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Some advances on the geometric non blow-up criteria of incompressible flows

Léo Agélas

*Department of Mathematics, IFP Energies Nouvelles, 1-4, avenue de Bois-Préau, F-92852
Rueil-Malmaison, France*

Abstract

One of the most challenging questions in fluid dynamics is whether the three-dimensional incompressible Navier-Stokes, Euler and two-dimensional Quasi-Geostrophic (2D QG) equations can develop a finite-time singularity from smooth initial data. The development of finite-time singularities was first investigated via energy methods which focus on global features or on point-wise Eulerian features of the flow. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy to overcome such shortcomings was established by focusing more on geometrical properties and flow structures, such as vortex tubes or vortex lines. However, all these recent geometric non blow-up criteria use the Lagrangian formulation of Incompressible Inviscid Flows, which requires much more computational effort than an Eulerian framework. In this paper, we improve these geometric non blow-up criteria while keeping an Eulerian setting by using Pshenichnyi's achievements in its investigation of necessary extremum conditions. By using also a dyadic unity partition of Littlewood decomposition, we established non blow-up criteria involving the dyadic BMO spaces known to be suitable for odd functions [86]. We thus obtained new insights on the growth of the maximum vorticity. Another highlight of our analysis is it allows to study the Navier-Stokes, Euler and 2D QG equations in a same framework. Our Eulerian geometric non blow-up criteria should give also new impetus to the numerical experiments due to their ease of implementation in comparison with Lagrangian geometric non blow-up criteria.

Keywords: Euler equations; Navier-Stokes equations; Quasi-Geostrophic equation; Finite time singularities; Geometric properties

Introduction

One of the most challenging questions in fluid dynamics is whether the incompressible 3D Navier-Stokes, Euler or 2D QG equations can develop a finite

Email address: `leo.agelas@ifpen.fr` (Léo Agélas)

time singularity from smooth and bounded initial data. We know already that
5 the blow up of smooth solutions to the Navier-Stokes and Euler equations is
controlled by the time integral of the maximum magnitude of the vorticity (see
[3, 66, 17], see also [10]). The Navier-Stokes and Euler equations describe the
motion of a fluid in the three-dimensional space. These fundamental equations
were derived over 250 years ago by Euler and since then have played a major role
10 in fluid dynamics. They have enriched many branches of mathematics, were in-
volved in many areas outside mathematical activity from weather prediction to
exploding supernova (see for instance the surveys [24],[2]) and present important
open physical and mathematical problems (see [24]). Regarding the 2D Quasi-
Geostrophic (2D QG) equation, it appears in atmospheric studies. It describes
15 the evolution of potential temperature u on the two dimensional boundary of a
rapidly rotating half space with small Rossby and Ekman numbers, for the case
of special solutions with constant potential vorticity in the interior and constant
buoyancy frequency (normalized to one), where equations in the bulk are com-
pressible Euler or Navier-Stokes equations coupled with temperature equation,
20 continuity equation, and equation of state.

In the case of Navier-Stokes equations, for a long time ago, a global weak
solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$ and $\nabla u \in L^2(\mathbb{R}^3 \times (0, \infty))^3$ was built by Leray
[79]. In particular, Leray introduced a notion of weak solutions for the Navier-
25 Stokes equations, and proved that, for every given $u_0 \in L^2(\mathbb{R}^3)^3$, there exists a
global weak solution $u \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))^3 \cap L^2([0, \infty[; \dot{H}^1(\mathbb{R}^3))^3$. Hopf has
proved the existence of a global weak solution in the general case \mathbb{R}^d , $d \geq 2$, [56].
Several ways are known to construct weak solutions ([42], [55], [43]), and mean-
while the regularity and the uniqueness of this weak solution has been known
30 for a long time ago for the two-dimensional case (see [78], [80], [82], [107]), in
the three-dimensional case the problem remains widely open in spite of great
efforts made. On the uniqueness many works have been done (see [41],[47],
[85],[81],[44]). Moreover, the uniqueness of the weak solutions follows if the reg-
ularity for weak solutions can be obtained. Many criteria have been obtained
35 on the regularity of weak solutions. In [96], it is proved that if u is a Leray-Hopf
weak solution belonging to $L^q(]0, T]; L^q(\mathbb{R}^3))^3$ with $\frac{2}{q} + \frac{3}{q} \leq 1$, $2 < p < \infty$, $3 <$
 $q < \infty$, then the solution $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. In [112] and [48], it is showed
that if u is a weak solution in $C([0, T]; L^3(\mathbb{R}^3))^3$, then $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$.
The limit case of $L^\infty([0, T]; L^3(\mathbb{R}^3))^3$ has been solved in [63]. Other criterion
40 regularity can be found in [52, 55, 47, 64, 67, 109, 7, 116, 19].

In the case of Euler Equations, in the two dimension case, uniqueness and
existence of classical solutions have been known for a long time ago (see [114,
115, 111, 36, 78]). Thus a natural question appears, namely what can be said
45 about the axisymmetric flow. The first results for the 3D case on the exis-
tence of classical solutions were obtained in the late sixties for axisymmetric
flow without swirl (see [108, 77]). However for the full three space dimensions,
little is known about smooth solutions apart from classical short-time existence

and uniqueness. Moreover, weak solutions are known to be badly behaved from
50 the point of view of Hadamard's well-posedness theory (see for instance the
surveys [35, 110]). Indeed, in [95], [99], non uniqueness of weak solutions has
been demonstrated. These solutions break both uniqueness and the energy con-
servation. Moreover, the energy was non-monotonous and even unbounded.
Later, it was shown in [33, 113] that there exist many weak solutions of the
55 Euler equations with bounded energy. However, the energy for these weak solu-
tions was not conserved. Recently it was shown in [34], that the non-increasing
energy criterion cannot be the key point in the uniqueness of weak solutions,
indeed for particular initial data, it has been proved that there exists infinitely
many bounded solutions which are strongly L^2 -continuous and satisfy the non
60 increasing of energy. Later in [105], this result has been improved for large
class of initial data. These recent results suggest that the non increasing energy
criterion is not fundamental for the uniqueness of solutions (see for instance
the surveys [35]). Further, another notion of weak solutions to the Euler equa-
tions was introduced in [83] called dissipative solutions and it was proven that
65 they coincide with classical Euler solutions, when those exist. Zero-viscosity
limits of Navier-Stokes equations have been shown to exist and to give Euler
solutions in some more generalized sense known as measure-valued Euler solu-
tions (see [36]). Considerable efforts have also been devoted to the study of
the regularity properties of the 3D Euler equations. The main difficulty in the
70 analysis lies in the presence of the nonlinear vortex stretching term and the
lack of a regularization mechanism. Despite these difficulties, a few important
partial results [3, 92, 39, 98, 20, 30, 46] have been obtained over the years which
have led to improved understanding of the regularity properties of the 3D Euler.

75 In the case of 2D QG equation, besides its direct physical significance [53, 89],
the 2D QG equation has very interesting features of resemblance to the 3D Euler
equation, being also an outstanding open problem of the finite time blow-up
issue. In particular, one can derive a necessary and sufficient blow-up condition
for the 2D QG equation similar to the well-known Beale-Kato-Majda (BKM)
80 criterion (Beale-Kato-Majda [3]). More precisely, the solution to the 2D QG
equation (3) becomes singular at time T^* if and only if $\int_0^{T^*} \|\nabla^\perp u(t)\|_{L^\infty} dt =$
 $+\infty$ (see [17]). Thus, $\nabla^\perp u$ plays a role similar to the vorticity ω in the 3D Euler
equations.

In the recent years, the 2D QG equation has been the focus of intense math-
85 ematical research [17, 26, 28, 9, 88, 25, 13], initiated by Constantin, Majda
and Tabak [17]. These latter showed that if the direction field $\xi = \nabla^\perp u / |\nabla^\perp u|$
and the velocity v remain smooth in a region, then no finite-time singularity is
possible in that region. A scenario for finite time blow up, a closing saddle, was
proposed and numerically investigated there. It was later proved by Cordoba
90 [26] that blow up does not happen in this scenario.

Besides the analytical results mentioned above, there also exists a sizable

literature focusing on the numerical search of a finite-time singularity for the 3D Euler equations and 2D QG equation (see [51, 94, 6, 46, 68, 69, 70, 71, 72, 73, 75, 90, 91, 17, 21, 25, 25]). Although finite-time singularities were frequently reported in numerical simulations of the Euler equations, most such singularities turned out to be either numerical artefacts or false predictions, as a result of either insufficient resolution or inadvertent data analysis procedure (see [100, 57, 59]).

Recently, it was shown in [106], a finite time blow up solution to an averaged three-dimensional Navier-Stokes equations of type $\partial_t u = \Delta u + \tilde{B}(u, u)$, where \tilde{B} is an averaged version of the Euler bilinear operator B , acting also on divergence free vector fields u and obeying as B to the cancellation property $\langle \tilde{B}(u, u), u \rangle = 0$. This result suggests that any successful method to affirmatively answer to the Existence and Smoothness problem must either use finer structure of B or else must rely crucially on some estimate or other property of the Euler bilinear operator B that is not shared by the averaged operator \tilde{B} . Such additional structure exists for instance, the Euler equation has a vorticity formulation involving only differential operators rather than pseudo-differential ones.

Indeed, recent studies indicate that the local geometric regularity of Lagrangian vortex filaments can lead to dynamic depletion of vortex stretching [18, 20, 30, 32, 58, 31]. In particular, the recent results obtained in [30, 32, 58, 31] show that geometric regularity of vortex lines, even in an extremely localized region containing the maximum vorticity can lead to depletion of nonlinear stretching, thus avoiding finite time singularity formation of the 3D Euler equations or 2D QG equation. Motivated by the work of Hou and Li [58], Hou and Lei [61] further proposed a new 3D model to study the stabilizing effect of convection. By constructing a new 3D model for axisymmetric flows with swirl, they showed that the convection term has a surprising stabilizing effect in the 3D incompressible Navier-Stokes equations and plays an essential role in depleting the vortex-stretching term, thus preventing the finite time blow up of the Euler equations.

Despite high hopes from an analytical point of view that these considerations will shed light on the true nature of vorticity accumulation, numerical results observing geometrical properties of Lagrangian vortex filaments are scarce. This comes from the fact that to monitor the Lagrangian evolution of vortex lines, this requires additional computational efforts.

Thus, in this paper, we improve in several ways the recent geometric criteria for non blow-up of the Navier-Stokes, Euler and 2D QG equations based on the local geometric regularity of the unit vorticity vector [20, 27, 17] and the geometrical properties of Lagrangian vortex filaments [30, 32, 31] while keeping an Eulerian setting. To get our first improvement in the sense of establishing geometric non blow-up criteria in an Eulerian setting, we had to overcome the obstruction that we do not know if there exists an isolated absolute maximum for the vorticity achieved along a smooth curve as it was assumed in Proposition 2.1 of [17]. Moreover, recent numerical experiments show that it is not always

the case (see [74], see also section 5.4.5 in [49]). We overcome this difficulty by
140 using a result of Pshenichnyi concerning directional derivatives of the function of
maximum and the structure of a set of supporting functionals [93]. The second
improvement comes from the fact that the recent non blow-up criteria involved
the maximum of the magnitude of the velocity in a localized region where the
position of maximum of vorticity is reached (see [20, 17, 30, 32]). But here, we
145 obtain a more accurate criterion involving the norm of the velocity in a local
dyadic BMO-type space defined on a localized region $\mathcal{R}(t)$ where the position
 $x(t)$ of maximum of vorticity is reached. Indeed, recent works show that dyadic
BMO spaces are better suitable than L^∞ and BMO spaces (see [86]). Then, our
second improvement is obtained by using a dyadic unity partition of Littlewood
150 decomposition for which the resulting dyadic annular rings are centered in $x(t)$
and by using also affine functions for which we use the fact that their rotational
is null. Then, we obtain a non blow-up criterion which involves for the velocity
only its mean oscillations on dyadic annular rings contained in a localized region
where the position of maximum of vorticity is reached. Hence, we bring a second
155 improvement since the non blow-up criteria given in [20, 17, 30, 32] required the
maximum of the magnitude of the velocity in a localized region. We obtain also
that the radius of the region $\mathcal{R}(t)$ is controlled by the norm of the velocity in
a dyadic BMO-type space. All these new results allow us to better understand
how non blow-up of solutions of Euler, Navier-Stokes and 2D QG equations can
160 hold even in the Kerr's critical case [73] (see our Remark 5.2 for more details).

Then, the paper is organized as follows:

- In section 1 , we give some notations and definitions.
- In section 2, we give a brief review of the recent geometric non blow-up
criteria and we also discuss about some improvements brought.
165 - In section 3, we recall some results about the local regularity of solutions
of Navier-Stokes, Euler and 2D QG equations.
- In section 4, we answer to an open question by obtaining a single expo-
nential bound on the H^r -norm of solution to Euler, Navier-Stokes or 2D
QG equations in terms of $\int_0^t \|\omega(s)\|_{\text{BMO}(\mathbb{R}^d)} ds$. From, this result, we infer
170 a lower bound on the blow-up rate of $\|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)}$.
- In section 5, we give our geometric properties for non blow-up of Navier-
Stokes, Euler and 2D QG equations in localized regions of maximum vorti-
city.
- In section 6, we go further in our improvements by giving geometric prop-
175 erties for non blow-up of Navier-Stokes, Euler and 2D QG equations eval-
uated only on the positions of maximum vorticity. Moreover, in the case
of Euler and 2D QG equations, our non blow-up criterion obtained is a
necessary and sufficient condition in preventing finite time blow-up of the
solutions.

- In section 7, by using the results obtained in section 6, we give some arguments in favor of a double-exponential in time growth of the maximum vorticity in the cases of Euler and 2D QG equations.

Let us now introduce the Navier-Stokes and Euler equations in \mathbb{R}^3 given by,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \\ \nabla \cdot u = 0, \end{cases} \quad (1)$$

in which $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ and $\nu \geq 0$ ($\nu = 0$ corresponds to the Euler equations) denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x, t) \in \mathbb{R}^3 \times [0, \infty[$ and the viscosity of the fluid,

with initial conditions,

$$u(x, 0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3, \quad (2)$$

where the initial data u_0 is a divergence free vector field on \mathbb{R}^3 .

Regarding the 2D QG equation (3) in \mathbb{R}^2 , it is given by

$$\begin{cases} \frac{\partial u}{\partial t} + v \cdot \nabla u = 0, \\ v = -\nabla^\perp (-\Delta)^{-\frac{1}{2}} u, \end{cases} \quad (3)$$

with initial data,

$$u(x, 0) = u_0. \quad (4)$$

Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. For v we have also the following representation

$$v = -R^\perp u, \quad (5)$$

185 where we have used the notation, $R^\perp u = (-R_2 u, R_1 u)$ with $R_j, j = 1, 2$, for the 2D Riesz transform defined by (see e.g. [104])

$$R_j(u)(x, t) = \frac{1}{2\pi} P.V \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x - y|^3} u(y, t) dy.$$

1. Some notations and definitions

In this section, we assume that $d \in \{1, 2, 3\}$.

