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A new path to the non blow-up of incompressible flows

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Abstract

One of the most challenging questions in fluid dynamics is whether the three-dimensional (3D) incompressible Navier-Stokes, 3D Euler and two-dimensional Quasi-Geostrophic (2D QG) equations can develop a finite-time singularity from smooth initial data. Recently, from a numerical point of view, Luo & Hou presented a class of potentially singular solutions to the Euler equations in a fluid with solid boundary Luo and Hou (2014a,b). Furthermore, in two recent papers Tao (2016a,b), Tao indicates a significant barrier to establishing global regularity for the 3D Euler and Navier-Stokes equations, in that any method for achieving this, must use the finer geometric structure of these equations. In this paper, we show that the singularity discovered by Luo & Hou which lies right on the boundary is not relevant in the case of the whole domain \mathbb{R}^3 . We reveal also that the translation and rotation invariance present in the Euler, Navier-Stokes and 2D QG equations are the key for the non blow-up in finite time of the solutions. The translation and rotation invariance of these equations combined with the anisotropic structure of regions of high vorticity allowed to establish a new geometric non blow-up criterion which yield us to the non blow-up of the solutions in all the Kerr's numerical experiments and to show that the potential mechanism of blow-up introduced in Brenner et al. (2016) cannot lead to the blow-up in finite time of solutions of Euler equations.

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1 **1. Introduction**

2 The Navier-Stokes and Euler equations describe the motion of a fluid in the
3 three-dimensional space. These fundamental equations were derived over 250
4 years ago by Euler and since then have played a major role in fluid dynamics.
5 They have enriched many branches of mathematics, were involved in many areas
6 outside mathematical activity from weather prediction to exploding supernova
7 (see for instance the surveys [Constantin \(2007\)](#), [Bardos and Titi \(2007\)](#)) and
8 present important open physical and mathematical problems (see [Constantin](#)
9 [\(2007\)](#)). Regarding the 2D Quasi-Geostrophic (2D QG) equation, it appears in
10 atmospheric studies. It describes the evolution of potential temperature u on
11 the two dimensional boundary of a rapidly rotating half space with small Rossby
12 and Ekman numbers, for the case of special solutions with constant potential
13 vorticity in the interior and constant buoyancy frequency (normalized to one),
14 where equations in the bulk are compressible Euler or Navier-Stokes equations
15 coupled with temperature equation, continuity equation, and equation of state.

16
17 In the case of Navier-Stokes equations, for a long time ago, a global weak
18 solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$ and $\nabla u \in L^2(\mathbb{R}^3 \times (0, \infty))^3$ was built by Leray
19 [Leray \(1934\)](#). In particular, Leray introduced a notion of weak solutions for the
20 Navier-Stokes equations, and proved that, for every given $u_0 \in L^2(\mathbb{R}^3)^3$, there
21 exists a global weak solution $u \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))^3 \cap L^2([0, \infty[; \dot{H}^1(\mathbb{R}^3))^3$.
22 Hopf has proved the existence of a global weak solution in the general case \mathbb{R}^d ,
23 $d \geq 2$, [Hopf \(1951\)](#). Meanwhile the regularity and the uniqueness of this weak
24 solution has been known for a long time ago for the two-dimensional case (see
25 [Ladyzhenskaya \(1969\)](#), [Lions and Prodi \(1959\)](#), [Lions \(1969\)](#), [Temam \(1977\)](#)),
26 in the three-dimensional case the problem remains widely open in spite of great
27 efforts made. On the uniqueness many works have been done (see [Furioli et al.](#)
28 [\(2000\)](#), [Giga \(1983\)](#), [Monniaux \(2000\)](#), [Lions \(1960\)](#), [Gallagher and Planchon \(2002\)](#)).
29 Concerning the regularity of weak solutions, in [Serrin \(1962\)](#), it is proved
30 that if u is a Leray-Hopf weak solution belonging to $L^q([0, T]; L^q(\mathbb{R}^3))^3$ with
31 $\frac{2}{q} + \frac{3}{q} \leq 1$, $2 < p < \infty$, $3 < q < \infty$, then the solution $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. In
32 [Von Wahl \(1986\)](#) and [Giga \(1986\)](#), it is showed that if u is a weak solution in
33 $C([0, T]; L^3(\mathbb{R}^3))^3$, then $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. The limit case of $L^\infty([0, T]; L^3(\mathbb{R}^3))^3$
34 has been solved in [Iskauriaza et al. \(2003\)](#). Other criterion regularity can also be
35 found in [He \(2002\)](#); [Heywood \(1988\)](#); [Giga \(1983\)](#); [Kato \(1984, 1990\)](#); [Beirão da Veiga](#)
36 [\(1995\)](#); [Chae and Choe \(1999\)](#); [Zhou \(2002\)](#); [Constantin and Fefferman \(1994\)](#).

37

38 In the case of Euler Equations, in the two dimension case, uniqueness and ex-
 39 istence of classical solutions have been known for a long time ago (see [Yudovich](#)
 40 [\(1963, 1995\)](#); [Vishik \(1999\)](#); [DiPerna and Majda \(1987\)](#); [Ladyzhenskaya \(1969\)](#)).
 41 However for the full three space dimensions, little is known about smooth solu-
 42 tions apart from classical short-time existence and uniqueness. Moreover, weak
 43 solutions are known to be badly behaved from the point of view of Hadamard’s
 44 well-posedness theory (see for instance the surveys [De Lellis and Székelyhidi](#)
 45 [\(2012\)](#); [Villani \(2008-2009\)](#)). Considerable efforts have been devoted to the
 46 study of the regularity properties of the 3D Euler equations. The main diffi-
 47 culty in the analysis lies in the presence of the nonlinear vortex stretching term
 48 and the lack of a regularization mechanism. Despite these difficulties, a few
 49 important partial results concerning the regularity of 3D Euler equations have
 50 been obtained over the years (see [Beale et al. \(1984\)](#); [Ponce \(1985\)](#); [Ferrari](#)
 51 [\(1993\)](#); [Shirota and Yanagisawa \(1993\)](#); [Constantin et al. \(1996\)](#); [Deng et al.](#)
 52 [\(2005\)](#); [Gibbon and Titi \(2013\)](#)).

53
 54 In the case of 2D QG equation, besides its direct physical significance [Held et al.](#)
 55 [\(1995\)](#); [Pedlosky \(1987\)](#), the 2D QG equation has very interesting features of re-
 56 semblance to the 3D Euler equation, being also an outstanding open problem of
 57 the finite time blow-up issue. In particular, one can derive a necessary and suffi-
 58 cient blow-up condition for the 2D QG equation similar to the well-known Beale-
 59 Kato-Majda (BKM) criterion (Beale-Kato-Majda [Beale et al. \(1984\)](#)). More
 60 precisely, the solution to the 2D QG equation (11) becomes singular at time T^*
 61 if and only if $\int_0^{T^*} \|\nabla^\perp u(t)\|_{L^\infty} dt = +\infty$ (see [Constantin et al. \(1994\)](#)). Thus,
 62 $\nabla^\perp u$ plays a role similar to the vorticity ω in the 3D Euler equations. In the
 63 recent years, the 2D QG equation has been the focus of intense mathemati-
 64 cal research [Constantin et al. \(1994\)](#); [Córdoba \(1998\)](#); [Córdoba and Fefferman](#)
 65 [\(2002\)](#); [Chae \(2003\)](#); [Ohkitani and Yamada \(1997\)](#); [Constantin et al. \(2012\)](#);
 66 [Chae et al. \(2012\)](#).

67
 68 Unfortunately despite of considerable efforts devoted to the regularity is-
 69 sue of the 3D Euler, 3D Navier-Stokes and 2D QG equations, standard scaling
 70 heuristics have long indicated to the experts that the identity energy, together
 71 with the harmonic analysis estimates available for the heat equation and for the
 72 Euler bilinear operator, are not sufficient by themselves if one wishes to improve
 73 the theory on the Cauchy problem for these equations. It seems crucial to use
 74 the specific structure of the nonlinear term in these equations, as well as the
 75 divergence free assumption. Indeed, some finite time blowup results have been
 76 established for various Navier-Stokes type equations (see [Montgomery-Smith](#)
 77 [\(2001\)](#); [Gallagher and Paicu \(2009\)](#); [Li and Sinai \(2008\)](#); [Plecháč and Sverák](#)
 78 [\(2003\)](#); [Katz and Pavlovic \(2005\)](#)). Nevertheless, for all of these Navier-Stokes
 79 type equations, the cancellation property of the Euler bilinear operator did not
 80 hold and for some, the energy identity did not hold (see [Montgomery-Smith](#)
 81 [\(2001\)](#); [Gallagher and Paicu \(2009\)](#); [Li and Sinai \(2008\)](#)).

82 However, recently it was shown also in [Tao \(2016a\)](#), a finite time blow up so-
 83 lution to an averaged three-dimensional Navier-Stokes equations of type $\partial_t u =$
 84 $\Delta u + \tilde{B}(u, u)$, where \tilde{B} is an averaged version of the Euler bilinear operator
 85 B , acting also on divergence free vector fields u and obeying as B to the can-
 86 cellation property $\langle \tilde{B}(u, u), u \rangle = 0$. This result suggests that any successful
 87 method to affirmatively answer to the Existence and Smoothness problem must
 88 either use finer structure of B or else must rely crucially on some estimate
 89 or other property of the Euler bilinear operator B that is not shared by the
 90 averaged operator \tilde{B} . Such additional structure exists for instance, the Euler
 91 equation has a vorticity formulation involving only differential operators rather
 92 than pseudo-differential ones.

93 However, even this vorticity formulation is not a barrier to get a finite time
 94 blow up solution. Indeed, it was shown in [Tao \(2016b\)](#), finite time blow-up solu-
 95 tions in the *class* of generalised Euler equations sharing with the Euler equation
 96 its main features such as: *vorticity formulation, energy conservation, Kelvin*
 97 *circulation theorem, vorticity-vector potential formulation viewed as the Gen-*
 98 *eralised Biot-Savart, function space estimates for the vector potential operator.*
 99 Then, it seems that there is no room left to establish global regularity of solu-
 100 tions of 3D Euler equations. However, as it is mentioned in [Tao \(2016b\)](#), there
 101 are two properties of the Euler equations which are not obeyed by the gener-
 102 alised Euler equations, namely translation invariance and rotation invariance.
 103 Further, these symmetries basically determined the usual Biot-Savart law (see
 104 [Kambe \(2003a,b\)](#)) which are thus not shared by the Generalised Euler equations
 105 introduced in [Tao \(2016b\)](#). Furthermore, as it was shown in [Constantin \(1994\)](#);
 106 [Constantin and Fefferman \(1994\)](#), the use of Biot-Savart law yield to rewrite
 107 the vorticity equation in the case of Euler ($\nu = 0$) and Navier-Stokes ($\nu > 0$)
 108 equations as follows:

$$\partial_t \omega + (\mathbf{v} \cdot \nabla) \omega - \nu \Delta \omega = \alpha \omega, \quad (1)$$

109 where

$$\alpha(x, t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} (\hat{y} \cdot \xi(x, t)) \det(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^3}, \quad (2)$$

110 with $\hat{y} = \frac{y}{|y|}$, $\xi = \frac{\omega}{|\omega|}$ and $\det(a, b, c)$ is the determinant of the matrix with
 111 columns a, b, c in that order. We thus notice from the expression of α that if the
 112 direction of the vorticity, ξ varies mildly within a small region around x , then
 113 the singularity of the integrand in (2) will be mild.

114 In this paper, we bring new insights which shed light on the mechanisms in-
 115 volved in the non blow-up of the solutions. We highlight through new geo-
 116 metric non blow-up criteria how the geometric regularity of the direction of
 117 vorticity combined with the anisotropic structure of the localized regions con-
 118 taining the positions where the maximum of the magnitude of the vorticity
 119 are reached, should prevent the formation of singularities. The novelty in
 120 the results of this paper lies on the use of the these two features in obtain-

121 ing geometric non blow-up criteria using the finer structure of the Euler bi-
122 linear operator B . Up to now, many progress had been made to better take
123 into account the geometrical properties and flow structures in the non blow-up
124 criteria (see e.g [Constantin and Fefferman \(1994\)](#); [Beirão da Veiga and Berselli](#)
125 [\(2009\)](#); [Berselli \(2009\)](#); [Constantin \(1994\)](#); [Constantin et al. \(1996\)](#); [Hou and Li](#)
126 [\(2008\)](#); [Deng et al. \(2005, 2006a,b\)](#)). However none of these non blow-up cri-
127 teria integrated both the geometric regularity of the direction of vorticity and
128 the anisotropic structure of localized regions containing the positions where the
129 maximum of the magnitude of the vorticity are reached. The most advanced
130 non blow-up criteria were given in [Deng et al. \(2005, 2006a,b\)](#) and were estab-
131 lished by using the Lagrangian formulation of the vorticity equation of the 3D
132 Euler and 2D QG equations.

133 However the results obtained in [Tao \(2016b\)](#) suggest that even the most ad-
134 vanced non blow-up criterion [Deng et al. \(2005, 2006a\)](#) do not capture the finest
135 structures of the the Euler bilinear operator B since it was shown in [Tao \(2016b\)](#)
136 that there exist generalised Euler equations sharing the same property than Euler
137 equations as the Lagrangian formulation for their vorticity equations and for
138 which their solutions blow up in finite time. Indeed, from [Deng et al. \(2005,](#)
139 [2006a\)](#), one can observe that the Deng-Hou-Yu non-blowup criterion can be ap-
140 plied to all the *class* of generalised Euler equations introduced in [Tao \(2016b\)](#).
141 Then, in order to bring new insights in the investigation of whether the 3D
142 incompressible Navier-Stokes, Euler and 2D QG equations can develop a finite-
143 time singularity from smooth initial data, it was crucial to establish new non
144 blow-up criteria which take into account the special structure of these equations
145 not shared by the Generalised Euler equations.

146 Then in our [Theorem 7.1](#), under mild assumptions based on the anisotropic
147 structure of regions of high vorticity, we show that the solutions of 3D Euler,
148 3D Navier-Stokes and 2D QG equations cannot blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t) \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right) dt < \infty,$$

149 where the functions \mathbf{A}_d and Ω satisfy:

$$\begin{aligned} \mathbf{A}_d(t) &\leq \|\nabla \xi(t)\|_\infty \\ \Omega(t) &= \frac{(T^* - t)^{-1}}{1 + \log^+((T^* - t)\|u(t)\|_\infty \mathbf{A}_0(t))} \\ \mathbf{A}_0(t) &\leq \|\nabla \xi(t)\|_\infty. \end{aligned}$$

150 Note that $\xi(t)$ is well defined only on $\mathcal{O}(t)$ the set of points x of \mathbb{R}^d where
151 $\omega(x, t) \neq 0$ and then $\|\nabla \xi(t)\|_\infty$ must be understood as $\|\nabla \xi(t)\|_{L^\infty(\mathcal{O}(t))}$.

152 In the case of 3D Euler equations and 2D QG equations by using their La-
153 grangian formulation, in [Theorem 7.2](#) we go further in the non blow-up criteria
154 by showing under mild assumptions based on the anisotropic structure of re-
155 gions of high vorticity, that their solutions do not blow up at a finite time T^*

156 if

$$\int_0^{T^*} \mathbf{A}_d(t) dt < \infty.$$

157 These results are obtained after a fine analysis of the term α defined by (2) com-
 158 bined with some results based on the anisotropic structure of regions of high
 159 vorticity. Our analysis starts by considering at each time $t \in]0, T^*[$ the regions
 160 containing the positions where the maximum of the magnitude of the vorticity
 161 are reached and shrinking to zero as time tends to T^* the alleged time of singular-
 162 ity. More precisely, these regions are balls of radius $\rho_0(t) = O((T^* - t)\|u(t)\|_\infty)$
 163 and of center the position of a point where the maximum of the magnitude of
 164 the vorticity is reached. Inside these regions, we then consider the regions of
 165 high vorticity for which the magnitude of the vorticity is greater than some
 166 function $\Omega(t)$ such that $\Omega(t) \gtrsim \frac{(T^* - t)^{-1}}{1 + \log^+(\rho_0(t)\|\nabla\xi(t)\|_\infty)}$.

167 In our analysis, to track in time the positions where the maximum of the mag-
 168 nitude of the vorticity is reached, we had to overcome the obstruction that
 169 we do not know if there exists an isolated absolute maximum for the vorticity
 170 achieved along a smooth curve in time as it was assumed in Proposition 2.1 of
 171 Constantin et al. (1994) and also in Deng et al. (2005, 2006a,b) (which assume
 172 that the position where the maximum of vorticity is reached, is advected with
 173 the flow). Moreover, recent numerical experiments show that it is not always the
 174 case (see Kerr and Bustamante (2012), see also section 5.4.5 in Grafke (2012)).
 175 We thus overcome this difficulty by using a result of Pshenichnyi concerning
 176 directional derivatives of the function of maximum and the structure of a set of
 177 supporting functionals Pshenichny (1971).

