



HAL
open science

An efficient quadrature rule on the cubed sphere

Brice Portelenelle, Jean-Pierre Croisille

► **To cite this version:**

Brice Portelenelle, Jean-Pierre Croisille. An efficient quadrature rule on the cubed sphere. 2016. hal-01380370

HAL Id: hal-01380370

<https://hal.science/hal-01380370>

Preprint submitted on 12 Oct 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AN EFFICIENT QUADRATURE RULE ON THE CUBED SPHERE

BRICE PORTELENELLE AND JEAN-PIERRE CROISILLE †‡

ABSTRACT. A new quadrature rule for functions defined on the sphere is introduced. The nodes are defined as the points of the Cubed Sphere. The associated weights are defined in analogy to the trapezoidal rule on each panel of the Cubed Sphere. The formula enjoys a symmetry property ensuring that a proportion of 7/8 of all Spherical Harmonics is integrated exactly. Based on the remaining Spherical Harmonics, it is possible to define modified weights giving an enhanced quadrature rule. Numerical results show that the new quadrature is competitive with classical rules of the literature. This second quadrature rule is believed to be of interest for applied mathematicians, physicists and engineers dealing with data located at the points of the Cubed Sphere.

Keywords: Quadrature rule on the sphere - Cubed Sphere - Spherical Harmonic.

Mathematics Subject Classification: 33C55, 41A55, 65D30, 65D32

1. INTRODUCTION

In this paper we consider quadrature rules for functions defined on the sphere. Let \mathbb{S}^2 be the unit sphere, and let $\mathbf{x} \in \mathbb{S}^2 \mapsto f(\mathbf{x})$ be a regular function. A quadrature rule $Q(f)$ is defined by

$$(1) \quad I(f) = \int_{\mathbb{S}^2} f(\mathbf{x}) d\sigma(\mathbf{x}) \simeq \sum_{p=1}^P w_p f(\mathbf{x}_p) = Q(f),$$

where $\mathbf{x}_p \in \mathbb{S}^2$ are the nodes and w_p are the weights. The quest for rules of the form (1) has been a longstanding topic of interest. The classical setup of the problem consists in finding a minimal number \mathcal{P} of nodes x_p with the associated weights w_p for (1) to be as exact as possible. More precisely, the problem is to determine the location on the sphere of a minimal number of nodes for the largest number of Spherical Harmonics to be exactly integrated. Recent works on this topic include [15, 10, 1, 7]. We also refer to the review paper [8] and the references therein.

Here, we are interested in a slightly different question. Over the past 20 years, the Cubed Sphere (see Fig.1) has become a popular spherical grid among researchers dealing with mathematical or physical models. In particular, in numerical climatology, the Cubed Sphere serves for various numerical schemes for time-dependent climate models on the sphere [11, 2, 16]. In this context, accurately evaluating averaged quantities over the sphere such as mass, momentum, energy or total vorticity is particularly important. This is in particular the case for the finite difference scheme introduced in [4, 5]. This scheme uses discrete unknowns located at the nodes of the Cubed Sphere. For this scheme, it is therefore natural to have at hand a quadrature rule with nodes \mathbf{x}_p selected as the gripoints of the Cubed Sphere.

To fully determine such a rule, it remains to identify a set of suitable weights w_p . A basic observation is that a particularly simple set of weights w_p , described hereafter, provides a rule (1), which is exact for a proportion of 7/8 of *all* Spherical Harmonics. Furthermore, for the remaining 1/8 Spherical Harmonics not integrated exactly, fourth order accuracy convergence of the rule was

Date: October 02, 2016.

This work was performed during the master degree thesis of the first author at the Univ. Lorraine, Math. Dept., Institut Elie Cartan de Lorraine in Metz, France, during the second semester of the academic year 2015-2016.

numerically obtained. This observation was the starting point of the present study. Building upon this first rule (called Q_a), we suggest a second rule Q_b , which keeps the 7/8 property and appears to be remarkably accurate.

The outline of the paper is as follows. Section 2 gives the notation for the Spherical Harmonics and the Cubed Sphere. In Section 3, our first quadrature rule, called Q_a , is introduced and the aforementioned 7/8 property is proved. This property is actually due to a set of combined symmetries shared by the Cubed Sphere and the Spherical Harmonics. In Section 4, we show how this property can serve to significantly enhance the selection of the weights associated with the nodes of the Cubed Sphere. This permits to define a second quadrature rule called Q_b , which again uses the Cubed Sphere points as quadrature nodes. The weights w_p are selected according to a specific least square problem. Numerical results show that the rule performs remarkably well compared to other spherical quadrature rules of the litterature.

2. NOTATION

Consider the reference Cartesian frame $\mathcal{R} = (\mathbf{O}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ in \mathbb{R}^3 . The unit sphere is $\mathbb{S}^2 = \{\mathbf{x}(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1\}$.

The Longitude-Latitude coordinate system is called (λ, θ) , with $-\pi \leq \lambda < \pi$ and $-\pi/2 \leq \theta \leq \pi/2$.

For $n \geq 0$ and m with $-n \leq m \leq n$, the Spherical Harmonic $Y_n^m : \mathbf{x} \in \mathbb{S}^2 \mapsto Y_n^m(\mathbf{x}) \in \mathbb{C}$ is defined by [9]:

$$(2) \quad Y_n^m(\mathbf{x}) = (-1)^m N_n^{|m|} P_n^{|m|}(\sin \theta) e^{im\lambda}.$$

For $0 \leq m \leq n$, the function $x \in [-1, 1] \mapsto P_n^m(x)$ is the *associated Legendre polynomial*. It is defined in terms of the Legendre polynomial $P_n(x) = \frac{d^n}{dx^n} \left((x^2 - 1)^n \right) / (2^n n!)$ by the relation:

$$(3) \quad P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x), \quad 0 \leq m \leq n, x \in [-1, 1].$$

The Spherical Harmonics form an orthonormal basis of the space $L^2(\mathbb{S}^2)$ and therefore

$$(4) \quad I(Y_n^m) = 0 \text{ if } (n, m) \neq (0, 0), \quad I(Y_0^0) = \sqrt{4\pi}.$$

The normalisation constant N_n^m in (2) is

$$(5) \quad N_n^m = \left(\frac{2n + 1}{4\pi} \frac{(n - m)!}{(n + m)!} \right)^{\frac{1}{2}}.$$

A quadrature rule is usually designed using the functions Y_n^m . The accuracy is measured by the order n such that the functions $Y_n^m(\mathbf{x})$, $|m| \leq n$, are integrated exactly.

Our next ingredient is a particular grid of \mathbb{S}^2 called the Cubed Sphere (see Fig. 1). The Cubed Sphere with parameter N consists of $6N^2 + 2$ points located on six panels, called \mathcal{P}^k , ($I \leq k \leq VI$). These six panels can be matched to the six faces of the cube $[-1, 1]^3$ embedding the sphere. This fact is the reason of the terminology *Cubed Sphere*.

