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Abstract

One of the most challenging questions in fluid dynamics is whether the three-dimensional incompressible Navier-Stokes, Euler and two-dimensional Quasi-Geostrophic (2D QG) equations can develop a finite-time singularity from smooth initial data. The development of finite-time singularities was first investigated via energy methods which focus on global features or on point-wise Eulerian features of the flow. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy to overcome such shortcomings was established by focusing more on geometrical properties and flow structures, such as vortex tubes or vortex lines. However, all these recent geometric non blow-up criteria use the Lagrangian formulation of Incompressible Inviscid Flows, which requires much more computational effort than an Eulerian framework. In this paper, we improve these geometric non blow-up criteria while keeping an Eulerian setting by using Pshenichnyi's achievements in its investigation of necessary extremum conditions. By using also a dyadic unity partition of Littlewood decomposition, we established non blow-up criteria involving the dyadic BMO spaces known to be suitable for odd functions [86]. We thus obtained new insights on the growth of the maximum vorticity. Another highlight of our analysis is it allows to study the Navier-Stokes, Euler and 2D QG equations in a same framework. Our Eulerian geometric non blow-up criteria should give also new impetus to the numerical experiments due to their ease of implementation in comparison with Lagrangian geometric non blow-up criteria.

Keywords: Euler equations; Navier-Stokes equations; Quasi-Geostrophic equation; Finite time singularities; Geometric properties

Introduction

One of the most challenging questions in fluid dynamics is whether the incompressible 3D Navier-Stokes, Euler or 2D QG equations can develop a finite

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time singularity from smooth and bounded initial data. We know already that the blow up of smooth solutions to the Navier-Stokes and Euler equations is controlled by the time integral of the maximum magnitude of the vorticity (see [3, 66, 17], see also [10]). The Navier-Stokes and Euler equations describe the motion of a fluid in the three-dimensional space. These fundamental equations were derived over 250 years ago by Euler and since then have played a major role in fluid dynamics. They have enriched many branches of mathematics, were involved in many areas outside mathematical activity from weather prediction to exploding supernova (see for instance the surveys [24],[2]) and present important open physical and mathematical problems (see [24]). Regarding the 2D Quasi-Geostrophic (2D QG) equation, it appears in atmospheric studies. It describes the evolution of potential temperature u on the two dimensional boundary of a rapidly rotating half space with small Rossby and Ekman numbers, for the case of special solutions with constant potential vorticity in the interior and constant buoyancy frequency (normalized to one), where equations in the bulk are compressible Euler or Navier-Stokes equations coupled with temperature equation, continuity equation, and equation of state.

In the case of Navier-Stokes equations, for a long time ago, a global weak solution $u \in L^\infty(0,\infty;L^2(\mathbb{R}^3))^3$ and $\nabla u \in L^2(\mathbb{R}^3\times(0,\infty))^3$ was built by Leray [79]. In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equations, and proved that, for every given $u_0 \in L^2(\mathbb{R}^3)^3$, there exists a global weak solution $u \in L^\infty([0,+\infty[;L^2(\mathbb{R}^3))^3 \cap L^2([0,\infty[;\dot{H}^1(\mathbb{R}^3))^3$. Hopf has proved the existence of a global weak solution in the general case $\mathbb{R}^d$, $d \geq 2$, [56]. Several ways are known to construct weak solutions ([42], [55], [43]), and meanwhile the regularity and the uniqueness of this weak solution has been known for a long time ago for the two-dimensional case (see [78], [80], [82], [107]), in the three-dimensional case the problem remains widely open in spite of great efforts made. On the uniqueness many works have been done (see [41],[47], [85],[81],[44]). Moreover, the uniqueness of the weak solutions follows if the regularity for weak solutions can be obtained. Many criteria have been obtained on the regularity of weak solutions. In [96], it is proved that if u is a Leray-Hopf weak solution belonging to $L^q(]0,T];L^q(\mathbb{R}^3))^3$ with $\frac{2}{q}+\frac{3}{q}\leq 1$, $2<p<\infty$, $3<q<\infty$, then the solution $u \in C^\infty(\mathbb{R}^3\times]0,T])^3$. In [112] and [48], it is showed that if u is a weak solution in $C([0,T];L^3(\mathbb{R}^3))^3$, then $u \in C^\infty(\mathbb{R}^3\times]0,T])^3$. The limit case of $L^\infty([0,T];L^3(\mathbb{R}^3))^3$ has been solved in [63]. Other criterion regularity can be found in [52, 55, 47, 64, 67, 109, 7, 116, 19].

In the case of Euler Equations, in the two dimension case, uniqueness and existence of classical solutions have been known for a long time ago (see [114, 115, 111, 36, 78]). Thus a natural question appears, namely what can be said about the axisymmetric flow. The first results for the 3D case on the existence of classical solutions were obtained in the late sixties for axisymmetric flow without swirl (see [108, 77]). However for the full three space dimensions, little is known about smooth solutions apart from classical short-time existence

and uniqueness. Moreover, weak solutions are known to be badly behaved from the point of view of Hadamard's well-posedness theory (see for instance the surveys [35, 110]). Indeed, in [95], [99], non uniqueness of weak solutions has been demonstrated. These solutions break both uniqueness and the energy conservation. Moreover, the energy was non-monotonous and even unbounded. Later, it was shown in [33, 113] that there exist many weak solutions of the Euler equations with bounded energy. However, the energy for these weak solutions was not conserved. Recently it was shown in [34], that the non-increasing energy criterion cannot be the key point in the uniqueness of weak solutions, indeed for particular initial data, it has been proved that there exists infinitely many bounded solutions which are strongly L^2-continuous and satisfy the non increasing of energy. Later in [105], this result has been improved for large class of initial data. These recent results suggest that the non increasing energy criterion is not fundamental for the uniqueness of solutions (see for instance the surveys [35]). Further, another notion of weak solutions to the Euler equations was introduced in [83] called dissipative solutions and it was proven that they coincide with classical Euler solutions, when those exist. Zero-viscosity limits of Navier-Stokes equations have been shown to exist and to give Euler solutions in some more generalized sense known as measure-valued Euler solutions (see [36]). Considerable efforts have also been devoted to the study of the regularity properties of the 3D Euler equations. The main difficulty in the analysis lies in the presence of the nonlinear vortex stretching term and the lack of a regularization mechanism. Despite these difficulties, a few important partial results [3, 92, 39, 98, 20, 30, 46] have been obtained over the years which have led to improved understanding of the regularity properties of the 3D Euler.