178 Our analysis led first to the non blow-up criterion given by our Theorem 5.1,
 179 namely, the solutions of 3D Euler, 3D Navier-Stokes and 2D QG equations
 180 cannot blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t)\pi(t) dt < \infty, \quad (3)$$

181 where the function π is given by:

$$\pi(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{0 < R \leq \rho_0(t)} \frac{1}{R^{d-1}} \int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz, \quad (4)$$

182 with $\Theta(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x,t)| = \|\omega(t)\|_\infty\}$ and $\mathcal{V}(t) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d; |\omega(z,t)| \geq$
 183 $\Omega(t)\}$.

184 In our Lemma 6.1, we thus derive a straightforward estimate of the function
 185 $\pi(t)$, that is

$$\pi(t) \leq 3\|\omega(t)\|_\infty \sup_{x \in \Theta(t)} |\mathcal{V}(t) \cap B(x, \rho_0(t))|^{\frac{1}{d}}. \quad (5)$$

186 Thanks to the non blow-up criterion (3) and (5), we show the non blow-up
 187 in finite time of the solutions of Euler equations for Kerr's numerical experi-
 188 ments Kerr (2005, 1998, 1997) without additional numerical tests as it was the

189 case in [Deng et al. \(2005, 2006a\)](#), just by using the anisotropic structure of re-
 190 gions of high vorticity whose the features are described in [Kerr \(2005, 1998,](#)
 191 [1997\)](#). Moreover, we show that the potential mechanism of blow-up introduced
 192 in [Brenner et al. \(2016\)](#) cannot lead to blow-up in finite time for Euler equa-
 193 tions. To go further in our estimate of the function π , we use some assumptions
 194 characterizing the anisotropic structure of regions of high vorticity whose the
 195 justifications are given at the begininig of subsection 7.2 and we show in Propo-
 196 sition 7.1 that

$$\pi(t) \lesssim 1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right), \quad (6)$$

197 which yield to Theorem 7.1.

198 In the case of Euler equations and 2D QG equations by using their Lagrangian
 199 formulation, after a fine and sharp analysis of the expression of the function π
 200 (4) led thanks to our Lemmata 7.2,7.3 and 7.4, in Proposition 7.2 we go further
 201 in the non blow-up criteria by showing under mild assumptions based on the
 202 anisotropic structure of regions of high vorticity, that

$$\pi(t) = O(1). \quad (7)$$

203 We emphasize that according the 'thickness' of the structure of regions of high
 204 vorticity that these two estimates (6) and (7) can be much better. Indeed
 205 from the analysis led in [Kuznetsov and Ruban \(2000\)](#); [Kuznetsov et al. \(2001\)](#)
 206 for the study of collapse of vortex lines and agrees with numerical experiments
 207 [Agafontsev et al. \(2015, 2017\)](#), we could expect that (see Remark 7.1)

$$\pi(t) \lesssim \Omega(t)^{-\frac{1}{2}},$$

208 and then obtain in this case, the non blow-up in finite time of the solutions of
 209 Euler equations if

$$\int_0^{T^*} \mathbf{A}_d(t) \Omega(t)^{-\frac{1}{2}} dt < \infty. \quad (8)$$

210 We point out also that our geometric non blow-up criterion reveals the role
 211 of the geometric structures of the Incompressible flows in the non blow-up in
 212 finite time of the solutions and presents the advantage to be established in an
 213 Eulerian setting in comparison with all the recent geometric non blow-up crite-
 214 ria [Constantin et al. \(1996\)](#); [Deng et al. \(2005, 2006a\)](#); [Constantin et al. \(1994\)](#)
 215 using the Lagrangian formulation of Incompressible Inviscid Flows, which re-
 216 quires much more computational effort as it is mentioned in [Grafke and Grauer](#)
 217 [\(2013\)](#) and in section 5.4.5 of [Grafke \(2012\)](#). Furthermore, due to the exis-
 218 tence of hyperbolic-saddle singularities suggested by the generation of strong
 219 fronts in geophysical/meteorology observations (see [Constantin et al. \(1994\)](#);
 220 [Córdoba \(1998\)](#)), and antiparallel vortex line pairing observed in numerical sim-
 221 ulations and physical experiments, it was important to take them into account
 222 in our geometric non blow-up criterion. This is performed thanks to the term
 223 $\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))$ (see (27),(28)) involved in the definition of the function
 224 \mathbf{A}_d given at (39).

225 Then, the paper is organized as follows:

- 226 • In section 2 , we give some notations and definitions.
- 227 • In section 3, we recall some results about the local regularity of solutions
228 of Navier-Stokes, Euler and 2D QG equations.
- 229 • In section 4, we give the reason for which we can assume for any time t
230 that $\|\omega(t)\|_\infty > 0$ without loss of generality.
- 231 • In section 5, in Theorem 5.1, we establish a new geometric criterion for
232 the non blow-up in finite time of the solutions of 3D Navier-Stokes, 3D
233 Euler and 2D QG equations. We show that their solutions cannot blow up
234 at a finite time T^* if $\int_0^{T^*} \mathbf{A}_d(t)\pi(t) dt < \infty$, where $\mathbf{A}_d(t)$ is based on the
235 regularity of the direction of the vorticity ξ in regions shrinking to zero as
236 time tends to T^* and containing the positions where the maximum of the
237 magnitude of the vorticity is reached (see definition of \mathbf{A}_d at (39)).
- 238 • In section 6, we show the non-blowup in finite time of the solutions of
239 the Euler equations in the numerical experiments considered these last
240 years, by using inequality (5) about the function π (4) and the anisotropic
241 structure of regions of high vorticity described in Kerr (2005, 1998, 1997).
- 242 • In section 7, we show the estimates (5), (6) and (7) concerning the function
243 π defined by (4), and obtain new non blow-up criteria in Theorems 7.1
244 and 7.2.

245 Let us now introduce the 3D Navier-Stokes and Euler equations given by,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \\ \nabla \cdot u = 0, \end{cases} \quad (9)$$

246 in which $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ and
247 $\nu \geq 0$ ($\nu = 0$ corresponds to the Euler equations) denote respectively the
248 unknown velocity field, the scalar pressure function of the fluid at the point
249 $(x, t) \in \mathbb{R}^3 \times [0, \infty[$ and the viscosity of the fluid,

250
251 with initial conditions,

$$u(x, 0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3, \quad (10)$$

252 where the initial data u_0 is a divergence free vector field on \mathbb{R}^3 .

253
254 Regarding the 2D QG equation in \mathbb{R}^2 , it is given by

$$\begin{cases} \frac{\partial u}{\partial t} + v \cdot \nabla u = 0, \\ v = \nabla^\perp (-\Delta)^{-\frac{1}{2}} u, \end{cases} \quad (11)$$

255 with initial data,

$$u(x, 0) = u_0. \quad (12)$$

256 Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. For v we have also the following representation

$$v = R^\perp u, \quad (13)$$

257 where we have used the notation, $R^\perp u = (-R_2 u, R_1 u)$ with $R_j, j = 1, 2$, for the
258 2D Riesz transform defined by (see e.g. [Stein \(1970\)](#))

$$R_j(u)(x, t) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x - y|^3} u(y, t) dy.$$

259 2. Some notations and definitions

260 In this section, we assume that $d \in \mathbb{N}$, $d \geq 2$.
261 For any vector $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we denote by $|x|$ the euclidean norm of
262 x given by $|x| = \sqrt{\sum_{i=1}^d |x_i|^2}$. For any $y \in \mathbb{R}^d$, $y \neq 0$, we denote by \hat{y} the unit
263 vector $\hat{y} = \frac{y}{|y|}$. For any m -dimensional subset A of \mathbb{R}^d , $1 \leq m \leq d$, we denote
264 by $|A|$ its measure. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of real square matrices of size
265 d . We denote by Id the identity matrix of $\mathcal{M}(\mathbb{R}^d)$. For any vector field v defined
266 from \mathbb{R}^d to \mathbb{R}^d , we denote by ∇v the gradient matrix of v , the matrix of $\mathcal{M}(\mathbb{R}^d)$
267 with ij -component, $\frac{\partial v_i}{\partial x_j}$ for all $1 \leq i, j \leq d$. For any real a , we denote by a^+
268 the real defined by $a^+ \stackrel{\text{def}}{=} \max(a, 0)$. For any function φ defined on $\mathbb{R}^d \times [0, +\infty[$,
269 for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on \mathbb{R}^d by $x \mapsto \varphi(x, t)$. We
270 denote by $C_c^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact
271 support in \mathbb{R}^d . We denote by BC the class of bounded and continuous functions
272 and by BC^m the class of bounded and m times continuously derivable functions.
273 For any $R > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B(x_0, R)$, the ball of \mathbb{R}^d of center x_0
274 and radius R . For any $R > 0$, we denote by B_R , the ball of \mathbb{R}^d of center 0 and
275 radius R .

276 We denote by div the differential operator given by, $\text{div} = \sum_{i=1}^d \frac{\partial}{\partial x_i}$.

277 We denote $A \lesssim B$, $B \gtrsim A$ or $A = O(B)$ the estimate $A \leq cB$ where $c > 0$ is an
278 absolute constant. If we need c to depend on a parameter, we shall indicate this
279 by subscripts, thus for instance $A \lesssim_s B$ denotes the estimate $A \leq c_s B$ for some c_s
280 depending on s . We use $A \sim B$ as shorthand for $A \lesssim B \lesssim A$.

281 For any $f \in L^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)^d$ or $L^p(\mathbb{R}^d)^{d \times d}$) with $1 \leq p \leq +\infty$, we
282 denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p -norm of f .

283 We denote by $H^s(\mathbb{R}^d)$ the Sobolev space $J^{-s}L^2(\mathbb{R}^d)$ where $J = (1 - \Delta)^{\frac{1}{2}}$. We
284 denote by $H_\sigma^s(\mathbb{R}^3)$ the Sobolev space $H_\sigma^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\psi \in H^s(\mathbb{R}^3)^3 : \text{div} \psi = 0\}$. In
285 order to unify our notations with the two dimensional case 2D QG, we denote
286 by $H_\sigma^s(\mathbb{R}^2)$ the Sobolev space $H^s(\mathbb{R}^2)$.

287 We denote by \mathbb{P} the well-known 3D matrix Leray's projection operator with
288 components,

$$\mathbb{P}_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Delta^{-1} = \delta_{i,j} - R_j R_k, \quad (14)$$

289 where R_j are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} = \frac{1}{4\pi} \frac{x_j}{|x|^4} \star$ (see
 290 [Stein \(1970\)](#) for more details), Δ^{-1} is the inverse of Laplace operator given by
 291 $\Delta^{-1} = -\frac{1}{4\pi|x|} \star$, with \star the convolution operator.

292 3. Local regularity of the solutions

293 In this section, we deal with the main result on local regularity of 3D Navier-
 294 Stokes and Euler equations in its general form. By introducing \mathbb{P} the matrix
 295 Leray operator, Euler equations (9)-(10) can be re-written as follows,

$$\frac{\partial u}{\partial t} + \mathbb{P}(u \cdot \nabla)u = 0, \quad (15)$$

296 with initial conditions,

$$u(0) = u_0. \quad (16)$$

297 For u solution of (15)-(16), $\omega = \nabla \times u$ the vorticity of u formally satisfies the
 298 vorticity equation,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega = 0, \quad (17)$$

299 with initial conditions,

$$\omega(0) = \omega_0,$$

300 where $\omega_0 = \nabla \times u_0$ is the vorticity of u_0 .

301 In the case of 2D QG equation, we get for u solution of (11), $\omega = \nabla^\perp u$ the
 302 vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v = 0, \quad (18)$$

303 with initial conditions,

$$\omega(0) = \omega_0,$$

304 where $\omega_0 = \nabla^\perp u_0$ is the vorticity of u_0 .

305 In the region where $|\omega| > 0$, we define ξ the direction of the vorticity by $\xi = \frac{\omega}{|\omega|}$.

306 *3.1. Local regularity for 3D Navier-Stokes or 3D Euler equations*

307 Assuming $u_0 \in H_\sigma^r(\mathbb{R}^3)$ with $r > \frac{5}{2}$, thanks to Theorem 3.5 in [Kato and Ponce](#)
 308 [\(1988\)](#), Theorem 1 in [Bourguignon and Brezis \(1974\)](#) (see also Theorem I in
 309 [Kato and Ponce \(1986\)](#) and the results obtained in [Beale et al. \(1984\)](#)), we de-
 310 duce that there exists a time $T > 0$ such that there exists a unique strong
 311 solution $u \in C([0, T[, H_\sigma^r(\mathbb{R}^3)) \cap C^1([0, T[, H_\sigma^{r-2}(\mathbb{R}^3))$ to the Navier-Stokes or
 312 Euler equations [\(15\)-\(16\)](#) and the energy equality holds for u , that means for
 313 all $t \in [0, T[$,

$$\|u(t)\|_2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2. \quad (19)$$

314 Moreover, if $u \notin C([0, T[, H_\sigma^r(\mathbb{R}^3))$, then we get (see [Beale et al. \(1984\)](#); [Kato and Ponce](#)
 315 [\(1988\)](#); [Kozono and Taniuchi \(2000\)](#)),

$$\int_0^T \|\omega(t)\|_\infty dt = +\infty. \quad (20)$$

316 Notice thanks to Remark 3.7 in [Kato and Ponce \(1988\)](#), in the case of Euler
 317 equations, we get in addition that $u \in C^1([0, T[, H_\sigma^{r-1}(\mathbb{R}^3))$. We retrieve the
 318 pressure p from the velocity u with the formula,

$$p = -\Delta^{-1} \operatorname{div}((u \cdot \nabla)u).$$

319 Furthermore, we get the local estimate [\(21\)](#). Indeed, thanks to remark 4.4 in
 320 [Kato and Ponce \(1988\)](#), we get

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ with } t_0 < t < T, \quad (21)$$

321 provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

322 *3.2. Local regularity for 2D QG equation*

323 This subsection is devoted to the local well-posedness of the 2D QG equation
 324 with a characterization of the maximal time existence of strong solutions. By
 325 using the same arguments as the proof of Proposition 4.2 in [Agélas \(2016\)](#),
 326 we get that the H^s -norm of u is controlled by the integral in time of the
 327 maximum magnitude of the vorticity of u . A such Proposition has been proved
 328 in [Constantin et al. \(1994\)](#) for any integer $s \geq 3$, but here we extend this result
 329 to all real $s > 2$. This improvement is obtained by using the logarithmic Sobolev
 330 inequality proved in [Kozono and Taniuchi \(2000\)](#); [Kato and Ponce \(1988\)](#) which
 331 requires only that $s > 2$ instead of using the one proved in [Beale et al. \(1984\)](#)
 332 as it is the case in [Constantin et al. \(1994\)](#) and which requires integer $s \geq 3$.
 333 Then by using the same arguments as the proof of Proposition 4.3 in [Agélas](#)
 334 [\(2016\)](#), we get the following result which gives an improvement in comparison
 335 with Theorem 2.1 in [Constantin et al. \(1994\)](#):

336 Assuming $u_0 \in H^r(\mathbb{R}^2)$ with $r > 2$, we get that there exists a time $T > 0$
 337 such that there exists a unique strong solution $u \in C([0, T[, H^r(\mathbb{R}^2))$ to the

338 2D QG equation (11)-(12) and the energy equality holds for u , that means for
 339 all $p \in [2, \infty]$ and $t \in [0, T[$,

$$\|u(t)\|_p = \|u_0\|_p. \quad (22)$$

340 Moreover, if $u \notin C([0, T], H^r(\mathbb{R}^2))$, then

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \quad (23)$$

341 Owing to $u \in C([0, T[, H^r(\mathbb{R}^2))$ and thanks to Lemma X4 in [Kato and Ponce](#)
 342 (1988), from 2D QG (11), we get $u \in C^1([0, T[, H^{r-1}(\mathbb{R}^2))$.

343 Similarly as in (21), we have

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ for } t_0 < t < T, \quad (24)$$

344 provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

345 4. Assumption on the maximum vorticity

346 Let $d \in \{2, 3\}$, $r > \frac{d}{2} + 1$ and $u_0 \in H_\sigma^r(\mathbb{R}^d)$. Let $T^* > 0$ be such that there
 347 exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler or 2D QG
 348 equations (9)-(10) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

349 Thanks to the results of the section 3, a such time T^* exists.

350 In this paper, we are concerned with the non blowup in finite time of the solu-
 351 tions u at times such T^* . Then, without loss of generality, in the whole of this
 352 paper, we consider only times of existence T^* such that for all $t \in [0, T^*[$,

$$\|\omega(t)\|_\infty > 0. \quad (25)$$

353 Indeed, let us assume that there exists $t_0 \in [0, T^*[$ such that $\|\omega(t_0)\|_\infty = 0$.

354 In the case of 2D QG equations (11), we get that $\omega(t_0) \equiv 0$ and then $\nabla u(t_0) \equiv 0$.
 355 Since $x \mapsto u(t_0, x)$ vanishes at infinity, then we get $u(t_0) \equiv 0$. Then by using
 356 inequality (24) concerning the local regularity, we deduce that $u(t) \equiv 0$ for all
 357 $t \in [t_0, T^*[$ and no blowup can occur at the time T^* .

