) - \eta[i])^2 + \mu \left( \alpha w[k] \|f\|_{L^2(\mathbb{R}^d)}^2 + (1 - \alpha) \|f\|_{BL^s(\mathbb{R}^d)}^2 \right). \quad (5.27)$$

We therefore need to estimate  $\mathbb{E} \|f_k^\eta\|_{L^2(\Omega)}^2$ . From Theorem 5.2 applied with  $p = q = x = 2$  and  $l = 0$  we obtain that for  $u \in H^s(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq C \left( h_{Y,\Omega}^s \|u\|_{H^s(\Omega)} + h_{Y,\Omega}^{d/2} \|u|_Y\|_2 \right).$$

Using the above inequality together with Proposition 5.2, we get that

$$\|f_k^\eta\|_{L^2(\Omega)}^2 \leq 2C \left( h_{Y,\Omega}^{2s} \|f_k^\eta\|_{H^s(\Omega)}^2 + n^{-1} \sum_{i=1}^n f_k^\eta(y_i)^2 \right).$$

As in [24], let us define the  $n \times n$  matrix  $\tilde{\Gamma}$  such that

$$\langle \tilde{\Gamma} z, z \rangle = \min_{\substack{u \in BL^s(\mathbb{R}^d) \\ u(y_i) = z[i]}} (1 - \alpha) \|u\|_{H^s(\mathbb{R}^d)}^2 + \alpha w[k] \|u\|_{L^2(\mathbb{R}^d)}^2. \quad (5.28)$$

The solution of problem (5.28) is a spline interpolating the data  $(y_i, z[i])_{i=1}^n$ . Using this matrix, we can write (5.27) as:

$$\min_{z \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n (z[i] - \eta[i])^2 + \mu \langle \tilde{\Gamma} z, z \rangle,$$

see [24, 23, 25] for details. Thus, the solution  $\hat{z} = (f_k^\eta(y_i))_{i=1}^n$  is obtained by:

$$\hat{z} = (\text{Id} + n\mu\tilde{\Gamma})^{-1}\eta.$$

By letting  $E_\mu = (\text{Id} + n\mu\tilde{\Gamma})^{-1}$ , we obtain

$$n^{-1} \sum_{i=1}^n f_k^\eta(y_i)^2 = n^{-1} \sum_{i=1}^n \hat{z}[i]^2 = n^{-1} \eta^T E_\mu^2 \eta$$

and

$$\begin{aligned} (1 - \alpha) |f_k^\eta|_{H^s(\mathbb{R}^d)}^2 + \alpha w[k] \|f_k^\eta\|_{L^2(\mathbb{R}^d)}^2 &= \hat{z}^T \tilde{\Gamma} \hat{z} = \eta^T E_\mu \tilde{\Gamma} E_\mu \eta \\ &= (n\mu)^{-1} \eta^T E_\mu (E_\mu^{-1} - \text{Id}) E_\mu \eta \\ &= (n\mu)^{-1} \eta^T (E_\mu - E_\mu^2) \eta. \end{aligned}$$

Thus

$$|f_k^\eta|_{H^s(\Omega)}^2 \leq |f_k^\eta|_{H^s(\mathbb{R}^d)}^2 \leq (n\mu)^{-1} \eta^T (E_\mu - E_\mu^2) \eta.$$

Using the fact that  $\eta$  has i.i.d. components with zero mean and variance  $\sigma^2$ , we get that,

$$\mathbb{E} \left[ n^{-1} \sum_{i=1}^n f_\lambda^\eta(y_i)^2 \right] = n^{-1} \sigma^2 \text{Tr}(E_\mu^2),$$

and on the other hand

$$\begin{aligned} \mathbb{E} |f_k^\eta|_{H^s(\Omega)}^2 &\leq (n\mu)^{-1} \sigma^2 (\text{Tr}(E_\mu) - \text{Tr}(E_\mu^2)) \\ &\leq (n\mu)^{-1} \sigma^2 \text{Tr}(E_\mu). \end{aligned}$$