For any vector $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we denote by $|x|$ the euclidean norm

190 of x given by $|x| = \sqrt{\sum_{i=1}^d |x_i|^2}$. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of real square

matrices of size d . For any vector field v defined from \mathbb{R}^d to \mathbb{R}^d , we denote by

∇v the gradient matrix of v , the matrix defined with ij -component, $\frac{\partial v_i}{\partial x_j}$ for all $1 \leq i, j \leq d$. For $d = 2$, for any scalar field v defined from \mathbb{R}^d to \mathbb{R} , we denote by $\text{curl } v$ the rotational $\text{curl } v = \nabla^\perp v$. For $d = 3$, for any vector field v defined from \mathbb{R}^d to \mathbb{R}^d , we denote by $\text{curl } v$ the rotational $\text{curl } v = \nabla \times v$. For any function φ defined on $\mathbb{R}^d \times [0, +\infty[$, for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on \mathbb{R}^d by $x \mapsto \varphi(x, t)$. We denote by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support in \mathbb{R}^d . We denote by BC the class of bounded and continuous functions and by BC^m the class of bounded and m times continuously derivable functions.

For any $R > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B(x_0, R)$, the ball of \mathbb{R}^d of center x_0 and radius R . For any $R > 0$, we denote by B_R , the ball of \mathbb{R}^d of center $(0, 0, 0)$ and radius R . We denote by $\text{BMO}(\mathbb{R}^d)$ the space of locally integrable functions of bounded mean oscillation equipped with the norm $\|f\|_{\text{BMO}(\mathbb{R}^d)} :=$

$$\sup_{x \in \mathbb{R}^d, R > 0} \frac{1}{|B(x, R)|} \int_{B(x, R)} |f - f_{B(x, R)}|, \text{ where } f_{B(x, R)} = \frac{1}{|B(x, R)|} \int_{B(x, R)} f.$$

We denote by div the differential operator given by, $\text{div} = \sum_{i=1}^d \frac{\partial}{\partial x_i}$.

We denote $A \lesssim D$, the estimate $A \leq cD$ where $c > 0$ is an absolute constant. For any $f \in L^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)^d$ or $L^p(\mathbb{R}^d)^{d \times d}$) with $1 \leq p \leq +\infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p -norm of f .

Given an absolutely integrable function $f \in L^1(\mathbb{R}^d)$, we define the Fourier transform $\hat{f} : \mathbb{R}^d \mapsto \mathbb{C}$ by the formula,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx,$$

and extend it to tempered distributions. We will use also the notation $\mathcal{F}(f)$ for the Fourier transform of f . We define also the inverse Fourier transform $\check{f} : \mathbb{R}^d \mapsto \mathbb{C}$ by the formula,

$$\check{f}(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} f(\xi) d\xi.$$

For $s \in \mathbb{R}$, we define the Sobolev norm $\|f\|_{H^s(\mathbb{R}^d)}$ of a tempered distribution $f : \mathbb{R}^d \mapsto \mathbb{R}$ by,

$$\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

and then we denote by $H^s(\mathbb{R}^d)$ the space of tempered distributions with finite $H^s(\mathbb{R}^d)$ norm, which matches when s is a non negative integer with the classical Sobolev space $H^k(\mathbb{R}^d)$, $k \in \mathbb{N}$. The Sobolev space $H^s(\mathbb{R}^d)$ can be written as $H^s(\mathbb{R}^d) = J^{-s} L^2(\mathbb{R}^d)$ where $J = (1 - \Delta)^{\frac{1}{2}}$.

For $s > -1$, we also define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

and then we denote by $\dot{H}^s(\mathbb{R}^d)$ the space of tempered distributions with finite $\dot{H}^s(\mathbb{R}^d)$ norm.

235 We denote by $H_\sigma^s(\mathbb{R}^3)$ the Sobolev space $H_\sigma^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\psi \in H^s(\mathbb{R}^3)^3 : \text{div} \psi = 0\}$. In order to unify our notations with the two dimensional case 2D QG, we denote by $H_\sigma^s(\mathbb{R}^2)$ the Sobolev space $H^s(\mathbb{R}^2)$.

Definition 1.1. We denote by \mathbb{P} the matrix Leray's projection operator with components,

$$\mathbb{P}_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Delta^{-1} = \delta_{i,j} - R_j R_k, \quad (6)$$

230 where R_j are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} = \frac{1}{4\pi} \frac{x_j}{|x|^4} \star$ (see [104] for more details), Δ^{-1} is the inverse of Laplace operator given by $\Delta^{-1} = -\frac{1}{4\pi|x|} \star$, with \star the convolution operator.

The operator \mathbb{P} , which acts on vector-valued functions, is a projection : $\mathbb{P}^2 = \mathbb{P}$, annihilates gradients and maps into solenoidal (divergence-free) vectors; it is a bounded operator on (vector-valued) L^q , $1 < q < \infty$ and commutes with translation. We can notice that the operator \mathbb{P} can be written under the form,

$$\mathbb{P} = \text{Id} - \nabla \Delta^{-1} \text{div}, \quad (7)$$

which yields to Helmholtz decomposition, indeed for all $v \in L^q(\mathbb{R}^3)^3$, $1 < q < \infty$,

$$\begin{aligned} v &= \mathbb{P}v + \nabla \psi, \text{ with } \text{div } \mathbb{P}v = 0, \\ \psi &= \Delta^{-1} \text{div } v. \end{aligned} \quad (8)$$

2. Brief review and improvement of the non blow-up criteria

235 Historically, non blow-up criteria for the incompressible Euler equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy for overcoming such shortcomings was established by focusing more on geometrical properties and flow structures (see e.g. [20, 27]), such as vortex tubes or vortex lines.

240 In particular, Constantin, Fefferman and Majda ([20]) showed that local geometric regularity of the unit vorticity vector can lead to depletion of the vortex stretching. They prove that if there is up to time T an $O(1)$ region in which the vorticity vector is smoothly directed, i.e., the maximum norm of $\nabla \xi$

(here $\xi = \frac{\omega}{|\omega|}$, ω the vorticity) in this region is L^2 integrable in time from 0
 245 to T , and the maximum norm of velocity in some $O(1)$ neighbourhood of this
 region is uniformly bounded in time, then no blow-up can occur in this region
 up to time T . This criterion takes into account the local structure of the flow
 and follows the evolution of vortex lines, but the (global) bound on the velocity
 250 makes this theorem hard to apply in simulations. Indeed, in the case of the
 collapse of a regular vortex tube, we expect that the dominating part of the
 velocity field is given by the rotational component of the velocity in the cross
 section normal to the direction of the vortex tube. As the vortex tube collapses,
 the rotational component of the velocity field may blow up proportional to the
 square root of the maximum vorticity from Kelvin's circulation theorem.

255 Then, a weaker restriction on the velocity field is presented by Cordoba and
 Fefferman [27]. They consider vortex tubes that are regular in a sense in an
 $O(1)$ region. With milder assumptions on the surrounding velocity it is shown
 that the tube cannot reach zero thickness in finite time.

260 However, all the existing theorems deal with $O(1)$ regions in which the vortic-
 ity vector is assumed to have some regularity, while in numerical computations,
 the regions that have such regularity and contain maximum vorticity are all
 shrinking with time.

Further, Kerr ([68], [75]) presented numerical evidence which suggested a
 finite-time singularity of the 3D Euler equations for two perturbed antiparallel
 265 vortex tubes which was consistent with the non blow-up criterion of Beale,
 Kato and Majda ([3]) and that of Constantin, Fefferman and Majda ([20]). His
 computations showed that the growth of the peak vorticity obeys the blow-up
 rate $(T-t)^{-1}$ and the velocity field was shown to blow up like $O(T-t)^{-\frac{1}{2}}$. Kerr
 argued that the curvature, κ , of the vortex lines and $\nabla\xi$ are likely bounded by
 270 $(T-t)^{-\frac{1}{2}}$ [75] and located in the inner region with distance $(T-t)^{\frac{1}{2}}$ away from
 the position where the maximum of vorticity is achieved.

Inspired by the work of Constantin, Fefferman and Majda ([20]), Deng, Hou
 and Yu ([30, 32]) recently showed that geometric regularity of Lagrangian vortex
 filaments, even in an extremely localized region containing the maximum of
 275 vorticity which may shrink with time, can lead to depletion of the nonlinear
 vortex stretching, thus avoiding finite time singularity formation of the 3D Euler
 equations. Moreover, they obtain that Kerr's blow-up scenario falls into the
 critical case of their local non blow-up criteria (Deng, Hou and Yu [30, 32]).

280 Despite high hopes from an analytical point of view that these considerations
 will shed light on the true nature of vorticity accumulation, numerical results
 observing geometrical properties of Lagrangian vortex filaments are scarce. This
 is primarily due to the fact that Eulerian quantities such as $\|\omega(t)\|_\infty$ are readily
 tractable in post-processing, while monitoring the Lagrangian evolution requires
 additional computational effort (see section 5.4.5 of [49]). On top of that, the ge-
 285 ometry of integral curves at an instance in time, though in principle computable
 in post-processing, as well as derived quantities such as their convergence and
 curvature, are quite inaccessible in comparison to simple Eulerian criteria (see
 [50], see also section 5.4.5 of [49]).

Then, in this paper, we proposed non blow-up Eulerian criteria as effective
 290 as those obtained by Deng, Hou and Yu [30, 32, 31].

2.1. Improvement of the non blow-up criteria of Navier-Stokes and Euler equations

Let us assume that $\|u(t)\|_{\mathcal{BM}\mathcal{O}_d(\mathbb{R}^d)}$ is bounded by $O(T-t)^{B-1}(1+\log(1+(T-t)^{-1}))^{-\epsilon}$ for some $0 \leq B \leq 1$ and $\epsilon > 0$ (see section 5.1 for the definition
 295 of $\mathcal{BM}\mathcal{O}_d(\mathbb{R}^d)$). Therefore, from our Corollary 5.2, we infer that if $\nabla\xi(t)$ is bounded by $O(T-t)^{-A_0}$ and the velocity $u(t)$ is bounded by $O(T-t)^{-A}$ in the ball centered on the position where the maximum of vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius $\rho(t)$ of type $O(T-t)^B$, then there is no blow-up of the solutions of Euler or Navier-Stokes equations at the time T provided that
 300 $A_0 + A < 1$, $A \geq 0$, $A_0 \geq 0$.

Our criterion presents some advantages in comparison with the Deng, Hou and Yu's non blow-up-criterion [30, 32] besides the fact that it is a criterion established in an Eulerian setting:

- It includes the Navier-Stokes equations which is not the case with the
 305 Lagrangian approach used in [30, 32] where their proof and results were valid only for Euler equations.
- We do not make assumption of type $L(t) \gtrsim (T-t)^{B'}$ on the length $L(t)$ of vortex line segments along which its maximum vorticity is comparable to $\|\omega(t)\|_\infty$. Moreover from (3.11) of [30], we expect that $L(t) \lesssim \int_t^T \|u(s)\|_\infty ds \lesssim (T-t)^B$, then we could expect that $B' \geq B$.
 310
- No blow-up of solutions of Euler equations in [30, 32] is ensured under the conditions $A + B' < 1$ and $-A_0 + B' \geq 0$, the case $A + B' = 1$ being obtained under some additional assumptions. However for the case $B' > 1 - A > A_0$, we can say nothing about the blow-up of solutions of
 315 Euler equations with Deng, Hou, Yu non blow-up criterion whereas with our criterion, we can ensure that there is no blow-up of the solutions of Euler equations at the time T .

Moreover, in Remark 5.2, we obtain the non blow-up of the solutions of Euler equations for the Kerr's critical case appearing in [59] under a very weak condition. In remark 5.3, we point out that with our criterion, we can even handle
 320 some blow-up scenarios of type II.

2.2. Improvement of the non blow-up criteria of 2D QG equations

From our Corollary 5.1, we found that if $\nabla\xi(t)$ is bounded by $O(T-t)^{-1} \log(1+(T-t)^{-1})^{-\beta}$, $\beta > 1$ in the ball \tilde{B}_t centered on the position where the maximum of vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius of type $O(T-t) \log(1+(T-t)^{-1})^\alpha$,
 325 $\alpha > 0$, then there is no blow-up of the solutions of the 2D QG equation at the time T .

Our criterion presents some advantages in comparison with the Deng, Hou and Yu's non blow-up-criterion [31], besides the fact that is a criterion in the
 330 Eulerian setting:

- we do not make assumptions of type $L(t) \gtrsim \frac{c_L}{\log \log \|\omega(t)\|_\infty}$ or $L(t) \gtrsim (T-t)^A$, $A < 1$ (if we assume that $\|\omega(t)\|_\infty \lesssim (T-t)^{-B}$ for some $B < +\infty$) on the length $L(t)$ of vortex line segments along which its maximum vorticity is comparable to the maximum vorticity in the whole 2D space.
- No blow-up of solutions of 2D QG equation in [31] is roughly speaking ensured under the condition that $\|\nabla \xi(t)\|_{L^\infty(B_t)} L(t) < +\infty$, where B_t is the ball centered on the position where the maximum vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius $L(t)$. However for the case $L(t) = O(T-t)^A$ with $A \geq 1$ and $\|\nabla \xi(t)\|_{L^\infty(\tilde{B}_t)} = O(T-t)^{-1} \log(1 + (T-t)^{-1})^{-\beta}$, $\beta > 1$ we can say nothing about the blow-up of solutions of 2D QG equation with Deng, Hou, Yu non blow-up criterion [31] whereas with our criterion, we can ensure that even in this case, there is no blow-up of the solutions of 2D QG equation at the time T .
- we obtain double exponential growth in time of $\|\omega(t)\|_\infty$ as soon as $\|\nabla \xi(t)\|_{L^\infty(\tilde{B}_t)}$ is bounded up to T whereas a such result is obtained in [31] under the condition that $\nabla \xi(t)$ is bounded up to T but in the ball centered on the position where the maximum vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius of type $O(1)$, which is much more restrictive than our condition.

As we can notice, all these non blow-up criteria required the local regularity of the velocity and the direction of vorticity in all a region or a vortex line containing the position where maximum of vorticity is reached. Then, in section 6 with our Theorem 6.1, we go further in our improvements by giving geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations evaluated only on the positions where the maximum of vorticity is reached.

3. Local regularity of solution of Navier-Stokes, Euler and 2D QG equations

In this section, we deal with the main result on local regularity of Navier-Stokes and Euler equations in its general form. By introducing \mathbb{P} the matrix Leray operator, Euler equations (1)-(2) can be re-written as follows,

$$\frac{\partial u}{\partial t} + \mathbb{P}(u \cdot \nabla)u = 0, \quad (9)$$

with initial conditions,

$$u(0) = u_0. \quad (10)$$

For u solution of (9)-(10), $\omega = \nabla \times u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega = 0, \quad (11)$$

with initial conditions,

$$\omega(0) = \omega_0, \quad (12)$$

where $\omega_0 = \nabla \times u_0$ is the vorticity of u_0 .

