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► **To cite this version:**

Léo Agélas. Some advances on the geometric non blow-up criteria of incompressible flows. 2017. hal-01380349v3

HAL Id: hal-01380349

<https://hal.science/hal-01380349v3>

Preprint submitted on 31 Aug 2017 (v3), last revised 22 Jan 2019 (v4)

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Some advances on the geometric non blow-up criteria of incompressible flows

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Abstract

One of the most challenging questions in fluid dynamics is whether the three-dimensional (3D) incompressible Navier-Stokes, Euler and two-dimensional Quasi-Geostrophic (2D QG) equations can develop a finite-time singularity from smooth initial data. Recently, from a numerical point of view, Luo & Hou presented a class of potentially singular solutions to the Euler equations in a fluid with solid boundary [70, 71]. Furthermore, in two recent papers [85, 86], Tao indicates a significant barrier to establishing global regularity for the three-dimensional Euler and Navier-Stokes equations, in that any method for achieving this must use the finer geometric structure of these equations. In this paper, we show that the singularity discovered by Luo & Hou which lies right on the boundary is not relevant in the case of the whole domain \mathbb{R}^3 . We reveal also that the translation and rotation invariance present in the Euler, Navier-Stokes and 2D QG equations and not shared by the averaged Navier-Stokes and generalised Euler equations introduced respectively in [85, 86], is the key for the non blow-up in finite time of the solutions. The translation and rotation invariance of these equations which characterize their special geometric structures allowed to establish a new geometric non blow-up criterion and to improve greatly the Beale-Kato-Madja regularity criterion.

Keywords: 3D Euler equations; 3D Navier-Stokes equations; 2D Quasi-Geostrophic equation; Finite time singularities; Geometric properties for non blow-up

Mathematics Subject Classification: 35Q30, 35Q31, 76B60, 76B65, 76B03

Introduction

One of the most challenging questions in fluid dynamics is whether the incompressible 3D Navier-Stokes, Euler or 2D QG equations can develop a finite

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time singularity from smooth and bounded initial data. We know already that
 5 the blow up of smooth solutions to the Navier-Stokes and Euler equations is
 controlled by the time integral of the maximum magnitude of the vorticity (see
 [3, 54, 15], see also [12]). The Navier-Stokes and Euler equations describe the
 motion of a fluid in the three-dimensional space. These fundamental equations
 were derived over 250 years ago by Euler and since then have played a major role
 10 in fluid dynamics. They have enriched many branches of mathematics, were in-
 volved in many areas outside mathematical activity from weather prediction to
 exploding supernova (see for instance the surveys [20],[2]) and present important
 open physical and mathematical problems (see [20]). Regarding the 2D Quasi-
 Geostrophic (2D QG) equation, it appears in atmospheric studies. It describes
 15 the evolution of potential temperature u on the two dimensional boundary of a
 rapidly rotating half space with small Rossby and Ekman numbers, for the case
 of special solutions with constant potential vorticity in the interior and constant
 buoyancy frequency (normalized to one), where equations in the bulk are com-
 pressible Euler or Navier-Stokes equations coupled with temperature equation,
 20 continuity equation, and equation of state.

In the case of Navier-Stokes equations, for a long time ago, a global weak
 solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$ and $\nabla u \in L^2(\mathbb{R}^3 \times (0, \infty))^3$ was built by Leray
 [65]. In particular, Leray introduced a notion of weak solutions for the Navier-
 25 Stokes equations, and proved that, for every given $u_0 \in L^2(\mathbb{R}^3)^3$, there exists a
 global weak solution $u \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))^3 \cap L^2([0, \infty[; \dot{H}^1(\mathbb{R}^3))^3$. Hopf has
 proved the existence of a global weak solution in the general case \mathbb{R}^d , $d \geq 2$, [44].
 Several ways are known to construct weak solutions ([32], [43], [33]), and mean-
 while the regularity and the uniqueness of this weak solution has been known
 30 for a long time ago for the two-dimensional case (see [64], [66], [68], [84]), in
 the three-dimensional case the problem remains widely open in spite of great
 efforts made. On the uniqueness many works have been done (see [31],[36],
 [72],[67],[34]). Concerning the regularity of weak solutions, in [80], it is proved
 that if u is a Leray-Hopf weak solution belonging to $L^q(]0, T]; L^q(\mathbb{R}^3))^3$ with
 35 $\frac{2}{q} + \frac{3}{q} \leq 1$, $2 < p < \infty$, $3 < q < \infty$, then the solution $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. In
 [89] and [37], it is showed that if u is a weak solution in $C([0, T]; L^3(\mathbb{R}^3))^3$, then
 $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. The limit case of $L^\infty([0, T]; L^3(\mathbb{R}^3))^3$ has been solved in
 [48]. Other criterion regularity can be found in [41, 43, 36, 51, 52, 4, 10, 92, 17].

40 In the case of Euler Equations, in the two dimension case, uniqueness and
 existence of classical solutions have been known for a long time ago (see [90,
 91, 88, 29, 64]). However for the full three space dimensions, little is known
 about smooth solutions apart from classical short-time existence and uniqueness.
 Moreover, weak solutions are known to be badly behaved from the point of view
 45 of Hadamard's well-posedness theory (see for instance the surveys [28, 87]). Fur-
 ther, from the notion of weak solutions to the Euler equations introduced in [69]
 called dissipative solutions, it was proven that they coincide with classical Euler
 solutions, when those exist. Zero-viscosity limits of Navier-Stokes equations

50 have been shown to exist and to give Euler solutions in some more generalized
 sense known as measure-valued Euler solutions (see [29]). Considerable efforts
 have also been devoted to the study of the regularity properties of the 3D Euler
 equations. The main difficulty in the analysis lies in the presence of the nonlin-
 ear vortex stretching term and the lack of a regularization mechanism. Despite
 these difficulties, a few important partial results [3, 77, 30, 83, 18, 25, 35] have
 55 been obtained over the years which have led to improved understanding of the
 regularity properties of the 3D Euler.

In the case of 2D QG equation, besides its direct physical significance [42, 74],
 the 2D QG equation has very interesting features of resemblance to the 3D Euler
 60 equation, being also an outstanding open problem of the finite time blow-up
 issue. In particular, one can derive a necessary and sufficient blow-up condition
 for the 2D QG equation similar to the well-known Beale-Kato-Majda (BKM)
 criterion (Beale-Kato-Majda [3]). More precisely, the solution to the 2D QG
 equation (5) becomes singular at time T^* if and only if $\int_0^{T^*} \|\nabla^\perp u(t)\|_{L^\infty} dt =$
 65 $+\infty$ (see [15]). Thus, $\nabla^\perp u$ plays a role similar to the vorticity ω in the 3D Euler
 equations.

In the recent years, the 2D QG equation has been the focus of intense math-
 ematical research [15, 22, 24, 11, 73, 21, 14], initiated by Constantin, Majda
 and Tabak [15]. These latter showed that if the direction field $\xi = \nabla^\perp u / |\nabla^\perp u|$
 70 and the velocity v remain smooth in a region, then no finite-time singularity is
 possible in that region. A scenario for finite time blow up, a closing saddle, was
 proposed and numerically investigated there. It was later proved by Cordoba
 [22] that blow up does not happen in this scenario.

75 Besides the analytical results mentioned above, there also exists a sizable lit-
 erature focusing on the numerical search of a finite-time singularity for the 3D
 Euler equations and 2D QG equation (see [40, 79, 9, 55, 56, 57, 58, 59, 60, 62,
 75, 76, 15, 19, 21]). Although finite-time singularities were frequently reported
 in numerical simulations of the Euler equations, most such singularities turned
 80 out to be either numerical artefacts or false predictions, as a result of either
 insufficient resolution or inadvertent data analysis procedure (see [81, 45, 47]).

Recently, it was shown also in [85], a finite time blow up solution to an aver-
 aged three-dimensional Navier-Stokes equations of type $\partial_t u = \Delta u + \tilde{B}(u, u)$,
 85 where \tilde{B} is an averaged version of the Euler bilinear operator B , acting also on
 divergence free vector fields u and obeying as B to the cancellation property
 $\langle \tilde{B}(u, u), u \rangle = 0$. This result suggests that any successful method to affirma-
 tively answer to the Existence and Smoothness problem must either use finer
 structure of B or else must rely crucially on some estimate or other property of
 90 the Euler bilinear operator B that is not shared by the averaged operator \tilde{B} .
 Such additional structure exists for instance, the Euler equation has a vorticity
 formulation involving only differential operators rather than pseudo-differential

ones.

95 However, even this vorticity formulation is not a barrier to get a finite time
 blow up solution. Indeed, it was shown in [86], finite time blowup solutions
 to the generalised Euler equations sharing with the Euler equation its main
 features such as:

- vorticity formulation,
- energy conservation,
- 100 • Kelvin circulation theorem,
- vorticity-vector potential formulation viewed as the Generalised Biot-Savart,
- function space estimates for the vector potential operator.

Then, it seems that there is no room left to establish global regularity of solutions of 3D Euler equations. However, as it is mentioned in [86], there are two properties of the Euler equations which are not obeyed by the generalised Euler equations, namely translation invariance and rotation invariance.