Each panel \mathcal{P}^k has a square like shape represented on Fig. 2. It is supplied with a coordinate system (ξ, η) , whose coordinate lines are great circle sections. The coordinate lines $\xi = 0$, (resp. $\eta = 0$) represent the vertical (resp. horizontal) equatorial line at the center of \mathcal{P}^k . The function φ^k is the bijective application defined by

$$(6) \quad \varphi^k : (\xi, \eta) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \mapsto \mathbf{x}(\xi, \eta) \in \mathcal{P}^k.$$

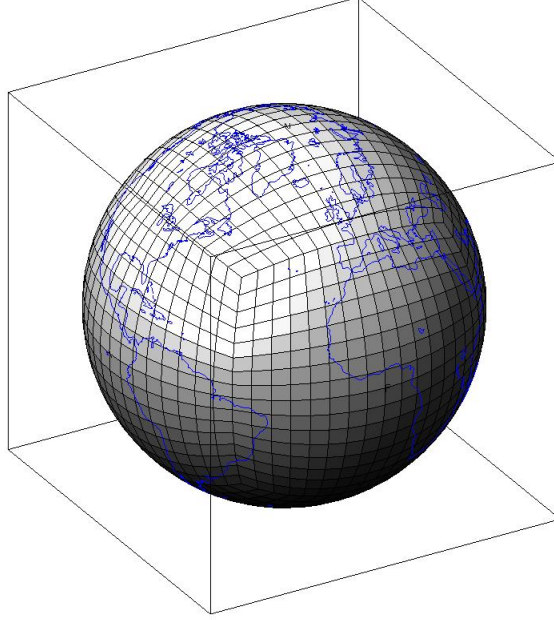


FIGURE 1. The Cubed Sphere with a grid parameter of $N = 16$. The total number of gridpoints is $6 \times N^2 + 2 = 1538$ in this case.

For example for the panel $\mathcal{P}^{(I)}$, the point $\mathbf{x}(x, y, z) = \varphi^{(I)}(\xi, \eta)$ satisfies the relations

$$(7) \quad \begin{cases} X = \tan(\xi), & Y = \tan(\eta), \\ X = \frac{y}{x}, & Y = \frac{z}{x}, \\ x^2 + y^2 + z^2 = 1. \end{cases}$$

Analog relations hold for the five other panels. On \mathcal{P}^k , the grid is defined by the $(N + 1)^2$ points

$$(8) \quad \mathbf{s}_{i,j}^k = \varphi^k(i\Delta\xi, j\Delta\eta), \quad -N/2 \leq i, j \leq N/2,$$

where $\Delta\xi = \Delta\eta = \pi/2N$.¹ Finally the function $(\xi, \eta) \in [-\frac{\pi}{4}, \frac{\pi}{4}] \mapsto G(\xi, \eta) \in \mathbb{M}_2(\mathbb{R})$ denotes the metric tensor defined by

$$(9) \quad G(\xi, \eta) = \begin{pmatrix} \frac{\partial x}{\partial \xi} \cdot \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} \\ \frac{\partial x}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \eta} \cdot \frac{\partial x}{\partial \eta} \end{pmatrix}.$$

The function $(\xi, \eta) \mapsto G(\xi, \eta)$ obviously does not depend on the panel. For, the panel \mathcal{P}^k is deduced from the panel $\mathcal{P}^{(I)}$ by a fixed rotation of the cube, and (9) is invariant under this transformation. This metric tensor will serve in the next section. We refer to [13] for other properties of the Cubed Sphere.

¹We assume for simplicity that N is an even integer.

3. A FIRST QUADRATURE RULE

3.1. Derivation of the rule. In this section we introduce our first quadrature rule called Q_a . It uses the points $\mathbf{s}_{i,j}^k$ of the Cubed Sphere as the nodes \mathbf{x}_p in (1). Let $f : \mathbb{S}^2 \mapsto \mathbb{R}$ be a regular function defined on \mathbb{S}^2 . Using the fact that the sphere \mathbb{S}^2 is decomposed into the set of the six panels \mathcal{P}^k , ($I \leq k \leq (VI)$), we have

$$(10) \quad \begin{aligned} I(f) &= \int_{\mathbb{S}^2} f(\mathbf{x}) d\sigma(\mathbf{x}) \\ &= \sum_{k=(I)}^{(VI)} \underbrace{\int_{\mathcal{P}^k} f(\mathbf{x}) d\sigma(\mathbf{x})}_{I_k(f)}. \end{aligned}$$

Using on \mathcal{P}^k the change of variables $\mathbf{x} = \varphi_k(\xi, \eta)$ gives by the chain rule [14],

$$(11) \quad I_k(f) = \int_{(\xi, \eta) \in [-\frac{\pi}{4}, \frac{\pi}{4}]^2} (f \circ \varphi_k)(\xi, \eta) \sqrt{|\det G(\xi, \eta)|} d\xi d\eta.$$

We denote $f_k = f \circ \varphi_k$. The integral $I_k(f)$ is approximated by $Q_k(f)$ defined by

$$(12) \quad Q_k(f) = \Delta\xi\Delta\eta \sum_{i,j=-N/2}^{N/2} c_{i,j} g_{i,j} f_k(\xi_i, \eta_j),$$

where the weights $g_{i,j}$ are given by

$$(13) \quad g_{i,j} = \sqrt{|\det G(\xi_i, \eta_j)|}.$$

Furthermore, the coefficients $c_{i,j}$ are defined as follows:

- The internal points (i, j) with $-N/2 < i, j < N/2$, displayed with circles on Fig 2 are counted with coefficient $c_{i,j} = 1$.
- The edge points (i, j) with $i = \pm N/2$ and $-N/2 < j < N/2$ or (i, j) with $j = \pm N/2$ and $-N/2 < i < N/2$, displayed as squares on Fig. 2, are multiplied by the coefficient $c_{i,j} = 1/2$.
- The four corner values $(i, j) \in \{(-N/2, -N/2), (-N/2, N/2), (N/2, -N/2), (N/2, N/2)\}$ displayed as pentagons on Fig. 2 are multiplied by the coefficient $c_{i,j} = 1/3$.

This convention is natural since each edge point belongs to two neighbor panels. It is therefore counted twice whence the coefficient $1/2$. Similarly, each corner point is counted three times, and this gives a coefficient $1/3$ at the panel level. The rule Q_a is finally defined by

$$(14) \quad I(f) \simeq Q_a(f) = \sum_{k=(I)}^{(VI)} Q_k(f).$$

Remark 3.1. In one dimension, the trapezoidal rule, expressed for $f(x)$ defined on $I = [0, 1]$, is given by:

$$(15) \quad \int_0^1 f(x) dx \simeq \frac{1}{N} \left(\frac{1}{2} f(0) + \sum_1^{N-1} f(i\Delta x) + \frac{1}{2} f(1) \right).$$

On the panel \mathcal{P}_k , the formula (11) can be interpreted as the tensor product of the rule (15) on $[-\pi/4, \pi/4] \times [-\pi/4, \pi/4]$ applied to the integrand in (11) except that the coefficient $1/3$ is used at the corner points, instead of $1/4$.

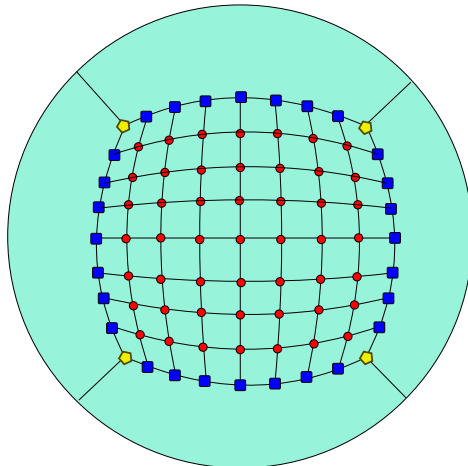


FIGURE 2. The points of a typical panel of the Cubed Sphere are classified in three categories: (i) Circles correspond to *internal* points; (ii) Squares correspond to *edge* points ; (iii) Pentagons correspond to *corner* points

3.2. Main result. Spherical Harmonics are commonly used to design quadrature rules (1). In the case where nodes and/or weights have particular spherical symmetries, one expects some specific subset of the Spherical Harmonics to be integrated exactly. Building quadrature rules on the sphere using invariance of the nodes under a subgroup of SO_3 has been used by several authors. We refer to [12] and the references therein. An important result in this direction is Sobolev's theorem, which states that a necessary and sufficient condition for a quadrature rule, invariant by a subgroup \mathcal{G} of $SO_3(\mathbb{R})$, to be exact up to degree n is to be exact for all \mathcal{G} -invariant Spherical Harmonics of degree $\leq n$, [8]. Our rules are not a direct consequence of this result and we will consider a different approach. In a first step, we identify without invoking the group of rotations of the cube, all Spherical Harmonics which are invariant under the rule Q_a .