In the case of 2D QG equation, we get for u solution of (3), $\omega = \nabla^\perp u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega - (\omega \cdot \nabla) v = 0, \quad (13)$$

with initial conditions,

$$\omega(0) = \omega_0, \quad (14)$$

where $\omega_0 = \nabla^\perp u_0$ is the vorticity of u_0 .

In the region where $|\omega| > 0$, we define ξ the direction of the vorticity by $\xi = \frac{\omega}{|\omega|}$.

360 3.1. Local regularity for Navier-Stokes or Euler equations

Assuming $u_0 \in H_\sigma^r(\mathbb{R}^3)$ with $r > \frac{5}{2}$, thanks to Theorem 3.5 and 4.7 in [66], Theorem 1 in [5] (see also Theorem I in [65] and the results obtained in [3]), we deduce that there exists a time $T > 0$ such that there exists an unique strong solution $u \in C([0, T[, H_\sigma^r(\mathbb{R}^3))$ to the Navier-Stokes or Euler equations (9)-(10) and the energy equality holds for u , that means for all $t \in [0, T[$,

$$\|u(t)\|_2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2. \quad (15)$$

Moreover, if $u \notin C([0, T], H_\sigma^r(\mathbb{R}^3))$, then we get (see [3, 66, 76]),

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \quad (16)$$

Owing to $u \in C([0, T[, H_\sigma^r(\mathbb{R}^3))$ and thanks to Lemma X4 in [66], from Navier-Stokes or Euler equations (9), we get $u \in C^1([0, T[, H^{r-1}(\mathbb{R}^3))^3$.

Moreover, for the Navier-Stokes equations, thanks to (15), we get the following estimate (for the proof see Appendix A and (43) in [23]), for all $t \in [0, T[$,

$$\int_0^t \|u(s)\|_\infty ds \leq C(\|u_0\|_2^2 + \sqrt{t}), \quad (17)$$

where $C > 0$ is a constant.

From Helmholtz decomposition (8), we retrieve the pressure p from the velocity u with the formula,

$$p = -\Delta^{-1} \operatorname{div}((u \cdot \nabla)u).$$

Furthermore, we get the local estimate (18). Indeed, thanks to remark 4.4 in [66], we get

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ with } t_0 < t < T, \quad (18)$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

365 *3.2. Local regularity for 2D QG equation*

This subsection is devoted to the local well-posedness of the 2D QG equation with a characterization of the maximal time existence of strong solutions. By using the same arguments as the proof of Proposition 4.2 in [1], we get that the H^s -norm of u is controlled by the integral in time of the maximum magnitude of the vorticity of u . A such Proposition has been proved in [17] for any integer 370 $s \geq 3$, but here we extend this result to all real $s > 2$. This improvement is obtained by using the logarithmic Sobolev inequality proved in [76, 66] which requires only that $s > 2$ instead of using the one proved in [3] as it is the case in [17] and which requires integer $s \geq 3$. Then by using the same arguments 375 as the proof of Proposition 4.3 in [1], we get the following result which gives an improvement in comparison with Theorem 2.1 in [17]:

Assuming $u_0 \in H^r(\mathbb{R}^2)$ with $r > 2$, we get that there exists a time $T > 0$ such that there exists an unique strong solution $u \in C([0, T[, H^r(\mathbb{R}^2))$ to the 2D QG equation (3)-(4) and the energy equality holds for u , that means for all $p \in [2, \infty]$ and $t \in [0, T[$,

$$\|u(t)\|_p = \|u_0\|_p. \quad (19)$$

Moreover, if $u \notin C([0, T], H^r(\mathbb{R}^2))$, then

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \quad (20)$$

Owing to $u \in C([0, T[, H^r(\mathbb{R}^2))$ and thanks to Lemma X4 in [66], from 2D QG (3), we get $u \in C^1([0, T[, H^{r-1}(\mathbb{R}^2))$.

Similarly as in (18), we have

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ for } t_0 < t < T, \quad (21)$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

4. A lower bound on blow up rate of the vorticity in the BMO-norm

The blow up of smooth solutions to the Navier-Stokes, Euler or 2D QG equations is controlled by BKM-type criteria formulated in terms of the time integral of the maximum magnitude of the vorticity (see [3, 66, 17]). And this criterion have been refined in [76] and [8] by replacing respectively the L^∞ norm of the vorticity by weaker norms close to the L^∞ norm, more precisely by the BMO space and the Triebel-Lizorkin space $\dot{F}_{\infty, \infty}^0$. In particular, in all these works, the authors obtained a double exponential bound for $\|u(t)\|_{H^r}$ in terms of $\int_0^t \|\omega(s)\|_X ds$ with $X = L^\infty, \text{BMO}$ or $\dot{F}_{\infty, \infty}^0$ as follows (see [3, 66, 76, 8, 17], see also [29])

$$\|u(t)\|_{H^r} \leq \|u_0\|_{H^r} \exp \left(\exp \left(C \int_0^t \|\omega(s)\|_X ds \right) \right). \quad (22)$$

It is an open question whether (22) is sharp in the sense that is it possible to get single exponential bound. This question is addressed in [29]. In this section, we answer to this question, more precisely by using a sharp version of logarithmic Sobolev inequality 23, we show in Proposition 4.1 that we need only one exponential in term of $\int_0^t \|\omega(s)\|_{\text{BMO}(\mathbb{R}^d)} ds$ to get an upper bound of $\|u(t)\|_{H^r}$ with $r > \frac{d}{2} + 1$. Then, from Proposition 4.1, we obtain a lower bound on the blow-up rate of the vorticity ω in the BMO–norm.

Let us first mention the following sharp version of logarithmic Sobolev inequality (2.6) in [87] obtained from Theorem 2.1 in [87] which states

$$\|\nabla f\|_{L^\infty(\mathbb{R}^d)} \lesssim 1 + \|\nabla \times f\|_{\text{BMO}(\mathbb{R}^d)} (\log^+ \|f\|_{W^{s,p}(\mathbb{R}^d)})^{\frac{1}{2}} \quad \text{with } p > 1 \text{ and } s > 1 + \frac{d}{p}, \quad (23)$$

where $\nabla \cdot f = 0$, $\nabla \times f$ is the vorticity of f and $\log^+ x = \max(0, \log x)$ for any $x > 0$.

Owing to (23), in the following Proposition 4.1, we give a single exponential bound for $\|u(t)\|_{H^r}$ in terms of $\int_0^t \|\omega(s)\|_{\text{BMO}} ds$.

Proposition 4.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$, $r > 1 + \frac{d}{2}$. Let $T^* > 0$ be such that there exists $u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d))$ solution to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4). Then there exists a real $c_0 > 0$ depending only on r such that for all $t \in [0, T^*[$*

$$\|u(t)\|_{H^r} \leq e^{(\eta_0 + c_0 \int_0^t \|\omega(\tau)\|_{\text{BMO}(\mathbb{R}^d)} d\tau)^2}, \quad (24)$$

with $\eta_0 = \sqrt{1 + \log(1 + \|u_0\|_{H^r})} + c_0 T^*$.

Proof. We set $\mathbf{v} = u$ in the case of Navier-Stokes or Euler equations and $\mathbf{v} = -R^\perp u$ with $\nu = 0$ in the case of 2D QG equation. Taking the L^2 inner product of the equations (1) or (3) with $J^r u$, one has,

$$\frac{1}{2} \frac{d}{dt} \|J^r u\|_2^2 + \nu \|\nabla J^r u\|_2^2 = - \int_{\mathbb{R}^d} J^r u [J^r ((\mathbf{v} \cdot \nabla) u) - (\mathbf{v} \cdot \nabla) J^r u], \quad (25)$$

where we have used the divergence free condition $\nabla \cdot \mathbf{v} = 0$. We apply Cauchy-Schwarz inequality to the term at the right hand side of equality (25) to obtain,

$$\frac{1}{2} \frac{d}{dt} \|J^r u\|_2^2 \leq \|J^r u\|_2 \|J^r ((\mathbf{v} \cdot \nabla) u) - (\mathbf{v} \cdot \nabla) J^r u\|_2. \quad (26)$$

Thanks to Lemma XI in [66], we obtain that for all $t \in [0, T^*[,$

$$\frac{1}{2} \frac{d}{dt} \|J^r u(t)\|_2^2 \lesssim_r (\|\nabla \mathbf{v}(t)\|_{L^\infty(\mathbb{R}^d)} \|J^r u(t)\|_2 + \|J^r \mathbf{v}(t)\|_2 \|\nabla u(t)\|_{L^\infty(\mathbb{R}^d)}) \|J^r u(t)\|_2. \quad (27)$$

Since Riesz transforms maps $\text{BMO}(\mathbb{R}^d)$ into itself and maps also $H^r(\mathbb{R}^d)$ into itself, then thanks to (23), from (27) we infer that for all $t \in [0, T^*[,$

$$\frac{d}{dt} \|u(t)\|_{H^r}^2 \lesssim_r (1 + \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)} (\log^+ \|u(t)\|_{H^r})^{\frac{1}{2}}) \|u(t)\|_{H^r}^2,$$

which yields to

$$\begin{aligned}
\frac{d}{dt} \|u(t)\|_{H^r} &\lesssim_r (1 + \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)} (\log(1 + \|u(t)\|_{H^r}))^{\frac{1}{2}}) \|u(t)\|_{H^r} \\
&\leq (1 + \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)}) (1 + (\log(1 + \|u(t)\|_{H^r}))^{\frac{1}{2}}) \|u(t)\|_{H^r} \\
&\leq \sqrt{2} (1 + \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)}) (1 + \log(1 + \|u(t)\|_{H^r}))^{\frac{1}{2}} \|u(t)\|_{H^r}.
\end{aligned}$$

Thus, we divide the inequality just above by $1 + \|u(t)\|_{H^r}$ to obtain for all $t \in [0, T^*[$ with $G(t) \stackrel{\text{def}}{=} 1 + \log(1 + \|u(t)\|_{H^r})$ the following inequality

$$\frac{d}{dt} G(t) \leq c_0 (1 + \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)}) G(t)^{\frac{1}{2}}, \quad (28)$$

where $c_0 > 0$ is a real depending only on r . After solving the Ordinary Differential Inequality (28), we obtain for all $t \in [0, T^*[$

$$\sqrt{G(t)} \leq \sqrt{G(0)} + 2c_0 \int_0^t (1 + \|\omega(s)\|_{\text{BMO}(\mathbb{R}^d)}) ds. \quad (29)$$

Then, from (29), we infer that for all $t \in [0, T^*[$,

$$\begin{aligned}
G(t) &\leq \left(\sqrt{G(0)} + 2c_0 \int_0^t (1 + \|\omega(s)\|_{\text{BMO}(\mathbb{R}^d)}) ds \right)^2 \\
&\leq \left(\sqrt{G(0)} + 2c_0 T^* + 2c_0 \int_0^t \|\omega(s)\|_{\text{BMO}(\mathbb{R}^d)} ds \right)^2
\end{aligned}$$

By using the definition of the function G , we deduce that for all $t \in [0, T^*[$,

$$\|u(t)\|_{H^r} \leq e^{\left(\sqrt{G(0)} + 2c_0 T^* + 2c_0 \int_0^t \|\omega(s)\|_{\text{BMO}(\mathbb{R}^d)} ds \right)^2},$$

with $G(0) = 1 + \log(1 + \|u_0\|_{H^r})$ which concludes the proof. \square

390 Now, thanks to Proposition 4.1, in Lemma 4.1 we show that if the solution blows up at a finite time T^* , then the vorticity in the BMO-norm blow up at least faster than $O(T^* - t) \left(\log \frac{T^*}{T^* - t} \right)^{\frac{1}{2}}$. To the best of our knowledge, this result is new. To get our Lemma 4.1, we borrow some arguments used for the proof of Theorem 1.1 in [12].

395 **Lemma 4.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 1$. Let $T^* > 0$ be such that there exists $u \in C([0, T^*[, H^r(\mathbb{R}^d))$ solution to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4). There exists a constant $c > 0$ such that if u blows up at the finite time T^* then we get*

$$\limsup_{t \rightarrow T^*} (T^* - t) \left(\log \frac{T^*}{T^* - t} \right)^{\frac{1}{2}} \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)} \geq c.$$

Proof. Let us assume that u blows up at the finite time T^* . Then, from the BKM's criteria (16) and (20), we have

$$\int_0^{T^*} \|\omega(t)\|_\infty dt = +\infty. \quad (30)$$

Let $M = \limsup_{t \rightarrow T^*} (T^* - t) \left(\log \frac{T^*}{T^* - t} \right)^{\frac{1}{2}} \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)}$. Let $c_0 > 0$ be the real depending only on r involved in Proposition 4.1. Let us show that $M < \frac{1}{c_0 \sqrt{8}}$ leads to a contradiction. Then, we suppose that $M < \frac{1}{c_0 \sqrt{8}}$. Hence, there exists $t_0 \in]0, T^*[$ such that

$$\sup_{t_0 < t < T^*} (T^* - t) \left(\log \frac{T^*}{T^* - t} \right)^{\frac{1}{2}} \|\omega(t)\|_{\text{BMO}(\mathbb{R}^d)} := M_0 < \frac{1}{c_0 \sqrt{8}}. \quad (31)$$

Further, thanks to Proposition 4.1 used with $u(t_0)$ as initial data, we infer that
400 for all $t \in [t_0, T^*[$,

$$\|u(t)\|_{H^r} \leq e^{\left(\eta(t_0) + c_0 \int_{t_0}^t \|\omega(\tau)\|_{\text{BMO}(\mathbb{R}^d)} d\tau \right)^2},$$

with $\eta(t_0) = \sqrt{1 + \log(1 + \|u(t_0)\|_{H^r})} + c_0(T^* - t_0)$. Then we get for all $t \in [t_0, T^*[$,

$$\|u(t)\|_{H^r} \leq e^{2\eta(t_0)^2 + 2c_0^2 \left(\int_{t_0}^t \|\omega(\tau)\|_{\text{BMO}(\mathbb{R}^d)} d\tau \right)^2}. \quad (32)$$

Then thanks to (31) and (32), we deduce that for all $t \in [t_0, T^*[$,

$$\|u(t)\|_{H^r} \leq e^{2\eta(t_0)^2} e^{2c_0^2 M_0^2 \left(\int_{t_0}^t (T^* - s)^{-1} \left(\log \frac{T^*}{T^* - s} \right)^{-\frac{1}{2}} ds \right)^2}. \quad (33)$$

However, for any $t \in [t_0, T^*[$, we have

$$\begin{aligned} \int_{t_0}^t (T^* - s)^{-1} \left(\log \frac{T^*}{T^* - s} \right)^{-\frac{1}{2}} ds &= \left[2 \left(\log \frac{T^*}{T^* - s} \right)^{\frac{1}{2}} \right]_{t_0}^t \\ &= 2 \left(\log \frac{T^*}{T^* - t} \right)^{\frac{1}{2}} - 2 \left(\log \frac{T^*}{T^* - t_0} \right)^{\frac{1}{2}} \\ &\leq 2 \left(\log \frac{T^*}{T^* - t} \right)^{\frac{1}{2}}. \end{aligned}$$

Then, from (33) we infer that for all $t \in [t_0, T^*[$,

$$\begin{aligned} \|u(t)\|_{H^r} &\leq e^{2\eta(t_0)^2} e^{8c_0^2 M_0^2 \log \frac{T^*}{T^* - t}} \\ &= e^{2\eta(t_0)^2} \left(\frac{T^*}{T^* - t} \right)^{8c_0^2 M_0^2}. \end{aligned} \quad (34)$$

Since $r > \frac{d}{2} + 1$, we get the following Sobolev embedding $H^r(\mathbb{R}^d) \hookrightarrow BC^1(\mathbb{R}^d)$ and hence we have that for all $t \in [0, T^*$, $\|\nabla u(t)\|_\infty \lesssim \|u(t)\|_{H^r}$. Therefore, thanks to (34), we deduce that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \lesssim e^{2\eta(t_0)^2} \left(\frac{T^*}{T^* - t} \right)^{8c_0^2 M_0^2}. \quad (35)$$

Since $8M_0^2 c_0^2 < 1$, then from (35) we infer that $\int_{t_0}^{T^*} \|\omega(t)\|_\infty dt < +\infty$ which leads to a contradiction with (30). Therefore, we deduce that $M \geq \frac{1}{c_0 \sqrt{8}}$, which concludes the proof. \square

405 5. Geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations in localized regions of maximum vorticity

In this section, we give new geometric properties for non blow-up of the Navier-Stokes, Euler and 2D QG equations based on the regularity of the velocity and the direction of the vorticity in extremely localized regions containing the maximum vorticity. Our results improve the ones obtained in [20, 17] but also the ones obtained in [30, 32, 31]. One of the highlights of our results is that they are established in an Eulerian setting, which is more tractable in numerical simulations in comparison with geometric non blow-up criteria given in Lagrangian settings as it is the case in [30, 32, 31].