In the case of 2D QG equation, besides its direct physical significance [53, 89], the 2D QG equation has very interesting features of resemblance to the 3D Euler equation, being also an outstanding open problem of the finite time blow-up issue. In particular, one can derive a necessary and sufficient blow-up condition for the 2D QG equation similar to the well-known Beale-Kato-Majda (BKM) criterion (Beale-Kato-Majda [3]). More precisely, the solution to the 2D QG equation (3) becomes singular at time T^* if and only if $\int_0^{T^*} \|\nabla^\perp u(t)\|_{L^\infty}\, dt = +\infty$ (see [17]). Thus, $\nabla^\perp u$ plays a role similar to the vorticity ω in the 3D Euler equations.

In the recent years, the 2D QG equation has been the focus of intense mathematical research [17, 26, 28, 9, 88, 25, 13], initiated by Constantin, Majda and Tabak [17]. These latter showed that if the direction field $\xi = \nabla^\perp u/|\nabla^\perp u|$ and the velocity v remain smooth in a region, then no finite-time singularity is possible in that region. A scenario for finite time blow up, a closing saddle, was proposed and numerically investigated there. It was later proved by Cordoba [26] that blow up does not happen in this scenario.

Besides the analytical results mentioned above, there also exists a sizable

literature focusing on the numerical search of a finite-time singularity for the 3D Euler equations and 2D QG equation (see [51, 94, 6, 46, 68, 69, 70, 71, 72, 73, 75, 90, 91, 17, 21, 25, 25]). Although finite-time singularities were frequently reported in numerical simulations of the Euler equations, most such singularities turned out to be either numerical artefacts or false predictions, as a result of either insufficient resolution or inadvertent data analysis procedure (see [100, 57, 59]).

Recently, it was shown in [106], a finite time blow up solution to an averaged three-dimensional Navier-Stokes equations of type $\partial_t u = \Delta u + \tilde{B}(u,u)$, where $\tilde{B}$ is an averaged version of the Euler bilinear operator B, acting also on divergence free vector fields u and obeying as B to the cancellation property $\langle \tilde{B}(u,u), u\rangle = 0$. This result suggests that any successful method to affirmatively answer to the Existence and Smoothness problem must either use finer structure of B or else must rely crucially on some estimate or other property of the Euler bilinear operator B that is not shared by the averaged operator $\tilde{B}$. Such additional structure exists for instance, the Euler equation has a vorticity formulation involving only differential operators rather than pseudo-differential ones.

Indeed, recent studies indicate that the local geometric regularity of Lagrangian vortex filaments can lead to dynamic depletion of vortex stretching [18, 20, 30, 32, 58, 31]. In particular, the recent results obtained in [30, 32, 58, 31] show that geometric regularity of vortex lines, even in an extremely localized region containing the maximum vorticity can lead to depletion of nonlinear stretching, thus avoiding finite time singularity formation of the 3D Euler equations or 2D QG equation. Motivated by the work of Hou and Li [58], Hou and Lei [61] further proposed a new 3D model to study the stabilizing effect of convection. By constructing a new 3D model for axisymmetric flows with swirl, they showed that the convection term has a surprising stabilizing effect in the 3D incompressible Navier-Stokes equations and plays an essential role in depleting the vortex-stretching term, thus preventing the finite time blow up of the Euler equations.

Despite high hopes from an analytical point of view that these considerations will shed light on the true nature of vorticity accumulation, numerical results observing geometrical properties of Lagrangian vortex filaments are scarce. This comes from the fact that to monitor the Lagrangian evolution of vortex lines, this requires additional computational efforts.

Thus, in this paper, we improve in several ways the recent geometric criteria for non blow-up of the Navier-Stokes, Euler and 2D QG equations based on the local geometric regularity of the unit vorticity vector [20, 27, 17] and the geometrical properties of Lagrangian vortex filaments [30, 32, 31] while keeping an Eulerian setting. To get our first improvement in the sense of establishing geometric non blow-up criteria in an Eulerian setting, we had to overcome the obstruction that we do not know if there exists an isolated absolute maximum for the vorticity achieved along a smooth curve as it was assumed in Proposition 2.1 of [17]. Moreover, recent numerical experiments show that it is not always

the case (see [74], see also section 5.4.5 in [49]). We overcome this difficulty by using a result of Pshenichnyi concerning directional derivatives of the function of maximum and the structure of a set of supporting functionals [93]. The second improvement comes from the fact that the recent non blow-up criteria involved the maximum of the magnitude of the velocity in a localized region where the position of maximum of vorticity is reached (see [20, 17, 30, 32]). But here, we obtain a more accurate criterion involving the norm of the velocity in a local dyadic BMO-type space defined on a localized region $\mathcal{R}(t)$ where the position $x(t)$ of maximum of vorticity is reached. Indeed, recent works show that dyadic BMO spaces are better suitable than L^∞ and BMO spaces (see [86]). Then, our second improvement is obtained by using a dyadic unity partition of Littlewood decomposition for which the resulting dyadic annular rings are centered in $x(t)$ and by using also affine functions for which we use the fact that their rotational is null. Then, we obtain a non blow-up criterion which involves for the velocity only its mean oscillations on dyadic annular rings contained in a localized region where the position of maximum of vorticity is reached. Hence, we bring a second improvement since the non blow-up criteria given in [20, 17, 30, 32] required the maximum of the magnitude of the velocity in a localized region. We obtain also that the radius of the region $\mathcal{R}(t)$ is controlled by the norm of the velocity in a dyadic BMO-type space. All these new results allow us to better understand how non blow-up of solutions of Euler, Navier-Stokes and 2D QG equations can hold even in the Kerr's critical case [73] (see our Remark 5.2 for more details).

Then, the paper is organized as follows:

- In section 1 , we give some notations and definitions.
- In section 2, we give a brief review of the recent geometric non blow-up criteria and we also discuss about some improvements brought.
- In section 3, we recall some results about the local regularity of solutions of Navier-Stokes, Euler and 2D QG equations.
- In section 4, we answer to an open question by obtaining a single exponential bound on the H^r-norm of solution to Euler, Navier-Stokes or 2D QG equations in terms of $\int_0^t \|\omega(s)\|_{\mathrm{BMO}(\mathbb{R}^d)}\, ds$. From, this result, we infer a lower bound on the blow-up rate of $\|\omega(t)\|_{\mathrm{BMO}(\mathbb{R}^d)}$.
- In section 5, we give our geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations in localized regions of maximum vorticity.
- In section 6, we go further in our improvements by giving geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations evaluated only on the positions of maximum vorticity. Moreover, in the case of Euler and 2D QG equations, our non blow-up criterion obtained is a necessary and sufficient condition in preventing finite time blow-up of the solutions.