358 By following step by step the proof of Lemma 4 given in [Deng et al. \(2005\)](#) but
 359 keeping the term $\|u(t)\|_{L^2(\mathbb{R}^d)}$ after using the Cauchy-Schwarz inequality, we
 360 obtain for all $t \in [0, T^*[$,

$$\begin{aligned} \|u(t)\|_\infty &\lesssim \|u(t)\|_2^{\frac{2}{d+2}} \|\omega(t)\|_\infty^{\frac{d}{d+2}} \\ &\leq \|u_0\|_2^{\frac{2}{d+2}} \|\omega(t)\|_\infty^{\frac{d}{d+2}}, \end{aligned} \quad (26)$$

361 where we have used (19) for the last inequality. Then thanks to (26) used with
 362 $d = 3$, we obtain that $\|u(t_0)\|_\infty \equiv 0$ which implies that $u(t_0) \equiv 0$. Then by using
 363 the inequality (21) of local regularity, we deduce $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and
 364 thus no blowup can occur at the time T^* .

365 5. Geometric properties for non blow-up of the solutions

366 Historically, non blow-up criteria for the incompressible Euler equations and
367 2D QG equations commonly focus on global features of the flow, such as norms
368 of the velocity or the vorticity fields. This comes at the disadvantage of neglect-
369 ing the structures and physical mechanisms of the flow evolution. A strat-
370 egy for overcoming such shortcomings was established by focusing more on
371 geometrical properties and flow structures (see e.g. [Constantin et al. \(1996\)](#);
372 [Cordoba and Fefferman \(2001\)](#)), such as vortex tubes or vortex lines.

373 In particular, in [Constantin et al. \(1996, 1994\)](#) the authors showed that local
374 geometric regularity of the unit vorticity vector can lead to depletion of the
375 vortex stretching. They prove that if there is up to time T an $O(1)$ region in
376 which the vorticity vector is smoothly directed, i.e., the maximum norm of $\nabla\xi$
377 (here $\xi = \frac{\omega}{|\omega|}$, ω the vorticity) in this region is L^2 integrable in time from 0
378 to T , and the maximum norm of velocity in some $O(1)$ neighbourhood of this
379 region is uniformly bounded in time, then no blow-up can occur in this region
380 up to time T .

381 However, this theorem dealt with $O(1)$ regions in which the vorticity vector is
382 assumed to have some regularity, while in numerical computations, the regions
383 that have such regularity and contain maximum vorticity are all shrinking with
384 time (see [Kerr \(1993, 1995, 1997, 1998\)](#); [Pelz \(1997, 2001\)](#)).

385 Inspired by the work of [Constantin et al. \(1996, 1994\)](#), in [Deng et al. \(2005,](#)
386 [2006a,b\)](#) the authors showed that geometric regularity of Lagrangian vortex
387 filaments, even in an extremely localized region containing the maximum of
388 vorticity which may shrink with time, can lead to depletion of the nonlinear
389 vortex stretching, thus avoiding finite time singularity formation of the 3D Euler
390 equations and 2D QG equations.

391 However, all the recent geometric constraints for non blow-up criteria of
392 Euler and 2D QG equations based on local geometric regularity of Lagrangian
393 vortex filaments [Deng et al. \(2005, 2006a,b\)](#) make the assumption that the po-
394 sition where the maximum of vorticity is reached, is advected with the flow,
395 however it is not always the case, as described in [Kerr and Bustamante \(2012\)](#)
396 (see also section 5.4.5 of [Grafke \(2012\)](#)).

397 Then in our Theorem 5.1, we establish in an Eulerian setting a new geo-
398 metric non blow-up criterion for the Navier-Stokes, Euler and 2D QG equations
399 based on the regularity of the direction of the vorticity in extremely localized
400 regions containing the positions where the maximum of the magnitude of the
401 vorticity are reached and shrinking to zero as time increase to some T^* the al-
402 leged time of singularity. Our Eulerian geometric non blow-up criterion should
403 give also new impetus to the numerical experiments due to their ease of imple-
404 mentation in comparison with Lagrangian geometric non blow-up criteria (see
405 [Grafke and Grauer \(2013\)](#), see also section 5.4.5 of [Grafke \(2012\)](#)). Moreover
406 our geometric non blow-up criterion is also valid for the Navier-Stokes equations
407 that is not the case for the existing geometric non blow-up criteria obtained in
408 [Constantin et al. \(1996\)](#); [Deng et al. \(2005, 2006a,b\)](#) based on a Lagrangian for-

409 mulation of Incompressible Inviscid Flows.

410 To obtain our Theorem 5.1, we begin with Lemma 5.1.

411 **Lemma 5.1.** *Let $d \in \mathbb{N}^*$, $T > 0$ and $f \in C([0, T]; BC(\mathbb{R}^d))$ such that*
 412 *$\inf_{t \in [0, T]} \|f(t)\|_\infty > 0$ and for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$. Then*
 413 *there exists $R > 0$ such that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$.*

414 *Proof.* We set $a = \inf_{t \in [0, T]} \|f(t)\|_\infty > 0$. Since $t \mapsto f(t)$ is a continuous function
 415 from the compact $[0, T]$ into the metric space $L^\infty(\mathbb{R}^d)$ then it is uniformly
 416 continuous. Hence, there exists $N \in \mathbb{N}^*$ such that for all $t, t' \in [0, T]$, $|t - t'| \leq \frac{T}{N}$
 417 we have $\|f(t) - f(t')\|_\infty \leq \frac{a}{4}$. We introduce the subdivision $\{t_i\}_{i \in \llbracket 0, N \rrbracket}$ of
 418 $[0, T]$ defined by $t_i = i \frac{T}{N}$ for $i \in \llbracket 0, N \rrbracket$. Since for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$
 419 as $|x| \rightarrow +\infty$, then for each $i \in \llbracket 0, N \rrbracket$, there exists $R_i > 0$ such that for all
 420 $|x| \geq R_i$, $|f(x, t_i)| \leq \frac{a}{4}$. We set $R = \max_{i \in \llbracket 0, N \rrbracket} R_i$. Let $t \in [0, T]$ then there exists
 421 $j \in \llbracket 0, N \rrbracket$ such that $|t - t_j| \leq \frac{T}{N}$ and hence for all $|x| \geq R \geq R_j$, we have
 422 $|f(x, t)| \leq |f(x, t) - f(x, t_j)| + |f(x, t_j)| \leq \frac{a}{2} \leq \frac{\|f(t)\|_\infty}{2}$. Then, we infer that
 423 for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$, which concludes the proof. \square

424 Before to prove Theorem 5.1, we need to introduce the following function
 425 \mathbf{D}_d defined from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R} with $d \in \{2, 3\}$ as follows: for $d = 3$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3) \text{Det}(a_1, a_2, a_3).$$

426 The Det in \mathbf{D}_d is the determinant of the matrix whose columns are the three
 427 unit column vectors a_1, a_2, a_3 . We observe that $\text{Det}(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$,
 428 then, we get

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3) a_1 \cdot (a_2 \times a_3). \quad (27)$$

429 and for $d = 2$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3^\perp) (a_2 \cdot a_3^\perp), \quad (28)$$

430 where for any $z = (z_1, z_2) \in \mathbb{R}^2$, $z^\perp = (-z_2, z_1)$. We can notice that for
 431 $d \in \{2, 3\}$ the function \mathbf{D}_d is linear from its second variable.

432 From (27) and (28) we get $\mathbf{D}_d(a_1, a_3, a_3) = 0$ then we deduce that for any
 433 $a_1, a_2, a_3 \in B(0, 1)$,

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq |a_2 - a_3|, \quad (29)$$

434 and we get also

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq 1. \quad (30)$$

435 Now, we turn to the proof of our Theorem.

436 **Theorem 5.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 437 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 438 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)). \quad (31)$$

439 Let ρ_0 be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\rho_0(t) \stackrel{\text{def}}{=} 36(T^* - t)\|u(t)\|_\infty. \quad (32)$$

440 Let \mathbf{A}_0 be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by:

$$\mathbf{A}_0(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho_0(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}, \quad (33)$$

441 where for any $t \in [0, T^*[$

$$\Theta(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}. \quad (34)$$

442 Let Ω be the function defined from $[0, T^*[$ to $]0, +\infty[$ by:

$$\Omega(t) \stackrel{\text{def}}{=} \frac{(T^* - t)^{-1}}{8(1 + \log^+(4\rho_0(t)\mathbf{A}_0(t)))}. \quad (35)$$

443 We introduce also the set of high vorticity regions defined for all $t \in [0, T^*[$ by

$$\mathcal{V}(t) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d; |\omega(z, t)| \geq \Omega(t)\}. \quad (36)$$

444 Let π be the function defined from $[0, T^*[$ to $]0, +\infty[$, for all $t \in [0, T^*[$ by

$$\pi(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{0 < R \leq \rho_0(t)} \frac{1}{R^{d-1}} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz. \quad (37)$$

445 Let ρ be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\rho(t) \stackrel{\text{def}}{=} 4(d+1)c_d(T^* - t)\pi(t), \quad (38)$$

446 where $c_d = \frac{3}{4\pi}$ if $d = 3$, $c_d = \frac{1}{2\pi}$ else.

447 Let \mathbf{A}_d be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}. \quad (39)$$

448 Then if there exists $t_1 \in [0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t)\pi(t) < +\infty, \quad (40)$$

449 then the solution u cannot blowup at the finite time T^* .

450 Moreover, we have for all $t \in [0, T^*[$ and $x \in \Theta(t)$,

$$\nabla|\omega|(x, t) = 0 \text{ and } \nabla \cdot \xi(x, t) = 0.$$

451 *Proof.* Let $0 < T < T^*$. We want first to apply Lemma 5.1 to the function ω ,
452 then we check that the hypotheses of the Lemma are satisfied.
453 Since $u \in C([0, T]; H^r(\mathbb{R}^d)) \cap C^1([0, T]; H^{r-2}(\mathbb{R}^d))$, then we infer that $\omega \in$
454 $C([0, T]; H^{r-1}(\mathbb{R}^d)) \cap C^1([0, T]; H^{r-3}(\mathbb{R}^d))$. Thanks to the Sobolev embedding
455 $H^s(\mathbb{R}^d) \hookrightarrow BC^m(\mathbb{R}^d)$ for $s > \frac{d}{2} + m$, $m \in \mathbb{N}$ and since $r > \frac{d}{2} + 3$ we deduce
456 that $\omega \in C([0, T]; BC^2(\mathbb{R}^d)) \cap C^1([0, T]; BC(\mathbb{R}^d))$. Thanks to (25), we get
457 that $\inf_{t \in [0, T]} \|\omega(t)\|_\infty > 0$. Moreover, since $\omega \in C([0, T]; H^{r-1}(\mathbb{R}^d))$ with $r >$
458 $\frac{d}{2} + 3$, we have for any $t \in [0, T]$, $|\omega(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$, the proof follows
459 immediately by using the density of $C_0^\infty(\mathbb{R}^d)$ in $H^{r-1}(\mathbb{R}^d)$ and the Sobolev
embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $r > \frac{d}{2} + 3$.

460 Then thanks to Lemma 5.1, there exists $R > 0$ such that for all $t \in [0, T]$,
461 $\|\omega(t)\|_\infty = \sup_{x \in B_R} |\omega(x, t)|$. Then for all $t \in [0, T]$, the set $\Theta(t)$ defined by (34)
462 can be rewritten as follows:
463

$$\Theta(t) = \{x \in B_R; |\omega(x, t)| = \|\omega(t)\|_\infty\}. \quad (41)$$

464 We introduce the direction of the vorticity $\xi = \frac{\omega}{|\omega|}$ defined on the non empty
465 open set $\mathcal{O} \stackrel{\text{def}}{=} \{(x, t) \in \mathbb{R}^d \times [0, T]; |\omega(x, t)| > 0\}$.

466 We set $\mathbf{v} = u$ in the case of 3D Navier-Stokes or 3D Euler equations and
467 $\mathbf{v} = R^\perp u$ with $\nu = 0$ in the case of 2D QG equation.

468 Then by multiplying (17) or (18) by ξ , we get that for all $(x, t) \in \mathcal{O}$,

$$\begin{aligned} \frac{\partial |\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla |\omega|(x, t) & - (\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) \\ & - \nu \Delta |\omega|(x, t) + \nu |\omega(x, t)| |\nabla \xi(x, t)|^2 = 0. \end{aligned} \quad (42)$$

469 We introduce the function φ defined for all $t \in [0, T]$ by

$$\varphi(t) \stackrel{\text{def}}{=} \sup_{x \in B_R} |\omega(x, t)|$$

470 and we search the expression of its derivative. For this, we use the main Theorem
471 obtained in Pshenichny (1971) or Theorem 1 in Borisenko and Minchenko (1992)
472 after verifying that the hypotheses of the Theorem are satisfied.

473 Since $\omega \in C([0, T]; BC^2(\mathbb{R}^d)) \cap C^1([0, T]; BC(\mathbb{R}^d))$, then we deduce that $|\omega| \in$
474 $BC(\mathcal{O})$, $\frac{\partial |\omega|}{\partial t} \in BC(\mathcal{O})$ and $\nabla^2 |\omega| \in BC(\mathcal{O})$. Since for any $t \in [0, T]$, $\Theta(t) \subset$
475 $\mathcal{O} \times \{t\}$, then, thanks to the results obtained in Pshenichny (1971) (see also
476 Theorem 1 in Borisenko and Minchenko (1992)), by using also (41) we obtain
477 the expression of the derivative of φ given for any $t \in [0, T]$ by,

$$\varphi'(t) = \sup_{x \in \Theta(t)} \frac{\partial |\omega|}{\partial t}(x, t). \quad (43)$$

478 Further for all $x \in \Theta(t) \subset B_R$, we have $|\omega(x, t)| = \varphi(t) = \|\omega(t)\|_\infty$, we thus
 479 infer that

$$\nabla|\omega|(x, t) = 0 \text{ and } \Delta|\omega|(x, t) \leq 0. \quad (44)$$

480 Therefore, we have for all $x \in \Theta(t)$,

$$\begin{aligned} \frac{\partial|\omega|}{\partial t}(x, t) &= \frac{\partial|\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla|\omega|(x, t) \\ &= (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) + \nu\Delta|\omega|(x, t) - \nu|\omega(x, t)||\nabla\xi(x, t)|^2 \\ &\leq (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \end{aligned} \quad (45)$$

481 where we have used (42) for the second equality and (44) for the last inequality.
 482 We can notice that we get equality for (45) in the case of 3D Euler or 2D
 483 QG equations, since for these equations we have not the terms $\nu\Delta|\omega|(x, t)$ and
 484 $\nu|\omega(x, t)||\nabla\xi(x, t)|^2$.