We now have to focus on the estimation of both  $\text{Tr}(E_\mu) = \sum_{i=1}^n (1 + n\mu\lambda_i(\tilde{\Gamma}))^{-1}$  and  $\text{Tr}(E_\mu^2) = \sum_{i=1}^n (1 + n\mu\lambda_i(\tilde{\Gamma}))^{-2}$ , where  $\lambda_i(\tilde{\Gamma})$  is the  $i$ -th eigenvalue of  $\tilde{\Gamma}$ . This will be achieved by analyzing the eigenvalues of the matrix  $\tilde{\Gamma}$ . This step is uneasy. Hopefully, Utreras analyzed the eigenvalues of the matrix  $\Gamma$  associated to thin-plate splines in [24]. Matrix  $\Gamma$  is defined in a similar way as (5.28):

$$\langle \Gamma z, z \rangle = \min_{\substack{u \in BL_s(\mathbb{R}^d) \\ u(y_i) = z[i]}} |u|_{H^s(\mathbb{R}^d)}^2. \quad (5.29)$$

One therefore has that  $(1 - \alpha)z^T \Gamma z \leq z^T \tilde{\Gamma} z$  for all  $z \in \mathbb{R}^N$ . Therefore the matrix  $\tilde{\Gamma} - (1 - \alpha)\Gamma$  is semi-definite positive. By virtue of Weyl's monotonicity theorem [28], we get that  $(1 - \alpha)\lambda_i(\Gamma) \leq \lambda_i(\tilde{\Gamma})$ . Hence we can bound the traces of the matrices  $E_\mu$  and  $E_\mu^2$  as follows

$$\begin{aligned} \text{Tr}(E_\mu) &\leq \sum_{i=1}^n (1 + (1 - \alpha)n\mu\lambda_i(\Gamma))^{-1}, \\ \text{Tr}(E_\mu^2) &\leq \sum_{i=1}^n (1 + (1 - \alpha)n\mu\lambda_i(\Gamma))^{-2}. \end{aligned}$$

It is shown in [24], that  $\gamma = \binom{s-1+d}{s-1}$  eigenvalues  $\lambda_i(\Gamma)$  are null and the others satisfy  $i^{2m/d}n^{-1} \lesssim \lambda_i(\Gamma) \lesssim i^{2m/d}n^{-1}$  for  $\gamma+1 \leq i \leq n$ . Following [24], it can be shown that both traces are bounded by quantities proportional to  $\mu^{-d/2s}$ . Thus, one has that

$$\mathbb{E}\|f_k^\eta\|_{L^2(\Omega)}^2 \lesssim \sigma^2(n^{-1}\mu^{-d/2s} + n^{-1}h_{Y,\Omega}^{2s}\mu^{-1}\mu^{-d/2s}).$$

Since  $\mu^{d/2s}n \geq 1$  and using Proposition 5.2 that gives  $n \lesssim h_{Y,\Omega}^{-d}$  we obtain that  $h_{Y,\Omega}^{2s}\mu \lesssim 1$ . Hence

$$\mathbb{E}\|f_k^\eta\|_{L^2(\Omega)}^2 \lesssim \sigma^2n^{-1}\mu^{-d/2s} (1 + h_{Y,\Omega}^{2s}\mu) \lesssim \sigma^2n^{-1}\mu^{-d/2s},$$

which completes the proof of Lemma 5.2.  $\square$

## 5.4 Proof of the main results

### Proof of Theorem 3.1

*Proof.* By equation (5.2):

$$\mathbb{E}\|\hat{H} - H\|_{HS}^2 \leq 2(\epsilon_d(N) + \epsilon_e(n)). \quad (5.30)$$

By Corollary (5.1)

$$\epsilon_d(N) \lesssim N^{-2r/d}. \quad (5.31)$$

Now, let us control  $\epsilon_e$ . Let  $F_N = F(\cdot)[1 : N]$ .

$$\epsilon_e(n) = \mathbb{E}\|\hat{F} - F_N\|_{\mathbb{R}^N \times L^2(\Omega)}^2 \quad (5.32)$$

$$= \sum_{k=1}^N \mathbb{E}\|\hat{F}(\cdot)[k] - F_N(\cdot)[k]\|_{L^2(\Omega)}^2 \quad (5.33)$$

$$\stackrel{(5.9)}{\lesssim} \sum_{k=1}^N \left( \mu \|F_N(\cdot)[k]\|_{H^s(\Omega)}^2 + n^{-1}\sigma^2\mu^{-d/2s} \right) \quad (5.34)$$

$$= \mu \|F_N\|_{\mathbb{R}^N \times H^s(\Omega)}^2 + Nn^{-1}\sigma^2\mu^{-d/2s}. \quad (5.35)$$