Further, these symmetries basically determined the usual Biot-Savart law. Indeed, in the theory of gauge fields for which gauge principle is applied to a free-field Lagrangian, requiring it to have a symmetry, i.e. the gauge invariance, the Euler equations are characterized by two relevant symmetry gauge groups: a translation group and a rotation group. Indeed, Hamilton's principle together with isentropic material variations, gauge principle and the gauge-covariant derivative $\nabla_t \mathbf{v} := D_t \mathbf{v} + \Omega \mathbf{v}$, with Ω a gauge field, lead to the Euler equations of motion (see [49, 50]). The form of the gauge-covariant derivative $\nabla_t \mathbf{v}$ is deduced on the basis of the gauge principle by requiring it to be invariant with respect to Galilei and translational transformations, which yields to $D_t \mathbf{v} = \partial_t \mathbf{v} + \frac{1}{2} \nabla |\mathbf{v}|^2$ and invariant with respect to rotational transformation $SO(3)$ which yields to $\Omega \mathbf{v} = \hat{\Omega} \times \mathbf{v}$ where $\hat{\Omega} = \nabla \times \mathbf{v}$ (see [49, 50] for more details). We thus obtain that the vorticity $\omega \stackrel{\text{def}}{=} \nabla \times \mathbf{v}$ is the gauge field with respect to the gauge group $SO(3)$. By taking the rotational of Equation $\omega = \nabla \times \mathbf{v}$ and by using the fact that $\nabla \cdot \mathbf{v} = 0$, we derive the usual Biot-Savart law, namely $\mathbf{v} = (-\Delta)^{-1} \nabla \times \omega$.

Thus, the special geometric structures of the Biot-Savart law are not shared by the Generalised Euler equations introduced in [86]. Then, in this paper, we exploit the special geometric structures of the Biot-Savart law to give strong arguments in favour of a non blow-up in finite time of the 3D Euler, 3D Navier-Stokes and 2D QG equations. Indeed, as it was shown in [16, 17], the use of Biot-Savart law leads to rewrite the vorticity equation in the case of Euler ($\nu = 0$) and Navier-Stokes ($\nu > 0$) equations as follows:

$$\partial_t \omega + (\mathbf{v} \cdot \nabla) \omega - \nu \Delta \omega = \alpha \omega, \quad (1)$$

where

$$\alpha(x, t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} (\hat{y} \cdot \xi(x, t)) \det(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^3}, \quad (2)$$

with $\hat{y} = \frac{y}{|y|}$, $\xi = \frac{\omega}{|\omega|}$ and $\det(a, b, c)$ is the determinant of the matrix with columns a, b, c in that order. We thus notice from the expression of α that if the direction of the vorticity, ξ varies mildly within a small region around x , then the singularity of the integrand in 2 will be mild. In this direction, some studies indicate that the geometric regularity of the direction of vorticity can lead to dynamic depletion of vortex stretching [17, 5, 6, 16, 18, 25, 27, 46, 26]. In particular, the recent results obtained in [25, 27, 46, 26] show that geometric regularity of vortex lines, even in an extremely localized region containing the maximum vorticity can lead to depletion of nonlinear stretching, thus avoiding finite time singularity formation of the 3D Euler equations or 2D QG equation.

But recently, a convincing numerical evidence for a singular solution to the Euler equations has been found in a fluid with periodic boundary condition along the axial direction and no-flow boundary condition on the solid wall [71] (see also [70]), for which the point of the potential singularity, which is also the point of the maximum vorticity, is always located at the solid boundary. We show however in this paper that such singularity can not exist in the whole domain \mathbb{R}^3 . Indeed, in the whole domain of \mathbb{R}^3 at any point of the maximum vorticity, $q_0 \in \mathbb{R}^3$, we get $\nabla|\omega|(q_0, t) = 0$ for any time t before the alleged time of singularity T^* , then this result combined with the fact that the vorticity ω is a divergence-free vector field, yields to $\nabla \cdot \xi(q_0, t) = 0$. However in [71], the presence of a solid boundary and the fact that q_0 the point of the maximum vorticity is always located on the solid boundary, prevent to get $\nabla|\omega|(q_0, t) = 0$ and this allows to get $\nabla \cdot \xi(q_0, t) \sim (T^* - t)^{-2.9165} \neq 0$ as it is observed in their numerical test. This latter is the main element used to invalidate the Deng-Hou-Yu non-blowup criterion [25, 27].

Thus, in this paper, we reveal strong depletion in the nonlinear vortex stretching term $\alpha\omega$ appearing in the vorticity equation (1) by using both the special geometric structures of the Biot-Savart law characterized by α (2) and the fact that at the position of maximum of vorticity in \mathbb{R}^3 , q_0 for a given time $t < T^*$, we get necessarily $\nabla|\omega|(q_0, t) = 0$ and $\nabla \cdot \xi(q_0, t) = 0$. In this paper, we establish new geometric criteria for non blow-up in finite time of the solutions of Navier-Stokes, Euler and 2D QG equations based on the regularity of the direction of the vorticity ξ in regions containing the positions where the maximum of the magnitude of the vorticity is reached and shrinking to zero as time tends to some T^* the alleged time of singularity.

Through our Theorem 4.1, we derive a regularity criterion which appears less restrictive than the Beale-Kato-Madja (BKM) regularity criterion type. This regularity criterion may stand as great a improvement over the usual BKM regularity criterion which states that if $\int_0^T \|\omega(t)\|_\infty dt < +\infty$ then the solution $u \in C([0, T]; H^r(\mathbb{R}^d))$ with $r > \frac{d}{2} + 3$ remains in $H^r(\mathbb{R}^d)$ up to time T . To get these results, we had to overcome the obstruction that we do not know if there exists an isolated absolute maximum for the vorticity achieved along a smooth

curve in time as it was assumed in Proposition 2.1 of [15] and also in [25, 27, 26] (which assume that the position where the maximum of vorticity is reached, is advected with the flow). Moreover, recent numerical experiments show that it is not always the case (see [61], see also section 5.4.5 in [38]). We thus overcome this difficulty by using a result of Pshenichnyi concerning directional derivatives of the function of maximum and the structure of a set of supporting functionals [78].

Our geometric non blow-up criterion reveals the role of the geometric structures of the Incompressible flows in the non blow-up in finite time of the solutions and presents the advantage to be established in an Eulerian setting in comparison with all the recent geometric non blow-up criteria [18, 25, 27, 15] using the Lagrangian formulation of Incompressible Inviscid Flows, which requires much more computational effort as it is mentioned in [39] and in section 5.4.5 of [38]. Furthermore, due to the existence of hyperbolic-saddle singularities suggested by the generation of strong fronts in geophysical/meteorology observations (see [15, 22]), and antiparallel vortex line pairing observed in numerical simulations and physical experiments, it was important to take them into account in our geometric non blow-up criterion. This is performed thanks to the term $\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))$ (see (20),(21)) involved in the definition of the function \mathbf{B}_d given at (62).

Then, the paper is organized as follows:

- In section 1 , we give some notations and definitions.
- In section 2, we recall some results about the local regularity of solutions of Navier-Stokes, Euler and 2D QG equations.
- In section 3, we give the reason for which we can assume for any time t that $\|\omega(t)\|_\infty > 0$ without loss of generality.
- In section 4, through Theorem 4.1 deduced from Proposition 4.1, we establish a new geometric criterion for the non blow-up in finite time of the solutions of 3D Navier-Stokes, 3D Euler and 2D QG equations. We show under a very weak assumption \mathbf{H} (emphasized in the section 5) that their solutions cannot blow up at a finite time T^* if $\int_0^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt < +\infty$, where \mathbf{B}_d is smaller than one and is based on the regularity of the direction of the vorticity ξ in regions shrinking to zero as time tends to T^* and containing the positions where the maximum of the magnitude of the vorticity is reached (see definition of \mathbf{B}_d at (62) using (20) and (21)).
- In section 5, through Lemma 5.1 we show that hypothesis \mathbf{H} appearing in the statement of Theorem 4.1 is a very weak assumption. Then, thanks to Theorem 4.1 and Lemma 5.1, in Corollary 5.1 we release our geometric criterion for the non blow-up in finite time of the solutions of 3D Navier-Stokes, 3D Euler and 2D QG equations. Then, we give some examples of non blow-up in finite time of the solutions of 3D Navier-Stokes, 3D Euler

and 2D QG equations by considering the most plausible blow-up rates for $\|\omega(t)\|_\infty$, $\|u(t)\|_\infty$ and $\|\nabla\xi(t)\|_\infty$.

Let us now introduce the 3D Navier-Stokes and Euler equations given by,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \\ \nabla \cdot u = 0, \end{cases} \quad (3)$$

in which $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ and $\nu \geq 0$ ($\nu = 0$ corresponds to the Euler equations) denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x, t) \in \mathbb{R}^3 \times [0, \infty[$ and the viscosity of the fluid,

with initial conditions,

$$u(x, 0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3, \quad (4)$$

190 where the initial data u_0 is a divergence free vector field on \mathbb{R}^3 .

Regarding the 2D QG equation (5) in \mathbb{R}^2 , it is given by

$$\begin{cases} \frac{\partial u}{\partial t} + v \cdot \nabla u = 0, \\ v = -\nabla^\perp (-\Delta)^{-\frac{1}{2}} u, \end{cases} \quad (5)$$

with initial data,

$$u(x, 0) = u_0. \quad (6)$$

Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. For v we have also the following representation

$$v = -R^\perp u, \quad (7)$$

where we have used the notation, $R^\perp u = (-R_2 u, R_1 u)$ with $R_j, j = 1, 2$, for the 2D Riesz transform defined by (see e.g. [82])

$$R_j(u)(x, t) = \frac{1}{2\pi} P.V \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x - y|^3} u(y, t) dy.$$

1. Some notations and definitions

195 In this section, we assume that $d \in \mathbb{N}$, $d \geq 2$.

For any vector $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we denote by $|x|$ the euclidean norm

of x given by $|x| = \sqrt{\sum_{i=1}^d |x_i|^2}$. For any subset A of \mathbb{R}^d , we denote by $|A|$ its

d -dimensional measure. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of real square matrices of size d . We denote by Id the identity matrix of $\mathcal{M}(\mathbb{R}^d)$. For any vector field

200 v defined from \mathbb{R}^d to \mathbb{R}^d , we denote by ∇v the gradient matrix of v , the matrix

of $\mathcal{M}(\mathbb{R}^d)$ with ij -component, $\frac{\partial v_i}{\partial x_j}$ for all $1 \leq i, j \leq d$. For any real a , we

denote by a^+ the real defined by $a^+ \stackrel{\text{def}}{=} \max(a, 0)$. For any function φ defined on $\mathbb{R}^d \times [0, +\infty[$, for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on \mathbb{R}^d by $x \mapsto \varphi(x, t)$. We denote by $C_c^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support in \mathbb{R}^d . We denote by BC the class of bounded and continuous functions and by BC^m the class of bounded and m times continuously derivable functions.