415 Then, this section is organized as follows:

- In the subsection 5.1, we establish in our main Theorem 5.1 a new bound for the maximum vorticity of 2D QG, Navier-Stokes and Euler equations.
- In the subsection 5.2, we use Theorem 5.1 and Lemma 4.1 to obtain the geometric properties for non blow-up of 2D QG equation in Corollary 5.1.
- 420 • In the subsection 5.3, we use Theorem 5.1 and Lemma 4.1 to obtain the geometric properties for non blow-up of Navier-Stokes and Euler equations in Corollary 5.2.

5.1. Estimate on the maximum vorticity for 2D QG, Navier-Stokes and Euler equations

425 In this subsection, in Theorem 5.1, we obtain a new estimate on the maximum vorticity for 2D QG, Navier-Stokes and Euler equations based on the regularity of the velocity and the gradient of the direction of the vorticity in a region shrinking to zero as time increase to some T^* the alleged time of singularity and containing the position where the maximum vorticity $\|\omega(t)\|_\infty$ is reached.

430 To establish our Theorem 5.1, we need to introduce dyadic BMO-type spaces and use Lemma 5.1.

Dyadic versions of the $BMO(\mathbb{R}^d)$ space are little wider than $BMO(\mathbb{R}^d)$. Moreover, in order to consider a possible blow-up solution for the 3D Navier-Stokes, Euler or 2D QG equations, handling dyadic versions of the $BMO(\mathbb{R}^d)$ space is more reasonable than the $BMO(\mathbb{R}^d)$ space for the reasons exposed in [86].

Then, we introduce $\mathcal{BMO}_d(\mathbb{R}^d)$ a dyadic version of the usual $BMO(\mathbb{R}^d)$ space, defined as the space of locally integrable functions f such that

$$\|f\|_{\mathcal{BMO}_d(\mathbb{R}^d)} \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^d} \sup_{j \in \mathbb{Z}} 2^{-jd} \int_{\mathcal{C}_j} |f(x+y) - \tilde{f}_j(x)| dy < +\infty,$$

where \mathcal{C}_j is the annular ring $\mathcal{C}_j \stackrel{\text{def}}{=} \{y \in \mathbb{R}^d; 2^{j-1} \leq |y| \leq 2^{j+1}\}$ and $f_j(x)$ the mean value $f_j(x) = \frac{1}{|\mathcal{C}_j|} \int_{\mathcal{C}_j} f(x+y) dy$.

For any F non-empty set of \mathbb{R}^d , we denote by $\mathcal{BMO}_d(F)$ the space of locally integrable functions f such that

$$\|f\|_{\mathcal{BMO}_d(F)} \stackrel{\text{def}}{=} \sup_{x \in F} \sup_{j \in \mathbb{Z}} 2^{-jd} \int_{\mathcal{C}_j} |f(x+y) - f_j(x)| dy < +\infty,$$

For any $\rho > 0$, we denote also by $\mathcal{BMO}_d(F, \rho)$ the space of locally integrable functions f such that

$$\|f\|_{\mathcal{BMO}_d(F, \rho)} \stackrel{\text{def}}{=} \sup_{x \in F} \sup_{j \in \mathbb{Z}; \mathcal{C}_j \subset B(0, \rho)} 2^{-jd} \int_{\mathcal{C}_j} |f(x+y) - f_j(x)| dy < +\infty.$$

Now, we turn to the proof of Lemma 5.1.

Lemma 5.1. *Let $d \in \mathbb{N}^*$, $T > 0$ and $f \in C([0, T]; BC(\mathbb{R}^d))$ such that $\inf_{t \in [0, T]} \|f(t)\|_\infty > 0$ and for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$. Then there exists $R > 0$ such that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$.*

Proof. We set $a = \inf_{t \in [0, T]} \|f(t)\|_\infty > 0$. Since the function $t \mapsto f(t)$ is a continuous function from the compact $[0, T]$ into the metric space $L^\infty(\mathbb{R}^d)$ then it is uniformly continuous. Therefore, there exists $N \in \mathbb{N}^*$ such that for all $t, t' \in [0, T]$, $|t - t'| \leq \frac{T}{N}$ we have $\|f(t) - f(t')\|_\infty \leq \frac{a}{4}$. We introduce the subdivision $\{t_i\}_{i \in [0, N]}$ of $[0, T]$ defined by $t_i = i \frac{T}{N}$ for $i \in [0, N]$. Since for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$, then for each $i \in [0, N]$, there exists $R_i > 0$ such that for all $|x| \geq R_i$, $|f(x, t_i)| \leq \frac{a}{4}$. We set $R = \max_{i \in [0, N]} R_i$. Let $t \in [0, T]$, then there exists $j \in [0, N]$ such that $|t - t_j| \leq \frac{T}{N}$, therefore for all $|x| \geq R$, we have $|f(x, t)| \leq |f(x, t) - f(x, t_j)| + |f(x, t_j)| \leq \frac{a}{2} \leq \frac{\|f(t)\|_\infty}{2}$. Then, we infer that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$, which concludes the proof. \square

Now, we turn to the proof of our Theorem.

Theorem 5.1. Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equations (3)-(4) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1(]0, T^*]; H^{r-1}(\mathbb{R}^d)).$$

Then for any continuous function ρ from $[0, T^*]$ to $[0, +\infty[$ such that ρ positive
 465 on $[0, T^*[$ and $\int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))}}{(1 + \log^+ \|\omega(t)\|_\infty)\rho(t)} dt < +\infty$, we have for all $t \in [0, T^*[$

$$\|\omega(t)\|_\infty \leq \exp\left(c_0(1 + \log^+ \|\omega_0\|_\infty) e^{c_0 \int_0^t \frac{a(s)(1+b(s))}{1+\log^+ \|\omega(s)\|_\infty} ds}\right),$$

where $c_0 > 0$ is a constant and for all $t \in [0, T^*[$

$$\begin{aligned} \Omega(t) &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}, \\ a(t) &\stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))}, \\ b(t) &\stackrel{\text{def}}{=} \min\left(\|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))}, \frac{\rho(t)\|\nabla u(t)\|_\infty}{1 + \log^+ \|\omega(t)\|_\infty}\right). \end{aligned} \quad (36)$$

Proof. Thanks to the results of the section 3, there exists effectively a time of existence $T^* > 0$ such that there exists a unique strong solution u to the Navier-Stokes, Euler and 2D QG equations (1)-(2) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1(]0, T^*]; H^{r-1}(\mathbb{R}^d)).$$

We can assume that for all $t \in [0, T^*[$,

$$\|\omega(t)\|_{L^\infty} > 0. \quad (37)$$

Otherwise there will exists $t_0 \in [0, T^*[$ such that $\|\omega(t_0)\|_\infty = 0$.

470 In the case of 2D QG equations (3), we get that $\omega(t_0) = 0$ and then $\nabla u(t_0) = 0$. Since $u(t_0, x)$ vanishes at infinity, then we will get $u(t_0) = 0$. Then by using inequality (21) concerning the local regularity, we deduce that $u(t) = 0$ for all $t \in [t_0, T^*[$ which allow us to conclude the proof.

475 In the case of Navier-Stokes or Euler equations (1), by following step by step the proof of Lemma 4 given in [30] but keeping the term $\|u(t)\|_{L^2(\mathbb{R}^3)}$ after using the Cauchy-Schwarz inequality, we obtain for all $t \in [0, T^*[$,

$$\begin{aligned} \|u(t)\|_\infty &\lesssim \|u(t)\|_2^{\frac{2}{5}} \|\omega(t)\|_\infty^{\frac{3}{5}} \\ &\leq \|u_0\|_2^{\frac{2}{5}} \|\omega(t)\|_\infty^{\frac{3}{5}}, \end{aligned}$$

where we have used (15) for the last inequality. Then, we will obtain that $\|u(t_0)\|_\infty = 0$ which will imply that $u(t_0) = 0$. Then by using the inequality (18) of local regularity, we will deduce $u(t) = 0$ for all $t \in [t_0, T^*[$ and this will

480 allows us to conclude the proof.

Let $0 < T < T^*$. We want to apply Lemma 5.1 to the function ω , then we check that the hypotheses of the Lemma are satisfied.

Since $u \in C([0, T]; H^r(\mathbb{R}^d))$, then we get $\omega \in C([0, T]; H^{r-1}(\mathbb{R}^d))$. Thanks to
 485 the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow BC^2(\mathbb{R}^d)$ for $r > \frac{d}{2} + 3$, we deduce that
 $\omega \in C([0, T]; BC^2(\mathbb{R}^d))$. Then thanks to (37), we infer that $\inf_{t \in [0, T]} \|\omega(t)\|_\infty > 0$.

Moreover, since $\omega \in C([0, T]; H^{r-1}(\mathbb{R}^d))$ with $r > \frac{d}{2} + 3$, we have for any
 $t \in [0, T]$, $|\omega(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$, the proof follows immediately by using
 the density of $C_0^\infty(\mathbb{R}^d)$ in $H^{r-1}(\mathbb{R}^d)$ and the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow$
 490 $L^\infty(\mathbb{R}^d)$ for $r > \frac{d}{2} + 3$.

Then thanks to Lemma 5.1, there exists $R > 0$ such that for all $t \in [0, T]$,
 $\|\omega(t)\|_\infty = \sup_{x \in B_R} |\omega(x, t)|$. Then for all $t \in [0, T]$, the set $\Omega(t)$ given by (36) can
 be rewritten as follows:

$$\Omega(t) = \{x \in B_R; |\omega(x, t)| = \|\omega(t)\|_\infty\}.$$

We introduce the direction of the vorticity $\xi = \frac{\omega}{|\omega|}$ defined on the non-empty
 495 open set $\mathcal{O} = \{(x, t) \in \mathbb{R}^d \times [0, T]; |\omega(x, t)| > 0\}$.

We set $\mathbf{v} = u$ in the case of Navier-Stokes or Euler equations and $\mathbf{v} = -R^\perp u$
 with $\nu = 0$ in the case of 2D QG equation.

Then by multiplying (11) or (13) by ξ , we get that for all $(x, t) \in \mathcal{O}$,

$$\begin{aligned} \frac{\partial |\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla |\omega|(x, t) & - (\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) \\ & - \nu \Delta |\omega|(x, t) + \nu |\omega(x, t)| |\nabla \xi(x, t)|^2 = 0. \end{aligned} \quad (38)$$

We introduce the function φ defined for all $t \in [0, T]$ by $\varphi(t) \stackrel{\text{def}}{=} \sup_{x \in B_R} |\omega(x, t)|$
 and we search the expression of its derivative. For this, we will use the main
 Theorem obtained in [93] or Theorem 1 in [4] and then we have to check that
 the hypotheses of the Theorem are satisfied.

Since $\omega \in C([0, T]; BC^2(\mathbb{R}^d))$, then from (11) and (13), we get

$\frac{\partial \omega}{\partial t} \in C([0, T]; BC(\mathbb{R}^d))$. Therefore, we infer that $|\omega| \in BC(\mathcal{O})$ and $\frac{\partial |\omega|}{\partial t} \in$
 $BC(\mathcal{O})$. Since for any $t \in [0, T]$, $\Omega(t) \subset \mathcal{O}$, then, thanks to the results obtained
 in [93] (see also Theorem 1 in [4]), we obtain the expression of the derivative of
 φ given for any $t \in [0, T]$ by,

$$\varphi'(t) = \sup_{x \in \Omega(t)} \frac{\partial |\omega|}{\partial t}(x, t). \quad (39)$$

Further for all $x \in \Omega(t) \subset B_R$, we have $|\omega(x, t)| = \varphi(t) = \|\omega(t)\|_\infty$, we thus infer that

$$\nabla|\omega|(x, t) = 0 \text{ and } \Delta|\omega|(x, t) \leq 0. \quad (40)$$

Therefore, we have for all $x \in \Omega(t)$,

$$\begin{aligned} \frac{\partial|\omega|}{\partial t}(x, t) &= \frac{\partial|\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla|\omega|(x, t) \\ &= (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) + \nu\Delta|\omega|(x, t) - \nu|\omega(x, t)| |\nabla\xi(x, t)|^2 \\ &\leq (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \end{aligned} \quad (41)$$

where we have used (38) for the second equality and (40) for the last inequality. We can notice that we get equality for (41) in the case of Euler or 2D QG equations, since for these equations we have not the terms $\nu\Delta|\omega|(x, t)$ and $\nu|\omega(x, t)| |\nabla\xi(x, t)|^2$.

Then using (41), from (39), we obtain,

$$\varphi'(t) \leq \sup_{x \in \Omega(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t),$$

which means that

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \sup_{x \in \Omega(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \quad (42)$$

where equality holds in the case of Euler or 2D QG equations.