This upper-bound allows to set the value of the regularization parameter  $\mu$  by balancing the two terms  $\mu \|F_N\|_{\mathbb{R}^N \times H^s(\Omega)}^2$  and  $Nn^{-1}\sigma^2\mu^{-d/2s}$ :

$$\mu \|F\|_{\mathbb{R}^N \times H^s(\Omega)}^2 \propto Nn^{-1}\sigma^2\mu^{-d/2s}. \quad (5.36)$$

This yields

$$\mu \propto \left( N\sigma^2n^{-1} \|F\|_{\mathbb{R}^N \times H^s(\Omega)}^{-2} \right)^{\frac{2s}{2s+d}}. \quad (5.37)$$

Plugging this value in the upper-bound of  $\epsilon_e(n)$  gives

$$\begin{aligned} \mu \|F\|_{\mathbb{R}^N \times H^s(\Omega)}^2 &\propto \left( N\sigma^2 n^{-1} \|F\|_{\mathbb{R}^N \times H^s(\Omega)}^{-2} \right)^{\frac{2s}{2s+d}} \|F\|_{\mathbb{R}^N \times H^s(\Omega)}^2, \\ &= (N\sigma^2 n^{-1})^{\frac{2s}{2s+d}} \|F\|_{\mathbb{R}^N \times H^s(\Omega)}^{\frac{2d}{2s+d}} \end{aligned} \quad (5.38)$$

and

$$\epsilon_e(n) \lesssim (N\sigma^2 n^{-1})^{\frac{2s}{2s+d}}. \quad (5.39)$$

□

### Proof of Theorem 3.1

*Proof.* To obtain this bound we use Theorem 3.1 and we balance the two terms so that:

$$N^{-2r/d} \propto (N\sigma^2 n^{-1})^{\frac{2s}{2s+d}} \quad (5.40)$$

This gives the choice  $N \propto (\sigma^{-2}n)^{\frac{2sd}{4rs+2rd+2sd}}$ . Replacing  $N$  by this value in bound (3.6) gives

$$\begin{aligned} N^{-2r/d} &\propto (\sigma^2 n^{-1})^{\frac{4rs}{4rs+2rd+2sd}}, \\ &= (\sigma^2 n^{-1})^{\frac{2q}{2q+d}}. \end{aligned} \quad (5.41)$$

□

### Proof of Theorem 3.2

*Proof.* In the proof, we first need to define an appropriate wavelet basis to characterize the fact that a function belongs to the Sobolev ball (for some constant  $A > 0$ )

$$H^s(\Omega, A) = \left\{ u \in L^2(\Omega), \|u\|_{H^s(\Omega)}^2 \leq A \right\},$$

through its wavelet coefficients. We assume that we are given a bi-orthogonal wavelet basis of  $L^2(\Omega)$  generated by a pair of dual scaling functions with compact supports and by tensor products. For a detailed description of (bi-orthogonal) wavelet basis construction, we refer to [8]. The scaling and wavelet functions at scale  $j$  (that is at resolution level  $2^j$ ) will be denoted by  $\phi_\lambda$  and  $\psi_\lambda$ , respectively, where the index  $\lambda$  summarizes both the usual scale and space parameters  $j$  and  $k$ . In other words, for  $d = 1$ , we set  $\lambda = (j, k)$  and denote  $\phi_{j,k}(\cdot) = 2^{j/2}\phi(2^j \cdot -k)$  and  $\psi_{j,k}(\cdot) = 2^{j/2}\psi(2^j \cdot -k)$ . For  $d \geq 2$ , the notation  $\psi_\lambda$  stands for the adaptation of scaling and wavelet functions to  $\Omega = [0, 1]^d$  (see [8], Chapter 2). The notation  $|\lambda| = j$  will be used to denote a wavelet at scale  $j$ , where  $j_0$  denotes the coarse level of approximation. In order to simplify the notation, as it is commonly used, we take  $j_0 = 0$ , and we write  $(\psi_\lambda)_{|\lambda|=-1}$  for  $(\phi_\lambda)_{|\lambda|=0}$ . Finally,  $|\lambda| < j_1$  denotes all wavelets at scales  $j$ , with  $-1 \leq j < j_1$ , and we use the notation