Observe that the weights $g_{i,j}$ in (13) do not depend on the panel \mathcal{P}^k . Furthermore they satisfy the following invariance properties for $-N/2 \leq i, j \leq N/2$:

- Invariance by the rotation r of angle $\pi/2$, i.e:

$$(16) \quad g_{i,j} = g_{-j,i},$$

- Invariance by the symmetry s with respect to the first diagonal, i.e.:

$$(17) \quad g_{i,j} = g_{j,i}.$$

Combining these two transformations, it is easily seen that the coefficients $g_{i,j}$ also are symmetric with respect to the second diagonal and with respect to the coordinate lines $i = 0$ and $j = 0$, respectively. Otherwise stated, for $-N/2 \leq i, j \leq N/2$:

$$(18) \quad g_{i,j} = g_{-j,-i} = g_{-i,j} = g_{i,-j} = g_{-i,-j}.$$

For example for $N = 8$, the coefficients $g_{i,j}$ can be arranged as shown in Table 1, where the symmetry properties (16), (17) and (18) are represented with letters. For a Cubed Sphere with parameter N , the number of independent weights is given by the integer $q_N = (N + 2)(N + 4)/8$. These weights correspond to the indices $0 \leq j \leq i \leq N/2$. They are displayed by boldface letters in Table 1.

$$(19) \quad \begin{array}{ccccccccc} \mathbf{a} & \mathbf{b} & \mathbf{d} & \mathbf{g} & \mathbf{k} & \mathbf{g} & \mathbf{d} & \mathbf{b} & \mathbf{a} \\ \mathbf{b} & \mathbf{c} & \mathbf{e} & \mathbf{h} & \mathbf{l} & \mathbf{h} & \mathbf{e} & \mathbf{c} & \mathbf{b} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{i} & \mathbf{m} & \mathbf{i} & \mathbf{f} & \mathbf{e} & \mathbf{d} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} & \mathbf{j} & \mathbf{n} & \mathbf{j} & \mathbf{i} & \mathbf{h} & \mathbf{g} \\ \mathbf{k} & \mathbf{l} & \mathbf{m} & \mathbf{n} & \mathbf{o} & \mathbf{n} & \mathbf{m} & \mathbf{l} & \mathbf{k} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} & \mathbf{j} & \mathbf{n} & \mathbf{j} & \mathbf{i} & \mathbf{h} & \mathbf{g} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{i} & \mathbf{m} & \mathbf{i} & \mathbf{f} & \mathbf{e} & \mathbf{d} \\ \mathbf{b} & \mathbf{c} & \mathbf{e} & \mathbf{h} & \mathbf{l} & \mathbf{h} & \mathbf{e} & \mathbf{c} & \mathbf{b} \\ \mathbf{a} & \mathbf{b} & \mathbf{d} & \mathbf{g} & \mathbf{k} & \mathbf{g} & \mathbf{d} & \mathbf{b} & \mathbf{a} \end{array}$$

TABLE 1. Parameters of the weights in a typical panel \mathcal{P}^k in the Cubed Sphere with parameter $N = 8$. The independent parameters are displayed in boldface. The number of independent parameters is $q_N = (N+2)(N+4)/8$. There is 15 independent parameters in this particular case.

| Cubed Sphere parameter N | 4 | 8 | 16 | 32 | 64 |
|--|----|-----|------|------|-------|
| Number of quadrature nodes in a panel $(= (N + 1)^2)$ | 25 | 81 | 289 | 1089 | 4225 |
| Number of quad. nodes on $\mathbb{S}^2(= 6N^2 + 2)$ | 98 | 386 | 1538 | 6146 | 24578 |
| Number of independent weights $q_N = (N + 2)(N + 4)/8$ | 6 | 15 | 45 | 153 | 561 |

TABLE 2. Number of quadrature nodes in function of the parameter N of the Cubed Sphere.

Table 2 displays typical values of the integer q_N and of the number of quadrature nodes for a series of values of N . Under the assumptions (16-17) on the weights, the following statement holds:

Proposition 3.1. *The quadrature rule (14) associated with the nodes of the Cubed Sphere $\mathbf{s}_{i,j}^k$ and with a set of weights $g_{i,j}^k$ satisfying (16-17) is exact for all Spherical Harmonics $Y_n^m(\mathbf{x})$ such that*

- n odd,
- n even and m not a multiple of 4.

Proof. The Spherical Harmonic Y_n^m is given by (see (2)):

$$(20) \quad Y_n^m(\mathbf{x}) = (-1)^m N_n^{|m|} P_n^{|m|}(\sin \theta) e^{im\lambda}, \quad n \geq 0, \quad -n \leq m \leq n$$

The exact integration property stated of Prop. 3.1 is based on the fact that non zero values of $Y_n^m(\mathbf{x})$ have their exact opposite on another part of the Cubed Sphere. This property is due to the symmetries (16, 17) of the weights $g_{i,j}$, and to the following relations:

$$(21) \quad Y_n^m(\lambda + \pi, \theta) = (-1)^m \times Y_n^m(\lambda, \theta),$$

$$(22) \quad Y_n^m(\lambda, -\theta) = (-1)^{n+m} \times Y_n^m(\lambda, \theta),$$

$$(23) \quad Y_n^m\left(\lambda + p\frac{\pi}{m}, \theta\right) = (-1)^p \times Y_n^m(\lambda, \theta).$$

with $n > 0$, $m \in \{-n, \dots, n\} \setminus \{0\}$, $p \in \mathbb{Z}$, and $(\lambda, \theta) \in [-\pi, \pi \times [-\frac{\pi}{2}; \frac{\pi}{2}]]$. Note that (21) is a particular case of (23).

We consider the four following cases. In each case, the approximate value $Q_a(Y_n^m)$ is compared to the exact integral $I(Y_n^m)$ in (4).

- **Case n odd.** In this case, if m is even, then the property (22) leads to

$$(24) \quad Q_{(I)}(Y_n^m) = Q_{(II)}(Y_n^m) = Q_{(III)}(Y_n^m) = Q_{(IV)}(Y_n^m) = 0,$$

due to opposite values of $Y_n^m(\lambda, \theta)$ for opposite values of θ , and to the properties $g_{i,j} = g_{i,-j}$ of the weights. Moreover, due to the relation (22) and using the fact that the weights $g_{i,j}$ are identical on the six panels, we have

$$(25) \quad Q_{(V)}(Y_n^m) = -Q_{(VI)}(Y_n^m).$$

This gives $Q_a(Y_n^m) = 0$.

If m is odd, then the relation (21) leads to the relations

$$(26) \quad Q_{(I)}(Y_n^m) = -Q_{(III)}(Y_n^m), \quad Q_{(II)}(Y_n^m) = -Q_{(IV)}(Y_n^m).$$

We have used the relation $g_{i,j} = g_{-i,j}$ and the fact that the weights $g_{i,j}$ are identical on each panel. Furthermore due to (21) and $g_{i,j} = g_{-i,-j}$ (symmetry of the weights with respect to the center of each panel) we find

$$(27) \quad Q_{(V)}(Y_n^m) = Q_{(VI)}(Y_n^m) = 0.$$

Summing up over the six panels yields $Q_a(Y_n^m) = 0$ (exact value).