For any $R > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B(x_0, R)$, the ball of \mathbb{R}^d of center x_0 and radius R . For any $R > 0$, we denote by B_R , the ball of \mathbb{R}^d of center 0 and radius R .

We denote by div the differential operator given by, $\text{div} = \sum_{i=1}^d \frac{\partial}{\partial x_i}$.

We denote $A \lesssim D$, the estimate $A \leq cD$ where $c > 0$ is an absolute constant. For any $f \in L^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)^d$ or $L^p(\mathbb{R}^d)^{d \times d}$) with $1 \leq p \leq +\infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p -norm of f .

We denote by $H^s(\mathbb{R}^d)$ the Sobolev space $J^{-s}L^2(\mathbb{R}^d)$ where $J = (1 - \Delta)^{\frac{1}{2}}$. We denote by $H_\sigma^s(\mathbb{R}^3)$ the Sobolev space $H_\sigma^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\psi \in H^s(\mathbb{R}^3)^3 : \text{div} \psi = 0\}$. In order to unify our notations with the two dimensional case 2D QG, we denote by $H_\sigma^s(\mathbb{R}^2)$ the Sobolev space $H^s(\mathbb{R}^2)$.

We denote by \mathbb{P} the well-known 3D matrix Leray's projection operator with components,

$$\mathbb{P}_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Delta^{-1} = \delta_{i,j} - R_j R_k, \quad (8)$$

where R_j are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} = \frac{1}{4\pi} \frac{x_j}{|x|^4} \star$ (see [82] for more details), Δ^{-1} is the inverse of Laplace operator given by $\Delta^{-1} = -\frac{1}{4\pi|x|} \star$, with \star the convolution operator.

2. Local regularity of solution of 3D Navier-Stokes, Euler and 2D QG equations

In this section, we deal with the main result on local regularity of 3D Navier-Stokes and Euler equations in its general form. By introducing \mathbb{P} the matrix Leray operator, Euler equations (3)-(4) can be re-written as follows,

$$\frac{\partial u}{\partial t} + \mathbb{P}(u \cdot \nabla)u = 0, \quad (9)$$

with initial conditions,

$$u(0) = u_0. \quad (10)$$

For u solution of (9)-(10), $\omega = \nabla \times u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega = 0, \quad (11)$$

with initial conditions,

$$\omega(0) = \omega_0,$$

where $\omega_0 = \nabla \times u_0$ is the vorticity of u_0 .

In the case of 2D QG equation, we get for u solution of (5), $\omega = \nabla^\perp u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v = 0, \quad (12)$$

with initial conditions,

$$\omega(0) = \omega_0,$$

where $\omega_0 = \nabla^\perp u_0$ is the vorticity of u_0 .

In the region where $|\omega| > 0$, we define ξ the direction of the vorticity by $\xi = \frac{\omega}{|\omega|}$.

2.1. Local regularity for 3D Navier-Stokes or 3D Euler equations

Assuming $u_0 \in H_\sigma^r(\mathbb{R}^3)$ with $r > \frac{5}{2}$, thanks to Theorem 3.5 in [54], Theorem 1 in [8] (see also Theorem I in [53] and the results obtained in [3]), we deduce that there exists a time $T > 0$ such that there exists a unique strong solution $u \in C([0, T[, H_\sigma^r(\mathbb{R}^3)) \cap C^1([0, T[, H_\sigma^{r-2}(\mathbb{R}^3))$ to the Navier-Stokes or Euler equations (9)-(10) and the energy equality holds for u , that means for all $t \in [0, T[$,

$$\|u(t)\|_2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2. \quad (13)$$

Moreover, if $u \notin C([0, T[, H_\sigma^r(\mathbb{R}^3))$, then we get (see [3, 54, 63]),

$$\int_0^T \|\omega(t)\|_\infty dt = +\infty. \quad (14)$$

Notice thanks to Remark 3.7 in [54], in the case of Euler equations, we get in addition that $u \in C^1([0, T[, H_\sigma^{r-1}(\mathbb{R}^3))$. We retrieve the pressure p from the velocity u with the formula,

$$p = -\Delta^{-1} \operatorname{div}((u \cdot \nabla)u).$$

Furthermore, we get the local estimate (15). Indeed, thanks to remark 4.4 in [54], we get

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \quad \text{with } t_0 < t < T, \quad (15)$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

235 *2.2. Local regularity for 2D QG equation*

This subsection is devoted to the local well-posedness of the 2D QG equation with a characterization of the maximal time existence of strong solutions. By using the same arguments as the proof of Proposition 4.2 in [1], we get that the H^s -norm of u is controlled by the integral in time of the maximum magnitude
 240 of the vorticity of u . A such Proposition has been proved in [15] for any integer $s \geq 3$, but here we extend this result to all real $s > 2$. This improvement is obtained by using the logarithmic Sobolev inequality proved in [63, 54] which requires only that $s > 2$ instead of using the one proved in [3] as it is the case in [15] and which requires integer $s \geq 3$. Then by using the same arguments
 245 as the proof of Proposition 4.3 in [1], we get the following result which gives an improvement in comparison with Theorem 2.1 in [15]:

Assuming $u_0 \in H^r(\mathbb{R}^2)$ with $r > 2$, we get that there exists a time $T > 0$ such that there exists a unique strong solution $u \in C([0, T[, H^r(\mathbb{R}^2))$ to the 2D QG equation (5)-(6) and the energy equality holds for u , that means for all $p \in [2, \infty]$ and $t \in [0, T[$,

$$\|u(t)\|_p = \|u_0\|_p. \quad (16)$$

Moreover, if $u \notin C([0, T], H^r(\mathbb{R}^2))$, then

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \quad (17)$$

Owing to $u \in C([0, T[, H^r(\mathbb{R}^2))$ and thanks to Lemma X4 in [54], from 2D QG (5), we get $u \in C^1([0, T[, H^{r-1}(\mathbb{R}^2))$. Similarly as in (15), we have

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ for } t_0 < t < T, \quad (18)$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

3. Assumption on the maximum vorticity without loss of generality

Let $d \in \{2, 3\}$, $r > \frac{d}{2} + 1$ and $u_0 \in H_\sigma^r(\mathbb{R}^d)$. Let $T^* > 0$ be such that there
 250 exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler or 2D QG equations (3)-(4) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Thanks to the results of the section 2, a such time T^* exists.

In this paper, we are concerned with the non blowup in finite time of the solutions u at times such T^* . Then, without loss of generality, in the whole of this paper, we consider only times of existence T^* such that for all $t \in [0, T^*[,$

$$\|\omega(t)\|_\infty > 0. \quad (19)$$

Indeed, let us assume that there exists $t_0 \in [0, T^*[$ such that $\|\omega(t_0)\|_\infty = 0$.

In the case of 2D QG equations (5), we get that $\omega(t_0) \equiv 0$ and then $\nabla u(t_0) \equiv 0$. Since $x \mapsto u(t_0, x)$ vanishes at infinity, then we get $u(t_0) \equiv 0$. Then by using inequality (18) concerning the local regularity, we deduce that $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and no blowup can occur at the time T^* .

In the case of 3D Navier-Stokes or 3D Euler equations (3), by following step by step the proof of Lemma 4 given in [25] but keeping the term $\|u(t)\|_{L^2(\mathbb{R}^3)}$ after using the Cauchy-Schwarz inequality, we obtain for all $t \in [0, T^*[$,

$$\begin{aligned} \|u(t)\|_\infty &\lesssim \|u(t)\|_2^{\frac{2}{5}} \|\omega(t)\|_\infty^{\frac{3}{5}} \\ &\leq \|u_0\|_2^{\frac{2}{5}} \|\omega(t)\|_\infty^{\frac{3}{5}}, \end{aligned}$$

where we have used (13) for the last inequality. Then, we obtain that $\|u(t_0)\|_\infty \equiv 0$ which implies that $u(t_0) \equiv 0$. Then by using the inequality (15) of local regularity, we deduce $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and thus no blowup can occur at the time T^* .

4. Geometric properties for non blow-up of 3D Navier-Stokes, 3D Euler and 2D QG equations in localized regions of maximum vorticity

Historically, non blow-up criteria for the incompressible Euler equations and 2D QG equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy for overcoming such shortcomings was established by focusing more on geometrical properties and flow structures (see e.g. [18, 23]), such as vortex tubes or vortex lines.

In particular, in [18, 15] the authors showed that local geometric regularity of the unit vorticity vector can lead to depletion of the vortex stretching. They prove that if there is up to time T an $O(1)$ region in which the vorticity vector is smoothly directed, i.e., the maximum norm of $\nabla \xi$ (here $\xi = \frac{\omega}{|\omega|}$, ω the vorticity)

in this region is L^2 integrable in time from 0 to T , and the maximum norm of velocity in some $O(1)$ neighbourhood of this region is uniformly bounded in time, then no blow-up can occur in this region up to time T .

However, this theorem dealt with $O(1)$ regions in which the vorticity vector is assumed to have some regularity, while in numerical computations, the regions that have such regularity and contain maximum vorticity are all shrinking with time (see [55, 56, 58, 59, 75, 76]).

Inspired by the work of [18, 15], in [25, 27, 26] the authors showed that geometric regularity of Lagrangian vortex filaments, even in an extremely localized region containing the maximum of vorticity which may shrink with time, can lead to depletion of the nonlinear vortex stretching, thus avoiding finite time singularity formation of the 3D Euler equations and 2D QG equations.