485 Then using (45), from (43), we obtain,

$$\varphi'(t) \leq \sup_{x \in \Theta(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t),$$

486 which means that

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \sup_{x \in \Theta(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \quad (46)$$

487 where equality holds in the case of 3D Euler or 2D QG equations. We use
 488 now the function α introduced in Constantin (1994); Constantin and Fefferman
 489 (1994) for the 3D Navier-Stokes or 3D Euler equations and in Constantin et al.
 490 (1994) for the 2D QG equation, defined for all $(x, t) \in \mathcal{O}$ by,

$$\alpha(x, t) = c_d P.V. \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d}, \quad (47)$$

491 where $\hat{y} = \frac{y}{|y|}$ and in the case of 3D Navier-Stokes or 3D Euler equations for
 492 which $d = 3$, $c_d = \frac{3}{4\pi}$ and in the case of 2D QG equation for which $d = 2$,
 493 $c_d = \frac{1}{2\pi}$. We use the fact that $|\omega(x+y, t)|\xi(x+y, t) = \omega(x+y, t)$ and the
 494 fact that \mathbf{D}_d is linear in comparison with its second variable, to rewrite (47) as
 495 follows:

$$\alpha(x, t) = c_d P.V. \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \omega(x+y, t), \xi(x, t)) \frac{dy}{|y|^d}. \quad (48)$$

496 By using the Biot-Savart law (see Chorin and Marsden (1993)) for which in the
 497 case of Euler and Navier-Stokes equations, we have

$$\mathbf{v}(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y) dy,$$

498 and in the case of 2D QG equations, we get an equivalent formula

$$\mathbf{v}(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|y|} \omega(x + y, t) dy,$$

499 we deduce as in [Constantin \(1994\)](#); [Constantin and Fefferman \(1994\)](#) and [Constantin et al.](#)
500 [\(1994\)](#) that for all $(x, t) \in \mathcal{O}$

$$(\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) = \alpha(x, t) |\omega(x, t)|.$$

501 Therefore, from [\(46\)](#), we deduce that for all $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt} \|\omega(t)\|_\infty &\leq \sup_{x \in \Theta(t)} \alpha(x, t) |\omega(x, t)| \\ &= \left(\sup_{x \in \Theta(t)} (\alpha(x, t)) \right) \|\omega(t)\|_\infty, \end{aligned} \quad (49)$$

502 where we have used the fact that for all $x \in \Theta(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty$. Let us
503 estimate now $\alpha(x, t)$ for any $t \in [0, T]$ and $x \in \Theta(t)$. For this purpose, let us
504 take $t \in [0, T]$ and $x \in \Theta(t)$, then we decompose the term $\alpha(x, t)$ as the sum of
505 three terms,

$$\alpha(x, t) = I_1 + I_2 + I_3 \quad (50)$$

506 where,

$$I_1 = c_d \int_{B(0, \min(\rho(t), \rho_0(t)))} \mathbf{D}_d(\hat{y}, \omega(x + y, t), \xi(x, t)) \frac{dy}{|y|^d}, \quad (51)$$

507

$$I_2 = c_d \int_{B(0, \rho_0(t)) \cap B(0, \min(\rho(t), \rho_0(t)))^c} \mathbf{D}_d(\hat{y}, \omega(x + y, t), \xi(x, t)) \frac{dy}{|y|^d} \quad (52)$$

508 and

$$I_3 = c_d \int_{B(0, \rho_0(t))^c} \mathbf{D}_d(\hat{y}, \omega(x + y, t), \xi(x, t)) \frac{dy}{|y|^d}. \quad (53)$$

509 Then, we estimate the three terms I_1, I_2 and I_3 . For the term I_1 , from [\(51\)](#) we
510 get

$$\begin{aligned} I_1 &= c_d \int_{B(0, \min(\rho(t), \rho_0(t)))} \frac{\mathbf{D}_d(\hat{y}, \xi(x + y, t), \xi(x, t))}{|y|} |\omega(x + y, t)| \frac{dy}{|y|^{d-1}} \\ &\leq c_d \mathbf{A}_d(t) \int_{B(0, \min(\rho(t), \rho_0(t)))} \frac{|\omega(x + y, t)|}{|y|^{d-1}} dy \\ &= c_d \mathbf{A}_d(t) \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)^c} \frac{|\omega(z, t)|}{|x - z|^{d-1}} dz \\ &+ c_d \mathbf{A}_d(t) \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z, t)|}{|x - z|^{d-1}} dz \\ &\leq c_d \mathbf{A}_d(t) \Omega(t) \int_{B(0, \rho(t))} \frac{dy}{|y|^{d-1}} + c_d \mathbf{A}_d(t) \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z, t)|}{|z - x|^{d-1}} dz. \end{aligned}$$

511 Furthermore, we have

$$\int_{B(0,\rho(t))} \frac{dy}{|y|^{d-1}} = |B(0,1)|\rho(t).$$

512 Therefore, by using the fact that $c_d|B(0,1)| \leq 1$ (since $|B(0,1)| = \frac{4\pi}{3}$ for $d = 3$
513 and $|B(0,1)| = \pi$ for $d = 2$), we deduce

$$I_1 \leq \mathbf{A}_d(t)\Omega(t)\rho(t) + \mathbf{A}_d(t)I_{1,1}, \quad (54)$$

514 where

$$I_{1,1} \stackrel{\text{def}}{=} c_d \int_{B(x,\min(\rho(t),\rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z,t)|}{|z-x|^{d-1}} dz. \quad (55)$$

515 Let $\varepsilon(t) \stackrel{\text{def}}{=} \frac{\pi(t)}{\|\omega(t)\|_\infty}$, then we have

$$I_{1,1} \leq c_d \int_{B(x,\varepsilon(t))} \frac{|\omega(z,t)|}{|z-x|^{d-1}} dz + c_d I_{1,2}, \quad (56)$$

516 with

$$I_{1,2} \stackrel{\text{def}}{=} \int_{B(x,\varepsilon(t))^c \cap B(x,\min(\rho(t),\rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z,t)|}{|z-x|^{d-1}} dz. \quad (57)$$

517 On one hand, we have

$$\begin{aligned} c_d \int_{B(x,\varepsilon(t))} \frac{|\omega(z,t)|}{|z-x|^{d-1}} dz &\leq c_d \|\omega(t)\|_\infty \int_{B(x,\varepsilon(t))} \frac{dz}{|z-x|^{d-1}} \\ &= c_d |B(0,1)| \|\omega(t)\|_\infty \varepsilon(t) \\ &\leq \pi(t). \end{aligned} \quad (58)$$

518 On the other hand, we estimate $I_{1,2}$.

519 If $\mathcal{V}(t) = \emptyset$ or $\min(\rho(t), \rho_0(t)) \leq \varepsilon(t)$ then from (57) we infer that $I_{1,2} = 0$.
520 Let us assume now that $\mathcal{V}(t) \neq \emptyset$ and $\min(\rho(t), \rho_0(t)) > \varepsilon(t)$. Then, from (57)
521 we get

$$\begin{aligned} I_{1,2} &= \int_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \left(\int_{\partial B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| \frac{d\gamma(z)}{|z-x|^{d-1}} \right) dR \\ &= \int_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \frac{1}{R^{d-1}} \left(\int_{\partial B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| d\gamma(z) \right) dR. \end{aligned} \quad (59)$$

522 We introduce the function F_t defined from $[0, \rho_0(t)]$ to $[0, +\infty[$ for all $0 \leq R \leq$
523 $\rho_0(t)$ by

$$F_t(R) \stackrel{\text{def}}{=} \int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz. \quad (60)$$

524 We observe that $\frac{dF_t}{dR}(R) = \int_{\partial B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| d\gamma(z)$, then from (59) we get
525 $I_{1,2} = \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{1}{R^{d-1}} \frac{dF_t}{dR}(R) dR$ and by using an integration by parts we
526 obtain

$$I_{1,2} = \frac{F_t(\min(\rho(t), \rho_0(t)))}{\min(\rho(t), \rho_0(t))^{d-1}} - \frac{F_t(\varepsilon(t))}{\varepsilon(t)^{d-1}} + (d-1) \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{F_t(R)}{R^d} dR. \quad (61)$$

527 From (60), we notice that $\pi(t) = \sup_{0 < R \leq \rho_0(t)} \frac{F_t(R)}{R^{d-1}}$, then from (61) we deduce

$$\begin{aligned} I_{1,2} &\leq \pi(t) + (d-1)\pi(t) \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{dR}{R} \\ &= \pi(t) \left(1 + (d-1) \log \left(\frac{\min(\rho(t), \rho_0(t))}{\varepsilon(t)} \right) \right). \end{aligned}$$

528 Therefore whatever the case we get

$$I_{1,2} \leq \pi(t) \left(1 + (d-1) \log^+ \left(\frac{\min(\rho(t), \rho_0(t))}{\varepsilon(t)} \right) \right). \quad (62)$$

529 Since $\min(\rho(t), \rho_0(t)) \leq \rho(t)$ and thanks to (38) we get

$$\begin{aligned} \frac{\min(\rho(t), \rho_0(t))}{\varepsilon(t)} &\leq 4(d+1)c_d(T^* - t)\|\omega(t)\|_\infty \\ &\leq 4(T^* - t)\|\omega(t)\|_\infty. \end{aligned}$$

530 Therefore from (62) we infer

$$I_{1,2} \leq \pi(t) \left(1 + (d-1) \log^+ (4(T^* - t)\|\omega(t)\|_\infty) \right). \quad (63)$$

531 Thanks to (55)-(58) and (63), we get

$$I_{1,1} \leq \pi(t) + c_d\pi(t)(1 + (d-1) \log^+ (4(T^* - t)\|\omega(t)\|_\infty)). \quad (64)$$

532 Then thanks to (64), from (54) we obtain

$$I_1 \leq \mathbf{A}_d(t)(\Omega(t)\rho(t) + \pi(t) + c_d\pi(t)(1 + (d-1) \log^+ (4(T^* - t)\|\omega(t)\|_\infty))). \quad (65)$$

533 By using the definition of the functions ρ and Ω , for which we have $\Omega(t)\rho(t) \leq$
534 $4c_d(d+1)\pi(t)$, from (65) we deduce

$$\begin{aligned} I_1 &\leq \mathbf{A}_d(t)\pi(t)(4c_d(d+1) + 1 + c_d(1 + (d-1) \log^+ (4(T^* - t)\|\omega(t)\|_\infty))) \\ &\leq 6\mathbf{A}_d(t)\pi(t)(1 + \log^+ (4(T^* - t)\|\omega(t)\|_\infty)). \end{aligned} \quad (66)$$

535 For the term I_2 given by (52), after using the change of variables $z = x + y$,
 536 we decompose I_2 as the sum of two terms $I_{2,1}$ and $I_{2,2}$ defined by

$$I_{2,1} = c_d \int_{B(x, \rho_0(t)) \cap B(x, \min(\rho(t), \rho_0(t)))^c \cap \mathcal{V}(t)^c} \mathbf{D}_d(\widehat{z-x}, \omega(z, t), \xi(x, t)) \frac{dz}{|z-x|^d} \quad (67)$$

537 and

$$I_{2,2} = c_d \int_{B(x, \rho_0(t)) \cap B(x, \min(\rho(t), \rho_0(t)))^c \cap \mathcal{V}(t)} \mathbf{D}_d(\widehat{z-x}, \omega(z, t), \xi(x, t)) \frac{dz}{|z-x|^d}. \quad (68)$$

538 Let us estimate the terms $I_{2,1}$ and $I_{2,2}$. For this purpose, we introduce the
 539 function γ defined for all $t \in [0, T^*[$ by $\gamma(t) = \min\left(\frac{1}{4A_0(t)\Omega(t)(T^*-t)}, \rho_0(t)\right)$.
 540 From (67), we observe

$$\begin{aligned} I_{2,1} &\leq c_d \int_{B(x, \rho_0(t)) \cap \mathcal{V}(t)^c} \mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+ \frac{|\omega(z, t)|}{|z-x|^d} dz \\ &\leq c_d \Omega(t) \int_{B(x, \rho_0(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz \\ &= c_d \Omega(t) \int_{B(x, \gamma(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz \\ &\quad + c_d \Omega(t) \int_{B(x, \rho_0(t)) \cap B(x, \gamma(t))^c} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz. \end{aligned} \quad (69)$$

541 Furthermore, on one hand, by (33) we have for all $z \in B(x, \gamma(t))$,

$$\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+ \leq A_0(t)|z-x|,$$

542 and hence, we obtain

$$\begin{aligned} \Omega(t) \int_{B(x, \gamma(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz &\leq A_0(t)\Omega(t) \int_{B(x, \gamma(t))} \frac{dz}{|z-x|^{d-1}} \\ &= |B(0, 1)| A_0(t)\Omega(t)\gamma(t) \\ &\leq \frac{|B(0, 1)|}{4(T^*-t)}, \end{aligned} \quad (70)$$

543 where we have used the definition of the function γ . On the other hand, since

544 $|\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))| \leq 1$, we get

$$\begin{aligned}
& \int_{B(x, \rho_0(t)) \cap B(x, \gamma(t))^c} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz \\
& \leq \int_{B(x, \rho_0(t)) \cap B(x, \gamma(t))^c} \frac{dz}{|z-x|^d} \\
& = |B(0, 1)| \int_{\gamma(t)}^{\rho_0(t)} \frac{ds}{s} \\
& = |B(0, 1)| \log \left(\frac{\rho_0(t)}{\gamma(t)} \right) \\
& = |B(0, 1)| \log^+ (4\rho_0(t)A_0(t)\Omega(t)(T^* - t)), \tag{71}
\end{aligned}$$

545 where we have used again the definition of the function γ . Owing to (70) and
546 (71), from (69), we deduce

$$I_{2,1} \leq c_d |B(0, 1)| \left(\frac{1}{4(T^* - t)} + \Omega(t) \log^+ (4\rho_0(t)A_0(t)\Omega(t)(T^* - t)) \right). \tag{72}$$

547 For the term $I_{2,2}$, if $\mathcal{V}(t) = \emptyset$ or if $\rho(t) \geq \rho_0(t)$ then from (68) we infer that
548 $I_{2,2} = 0$. Let us assume now that $\mathcal{V}(t) \neq \emptyset$ and $\rho(t) < \rho_0(t)$. Then, from (68)
549 we get

$$\begin{aligned}
I_{2,2} & = c_d \int_{\rho(t)}^{\rho_0(t)} \left(\int_{\partial B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| \frac{d\gamma(z)}{|z-x|^d} \right) dR \\
& = c_d \int_{\rho(t)}^{\rho_0(t)} \frac{1}{R^d} \left(\int_{\partial B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| d\gamma(z) \right) dR. \tag{73}
\end{aligned}$$

550 By using (60) and an integration by parts we obtain

$$I_{2,2} = c_d \left(\frac{F_t(\rho_0(t))}{\rho_0(t)^d} - \frac{F_t(\rho(t))}{\rho(t)^d} + d \int_{\rho(t)}^{\rho_0(t)} \frac{F_t(R)}{R^{d+1}} dR \right). \tag{74}$$

551 Since $\pi(t) = \sup_{0 < R \leq \rho_0(t)} \frac{F_t(R)}{R^{d-1}}$, then from (74) we deduce

$$\begin{aligned}
I_{2,2} & \leq c_d \left(\frac{\pi(t)}{\rho_0(t)} + d\pi(t) \int_{\rho(t)}^{\rho_0(t)} \frac{dR}{R^2} \right) \\
& \leq c_d(d+1) \frac{\pi(t)}{\rho(t)},
\end{aligned}$$

552 where we have used the fact that $\rho(t) < \rho_0(t)$. Therefore whatever the case, we
553 obtain that

$$I_{2,2} \leq c_d(d+1) \frac{\pi(t)}{\rho(t)}. \tag{75}$$

554 By using (38) the definition of the function ρ , from (75) we deduce

$$I_{2,2} \leq \frac{1}{4(T^* - t)}. \tag{76}$$

555 Owing to (72) and (76), since also that $I_2 = I_{2,1} + I_{2,2}$ and $c_d|B(0,1)| \leq 1$ we
 556 thus obtain

$$I_2 \leq \frac{1}{2(T^* - t)} + \Omega(t) \log^+ (4\rho_0(t)A_0(t)\Omega(t)(T^* - t)). \quad (77)$$

557 For the term I_3 given by (53), to obtain a precise non blowup criterion for 3D
 558 the Navier-Stokes, 3D Euler and 2D QG equations that could be used easily in
 559 numerical experiments, it was important to explicit the constant involved in the
 560 estimate of the term I_3 . For this purpose, we deal first with the case of the 3D
 561 the Navier-Stokes and 3D Euler equations, then after we consider the case of
 562 the 2D QG equations.