- **Case n even and m odd.** In this case, $n + m$ is odd, and then (22) shows that $Q_k(Y_n^m) = 0$ for the panels $k = (I), (II), (III)$ and (IV) . This is due to the fact that the weights $g_{i,j}$ have the property $g_{i,j} = g_{i,-j}$. For the panels $k = (V)$ and $k = (VI)$, we have $Q_k(Y_n^m) = 0$ due to (21) and $g_{i,j} = g_{-i,-j}$. Therefore $Q_a(Y_n^m) = 0$.
- **Case n even, m even and $m \not\equiv 0[4]$.** In this case, $Q_k(Y_n^m) = 0$ for the panels $k = (V)$ and $k = (VI)$. This is due to (23) (take $p = \frac{m}{2}$ in this formula) and to the fact that the weights $g_{i,j}$ satisfy (16). Moreover, again using (23) with $p = \frac{m}{2}$, one has $Q_{(I)}(Y_n^m) = -Q_{(II)}(Y_n^m)$ and $Q_{(III)}(Y_n^m) = -Q_{(IV)}(Y_n^m)$. Note in addition that $Q_{(I)}(Y_n^m) = Q_{(III)}(Y_n^m)$. This yields $Q_a(Y_n^m) = 0$.
- **Case n even, m even and $m \equiv 0[4]$.** Note that Y_0^0 belongs to this case. In this case,

$$(28) \quad Q_{(I)}(Y_n^m) = Q_{(II)}(Y_n^m) = Q_{(III)}(Y_n^m) = Q_{(IV)}(Y_n^m).$$

This is due to the property (23). In addition, due to (22),

$$(29) \quad Q_{(V)}(Y_n^m) = Q_{(VI)}(Y_n^m),$$

In this case, a possible cancellation within the terms of the approximate integral $Q_a(Y_n^m)$ could only occur between the contribution of panels (I), (II), (III) and (IV) on the one hand and the contribution of panels (V) and (VI) on the other hand. Such a cancellation does not occur in general. This explains why $Q_a(Y_n^m)$ is not exact in this case. \square

Remark 3.2. The former three cases describe all together a proportion of (asymptotically) 7/8 of all Spherical Harmonics. The latter case corresponds to the remaining 1/8 Spherical Harmonics.

Remark 3.3. One may wonder if the Spherical Harmonics of this latter category (the remaining 1/8 Spherical Harmonics) are effectively inexactly integrated by the rule Q_a for all integers n even and $m \equiv 0[4]$. This question is unanswered for now. In fact, it happens that the function Y_2^0 ($(n, m) = (2, 0)$) is exactly integrated by the rule Q_a . Prop. 3.1 therefore only describes a sufficient condition for the exactness of the rule Q_a .

3.3. Numerical results. In this section we show numerical results obtained with the rule Q_a . The functions f_1, f_2, f_3 and f_4 that are used to test numerical quadrature formula are the following, [7, 3].

$$(30) \quad \left\{ \begin{array}{l} f_1(x, y, z) = 1 + x + y^2 + x^2y + x^4 + y^5 + x^2y^2z^2, \\ f_2(x, y, z) = 0.75e^{-(9x-2)^2/4-(9y-2)^2/4-(9z-2)^2/4} \\ \quad + 0.75e^{-(9x+1)^2/49-(9y+1)/10-(9z+1)/10} \\ \quad + 0.5e^{-(9x-7)^2/4-(9y-3)^3/4-(9z-5)^2/4} \\ \quad - 0.2e^{-(9x-4)^2-(9y-7)^2-(9z-5)^2}, \\ f_3(x, y, z) = (1 + \tanh(-9x - 9y + 9z))/9, \\ f_4(x, y, z) = (1 + \text{sign}(-9x - 9y + 9z))/9. \end{array} \right.$$

The exact values are given by:

$$(31) \quad \left\{ \begin{array}{l} \int_{\mathbb{S}^2} f_1 = \frac{216\pi}{35}, \\ \int_{\mathbb{S}^2} f_2 = 6.6961822200736179523\dots, \\ \int_{\mathbb{S}^2} f_3 = \frac{4\pi}{9}, \\ \int_{\mathbb{S}^2} f_4 = \frac{4\pi}{9}. \end{array} \right.$$

The numerical results for several values of N (the total numbers of points for the discretisation of \mathbb{S}^2 are precised) are displayed in Table 3. According to [7], we display the accuracy for the four test functions in (30), by retaining the worst case among 1000 random solid rotations operated on the argument (x, y, z) of these functions. On Fig. 3 the convergence rate is numerically evaluated as fourth order. The accuracy is excellent, compared to results reported elsewhere. For example, for a Cubed Sphere parameter $N \leq 12$, which corresponds to a number of quadrature nodes on the sphere of $6N^2 + 2 \leq 1000$, the accuracy is of the same order as the one obtained with optimally selected quadrature nodes [7].

Remark 3.4. A proof of the fourth order accuracy of the rule Q_a is not straightforward. In particular, it cannot be simply deduced from the accuracy of the trapezoidal rule in one dimension, since the rule Q_a is not exactly a tensor product rule on each panel, see Remark 3.1.

4. A SECOND QUADRATURE RULE

4.1. General derivation. As shown in Prop. 3.1, the quadrature rule Q_a in (14) is exact for a proportion of 7/8 of all Spherical Harmonics. However, it is not exact for the remaining 1/8 Spherical Harmonics. It is numerically observed to be fourth-order accurate, (see Fig. 3 and Remark 3.4). In this section we suggest a second quadrature rule, called Q_b , whose nodes again are the points of the Cubed Sphere. It is designed as a perturbation of Q_a . Similarly to Q_a , the rule Q_b is expressed as a sum of contributions of each panel \mathcal{P}^k

$$(32) \quad I(f) \simeq Q_b(f) = \sum_{k=(I)}^{(VI)} Q_{k,\varepsilon}(f),$$

| N | Number of nodes | $ I(f_1) - Q_a(f_1) $ | $ I(f_2) - Q_a(f_2) $ | $ I(f_3) - Q_a(f_3) $ | $ I(f_4) - Q_a(f_4) $ |
|-----|-----------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 4 | 98 | 1.623(-2) | 1.721(-2) | 1.114(-3) | 1.114(-3) |
| 6 | 218 | 2.900(-3) | 2.638(-3) | 2.170(-4) | 2.170(-4) |
| 8 | 386 | 9.849(-4) | 8.320(-4) | 6.829(-5) | 6.829(-5) |
| 10 | 602 | 4.008(-4) | 2.157(-4) | 2.790(-5) | 2.790(-5) |
| 12 | 866 | 1.900(-4) | 7.791(-5) | 1.344(-5) | 1.344(-5) |
| 14 | 1178 | 1.017(-4) | 3.810(-5) | 7.247(-6) | 7.247(-6) |
| 16 | 1538 | 5.828(-5) | 2.080(-5) | 4.245(-6) | 4.245(-6) |
| 32 | 6146 | 3.747(-6) | 1.339(-6) | 2.650(-7) | 2.650(-7) |
| 64 | 24578 | 2.258(-7) | 8.089(-8) | 1.656(-8) | 1.656(-8) |

TABLE 3. Accuracy of the quadrature rule Q_a for the test functions f_1 , f_2 , f_3 and f_4 in (30). The result corresponds to the worst case among 1000 randomly selected solid rotations operated on the argument (x, y, z) of the functions f_1 , f_2 , f_3 and f_4 .