- In section 7, by using the results obtained in section 6, we give some arguments in favor of a double-exponential in time growth of the maximum vorticity in the cases of Euler and 2D QG equations.

Let us now introduce the Navier-Stokes and Euler equations in $\mathbb{R}^3$ given by,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

in which $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$, $p = p(x,t) \in \mathbb{R}$ and $\nu \geq 0$ ($\nu = 0$ corresponds to the Euler equations) denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x,t) \in \mathbb{R}^3 \times [0, \infty[$ and the viscosity of the fluid,

with initial conditions,

$$u(x,0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3, \tag{2}$$

where the initial data u_0 is a divergence free vector field on $\mathbb{R}^3$.

Regarding the 2D QG equation (3) in $\mathbb{R}^2$, it is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + v \cdot \nabla u = 0, \\ v = -\nabla^{\perp}(-\Delta)^{-\frac{1}{2}} u, \end{cases} \tag{3}$$

with initial data,

$$u(x,0) = u_0. \tag{4}$$

Here $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. For v we have also the following representation

$$v = -R^{\perp} u, \tag{5}$$

where we have used the notation, $R^{\perp}u = (-R_2 u, R_1 u)$ with $R_j, j = 1, 2$, for the 2D Riesz transform defined by (see e.g. [104])

$$R_j(u)(x,t) = \frac{1}{2\pi} P.V \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x-y|^3} u(y,t)\, dy.$$

1. Some notations and definitions

In this section, we assume that $d \in \{1, 2, 3\}$.
For any vector $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$, we denote by $|x|$ the euclidean norm of x given by $|x| = \sqrt{\sum_{i=1}^{d} |x_i|^2}$. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of real square matrices of size d. For any vector field v defined from $\mathbb{R}^d$ to $\mathbb{R}^d$, we denote by

∇v the gradient matrix of v, the matrix defined with $ij-$component, $\dfrac{\partial v_i}{\partial x_j}$ for all $1 \leq i, j \leq d$. For $d = 2$, for any scalar field v defined from $\mathbb{R}^d$ to $\mathbb{R}$, we denote by $\operatorname{curl} v$ the rotational $\operatorname{curl} v = \nabla^\perp v$. For $d = 3$, for any vector field v defined from $\mathbb{R}^d$ to $\mathbb{R}^d$, we denote by $\operatorname{curl} v$ the rotational $\operatorname{curl} v = \nabla \times v$. For any function φ defined on $\mathbb{R}^d \times [0, +\infty[$, for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on $\mathbb{R}^d$ by $x \longmapsto \varphi(x, t)$. We denote by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$. We denote by BC the class of bounded and continuous functions and by BC^m the class of bounded and m times continuously derivable functions.
For any $R > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B(x_0, R)$, the ball of $\mathbb{R}^d$ of center x_0 and radius R. For any $R > 0$, we denote by B_R, the ball of $\mathbb{R}^d$ of center $(0, 0, 0)$ and radius R. We denote by $\mathrm{BMO}(\mathbb{R}^d)$ the space of locally integrable functions of bounded mean oscillation equipped with the norm $\|f\|_{\mathrm{BMO}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d, R>0} \dfrac{1}{|B(x,R)|} \displaystyle\int_{B(x,R)} |f - f_{B(x,R)}|$, where $f_{B(x,R)} = \dfrac{1}{|B(x,R)|} \displaystyle\int_{B(x,R)} f$.
We denote by div the differential operator given by, $\operatorname{div} = \displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial x_i}$.
We denote $A \lesssim D$, the estimate $A \leq c\, D$ where $c > 0$ is an absolute constant.
For any $f \in L^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)^d$ or $L^p(\mathbb{R}^d)^{d\times d}$) with $1 \leq p \leq +\infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p-norm of f.

Given an absolutely integrable function $f \in L^1(\mathbb{R}^d)$, we define the Fourier transform $\hat{f} : \mathbb{R}^d \longmapsto \mathbb{C}$ by the formula,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\, dx,$$

and extend it to tempered distributions. We will use also the notation $\mathcal{F}(f)$ for the Fourier transform of f. We define also the inverse Fourier transform $\check{f} : \mathbb{R}^d \longmapsto \mathbb{C}$ by the formula,

$$\check{f}(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(\xi)\, d\xi.$$

For $s \in \mathbb{R}$, we define the Sobolev norm $\|f\|_{H^s(\mathbb{R}^d)}$ of a tempered distribution $f : \mathbb{R}^d \longmapsto \mathbb{R}$ by,

$$\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi \right)^{\frac{1}{2}},$$

and then we denote by $H^s(\mathbb{R}^d)$ the space of tempered distributions with finite $H^s(\mathbb{R}^d)$ norm, which matches when s is a non negative integer with the classical Sobolev space $H^k(\mathbb{R}^d)$, $k \in \mathbb{N}$. The Sobolev space $H^s(\mathbb{R}^d)$ can be written as $H^s(\mathbb{R}^d) = J^{-s} L^2(\mathbb{R}^d)$ where $J = (1 - \Delta)^{\frac{1}{2}}$.

For $s > -1$, we also define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

and then we denote by $\dot{H}^s(\mathbb{R}^d)$ the space of tempered distributions with finite $\dot{H}^s(\mathbb{R}^d)$ norm.

We denote by $H^s_\sigma(\mathbb{R}^3)$ the Sobolev space $H^s_\sigma(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\psi \in H^s(\mathbb{R}^3)^3 : \text{div}\psi = 0\}$. In order to unify our notations with the two dimensional case 2D QG, we denote by $H^s_\sigma(\mathbb{R}^2)$ the Sobolev space $H^s(\mathbb{R}^2)$.

Definition 1.1. *We denote by $\mathbb{P}$ the matrix Leray's projection operator with components,*

$$\mathbb{P}_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Delta^{-1} = \delta_{i,j} - R_j R_k, \tag{6}$$

where R_j are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}} = \dfrac{1}{4\pi} \dfrac{x_j}{|x|^4} \star$ (see [104] for more details), Δ^{-1} is the inverse of Laplace operator given by $\Delta^{-1} = -\dfrac{1}{4\pi|x|} \star$, with $\star$ the convolution operator.