We use now the function α introduced in [19] for the Navier-Stokes or Euler equations and in [17] for the 2D QG equation, defined for all $(x, t) \in \mathcal{O}$ by,

$$\alpha(x, t) = c_d P.V. \int_{\mathbb{R}^d} D(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d}, \quad (43)$$

where $\hat{y} = \frac{y}{|y|}$ and

505 In the case of Navier-Stokes or Euler equations c_d and D are given by

$$c_d = \frac{3}{4\pi},$$

and

$$D(a_1, a_2, a_3) = (a_1 \cdot a_3) \text{Det}(a_1, a_2, a_3).$$

The Det in D is the determinant of the matrix whose columns are the three unit column vectors a_1, a_2, a_3 . We observe that $\text{Det}(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$, then, we get

$$D(a_1, a_2, a_3) = (a_1 \cdot a_3) a_1 \cdot (a_2 \times a_3). \quad (44)$$

In the case of 2D QG equations, c_d and D are given by

$$c_d = 1,$$

and

$$D(a_1, a_2, a_3) = (a_1 \cdot a_3^\perp) (a_2 \cdot a_3^\perp), \quad (45)$$

where for any $z = (z_1, z_2) \in \mathbb{R}^2$, $z^\perp = (-z_2, z_1)$.

We can notice that D is linear from its second variable. From [19] and [17], we get that for all $x \in \mathcal{O}$

$$(\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) = \alpha(x, t) |\omega(x, t)|.$$

Therefore, from (42), we deduce,

$$\begin{aligned} \frac{d}{dt} \|\omega(t)\|_\infty &\leq \sup_{x \in \Omega(t)} \alpha(x, t) |\omega(x, t)| \\ &= \left(\sup_{x \in \Omega(t)} \alpha(x, t) \right) \|\omega(t)\|_\infty, \end{aligned} \quad (46)$$

where we have used the fact that for all $x \in \Omega(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty$.

We take a smooth cutoff function $\chi : [0, +\infty[\rightarrow [0, 1]$ such that $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. We introduce the function γ defined for all $t \in [0, T^*[$ by $\gamma(t) = \frac{1}{4} \min \left(\frac{1}{\|\omega(t)\|_\infty}, \rho(t) \right)$.

Let us estimate now $|\alpha(x, t)|$ for any $t \in [0, T]$ and $x \in \Omega(t)$. We bound $|\alpha(x, t)|$ by the sum of three terms

$$|\alpha(x, t)| \leq |I_1| + |I_2| + |I_3| \quad (47)$$

with,

$$I_1 = c_d \int_{\mathbb{R}^d} \chi \left(\frac{|y|}{\gamma(t)} \right) D(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d}, \quad (48)$$

$$I_2 = c_d \int_{\mathbb{R}^d} \left(1 - \chi \left(\frac{|y|}{\gamma(t)} \right) \right) \chi \left(\frac{8|y|}{\rho(t)} \right) D(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d} \quad (49)$$

and

$$I_3 = c_d \int_{\mathbb{R}^d} \left(1 - \chi \left(\frac{|y|}{\gamma(t)} \right) \right) \left(1 - \chi \left(\frac{8|y|}{\rho(t)} \right) \right) D(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d}. \quad (50)$$

Then, we estimate each term I_1 , I_2 , I_3 . From (44) and (45), we observe that for all $y \in B \left(0, \frac{\rho(t)}{2} \right)$

$$|D(\hat{y}, \xi(x+y, t), \xi(x, t))| \leq |\xi(x+y, t) - \xi(x, t)| \leq \|\nabla \xi(t)\|_{L^\infty(B(x, \frac{\rho(t)}{2}))} |y|. \quad (51)$$

Since $B(0, 2\gamma(t)) \subset B(0, \frac{\rho(t)}{2})$, then using (51), for the first term I_1 , we obtain,

$$\begin{aligned} |I_1| &\lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \frac{\rho(t)}{2}))} \|\omega(t)\|_\infty \gamma(t) \\ &\leq \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))}. \end{aligned} \quad (52)$$

Furthermore, we have $|\omega(x + y, t)| = \omega(x + y, t) \cdot \xi(x + y, t)$ and $\omega(x + y, t) = \operatorname{curl}_y u(x + y, t)$. Then the second term I_2 can be rewritten under the form

$$I_2 = c_d \int_{\mathbb{R}^d} \psi(x, y, t) \cdot \operatorname{curl}_y u(x + y, t) dy$$

where

$$\psi(x, y, t) = \frac{1}{|y|^d} \left(1 - \chi \left(\frac{|y|}{\gamma(t)} \right) \right) \chi \left(\frac{8|y|}{\rho(t)} \right) D(\hat{y}, \xi(x + y, t), \xi(x, t)) \xi(x + y, t).$$

To get a fine estimate of I_2 , we introduce a dyadic unity partition of Littlewood-Paley decomposition. Let $\Psi : [0, +\infty[\mapsto [0, 1]$ be a smooth function such that $\Psi \equiv 1$ on $[0, 1]$, $\Psi \equiv 0$ on $[2, +\infty[$ and Ψ is strictly decreasing on $[1, 2]$. For $r \geq 0$, let $\Phi(r) := \Psi(r) - \Psi(2r)$, so that Φ is a smooth bump function supported on $[1/2, 2]$ satisfying for any $r > 0$, $\sum_{j \in \mathbb{Z}} \Phi(2^{-j}r) = 1$. We thus introduce the radial

maps $\phi : \mathbb{R}^d \mapsto [0, 1]$ defined for all $y \in \mathbb{R}^d$ by $\phi(y) = \Phi(|y|)$. Then ϕ is a non-negative smooth radial function compactly supported in $\{y \in \mathbb{R}^d; 1/2 \leq |y| \leq 2\}$ such that for all $y \in \mathbb{R}^d \setminus \{0\}$, $\sum_{j \in \mathbb{Z}} \phi(2^{-j}y) = 1$.

For any $j \in \mathbb{Z}$, we introduce the annular ring $\mathcal{C}_j = \{y \in \mathbb{R}^d; 2^{j-1} \leq |y| \leq 2^{j+1}\}$ and we set $\phi_j = \phi(2^{-j}\cdot)$. We notice that ϕ_j is supported on \mathcal{C}_j . We introduce also the annular ring $\mathcal{C}(t) = \left\{ y \in \mathbb{R}^d; \gamma(t) \leq |y| \leq \frac{\rho(t)}{4} \right\}$. We can notice that $\psi(x, \cdot, t)$ is supported on $\mathcal{C}(t)$. Then, we observe,

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) |\psi(x, y, t)| |\operatorname{curl}_y u(x + y, t)| dy &\leq \|\omega(t)\|_\infty \sum_{j \in \mathbb{Z}} \int_{\mathcal{C}(t)} \phi_j(y) \frac{dy}{|y|^d} \\ &\leq \frac{\|\omega(t)\|_\infty}{\gamma(t)^d} \sum_{j \in \mathbb{Z}} \int_{\mathcal{C}(t)} \phi_j(y) dy \\ &= \frac{\|\omega(t)\|_\infty}{\gamma(t)^d} \int_{\mathcal{C}(t)} \sum_{j \in \mathbb{Z}} \phi_j(y) dy \\ &= \frac{\|\omega(t)\|_\infty}{\gamma(t)^d} \int_{\mathcal{C}(t)} dy \\ &\leq \|\omega(t)\|_\infty \left(\frac{\rho(t)}{4\gamma(t)} \right)^d |B(0, 1)|. \end{aligned}$$

For any $t \in [0, T]$, we infer $\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) |\psi(x, y, t)| |\operatorname{curl}_y u(x + y, t)| dy < +\infty$.

Hence, we can write I_2 as follows

$$I_2 = c_d \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) \psi(x, y, t) \cdot \operatorname{curl}_y u(x + y, t) dy.$$

For any $j \in \mathbb{Z}$, we introduce $u_j(x, t) = \frac{1}{|\mathcal{C}_j|} \int_{\mathcal{C}_j} u(x + y, t)$. Then I_2 can be rewritten as follows,

$$I_2 = c_d \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) \psi(x, y, t) \cdot \operatorname{curl}_y (u(x + y, t) - u_j(x, t)) dy.$$

Then, by using an integration by part and after elementary computations where we use the fact that $\chi'(r)$ is compactly supported in $[1, 2]$, we deduce

$$|I_2| \lesssim \sum_{j \in \mathbb{Z}} \int_{\mathcal{A}_j(t)} f_j(x, y, t) |u(x + y, t) - u_j(x, t)| dy, \quad (53)$$

where,

$$f_j(x, y, t) = \left(\frac{2^{-j}}{|y|^d} + \frac{1}{|y|^{d+1}} \right) \mathcal{P}(\xi(x + y, t), \xi(x, t)) + \frac{|\nabla \xi(x + y, t)|}{|y|^d}, \quad (54)$$

$\mathcal{A}_j(t) = \mathcal{C}_j \cap \mathcal{C}(t)$ and $\mathcal{P}(\xi(x + y, t), \xi(x, t)) = |\xi(x + y, t) \times \xi(x, t)|$ in the case of Navier-Stokes or Euler equations and $\mathcal{P}(\xi(x + y, t), \xi(x, t)) = |\xi(x + y, t) \cdot \xi^\perp(x, t)|$ in the case of 2D QG equation. Further, we get for all $y \in B(0, \frac{\rho(t)}{4})$

$$\mathcal{P}(\xi(x + y, t), \xi(x, t)) \leq |\xi(x + y, t) - \xi(x, t)| \leq \|\nabla \xi(t)\|_{L^\infty(B(x, \frac{\rho(t)}{4}))} |y|. \quad (55)$$

Then, owing to (55), from (54), we get for all $y \in B(0, \frac{\rho(t)}{4})$,

$$f_j(x, y, t) \leq \left(\frac{2^{-j}}{|y|^{d-1}} + \frac{2}{|y|^d} \right) \|\nabla \xi(t)\|_{L^\infty(B(x, \frac{\rho(t)}{4}))}. \quad (56)$$

Since for all $y \in \mathcal{C}_j$, we have $\frac{1}{|y|} \leq 2^{-(j-1)}$, then owing to (56), from 53 we deduce

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} \sum_{j \in \mathbb{Z}} 2^{-jd} \int_{\mathcal{A}_j(t)} |u(x + y, t) - u_j(x, t)| dy. \quad (57)$$

However the set $\mathcal{A}_j(t)$ is empty if $\frac{\rho(t)}{4} < 2^{j-1}$ or if $\gamma(t) > 2^{j+1}$, which means

that the set $\mathcal{A}_j(t)$ is empty if $\frac{\log(\frac{\rho(t)}{4})}{\log 2} < j - 1$ or if $\frac{\log(\gamma(t))}{\log 2} > j + 1$. Therefore,

if the set $\mathcal{A}_j(t)$ is non-empty then we get $\frac{\log(\gamma(t))}{\log 2} - 1 \leq j \leq 1 + \frac{\log(\frac{\rho(t)}{4})}{\log 2}$. Then

by introducing the interval $I(t) = \left[\frac{\log(\gamma(t))}{\log 2} - 1, \frac{\log(\frac{\rho(t)}{4})}{\log 2} + 1 \right]$, from (57) we obtain,

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} \sum_{j \in I(t)} 2^{-jd} \int_{\mathcal{C}_j} |u(x + y, t) - u_j(x, t)| dy, \quad (58)$$

where we have used also the fact that $\mathcal{A}_j(t) \subset \mathcal{C}_j$. On one hand, for any $j \in I(t)$, we have $\mathcal{C}_j \subset B(0, \rho(t))$, then we get

$$2^{-jd} \int_{\mathcal{C}_j} |u(x+y, t) - u_j(x, t)| dy \leq \|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))}. \quad (59)$$

On the other hand, thanks to Poincare-Wirtinger inequality, we get,

$$\begin{aligned} 2^{-jd} \int_{\mathcal{C}_j} |u(x+y, t) - u_j(x, t)| dy &\lesssim 2^j \left(2^{-jd} \int_{\mathcal{C}_j} |\nabla u(x+y, t)| dy \right) \\ &\lesssim 2^j \|\nabla u(t)\|_\infty. \end{aligned} \quad (60)$$

On one hand, thanks to (59), from (58) we have,

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} \|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))} \sum_{j \in I(t)} 1, \quad (61)$$

moreover we notice that

$$\begin{aligned} \sum_{j \in I(t)} 1 &\leq 3 + \frac{\log(\frac{\rho(t)}{4\gamma(t)})}{\log 2} \\ &\leq \max\left(3, \frac{1}{\log 2}\right) (1 + \log^+(\rho(t)\|\omega(t)\|_\infty)). \end{aligned} \quad (62)$$

On the other hand, thanks to (60), from (58) we have

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} \|\nabla u(t)\|_\infty \sum_{j \in I(t)} 2^j, \quad (63)$$

moreover we notice that

$$\begin{aligned} \sum_{j \in I(t)} 2^j &\leq \sum_{j \leq \frac{\log(\frac{\rho(t)}{4})}{\log 2} + 1} 2^j \\ &= \sum_{\ell \geq -\frac{\log(\frac{\rho(t)}{4})}{\log 2} - 1} 2^{-\ell} \\ &\leq 2 \times 2^{\frac{\log(\frac{\rho(t)}{4})}{\log 2} + 1} = \rho(t). \end{aligned} \quad (64)$$

Then, from (61)-(64) we obtain,

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} J_0(t), \quad (65)$$

with $J_0(t) = \min(\|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))} (1 + \log^+(\rho(t)\|\omega(t)\|_\infty)), \rho(t)\|\nabla u(t)\|_\infty)$. Since $0 < \rho(t) \leq \rho_0$ with $\rho_0 = \|\rho\|_{L^\infty([0, T^*])}$, then from (65), we deduce that,

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} J_1(t), \quad (66)$$

530 with $J_1(t) = \min(\|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))} (1 + \log^+ \|\omega(t)\|_\infty), \rho(t) \|\nabla u(t)\|_\infty)$, where ρ_0 have been fixed as a constant.

For the last term I_3 , we use the fact that $|\omega(x+y, t)|\xi(x+y, t) = \omega(x+y, t)$ and the fact that D is linear in comparison with its second variable, then we rewrite I_3 as follows

$$I_3 = c_d \int_{\mathbb{R}^d} \Lambda(x, y, t) dy,$$

535 where $\Lambda(x, y, t) = \left(1 - \chi\left(\frac{|y|}{\gamma(t)}\right)\right) \left(1 - \chi\left(\frac{8|y|}{\rho(t)}\right)\right) D(\hat{y}, \omega(x+y, t), \xi(x, t)) \frac{1}{|y|^d}$.

We observe that

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) |\Lambda(x, y, t)| dy &\leq \sum_{j \in \mathbb{Z}} \int_{|y| \geq \frac{\rho(t)}{8}} \phi_j(y) \frac{|\omega(x+y, t)|}{|y|^d} dy \\ &= \int_{|y| \geq \frac{\rho(t)}{8}} \sum_{j \in \mathbb{Z}} \phi_j(y) \frac{|\omega(x+y, t)|}{|y|^d} dy \\ &= \int_{|y| \geq \frac{\rho(t)}{8}} \frac{|\omega(x+y, t)|}{|y|^d} dy \\ &\lesssim \frac{\|\omega(t)\|_2}{\rho(t)^{\frac{d}{2}}}, \end{aligned}$$

where we have used Cauchy-Schwarz inequality to obtain the last inequality.