$\tilde{\psi}_\lambda$  to denote the dual wavelet basis of  $\psi_\lambda$ . Now, assume that a function  $u \in L^2(\Omega)$  admits the wavelet decomposition

$$u(y) = \sum_{j=-1}^{+\infty} \sum_{|\lambda|=j} c[\lambda] \psi_\lambda(y)$$

where the  $c[\lambda]$ 's are real coefficients satisfying  $c[\lambda] = \langle u, \tilde{\psi}_\lambda \rangle_{L^2(\Omega)}$ . It is well known that wavelet coefficients may be used to characterize the smoothness of functions. For instance, by Theorem 3.10.5 in [8] and using the fact that the Besov space  $B_{2,2}^s(\Omega)$  is equal to the Sobolev space  $H^s(\Omega)$  (see e.g. Remark 3.2.4 in [8]), it follows that, under appropriate assumptions on the scaling function  $\phi$  and its dual version (see e.g. those of Theorem 3.10.5 in [8]),

$$u \in H^s(\Omega, A) \iff \sum_{j=-1}^{+\infty} \sum_{|\lambda|=j} 2^{2js} |c[\lambda]|^2 \leq C(A, s, \Omega) \quad (5.42)$$

for some constant  $C(A, s, \Omega) > 0$  depending only on  $A$ ,  $s$  and  $\Omega$ . Throughout the proof, it is assumed that the bi-orthogonal wavelet basis is chosen such that the wavelet characterization of Sobolev norms (5.42) is satisfied. In particular, we assume that  $\psi$  possesses  $s$  vanishing moments.

The arguments to prove the lower bound (3.9) are based on the standard Assouad's cube technique (see, e.g. [22], Chapter 2, Section 2.7.2). Consider the following SVIR test functions

$$S_v(x, y) = \mu_{k_1, j_1} \sum_{k=1}^{k_1} \sum_{|\lambda| < j_1} v[k, \lambda] \phi_k(x) \psi_\lambda(y), \quad \forall (x, y) \in \Omega \times \Omega,$$

where  $v = (v[k, \lambda])_{k \leq k_1, |\lambda| < j_1} \in \mathcal{V} := \{1, -1\}^{k_1 2^{j_1 d}}$ , and  $\mu_{k_1, j_1}$  is a positive sequence of reals satisfying the condition

$$\mu_{k_1, j_1} = ck_1^{-1/2} 2^{-j_1 d/2} \min(k_1^{-r/d}, 2^{-j_1 s}), \quad (5.43)$$

for some constant  $c > 0$  not depending on  $k_1$  and  $j_1$ .

Let us first discuss the choice of the constant  $c$  in (5.43). For any  $x \in \Omega$  and  $v \in \mathcal{V}$  one has that

$$\begin{aligned} \mu_{k_1, j_1}^2 \sum_{j=-1}^{j_1-1} \sum_{|\lambda|=j} 2^{2js} \left( \sum_{k=1}^{k_1} v[k, \lambda] \phi_k(x) \right)^2 &\leq \mu_{k_1, j_1}^2 \sum_{j=-1}^{j_1-1} \sum_{|\lambda|=j} 2^{2js} \left( \sum_{k=1}^{k_1} |\phi_k(x)| \right)^2 \\ &\leq \mu_{k_1, j_1}^2 \sum_{j=-1}^{j_1-1} 2^{j(2s+d)} C_\phi k_1 \leq C_\phi \mu_{k_1, j_1}^2 k_1 2^{j_1(2s+d)} \end{aligned}$$

where the last inequalities follow from Assumption (3.8) and the fact that the number of wavelets at scale  $j$  is  $2^{jd}$ . Hence, by the wavelet characterization of Sobolev norms (5.42) and the condition (5.43) on  $\mu_{k_1, j_1}$ , it follows that if  $c^2 \leq C(A_2, s, \Omega) C_\phi^{-1}$  then

$$\sup_{x \in \Omega} \|S_v(x, \cdot)\|_{H^s(\Omega)}^2 \leq A_2,$$



for any  $v \in \mathcal{V}$ . Similarly, for any  $y \in \Omega$  and  $v \in \mathcal{V}$  one has that