The operator $\mathbb{P}$, which acts on vector-valued functions, is a projection : $\mathbb{P}^2 = \mathbb{P}$, annihilates gradients and maps into solenoidal (divergence-free) vectors; it is a bounded operator on (vector-valued) L^q, $1 < q < \infty$ and commutes with translation. We can notice that the operator $\mathbb{P}$ can be written under the form,

$$\mathbb{P} = \text{Id} - \nabla \Delta^{-1} \text{div}, \tag{7}$$

which yields to Helmholtz decomposition, indeed for all $v \in L^q(\mathbb{R}^3)^3$, $1 < q < \infty$,

$$\begin{array}{ll} v & = \mathbb{P}v + \nabla\psi, \text{ with div } \mathbb{P}v = 0, \\ \psi & = \Delta^{-1} \text{div } v. \end{array} \tag{8}$$

2. Brief review and improvement of the non blow-up criteria

Historically, non blow-up criteria for the incompressible Euler equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy for overcoming such shortcomings was established by focusing more on geometrical properties and flow structures (see e.g. [20, 27]), such as vortex tubes or vortex lines.

In particular, Constantin, Fefferman and Majda ([20]) showed that local geometric regularity of the unit vorticity vector can lead to depletion of the vortex stretching. They prove that if there is up to time T an $O(1)$ region in which the vorticity vector is smoothly directed, i.e., the maximum norm of $\nabla\xi$

(here $\xi = \frac{\omega}{|\omega|}$, ω the vorticity) in this region is L^2 integrable in time from 0 to T, and the maximum norm of velocity in some $O(1)$ neighbourhood of this region is uniformly bounded in time, then no blow-up can occur in this region up to time T. This criterion takes into account the local structure of the flow and follows the evolution of vortex lines, but the (global) bound on the velocity makes this theorem hard to apply in simulations. Indeed, in the case of the collapse of a regular vortex tube, we expect that the dominating part of the velocity field is given by the rotational component of the velocity in the cross section normal to the direction of the vortex tube. As the vortex tube collapses, the rotational component of the velocity field may blow up proportional to the square root of the maximum vorticity from Kelvin's circulation theorem.

Then, a weaker restriction on the velocity field is presented by Cordoba and Fefferman [27]. They consider vortex tubes that are regular in a sense in an $O(1)$ region. With milder assumptions on the surrounding velocity it is shown that the tube cannot reach zero thickness in finite time.

However, all the existing theorems deal with $O(1)$ regions in which the vorticity vector is assumed to have some regularity, while in numerical computations, the regions that have such regularity and contain maximum vorticity are all shrinking with time.

Further, Kerr ([68], [75]) presented numerical evidence which suggested a finite-time singularity of the 3D Euler equations for two perturbed antiparallel vortex tubes which was consistent with the non blow-up criterion of Beale, Kato and Majda ([3]) and that of Constantin, Fefferman and Majda ([20]). His computations showed that the growth of the peak vorticity obeys the blow-up rate $(T-t)^{-1}$ and the velocity field was shown to blow up like $O(T-t)^{-\frac{1}{2}}$. Kerr argued that the curvature, κ, of the vortex lines and $\nabla\xi$ are likely bounded by $(T-t)^{-\frac{1}{2}}$ [75] and located in the inner region with distance $(T-t)^{\frac{1}{2}}$ away from the position where the maximum of vorticity is achieved.

Inspired by the work of Constantin, Fefferman and Majda ([20]), Deng, Hou and Yu ([30, 32]) recently showed that geometric regularity of Lagrangian vortex filaments, even in an extremely localized region containing the maximum of vorticity which may shrink with time, can lead to depletion of the nonlinear vortex stretching, thus avoiding finite time singularity formation of the 3D Euler equations. Moreover, they obtain that Kerr's blow-up scenario falls into the critical case of their local non blow-up criteria (Deng, Hou and Yu [30, 32]).

Despite high hopes from an analytical point of view that these considerations will shed light on the true nature of vorticity accumulation, numerical results observing geometrical properties of Lagrangian vortex filaments are scarce. This is primarily due to the fact that Eulerian quantities such as $\|\omega(t)\|_\infty$ are readily tractable in post-processing, while monitoring the Lagrangian evolution requires additional computational effort (see section 5.4.5 of [49]). On top of that, the geometry of integral curves at an instance in time, though in principle computable in post-processing, as well as derived quantities such as their convergence and curvature, are quite inaccessible in comparison to simple Eulerian criteria (see [50], see also section 5.4.5 of [49]).

Then, in this paper, we proposed non blow-up Eulerian criteria as effective as those obtained by Deng, Hou and Yu [30, 32, 31].

2.1. Improvement of the non blow-up criteria of Navier-Stokes and Euler equations

Let us assume that $\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\mathbb{R}^d)}$ is bounded by $O(T-t)^{B-1}(1+\log(1+(T-t)^{-1}))^{-\epsilon}$ for some $0 \leq B \leq 1$ and $\epsilon > 0$ (see section 5.1 for the definition of $\mathcal{BMO}_{\mathrm{d}}(\mathbb{R}^d)$). Therefore, from our Corollary 5.2, we infer that if $\nabla\xi(t)$ is bounded by $O(T-t)^{-A_0}$ and the velocity $u(t)$ is bounded by $O(T-t)^{-A}$ in the ball centered on the position where the maximum of vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius $\rho(t)$ of type $O(T-t)^B$, then there is no blow-up of the solutions of Euler or Navier-Stokes equations at the time T provided that $A_0 + A < 1$, $A \geq 0$, $A_0 \geq 0$.

Our criterion presents some advantages in comparison with the Deng, Hou and Yu's non blow-up-criterion [30, 32] besides the fact that it is a criterion established in an Eulerian setting:

- It includes the Navier-Stokes equations which is not the case with the Lagrangian approach used in [30, 32] where their proof and results were valid only for Euler equations.
- We do not make assumption of type $L(t) \gtrsim (T-t)^{B'}$ on the length $L(t)$ of vortex line segments along which its maximum vorticity is comparable to $\|\omega(t)\|_\infty$. Moreover from (3.11) of [30], we expect that $L(t) \lesssim \int_t^T \|u(s)\|_\infty \, ds \lesssim (T-t)^B$, then we could expect that $B' \geq B$.
- No blow-up of solutions of Euler equations in [30, 32] is ensured under the conditions $A + B' < 1$ and $-A_0 + B' \geq 0$, the case $A + B' = 1$ being obtained under some additional assumptions. However for the case $B' > 1 - A > A_0$, we can say nothing about the blow-up of solutions of Euler equations with Deng, Hou, Yu non blow-up criterion whereas with our criterion, we can ensure that there is no blow-up of the solutions of Euler equations at the time T.