563 **In the case of the 3D Euler or 3D Navier-Stokes equations** for which
 564 $d = 3$, we get $\mathbf{D}_3(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)) \det(\hat{y}, \omega(x+y, t), \xi(x, t))$.
 565 Since $\det(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\xi(x, t) \times \hat{y}) \cdot \omega(x+y, t)$ and $\omega(x+y, t) =$
 566 $\nabla_y \times u(x+y, t)$, we deduce

$$\mathbf{D}_3(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y}) \cdot \nabla_y \times u(x+y, t).$$

567 Then, after using an integration by parts, from (53), we deduce,

$$\begin{aligned} I_3 &= c_3 \int_{B(0, \rho_0(t))^c} \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \cdot u(x+y, t) dy \\ &\quad - c_3 \int_{\partial B(0, \rho_0(t))} \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \cdot \hat{y} \times u(x+y, t) d\gamma(y). \end{aligned} \quad (78)$$

568 After setting $\psi(y) \equiv \frac{(\hat{y} \cdot \xi(x, t))}{|y|^3}$ and $\mathbf{V}(y) \equiv (\xi(x, t) \times \hat{y})$, by using the following
 569 vectorial identity $\nabla \times (\psi \mathbf{V}) = \nabla \psi \times \mathbf{V} + (\nabla \times \mathbf{V})\psi$, we obtain after elementary
 570 computations, that for all $y \neq 0$,

$$\left| \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \right| \leq \left| \nabla \left(\frac{\hat{y}}{|y|^3} \right) \right| + \frac{|\nabla_y \times (\xi(x, t) \times \hat{y})|}{|y|^3}.$$

571 We have $\left| \nabla \left(\frac{\hat{y}}{|y|^3} \right) \right| \leq \frac{3}{|y|^4}$. Furthermore, we have $\nabla_y \times (\xi(x, t) \times \hat{y}) = (\nabla_y \cdot$
 572 $\hat{y})\xi(x, t) - (\xi(x, t) \cdot \nabla_y)\hat{y}$ and then we deduce $|\nabla_y \times (\xi(x, t) \times \hat{y})| \leq |\nabla \cdot \hat{y}| + |\nabla \hat{y}| \leq$
 573 $\frac{3}{|y|}$. After gathering these results, we obtain that for all $y \neq 0$,

$$\left| \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \right| \leq \frac{6}{|y|^4}.$$

574 Therefore, from (78) we obtain

$$I_3 \leq 6c_3 \int_{B(0, \rho_0(t))^c} \frac{|u(x+y, t)|}{|y|^4} dy + c_3 \int_{\partial B(0, \rho_0(t))} |u(x+y, t)| \frac{dy}{|y|^3} d\gamma(y). \quad (79)$$

575 **In the case of 2D QG equations** for which $d = 2$, we get $\mathbf{D}_2(\hat{y}, \omega(x +$
576 $y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)^\perp)(\omega(x + y, t) \cdot \xi(x, t)^\perp)$. Since $\omega(x + y, t) = \nabla_y^\perp u(x + y, t)$,
577 we deduce

$$\mathbf{D}_2(\hat{y}, \omega(x + y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp \cdot \nabla_y^\perp u(x + y, t).$$

578 Then, after using an integration by parts, from (53), we deduce,

$$\begin{aligned} I_3 &= -c_2 \int_{B(0, \rho_0(t))^c} \nabla_y^\perp \cdot \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \right) u(x + y, t) dy \\ &\quad + c_2 \int_{\partial B(0, \rho_0(t))} \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \right) \cdot y^\perp u(x + y, t). \end{aligned} \quad (80)$$

579 After setting $\psi(y) \equiv \frac{(\hat{y} \cdot \xi(x, t)^\perp)}{|y|^2}$ and $\mathbf{V}(y) \equiv \xi(x, t)^\perp$, by using the following
580 vectorial identity $\text{curl}(\psi \mathbf{V}) = \nabla^\perp \psi \cdot \mathbf{V} + \psi \text{curl} \mathbf{V}$, we obtain after elementary
581 computations, that for all $y \neq 0$,

$$\begin{aligned} \left| \nabla_y^\perp \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \right) \right| &\leq \left| \nabla \left(\frac{\hat{y}}{|y|^2} \right) \right| \\ &\leq \frac{2}{|y|^3}. \end{aligned}$$

582 Therefore, from (80) we obtain

$$I_3 \leq 2c_2 \int_{B(0, \rho_0(t))^c} \frac{|u(x + y, t)|}{|y|^3} dy + c_2 \int_{\partial B(0, \rho_0(t))} \frac{|u(x + y, t)|}{|y|^2} d\gamma(y). \quad (81)$$

583 Therefore, whatever the case considered, 3D Navier-Stokes, 3D Euler or 2D QG
584 equations, from (79) and (81) we get

$$I_3 \leq d(d-1)c_d \int_{B(0, \rho_0(t))^c} \frac{|u(x + y, t)|}{|y|^{d+1}} dy + c_d \int_{\partial B(0, \rho_0(t))} \frac{|u(x + y, t)|}{|y|^d} d\gamma(y). \quad (82)$$

585 Then from (82), we obtain

$$\begin{aligned} I_3 &\leq c_d \|u(t)\|_\infty \left(d(d-1) \int_{B(0, \rho_0(t))^c} \frac{dy}{|y|^{d+1}} + \int_{\partial B(0, \rho_0(t))} \frac{d\gamma(y)}{|y|^d} \right) \\ &= c_d \|u(t)\|_\infty \left(d(d-1) |B(0, 1)| \int_{\rho_0(t)}^{+\infty} \frac{ds}{s^2} + \frac{|\partial B(0, 1)|}{\rho_0(t)} \right) \\ &\leq 9 \frac{\|u(t)\|_\infty}{\rho_0(t)}. \end{aligned} \quad (83)$$

586 Then, owing to (66), (77) and (83), from (50) we deduce that for any $t \in [0, T]$
587 and $x \in \Theta(t)$,

$$\begin{aligned} \alpha(x, t) &\leq 6\mathbf{A}_d(t) \pi(t) (1 + \log^+(4(T^* - t) \|\omega(t)\|_\infty)) \\ &\quad + \frac{1}{2(T^* - t)} + \Omega(t) \log^+(4\rho_0(t) A_0(t) \Omega(t) (T^* - t)) + 9 \frac{\|u(t)\|_\infty}{\rho_0(t)}. \end{aligned} \quad (84)$$

588 By using (32), we deduce that for any $t \in [0, T]$ and $x \in \Theta(t)$,

$$\begin{aligned} \alpha(x, t) &\leq 6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)) \\ &\quad + \frac{3}{4(T^* - t)} + \Omega(t) \log^+(4\rho_0(t)A_0(t)\Omega(t)(T^* - t)). \end{aligned} \quad (85)$$

589 Furthermore, thanks to (35) we get $(T^* - t)\Omega(t) \log^+(4\rho_0(t)\mathbf{A}_0(t)) \leq \frac{1}{8}$ and
590 $(T^* - t)\Omega(t) < 1$ which implies that $\log^+((T^* - t)\Omega(t)) = 0$ and hence we obtain
591 that for all $t \in [0, T^*]$,

$$\begin{aligned} (T^* - t)\Omega(t) \log^+(4\rho_0(t)\mathbf{A}_0(t)\Omega(t)(T^* - t)) &\leq (T^* - t)\Omega(t)(\log^+(4\rho_0(t)\mathbf{A}_0(t)) + \log^+(\Omega(t)(T^* - t))) \\ &= (T^* - t)\Omega(t) \log^+(4\rho_0(t)\mathbf{A}_0(t)) \\ &\leq \frac{1}{8}. \end{aligned}$$

592 Therefore from (85), we deduce that for any $t \in [0, T]$ and $x \in \Theta(t)$,

$$\alpha(x, t) \leq 6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)) + \frac{7}{8(T^* - t)}. \quad (86)$$

593 Then from (49) we deduce

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \left(6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)) + \frac{7}{8(T^* - t)} \right) \|\omega(t)\|_\infty, \quad (87)$$

594 which is valid for all $t \in [0, T]$ and $T < T^*$ and then inequality (87) is valid for
595 all $t \in [0, T^*]$. Let $t_0 \in [0, T^*]$ such that

$$4(T^* - t_0) \leq 1. \quad (88)$$

596 Then we get that for all $t \in [t_0, T^*]$, $4(T^* - t) \leq 1$ and hence

$$\begin{aligned} \log^+(4(T^* - t)\|\omega(t)\|_\infty) &\leq \log^+(4(T^* - t)) + \log^+\|\omega(t)\|_\infty \\ &= \log^+\|\omega(t)\|_\infty. \end{aligned} \quad (89)$$

597 Owing to (89), from (87) we deduce that for all $t \in [t_0, T^*]$

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \left(6\mathbf{A}_d(t)\pi(t)(1 + \log^+\|\omega(t)\|_\infty) + \frac{7}{8(T^* - t)} \right) \|\omega(t)\|_\infty. \quad (90)$$

598 Thanks to Grönwall inequality, from (90) we deduce that for all $t \in [t_0, T^*]$,

$$\begin{aligned} \|\omega(t)\|_\infty &\leq \|\omega(t_0)\|_\infty e^{\int_{t_0}^t (6\mathbf{A}_d(\tau)\pi(\tau)(1 + \log^+\|\omega(\tau)\|_\infty) + \frac{7}{8(T^* - \tau)}) d\tau} \\ &= \left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \|\omega(t_0)\|_\infty e^{\int_{t_0}^t 6\mathbf{A}_d(\tau)\pi(\tau)(1 + \log^+\|\omega(\tau)\|_\infty) d\tau}. \end{aligned} \quad (91)$$

599 Since the function $z \mapsto \log^+(z)$ is non-decreasing on $]0, +\infty[$, then after applying
600 the function $1 + \log^+$ to the inequality (91) and using the fact that $\log^+(ab) \leq$
601 $\log^+ a + \log^+ b$, we thus obtain that for all $t \in [t_0, T^*[$

$$\begin{aligned} 1 + \log^+ \|\omega(t)\|_\infty &\leq 1 + \log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right) + \log^+ \|\omega(t_0)\|_\infty \\ &\quad + 6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) (1 + \log^+ \|\omega(\tau)\|_\infty) d\tau. \end{aligned} \quad (92)$$

602 Since the function $t \mapsto \log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right)$ is increasing over $[t_0, T^*[$ then thanks
603 to Gronwall Lemma, by (92) we deduce that for all $t \in [t_0, T^*[$,

$$1 + \log^+ \|\omega(t)\|_\infty \leq \left(\log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right) + 1 + \log^+ \|\omega(t_0)\|_\infty \right) e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau}. \quad (93)$$

604 Since $\log \|\omega(t)\|_\infty \leq 1 + \log^+ \|\omega(t)\|_\infty$, then from (93) we infer that for all
605 $t \in [t_0, T^*[$,

$$\begin{aligned} \|\omega(t)\|_\infty &\leq \exp \left(\left(\log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right) + 1 + \log^+ \|\omega(t_0)\|_\infty \right) e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau} \right) \\ &= \exp \left((1 + \log^+ \|\omega(t_0)\|_\infty) e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau} \right) \left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau}. \end{aligned} \quad (94)$$

606 Let us assume that there exists $t_1 \in [0, T^*[$ such that $\int_{t_1}^{T^*} \mathbf{A}_d(\tau) \pi(\tau) d\tau <$
607 $+\infty$. Then, in addition of (88), we choose $t_0 \in [t_1, T^*[$ such that $M_{t_0} \stackrel{\text{def}}{=} \int_{t_0}^{T^*} \mathbf{A}_d(\tau) \pi(\tau) d\tau <$
608 $\frac{1}{6} \log \left(\frac{8}{7} \right)$. We thus get for all $t \in [t_0, T^*[$,

$$\frac{7}{8} e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau} \leq \frac{7}{8} e^{6M_{t_0}} < 1.$$

609 Therefore with $\eta_{t_0} \stackrel{\text{def}}{=} \frac{7}{8} e^{6M_{t_0}}$, from (94) we deduce that for all $t \in [t_0, T^*[$

$$\|\omega(t)\|_\infty \leq \exp \left((1 + \log^+ \|\omega(t_0)\|_\infty) e^{6M_{t_0}} \right) \left(\frac{T^* - t_0}{T^* - t} \right)^{\eta_{t_0}}. \quad (95)$$

610 Since $\eta_{t_0} < 1$, from (95) we thus deduce that $\int_{t_0}^{T^*} \|\omega(t)\|_\infty dt < +\infty$. Since $u \in$
611 $C([0, T^*]; H_\sigma^r(\mathbb{R}^d))$ and thanks to the Sobolev embedding $H^r(\mathbb{R}^d) \hookrightarrow BC^3(\mathbb{R}^d)$
612 due to $r > \frac{d}{2} + 3$, we infer that $\omega \in C([0, T^*]; BC^2(\mathbb{R}^d))$ which implies that
613 $\int_0^{t_0} \|\omega(t)\|_\infty dt < +\infty$. Therefore we deduce that $\int_0^{T^*} \|\omega(t)\|_\infty dt < +\infty$.
614 If u blows up at the finite time T^* then thanks to (20) and (23) we have

615 $\int_0^{T^*} \|\omega(t)\|_\infty dt = +\infty$ which leads to a contradiction. Then, we deduce that u
616 cannot blow up at the time T^* which concludes the first part of proof.
617 Thanks to (44), we have already $\nabla|\omega|(x, t) = 0$. Since $\nabla \cdot \omega = 0$ and $\omega = |\omega|\xi$,
618 then we get

$$0 = \nabla \cdot \omega = |\omega|\nabla \cdot \xi + \xi \cdot \nabla|\omega|. \quad (96)$$

619 However, for all $x \in \Theta(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty > 0$ and from (44), we have
620 $\nabla|\omega|(x, t) = 0$. Therefore, from (96), we deduce that for all $t \in [0, T]$ and
621 $x \in \Theta(t)$,

$$\nabla \cdot \xi(x, t) = 0, \quad (97)$$

622 which completes the proof. \square

623 6. No blow up in finite time for numerical experiments

624 In this section, we show the non-blowup in finite time of the solutions of the
625 3D Euler equations in the numerical experiments considered these last years.

626 First, we emphasize that the singularity discovered in Luo and Hou (2014b)
627 which lies right on the boundary is not relevant in the case of the whole do-
628 main \mathbb{R}^3 . Indeed recently, the authors found a convincing numerical evidence
629 for a singular solution to the Euler equations in a fluid with periodic boundary
630 condition along the axial direction and no-flow boundary condition on the solid
631 wall Luo and Hou (2014b) (see also Luo and Hou (2014a)), for which the point
632 of the potential singularity, which is also the point of the maximum vorticity,
633 is always located at the solid boundary. However thanks to Theorem 5.1, we
634 deduce that such singularity can not exist in the whole domain \mathbb{R}^3 . Indeed, in
635 the whole domain of \mathbb{R}^3 at any point of the maximum vorticity, $q_0 \in \mathbb{R}^3$, thanks
636 to Theorem 5.1 we get $\nabla|\omega|(q_0, t) = 0$ for any time t before the alleged time of
637 singularity T^* , then this result combined with the fact that the vorticity ω is a
638 divergence-free vector field, yields to get $\nabla \cdot \xi(q_0, t) = 0$ in Theorem 5.1. However
639 in Luo and Hou (2014b), the presence of a solid boundary and the fact that q_0
640 the point of the maximum vorticity is always located on the solid boundary, pre-
641 vent to get $\nabla|\omega|(q_0, t) = 0$ and this allows to get $\nabla \cdot \xi(q_0, t) \sim (T^* - t)^{-2.9165} \neq 0$
642 as it is observed in their numerical test. This latter is the main element used to
643 invalidate the Deng-Hou-Yu non-blowup criterion Deng et al. (2005, 2006a).

644
645 There have been many computational attempts to find finite-time singulari-
646 ties of the 3D Euler and Navier-Stokes equations: see, e.g Melander and Hussain
647 (1989); Kerr and Hussain (1989); Pumir and Siggia (1990); Kerr (1993); Grauer et al.
648 (1998); Hou and Li (2006); Kerr (2005). One example that has been studied ex-
649 tensively in these numerical investigations is the interaction of two perturbed
650 anti-parallel vortex tubes. All the subsequent calculations assumed an anti-parallel
651 geometry, for which there are two symmetry planes. One in $y - z$ is between the
652 vortices and was called the 'dividing plane'. The other in $x - z$ is at the position
653 of maximum perturbation and was called the 'symmetry plane'. The difficulty
654 faced in each computational attempts cited was to find a better initial condition

655 within this geometry (see Kerr (2006)). From these computational attempts, a
 656 numerical controversy takes place around the question to know whether or not
 657 there is finite-time blow-up of the solutions of Euler equations (see Kerr (2006)).

658 In this section, we propose an answer to this controversy by using our Theo-
 659 rem 5.1. By using the anisotropic structure of regions of high vorticity described
 660 in Kerr (2005, 1998), we show straightforward thanks to our Theorem 5.1 that
 661 the solutions of Euler equations cannot blow up in finite time in these numerical
 662 experiments Kerr (2005, 1998, 1997).

663 For this purpose, we give a first bound of the function π defined by (37) in the
 664 following Lemma. The bound given in Lemma 6.1 of the function π is not a
 665 sharp bound but obtained without assumptions.

666 **Lemma 6.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such
 667 that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler
 668 equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

669 Under the definitions (32)-(37) in the Theorem 5.1, we have the following esti-
 670 mate: for all $t \in [0, T^*[$

$$\pi(t) \leq 3\|\omega(t)\|_\infty \sup_{x \in \Theta(t)} |\mathcal{V}(t) \cap B(x, \rho_0(t))|^{\frac{1}{d}}.$$

671 *Proof.* For any $t \in [0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$, we get

$$\frac{1}{R^{d-1}} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \leq \frac{1}{R^{d-1}} \|\omega(t)\|_\infty |\mathcal{V}(t) \cap B(x, R)|.$$

672 Furthermore, we have

$$\begin{aligned} |\mathcal{V}(t) \cap B(x, R)| &= |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} |\mathcal{V}(t) \cap B(x, R)|^{\frac{d-1}{d}} \\ &\leq |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} |B(x, R)|^{\frac{d-1}{d}} \\ &= |B(0, 1)|^{\frac{d-1}{d}} |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} R^{d-1} \\ &\leq 3 |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} R^{d-1} \end{aligned}$$

673 where we have used the fact that $|B(0, 1)| = \left(\frac{4\pi}{3}\right)^{\frac{2}{3}}$ if $d = 3$ or $|B(0, 1)| = \pi^{\frac{1}{2}}$
 674 if $d = 2$.