290 However, all the recent geometric constraints for non blow-up criteria of Euler and 2D QG equations based on local geometric regularity of Lagrangian vortex filaments [25, 27, 26] make the assumption that the position where the maximum of vorticity is reached, is advected with the flow, however it is not always the case, as described in [61] (see also section 5.4.5 of [38]).

295 Then in our Theorem 4.1 obtained from our Proposition 4.1, we establish in an Eulerian setting a new geometric non blow-up criterion for the Navier-Stokes, Euler and 2D QG equations based on the regularity of the direction of the vorticity in extremely localized regions containing the position of maximum vorticity and shrinking to zero as time increase to some T^* the alleged time of singularity. Our Eulerian geometric non blow-up criterion should give also
300 new impetus to the numerical experiments due to their ease of implementation in comparison with Lagrangian geometric non blow-up criteria (see [39], see also section 5.4.5 of [38]). Moreover our geometric non blow-up criterion is valid also for the Navier-Stokes equations that is not the case for the existing
305 geometric non blow-up criteria obtained in [18, 25, 27, 26] based on a Lagrangian formulation of Incompressible Inviscid Flows.

We thus begin with Lemma 4.1.

Lemma 4.1. *Let $d \in \mathbb{N}^*$, $T > 0$ and $f \in C([0, T]; BC(\mathbb{R}^d))$ such that $\inf_{t \in [0, T]} \|f(t)\|_\infty > 0$ and for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$. Then there exists $R > 0$ such that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$.*

310 *Proof.* We set $a = \inf_{t \in [0, T]} \|f(t)\|_\infty > 0$. Since $t \mapsto f(t)$ is a continuous function from the compact $[0, T]$ into the metric space $L^\infty(\mathbb{R}^d)$ then it is uniformly continuous. Hence, there exists $N \in \mathbb{N}^*$ such that for all $t, t' \in [0, T]$, $|t - t'| \leq \frac{T}{N}$ we have $\|f(t) - f(t')\|_\infty \leq \frac{a}{4}$. We introduce the subdivision $\{t_i\}_{i \in \llbracket 0, N \rrbracket}$ of
315 $[0, T]$ defined by $t_i = i \frac{T}{N}$ for $i \in \llbracket 0, N \rrbracket$. Since for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$, then for each $i \in \llbracket 0, N \rrbracket$, there exists $R_i > 0$ such that for all $|x| \geq R_i$, $|f(x, t_i)| \leq \frac{a}{4}$. We set $R = \max_{i \in \llbracket 0, N \rrbracket} R_i$. Let $t \in [0, T]$ then there exists $j \in \llbracket 0, N \rrbracket$ such that $|t - t_j| \leq \frac{T}{N}$ and hence for all $|x| \geq R \geq R_j$, we have
 $|f(x, t)| \leq |f(x, t) - f(x, t_j)| + |f(x, t_j)| \leq \frac{a}{2} \leq \frac{\|f(t)\|_\infty}{2}$. Then, we infer that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$, which concludes the proof. \square
320

Before to prove Proposition 4.1, we need to introduce the following function \mathbf{D}_d defined from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R} with $d \in \{2, 3\}$ as follows: for $d = 3$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3) \text{Det}(a_1, a_2, a_3).$$

The Det in \mathbf{D}_d is the determinant of the matrix whose columns are the three unit column vectors a_1, a_2, a_3 . We observe that $\text{Det}(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$, then, we get

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3) a_1 \cdot (a_2 \times a_3). \quad (20)$$

and for $d = 2$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3^\perp)(a_2 \cdot a_3^\perp), \quad (21)$$

where for any $z = (z_1, z_2) \in \mathbb{R}^2$, $z^\perp = (-z_2, z_1)$. We can notice that for $d \in \{2, 3\}$ the function \mathbf{D}_d is linear from its second variable.

From (20) and (21) we get $\mathbf{D}_d(a_1, a_3, a_3) = 0$ then we deduce that for any $a_1, a_2, a_3 \in B(0, 1)$,

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq |a_2 - a_3|, \quad (22)$$

and we get also

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq 1. \quad (23)$$

Before to prove Proposition 4.1, we need also the following Lemma.

Lemma 4.2. *Let $T > 0$, $a \in L^1([0, T], \mathbb{R}^+)$ and $b \in L^1([0, T], \mathbb{R}^+)$. Let $X \in C([0, T], \mathbb{R}^+) \cap W^{1,1}([0, T], \mathbb{R}^+)$ satisfying for a.e $t \in [0, T]$*

$$\frac{d}{dt}X(t) \leq a(t)X(t) + b(t)X(t)(1 + \log^+ X(t)). \quad (24)$$

Then for any $t_0 \in [0, T[$ we get that for all $t \in [t_0, T]$

$$X(t) \leq e^{(1+\log^+ X(t_0) + \int_{t_0}^t a(s) ds)} e^{\int_{t_0}^t b(\tau) d\tau}.$$

Proof. We multiply inequality (24) by $\mathbf{1}_{\{X(t) \geq 1\}}$, to obtain with $Y \equiv \max(X, 1)$, for a.e $t \in [0, T]$,

$$\frac{d}{dt}Y(t) \leq a(t)Y(t) + b(t)Y(t)(1 + \log Y(t)). \quad (25)$$

Since $Y \geq 1$, we can divide inequality (25) by $Y(t)$ to obtain for a.e $t \in [0, T]$,

$$\frac{d}{dt}(1 + \log Y(t)) \leq a(t) + b(t)(1 + \log Y(t)). \quad (26)$$

³²⁵ Thanks to Gronwall inequality, from (26) it is inferred that for any $t_0 \in [0, T[$ and for all $t \in [t_0, T]$

$$1 + \log Y(t) \leq (1 + \log Y(t_0))e^{\int_{t_0}^t b(\tau) d\tau} + \int_{t_0}^t a(s)e^{\int_s^t b(\tau) d\tau} ds,$$

which yields to

$$1 + \log^+ X(t) \leq (1 + \log^+ X(t_0))e^{\int_{t_0}^t b(\tau) d\tau} + \int_{t_0}^t a(s)e^{\int_s^t b(\tau) d\tau} ds. \quad (27)$$

Since $1 + \log^+ X(t) \geq \log X(t)$, we thus deduce that for all $t \in [t_0, T]$,

$$\begin{aligned} X(t) &\leq e^{(1+\log^+ X(t_0))e^{\int_{t_0}^t b(\tau) d\tau} + \int_{t_0}^t a(s)e^{\int_s^t b(\tau) d\tau} ds} \\ &\leq e^{(1+\log^+ X(t_0) + \int_{t_0}^t a(s) ds)} e^{\int_{t_0}^t b(\tau) d\tau}, \end{aligned}$$

which concludes the proof. \square

Now, we turn to the proof of our Proposition.

330 **Proposition 4.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (3)-(4) or 2D QG equations (5)-(6) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

335 *Let $n \geq 1$ be a real. Let ρ be a bounded function defined on $[0, T^*]$ and not vanishing on $[0, T^*[$ for which we set $\rho_\infty \stackrel{\text{def}}{=} \|\rho\|_{L^\infty([0, T^*])}$. Then we get that for any $t_0 \in [0, T^*[$ and for all $t \in [t_0, T^*[$*

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega(t_0)\|_\infty + \int_{t_0}^t (nv_d \mathbf{A}_{d,n}(s) + \gamma_d \mathbf{C}_d(s)) ds) e^{\int_{t_0}^t v_d C_{n,\rho} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau}},$$

with

$$\begin{aligned} \mathbf{A}_{d,n}(t) &\stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0, 1/\|\omega(t)\|_\infty^n) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|^{\frac{1}{n}}} \\ \mathbf{B}_d(t) &\stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+, \\ \mathbf{C}_d(t) &\stackrel{\text{def}}{=} \min \left(\frac{\|u(t)\|_\infty}{\rho(t)}, \frac{\|u_0\|_2}{\rho(t)^{\frac{d}{2}+1}} \right), \end{aligned} \quad (28)$$

where $v_d = \pi$ if $d = 2$ else $v_d = 1$, $\gamma_d = 10\pi$ if $d = 2$ else $\gamma_d = 18$, $C_{n,\rho} = \log^+ \rho_\infty + n$ and $\Omega(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}$. Moreover, we have for all $t \in [0, T^*[$ and $x \in \Omega(t)$,

$$\nabla|\omega|(x, t) = 0 \text{ and } \nabla \cdot \xi(x, t) = 0.$$

340 *Proof.* Let $0 \leq t_0 < T < T^*$. We want to apply Lemma 4.1 to the function ω , then we check that the hypotheses of the Lemma are satisfied.

Since $u \in C([0, T]; H^r(\mathbb{R}^d)) \cap C^1([0, T]; H^{r-2}(\mathbb{R}^d))$, then we infer that $\omega \in C([0, T]; H^{r-1}(\mathbb{R}^d)) \cap C^1([0, T]; H^{r-3}(\mathbb{R}^d))$. Thanks to the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow BC^m(\mathbb{R}^d)$ for $s > \frac{d}{2} + m$, $m \in \mathbb{N}$ and since $r > \frac{d}{2} + 3$ we deduce that $\omega \in C([0, T]; BC^2(\mathbb{R}^d)) \cap C^1([0, T]; BC(\mathbb{R}^d))$. Thanks to (19), we get

345 that $\inf_{t \in [0, T]} \|\omega(t)\|_\infty > 0$. Moreover, since $\omega \in C([0, T]; H^{r-1}(\mathbb{R}^d))$ with $r >$

$\frac{d}{2} + 3$, we have for any $t \in [0, T]$, $|\omega(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$, the proof follows immediately by using the density of $C_0^\infty(\mathbb{R}^d)$ in $H^{r-1}(\mathbb{R}^d)$ and the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $r > \frac{d}{2} + 3$.