Moreover, in Remark 5.2, we obtain the non blow-up of the solutions of Euler equations for the Kerr's critical case appearing in [59] under a very weak condition. In remark 5.3, we point out that with our criterion, we can even handle some blow-up scenarios of type II.

2.2. Improvement of the non blow-up criteria of 2D QG equations

From our Corollary 5.1, we found that if $\nabla\xi(t)$ is bounded by $O(T-t)^{-1}\log(1+(T-t)^{-1})^{-\beta}$, $\beta > 1$ in the ball $\widetilde{B}_t$ centered on the position where the maximum of vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius of type $O(T-t)\log(1+(T-t)^{-1})^{\alpha}$, $\alpha > 0$, then there is no blow-up of the solutions of the 2D QG equation at the time T.

Our criterion presents some advantages in comparison with the Deng, Hou and Yu's non blow-up-criterion [31], besides the fact that is a criterion in the Eulerian setting:

- we do not make assumptions of type $L(t) \gtrsim \frac{c_L}{\log\log\|\omega(t)\|_\infty}$ or $L(t) \gtrsim (T-t)^A$, $A < 1$ (if we assume that $\|\omega(t)\|_\infty \lesssim (T-t)^{-B}$ for some $B < +\infty$) on the length $L(t)$ of vortex line segments along which its maximum vorticity is comparable to the maximum vorticity in the whole 2D space.

- No blow-up of solutions of 2D QG equation in [31] is roughly speaking ensured under the condition that $\|\nabla\xi(t)\|_{L^\infty(B_t)}\, L(t) < +\infty$, where B_t is the ball centered on the position where the maximum vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius $L(t)$. However for the case $L(t) = O(T-t)^A$ with $A \geq 1$ and $\|\nabla\xi(t)\|_{L^\infty(\widetilde{B}_t)} = O(T-t)^{-1}\log(1+(T-t)^{-1})^{-\beta}$, $\beta > 1$ we can say nothing about the blow-up of solutions of 2D QG equation with Deng, Hou, Yu non blow-up criterion [31] whereas with our criterion, we can ensure that even in this case, there is no blow-up of the solutions of 2D QG equation at the time T.

- we obtain double exponential growth in time of $\|\omega(t)\|_\infty$ as soon as $\|\nabla\xi(t)\|_{L^\infty(\widetilde{B}_t)}$ is bounded up to T whereas a such result is obtained in [31] under the condition that $\nabla\xi(t)$ is bounded up to T but in the ball centered on the position where the maximum vorticity $\|\omega(t)\|_\infty$ is achieved, with a radius of type $O(1)$, which is much more restrictive than our condition.

As we can notice, all these non blow-up criteria required the local regularity of the velocity and the direction of vorticity in all a region or a vortex line containing the position where maximum of vorticity is reached. Then, in section 6 with our Theorem 6.1, we go further in our improvements by giving geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations evaluated only on the positions where the maximum of vorticity is reached.

3. Local regularity of solution of Navier-Stokes, Euler and 2D QG equations

In this section, we deal with the main result on local regularity of Navier-Stokes and Euler equations in its general form. By introducing $\mathbb{P}$ the matrix Leray operator, Euler equations (1)-(2) can be re-written as follows,

$$\frac{\partial u}{\partial t} + \mathbb{P}(u\cdot\nabla)u = 0, \tag{9}$$

with initial conditions,

$$u(0) = u_0. \tag{10}$$

For u solution of (9)-(10), $\omega = \nabla\times u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u - \nu\Delta\omega = 0, \tag{11}$$

with initial conditions,

$$\omega(0) = \omega_0, \tag{12}$$

where $\omega_0 = \nabla \times u_0$ is the vorticity of u_0.
In the case of 2D QG equation, we get for u solution of (3), $\omega = \nabla^\perp u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v = 0, \tag{13}$$

with initial conditions,

$$\omega(0) = \omega_0, \tag{14}$$

where $\omega_0 = \nabla^\perp u_0$ is the vorticity of u_0.
In the region where $|\omega| > 0$, we define ξ the direction of the vorticity by $\xi = \dfrac{\omega}{|\omega|}$.

3.1. Local regularity for Navier-Stokes or Euler equations

Assuming $u_0 \in H^r_\sigma(\mathbb{R}^3)$ with $r > \dfrac{5}{2}$, thanks to Theorem 3.5 and 4.7 in [66], Theorem 1 in [5] (see also Theorem I in [65] and the results obtained in [3]), we deduce that there exists a time $T > 0$ such that there exists an unique strong solution $u \in C([0, T[, H^r_\sigma(\mathbb{R}^3))$ to the Navier-Stokes or Euler equations (9)-(10) and the energy equality holds for u, that means for all $t \in [0, T[$,

$$\|u(t)\|_2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 \, ds = \|u_0\|_2. \tag{15}$$

Moreover, if $u \notin C([0, T], H^r_\sigma(\mathbb{R}^3))$, then we get (see [3, 66, 76]),

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \tag{16}$$

Owing to $u \in C([0, T[, H^r_\sigma(\mathbb{R}^3))$ and thanks to Lemma X4 in [66], from Navier-Stokes or Euler equations (9), we get $u \in C^1(]0, T[, H^{r-1}(\mathbb{R}^3))^3$.
Moreover, for the Navier-Stokes equations, thanks to (15), we get the following estimate (for the proof see Appendix A and (43) in [23]), for all $t \in [0, T[$,

$$\int_0^t \|u(s)\|_\infty \, ds \leq C(\|u_0\|_2^2 + \sqrt{t}), \tag{17}$$

where $C > 0$ is a constant.