675 Then, we deduce for that any $t \in [0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$,

$$\frac{1}{R^{d-1}} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \leq 3\|\omega(t)\|_\infty |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}}. \quad (98)$$

676 Owing to (98), we thus conclude the proof. \square

677 Now, we can show straightforward thanks to Theorem 5.1 and Lemma 6.1
678 that the solutions of Euler equations cannot blow up in finite time in the numerical experiments Kerr (2005, 1998, 1997).
679 For this purpose, we recall that in the numerical experiments Kerr (2005, 1998),
680 the author show that the blow-up rates at some time T^* the alleged time of singularity, to be considered for $\|\omega(t)\|_\infty$, $\|u(t)\|_\infty$ and $\|\nabla\xi(t)\|_\infty$ in Kerr (2005, 1998) are

$$\|\omega(t)\|_\infty \sim (T^* - t)^{-1}, \|u(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}} \text{ and } \|\nabla\xi(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}}, \quad (99)$$

684 for time $t \in [t_0, T^*[$ with $t_0 \in [0, T^*[$ sufficiently close to T^* . Moreover for time
685 $t \in [t_0, T^*[$ with t_0 sufficiently close to T^* , the author showed that the support
686 of the maximum vorticity

$$\mathcal{E}(t) = \{x \in \mathbb{R}^3, |\omega(x, t)| \sim \|\omega(t)\|_\infty\}$$

687 is characterized by two length scales $(T^* - t)$ and $(T^* - t)^{\frac{1}{2}}$ and its volume is
688 bounded by

$$|\mathcal{E}(t)| \lesssim (T^* - t)^2. \quad (100)$$

689 Thanks to the blow-up rates (99), from (32) we get that for all $t \in [t_0, T^*[$

$$\rho_0(t) \sim (T^* - t)^{\frac{1}{2}}.$$

690 From (33), thanks again to (99) we have that for all $t \in [t_0, T^*[$

$$\begin{aligned} \mathbf{A}_0(t) &\leq \sup_{x \in \Theta(t)} \|\nabla\xi(t)\|_{L^\infty(B(x, \rho_0(t)))} \\ &\lesssim (T^* - t)^{-\frac{1}{2}}. \end{aligned}$$

691 Then we deduce that for all $t \in [t_0, T^*[$

$$\mathbf{A}_0(t)\rho_0(t) \lesssim 1. \quad (101)$$

692 Therefore thanks to (101), from the definition (35) of the function Ω , we deduce
693 for all $t \in [t_0, T^*[$

$$\Omega(t) \gtrsim (T^* - t)^{-1}. \quad (102)$$

694 Owing to (102) and since for all $t \in [t_0, T^*[$, $\|\omega(t)\|_\infty \sim (T^* - t)^{-1}$ thanks to
695 (99), for the set $\mathcal{V}(t)$ defined by (36) we deduce that for all $t \in [t_0, T^*[$

$$\mathcal{V}(t) = \{x \in \mathbb{R}^3, |\omega(x, t)| \sim \|\omega(t)\|_\infty\}.$$

696 Then thanks to (100) we get that for all $t \in [t_0, T^*[$

$$|\mathcal{V}(t)| \lesssim (T^* - t)^2. \quad (103)$$

697 Thanks to Lemma 6.1, inequality (103) and the fact that $\|\omega(t)\|_\infty \sim (T^* - t)^{-1}$
698 thanks to (99), for the function π defined by (37), we get that for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim (T^* - t)^{-\frac{1}{3}}. \quad (104)$$

700 Furthermore, thanks to (99), for the function \mathbf{A}_3 defined by (39) for $d = 3$, we
 701 get that for all $t \in [t_0, T^*[$

$$\mathbf{A}_3(t) \leq \|\nabla \xi(t)\|_\infty \lesssim (T^* - t)^{-\frac{1}{2}}. \quad (105)$$

702 Owing to (104) and (105) we deduce that for all $t \in [t_0, T^*[$

$$\mathbf{A}_3(t)\pi(t) \lesssim (T^* - t)^{-\frac{5}{6}}. \quad (106)$$

703 Then, we deduce

$$\int_{t_0}^{T^*} \mathbf{A}_3(t)\pi(t) \lesssim (T^* - t_0)^{\frac{1}{6}} < +\infty. \quad (107)$$

704 Therefore, thanks to (107) and Theorem 5.1, we deduce that the solutions of
 705 the Euler equations considered for the numerical experiments Kerr (2005, 1998)
 706 cannot blow-up in finite time at the alleged time of singularity T^* . If one con-
 707 siders the plausible scenario of blow up proposed in Brenner et al. (2016), one
 708 observed that we get also the blow-up rates (99) and the estimate (100) (see
 709 (Hormoz and Brenner, 2012, section 4)), hence the potential mechanism pro-
 710 posed for the blow-up in finite time of solutions of Euler equations in Brenner et al.
 711 (2016) cannot in fact lead to the blow-up in finite time of the solutions of Euler
 712 equations.

713

714 7. Toward the non blowup in finite time of the solutions

715 In this section, under mild assumptions deriving from the structure of the
 716 regions of high vorticity, we obtain the non-blowup in finite time at some time
 717 T^* of the solutions of 2D QG, 3D Euler and 3D Navier-Stokes equations in the
 718 case where

$$\|\nabla \xi(t)\|_\infty \sim (T^* - t)^{-\gamma_\xi}, \quad 0 \leq \gamma_\xi < 1.$$

719 In the previous section, we have outlined that the estimate obtained in Lemma
 720 6.1 for the function π defined by (37) is not sharp, then in the subsection 7.2 we
 721 propose a better estimate for the function π and go further in the non blow-up
 722 criteria. However, before to deal with new non blow-up criteria in the subsection
 723 7.2 we need to introduce in the subsection 7.1 the Lagrangian flow map X and
 724 the definitions of vortex lines and vortex tubes in order to justify the assumption
 725 (121) used in Proposition 7.1 and for their use in Lemmata 7.2, 7.3, 7.4 and in
 726 the Proposition 7.2.

727 7.1. Lagrangian flow map, vortex lines and vortex tubes

728 Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 2$. Let $T^* > 0$ be such that there
 729 exists a unique strong solution u to the 3D Euler equations (9)-(10) or 2D QG
 730 equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-1}(\mathbb{R}^d)). \quad (108)$$

731 Solutions in this class exist thanks to section 3. We set $\mathbf{v} = u$ in the case of 3D
 732 Euler equations and $\mathbf{v} = R^1 u$ in the case of 2D QG equation.

733 Owing to $u \in C([0, T^*[, H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-1}(\mathbb{R}^d))$ with $r > \frac{d}{2} + 2$
 734 and thanks to the L^2 -boundedness of the Riesz transforms, we infer that
 735 $\mathbf{v} \in C([0, T^*[, H^r(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-1}(\mathbb{R}^d))$ with $r > \frac{d}{2} + 2$. Then by using
 736 the Sobolev embedding, $H^m(\mathbb{R}^d) \hookrightarrow BC^m(\mathbb{R}^d)$ with $n = [m - \frac{d}{2}]$ and $m > \frac{d}{2}$, we
 737 deduce that for any $0 < T < T^*$,

$$u \in BC^1(\mathbb{R}^d \times [0, T]), \quad \nabla u \in C([0, T^*]; BC^1(\mathbb{R}^d)) \quad (109)$$

738 and

$$\mathbf{v} \in BC^1(\mathbb{R}^d \times [0, T]), \quad \nabla \mathbf{v} \in C([0, T^*]; BC^1(\mathbb{R}^d)). \quad (110)$$

739 In Proposition 7.1, in the case of 3D Euler equations and 2D QG equations, we
 740 give an estimate of the function π defined by (37). For this purpose, we need to
 741 give the definition of a vortex line and recall some results about the Lagrangian
 742 flow map.

743 We thus introduce the flow map $X(\alpha, \tau, t)$ the particle path that passes by
 744 $\alpha \in \mathbb{R}^d$ at time $\tau \in [0, T^*[$. That is for $\tau \in [0, T^*[$ fixed, $X(\alpha, \tau, t)$ solves on
 745 $[0, T^*[$ (see chapter 4 in Majda and Bertozzi (2002) for more details on the flow
 746 map X)

$$\begin{aligned} \frac{\partial X(\alpha, \tau, t)}{\partial t} &= \mathbf{v}(X(\alpha, \tau, t), t), \\ X(\alpha, \tau, \tau) &= \alpha \in \mathbb{R}^d, \end{aligned} \quad (111)$$

747 Thanks to Cauchy-Lipschitz Theorem (see Theorems 2.2 and 2.13 in Teschl
 748 (2012)), for any $\alpha \in \mathbb{R}^d$ and $\tau \in [0, T^*[$ thanks to (110) we get that there
 749 exists a unique solution $X(\alpha, \tau, \cdot) \in C^1([0, T^*[$ to equation (111). For all
 750 $t \in [0, T^*[$ and $\tau \in [0, T^*[$, the map $X(\cdot, \tau, t)$ defined by equation (111) is a
 751 volume preserving C^1 -diffeomorphism from \mathbb{R}^d on itself. Indeed thanks to (110)
 752 and the Theorems 2.2, 2.10 and 2.13 in Teschl (2012), we deduce that for any
 753 $t \in [0, T^*[$ and $\tau \in [0, T^*[$, $X(\cdot, \tau, t)$ is a C^1 -diffeomorphism from \mathbb{R}^d on itself
 754 with inverse $X(\cdot, t, \tau)$, we notice also $X \in C^1(\mathbb{R}^d \times [0, T^*] \times [0, T^*])$. Moreover
 755 for any $t \in [0, T^*[$ and $\tau \in [0, T^*[$, $X(\cdot, \tau, t)$ is a volume preserving mapping
 756 thanks to Proposition 1.4 in Majda and Bertozzi (2002), for which we get

$$\det(\nabla_\alpha X(\alpha, \tau, t)) = 1. \quad (112)$$

757 Furthermore, we have for all $\tau \in [0, T^*[$, $t \in [0, T^*[$ and $\alpha \in \mathbb{R}^d$ (see Proposition
 758 1.8 in Majda and Bertozzi (2002) or Proposition page 24 in Chorin and Marsden
 759 (1993) for Euler equations and see Deng et al. (2006b) for 2D QG equation),

$$\omega(X(\alpha, \tau, t), t) = \nabla_\alpha X(\alpha, \tau, t) \omega(\alpha, \tau). \quad (113)$$

760

761 Recall that a vortex line in a fluid is an arc on an integral curve of the vorticity
 762 $\omega(x, t)$ for fixed t , and a vortex tube is a tubular neighborhood in \mathbb{R}^d , $d \in \{2,$

763 arising as a union of vortex lines. In what follows, we give a parametrization of
 764 vortex lines and vortex tubes.

765 We set $\mathcal{O} \stackrel{\text{def}}{=} \{(x, \tau') \in \mathbb{R}^d \times [0, T^*]; |\omega(x, \tau')| > 0\}$ and for any $t \in [0, T^*[$,
 766 $\mathcal{O}(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| > 0\}$. From (25), we get that for any $t \in [0, T^*[$, $\mathcal{O}(t)$
 767 is nonempty. Thanks to (109), we get that ω is continuous on $\mathbb{R}^d \times [0, T^*[$ and
 768 then we deduce that \mathcal{O} is an open subset of $\mathbb{R}^d \times [0, T^*[$ and also that for all
 769 $t \in [0, T^*[$, $\mathcal{O}(t)$ is an open subset of \mathbb{R}^d . Notice thanks again to (109) that ξ and
 770 $\nabla \xi$ are continuous on \mathcal{O} . Then, we get that for all $t \in [0, T^*[$, $\xi(\cdot, t) \in C^1(\mathcal{O}(t))$.
 771 Then, for all $t \in [0, T^*[$ and $\alpha \in \mathcal{O}(t)$, we denote by $\mathbf{x}_t(\alpha, \cdot) : \mathfrak{J}_{\alpha, t} \mapsto \mathbb{R}^d$ the
 772 vortex line that passes through α at the time t and defined by the ordinary
 773 differential equation:

$$\begin{aligned} \frac{\partial \mathbf{x}_t(\alpha, s)}{\partial s} &= \xi(\mathbf{x}_t(\alpha, s), t), \\ \mathbf{x}_t(\alpha, 0) &= \alpha. \end{aligned} \tag{114}$$

774 The set $\mathfrak{J}_{\alpha, t} \subset \mathbb{R}$ not reduced to $\{0\}$ denotes the maximal interval of existence of
 775 the unique solution $\mathbf{x}_t(\alpha, \cdot)$ of (114), this is ensured thanks to Cauchy-Lipschitz
 776 Theorem (see e.g Theorems 2.2 and 2.13 in Teschl (2012)). For any $t \in [0, T^*[$,
 777 we introduce $U_t = \{(\alpha, s) \in \mathcal{O}(t) \times \mathbb{R}; \mathbf{x}_t(\alpha, s) \in \mathcal{O}(t)\}$ the set of definition
 778 of the function \mathbf{x}_t . For any $t \in [0, T^*[$ since $\xi(\cdot, t) \in C^1(\mathcal{O}(t))$, then from
 779 Theorem 2.9 in Teschl (2012), we get that \mathbf{x}_t is continuous on U_t . We notice
 780 that $U_t = \mathbf{x}_t^{-1}(\mathcal{O}(t))$ and hence we obtain that U_t is an open subset of $\mathcal{O}(t) \times \mathbb{R}$.
 781 From Theorem 2.10 in Teschl (2012), we obtain that

$$\mathbf{x}_t \in C^1(U_t). \tag{115}$$

782 Any vortex tube \mathfrak{T} at a time $t \in [0, T^*[$ can be defined as $\mathfrak{T} \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in$
 783 $\mathcal{A}_0, s \in I_{\alpha, t} \subset \mathfrak{J}_{\alpha, t}\}$ where \mathcal{A}_0 is a bounded smooth surface (resp. curve) of
 784 \mathbb{R}^3 (resp. of \mathbb{R}^2) and for each $\alpha \in \mathcal{A}_0$, $I_{\alpha, t}$ is an interval of \mathbb{R} containing 0.

785 7.2. Anisotropic structure for the improvement of non blow-up criteria

786 In this subsection, in Proposition 7.1 we propose to show that the function
 787 $\pi(t)$ defined by (37) involved Theorem 5.1 is bounded by $C \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right)$
 788 by using assumptions related to the anisotropic scaling in the collapse of regions
 789 of high vorticity containing the positions of the maximum vorticity. In Propo-
 790 sition 7.2, in the case of the Euler and 2D QG equations, we improve logarith-
 791 mically the result obtained in Proposition 7.1 by showing that the function π is
 792 bounded.

793 These results come from the special feature of the structure of regions of high
 794 vorticity surrounding the peak of vorticity $\{y \in \mathbb{R}^d; |\omega(y, t)| \gtrsim \|\omega(t)\|_\infty\}$ ob-
 795 served in the numerical experiments Kerr (1998, 2005); Agafontsev et al. (2015,
 796 2017) and from analytical models (Agafontsev et al., 2017, section 3),
 797 (Majda and Bertozzi, 2002, sections 1.4 and 1.5), namely they are pancake-
 798 like structure characterized by two length scales whose one of it is bounded by
 799 $O \left(\frac{1}{\|\omega(t)\|_\infty} \right)$ and plays the role of the thickness of the pancake-like structure.

800 This suggests that for any $t \in [t_0, T^*[$ with $t_0 \in [0, T^*[$, $x \in \Theta(t)$, $\lambda \geq \Omega(t)$ the
801 set $\{y \in \mathbb{R}^d; |\omega(y, t)| \geq \lambda\}$ may be characterized by three length scales whose
802 one of them is of order $\frac{1}{\lambda}$, where Ω and Θ are respectively defined by (35) and
803 (34). Since for any $0 < R \leq \rho_0(t)$ the set $\mathcal{V}_{\lambda, R}(t) \stackrel{\text{def}}{=} \{y \in B(x, R); |\omega(y, t)| \geq$
804 $\lambda\} \subset B(x, R)$, we thus expect that the set $\mathcal{V}_{\lambda, R}(t)$ may have two of its length
805 scales of order R and the third one of order $\frac{1}{\lambda}$. Then we expect that for for any
806 $t \in [t_0, T^*[$, $x \in \Theta(t)$, $\lambda \geq \Omega(t)$ and $0 < R \leq \rho_0(t)$,

$$|\mathcal{V}_{\lambda, R}(t)| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}. \quad (116)$$

807 In Lemma 7.1 we give an argument in favour of the assumption (116) in the
808 case of 3D Euler equations and 2D QG equation by using their Lagrangian
809 structure. In Lemma 7.1, the property (P1) expresses the fact that we expect
810 the length of any segment of a vortex line included in the structure $\mathcal{V}_{\lambda, R}(t)$
811 is bounded by $O(R)$, since $\mathcal{V}_{\lambda, R}(t) \subset B(x, R)$. Furthermore property (P2)
812 expresses the pancake structure of regions of high vorticity observed in numerical
813 experiments. Indeed for the case of 3D Euler equations, if one assumes that
814 the set $\mathcal{V}_{\lambda, R}^0(t) \stackrel{\text{def}}{=} X^{-1}(\mathcal{V}_{\lambda, R}(t), t_0, t)$ is characterized by three length scales
815 $\ell_1^0, \ell_2^0, \ell_3^0$ associated to three main directions orthogonal between them pairwise,
816 then we should have for one of these length scales $\ell_1^0 \lesssim \ell_{\lambda, R}^0(t)$ or $\ell_2^0 \lesssim \ell_{\lambda, R}^0(t)$ or
817 $\ell_3^0 \lesssim \ell_{\lambda, R}^0(t)$ (117). Let us say that $\ell_1^0 \lesssim \ell_{\lambda, R}^0(t)$. Assuming that during the time
818 between t_0 and $t \in]t_0, T^*[$ the set $\mathcal{V}_{\lambda, R}^0(t)$ becomes a pancake-shaped structure,
819 we thus expect that $\ell_2^0 \lesssim R$ or $\ell_3^0 \lesssim R$ since $\mathcal{V}_{\lambda, R}(t) \subset B(x, R)$. Let us say that
820 $\ell_2^0 \lesssim R$. For the last length scale ℓ_3^0 , we just expect that $\ell_3^0 = O(1)$.
821 In the case of 3D Euler equations, we thus expect that $|\mathcal{V}_{\lambda, R}^0(t_0)| \lesssim \ell_{\lambda, R}^0(t_0)R$.
822 In the case of 2D QG equations, we will have only the two length scales ℓ_1^0 and
823 ℓ_3^0 and then we expect that $|\mathcal{V}_{\lambda, R}^0(t_0)| \lesssim \ell_{\lambda, R}^0(t_0)$. Then Lemma 7.1 gives an
824 explanation of assumption (116).