$$\mu_{k_1, j_1}^2 \sum_{k=1}^{k_1} w[k] \left( \sum_{j=-1}^{j_1-1} \sum_{|\lambda|=j} v[k, \lambda] \psi_\lambda(y) \right)^2 \leq c_1 \mu_{k_1, j_1}^2 \sum_{k=1}^{k_1} (1+k^2)^{r/d} \left( \sum_{j=-1}^{j_1-1} \sum_{|\lambda|=j} |\psi_\lambda(y)| \right)^2, \quad (5.44)$$

using the assumption that  $w[k] \leq c_1(1+k^2)^{r/d}$ . Let us define the set

$$I_j(y) = \{\lambda : |\lambda| = j \text{ and } \psi_\lambda(y) \neq 0\}.$$

Since, the wavelet  $\psi_\lambda$  is compactly supported, one has that the cardinality of  $I_j(y)$  is bounded by a constant  $D_s > 0$  that is independent of  $j$  and  $y$ . Thus, using the relation  $\|\psi_\lambda\|_\infty \leq C_\infty 2^{jd/2}$  (for some constant  $C_\infty > 0$ ) for any  $\lambda$  at scale  $j$ , we obtain from inequality (5.44) that,

$$\begin{aligned} \mu_{k_1, j_1}^2 \sum_{k=1}^{k_1} w[k] \left( \sum_{j=-1}^{j_1-1} \sum_{|\lambda|=j} v[k, \lambda] \psi_\lambda(y) \right)^2 &\leq c_1 \mu_{k_1, j_1}^2 \sum_{k=1}^{k_1} (1+k^2)^{r/d} \left( \sum_{j=-1}^{j_1-1} \sum_{\lambda \in I_j(y)} |\psi_\lambda(y)| \right)^2 \\ &\leq c_1 D_s^2 C_\infty^2 \mu_{k_1, j_1}^2 \sum_{k=1}^{k_1} (1+k^2)^{r/d} \left( \sum_{j=-1}^{j_1-1} 2^{jd/2} \right)^2, \\ &\leq c_1 D_s^2 C_\infty^2 \mu_{k_1, j_1}^2 k_1^{2r/d+1} 2^{j_1 d} \end{aligned}$$

Hence, by definition of the space  $\mathcal{E}^r(\Omega)$  and the condition (5.43) on  $\mu_{k_1, j_1}$ , it follows that if  $c^2 \leq A_1 c_1^{-1} D_s^{-2} C_\infty^{-2}$  then

$$\sup_{y \in \Omega} \|S_v(\cdot, y)\|_{\mathcal{E}^r(\Omega)}^2 \leq A_1,$$

for any  $v \in \mathcal{V}$ . Therefore, we have shown that if the constant  $c$  in (5.43) is chosen sufficiently small, then the operator  $H_v$  with SVIR function  $S_v$  belongs to the ball  $\mathcal{E}^{r,s}(\Omega, A_1, A_2)$  for any  $v \in \mathcal{V}$ . In the rest of the proof, it is thus assumed that the constant  $c$  is chosen sufficiently small to satisfy such property on the  $H_v$ 's.

In what follows, we use the notation  $\mathbb{E}_{H_v}$  to denote expectation with respect to the distribution  $\mathbb{P}_{H_v}$  of the random process  $F^\epsilon = (F_1^\epsilon, \dots, F_n^\epsilon)$  obtained from model (2.9) under the hypothesis that  $S = S_v$  where  $S_v$  is the SVIR function of the operator  $H_v$ .