350 Then thanks to Lemma 4.1, there exists $R > 0$ such that for all $t \in [0, T]$, $\|\omega(t)\|_\infty = \sup_{x \in B_R} |\omega(x, t)|$. Then for all $t \in [0, T]$, the set $\Omega(t)$ defined just above

(50) can be rewritten as follows:

$$\Omega(t) = \{x \in B_R; |\omega(x, t)| = \|\omega(t)\|_\infty\}.$$

We introduce the direction of the vorticity $\xi = \frac{\omega}{|\omega|}$ defined on the non empty open set $\mathcal{O} \stackrel{\text{def}}{=} \{(x, t) \in \mathbb{R}^d \times [0, T]; |\omega(x, t)| > 0\}$.

We set $\mathbf{v} = u$ in the case of 3D Navier-Stokes or 3D Euler equations and $\mathbf{v} = -R^\perp u$ with $\nu = 0$ in the case of 2D QG equation.

Then by multiplying (11) or (12) by ξ , we get that for all $(x, t) \in \mathcal{O}$,

$$\begin{aligned} \frac{\partial|\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla|\omega|(x, t) & - (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) \\ & - \nu\Delta|\omega|(x, t) + \nu|\omega(x, t)||\nabla\xi(x, t)|^2 = 0. \end{aligned} \quad (29)$$

We introduce the function φ defined for all $t \in [0, T]$ by $\varphi(t) \stackrel{\text{def}}{=} \sup_{x \in B_R} |\omega(x, t)|$ and we search the expression of its derivative. For this, we use the main Theorem obtained in [78] or Theorem 1 in [7] after verifying that the hypotheses of the Theorem are satisfied.

Since $\omega \in C([0, T]; BC^2(\mathbb{R}^d)) \cap C^1([0, T]; BC(\mathbb{R}^d))$, then we deduce that $|\omega| \in BC(\mathcal{O})$, $\frac{\partial|\omega|}{\partial t} \in BC(\mathcal{O})$ and $\nabla^2|\omega| \in BC(\mathcal{O})$. Since for any $t \in [0, T]$, $\Omega(t) \subset \mathcal{O}$, then, thanks to the results obtained in [78] (see also Theorem 1 in [7]), we obtain the expression of the derivative of φ given for any $t \in [0, T]$ by,

$$\varphi'(t) = \sup_{x \in \Omega(t)} \frac{\partial|\omega|}{\partial t}(x, t). \quad (30)$$

Further for all $x \in \Omega(t) \subset B_R$, we have $|\omega(x, t)| = \varphi(t) = \|\omega(t)\|_\infty$, we thus infer that

$$\nabla|\omega|(x, t) = 0 \text{ and } \Delta|\omega|(x, t) \leq 0. \quad (31)$$

Therefore, we have for all $x \in \Omega(t)$,

$$\begin{aligned} \frac{\partial|\omega|}{\partial t}(x, t) & = \frac{\partial|\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla|\omega|(x, t) \\ & = (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) + \nu\Delta|\omega|(x, t) - \nu|\omega(x, t)||\nabla\xi(x, t)|^2 \\ & \leq (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \end{aligned} \quad (32)$$

where we have used (29) for the second equality and (31) for the last inequality. We can notice that we get equality for (32) in the case of 3D Euler or 2D QG equations, since for these equations we have not the terms $\nu\Delta|\omega|(x, t)$ and $\nu|\omega(x, t)||\nabla\xi(x, t)|^2$.

Then using (32), from (30), we obtain,

$$\varphi'(t) \leq \sup_{x \in \Omega(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t),$$

which means that

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \sup_{x \in \Omega(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \quad (33)$$

where equality holds in the case of 3D Euler or 2D QG equations.

We use now the function α introduced in [16, 17] for the 3D Navier-Stokes or 3D Euler equations and in [15] for the 2D QG equation, defined for all $(x, t) \in \mathcal{O}$ by,

$$\alpha(x, t) = c_d P.V. \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d}, \quad (34)$$

where $\hat{y} = \frac{y}{|y|}$ and in the case of 3D Navier-Stokes or 3D Euler equations for which $d = 3$, $c_d = \frac{3}{4\pi}$ and in the case of 2D QG equation for which $d = 2$, $c_d = 1$.

By using the Biot-Savart law (see [13]) for which in the case of Euler and Navier-Stokes equations, we have

$$\mathbf{v}(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y) dy,$$

and in the case of 2D QG equations, we get an equivalent formula

$$\mathbf{v}(x, t) = - \int_{\mathbb{R}^2} \frac{1}{|y|} \omega(x+y, t) dy,$$

we deduce as in [16, 17] and [15] that for all $(x, t) \in \mathcal{O}$

$$(\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) = \alpha(x, t) |\omega(x, t)|.$$

Therefore, from (33), we deduce that for all $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt} \|\omega(t)\|_\infty &\leq \sup_{x \in \Omega(t)} \alpha(x, t) |\omega(x, t)| \\ &= \left(\sup_{x \in \Omega(t)} (\alpha(x, t)) \right) \|\omega(t)\|_\infty, \end{aligned} \quad (35)$$

where we have used the fact that for all $x \in \Omega(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty$. We take a smooth cutoff function $\chi : [0, +\infty[\rightarrow [0, 1]$ such that $|\chi'| \leq \frac{3}{2}$, $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$.

Let us estimate now $\alpha(x, t)$ for any $t \in [t_0, T]$ and $x \in \Omega(t)$. We decompose $\alpha(x, t)$ into two terms,

$$\alpha(x, t) = I_1 + I_2 \quad (36)$$

with,

$$I_1 = c_d \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) \chi\left(\frac{2|y|}{\rho(t)}\right) |\omega(x+y, t)| \frac{dy}{|y|^d}, \quad (37)$$

and

$$I_2 = c_d \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) \left(1 - \chi\left(\frac{2|y|}{\rho(t)}\right)\right) |\omega(x+y, t)| \frac{dy}{|y|^d}. \quad (38)$$

Then, we bound the two terms I_1 and I_2 . For the term I_1 , from (37) we get

$$I_1 = I_{1,1} + I_{1,2}, \quad (39)$$

with

$$I_{1,1} = c_d \int_{B(0,1/\|\omega(t)\|_\infty^n)} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) \chi\left(\frac{2|y|}{\rho(t)}\right) |\omega(x+y, t)| \frac{dy}{|y|^d} \quad (40)$$

and

$$I_{1,2} = c_d \int_{B(0,1/\|\omega(t)\|_\infty^n)^c} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) \chi\left(\frac{2|y|}{\rho(t)}\right) |\omega(x+y, t)| \frac{dy}{|y|^d}. \quad (41)$$

For the term $I_{1,1}$, on one hand we get

$$\begin{aligned} I_{1,1} &\leq c_d \int_{B(0,1/\|\omega(t)\|_\infty^n)} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|^{\frac{1}{n}}} |\omega(x+y, t)| \frac{dy}{|y|^{d-\frac{1}{n}}} \\ &\leq c_d \mathbf{A}_{d,n}(t) \|\omega(t)\|_\infty \int_{B(0,1/\|\omega(t)\|_\infty^n)} \frac{dy}{|y|^{d-\frac{1}{n}}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{B(0,1/\|\omega(t)\|_\infty^n)} \frac{dy}{|y|^{d-\frac{1}{n}}} &= |B(0,1)| \int_0^{1/\|\omega(t)\|_\infty^n} \frac{ds}{s^{1-\frac{1}{n}}} \\ &= \frac{n|B(0,1)|}{\|\omega(t)\|_\infty}. \end{aligned}$$

Therefore, with the fact that $v_d = c_d |B(0,1)|$ (since $|B(0,1)| = \frac{4\pi}{3}$ for $d = 3$ and $|B(0,1)| = \pi$ for $d = 2$), we deduce

$$I_{1,1} \leq n v_d \mathbf{A}_{d,n}(t). \quad (42)$$

370 For the term $I_{1,2}$, on one hand, from (41) we get

$$\begin{aligned} I_{1,2} &\leq c_d \int_{y \in \mathbb{R}^d; 1/\|\omega(t)\|_\infty^n \leq |y| \leq \rho(t)} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+ |\omega(x+y, t)| \frac{dy}{|y|^d} \\ &\leq c_d \mathbf{B}_d(t) \|\omega(t)\|_\infty \int_{y \in \mathbb{R}^d; 1/\|\omega(t)\|_\infty^n \leq |y| \leq \rho(t)} \frac{dy}{|y|^d} \\ &= c_d |B(0,1)| \mathbf{B}_d(t) \|\omega(t)\|_\infty \log^+(\rho(t) \|\omega(t)\|_\infty^n). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} \log^+(\rho(t) \|\omega(t)\|_\infty^n) &\leq \log^+(\rho_\infty \|\omega(t)\|_\infty^n) \\ &\leq \log^+ \rho_\infty + n \log^+ \|\omega(t)\|_\infty \\ &\leq C_{n,\rho} (1 + \log^+ \|\omega(t)\|_\infty), \end{aligned}$$

where $C_{n,\rho} = \log^+ \rho_\infty + n$. Then we deduce

$$I_{1,2} \leq v_d C_{n,\rho} \mathbf{B}_d(t) \|\omega(t)\|_\infty (1 + \log^+ \|\omega(t)\|_\infty). \quad (43)$$

By using (42) and (43), from (39) we deduce

$$I_1 \leq n v_d \mathbf{A}_{d,n}(t) + v_d C_{n,\rho} \mathbf{B}_d(t) \|\omega(t)\|_\infty (1 + \log^+ \|\omega(t)\|_\infty). \quad (44)$$

For the term I_2 , we use the fact that $|\omega(x+y, t)| \xi(x+y, t) = \omega(x+y, t)$ and the fact that \mathbf{D}_d is linear in comparison with its second variable, then we rewrite I_2 to obtain from (38),

$$I_2 = c_d \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \omega(x+y, t), \xi(x, t)) \left(1 - \chi\left(\frac{2|y|}{\rho(t)}\right)\right) \frac{1}{|y|^d} dy. \quad (45)$$

To obtain a precise non blowup criterion for 3D the Navier-Stokes, 3D Euler and 2D QG equations that could be used easily in numerical experiments, it was important to explicit the constant involved in the estimate of the term I_2 .
375 For this purpose, we deal first with the case of the 3D the Navier-Stokes and 3D Euler, then after we consider the case of the 2D QG equations.