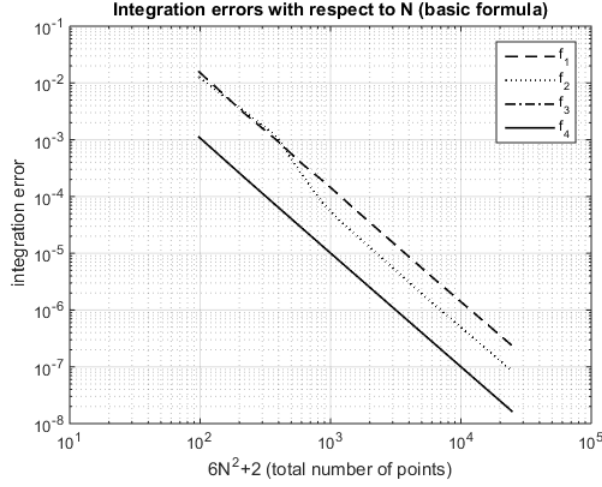


FIGURE 3. Quadrature error for the test functions f_1 , f_2 , f_3 and f_4 obtained with rule Q_a in function of the total number $6N^2 + 2$ nodes. Fourth-order accuracy is observed in the four cases. The worst case among 1000 randomly selected solid rotations on the argument (x, y, z) of the functions is displayed.

where the contribution of the panel \mathcal{P}^k is

$$\begin{aligned}
 (33) \quad Q_{k,\varepsilon}(f) &= \Delta\xi\Delta\eta \sum_{i,j=-N/2}^{N/2} c_{i,j}(g_{i,j} + \varepsilon_{i,j})f_k(\xi_i, \eta_j), \\
 &= Q_k(f) + \Delta\xi\Delta\eta \sum_{i,j=-N/2}^{N/2} c_{i,j}\varepsilon_{i,j}f_k(\xi_i, \eta_j).
 \end{aligned}$$

Therefore the quadrature rule $Q_b(f)$ is sought in the form of a perturbation of Q_a as:

$$(34) \quad Q_b(f) = Q_a(f) + \Delta\xi\Delta\eta \sum_{k=(I)}^{(VI)} \sum_{i,j=-N/2}^{N/2} c_{i,j}\varepsilon_{i,j}f_k(\xi_i,\eta_j).$$

If the symmetries (16) and (17) are imposed to the weights $\varepsilon_{i,j}$, then by Prop. 3.1, the rule Q_b is exact for the same Spherical Harmonics than the rule Q_a . Let us call \mathcal{I} the set of indexes (i,j) such that $0 \leq j \leq i \leq N/2$. This set is represented in Table 1 in boldface letters. Recall that $|\mathcal{I}| = q_N = (N+2)(N+4)/8$. The term $Q_k(f)$ in (12) can be expressed as

$$(35) \quad Q_k(f) = \Delta\xi\Delta\eta \sum_{(i,j) \in \mathcal{I}} g_{i,j} \hat{f}_{i,j}^k.$$

where $\hat{f}_{i,j}^k$ denotes, for $(i,j) \in \mathcal{I}$ and $(I) \leq k \leq (VI)$

$$(36) \quad \hat{f}_{i,j}^k = c_{i,j} \sum_{(i',j')/g_{i',j'}=g_{i,j}} f_k(\xi_{i'},\eta_{j'}).$$

With this notation, (33) can be expressed as

$$(37) \quad Q_{k,\varepsilon}(f) = \Delta\xi\Delta\eta \sum_{(i,j) \in \mathcal{I}} (g_{i,j} + \varepsilon_{i,j}) \hat{f}_{i,j}^k.$$

4.2. Determining $\varepsilon_{i,j}$. The question is now to evaluate suitable values of the q_N unknowns $(\varepsilon_{i,j})_{(i,j) \in \mathcal{I}}$. We denote by $\psi_1 = Y_0^0$, $\psi_2 = Y_2^0$, $\psi_3 = Y_4^0$, $\psi_4 = Y_4^4$, \dots , the sequence of the Spherical Harmonics Y_n^m with n even and $m \geq 0$, $m = 0[4]$. This is the set of $1/8$ of all Spherical Harmonics not integrated exactly by the rule Q_a . The set of equations determining the values $\varepsilon_{i,j}$ is defined as

$$(38) \quad Q_b(\psi_l) = I(\psi_l), \quad 1 \leq l \leq p_N.$$

The integer p_N , to be determined, is the number of Spherical Harmonics of the preceding form taken in account in (38). Using (34), equation (38) is recast in the form:

$$(39) \quad \sum_{(i,j) \in \mathcal{I}} \left(\Delta\xi\Delta\eta \sum_{k=(I)}^{(VI)} (\hat{\psi}_l)_{i,j}^k \right) \varepsilon_{i,j} = I(\psi_l) - Q_a(\psi_l), \quad 1 \leq l \leq p_N.$$

Lemma 4.1. For each $l \leq 0$, $(I) \leq k \leq (VI)$ and $(i,j) \in \mathcal{I}$,

$$(40) \quad I(\psi_l) - Q_a(\psi_l) \in \mathbb{R}$$

and

$$(41) \quad (\hat{\psi}_l)_{i,j}^k \in \mathbb{R}.$$

Proof. Let $\psi_l = Y_n^m$ with $n = 2n'$, $m = 4m'$, $0 \leq m \leq n$. The imaginary part of $Y_n^m(\mathbf{x})$ is called $H_n^m(\mathbf{x})$. According to (2), $H_n^m(\mathbf{x})$ is expressed as

$$(42) \quad H_n^m(\lambda, \theta) = N_n^m P_n^m(\sin \theta) \sin(m\lambda).$$

The imparity with respect to the longitude variable λ gives

$$(43) \quad H_n^m(-\lambda, \theta) = -H_n^m(\lambda, \theta),$$

and to the relation $g_{-i,j} = g_{i,j}$ imposes that the discrete integral of H_n^m on the panel $\mathcal{P}^{(k)}$ with $k = (I)$ is

$$(44) \quad Q_k(H_n^m) = 0.$$

Furthermore the relation (23) with $p = \pm 2m'$, $m = 4m'$ gives the relation (44) on the two panels $k = (II)$ and $k = (IV)$. Next, (21) yields (44) for the panel $k = (III)$. Consider now the panel $k = (V)$. The value $Q_{(V)}(H_m^n)$ can be decomposed in four terms each of them corresponding to an angular sector, in the form

$$\begin{aligned}
 Q_{(V)}(H_m^n) &= \sum_{i,j=-N/2}^{N/2} c_{i,j} g_{i,j}(H_m^n)_{(V)}(\xi_i, \eta_j) \\
 (45) \quad &= \sum_{(i,j), j < -|i|} c_{i,j} g_{i,j}(H_m^n)_{(V)}(\xi_i, \eta_j) + \sum_{(i,j), i \geq |j|} c_{i,j} g_{i,j}(H_m^n)_{(V)}(\xi_i, \eta_j) \\
 &+ \sum_{(i,j), j \geq |i|} c_{i,j} g_{i,j}(H_m^n)_{(V)}(\xi_i, \eta_j) + \sum_{(i,j), i < -|j|} c_{i,j} g_{i,j}(H_m^n)_{(V)}(\xi_i, \eta_j)
 \end{aligned}$$