The minimax risk

$$\mathcal{R}_{\sigma^2, n} := \inf_{\hat{H}} \sup_{H \in \mathcal{H}^{r,s}(\Omega, A_1, A_2)} \mathbb{E} \left\| \hat{H} - H \right\|_{HS}^2$$

can be bounded from below as follows

$$\mathcal{R}_{\sigma^2, n} \geq \inf_{\hat{H}} \sup_{v \in \mathcal{V}} \mathbb{E}_{H_v} \left\| \hat{H} - H_v \right\|_{HS}^2.$$

Since  $\left\| \hat{H} - H_v \right\|_{HS}^2 = \left\| \hat{S} - S_v \right\|_{L^2(\Omega \times \Omega)}^2$  it follows from orthonormality of the basis  $(\phi_k)_{k \geq 1}$  and by the Riesz stability property for bi-orthogonal wavelet bases (see e.g. inequality (7.156) in

[15]), that there exists a constant  $c_\psi > 0$  such that

$$\left\| \hat{H} - H_v \right\|_{HS}^2 \geq c_\psi \sum_{k=1}^{k_1} \sum_{|\lambda| < j_1} |\hat{\alpha}[k, \lambda] - \mu_{k_1, j_1} v[k, \lambda]|^2 \text{ where } \hat{\alpha}[k, \lambda] = \int_{\Omega \times \Omega} \hat{S}(x, y) \phi_k(x) \tilde{\psi}_\lambda(y) dx dy.$$

Therefore, the minimax risk satisfies the following inequality

$$\mathcal{R}_{\sigma^2, n} \geq \inf_{\hat{H}} \sup_{v \in \mathcal{V}} c_\psi \sum_{k=1}^{k_1} \sum_{|\lambda| < j_1} \mathbb{E}_{H_v} |\hat{\alpha}[k, \lambda] - \mu_{k_1, j_1} v[k, \lambda]|^2.$$

Then, define

$$\hat{v}[k, \lambda] := \arg \min_{v \in \{-1, 1\}} |\hat{\alpha}[k, \lambda] - \mu_{k_1, j_1} v|,$$

and remark that the triangular inequality and the definition of  $\hat{v}[k, \lambda]$  imply that

$$\mu_{k_1, j_1} |\hat{v}[k, \lambda] - v[k, \lambda]| \leq 2 |\hat{\alpha}[k, \lambda] - \mu_{k_1, j_1} v[k, \lambda]|,$$

which yields

$$\begin{aligned} \mathcal{R}_{\sigma^2, n} &\geq \inf_{\hat{H}} \sup_{v \in \mathcal{V}} \frac{c_\psi \mu_{k_1, j_1}^2}{4} \sum_{k=1}^{k_1} \sum_{|\lambda| < j_1} \mathbb{E}_{H_v} |\hat{v}[k, \lambda] - v[k, \lambda]|^2 \\ &\geq \inf_{\hat{H}} \frac{c_\psi \mu_{k_1, j_1}^2}{4} \frac{1}{\#\mathcal{V}} \sum_{v \in \mathcal{V}} \sum_{k=1}^{k_1} \sum_{|\lambda| < j_1} \mathbb{E}_{H_v} |\hat{v}[k, \lambda] - v[k, \lambda]|^2, \end{aligned}$$

where  $\#\mathcal{V}$  denotes the cardinality of the finite set  $\mathcal{V}$ .

For a given pair  $[k, \lambda]$  of indices and any  $v \in \mathcal{V}$ , we define the vector  $v^{(k, \lambda)} \in \mathcal{V}$  having all its components equal to  $v$  except the  $[k, \lambda]$ -th element. Moreover, to simplify the notation, we let  $\sum_{k, \lambda}$  denote the summation  $\sum_{k=1}^{k_1} \sum_{|\lambda| < j_1}$ . Then

$$\begin{aligned} \mathcal{R}_{\sigma^2, n} &\geq \inf_{\hat{H}} \frac{c_\psi \mu_{k_1, j_1}^2}{4} \frac{1}{\#\mathcal{V}} \sum_{k, \lambda} \sum_{v \in \mathcal{V} : v[k, \lambda] = 1} \left( \mathbb{E}_{H_v} |\hat{v}[k, \lambda] - v[k, \lambda]|^2 + \mathbb{E}_{H_{v^{(k, \lambda)}}} \left| \hat{v}[k, \lambda] - v^{(k, \lambda)}[k, \lambda] \right|^2 \right) \\ &\geq \inf_{\hat{H}} \frac{c_\psi \mu_{k_1, j_1}^2}{4} \frac{1}{\#\mathcal{V}} \sum_{k, \lambda} \sum_{v \in \mathcal{V} : v[k, \lambda] = 1} \mathbb{E}_{H_v} \left( |\hat{v}[k, \lambda] - v[k, \lambda]|^2 + \left| \hat{v}[k, \lambda] - v^{(k, \lambda)}[k, \lambda] \right|^2 \frac{d\mathbb{P}_{H_{v^{(k, \lambda)}}}}{d\mathbb{P}_{H_v}}(F^\epsilon) \right). \end{aligned}$$

where  $\frac{d\mathbb{P}_{H_{v^{(k, \lambda)}}}}{d\mathbb{P}_{H_v}}(F^\epsilon)$  is the log-likelihood ratio between the hypothesis  $H_{v^{(k, \lambda)}} : S = S_{v^{(k, \lambda)}}$  and the hypothesis  $H_v : S = S_v$  in model (2.9).