In the case of the 3D Euler or 3D Navier-Stokes equations for which $d = 3$, we get $\mathbf{D}_3(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)) \det(\hat{y}, \omega(x+y, t), \xi(x, t))$.
380 Since $\det(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\xi(x, t) \times \hat{y}) \cdot \omega(x+y, t)$ and $\omega(x+y, t) = \nabla_y \times u(x+y, t)$, we deduce

$$\mathbf{D}_3(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)) (\xi(x, t) \times \hat{y}) \cdot \nabla_y \times u(x+y, t).$$

Then, after using an integration by parts, from (45), we deduce,

$$I_2 = c_3 \int_{\mathbb{R}^3} \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t)) (\xi(x, t) \times \hat{y})}{|y|^3} \left(1 - \chi\left(\frac{2|y|}{\rho(t)}\right)\right) \right) \cdot u(x+y, t) dy. \quad (46)$$

After setting $\psi(y) \equiv \frac{(\hat{y} \cdot \xi(x, t))}{|y|^3} \left(1 - \chi\left(\frac{2|y|}{\rho(t)}\right)\right)$ and $\mathbf{V}(y) \equiv (\xi(x, t) \times \hat{y})$, by using the following vectorial identity $\nabla \times (\psi \mathbf{V}) = \nabla \psi \times \mathbf{V} + (\nabla \times \mathbf{V}) \psi$, we obtain after elementary computations, that for all $y \neq 0$,

$$\begin{aligned} & \left| \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t)) (\xi(x, t) \times \hat{y})}{|y|^3} \left(1 - \chi\left(\frac{2|y|}{\rho(t)}\right)\right) \right) \right| \\ & \leq \left(\left| \nabla \left(\frac{\hat{y}}{|y|^3} \right) \right| + \frac{2}{\rho(t)|y|^3} \left| \chi' \left(\frac{2|y|}{\rho(t)} \right) \right| + \frac{|\nabla_y \times (\xi(x, t) \times \hat{y})|}{|y|^3} \right) \mathbf{1}_{\{|y| \geq \frac{\rho(t)}{2}\}}. \end{aligned}$$

385 We have $\left| \nabla \left(\frac{\hat{y}}{|y|^3} \right) \right| \leq \frac{3}{|y|^4}$. Since the function χ' is compactly supported in $[1, 2]$ and $|\chi'| \leq \frac{3}{2}$, we get $\frac{1}{\rho(t)} \left| \chi' \left(\frac{2|y|}{\rho(t)} \right) \right| \leq \frac{3}{2|y|}$. Furthermore, we have $\nabla_y \times (\xi(x, t) \times \hat{y}) = (\nabla_y \cdot \hat{y}) \xi(x, t) - (\xi(x, t) \cdot \nabla_y) \hat{y}$ and then we deduce $|\nabla_y \times$

$(\xi(x, t) \times \hat{y})| \leq |\nabla \cdot \hat{y}| + |\nabla \hat{y}| \leq \frac{3}{|y|}$. After gathering these results, we obtain that for all $y \neq 0$,

$$\left| \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \left(1 - \chi \left(\frac{2|y|}{\rho(t)} \right) \right) \right) \right| \leq \frac{9}{|y|^4} \mathbf{1}_{\{|y| \geq \frac{\rho(t)}{2}\}}.$$

Therefore, from (46) we obtain

$$I_2 \leq 9c_3 \int_{y \in \mathbb{R}^3, |y| \geq \frac{\rho(t)}{2}} \frac{|u(x+y, t)|}{|y|^4} dy. \quad (47)$$

³⁹⁰ **In the case of 2D QG equations** for which $d = 2$, we get $\mathbf{D}_2(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)^\perp)(\omega(x+y, t) \cdot \xi(x, t)^\perp)$. Since $\omega(x+y, t) = \nabla_y^\perp u(x+y, t)$, we deduce

$$\mathbf{D}_2(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp \cdot \nabla_y^\perp u(x+y, t).$$

Then, after using an integration by parts, from (45), we deduce,

$$I_2 = -c_2 \int_{\mathbb{R}^2} \operatorname{curl}_y \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \left(1 - \chi \left(\frac{2|y|}{\rho(t)} \right) \right) \right) u(x+y, t) dy, \quad (48)$$

where for any vector field $F = (F_1, F_2)$ from \mathbb{R}^2 to \mathbb{R}^2 , $\operatorname{curl} F = -\partial_2 F_1 + \partial_1 F_2$.

After setting $\psi(y) \equiv \frac{(\hat{y} \cdot \xi(x, t)^\perp)}{|y|^2} \left(1 - \chi \left(\frac{2|y|}{\rho(t)} \right) \right)$ and $\mathbf{V}(y) \equiv \xi(x, t)^\perp$, by

³⁹⁵ using the following vectorial identity $\operatorname{curl}(\psi \mathbf{V}) = \nabla^\perp \psi \cdot \mathbf{V} + \psi \operatorname{curl} \mathbf{V}$, we obtain after elementary computations, that for all $y \neq 0$,

$$\begin{aligned} & \left| \operatorname{curl}_y \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \left(1 - \chi \left(\frac{2|y|}{\rho(t)} \right) \right) \right) \right| \\ & \leq \left(\left| \nabla \left(\frac{\hat{y}}{|y|^2} \right) \right| + \frac{2}{\rho(t)|y|^2} \left| \chi' \left(\frac{2|y|}{\rho(t)} \right) \right| \right) \mathbf{1}_{\{|y| \geq \frac{\rho(t)}{2}\}}. \end{aligned}$$

We have $\left| \nabla \left(\frac{\hat{y}}{|y|^2} \right) \right| \leq \frac{2}{|y|^3}$. Since the function χ' is compactly supported in $[1, 2]$ and $|\chi'| \leq \frac{3}{2}$, we get $\frac{1}{\rho(t)} \left| \chi' \left(\frac{2|y|}{\rho(t)} \right) \right| \leq \frac{3}{2|y|}$. We thus deduce that for all $y \neq 0$,

$$\left| \operatorname{curl}_y \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \left(1 - \chi \left(\frac{2|y|}{\rho(t)} \right) \right) \right) \right| \leq \frac{5}{|y|^3}.$$

Therefore, from (48) we obtain

$$I_2 \leq 5c_2 \int_{y \in \mathbb{R}^2, |y| \geq \frac{\rho(t)}{2}} \frac{|u(x+y, t)|}{|y|^3} dy. \quad (49)$$

Therefore, whatever the case considered, 3D Navier-Stokes, 3D Euler or 2D QG equations, from (47) and (49) we deduce

$$I_2 \leq \eta_d c_d \int_{y \in \mathbb{R}^d, |y| \geq \frac{\rho(t)}{2}} \frac{|u(x+y, t)|}{|y|^{d+1}} dy, \quad (50)$$

where $\eta_d = 9$ if $d = 3$ and $\eta_d = 5$ if $d = 2$. Then from (50), we obtain

$$\begin{aligned} I_2 &\leq \eta_d c_d |B(0, 1)| \|u(t)\|_\infty \int_{\frac{\rho(t)}{2}}^\infty \frac{ds}{s^2} \\ &= 2\eta_d v_d \frac{\|u(t)\|_\infty}{\rho(t)}. \end{aligned} \quad (51)$$

Furthermore, thanks to the Cauchy-Schwarz inequality and the energy equalities (13), (16), from (50) we infer also

$$\begin{aligned} I_2 &\leq \eta_d c_d \|u_0\|_2 \left(\int_{y \in \mathbb{R}^d, |y| \geq \frac{\rho(t)}{2}} \frac{dy}{|y|^{2d+2}} \right)^{\frac{1}{2}} \\ &= \frac{2^{\frac{d}{2}+1} \eta_d c_d |B(0, 1)|^{\frac{1}{2}} \|u_0\|_2}{(d+2)^{\frac{1}{2}} \rho(t)^{\frac{d}{2}+1}}. \end{aligned} \quad (52)$$

Therefore owing to (51) and (52) we get that for all $t \in [t_0, T]$

$$I_2 \leq \min \left(\gamma_d \frac{\|u(t)\|_\infty}{\rho(t)}, \lambda_d \frac{\|u_0\|_2}{\rho(t)^{\frac{d}{2}+1}} \right), \quad (53)$$

with $\gamma_d \stackrel{\text{def}}{=} 2\eta_d v_d$ and $\lambda_d = \frac{2^{\frac{d}{2}+1} \eta_d c_d |B(0, 1)|^{\frac{1}{2}}}{(d+2)^{\frac{1}{2}}}$. We observe that $\gamma_d = 18$ if $d = 3$ and $\gamma_d = 10\pi$ if $d = 2$; $\lambda_d = 18 \left(\frac{6}{5\pi}\right)^{\frac{1}{2}}$ if $d = 3$ and $\lambda_d = 10\pi$ if $d = 2$. Hence, we get $\lambda_d \leq \gamma_d$. Therefore, from (53) we get

$$I_2 \leq \gamma_d \min \left(\frac{\|u(t)\|_\infty}{\rho(t)}, \frac{\|u_0\|_2}{\rho(t)^{\frac{d}{2}+1}} \right). \quad (54)$$