From Helmholtz decomposition (8), we retrieve the pressure p from the velocity u with the formula,

$$p = -\Delta^{-1}\mathrm{div}((u \cdot \nabla)u).$$

Furthermore, we get the local estimate (18). Indeed, thanks to remark 4.4 in [66], we get

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ with } t_0 < t < T, \tag{18}$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

3.2. Local regularity for 2D QG equation

This subsection is devoted to the local well-posedness of the 2D QG equation with a characterization of the maximal time existence of strong solutions. By using the same arguments as the proof of Proposition 4.2 in [1], we get that the H^s-norm of u is controlled by the integral in time of the maximum magnitude of the vorticity of u. A such Proposition has been proved in [17] for any integer $s \geq 3$, but here we extend this result to all real $s > 2$. This improvement is obtained by using the logarithmic Sobolev inequality proved in [76, 66] which requires only that $s > 2$ instead of using the one proved in [3] as it is the case in [17] and which requires integer $s \geq 3$. Then by using the same arguments as the proof of Proposition 4.3 in [1], we get the following result which gives an improvement in comparison with Theorem 2.1 in [17]:

Assuming $u_0 \in H^r(\mathbb{R}^2)$ with $r > 2$, we get that there exists a time $T > 0$ such that there exists an unique strong solution $u \in C([0, T[, H^r(\mathbb{R}^2))$ to the 2D QG equation (3)-(4) and the energy equality holds for u, that means for all $p \in [2, \infty]$ and $t \in [0, T[$,

$$\|u(t)\|_p = \|u_0\|_p. \tag{19}$$

Moreover, if $u \notin C([0, T], H^r(\mathbb{R}^2))$, then

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \tag{20}$$

Owing to $u \in C([0, T[, H^r(\mathbb{R}^2))$ and thanks to Lemma X4 in [66], from 2D QG (3), we get $u \in C^1(]0, T[, H^{r-1}(\mathbb{R}^2))$.
Similarly as in (18), we have

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ for } t_0 < t < T, \tag{21}$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

4. A lower bound on blow up rate of the vorticity in the BMO−norm

The blow up of smooth solutions to the Navier-Stokes, Euler or 2D QG equations is controlled by BKM-type criteria formulated in terms of the time integral of the maximum magnitude of the vorticity (see [3, 66, 17]). And this criterion have been refined in [76] and [8] by replacing respectively the L^∞ norm of the vorticity by weaker norms close to the L^∞ norm, more precisely by the BMO space and the Triebel-Lizorkin space $\dot{F}^0_{\infty,\infty}$. In particular, in all these works, the authors obtained a double exponential bound for $\|u(t)\|_{H^r}$ in terms of $\int_0^t \|\omega(s)\|_X \, ds$ with $X = L^\infty, \text{BMO}$ or $\dot{F}^0_{\infty,\infty}$ as follows (see [3, 66, 76, 8, 17], see also [29])

$$\|u(t)\|_{H^r} \leq \|u_0\|_{H^r} \exp\left(\exp\left(C \int_0^t \|\omega(s)\|_X \, ds\right)\right). \tag{22}$$

It is an open question whether (22) is sharp in the sense that is it possible to get single exponential bound. This question is addressed in [29]. In this section, we answer to this question, more precisely thanks to our Lemma 4.1, we show in Proposition 4.2 that we need only one exponential in term of $\int_0^t \|\omega(s)\|_{\mathrm{BMO}(\mathbb{R}^d)}\, ds$ to get an upper bound of $\|u(t)\|_{H^r}$ with $r > \frac{d}{2} + 1$. Then, from Proposition 4.2, we obtain a lower bound on the blow-up rate of the vorticity ω in the BMO$-$norm.

Thus, we begin with the proof of Lemma 4.1. To get Lemma 4.1, similarly as Proposition 2.1 in [76], we use the following proposition due to Coifman-Meyer [Chapter V. Proposition 2 of [16]].

Proposition 4.1. *Let $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \backslash (0,0))$ satisfy*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}, \quad (\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d \backslash (0,0)$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. Suppose that $\sigma(0,\eta) = 0$. Then the bilinear operator $\sigma(D)(\cdot,\cdot)$ defined by

$$\sigma(D)(a,h)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} \sigma(\xi,\eta)\hat{a}(\xi)\hat{h}(\eta) d\xi\, d\eta, \quad x \in \mathbb{R}^d,$$

satisfies

$$\|\sigma(D)(a,h)\|_p \leq C_{d,p}\|a\|_{\mathrm{BMO}(\mathbb{R}^d)}\|h\|_p.$$

By using exactly the same arguments as the ones given for the proof of Lemma XI in [66] where we noticed that we could replace the Theorem X2 in [66] used for the proof by our Proposition 4.1, we obtained the following Lemma

Lemma 4.1. *If $s > 0$ and $1 < p < \infty$ then for all $f \in H^{s-1}(\mathbb{R}^d)$ with $\nabla f \in \mathrm{BMO}(\mathbb{R}^d)$ and $g \in H^{s-1}(\mathbb{R}^d) \cap \mathrm{BMO}(\mathbb{R}^d)$, we get*

$$\|J^s(fg) - fJ^s g\|_p \lesssim \|\nabla f\|_{\mathrm{BMO}(\mathbb{R}^d)}\|J^{s-1}g\|_p + \|J^s f\|_p \|g\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

where $J = (1-\Delta)^{\frac{1}{2}}$.

Owing to Lemma 4.1, in the following Proposition 4.2, we give a single exponential bound for $\|u(t)\|_{H^r}$ in terms of $\int_0^t \|\omega(s)\|_{\mathrm{BMO}}\, ds$.

Proposition 4.2. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$, $r > 1 + \frac{d}{2}$. Let $T^* > 0$ be such that there exists $u \in C([0,T^*[; H^r_\sigma(\mathbb{R}^d))$ solution to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4). Then there exists a constant $c_0 > 0$ such that for all $t \in [0,T^*[$*

$$\|u(t)\|_{H^r} \leq \|u_0\|_{H^r} e^{c_0 \int_0^t \|\omega(\tau)\|_{\mathrm{BMO}(\mathbb{R}^d)} d\tau}. \tag{23}$$

Proof. We set $\boldsymbol{v} = u$ in the case of Navier-Stokes or Euler equations and $\boldsymbol{v} = -R^\perp u$ with $\nu = 0$ in the case of 2D QG equation. Taking the L^2 inner product of the equations (1) or (3) with $J^r u$, one has,

$$\frac{1}{2}\frac{d}{dt}\|J^r u\|_2^2 + \nu\|\nabla J^r u\|_2^2 = -\int_{\mathbb{R}^d} J^r u\, [J^r((\boldsymbol{v}\cdot\nabla)u) - (\boldsymbol{v}\cdot\nabla)J^r u], \tag{24}$$

where we have used the divergence free condition $\nabla \cdot \boldsymbol{v} = 0$. We apply Cauchy-Schwarz inequality to the term at the right hand side of equality (24) to obtain,

$$\frac{1}{2}\frac{d}{dt}\|J^r u\|_2^2 \leq \|J^r u\|_2 \, \|J^r((\boldsymbol{v}\cdot\nabla)u) - (\boldsymbol{v}\cdot J^r\nabla)u\|_2 \,. \tag{25}$$