Since  $v^{(k, \lambda)}[k, \lambda] = -v[k, \lambda]$  and  $\hat{v}[k, \lambda] \in \{-1, 1\}$ , one has that, for any  $0 < \delta < 1$ ,

$$\begin{aligned} \mathcal{R}_{\sigma^2, n} &\geq 4c_\psi \mu_{k_1, j_1}^2 \frac{1}{\#\mathcal{V}} \sum_{k, \lambda} \sum_{v \in \mathcal{V} : v[k, \lambda] = 1} \mathbb{E}_{H_v} \left( \min \left( 1, \frac{d\mathbb{P}_{H_{v^{(k, \lambda)}}}}{d\mathbb{P}_{H_v}}(F^\epsilon) \right) \right) \\ &\geq 4c_\psi \delta \mu_{k_1, j_1}^2 \frac{1}{\#\mathcal{V}} \sum_{k, \lambda} \sum_{v \in \mathcal{V} : v[k, \lambda] = 1} \mathbb{P}_{H_v} \left( \frac{d\mathbb{P}_{H_{v^{(k, \lambda)}}}}{d\mathbb{P}_{H_v}}(F^\epsilon) > \delta \right), \end{aligned} \tag{5.45}$$

by Markov's inequality. Thanks to the Girsanov's formula (see e.g. Lemma A.5 in [22]), one has that, under the hypothesis that  $S = S_v$  in model (2.9):

$$\log \left( \frac{d\mathbb{P}_{H_{v^{(k,\lambda)}}}}}{d\mathbb{P}_{H_v}}(F^\epsilon) \right) = \sum_{i=1}^n \sum_{\ell=1}^{+\infty} \left( \sigma^{-1} \langle S_{v^{(k,\lambda)}}(\cdot, y_i) - S_v(\cdot, y_i), \phi_\ell \rangle \eta_{i,\ell} - \frac{\sigma^{-2}}{2} |\langle S_{v^{(k,\lambda)}}(\cdot, y_i) - S_v(\cdot, y_i), \phi_l \rangle|^2 \right)$$

where the  $\eta_{i,\ell}$ 's are iid standard Gaussian variables. By definition of  $v^{(k,\lambda)}$  and for  $v[k, \lambda] = 1$  one has that, for each  $1 \leq i \leq n$  and  $1 \leq k \leq k_1$ ,

$$\langle S_{v^{(k,\lambda)}}(\cdot, y_i) - S_v(\cdot, y_i), \phi_\ell \rangle = \begin{cases} -2\mu_{k_1, j_1} \psi_\lambda(y_i) & \text{if } \ell = k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the random variable  $Z_{k,\lambda} := \log \left( \frac{d\mathbb{P}_{H_{v^{(k,\lambda)}}}}}{d\mathbb{P}_{H_v}}(F^\epsilon) \right)$  is Gaussian with mean  $\theta_\lambda$  and variance  $\gamma_\lambda^2$  satisfying

$$\theta_\lambda = -2\sigma^{-2} \mu_{k_1, j_1}^2 \sum_{i=1}^n \psi_\lambda^2(y_i) \text{ and } \gamma_\lambda^2 = 4\sigma^{-2} \mu_{k_1, j_1}^2 \sum_{i=1}^n \psi_\lambda^2(y_i) = -2\theta_\lambda,$$

under the hypothesis that  $S = S_v$  in model (2.9). Hence, using that  $\theta_\lambda$  is negative, one has that