Then, we infer that for any $t \in [0, T]$, $\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) |\Lambda(x, y, t)| dy < +\infty$.

Then, we can write I_3 as follows

$$I_3 = c_d \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y) \left(1 - \chi\left(\frac{|y|}{\gamma(t)}\right)\right) \left(1 - \chi\left(\frac{8|y|}{\rho(t)}\right)\right) D(\hat{y}, \omega(x+y, t), \xi(x, t)) \frac{dy}{|y|^d}. \quad (67)$$

However $\omega(x+y, t) = \operatorname{curl}_y u(x+y, t) = \operatorname{curl}_y (u(x+y, t) - u_j(x, t))$, then from (67) by using an integration by part and after elementary computations where we use the fact that $\chi'(r)$ is compactly supported in $[1, 2]$, we deduce

$$|I_3| \lesssim \sum_{j \in \mathbb{Z}} \int_{\mathcal{B}_j(t)} \left(\frac{2^{-j}}{|y|^d} + \frac{1}{|y|^{d+1}} \right) |u(x+y, t) - u_j(x, t)| dy, \quad (68)$$

where $\mathcal{B}_j(t) = \{y \in \mathcal{C}_j; |y| \geq \frac{\rho(t)}{8}\}$. Then from (68), we deduce

$$\begin{aligned} |I_3| &\lesssim \sum_{j \in \mathbb{Z}, 2^{j-1} \geq \frac{\rho(t)}{8}} 2^{-j} 2^{-dj} \int_{\mathcal{C}_j} |u(x+y, t) - u_j(x, t)| dy \\ &\leq \|u(t)\|_{\mathcal{BMO}_d(\Omega(t))} \sum_{j \in \mathbb{Z}, 2^{j-1} \geq \frac{\rho(t)}{8}} 2^{-j} \\ &\leq \frac{8\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))}}{\rho(t)}. \end{aligned} \quad (69)$$

We set

$$\begin{aligned}
a(t) &\stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))}, \\
b(t) &\stackrel{\text{def}}{=} \min \left(\|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))}, \frac{\rho(t) \|\nabla u(t)\|_\infty}{1 + \log^+ \|\omega(t)\|_\infty} \right) \\
c(t) &\stackrel{\text{def}}{=} \frac{\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))}}{(1 + \log^+ \|\omega(t)\|_\infty) \rho(t)}.
\end{aligned} \tag{70}$$

Then, owing to (52), (66) and (69), from (47) we deduce that there exists a constant $C_1 > 0$ such that for any $t \in [0, T]$ and $x \in \Omega(t)$,

$$|\alpha(x, t)| \leq C_1(a(t)(1 + b(t)) + c(t))(1 + \log^+ \|\omega(t)\|_\infty). \tag{71}$$

Then from (46), using (71), we deduce that for all $t \in [0, T]$,

$$\frac{d}{dt} \|\omega(t)\|_\infty \leq C_1(a(t) + a(t)b(t) + c(t))(1 + \log^+ \|\omega(t)\|_\infty) \|\omega(t)\|_\infty. \tag{72}$$

After applying Gronwall inequality to (72), we obtain that for all $t \in [0, T]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp \left(C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau))(1 + \log^+ \|\omega(\tau)\|_\infty) d\tau \right). \tag{73}$$

We take the Logarithm of inequality (73), to obtain

$$\log \|\omega(t)\|_\infty \leq \log \|\omega_0\|_\infty + C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau))(1 + \log^+ \|\omega(\tau)\|_\infty) d\tau. \tag{74}$$

Then, from (74) we deduce that for all $t \in [0, T]$,

$$\begin{aligned}
1 + \log^+ \|\omega(t)\|_\infty &\leq 1 + \log^+ \|\omega_0\|_\infty \\
&\quad + C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau))(1 + \log^+ \|\omega(\tau)\|_\infty) d\tau.
\end{aligned} \tag{75}$$

Then thanks to Gronwall inequality, we deduce that for all $t \in [0, T]$,

$$1 + \log^+ \|\omega(t)\|_\infty \leq (1 + \log^+ \|\omega_0\|_\infty) e^{C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau)) d\tau}. \tag{76}$$

Plugging (76) into (73), we obtain

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp \left((1 + \log^+ \|\omega_0\|_\infty) C_1 \int_0^t \psi(\tau) e^{C_1 \int_0^\tau \psi(s) ds} d\tau \right). \tag{77}$$

where for all $t \in [0, T]$,

$$\psi(t) = a(t) + a(t)b(t) + c(t). \tag{78}$$

Further, we have

$$C_1 \int_0^t \psi(\tau) e^{C_1 \int_0^\tau \psi(s) ds} d\tau = e^{C_1 \int_0^t \psi(s) ds} - 1. \tag{79}$$

By plugging (79) into (77), we deduce that for all $t \in [0, T]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{-(1+\log^+ \|\omega_0\|_\infty)} \exp\left((1+\log^+ \|\omega_0\|_\infty) e^{C_1 \int_0^t \psi(s) ds}\right). \quad (80)$$

We notice that $\|\omega_0\|_\infty e^{-(1+\log^+ \|\omega_0\|_\infty)} \leq 1$. Then from (80), we get for all $t \in [0, T]$,

$$\|\omega(t)\|_\infty \leq \exp\left((1+\log^+ \|\omega_0\|_\infty) e^{C_1 \int_0^t \psi(s) ds}\right), \quad (81)$$

which is valid for all $0 < T < T^*$. Since by assumption $\int_0^{T^*} c(t) dt < +\infty$, then
 540 owing to (78) and (81), we deduce that there exists a real $C_2 > 0$ such that for all $t \in [0, T^*[$

$$\|\omega(t)\|_\infty \leq \exp\left(C_2(1+\log^+ \|\omega_0\|_\infty) e^{C_2 \int_0^t (a(s)+a(s)b(s)) ds}\right),$$

which concludes the proof. \square

Now, let us deal with the geometric properties for non-blow-up of 2D QG, Navier-Stokes and Euler equations obtained as an immediate consequence of
 545 Theorem 5.1 combined with the BKM blow-up criteria (16) and (20).

5.2. Geometric property for non blow-up of 2D QG equation

In this subsection, we give a geometric property for non blow-up of 2D QG equation based on the regularity of the direction of the vorticity. Then in the
 550 remark 5.1, we revisit the hyperbolic saddle scenario for the finite time blow-up of 2D QG equation introduced in [17].

We want to use Theorem 5.1 for the 2D QG equation. Then, let $u_0 \in H^r(\mathbb{R}^2)$ with $r > 4$ and $T^* > 0$ be such that there exists $u \in C([0, T^*]; H^r(\mathbb{R}^2))$ solution to the 2D QG equation (3)-(4). Thanks to (19) used with $p = \infty$, we get that for all $t \in [0, T^*[$,

$$\|u(t)\|_\infty = \|u_0\|_\infty. \quad (82)$$

Let us assume that u blows up at the finite time T^* . Let us take a continuous function ρ from $[0, T^*]$ to $[0, +\infty[$ such that ρ positive on $[0, T^*[$ and

$$\int_0^{T^*} \frac{1}{(1+\log^+ \|\omega(t)\|_\infty)\rho(t)} dt < +\infty. \text{ Then, owing to (82), we deduce}$$

$$\begin{aligned} \int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))}}{(1+\log^+ \|\omega(t)\|_\infty)\rho(t)} dt &\leq 2 \int_0^{T^*} \frac{\|u(t)\|_\infty}{(1+\log^+ \|\omega(t)\|_\infty)\rho(t)} dt \\ &\leq 2 \int_0^{T^*} \frac{\|u_0\|_\infty}{(1+\log^+ \|\omega(t)\|_\infty)\rho(t)} dt < +\infty. \end{aligned} \quad (83)$$

Let b be the function introduced in (36), then owing (82), we get for all $t \in [0, T^*[$, $b(t) \leq 2\|u_0\|_\infty$. Then, owing to (83), thanks to Theorem 5.1 combined with the BKM blow-up criterion (20), we obtain the following Corollary

Corollary 5.1. *Let $u_0 \in H^r(\mathbb{R}^2)$ with $r > 4$. Let $T^* > 0$ be such that there exists $u \in C([0, T^*]; H^r(\mathbb{R}^2))$ solution to the 2D QG equation (3)-(4). Let ρ be a continuous function from $[0, T^*]$ to $[0, +\infty[$ such that ρ positive on $[0, T^*[$ and*

$$\int_0^{T^*} \frac{dt}{(1 + \log^+ \|\omega(t)\|_\infty)\rho(t)} < +\infty. \text{ If we have}$$

$$\int_0^{T^*} a(t) dt < +\infty, \quad (84)$$

then the solution u cannot blow up at the finite time T^ which means that $u \in C([0, T^*]; H^r(\mathbb{R}^2))$, where for all $t \in [0, T^*[$*

$$a(t) \stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} \quad (85)$$

and $\Omega(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2; |\omega(x, t)| = \|\omega(t)\|_\infty\}$. Moreover we have for all $t \in [0, T^[$*

$$\|\omega(t)\|_\infty \leq \exp\left(c(1 + \log^+ \|\omega_0\|_\infty)e^{c(1 + \|\omega_0\|_\infty)} \int_0^t a(s) ds\right), \quad (86)$$

where $c > 0$ is a constant.

Remark 5.1. *For Corollary 5.1, in the case where we suppose that $\|\omega(t)\|_\infty$ blows up like $O\left(\frac{1}{(T-t)^\gamma}\right)$, with $\gamma > 0$ (which is in agreement with Lemma 4.1), we can take $\rho(t) = (T^* - t) \log\left(1 + \frac{1}{T^* - t}\right)^\alpha$ with $\alpha > 0$. Then, we can observe that our non blow-up criterion (84) for the 2D QG equation is an improvement of the one given in [17] and also an improvement of the one given in [31]. Further, the numerical experiments performed in [31] on the hyperbolic saddle test case (a candidate for finite-time blow-up suggested in [17]), indicate that the region of large vorticity ω in magnitude and the region of large $\nabla \xi$ in magnitude are disjoint (see Fig.2-Fig.5 in [31]) and shrink to zero as the time tends to the alleged time of singularity. This suggests that $\nabla \xi$ remains bounded in regions where $|\omega|$ achieves its maximum and these regions shrink to zero as the time tends to the alleged time of singularity T^* . Thus, if these regions are contained in balls of radius of type $O(\rho(t))$ then we obtain that $a(t)$ remains bounded up to the alleged time of singularity T^* and therefore from Corollary 5.1, we will obtain that the solution u cannot blow up at the time T^* . From (86), we will infer also that the maximum of the vorticity $\|\omega(t)\|_\infty$ is bounded by double exponential, which is consistent with the recent numerical experiments [88, 21, 22, 25].*

5.3. Geometric property for non blow-up of Navier-Stokes and Euler equations

In this subsection, we give a geometric property for non blow-up of Navier-Stokes and Euler equations based on the regularity of the velocity and the direction of the vorticity. Thanks to Theorem 5.1 and the BKM criterion 16, we obtain the following Corollary.

580 **Corollary 5.2.** Let $u_0 \in H_\sigma^r(\mathbb{R}^3)$ with $r > \frac{9}{2}$. Let $T^* > 0$ be such that there exists $u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^3))$ solution to the Navier-Stokes and Euler equations (1)-(2). Let ρ be a continuous function from $[0, T^*]$ to $[0, +\infty[$ such that ρ positive on $[0, T^*[$ and $\int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))}}{(1 + \log^+ \|\omega(t)\|_\infty)\rho(t)} dt < +\infty$. If we have

$$\int_0^{T^*} a(t)(1 + b(t)) dt < +\infty,$$

then the solution u cannot blow up at the finite time T^* which means that $u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^3)^3)$, where for all $t \in [0, T^*[$,

$$\begin{aligned} \Omega(t) &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^3; |\omega(x, t)| = \|\omega(t)\|_\infty\}, \\ a(t) &\stackrel{\text{def}}{=} \sup_{x \in \Omega(t)} \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))} \\ b(t) &\stackrel{\text{def}}{=} \min \left(\|u(t)\|_{\mathcal{BMO}_d(\Omega(t), \rho(t))}, \frac{\rho(t)\|\nabla u(t)\|_\infty}{1 + \log^+ \|\omega(t)\|_\infty} \right). \end{aligned} \quad (87)$$

Moreover, we have for all $t \in [0, T^*]$

$$\|\omega(t)\|_\infty \leq \exp \left(c_0(1 + \log^+ \|\omega_0\|_\infty) e^{c_0 \int_0^t a(s)(1+b(s)) ds} \right),$$

where $c_0 > 0$ is a constant.

Remark 5.2. Most of numerical experiments show that if u blows up at a finite time T^* then we get the following blow-up rates $\|\omega(t)\|_\infty \sim O(T^* - t)^{-1}$ (which is in agreement with Lemma 4.1), $\|\nabla u(t)\|_\infty \sim O(T^* - t)^{-1}$ (which is in agreement with Theorem 1.1 in [12]), $\|\nabla \xi(t)\|_\infty \sim O(T^* - t)^{-\frac{1}{2}}$, $\|u(t)\|_\infty \sim O(T^* - t)^{-\frac{1}{2}}$ (see for instance [73]). Owing to these blow-up rates, we expect that the blow-up rate of $\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))}$ be slightly smaller than the one of $\|u(t)\|_\infty$ such that we get

$$\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))} \lesssim \frac{(T - t)^{-\frac{1}{2}}}{(1 + \log(1 + \log(1 + \frac{1}{T^* - t})))^{2\beta}} \text{ with } \beta > 1. \quad (88)$$

585 Indeed from the sharp version of logarithmic inequality (2.5) of [87] and thanks also to Bernstein's inequality, we infer that

$$\|u(t)\|_\infty \lesssim 1 + \|u(t)\|_{\text{BMO}(\mathbb{R}^3)} (\log^+(\|\omega(t)\|_\infty + \|u(t)\|_2))^{\frac{1}{2}},$$

where the exponent $\frac{1}{2}$ is optimal. By using the energy inequality (15), we get $\|u(t)\|_\infty \lesssim 1 + \|u(t)\|_{\text{BMO}(\mathbb{R}^3)} (\log^+(\|\omega(t)\|_\infty + \|u_0\|_2))^{\frac{1}{2}}$ which implies that $\|u(t)\|_\infty \lesssim (1 + \log^+ \|u_0\|_2)(1 + \|u(t)\|_{\text{BMO}(\mathbb{R}^3)}(1 + \log^+ \|\omega(t)\|_\infty)^{\frac{1}{2}})$. Then we get that there exists a constant $c_0 > 0$ such that

$$\frac{\|u(t)\|_\infty}{(1 + \log^+ \|\omega(t)\|_\infty)^{\frac{1}{2}}} \leq c_0(1 + \|u(t)\|_{\text{BMO}(\mathbb{R}^3)}). \quad (89)$$

We get also

$$\begin{aligned} \frac{\|u(t)\|_\infty}{(1 + \log^+ \|\omega(t)\|_\infty)^{\frac{1}{2}}} &\lesssim \frac{\|u(t)\|_\infty}{(1 + \log(1 + \frac{1}{T^* - t}))^{\frac{1}{2}}} \\ &\lesssim \frac{(T^* - t)^{-\frac{1}{2}}}{(1 + \log(1 + \frac{1}{T^* - t}))^{\frac{1}{2}}}. \end{aligned} \quad (90)$$

Moreover $\text{BMO}(\mathbb{R}^3) \hookrightarrow \mathcal{BMO}_d(\mathbb{R}^3) \hookrightarrow \mathcal{BMO}_d(\Omega(t))$. Since the exponent $\frac{1}{2}$ is optimal in inequality (89) and thanks to Proposition 5.1 in [86] then we could expect that

$$\|u(t)\|_{\mathcal{BMO}_d(\Omega(t))} \lesssim \frac{(T^* - t)^{-\frac{1}{2}}}{(1 + \log(1 + \frac{1}{T^* - t}))^{\frac{1}{2}}}.$$

But, here we require only that (88) holds.