The first term vanishes, due to (43) and the relation $g_{-i,j} = g_{i,j}$. The relations (22) and (23) imply that the three other terms also vanish. The argument is similar for the panel $k = (VI)$. An analog argument shows that the imaginary part of $\hat{\psi}_{l,i,j}^k \in \mathbb{R}$ for all $l \geq 0$, $(i,j) \in \mathcal{I}$ and $(I) \leq k \leq (VI)$ also vanish. Therefore $Q_a(\psi_l) \in \mathbb{R}$ and since $I(\psi_l) \in \mathbb{R}$, (40) holds. \square

The set of relations (39) forms a linear system

$$(46) \quad A\boldsymbol{\varepsilon} = \mathbf{b},$$

Due to Lemma 4.1, the matrix A and the right-hand side \mathbf{b} are real. The unknown vector is

$$(47) \quad \boldsymbol{\varepsilon} = (\varepsilon_{i,j})_{(i,j) \in \mathcal{I}} \in \mathbb{R}^{q_N}.$$

The integer p_N is the number of Spherical Harmonics incorrectly integrated by the rule Q_a , which are taken in account in (46). A relevant value of this integer has been determined numerically as follows. On Fig. 4 the error of the rule Q_b is represented in function of the number of lines p_N in (39) for Cubed Sphere parameters $N = 4$, $N = 8$ and $N = 16$, respectively. If we withdraw several pikes denoting a superconvergence effect, the curves can be considered as describing a monotonic increasing behaviour. In view of these results, we have selected the parameter value $p_N = N^2/4$ as a compromise.

With this choice, the matrix A in (46) is rectangular with

$$(48) \quad A \in \mathbb{M}_{p_N, q_N}(\mathbb{R}), \quad p_N = N^2/4, \quad q_N = (N+2)(N+4)/8.$$

For $N \geq 8$, $p_N = N^2/4 > q_N \simeq N^2/8$, thus the system (46) is in general overdetermined. We solve (46) using the pseudo-inverse of Moore-Penrose [6, Chap. 3, pp. 57sqj]. We have used the *matlab* routine *pinv*.

Remark 4.2. Since the pseudo-inverse is used to solve the rectangular system (46), the identity (38) for the Spherical Harmonic ψ_l , $1 \leq l \leq p_N$ is not exactly satisfied, but only approximately. Therefore, strictly speaking, the rule Q_b does not integrate all Spherical Harmonics up to a certain order. This is in contrast to the usual setting stated in [12], [8], [7]. However in practice, the accuracy of the rule Q_b is remarkably good as will be seen in Section 4.5.

Remark 4.3. As mentionned in Remark 3.3, the function Y_2^0 is exactly integrated by the rule Q_a . Therefore this function could be withdrawn from the series of functions ψ_l in (38). It corresponds to a null line in (46). However keeping it does not prevent the generalized inverse to be used to solve the linear system.

Remark 4.4. An important question is whether or not the matrix A in (46) is full rank. This problem is open for the moment. Preliminary numerical experiments suggest that the matrix A may be not full rank for small values of the parameter N , which means for a coarse Cubed Sphere ($N \leq 12$), and

becomes full rank for larger values of N . This question requires further theoretical and numerical investigation. In practice however, we never observed so far numerical ill-conditioning effects of the matrix A when solving the linear system (46).

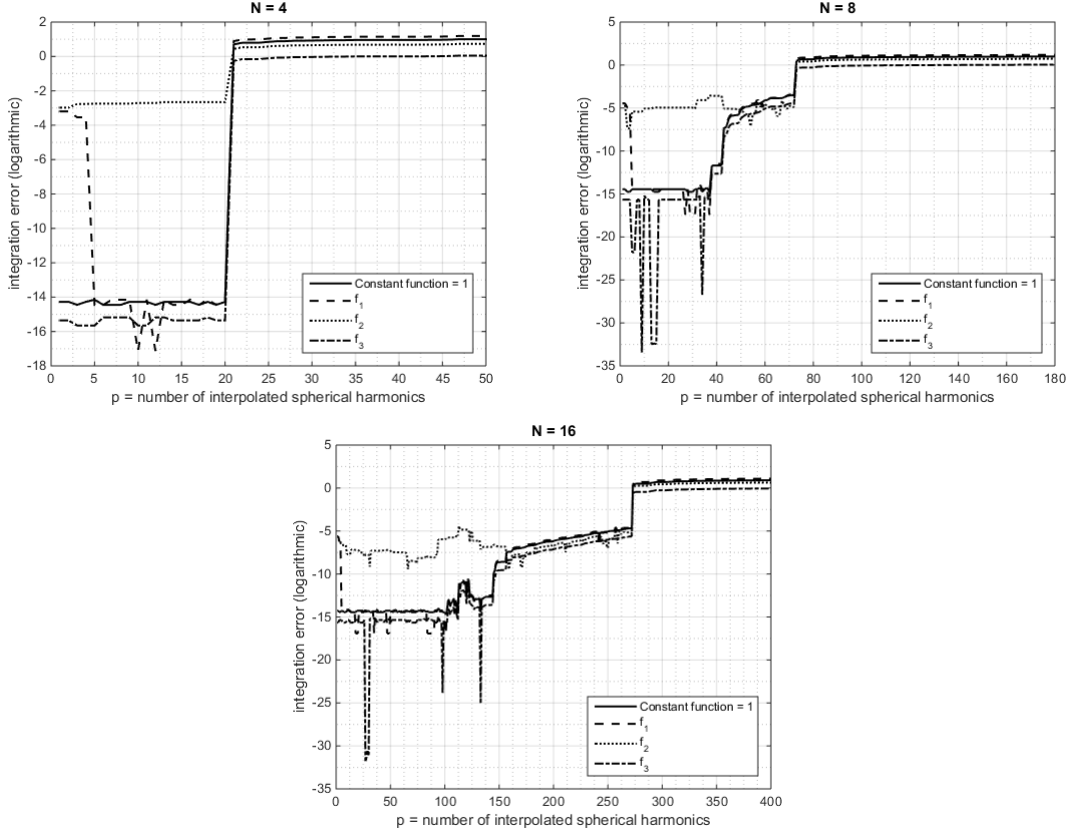


FIGURE 4. Quadrature error in Log scale for the test functions f_1 , f_2 and f_3 in function of the number p_N of the Spherical Harmonics taken in account in (38) defining the rule Q_b .

4.3. Magnitude of the parameters $\varepsilon_{i,j}$. In Fig. 5, we report the magnitude of the coefficients $\varepsilon_{i,j} \in \mathbb{R}$, solution of the equation (46). They are observed to be very small, with respect to the metric tensor terms $g_{i,j}$, which belongs to $[0.5, 1]$. In fact, as shown in Fig. 5, the maximum of $|\varepsilon_{i,j}|$ decreases according to $O\left(\frac{1}{N^2}\right)$ when N increases. It is why the rule Q_b can be considered as a perturbation of the rule Q_a .

4.4. Efficiency of the rule Q_b . In this section, we numerically evaluate the efficiency of the rule Q_b on several Spherical Harmonics. Since the rule Q_b possesses the 7/8 property, we only look at how Q_b behaves when approximately integrating a Spherical harmonics $Y_{n_0}^{m_0}$ with n_0 even and m_0 , a multiple of 4, are fixed integers, (see Prop. 3.1).