Thanks to Lemma 4.1, we obtain that for all $t \in [0, T^*[$,

$$\frac{1}{2}\frac{d}{dt}\|J^r u(t)\|_2^2 \lesssim (\|\nabla \boldsymbol{v}(t)\|_{\mathrm{BMO}(\mathbb{R}^d)}\|J^r u(t)\|_2 + \|J^r \boldsymbol{v}(t)\|_2 \|\nabla u(t)\|_{\mathrm{BMO}(\mathbb{R}^d)})\|J^r u(t)\|_2. \tag{26}$$

Since Riesz transforms maps $\mathrm{BMO}(\mathbb{R}^d)$ into itself and maps also $H^r(\mathbb{R}^d)$ into itself, from (26) we deduce that for all $t \in [0, T^*[$,

$$\frac{d}{dt}\|u(t)\|_{H^r}^2 \lesssim \|\omega(t)\|_{\mathrm{BMO}(\mathbb{R}^d)}\|u(t)\|_{H^r}^2.$$

Thus, thanks to Gronwall inequality, we infer (23), which concludes the proof. □

Now, thanks to Proposition 4.2, in Lemma 4.2 we show that if the solution blows up at a finite time T^*, then the vorticity in the BMO-norm will blow up as least faster than $O(T^* - t)$. To get our Lemma 4.2, we borrow some arguments used for the proof of Theorem 1.1 in [12].

Lemma 4.2. *Let $d \in \{2, 3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \dfrac{d}{2} + 1$. Let $T^* > 0$ be such that there exists $u \in C([0, T^*[, H^r(\mathbb{R}^d))$ solution to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4). There exists a constant $c > 0$ such that if u blows up at the finite time T^* then we get*

$$\limsup_{t \to T^*} (T^* - t)\|\omega(t)\|_{\mathrm{BMO}(\mathbb{R}^d)} \geq c.$$

Proof. Let us assume that u blows up at the finite time T^*. Then, from the BKM's criteria (16) and (20), we have

$$\int_0^{T^*} \|\omega(t)\|_\infty \, dt = +\infty. \tag{27}$$

Let $M = \limsup_{t \to T^*} (T^* - t)\|\omega(t)\|_{\mathrm{BMO}(\mathbb{R}^d)}$. Let $c_0 > 0$ be the constant involved in Proposition 4.2. Let us show that $M < \dfrac{1}{c_0}$ leads to a contradiction. Then, we suppose that $M < \dfrac{1}{c_0}$. Hence, there exists $t_0 \in]0, T^*[$ such that

$$\sup_{t_0 < t < T^*} (T^* - t)\|\omega(t)\|_{\mathrm{BMO}(\mathbb{R}^d)} := M_0 < \frac{1}{c_0}. \tag{28}$$

Further, thanks to Proposition 4.2 used with $u(t_0)$ as initial data, we infer that for all $t \in [t_0, T^*[$,

$$\|u(t)\|_{H^r} \leq \|u(t_0)\|_{H^r} e^{c_0 \int_{t_0}^t \|\omega(\tau)\|_{\mathrm{BMO}(\mathbb{R}^d)} d\tau}. \tag{29}$$

Then thanks to (28) and (29), we deduce that for all $t \in [t_0, T^*[$,

$$\begin{aligned} \|u(t)\|_{H^r} &\leq \|u(t_0)\|_{H^r} e^{c_0 M_0 \int_{t_0}^t (T^*-s)^{-1}\, ds} \\ &= \|u(t_0)\|_{H^r} \left(\frac{T^*-t_0}{T^*-t}\right)^{M_0 c_0}. \end{aligned} \tag{30}$$

Since $r > \dfrac{d}{2} + 1$, we get the following Sobolev embedding $H^r(\mathbb{R}^d) \hookrightarrow BC^1(\mathbb{R}^d)$ and hence we have that for all $t \in [0, T^*$, $\|\nabla u(t)\|_\infty \lesssim \|u(t)\|_{H^r}$. Therefore, from (30), we deduce that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \lesssim \|u(t_0)\|_{H^r} \left(\frac{T^*-t_0}{T^*-t}\right)^{M_0 c_0}. \tag{31}$$

Since $M_0 c_0 < 1$, then from (31) we infer that $\displaystyle\int_{t_0}^{T^*} \|\omega(t)\|_\infty\, dt < +\infty$ which leads to a contradiction with (27). Therefore, we deduce that $M \geq \dfrac{1}{c_0}$, which concludes the proof. □

5. Geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations in localized regions of maximum vorticity

In this section, we give new geometric properties for non blow-up of the Navier-Stokes, Euler and 2D QG equations based on the regularity of the velocity and the direction of the vorticity in extremely localized regions containing the maximum vorticity. Our results improve the ones obtained in [20, 17] but also the ones obtained in [30, 32, 31]. One of the highlights of our results is that they are established in an Eulerian setting, which is more tractable in numerical simulations in comparison with geometric non blow-up criteria given in Lagrangian settings as it is the case in [30, 32, 31].

Then, this section is organized as follows:

- In the subsection 5.1, we establish in our main Theorem 5.1 a new bound for the maximum vorticity of 2D QG, Navier-Stokes and Euler equations.
- In the subsection 5.2, we use Theorem 5.1 and Lemma 4.2 to obtain the geometric properties for non blow-up of 2D QG equation in Corollary 5.1.
- In the subsection 5.3, we use Theorem 5.1 and Lemma 4.2 to obtain the geometric properties for non blow-up of Navier-Stokes and Euler equations in Corollary 5.2.

5.1. Estimate on the maximum vorticity for 2D QG, Navier-Stokes and Euler equations

In this subsection, in Theorem 5.1, we obtain a new estimate on the maximum vorticity for 2D QG, Navier-Stokes and Euler equations based on the

regularity of the velocity and the gradient of the direction of the vorticity in a region shrinking to zero as time increase to some T^* the alleged time of singularity and containing the position where the maximum vorticity $\|\omega(t)\|_\infty$ is reached.

To establish our Theorem 5.1, we need to introduce dyadic BMO-type spaces and use Lemma 5.1.