590 Let us assume that u blows up at the finite time T^* with these blow-up rates which correspond to the Kerr's critical case studied by Deng et al. in [30, 32]. Then, owing to (88), from Corollary 5.2 we can take the function ρ as $\rho(t) = \frac{(T^* - t)^{\frac{1}{2}}}{(1 + \log(1 + \log(1 + \frac{1}{T^* - t})))^\beta}$. From (87) of Corollary 5.2, we get $a(t) \lesssim (T^* - t)^{-\frac{1}{2}}$ and

$$\begin{aligned} b(t) &\leq \frac{\rho(t)\|\nabla u(t)\|_\infty}{1 + \log^+ \|\omega(t)\|_\infty} \\ &\lesssim \frac{(T^* - t)^{-\frac{1}{2}}}{(1 + \log(1 + \frac{1}{T^* - t}))(1 + \log(1 + \log(1 + \frac{1}{T^* - t})))^\beta}. \end{aligned}$$

595 Since $\beta > 1$, we thus infer that $\int_0^{T^*} a(t)(1 + b(t)) dt < +\infty$ and thanks to Corollary 5.2, we deduce that in fact u cannot blow up at the time T^* .

Then, we have shown with our criterion that for the Kerr's critical case, we obtain the non blow-up of the solutions of Navier-Stokes or Euler equations provided that the very weak assumption (88) holds.

600 **Remark 5.3.** From Proposition 5.1 in [86], we notice that our criterion given in Corollary 5.2 can reveal the non blow-up of solutions of Navier-Stokes or Euler equations even in some cases of possible blow-up of type II.

6. Geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations evaluated only on the positions of maximum vorticity

605

In this section, we improve the geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations obtained in our Corollaries 5.1 and

5.2, but also the ones obtained in [30, 32]. Moreover, our geometric properties for non blow-up is given in an Eulerian setting.

610 Up to now, most of recent geometric constraints for non blow-up criteria of Navier-Stokes, Euler and 2D QG equations required the local regularity of the velocity and the direction of vorticity in all a region containing the position where maximum of vorticity is reached, this latter could shrink to zero as time increases. Therefore, these non blow-up criteria are limited due to the geometric
615 regularity required in all a given region $\mathcal{R}(t)$ containing the positions where the maximum of vorticity is reached. Indeed it could happen that in the region $\mathcal{R}(t)$ we have not enough regularity for the velocity or the direction of vorticity to conclude non blow-up of the solutions of Navier-Stokes, Euler or 2D QG equations, but sufficiently regularity in a region strictly included in $\mathcal{R}(t)$ to
620 prevent the non blow-up of these solutions.

Moreover, the recent geometric constraints for non blow-up criteria of Navier-Stokes, Euler and 2D QG equations based on the local geometric regularity of the Lagrangian vortex filaments [30, 32, 31] make the assumption that the position where the maximum of vorticity is reached, is advected with the flow,
625 however it is not always the case, as described in [74] (see also section 5.4.5 of [49]).

Then, in our Theorem 6.1, we obtain again some improvements of the non blow-up criteria for Navier-Stokes, Euler and 2D QG equations. Indeed we obtain geometric properties for non blow-up of Navier-Stokes, Euler and 2D
630 QG equations evaluated only on the positions where the maximum of vorticity are reached.

Moreover, in the case of Euler and 2D QG equations, the non blow-up criterion obtained is a necessary and sufficient condition in preventing finite time blow-up of the solutions.

635 Later, from our Theorem 6.1, we derive the Corollary 6.1 where we improve our non blow-up criterion logarithmically.

Theorem 6.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-1}(\mathbb{R}^d)).$$

640 Moreover for all $t \in [0, T^*]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s) ds},$$

where equality holds in the case of Euler and 2D QG equations and for all $t \in [0, T^*]$,

$$\begin{aligned} \Phi(t) &= \sup_{x \in \Omega(t)} -(\nabla \cdot \mathbf{v}_{\xi^\perp} + \mathbf{v}_{\xi^\perp} \cdot (\xi \cdot \nabla) \xi)(x, t), \\ \mathbf{v}_{\xi^\perp}(x, t) &= (\mathbf{v} - (\mathbf{v} \cdot \xi) \xi)(x, t) \\ \Omega(t) &= \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}, \end{aligned}$$

with $\mathbf{v} \equiv u$ in the case of Navier-Stokes or Euler equations, and $\mathbf{v} \equiv -R^\perp u$ in the case of 2D QG equation.

645 If we have,

$$\limsup_{t \rightarrow T^*} \int_0^t \Phi(s) ds < +\infty$$

then the solution u cannot blow up at the finite time T^* .

Proof. Thanks to the section 3, we have effectively that there exists a time of existence strictly positive $T^* > 0$ such that there exists a unique strong solution u to the Navier-Stokes, Euler or 2D QG equations (3)-(4) in the class

$$u \in C([0, T^*]; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-1}(\mathbb{R}^d)).$$

650 Similarly as for the proof of Theorem 5.1, we assume that for all $t \in [0, T^*[, \|\omega(t)\|_\infty > 0$.

We introduce the non-empty set $\mathcal{O} \stackrel{\text{def}}{=} \{(x, t) \in \mathbb{R}^d \times [0, T^*]; |\omega(x, t)| > 0\}$. We introduce also the direction of the vorticity $\xi = \frac{\omega}{|\omega|}$ defined on \mathcal{O} .

Let $0 < T < T^*$. From (42), we get that for all $t \in [0, T]$

$$\frac{d}{dt} \|\omega(t)\|_\infty \leq \sup_{x \in \Omega(t)} (\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t), \quad (91)$$

where equality holds in the case of Euler and 2D QG equations.

Since for any $(x, t) \in \mathcal{O}$, $(\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) = (\xi(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) |\omega(x, t)|$, then, from (91), we deduce that for all $t \in [0, T]$

$$\begin{aligned} \frac{d}{dt} \|\omega(t)\|_\infty &\leq \sup_{x \in \Omega(t)} (\xi(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) |\omega(x, t)| \\ &= \left(\sup_{x \in \Omega(t)} (\xi(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) \right) \|\omega(t)\|_\infty, \end{aligned} \quad (92)$$

where we have used the fact that for all $x \in \Omega(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty$. We notice,

$$(\xi \cdot \nabla) \mathbf{v} \cdot \xi = \xi \cdot \nabla(\mathbf{v} \cdot \xi) - \mathbf{v} \cdot ((\xi \cdot \nabla) \xi). \quad (93)$$

Since $\nabla \cdot \omega = 0$ and $\omega = |\omega| \xi$, then we get

$$|\omega| \nabla \cdot \xi + \xi \cdot \nabla |\omega| = 0. \quad (94)$$

However, for all $x \in \Omega(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty > 0$ and similarly as (40), we have $\nabla |\omega(x, t)| = 0$. Therefore, from (94), we deduce that for all $t \in [0, T]$ and $x \in \Omega(t)$,

$$\nabla \cdot \xi(x, t) = 0. \quad (95)$$

Owing to (95), we get that for all $t \in [0, T]$ and $x \in \Omega(t)$,

$$\xi(x, t) \cdot \nabla(\mathbf{v} \cdot \xi)(x, t) = \nabla \cdot ((\mathbf{v} \cdot \xi) \xi)(x, t). \quad (96)$$

We introduce \mathbf{v}_{ξ^\perp} the normal component to ξ of the velocity \mathbf{v}

$$\mathbf{v}_{\xi^\perp} = \mathbf{v} - (\mathbf{v} \cdot \xi)\xi. \quad (97)$$

Notice that $\mathbf{v}_{\xi^\perp} \cdot \xi = 0$, which means that \mathbf{v}_{ξ^\perp} belongs to ξ^\perp the plan orthogonal to ξ . Since $\nabla \cdot \mathbf{v} = 0$, then, we deduce that $\nabla \cdot \mathbf{v}_{\xi^\perp} = -\nabla \cdot ((\mathbf{v} \cdot \xi)\xi)$. Therefore from (96), we get that for all $t \in [0, T]$ and $x \in \Omega(t)$,

$$\xi(x, t) \cdot \nabla(\mathbf{v} \cdot \xi)(x, t) = -\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t). \quad (98)$$

Then using (93) and (98), we infer that for all $t \in [0, T]$ and $x \in \Omega(t)$,

$$(\xi(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) = -\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t) - \mathbf{v}(x, t) \cdot ((\xi(x, t) \cdot \nabla)\xi(x, t)). \quad (99)$$

Since $\mathbf{v} = (\mathbf{v} \cdot \xi)\xi + \mathbf{v}_{\xi^\perp}$, then we observe

$$\begin{aligned} \mathbf{v} \cdot ((\xi \cdot \nabla)\xi) &= (\mathbf{v} \cdot \xi) (\xi \cdot (\xi \cdot \nabla)\xi) + \mathbf{v}_{\xi^\perp} \cdot ((\xi \cdot \nabla)\xi) \\ &= (\mathbf{v} \cdot \xi) \xi \cdot \nabla|\xi|^2 + \mathbf{v}_{\xi^\perp} \cdot ((\xi \cdot \nabla)\xi) \\ &= \mathbf{v}_{\xi^\perp} \cdot ((\xi \cdot \nabla)\xi), \end{aligned}$$

where for the last equality, we have used the fact that $\nabla|\xi|^2 = 0$ since $|\xi| = 1$. Then, from (99), we deduce that for all $t \in [0, T]$ and $x \in \Omega(t)$,

$$(\xi(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) = -\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t) - \mathbf{v}_{\xi^\perp}(x, t) \cdot ((\xi(x, t) \cdot \nabla)\xi(x, t)). \quad (100)$$

Therefore using (100), from (92), we have for all $t \in [0, T]$,

$$\frac{d}{dt} \|\omega(t)\|_\infty \leq \left(\sup_{x \in \Omega(t)} -(\nabla \cdot \mathbf{v}_{\xi^\perp} + \mathbf{v}_{\xi^\perp} \cdot (\xi \cdot \nabla)\xi)(x, t) \right) \|\omega(t)\|_\infty. \quad (101)$$

Thanks to Gronwall inequality used in its differential form, we get for all $t \in [0, T]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s) ds}, \quad (102)$$

where $\Phi(s) = \sup_{x \in \Omega(s)} -(\nabla \cdot \mathbf{v}_{\xi^\perp} + \mathbf{v}_{\xi^\perp} \cdot (\xi \cdot \nabla)\xi)(x, s)$. Notice that for (102) equality holds in the case of Euler and 2D QG equations. Since inequality (102) is valid for all $0 < T < T^*$, then we infer that for all $t \in [0, T^*]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s) ds}, \quad (103)$$

655 which concludes the first part of the proof.

We assume now that $\limsup_{t \rightarrow T^*} \int_0^t \Phi(s) ds < +\infty$, then from (103), we get

$$\limsup_{t \rightarrow T^*} \|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp \left(\limsup_{t \rightarrow T^*} \int_0^t \Phi(s) ds \right) < +\infty.$$

However if u blows up at the time T^* then thanks to (16) and (20) we have

$\int_0^{T^*} \|\omega(t)\|_\infty dt = +\infty$ which leads to a contradiction with the fact that $\limsup_{t \rightarrow T^*} \|\omega(t)\|_\infty < +\infty$. Then, we deduce that u cannot blow up at the time T^* which completes the proof. □

From Theorem 6.1, we obtain the following Corollary,

Corollary 6.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1(]0, T^*]; H^{r-1}(\mathbb{R}^d)).$$

Moreover for all $t \in [0, T^*[$,

$$\|\omega(t)\|_\infty \leq \exp\left(\left(1 + \log^+ \|\omega_0\|_\infty\right) \exp\left(\int_0^t \frac{|\Phi(s)|}{1 + \log^+ \|\omega(s)\|_\infty} ds\right)\right),$$

where for all $t \in [0, T^*[$,

$$\begin{aligned} \Phi(t) &= \sup_{x \in \Omega(t)} -(\nabla \cdot \mathbf{v}_{\xi^\perp} + \mathbf{v}_{\xi^\perp} \cdot (\xi \cdot \nabla) \xi)(x, t), \\ \mathbf{v}_{\xi^\perp}(x, t) &= (\mathbf{v} - (\mathbf{v} \cdot \xi) \xi)(x, t) \\ \Omega(t) &= \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}, \end{aligned}$$

with $\mathbf{v} \equiv u$ in the case of Navier-Stokes or Euler equations, and $\mathbf{v} \equiv -R^\perp u$ in the case of 2D QG equation.

Furthermore, if we have,

$$\int_0^{T^*} \frac{|\Phi(s)|}{1 + \log^+ \|\omega(s)\|_\infty} ds < +\infty$$

then the solution u cannot blow up at the finite time T^* .