We observe on Fig. 6 what is expected: when the Spherical Harmonic $Y_{n_0}^{m_0}$ is not taken in account in the matrix A in (46), formulas Q_a and Q_b give indiscernable results. In the contrary, from the

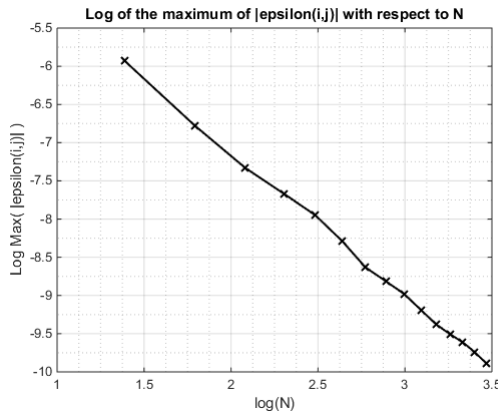


FIGURE 5. Maximum of the parameters $|\varepsilon_{i,j}|$ in (46) with respect to N , the parameter of the Cubed Sphere. The magnitude of $(\varepsilon_{i,j})_{(i,j) \in \mathcal{I}}$ behaves numerically as $1/N^2$.

moment that $Y_{n_0}^{m_0}$ is taken in account in the matrix A , the rule Q_b calculates $I(Y_{n_0}^{m_0})$ up to machine accuracy. Therefore, the fact that the linear system (46) is only approximately satisfied by the least square solution ε does not prevent to reach the machine accuracy in computing $I(Y_{n_0}^{m_0})$, and to keep preserving this accuracy when N increases.

Note however that the transition between the accuracy of the rule Q_a and the rule Q_b is less abrupt for Y_{40}^{16} than for Y_{16}^8 and Y_{24}^4 . This transition is non monotonic and indeed, for a Cubed Sphere parameter $N = 16$, the error with $Q_b(Y_{24}^{16})$ is slightly greater than with $Q_a(Y_{24}^{16})$. This is probably due to the least square method, which solves "in the mean" (38) and gives a value of ε with a lack of accuracy for Y_{40}^{16} for the particular value $N = 16$. Note that this phenomenon is unusual among all the Spherical Harmonics that have been tested. It does not impact the efficiency of the rule Q_b , as will be seen in Section 4.5. In the contrary, we also observe the opposite behaviour of rule Q_b compared to rule Q_a : for a Spherical Harmonic $Y_{n_0}^{m_0}$ not taken in account in (38), a more accurate value with $Q_b(Y_{n_0}^{m_0})$ than with $Q_a(Y_{n_0}^{m_0})$ can be obtained.

Overall, it is observed that the rules Q_a and Q_b give similar results for the Spherical Harmonics not taken in account in the interpolation in (39), (or equivalently, in the matrix A in (46)). Furthermore, the rule Q_b permits to integrate up to machine accuracy the Spherical Harmonics taken in account in (39). The greater is N , the more Spherical Harmonics can be taken in account in the matrix A . The former observation (for Spherical Harmonics not taken in account) is important in practice, since it indicates that the rule Q_b preserves the good convergence properties of the rule Q_a when N increases for all the functions Y_m^n with $n \geq n_0$, a fixed value.

4.5. Numerical results. In Table 4, numerical results obtained with the rule Q_b (32) applied to the set of test functions (30) are reported. As in Table 3, the worst case among 1000 randomly selected solid rotations is retained. According to results displayed in Table 4 and Fig. 7, the quadrature rule Q_b is much more accurate than Q_a . Indeed, for a very coarse Cubed Sphere with parameter $N \simeq 6$, the machine accuracy is reached ($\simeq 10^{-15}$) for the functions f_1 , f_3 and f_4 . The remarkable accuracy for f_1 , f_3 and f_4 can be explained as follows. Since f_1 is a polynomial, only the first Spherical Harmonics are useful, and therefore when the parameter p_N is such that $p_N \geq 9$, all Spherical Harmonics entering the decomposition of f_1 are integrated exactly by the rule Q_b . The functions f_3 and f_4 illustrate the accuracy power of the rule Q_b , which is due to the symmetry

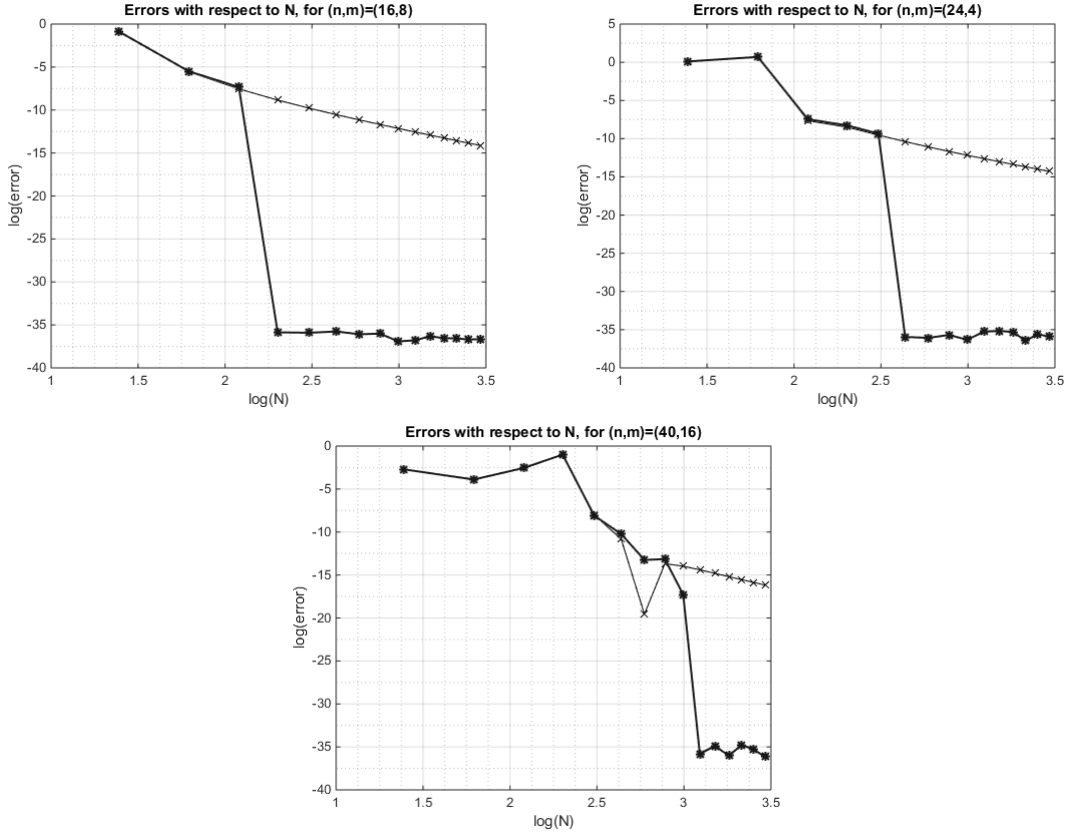


FIGURE 6. Integration errors in function of N , the parameter of the Cubed Sphere, for the three Spherical harmonics Y_{16}^8 , Y_{24}^4 and Y_{40}^{16} . The LogLog scale is used. The thin curve with \times symbols corresponds to the rule Q_a . The bold curve with $*$ symbols corresponds to the rule Q_b .

properties (16, 17, 18) combined with the good integration of Y_0^0 . Indeed, as mentioned in [3], the Spherical Harmonics entering the decomposition of f_3 are limited to Y_0^0 and Y_n^m with n odd. Thus it is sufficient for a rule to integrate exactly Y_0^0 since the Spherical Harmonics with an odd degree are exactly integrated thanks to the symmetries of the weights (7/8 property). The same observation holds for f_4 . Consequently, from the moment that $p_N \geq 1$, the rule Q_b already gives much better results than the rule Q_a simply because Y_0^0 is the first Spherical Harmonic taken in account in the matrix A .