825 **Lemma 7.1.** *Under the definitions (31)-(35) in the Theorem 5.1, we assume*
826 *that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $0 < R \leq \rho_0(t)$*
827 *and $\lambda \geq \Omega(t)$ the sets $\mathcal{V}_{\lambda, R}(t) \stackrel{\text{def}}{=} \{y \in B(x, R); |\omega(y, t)| \geq \lambda\}$ and $\mathcal{V}_{\lambda, R}^0(t) \stackrel{\text{def}}{=}$
828 $X^{-1}(\mathcal{V}_{\lambda, R}(t), t_0, t)$ satisfy:*

829 (P1) $|L_t \cap \mathcal{V}_{\lambda, R}(t)| \lesssim R$ for any vortex line L_t at time t ,

830 (P2) $|\mathcal{V}_{\lambda, R}^0(t)| \lesssim \ell_{\lambda, R}^0(t)R^{d-2}$ where

$$\ell_{\lambda, R}^0(t) \stackrel{\text{def}}{=} \sup_{L_{t_0} \subset \mathcal{I}(t_0)} |L_{t_0} \cap \mathcal{V}_{\lambda, R}^0(t)| \quad (117)$$

831 and $\mathcal{I}(t_0)$ denotes the set of all vortex lines L_{t_0} at time t_0 .

832 Then, we get that for any $t \in [t_0, T^*[$

$$|\mathcal{V}_{\lambda, R}(t)| \lesssim \|\omega(t_0)\|_{\infty} \frac{R^{d-1}}{\lambda}.$$

833 *Proof.* Let us take $t \in [t_0, T^*[$. If $|\mathcal{V}_{\lambda,R}(t)| = 0$ the proof follows immedi-
 834 ately. Then we assume that $|\mathcal{V}_{\lambda,R}(t)| > 0$. We have $|\mathcal{V}_{\lambda,R}(t)| = |\mathcal{V}_{\lambda,R}^0(t)|$ since
 835 $X(\cdot, t_0, t)$ is a volume preserving C^1 -diffeomorphism from \mathbb{R}^d to \mathbb{R}^d , then we
 836 get $|\mathcal{V}_{\lambda,R}^0(t)| > 0$ and Property (P2) yields to

$$|\mathcal{V}_{\lambda,R}(t)| \lesssim \ell_{\lambda,R}^0(t) R^{d-2}. \quad (118)$$

837 Furthermore for any vortex line L_{t_0} at time t_0 , we notice $X(L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t), t_0, t) =$
 838 $L_t \cap \mathcal{V}_{\lambda,R}(t)$ where L_t is the vortex line at time t defined by $L_t = X(L_{t_0}, t_0, t)$.
 839 Then, for any vortex line L_{t_0} at time t_0 such that $|L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t)| > 0$, by using
 840 the equation just below of (3.12) in Deng et al. (2005) and the definition of the
 841 set $\mathcal{V}_{\lambda,R}(t)$, we obtain

$$\frac{|L_t \cap \mathcal{V}_{\lambda,R}(t)|}{|L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t)|} \geq \frac{\lambda}{\|\omega(t_0)\|_\infty}. \quad (119)$$

842 By using Property (P1), for any vortex line L_{t_0} at time t_0 such that $|L_{t_0} \cap$
 843 $\mathcal{V}_{\lambda,R}^0(t)| > 0$, we deduce from (119) that $\frac{R}{|L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t)|} \gtrsim \frac{\lambda}{\|\omega(t_0)\|_\infty}$ which
 844 implies

$$\frac{R}{\ell_{\lambda,R}^0(t)} \gtrsim \frac{\lambda}{\|\omega(t_0)\|_\infty}. \quad (120)$$

845 Therefore by using (120), from (118) we thus infer $|\mathcal{V}_{\lambda,R}(t)| \lesssim \|\omega(t_0)\|_\infty \frac{R^{d-1}}{\lambda}$,
 846 which concludes the proof. \square

847 Thanks to assumption (116), we obtain the following Proposition 7.1.

848 **Proposition 7.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 849 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 850 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

851 *Under the definitions (32)-(37) in the Theorem 5.1, we assume that there exists*
 852 *$t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$, we get that for all $\lambda \geq \Omega(t)$*
 853 *and $0 < R \leq \rho_0(t)$*

$$|\{y \in B(x, R); |\omega(y, t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}. \quad (121)$$

854 *Then we get that for all $t \in [t_0, T^*[$*

$$\pi(t) \lesssim_{t_0} 1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right).$$

855 *Proof.* We have for all $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$

$$\begin{aligned}
\int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz &= \int_{B(x,R) \cap \mathcal{V}(t)} \int_0^{|\omega(z,t)|} d\lambda dz \\
&= \Omega(t) |B(x,R) \cap \mathcal{V}(t)| + \int_{B(x,R) \cap \mathcal{V}(t)} \int_{\Omega(t)}^{|\omega(z,t)|} d\lambda dz \\
&= \Omega(t) |B(x,R) \cap \mathcal{V}(t)| + \int_{\{z \in B(x,R), \Omega(t) < \lambda < |\omega(z,t)|\}} d\lambda dz \\
&= \Omega(t) |B(x,R) \cap \mathcal{V}(t)| \\
&+ \int_{[\Omega(t), \|\omega(t)\|_\infty]} |\{z \in B(x,R); |\omega(z,t)| > \lambda\}| d\lambda,
\end{aligned}$$

856 where we have used Fubini-Tonelli Theorem. Thanks to (121) we deduce that
857 for all $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$

$$\begin{aligned}
\int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz &\lesssim_{t_0} R^{d-1} \left(1 + \int_{[\Omega(t), \|\omega(t)\|_\infty]} \frac{d\lambda}{\lambda} \right) \\
&= R^{d-1} \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right).
\end{aligned} \tag{122}$$

858 Owing to (122), we thus conclude the proof. \square

859 **Remark 7.1.** *The analysis led in Kuznetsov and Ruban (2000); Kuznetsov et al.*
860 *(2001) for the study of collapse of vortex lines and agrees with numerical exper-*
861 *iments Agafontsev et al. (2015, 2017) suggests that the thickness of the regions*
862 *of high vorticity $\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}$ is $\frac{1}{\lambda^{\frac{3}{2}}}$ and since these regions are*
863 *included in the ball $B(x,R)$, we expect that $|\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}| \lesssim_{t_0}$*
864 *$\frac{R^{d-1}}{\lambda^{\frac{3}{2}}}$. Then under this assumption and by using the same arguments as previ-*
865 *ously, we obtain*

$$\pi(t) \lesssim \Omega(t)^{-\frac{1}{2}}.$$

866 Thanks to Theorem 5.1 and Proposition 7.1 we obtain Theorem 7.1.

867 **Theorem 7.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
868 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
869 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

870 *Under the definitions (32)-(35) in the Theorem 5.1, we assume that there exists*
871 *$t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$, we get that for all $\lambda \geq \Omega(t)$*
872 *and $0 < R \leq \rho_0(t)$*

$$|\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}.$$

873 Then if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t) \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right) dt < +\infty,$$

874 then the solution u cannot blowup at the finite time T^* where

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}$$

$$\rho(t) = O \left((T^* - t) \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right) \right).$$

875 **Remark 7.2.** Under the considerations of Remark 7.1, the non blow-up of the
876 solutions of Euler equations is obtained if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t) \Omega(t)^{-\frac{1}{2}} dt < +\infty.$$

877 Now, in the case of Euler equations and 2D QG equations by using their
878 Lagrangian formulation, after a fine and sharp analysis of the expression of π
879 (37) we go further in the non blow-up criteria by showing in Proposition 7.2
880 under mild assumptions based on the anisotropic structure of regions of high
881 vorticity, that $\pi(t) = O(1)$. For this purpose, we need the Lemmata 7.2, 7.3
882 and 7.4.

883 **Lemma 7.2.** Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such
884 that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler
885 equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

886 Let \mathcal{A}_0 be a smooth surface of \mathbb{R}^3 with boundary if $d = 3$ or a curve of \mathbb{R}^2 if
887 $d = 2$. For any $t_0 \in [0, T^*[$, let $\mathcal{A}(t_0, t) \stackrel{\text{def}}{=} X(\mathcal{A}_0, t_0, t)$ be the evolution of \mathcal{A}_0
888 through the flow map X from the time t_0 to t , for any $t \in [0, T^*[$. For any
889 $t_0 \in [0, T^*[$ and $t \in [0, T^*[$, let

$$\Gamma(t_0, t) \stackrel{\text{def}}{=} \int_{\mathcal{A}(t_0, t)} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y),$$

890 where $\mathbf{n}_t(\cdot)$ denotes a unit normal vector of $\mathcal{A}(t_0, t)$.

891 Then for any $t_0 \in [0, T^*[$ we get that $\Gamma(t_0, \cdot)$ is constant over $[0, T^*[$, that is to
892 say for all $t \in [t_0, T^*[$,

$$\Gamma(t_0, t) = \Gamma(t_0, t_0).$$

893 *Proof.* Thanks to Lemma 5 and Remark 3 of Schmidt and Schulz (2010) (see
894 also (4.9) chapter 9 in Delfour and Zolésio (2011)) and Lemma 7 of Schmidt and Schulz
895 (2010), we infer that for any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\int_{\mathcal{A}(t_0, t)} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) = \int_{\mathcal{A}_0} |\omega(X(\alpha, t_0, t), t) \cdot (\nabla X(\alpha, t_0, t))^{-T} \mathbf{n}(\alpha)| d\sigma(\alpha),$$

896 where \mathbf{n} is a unit normal vector of \mathcal{A}_0 . Thanks to (113), we infer that for any
 897 $\alpha \in \mathcal{A}_0$, $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\omega(X(\alpha, t_0, t), t) \cdot (\nabla X(\alpha, t_0, t))^{-T} \mathbf{n}(\alpha) = \omega(\alpha, t_0) \cdot \mathbf{n}(\alpha).$$

898 Therefore, we thus obtain that for any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\int_{\mathcal{A}(t_0, t)} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) = \int_{\mathcal{A}_0} |\omega(\alpha, t_0) \cdot \mathbf{n}(\alpha)| d\sigma(\alpha),$$

899 which concludes the proof. \square

900 **Lemma 7.3.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such
 901 that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler
 902 equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

903 *Let $t \in [0, T^*[$ and \mathfrak{T}_t a vortex tube at this time. Let \mathcal{A}_t and \mathcal{B}_t be two connected
 904 smooth orientable surfaces of \mathbb{R}^3 (resp. curves of \mathbb{R}^2 if $d = 2$) with boundary
 905 such their boundary encircle the vortex tube \mathfrak{T}_t and such that any vortex line of
 906 the vortex tube \mathfrak{T}_t intersects both \mathcal{A}_t and \mathcal{B}_t once each of them.*

907 *Then we get*

$$\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) = \int_{\mathcal{B}_t} |\omega(y, t) \cdot \tilde{\mathbf{n}}_t(y)| d\tilde{\sigma}(y),$$

908 *where \mathbf{n}_t and $\tilde{\mathbf{n}}_t$ are respectively the unit normal vector varying smoothly on the
 909 surfaces (resp. curves if $d = 2$) \mathcal{A}_t and \mathcal{B}_t , oriented to be outward to the portion
 910 of the tube \mathfrak{T}_t delimited by \mathcal{A}_t and \mathcal{B}_t .*

911 *Proof.* For any $x \in \mathcal{A}_t$, we denote by $\mathfrak{T}_t(x)$ the vortex line passing through x
 912 at time t , we get $\mathfrak{T}_t(x) \subset \mathfrak{T}_t$ and there exists an unique $y_{x,t} \in \mathcal{B}_t$ such that
 913 $\mathfrak{T}_t(x) \cap \mathcal{B}_t = \{y_{x,t}\}$. We thus introduce the function Φ_t defined from \mathcal{A}_t to
 914 \mathcal{B}_t for all $x \in \mathcal{A}_t$ by $\Phi_t(x) = y_{x,t}$. Since any vortex line of the vortex tube
 915 \mathfrak{T}_t intersects both \mathcal{A}_t and \mathcal{B}_t once each of them and since also \mathcal{A}_t and \mathcal{B}_t are
 916 smooth surfaces (smooth curves if $d = 2$), we infer thanks also to (115) that the
 917 function Φ_t is a homeomorphism from \mathcal{A}_t to \mathcal{B}_t .

918 We introduce the pairwise disjoint subsets of \mathcal{A}_t , namely $\mathcal{A}_t^+ \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y, t) \cdot$
 919 $\mathbf{n}_t(y) > 0\}$, $\mathcal{A}_t^- \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y, t) \cdot \mathbf{n}_t(y) < 0\}$ and $\mathcal{A}_t^0 \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y, t) \cdot$
 920 $\mathbf{n}_t(y) = 0\}$. By the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow BC^{m_r}(\mathbb{R}^d)$, $m_r =$
 921 $[r - 1 - \frac{d}{2}] \geq 2$, we get $\omega(t) \in BC^{m_r}(\mathbb{R}^d)$. Then \mathcal{A}_t^+ and \mathcal{A}_t^- are open subsets of
 922 the surface (curve if $d = 2$) \mathcal{A}_t and thus they are also smooth surfaces (smooth

923 curves if $d = 2$). On one hand, we have

$$\begin{aligned}
\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) &= \int_{\mathcal{A}_t^+} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) + \int_{\mathcal{A}_t^-} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) \\
&\quad + \int_{\mathcal{A}_t^0} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) \\
&= \int_{\mathcal{A}_t^+} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y) - \int_{\mathcal{A}_t^-} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y),
\end{aligned} \tag{123}$$

924 where we have used the definition of the sets \mathcal{A}_t^+ , \mathcal{A}_t^- and \mathcal{A}_t^0 . Since Φ_t is a
925 homeomorphism from \mathcal{A}_t to \mathcal{B}_t , we get that $\Phi_t(\mathcal{A}_t^+) \subset \mathcal{B}_t$, $\Phi_t(\mathcal{A}_t^-) \subset \mathcal{B}_t$ and
926 $\Phi_t(\mathcal{A}_t^+) \cap \Phi_t(\mathcal{A}_t^-) = \emptyset$. On the other hand, thanks to Helmholtz's first vortex
927 Theorem (see e.g (Wu, 2018, chapter 2)), we have

$$\int_{\mathcal{A}_t^+} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y) = - \int_{\Phi_t(\mathcal{A}_t^+)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y), \tag{124}$$

928 and

$$\int_{\mathcal{A}_t^-} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y) = - \int_{\Phi_t(\mathcal{A}_t^-)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y). \tag{125}$$

929 Then owing to (124) and (125), from (123) we deduce

$$\begin{aligned}
\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) &= - \int_{\Phi_t(\mathcal{A}_t^+)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y) \\
&\quad + \int_{\Phi_t(\mathcal{A}_t^-)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y),
\end{aligned}$$

930 which implies

$$\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) \leq \int_{\mathcal{B}_t} |\omega(y, t) \cdot \tilde{\mathbf{n}}_t(y)| d\tilde{\sigma}(y). \tag{126}$$

931 It remains to show that

$$\int_{\mathcal{B}_t} |\omega(y, t) \cdot \tilde{\mathbf{n}}_t(y)| d\tilde{\sigma}(y) \leq \int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y). \tag{127}$$