Proof. Indeed thanks to Theorem 6.1, we get that for all $t \in [0, T^*[$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s) ds},$$

which implies that

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t |\Phi(s)| ds}. \tag{104}$$

We observe that we can rewrite inequality (104) as follows

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \psi(s)(1 + \log^+ \|\omega(s)\|_\infty) ds}. \tag{105}$$

where $\psi(s) = \frac{|\Phi(s)|}{1+\log^+ \|\omega(s)\|_\infty}$. We take the logarithm of inequality (105) to obtain,

$$\log(\|\omega(t)\|_\infty) \leq \log(\|\omega_0\|_\infty) + \int_0^t \psi(s)(1 + \log^+(\|\omega(s)\|_\infty)) ds,$$

which yields to

$$1 + \log^+ \|\omega(t)\|_\infty \leq 1 + \log^+ \|\omega_0\|_\infty + \int_0^t \psi(s)(1 + \log^+ \|\omega(s)\|_\infty) ds. \quad (106)$$

Then thanks to Gronwall inequality used in its integral form, we deduce that for all $t \in [0, T^*[$,

$$1 + \log^+ \|\omega(t)\|_\infty \leq (1 + \log^+ \|\omega_0\|_\infty) e^{\int_0^t \psi(s) ds}. \quad (107)$$

By plugging inequality (107) into (105), we obtain,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{(1+\log^+ \|\omega_0\|_\infty) \int_0^t \psi(s) e^{\int_0^s \psi(\tau) d\tau} ds}. \quad (108)$$

However $\int_0^t \psi(s) e^{\int_0^s \psi(\tau) d\tau} ds = e^{\int_0^t \psi(\tau) d\tau} - 1$. Hence, from (108), we obtain,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{-(1+\log^+ \|\omega_0\|_\infty)} e^{(1+\log^+ \|\omega_0\|_\infty) e^{\int_0^t \psi(\tau) d\tau}} ds. \quad (109)$$

We notice that $\|\omega_0\|_\infty e^{-(1+\log^+ \|\omega_0\|_\infty)} \leq 1$, then from (109), we get

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega_0\|_\infty) e^{\int_0^t \psi(\tau) d\tau}} ds, \quad (110)$$

675 which concludes the first part of the proof. We assume now that $\int_0^{T^*} \psi(s) ds < +\infty$, then from (110), we get

$$\limsup_{t \rightarrow T^*} \|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega_0\|_\infty) e^{\int_0^{T^*} \psi(\tau) d\tau}} ds < +\infty.$$

However if u blows up at the time T^* then thanks to (16) and (20) we have

$\int_0^{T^*} \|\omega(t)\|_\infty dt = +\infty$ which leads to a contradiction with the fact that $\limsup_{t \rightarrow T^*} \|\omega(t)\|_\infty < +\infty$. Then, we deduce that u cannot blow up at the time T^*

680 which completes the proof. □

7. Some arguments in favor of a double-exponential in time growth of the maximum vorticity

In this section, we give some heuristic arguments combined with Corollary (6.1) to show some evidence in favor of a double-exponential in time growth of

the maximum vorticity in the cases of Euler and 2D QG equations. For this, we take T^* as the maximal time of existence of solutions. We assume that $T^* < +\infty$, which means that the solutions blow up at the time T^* and we assume also that

$$\liminf_{t \rightarrow T^*} \|\omega(t)\|_\infty = +\infty. \quad (111)$$

We introduce $\kappa \stackrel{\text{def}}{=} |(\xi \cdot \nabla)\xi|$ which seen as the curvature of vortex lines (see [30, 31]).

We highlight now the stabilizing effect of the convection term by using the Lagrangian formulation of the vorticity equation (see [60][Section 2]). Indeed, when we consider the convection term together with the vortex stretching term of the vorticity equation, we preserve the Lagrangian structure of the solution $\omega(X(\alpha, \tau, t), t) = \nabla_\alpha X(\alpha, \tau, t)\omega(\alpha, \tau)$ [84, 15, 31] where $X(\alpha, \tau, t)$ is the particle path that passes α at time τ and defined by $\frac{\partial}{\partial t}X(\alpha, \tau, t) = \mathbf{v}(X(\alpha, \tau, t), t)$, $X(\alpha, \tau, \tau) = \alpha$. Therefore, we observe that the vorticity can increase in time only through the dynamic deformation of the Lagrangian flow map X . Due to the divergence-free property of the velocity field \mathbf{v} , the flow map is volume-preserving which means that $\text{Det}(\nabla_\alpha X(\alpha, \tau, t)) \equiv 1$ (see [84]), thus as vorticity increases dynamically, the hyper-rectangle spanned by the d vectors, $\frac{\partial X}{\partial \alpha_i}, 1 \leq i \leq d$ will experience severe deformation and become flattened dynamically.

Therefore at the region of high vorticity, a vortex tube will experience tremendous core deformation and become severely flattened which is in agreement with the numerical experiments [60, 57, 74], then thanks to (111) this suggests that the curvature $\kappa(x, t)$ for all $x \in \Omega(t)$ and $t \in [t_0, T^*[$ may remain uniformly bounded up to T^* , for some t_0 sufficiently close to T^* . The time $t_0 \in [0, T^*[$ is chosen also such that for all $t \in [t_0, T^*[$, $\|\omega(t)\|_\infty \geq \|\omega(t_0)\|_\infty$. Then, we expect that for all $t \in [t_0, T^*[$,

$$\sup_{x \in \Omega(t)} \kappa(x, t) \leq C_0, \quad (112)$$

where $C_0 > 0$ is a constant.

7.1. Heuristic arguments for Euler equations in favor of a double-exponential in time growth of the maximum vorticity $\|\omega(t)\|_\infty$

Let $x \in \Omega(t)$ and $t \in]t_0, T^*[$. Since the flow map X is a C^1 -diffeomorphism from \mathbb{R}^3 to \mathbb{R}^3 (see [84]), there exists an unique $x_0 \in \mathbb{R}^3$ such that $x_0 = X(x_0, t_0, t)$ and $x = X(x_0, t_0, t)$. Let $R(t) \stackrel{\text{def}}{=} O(\kappa^{-1}(x, t)) \ll \kappa^{-1}(x, t)$ (R depends also on x but for simplicity in the notation, we keep only t), $\kappa^{-1}(x, t)$ is in fact the radius of curvature of the vortex filament passing through x at the time t . Let $\mathcal{C}_{R(t)}(t_0)$ be a vortex tube at the time t_0 containing x_0 of length ϵ such that any of its cross sections is assumed to be similar to a disk of radius $O(R(t))$. For any $s \in [t_0, t]$, we consider the vortex tube $\mathcal{C}_{R(t)}(s) \stackrel{\text{def}}{=} X(\mathcal{C}_{R(t)}(t_0), t_0, s)$ with its

710 length denoted $\ell(s)$ and whose any of its cross sections is assumed to be similar
to a disk of radius $O(\delta(s))$ for some $\delta(s) > 0$. We observe that $x \in \mathcal{C}_{R(t)}(t)$ and
due to the vortex stretching, we can assume that $\delta(t) \leq R(t)$.

Following numerical experiments (see [73]), the blow-up of the solution takes
place progressively in a anisotropic collapsing region characterized at the time
715 t by two length scales similar to $\delta(t)$ and $R(t)$. This region at the time t
is identified to the vortex tube $\mathcal{C}_R(t) \cap B(x, R(t))$. Then, since the very high
vorticities at the time t are concentrated in the vortex tube $\mathcal{C}_{R(t)}(t) \cap B(x, R(t))$,
we make the assumption that the velocity at the time t in the vortex tube
 $\mathcal{C}_{R(t)}(t) \cap B(x, R(t))$ is mainly induced by itself.

Then, by using the arguments involved in section 13.2 pp. 322 in [37], from
(13.25) of [37] with $\epsilon = R(t)$, we get that for all $y \in \mathcal{C}_{R(t)}(t) \cap B(x, R(t))$

$$|v_{\xi^\perp}(y, t)| \lesssim |\Gamma(t)| \kappa(x, t) (1 + \log(R(t)/\delta(t))), \quad (113)$$

where $\Gamma(t)$ is the Kelvin's circulation obtained from any boundary of a cross
section of the vortex tube $\mathcal{C}_{R(t)}(t)$. Let us give an estimate of $\frac{R(t)}{\delta(t)}$. Thanks
to the conservation of the volume of the vortex tubes $\mathcal{C}_{R(t)}(s)$, $s \in [t_0, t]$ during
the transport by the flow map X , we get $|\mathcal{C}_{R(t)}(t_0)| = |\mathcal{C}_{R(t)}(t)|$ and therefore
 $R(t)^3 \lesssim \delta(t)^2 \ell(t)$. Moreover from [84, 15], we assume that the stretching of the
vortex tube $\mathcal{C}_{R(t)}(t_0)$ experienced during the flow gives us $\frac{\ell(t)}{R(t)} \lesssim \frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}$.

Then we deduce that $R(t)/\delta(t) \lesssim \left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty} \right)^{\frac{1}{2}}$. Then from (113), we get for
all $y \in \mathcal{C}_{R(t)}(t) \cap B(x, R(t))$

$$|v_{\xi^\perp}(y, t)| \lesssim |\Gamma(t)| \kappa(x, t) \left(1 + \log \left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty} \right) \right). \quad (114)$$

Thanks to Kelvin's circulation Theorem, we get $|\Gamma(t)| = |\Gamma(t_0)| \lesssim R(t) \|u(t_0)\|_\infty$.
Then, from (114), we get for all $y \in \mathcal{C}_{R(t)}(t) \cap B(x, R(t))$

$$|v_{\xi^\perp}(y, t)| \lesssim R(t) \|u(t_0)\|_\infty \kappa(x, t) \left(1 + \log \left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty} \right) \right). \quad (115)$$

Since $R(t) = O(\kappa^{-1}(x, t))$, then from (115) we deduce for all $y \in \mathcal{C}_{R(t)}(t) \cap$
 $B(x, R(t))$

$$|v_{\xi^\perp}(x, t)| \lesssim \|u(t_0)\|_\infty \left(1 + \log \left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty} \right) \right). \quad (116)$$

720 By assuming that,

$$|\nabla \cdot v_{\xi^\perp}(x, t)| \lesssim \left| \frac{1}{|B(x, R(t))|} \int_{B(x, R(t))} \nabla \cdot v_{\xi^\perp}(y, t) dy \right|,$$

we deduce thanks to Stokes Theorem that

$$\begin{aligned} |\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t)| &\lesssim \frac{|S(x, R(t))|}{|B(x, R(t))|} \|v_{\xi^\perp}(t)\|_{L^\infty(B(x, R(t)))} \\ &= \frac{3}{R(t)} \|v_{\xi^\perp}(t)\|_{L^\infty(B(x, R(t)))} \end{aligned}$$

where $S(x, R(t))$ is the sphere of center x and radius $R(t)$. Since the very high vorticities at the time t are concentrated in the vortex tube $\mathcal{C}_{R(t)}(t) \cap B(x, R(t))$, we expect that $\|v_{\xi^\perp}(t)\|_{L^\infty(B(x, R(t)))} \lesssim \|v_{\xi^\perp}(t)\|_{L^\infty(\mathcal{T}(t))}$, where $\mathcal{T}(t) = \mathcal{C}_{R(t)}(t) \cap B(x, R(t))$. Then using again (115), we obtain

$$|\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t)| \lesssim \|u(t_0)\|_\infty \kappa(x, t) \left(1 + \log \left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty} \right)\right). \quad (117)$$

Then, owing to (116) and (117), from the function Φ involved in Corollary 6.1, we get

$$\int_{t_0}^t \frac{|\Phi(s)|}{1 + \log^+ \|\omega(s)\|_\infty} ds \lesssim \|u(t_0)\|_\infty (1 + |\log(\|\omega(t_0)\|_\infty)|) \int_{t_0}^t \sup_{x \in \Omega(s)} k(x, s) ds.$$

Then thanks to Corollary 6.1 used with the initial data $u(t_0)$, we deduce that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1 + \log^+(\|\omega(t_0)\|_\infty)) e^{\int_{t_0}^t f(s) ds}}, \quad (118)$$

where $f(s) = C \|u(t_0)\|_\infty (1 + |\log(\|\omega(t_0)\|_\infty)|) \sup_{x \in \Omega(s)} k(x, s)$ with $C > 0$ a constant.

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Owing to (112), from (118), we deduce that there exists a real $C > 0$ depending only on $u(t_0)$ such that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1 + \log^+ \|\omega(t_0)\|_\infty) e^{C(t-t_0)}},$$

which indicates that we can expect a double-exponential in time growth of the maximum vorticity and then no blow-up is possible at T^* .

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7.2. Heuristic arguments for 2D QG equations in favor of a double-exponential in time growth of the maximum vorticity $\|\omega(t)\|_\infty$

Let us assume that for all $t \in [t_0, T^*[$,

$$|\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t)| \lesssim \left| \frac{1}{|B(x, \rho(t))|} \int_{B(x, \rho(t))} \nabla \cdot \mathbf{v}_{\xi^\perp}(y, t) dy \right|, \quad (119)$$

where ρ is a continuous function on $[0, T^*]$ such that $\int_0^{T^*} \frac{dt}{\rho(t)} < +\infty$. Then thanks to Stokes's Theorem, we deduce that for all $t \in [t_0, T^*[$

$$\begin{aligned} \left| \frac{1}{|B(x, \rho(t))|} \int_{B(x, \rho(t))} \nabla \cdot \mathbf{v}_{\xi^\perp}(y, t) dy \right| &\leq \frac{|S(x, \rho(t))|}{|B(x, \rho(t))|} \|\mathbf{v}(t)\|_\infty \\ &\lesssim \frac{1}{\rho(t)} \|\mathbf{v}(t)\|_\infty. \end{aligned}$$

Then from (119), we infer

$$|\nabla \cdot \mathbf{v}_{\xi^\perp}(x, t)| \lesssim \frac{1}{\rho(t)} \|\mathbf{v}(t)\|_\infty. \quad (120)$$

Thanks to the result obtained in [26], we get that there exists $C(u_0) > 0$ depending continuously only on u_0 such that

$$\|\mathbf{v}(t)\|_\infty \leq C(u_0)(1 + \log^+ \|\omega(t)\|_\infty). \quad (121)$$

Then thanks to (120) and (121) for all $t \in [t_0, T^*[$, we infer

$$\int_{t_0}^t \sup_{x \in \Omega(s)} \frac{|\nabla \cdot \mathbf{v}_{\xi^\perp}(x, s)|}{1 + \log^+ \|\omega(s)\|_\infty} ds \lesssim C(u_0) \int_{t_0}^{T^*} \frac{ds}{\rho(s)} < +\infty. \quad (122)$$

Then, owing to (122) and (121), from the function Φ involved in Corollary 6.1, we get

$$\int_{t_0}^t \frac{|\Phi(s)|}{1 + \log^+ \|\omega(s)\|_\infty} ds \lesssim C(u_0) \left(1 + \int_{t_0}^t \sup_{x \in \Omega(s)} k(x, s) ds \right).$$

Then thanks to Corollary 6.1 used with the initial data $u(t_0)$, we deduce that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1 + \log^+ \|\omega(t_0)\|_\infty) e^{CC(u_0) + \int_{t_0}^t f(s) ds}}, \quad (123)$$

where $f(s) = CC(u_0) \sup_{x \in \Omega(s)} k(x, s)$ with $C > 0$ a constant.

Owing to (112), from (123), we deduce that there exists a real $C_1(u_0) > 0$ depending only on u_0 such that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1 + \log^+ \|\omega(t_0)\|_\infty) e^{CC(u_0) + C_1(u_0)(t - t_0)}},$$

which indicates that we can expect a double-exponential in time growth of the maximum vorticity and then no blow-up is possible at T^* .

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