In the case of f_2 , the improvement is less drastic. This is due to the fact that there are Spherical Harmonics entering the decomposition in Spherical Harmonics of f_2 , which are not taken in account in the matrix A in (46). Therefore the error decreases progressively. However the decreasing rate is remarkably fast: the slope shown in Fig. 7 is numerically evaluated as -13 , instead of -4 for the rule Q_a (see Fig. 3). Overall, as observed in Table 4, the level of accuracy obtained with the rule Q_b are excellent when compared with quadrature rules using optimally selected set of nodes, [7].

| N | Number of nodes | p_N | $ I(f_1) - Q_b(f_1) $ | $ I(f_2) - Q_b(f_2) $ | $ I(f_3) - Q_b(f_3) $ | $ I(f_4) - Q_b(f_4) $ |
|-----|-----------------|-------|-----------------------|-----------------------|-----------------------|-----------------------|
| 4 | 98 | 4 | 1.646(-4) | 1.376(-2) | 6.661(-16) | 6.661(-16) |
| 6 | 218 | 9 | 7.105(-15) | 2.664(-3) | 6.661(-16) | 4.441(-16) |
| 8 | 386 | 16 | 3.553(-15) | 8.085(-4) | 4.441(-16) | 2.220(-16) |
| 10 | 602 | 25 | 1.066(-14) | 1.067(-4) | 6.661(-16) | 4.441(-16) |
| 12 | 866 | 36 | 3.553(-15) | 1.270(-5) | 2.220(-16) | 4.441(-16) |
| 14 | 1178 | 49 | 3.553(-15) | 1.272(-6) | 4.441(-16) | 2.220(-16) |
| 16 | 1538 | 64 | 1.066(-14) | 8.212(-8) | 6.661(-16) | 4.441(-16) |
| 32 | 6146 | 256 | 7.105(-15) | 3.610(-13) | 6.661(-16) | 4.441(-16) |
| 64 | 24578 | 1024 | 7.105(-15) | 2.000(-15) | 4.441(-16) | 6.661(-16) |

TABLE 4. Accuracy of the quadrature rule Q_b for the test functions f_1 , f_2 , f_3 and f_4 in (30). The machine accuracy is reached from a number of quadrature nodes as low as 218 for the functions f_1 , f_3 and f_4 . The results correspond to the worst case among 1000 randomly selected solid rotations applied to the arguments (x, y, z) of the functions f_1 , f_2 , f_3 and f_4 .

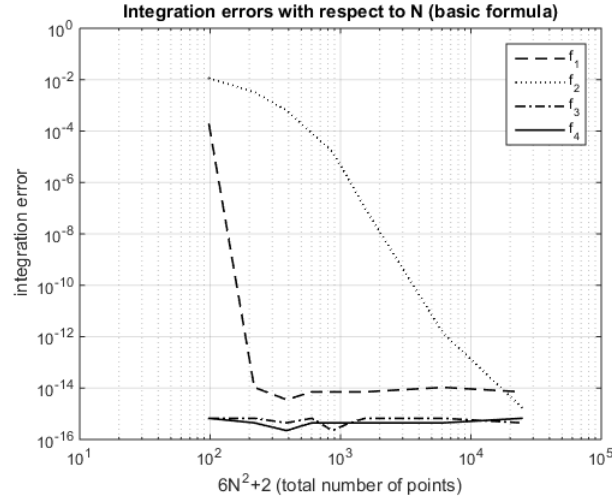


FIGURE 7. Quadrature error for the test functions f_1 , f_2 , f_3 and f_4 obtained with the rule Q_b in function of the number $6N^2 + 2$ of quadrature nodes on the sphere.

5. CONCLUSION

A new approach to numerical quadrature on the sphere is introduced. The quadrature nodes are the points of the Cubed Sphere. Two rules, called Q_a and Q_b , are introduced. Both of them satisfy the 7/8 property which states that 7/8 of all Spherical Harmonics are integrated exactly. The first rule Q_a is an analog of the trapezoidal rule on the sphere. The second rule Q_b , is an enhancement of the rule Q_a , and provides a remarkable accuracy on a set of test functions [7, 3]. The rule Q_b is efficient not only when applied to regular functions, but also when applied to irregular functions as

the function f_4 in (30), which uses the *sign* function in its definition. Further studies are ongoing to better understand the mathematical properties of the rules Q_a and Q_b .

REFERENCES

- [1] C. Ahrens and G. Beylkin. Rotationally invariant quadratures for the sphere. *Proceedings of the Royal Society of London A*, 465(2110):3103–3125, 2009.
- [2] L. Bao, R. D. Nair, and H. M. Tufo. A mass and momentum flux-form high-order discontinuous galerkin shallow water model on the cubed-sphere. *J. Comput. Phys.*, to appear, 2014.
- [3] C. H. L. Beentjes. Quadrature on a spherical surface. Working note available on the website <http://people.maths.ox.ac.uk/beentjes/Essays/>, 2015.
- [4] J.-P. Croisille. Hermitian compact interpolation on the Cubed-Sphere grid. *Jour. Sci. Comp.*, 57,1:193–212, 2013.
- [5] J.-P. Croisille. Hermitian approximation of the spherical divergence on the Cubed-Sphere. *Jour. Comp. App. Math.*, 280:188–201, 2015.
- [6] P. Deuffhard and A. Hohmann. *Numerical Analysis in Modern Scientific Computing. An Introduction*. Number 43 in TAM. Springer-Verlag, 2cd edition, 2003.
- [7] B. Fornberg and J.M. Martel. On spherical harmonics based numerical quadrature over the surface of a sphere. *Advances in Computational Mathematics*, 40(5-6):1169–1184, 2014.
- [8] K. Hesse, I.H. Sloan, and R. S. Womersley. Chap. 40: Numerical Integration on the Sphere. In W. Freeden, M. Z. Nashed, and T. Sonar, editors, *Handbook of Geomathematics*. Springer, 2010.
- [9] M. N. Jones. *Spherical Harmonics and Tensors for classical field theory*. Research Studies Press, 1985.
- [10] J. Keiner and D. Potts. Fast evaluation of quadrature formulae on the sphere. *Mathematics of computation*, 77(261):397–419, 2008.
- [11] P. Lauritzen, R. D. Nair, and P. A. Ullrich. A conservative semi-Lagrangian multi-tracer transport scheme (CSLAM) on the cubed sphere grid. *J. Comput. Phys.*, 229:1401–1424, 2010.
- [12] A.D. McLaren. Optimal numerical integration on a sphere. *Mathematics of Computation*, 17(84):361–383, 1963.
- [13] C. Ronchi, R. Iacono, and P. S. Paolucci. The Cubed Sphere: A new method for the solution of partial differential equations in spherical geometry. *J. Comput. Phys.*, 124:93–114, 1996.
- [14] J. G. Simmonds. *A Brief on Tensor Analysis*. Undergraduate Texts in Math. Springer, 2cd edition, 1994.
- [15] I.H. Sloan and R. S. Womersley. Extremal systems of points and numerical integration on the sphere. *Advances in Computational Mathematics*, 21(1-2):107–125, 2004.
- [16] P. A. Ullrich, C. Jablonowski, and B. van Leer. High order finite-volume methods for the shallow-water equations on the sphere. *J. Comput. Phys.*, 229:6104–6134, 2010.

†UNIVERSITÉ DE LORRAINE, DÉPARTEMENT DE MATHÉMATIQUES, F-57045 METZ, FRANCE, ‡C.N.R.S., INSTITUT ELIE CARTAN DE LORRAINE, UMR 7502, F-57045 METZ, FRANCE
E-mail address: brice.portelenelle@ensimag.grenoble-inp.fr
E-mail address: jean-pierre.croisille@univ-lorraine.fr