932 By introducing the pairwise disjoint subsets of \mathcal{B}_t , namely $\mathcal{B}_t^+ \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y, t) \cdot$
933 $\tilde{\mathbf{n}}_t(y) > 0\}$, $\mathcal{B}_t^- \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) < 0\}$ and $\mathcal{B}_t^0 \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y, t) \cdot$
934 $\tilde{\mathbf{n}}_t(y) = 0\}$ and using the fact that Φ_t^{-1} is a homeomorphism from \mathcal{B}_t to \mathcal{A}_t ,
935 we deduce with the same arguments used to get (126), inequality (127). Then,
936 owing to (126) and (127) we conclude the proof. \square

937 Before to turn to the proof of Lemma 7.4, Proposition 7.2 and Theorem 7.2,
938 we need to introduce some definitions. Let $r > \frac{d}{2} + 3$ and $T^* > 0$ be such

939 that there exists an unique strong solution u to the 3D Navier-Stokes, 3D Euler
 940 equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

941 For any $t \in [0, T^*[$ and any vortex tube \mathfrak{T}_t at time t , we define by
 942 $\mathcal{S}(\mathfrak{T}_t)$ the set of the connected smooth orientable surfaces of \mathbb{R}^3 (curves of \mathbb{R}^2
 943 if $d = 2$) with boundary that is intersected only once by any vortex line of \mathfrak{T}_t
 944 and such that their boundary encircle the vortex tube \mathfrak{T}_t .
 945 We define also the function $\Gamma_{\mathfrak{T}_t}$ defined from $\mathcal{S}(\mathfrak{T}_t)$ to $[0, +\infty[$ for all $\mathcal{A} \in \mathcal{S}(\mathfrak{T}_t)$
 946 by

$$\Gamma_{\mathfrak{T}_t}(\mathcal{A}) \stackrel{\text{def}}{=} \int_{\mathcal{A}} |\omega(y, t) \cdot \mathbf{n}(y)| d\sigma(y). \quad (128)$$

947 Thanks to Lemma 7.3, we deduce that for any $t \in [0, T^*[$ and any vortex tube
 948 \mathfrak{T}_t at time t

$$\Gamma_{\mathfrak{T}_t} \text{ is constant over } \mathcal{S}(\mathfrak{T}_t). \quad (129)$$

949 Owing to (129), for any $t \in [0, T^*[$ and any vortex tube \mathfrak{T}_t at time t , we define
 950 $\Gamma_{\text{abs}}(\mathfrak{T}_t)$ that we call the absolute strength of the vortex tube \mathfrak{T}_t by

$$\Gamma_{\text{abs}}(\mathfrak{T}_t) \stackrel{\text{def}}{=} \Gamma_{\mathfrak{T}_t}(\mathcal{A}_0), \quad (130)$$

951 with \mathcal{A}_0 an arbitrary element of $\mathcal{S}(\mathfrak{T}_t)$. As vortex tube moves with the fluid
 952 characterized by the flow map X (thanks to Helmholtz's first vortex Theorem),
 953 then for any vortex tube \mathfrak{T}_t at a time $t \in [0, T^*[$, we deduce that $X(\mathfrak{T}_t, t, \tau)$ is
 954 a vortex tube at time τ for any $\tau \in [0, T^*[$.

955 Thanks to Lemma 7.2, we infer that for any $t \in [0, T^*[$ and any vortex tube \mathfrak{T}_t
 956 at time t ,

$$\Gamma_{\text{abs}}(\mathfrak{T}_t) = \Gamma_{\text{abs}}(X(\mathfrak{T}_t, t, \tau)) \text{ for any } \tau \in [0, T^*[, \quad (131)$$

957 which means that the absolute strength of any vortex tube \mathfrak{T}_t at a time $t \in$
 958 $[0, T^*[$ moving with the fluid does not change with the time.

959 **Lemma 7.4.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such
 960 that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler
 961 equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

962 *Let $t \in [0, T^*[$ and \mathfrak{T}_t a vortex tube at time t defined by $\mathfrak{T}_t \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in$
 963 $\mathcal{A}_t, s \in J_t\}$ with \mathcal{A}_t a connected smooth orientable surface of \mathbb{R}^3 (curve of \mathbb{R}^2 if
 964 $d = 2$) with boundary and J_t an interval of \mathbb{R} containing 0 such that*

$$\bullet J_t \subset \bigcap_{\alpha \in \mathcal{A}_t} \mathfrak{J}_{\alpha, t},$$

965

966 *any vortex line of the tube \mathfrak{T}_t intersects \mathcal{A}_t only once, i.e*

$$\forall \beta \in \mathcal{A}_t, \{\mathbf{x}_t(\beta, s); s \in \mathfrak{J}_{\beta, t}\} \cap \mathcal{A}_t = \{\beta\}. \quad (132)$$

967 Then we have

$$\int_{\mathfrak{T}_t} |\omega(z, t)| dz = |J_t| \Gamma_{\text{abs}}(\mathfrak{T}_t).$$

968 *Proof.* If $J_t = 0$ then the result follows immediately. Therefore we assume that
 969 $J_t \neq \{0\}$. For any $s \in J_t$, we define the smooth surface of \mathbb{R}^3 (curve of \mathbb{R}^2 if
 970 $d = 2$) with boundary,

$$\mathcal{A}_t(s) \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_t\}.$$

971 Due to the definition of the vortex tube \mathfrak{T}_t , we get that for any $s \in J_t$, the
 972 boundary of $\mathcal{A}_t(s)$ encircles the vortex tube \mathfrak{T}_t . Thanks to (114), we get

$$\int_{\mathfrak{T}_t} |\omega(z, t)| dz = \int_{s \in J_t} \int_{\mathcal{A}_t(s)} |\omega(y, t)| |\mathbf{n}_t(s) \cdot \xi(y, t)| d\sigma(\alpha) ds,$$

973 where $\mathbf{n}_t(s)$ is a unit normal vector of $\mathcal{A}_t(s)$. Since $\omega(y, t) = |\omega(y, t)|\xi(y, t)$ then
 974 we obtain

$$\begin{aligned} \int_{\mathfrak{T}_t} |\omega(z, t)| dz &= \int_{s \in J_t} \int_{\mathcal{A}_t(s)} |\omega(y, t) \cdot \mathbf{n}_t(s)| d\sigma(\alpha) ds \\ &= \int_{s \in J_t} \Gamma_{\mathfrak{T}_t}(\mathcal{A}_t(s)) ds. \end{aligned} \quad (133)$$

975 We show now that for any $s_0 \in J_t$, any vortex line of the vortex tube \mathfrak{T}_t
 976 intersects $\mathcal{A}_t(s_0)$ only once. For this purpose, let $\alpha_1 \in \mathcal{A}_t(s_0)$. Thanks to
 977 Cauchy-Lipschitz Theorem (see e.g Theorem 2.2 in Teschl (2012)) used for (114),
 978 we deduce that there exists an unique $\beta_1 \in \mathcal{A}_t$ such that $\alpha_1 = \mathbf{x}_t(\beta_1, s_0)$.
 979 Suppose for a contradiction that

$$\{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t(s_0) \neq \{\alpha_1\}.$$

980 Then there exists $\alpha_2 \neq \alpha_1$ such that $\alpha_2 \in \{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t(s_0)$. There-
 981 fore we get that $\alpha_2 = \mathbf{x}_t(\beta_1, s_2)$ with $s_2 \in \mathfrak{J}_{\beta_1, t}$, $s_2 \neq s_0$ since $\alpha_2 \neq \alpha_1$. We
 982 get also that there exists an unique $\beta_2 \in \mathcal{A}_t$ such that $\alpha_2 = \mathbf{x}_t(\beta_2, s_0)$ where
 983 $\beta_2 \neq \beta_1$ since $\alpha_2 \neq \alpha_1$. We thus infer that

$$\mathbf{x}_t(\beta_1, s_2) = \mathbf{x}_t(\beta_2, s_0). \quad (134)$$

984 By the maximality of $\mathfrak{J}_{\beta_1, t}$, from (134) we infer that $s_2 - s_0 \in \mathfrak{J}_{\beta_1, t}$ and $\beta_2 =$
 985 $\mathbf{x}_t(\beta_1, s_2 - s_0)$ which implies $\{\beta_1, \beta_2\} \subset \{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t$. This latter
 986 contradicts (132). Therefore, we deduce that

$$\{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t(s_0) = \{\alpha_1\}.$$

987 This means that the vortex line of the vortex tube \mathfrak{T}_t passing through $\alpha_1 \in$
 988 $\mathcal{A}_t(s_0)$ intersects $\mathcal{A}_t(s_0)$ only once, which matches to our desired result. Then
 989 we get that for any $s \in J_t$, $\mathcal{A}_t(s) \in \mathcal{S}(\mathfrak{T}_t)$ and hence thanks to (129) and (130),
 990 from (133) we obtain $\int_{\mathfrak{T}_t} |\omega(z, t)| dz = |J_t| \Gamma_{\text{abs}}(\mathfrak{T}_t)$. Then we conclude the
 991 proof. \square

992 **Proposition 7.2.** Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be
 993 such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D
 994 Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

995 Under the definitions (32)-(37) in the Theorem 5.1, we assume that there exists
 996 $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$ there
 997 exists a vortex tube $\mathfrak{V}_{x,t}^R$ defined by $\mathfrak{V}_{x,t}^R \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_{x,t}^R, s \in I_{x,t}^R\}$ with
 998 $\mathcal{A}_{x,t}^R$ a connected smooth orientable surface of \mathbb{R}^3 (curve of \mathbb{R}^2 if $d = 2$) and $I_{x,t}^R$
 999 an interval of \mathbb{R} containing 0 such that:

1000 (P1) $\mathcal{V}(t) \cap B(x, R) \subset \mathfrak{V}_{x,t}^R$.

1001 (P2) any vortex line of the tube $\mathfrak{V}_{x,t}^R$ intersects $\mathcal{A}_{x,t}^R$ only once.

1002 (P3) $|I_{x,t}^R| \lesssim R$

1003 (P4) $\Gamma_{\text{abs}}(\mathfrak{V}_{x,t}^R) \lesssim \bar{\mathbf{v}}(t_0) R^{d-2}$ where $\bar{\mathbf{v}}(t_0) > 0$ is a real which depend only on
 1004 t_0 (and have the characteristic of a velocity).

1005 Then we get that for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim \bar{\mathbf{v}}(t_0).$$

1006 *Proof.* Let $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$. Thanks to property (P1)
 1007 we have

$$\int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \leq \int_{\mathfrak{V}_{x,t}^R} |\omega(z, t)| dz. \quad (135)$$

1008 Furthermore, thanks to property (P2) and Lemma 7.4 we get

$$\int_{\mathfrak{V}_{x,t}^R} |\omega(z, t)| dz = |I_{x,t}^R| \Gamma_{\text{abs}}(\mathfrak{V}_{x,t}^R). \quad (136)$$

1009 Thanks to the properties (P3) and (P4), from (136) we deduce

$$\int_{\mathfrak{V}_{x,t}^R} |\omega(z, t)| dz \lesssim R^{d-1} \bar{\mathbf{v}}(t_0). \quad (137)$$

1010 Owing to (137), from (135) we infer

$$\int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \lesssim R^{d-1} \bar{\mathbf{v}}(t_0). \quad (138)$$

1011 From the definition (37) of the function π , thanks to (138) we thus deduce that
 1012 for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim \bar{\mathbf{v}}(t_0),$$

1013 which concludes the proof. \square

1014 In the two following Remarks, we give explicit values for $\bar{\mathbf{v}}(t_0)$.

1015 **Remark 7.3.** *In the case of 2D QG equation for which $d = 2$, we have that for*
 1016 *all $t \in [t_0, T^*]$,*

$$\pi(t) \lesssim \|u_0\|_\infty,$$

1017 *if we replace the hypothesis (P4) by the assumption that the real-valued function*
 1018 *$\omega(\cdot, t) \cdot \mathbf{n}$ keeps a constant sign over $\mathcal{A}_{x,t}^R$, where \mathbf{n} is a unit normal vector*
 1019 *varying smoothly on $\mathcal{A}_{x,t}^R$.*

1020 *Indeed in this case, we get $\int_{\mathcal{A}_{x,t}^R} |\omega(\alpha, t) \cdot \mathbf{n}(\alpha)| d\alpha = \left| \int_{\mathcal{A}_{x,t}^R} \omega(\alpha, t) \cdot \mathbf{n}(\alpha) d\alpha \right|$*
 1021 *and furthermore thanks to Stokes Theorem we have $\int_{\mathcal{A}_{x,t}^R} \omega(\alpha, t) \cdot \mathbf{n}(\alpha) d\alpha =$*
 1022 *$u(\alpha_2, t) - u(\alpha_1, t)$ where α_2 and α_1 are the two endpoints of the line segment*
 1023 *$\mathcal{A}_{x,t}^R$. We thus infer $\int_{\mathcal{A}_{x,t}^R} |\omega(\alpha, t) \cdot \mathbf{n}(\alpha)| d\alpha \leq 2\|u(t)\|_\infty = 2\|u_0\|_\infty$ thanks to*
 1024 *(22) and then we take $\bar{\mathbf{v}}(t_0) = \|u_0\|_\infty$. Then with the properties (P1)-(P3), we*
 1025 *thus obtain that for all $t \in [t_0, T^*]$, $\pi(t) \lesssim \|u_0\|_\infty$.*

1026 **Remark 7.4.** *For any $t \in [t_0, T^*]$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$ let us assume*
 1027 *that there exists $t_1 \in [0, t_0]$ depending on t, x and R such that for the vortex tube*
 1028 *$\mathfrak{I}_{x,t_1}^R \stackrel{\text{def}}{=} X(\mathfrak{I}_{x,t}^R, t, t_1)$ at time t_1 we have*

$$\inf_{\mathcal{A} \in \mathcal{S}(\mathfrak{I}_{x,t_1}^R)} |\mathcal{A}| \lesssim R^{d-2},$$

1029 *then Property (P4) holds with $\bar{\mathbf{v}}(t_0) = \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])}$. Indeed thanks to (131)*
 1030 *we have*

$$\mathbf{\Gamma}_{\text{abs}}(\mathfrak{I}_{x,t}^R) = \mathbf{\Gamma}_{\text{abs}}(\mathfrak{I}_{x,t_1}^R).$$

1031 *Furthermore, thanks to (129) and (130), we deduce that*

$$\begin{aligned} \mathbf{\Gamma}_{\text{abs}}(\mathfrak{I}_{x,t_1}^R) &= \inf_{\mathcal{A} \in \mathcal{S}(\mathfrak{I}_{x,t_1}^R)} \int_{\mathcal{A}} |\omega(y, t_1) \cdot \mathbf{n}(y)| d\sigma(y) \\ &\leq \|\omega(t_1)\|_\infty \inf_{\mathcal{A} \in \mathcal{S}(\mathfrak{I}_{x,t_1}^R)} |\mathcal{A}| \\ &\lesssim \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])} R^{d-2}. \end{aligned}$$

1032 *Therefore, we deduce that $\mathbf{\Gamma}_{\text{abs}}(\mathfrak{I}_{x,t}^R) \lesssim \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])} R^{d-2}$ which matches*
 1033 *with Property (P4) for $\bar{\mathbf{v}}(t_0) = \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])}$.*

1034 Then thanks to Theorem 5.1 and Proposition 7.2 we deduce Theorem 7.2.

1035 **Theorem 7.2.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 1036 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 1037 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

1038 Under the definitions (32)-(36) in the Theorem 5.1, we assume that there exists
 1039 $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$ there
 1040 exists a vortex tube $\mathfrak{T}_{x,t}^R$ defined by $\mathfrak{T}_{x,t}^R \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_{x,t}^R, s \in I_{x,t}^R\}$ with
 1041 $\mathcal{A}_{x,t}^R$ a connected smooth orientable surface of \mathbb{R}^3 (curve of \mathbb{R}^2 if $d = 2$) and $I_{x,t}^R$
 1042 an interval of \mathbb{R} containing 0 such that:

1043 (P1) $\mathcal{V}(t) \cap B(x, R) \subset \mathfrak{T}_{x,t}^R$.

1044 (P2) $|I_{x,t}^R| \lesssim R$

1045 (P3) any vortex line of the tube $\mathfrak{T}_{x,t}^R$ intersects $\mathcal{A}_{x,t}^R$ only once.

1046 (P4) $\Gamma_{\text{abs}}(\mathfrak{T}_{x,t}^R) \lesssim \bar{\nu}(t_0)R^{d-2}$ where $\bar{\nu}(t_0) > 0$ is a real depending only on t_0
 1047 (and have the characteristic of a velocity).

1048 Then if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t) dt < +\infty,$$

1049 then the solution u cannot blowup at the finite time T^* with

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}$$

$$\rho(t) = O((T^* - t)\bar{\nu}(t_0)).$$

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