$$\mathbb{P}_{H_v}(Z_{k,\lambda} \geq 3\theta_\lambda) = \mathbb{P}_{H_v} \left( \frac{Z_{k,\lambda} - \theta_\lambda}{\sqrt{2|\theta_\lambda|}} \geq -\sqrt{2|\theta_\lambda|} \right) \geq \frac{1}{2},$$

by symmetry of the standard Gaussian distribution. Hence, inserting the above inequality into (5.45) with  $\delta = \exp(3\theta_\lambda)$ , it implies that

$$\mathcal{R}_{\sigma^2, n} \geq c_\psi \exp(3\theta_\lambda) \mu_{k_1, j_1}^2 k_1 2^{dj_1} \tag{5.46}$$

$$= c_\psi \exp(3\theta_\lambda) c^2 \min(k_1^{-r/d}, 2^{-j_1 s}). \tag{5.47}$$

By setting  $k_1 = k_1^{(\sigma^2, n)}$  and  $j_1 = j_1^{(\sigma^2, n)}$  with

$$k_1^{(\sigma^2, n)} = \lfloor (\sigma^2 n^{-1})^{-\frac{q}{(2q+d)r/d}} \rfloor \text{ and } 2^{j_1^{(\sigma^2, n)}} = \lfloor (\sigma^2 n^{-1})^{-\frac{q}{(2q+d)s}} \rfloor, \tag{5.48}$$

we get

$$\mathcal{R}_{\sigma^2, n} \geq c_\psi \exp(3\theta_\lambda) c^2 (\sigma^2 n^{-1})^{\frac{q}{2q+d}}. \tag{5.49}$$

It now remains to show that the constant  $\theta_\lambda$  is bounded from below, independently of  $\sigma$  and  $n$ .

The idea is to remark that  $\frac{1}{n} \sum_{i=1}^n \psi_\lambda^2(y_i)$  behaves like a Riemann integral of  $\psi_\lambda$  and should therefore be bounded by a constant since  $\|\psi_\lambda\|_2 = 1$ . This statement can be proved using the following reasoning. Since vector  $Y = (y_1, \dots, y_n)$  of PSFs locations satisfies the quasi-uniformity condition  $h_{Y, \Omega} \leq Bq_{Y, \Omega}$ , we get from Proposition 5.2 that the separation distance  $q_{Y, \Omega}$  satisfies  $q_{Y, \Omega}^d \geq B_1 n^{-1}$  for some constant  $B_1$ . Now, the support of wavelet  $\psi_\lambda$  is contained

in a hypercube of volume proportional to  $2^{-d|\lambda|}$ . Hence, the number of locations  $y_i$  in  $\text{supp}(\psi_\lambda)$  is bounded above by  $2^{-d|\lambda|}/q_{Y,\Omega}^d \leq B_2 n 2^{-d|\lambda|}$  for some constant  $B_2$ . To conclude, remark that  $\|\psi_\lambda\|_\infty = 2^{d|\lambda|/2} \|\psi\|_\infty$ , hence:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \psi_\lambda(y_i)^2 &= \frac{1}{n} \sum_{y_i \in \text{supp}(\psi_\lambda)} \psi_\lambda(y_i)^2 \\ &\leq \frac{1}{n} B_2 n 2^{-d|\lambda|} \|\psi_\lambda\|_\infty^2 \\ &\leq B_2 \|\psi\|_\infty^2 =: B_3. \end{aligned}$$

This implies that

$$\theta_\lambda \geq -2B_3 c_1, \text{ for all } \lambda < j_1^{(\sigma^2, n)}.$$

Hence, inserting the above inequality into (5.47), it implies that

$$\mathcal{R}_{\sigma^2, n} \geq c_\psi \exp(-6B_3 c_1) \mu_{k_1^{(\sigma^2, n)}, j_1^{(\sigma^2, n)}}^2 k_1^{(\sigma^2, n)} 2^{d j_1^{(\sigma^2, n)}}.$$

Using the expressions of  $j_1^{(\sigma^2, n)}$  and  $k_1^{(\sigma^2, n)}$  given in (5.48), together with (5.43), we finally obtain that there exists a constant  $c_0 > 0$ , that does not depend on  $\frac{\sigma^2}{n}$ , such that

$$\mathcal{R}_{\sigma^2, n} \geq c_0 (\sigma^2 n^{-1})^{\frac{2q}{2q+d}},$$

completing the proof of the theorem. □

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