Then, owing to (44) and (54), from (36) we deduce that for any $t \in [t_0, T]$ and $x \in \Omega(t)$,

$$\begin{aligned} \alpha(x, t) &\leq n v_d \mathbf{A}_{d,n}(t) + v_d C_{n,\rho} \mathbf{B}_d(t) \|\omega(t)\|_\infty (1 + \log^+ \|\omega(t)\|_\infty) \\ &\quad + \gamma_d \min \left(\frac{\|u(t)\|_\infty}{\rho(t)}, \frac{\|u_0\|_2}{\rho(t)^{\frac{d}{2}+1}} \right). \end{aligned} \quad (55)$$

Then from (35), after using (55) and (28), we obtain that for all $t \in [t_0, T]$,

$$\frac{d}{dt} \|\omega(t)\|_\infty \leq (n v_d \mathbf{A}_{d,n}(t) + \gamma_d \mathbf{C}_d(t)) \|\omega(t)\|_\infty + v_d C_{n,\rho} \mathbf{B}_d(t) \|\omega(t)\|_\infty^2 (1 + \log^+ \|\omega(t)\|_\infty). \quad (56)$$

400 Thanks to Lemma 4.2, we obtain that for any $t_0 \in [0, T[$ for all $t \in [t_0, T]$,

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega(t_0)\|_\infty + \int_{t_0}^t (nv_d \mathbf{A}_{d,n}(s) + \gamma_d \mathbf{C}_d(s)) ds)} e^{\int_{t_0}^t v_d C_{n,\rho} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau},$$

which concludes the first part of the proof. Thanks to (31), we have already $\nabla|\omega|(x, t) = 0$. Since $\nabla \cdot \omega = 0$ and $\omega = |\omega|\xi$, then we get

$$0 = \nabla \cdot \omega = |\omega| \nabla \cdot \xi + \xi \cdot \nabla |\omega|. \quad (57)$$

However, for all $x \in \Omega(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty > 0$ and from (31), we have $\nabla|\omega|(x, t) = 0$. Therefore, from (57), we deduce that for all $t \in [0, T]$ and $x \in \Omega(t)$,

$$\nabla \cdot \xi(x, t) = 0, \quad (58)$$

which completes the proof. \square

We define on $[0, T^*]$ the function ρ such that:

$$\begin{aligned} \text{If } \limsup_{t \rightarrow T^*} (T^* - t)^{\frac{d}{d+2}} \|u(t)\|_\infty < +\infty \\ \text{then for all } t \in [0, T^*], \rho(t) = \frac{1}{2\gamma_d} \|u(t)\|_\infty (T^* - t). \end{aligned} \quad (59)$$

$$\begin{aligned} \text{Otherwise if } \limsup_{t \rightarrow T^*} (T^* - t)^{\frac{d}{d+2}} \|u(t)\|_\infty = +\infty \\ \text{then for all } t \in [0, T^*], \rho(t) = \frac{1}{(2\gamma_d \|u_0\|_2)^{\frac{2}{d+2}}} (T^* - t)^{\frac{2}{d+2}}, \end{aligned} \quad (60)$$

with $\gamma_d = 10\pi$ if $d = 2$ else $\gamma_d = 18$.

Remark 4.1. *In the case of the 2D QG equations, thanks to (16) used with $p = \infty$, from (59) we get that the function ρ is defined for all $t \in [0, T^*]$ by*
 405 $\rho(t) = \frac{1}{20\pi} \|u_0\|_\infty (T^* - t)$.

Thanks to Proposition 4.1, we obtain our Theorem.

Theorem 4.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (3)-(4) or 2D QG equations (5)-(6) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Let ρ be the bounded function defined on $[0, T^]$ by (59) and (60). If there exists $n \geq 1$ and $t_1 \in [0, T^*[$ such that*

$$\int_{t_1}^{T^*} \mathbf{A}_{d,n}(t) dt < +\infty \text{ and } \int_{t_1}^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt < +\infty, \quad (61)$$

then the solution u cannot blowup at the finite time T^* , where for all $t \in [t_1, T^*[$

$$\mathbf{A}_{d,n}(t) \stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0,1/\|\omega(t)\|_\infty) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|^{\frac{1}{n}}} \quad (62)$$

$$\mathbf{B}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0,\rho(t)) \setminus \{0\}} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+,$$

410 with $\Omega(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}$.

Moreover, we have for all $t \in [0, T^*[$ and $x \in \Omega(t)$,

$$\nabla|\omega|(x, t) = 0 \text{ and } \nabla \cdot \xi(x, t) = 0.$$

Proof. Let us assume that there exists $n \geq 1$ and $t_1 \in [0, T^*[$ such that (61) holds. We introduce the function \mathbf{C}_d defined on $[0, T^*[$ by $\mathbf{C}_d(t) \stackrel{\text{def}}{=} \min\left(\frac{\|\omega(t)\|_\infty}{\rho(t)}, \frac{\|\omega_0\|_2}{\rho(t)^{\frac{d}{2}+1}}\right)$ for all $t \in [0, T^*[$. Thanks to Proposition 4.1, we get that for any $t_0 \in [0, T^*[$ and for all $t \in [t_0, T^*[$

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega(t_0)\|_\infty + \int_{t_0}^t (nv_d \mathbf{A}_{d,n}(s) + \gamma_d \mathbf{C}_d(s)) ds) e^{\int_{t_0}^t v_d C_{n,\rho} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau}}, \quad (63)$$

where $v_d = \pi$ if $d = 2$ else $v_d = 1$, $\gamma_d = 10\pi$ if $d = 2$ else $\gamma_d = 18$, $C_{n,\rho} = \log^+ \rho_\infty + n$. By using the definition of the function ρ given by (59) and (60), we observe that for any $t_0 \in [0, T^*[$ and for all $t \in [t_0, T^*[$

$$\begin{aligned} \int_{t_0}^t \mathbf{C}_d(s) ds &= \frac{1}{2\gamma_d} \int_{t_0}^t \frac{ds}{T^* - s} \\ &= \frac{1}{2\gamma_d} \log\left(\frac{T^* - t_0}{T^* - t}\right). \end{aligned} \quad (64)$$

By using (64), from (63) we deduce that for any $t_0 \in [0, T^*[$ and for all $t \in [t_0, T^*[$

$$\|\omega(t)\|_\infty \leq \psi(t_0, t) \left(\frac{T^* - t_0}{T^* - t}\right)^{\frac{1}{2} e^{\int_{t_0}^t v_d C_{n,\rho} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau}}, \quad (65)$$

where $\psi(t_0, t) \equiv e^{(1+\log^+ \|\omega(t_0)\|_\infty + \int_{t_0}^t nv_d \mathbf{A}_{d,n}(s) ds) e^{\int_{t_0}^t v_d C_{n,\rho} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau}}$.

Thanks to (61), we infer that $\lim_{t_0 \rightarrow T^*} \int_{t_0}^{T^*} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau = 0$ and hence there exists $t_2 \in [t_1, T^*[$ such that

$$\int_{t_2}^{T^*} \mathbf{B}_d(\tau) \|\omega(\tau)\|_\infty d\tau \leq \frac{\log(\frac{3}{2})}{v_d C_{n,\rho}}. \quad (66)$$

Owing to (66), by using (65) with $t_0 = t_2$ we obtain that for all $t \in [t_2, T^*[$,

$$\begin{aligned} \|\omega(t)\|_\infty &\leq \psi(t_2, t) \left(\frac{T^* - t_2}{T^* - t}\right)^{\frac{3}{4}} \\ &\leq M \left(\frac{T^* - t_2}{T^* - t}\right)^{\frac{3}{4}}, \end{aligned} \quad (67)$$

where $M \stackrel{\text{def}}{=} \psi(t_2, T^*) < +\infty$ thanks again to (61). After integrating inequality (67) over $[t_2, T]$ with $T \in [t_2, T^*[$, we obtain

$$\begin{aligned} \int_{t_2}^T \|\omega(t)\|_\infty dt &\leq 4M(T^* - t_2)^{\frac{3}{4}}(T - t_2)^{\frac{1}{4}} \\ &\leq 4M(T^* - t_2). \end{aligned}$$

Therefore we deduce that $\int_{t_2}^{T^*} \|\omega(t)\|_\infty dt \leq 4M(T^* - t_2) < +\infty$. Since $u \in$
415 $C([0, T^*]; H_\sigma^r(\mathbb{R}^d))$ and thanks to the Sobolev embedding $H^r(\mathbb{R}^d) \hookrightarrow BC^3(\mathbb{R}^d)$
due to $r > \frac{d}{2} + 3$, we infer that $\omega \in C([0, T^*]; BC^2(\mathbb{R}^d))$ which implies that
 $\int_0^{t_2} \|\omega(t)\|_\infty dt < +\infty$. Therefore we get $\int_0^{T^*} \|\omega(t)\|_\infty dt < +\infty$. If u blows up
at the finite time T^* then thanks to (14) and (17) we have $\int_0^{T^*} \|\omega(t)\|_\infty dt =$
 $+\infty$ which leads to a contradiction. Then, we deduce that u cannot blow up at
420 the time T^* which concludes the first part of proof. Thanks again to Proposition
4.1, we get that for all $t \in [0, T^*[$ and $x \in \Omega(t)$,

$$\nabla|\omega|(x, t) = 0 \text{ and } \nabla \cdot \xi(x, t) = 0,$$

which completes the proof. \square

5. Some applications of our geometric non blow-up criterion

In this section, through Lemma 5.1 we show that the condition $\int_{t_1}^{T^*} \mathbf{A}_{d,n}(t) dt <$
425 $+\infty$ appearing in the statement of Theorem 4.1 is a very weak assumption.
Then, thanks to Theorem 4.1 and Lemma 5.1, in Corollary 5.1 we release our
geometric criterion for the non blow-up in finite time of the solutions of 3D
Navier-Stokes, 3D Euler and 2D QG equations. First, thanks to Theorem 4.1,
we emphasize that the singularity discovered by Luo & Hou in [71] which lies
430 right on the boundary is not relevant in the case of the whole domain \mathbb{R}^3 . In-
deed, thanks to Theorem 4.1, in the whole domain of \mathbb{R}^3 at any point of the
maximum vorticity, $q_0 \in \mathbb{R}^3$, we get $\nabla|\omega|(q_0, t) = 0$ and $\nabla \cdot \xi(q_0, t) = 0$ for
any any time t before the alleged time of singularity T^* . However in [71], the
presence of a solid boundary and the fact that q_0 the point of the maximum
435 vorticity is always located on the solid boundary, prevent to get $\nabla|\omega|(q_0, t) = 0$
and this allows to get $\nabla \cdot \xi(q_0, t) \sim (T^* - t)^{-2.9165} \neq 0$ as it is observed in their
numerical test. This latter is the main element used to invalidate the Deng-
Hou-Yu non-blowup criterion [25, 27].

440 Under the assumption of a blow-up in finite time at T^* of the solution u of
3D Navier-Stokes, 3D Euler or 2D QG equations for which we suppose that
(68) holds, then thanks to Lemma 5.1, we obtain that the term $\mathbf{A}_{d,n}(t)$ defined
in (62) or again in (69), is L^1 -integrable over the time interval $[0, T^*]$,

445 which indicates that the condition $\int_{t_1}^{T^*} \mathbf{A}_{d,n}(t) dt < +\infty$ for $t_1 \in [0, T^*[$ is not a restrictive assumption.

Lemma 5.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (3)-(4) or 2D QG equations (5)-(6) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Suppose that there exist $\gamma_0 > 0, c_0 > 0, \gamma_1 \geq 0, c_1 > 0, n \geq 1$ such that $n > \frac{\gamma_0 + \gamma_1 - 1}{\gamma_0}$ and $t_0 \in [0, T^[$ such that for all $t \in [t_0, T^*[$,*

$$\|\omega(t)\|_\infty \geq c_0(T^* - t)^{-\gamma_0} \text{ and } \sup_{x \in \Omega(t)} \|\nabla \xi(t)\|_{L^\infty(B(x, 1/\|\omega(t)\|_\infty^n))} \leq c_1(T^* - t)^{-\gamma_1}, \quad (68)$$

with $\Omega(t) = \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}$. Then we get

$$\int_{t_0}^{T^*} \mathbf{A}_{d,n}(t) dt < +\infty,$$

where

$$\mathbf{A}_{d,n}(t) \stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0, 1/\|\omega(t)\|_\infty^n) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|^{\frac{1}{n}}}. \quad (69)$$

Proof. Thanks to (22), for all $t \in [t_0, T^*[$ and for all $x \in \Omega(t)$, we get for any $y \in B(0, 1/\|\omega(t)\|_\infty^n) \setminus \{0\}$,

$$|\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))| \leq |\xi(x+y, t) - \xi(x, t)| \leq \|\nabla \xi(t)\|_{L^\infty(B(x, 1/\|\omega(t)\|_\infty^n))} |y|. \quad (70)$$

Then from (70) we get that for any $y \in B(0, 1/\|\omega(t)\|_\infty^n) \setminus \{0\}$,

$$\begin{aligned} \frac{|\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))|}{|y|^{\frac{1}{n}}} &\leq \|\nabla \xi(t)\|_{L^\infty(B(x, 1/\|\omega(t)\|_\infty^n))} |y|^{1-\frac{1}{n}} \\ &\leq \frac{\|\nabla \xi(t)\|_{L^\infty(B(x, 1/\|\omega(t)\|_\infty^n))}}{\|\omega(t)\|_\infty^{n-1}}. \end{aligned} \quad (71)$$

Then owing to (71), from (69) we infer that for all $t \in [t_0, T^*[$,

$$\mathbf{A}_{d,n}(t) \leq \frac{c_1}{c_0^{n-1}} (T^* - t)^{-(\gamma_1 - \gamma_0(n-1))}. \quad (72)$$

450 Since $n > \frac{\gamma_0 + \gamma_1 - 1}{\gamma_0}$, we get $\gamma_1 - \gamma_0(n-1) < 1$. Therefore from (72) we deduce

$$\int_{t_0}^{T^*} \mathbf{A}_{d,n}(t) dt < +\infty,$$

which concludes the proof. \square

Owing to Lemma 5.1, from Theorem 4.1 we get the following Corollary 5.1.

Corollary 5.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be
455 such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D
Euler equations (3)-(4) or 2D QG equations (5)-(6) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

*Suppose that there exist $\gamma_0 > 0, c_0 > 0, \gamma_1 \geq 0, c_1 > 0, n \geq 1$ such that
460 $n > \frac{\gamma_0 + \gamma_1 - 1}{\gamma_0}$ and $t_0 \in [0, T^*]$ such that for all $t \in [t_0, T^*]$,*

$$\|\omega(t)\|_\infty \geq c_0(T^* - t)^{-\gamma_0} \text{ and } \sup_{x \in \Omega(t)} \|\nabla \xi(t)\|_{L^\infty(B(x, 1/\|\omega(t)\|_\infty^n))} \leq c_1(T^* - t)^{-\gamma_1},$$

*with $\Omega(t) = \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}$. Then the solution u blows up at the
460 finite time T^* (i.e $\limsup_{t \rightarrow T^*} \|u(t)\|_{H^r(\mathbb{R}^d)} = +\infty$) if and only if*

$$\int_{t_0}^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt = +\infty,$$

where

$$\mathbf{B}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+, \quad (73)$$

with ρ the function defined on $[0, T^*]$ by (59) and (60).

Proof. Thanks to Lemma 5.1, we infer

$$\int_{t_0}^{T^*} \mathbf{A}_{d,n}(t) dt < +\infty. \quad (74)$$

where $\mathbf{A}_{d,n}$ is the function defined by (69). Let us assume that u blows up at the
finite time T^* . If $\int_{t_0}^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt < +\infty$, owing to (74) we infer thanks
to Theorem 4.1 that the solution u cannot blowup at the finite time T^* which
465 leads to a contradiction and hence $\int_{t_0}^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt = +\infty$. Let us assume
now that $\int_{t_0}^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt = +\infty$. Thanks to (23), from (73), we deduce
that for all $t \in [t_0, T^*]$, $0 \leq \mathbf{B}_d(t) \leq 1$ which implies that $\int_{t_0}^{T^*} \|\omega(t)\|_\infty dt \geq$
 $\int_{t_0}^{T^*} \mathbf{B}_d(t) \|\omega(t)\|_\infty dt = +\infty$. Then, thanks to (14) and (17), we deduce that
the solution u blows up at the finite time T^* . Then, we conclude the proof. \square

470 Let us consider the function Φ_γ defined for all $s \in [0, 1]$ by

$$\Phi_\gamma(s) = \left(\log \log \left(\frac{e^{e^1}}{s} \right) \right)^{-\gamma} \left(\log \left(\frac{e^{e^1}}{s} \right) \right)^{-1} \quad \text{with } \gamma > 1.$$

Then, we get that for any $0 < \lambda \leq 1$,

$$\begin{aligned} \int_0^\lambda \frac{\Phi_\gamma(s)}{s} ds &= \left[\frac{\left(\log \log \left(\frac{e^{e^1}}{s} \right) \right)^{-\gamma+1}}{\gamma - 1} \right]_0^\lambda \\ &= \frac{\left(\log \log \left(\frac{e^{e^1}}{\lambda} \right) \right)^{-\gamma+1}}{\gamma - 1} < +\infty. \end{aligned} \quad (75)$$

Thus if one considers for the 3D Navier-Stokes and 3D Euler equations, the most plausible blow-up rates at a finite time T^* for $\|\omega(t)\|_\infty$, $\|u(t)\|_\infty$ and $\|\nabla\xi(t)\|_\infty$ namely (see [58, 59, 60, 61, 62]),

$$\|\omega(t)\|_\infty \sim (T^* - t)^{-1}, \quad \|\nabla\xi(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}} \quad \text{and} \quad \|u(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}},$$

475 then thanks to Corollary 5.1 and (75), the solution u cannot blow up at the finite time T^* if $\mathbf{B}_d(t) \leq \Phi_\gamma(T^* - t)$ for all $t \in [t_0, T^*[$ for some $t_0 \in [0, T^*[$ such that $T^* - t_0 \leq 1$. We recall that $\mathbf{B}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \mathbf{D}_d(\hat{y}, \xi(x + y, t), \xi(x, t))^+$ with ρ defined by (59) since $\|u(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}}$ which implies that $\rho(t) \sim (T^* - t)^{\frac{1}{2}}$.

480 For the 2D QG equations, the numerical experiments performed in [26] on the hyperbolic saddle test case (a candidate for finite-time blow-up suggested in [15]), indicate that the region of large vorticity ω in magnitude and the region of large $\nabla\xi$ in magnitude are disjoint (see Fig.2-Fig.5 in [26]) and shrink to zero as the time tends to the alleged time of singularity. This suggests that $\nabla\xi$ 485 remains bounded in regions where $|\omega|$ achieves its maximum and these regions shrink to zero as the time tends to the alleged time of singularity T^* . If these regions are contained in balls of radius of type $O(\rho(t))$, with $\rho(t)$ given by $\rho(t) = \frac{1}{20\pi} \|u_0\|_\infty (T^* - t)$ thanks to Remark 4.1, we will get $\mathbf{B}_d(t) \lesssim \rho(t)$ and hence thanks to Corollary 5.1, we will obtain that the solution u cannot blow up at the finite time T^* if $\int_{t_0}^{T^*} (T^* - t) \|\omega(t)\|_\infty < +\infty$ for some $t_0 \in [0, T^*[$. 490

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