Dyadic versions of the $\mathrm{BMO}(\mathbb{R}^d)$ space are little wider than $\mathrm{BMO}(\mathbb{R}^d)$. Moreover, in order to consider a possible blow-up solution for the 3D Navier-Stokes, Euler or 2D QG equations, handling dyadic versions of the $\mathrm{BMO}(\mathbb{R}^d)$ space is more reasonable than the $\mathrm{BMO}(\mathbb{R}^d)$ space for the reasons exposed in [86].

Then, we introduce $\mathcal{BMO}_{\mathrm{d}}(\mathbb{R}^d)$ a dyadic version of the usual $\mathrm{BMO}(\mathbb{R}^d)$ space, defined as the space of locally integrable functions f such that

$$\|f\|_{\mathcal{BMO}_{\mathrm{d}}(\mathbb{R}^d)} \stackrel{\mathrm{def}}{=} \sup_{x\in\mathbb{R}^d}\sup_{j\in\mathbb{Z}} 2^{-jd}\int_{\mathcal{C}_j} |f(x+y)-\tilde{f}_j(x)|\,dy < +\infty,$$

where $\mathcal{C}_j$ is the annular ring $\mathcal{C}_j \stackrel{\mathrm{def}}{=} \{y\in\mathbb{R}^d; 2^{j-1}\le|y|\le 2^{j+1}\}$ and $f_j(x)$ the mean value $f_j(x) = \dfrac{1}{|\mathcal{C}_j|}\displaystyle\int_{\mathcal{C}_j} f(x+y)\,dy$.

For any F non-empty set of $\mathbb{R}^d$, we denote by $\mathcal{BMO}_{\mathrm{d}}(F)$ the space of locally integrable functions f such that

$$\|f\|_{\mathcal{BMO}_{\mathrm{d}}(F)} \stackrel{\mathrm{def}}{=} \sup_{x\in F}\sup_{j\in\mathbb{Z}} 2^{-jd}\int_{\mathcal{C}_j} |f(x+y)-f_j(x)|\,dy < +\infty,$$

For any $\rho>0$, we denote also by $\mathcal{BMO}_{\mathrm{d}}(F,\rho)$ the space of locally integrable functions f such that

$$\|f\|_{\mathcal{BMO}_{\mathrm{d}}(F,\rho)} \stackrel{\mathrm{def}}{=} \sup_{x\in F}\sup_{j\in\mathbb{Z};\mathcal{C}_j\subset B(0,\rho)} 2^{-jd}\int_{\mathcal{C}_j} |f(x+y)-f_j(x)|\,dy < +\infty.$$

Now, we turn to the proof of Lemma 5.1.

Lemma 5.1. *Let $d\in\mathbb{N}^*$, $T>0$ and $f\in C([0,T];BC(\mathbb{R}^d))$ such that $\inf\limits_{t\in[0,T]}\|f(t)\|_\infty>0$ and for any $t\in[0,T]$, $|f(x,t)|\to 0$ as $|x|\to+\infty$. Then there exists $R>0$ such that for all $t\in[0,T]$, $\|f(t)\|_\infty = \sup\limits_{x\in B_R}|f(x,t)|$.*

Proof. We set $a = \inf\limits_{t\in[0,T]}\|f(t)\|_\infty > 0$. Since the function $t\mapsto f(t)$ is a continuous function from the compact $[0,T]$ into the metric space $L^\infty(\mathbb{R}^d)$ then it is uniformly continuous. Therefore, there exists $N\in\mathbb{N}^*$ such that for all $t,t'\in[0,T]$, $|t-t'|\le\frac{T}{N}$ we have $\|f(t)-f(t')\|_\infty\le\dfrac{a}{4}$. We introduce the subdivision $\{t_i\}_{\{i\in[\![0,N]\!]\}}$ of $[0,T]$ defined by $t_i = i\frac{T}{N}$ for $i\in[\![0,N]\!]$. Since for any $t\in[0,T]$, $|f(x,t)|\to 0$ as $|x|\to+\infty$, then for each $i\in[\![0,N]\!]$, there exists $R_i>0$ such that for all $|x|\ge R_i$, $|f(x,t_i)|\le\dfrac{a}{4}$. We set $R = \max\limits_{i\in[\![0,N]\!]} R_i$. Let

$t \in [0,T]$, then there exists $j \in [\![0,N]\!]$ such that $|t-t_j| \leq \frac{T}{N}$, therefore for all $|x| \geq R$, we have $|f(x,t)| \leq |f(x,t)-f(x,t_j)| + |f(x,t_j)| \leq \frac{a}{2} \leq \frac{\|f(t)\|_\infty}{2}$. Then, we infer that for all $t \in [0,T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x,t)|$, which concludes the proof. □

Now, we turn to the proof of our Theorem.

Theorem 5.1. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2}+3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equations (3)-(4) in the class*

$$u \in C([0,T^*[;H^r_\sigma(\mathbb{R}^d)) \cap C^1(]0,T^*[;H^{r-1}(\mathbb{R}^d)).$$

Then for any continuous function ρ from $[0,T^*]$ to $[0,+\infty[$ such that ρ positive on $[0,T^*[$ and $\displaystyle\int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t))}}{(1+\log^+\|\omega(t)\|_\infty)\rho(t)}\,dt < +\infty$, we have for all $t \in [0,T^*[$

$$\|\omega(t)\|_\infty \leq \exp\left(c_0(1+\log^+\|\omega_0\|_\infty)e^{c_0\int_0^t \frac{a(s)(1+b(s))}{1+\log^+\|\omega(s)\|_\infty}\,ds}\right),$$

where $c_0 > 0$ is a constant and for all $t \in [0,T^*[$

$$\begin{aligned} \Omega(t) &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x,t)| = \|\omega(t)\|_\infty\}, \\ a(t) &\stackrel{\text{def}}{=} \sup_{x\in\Omega(t)} \|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))}, \\ b(t) &\stackrel{\text{def}}{=} \min\left(\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t),\rho(t))}, \frac{\rho(t)\|\nabla u(t)\|_\infty}{1+\log^+\|\omega(t)\|_\infty}\right). \end{aligned} \tag{32}$$

Proof. Thanks to the results of the section 3, there exists effectively a time of existence $T^* > 0$ such that there exists a unique strong solution u to the Navier-Stokes, Euler and 2D QG equations (1)-(2) in the class

$$u \in C([0,T^*[;H^r_\sigma(\mathbb{R}^d)) \cap C^1(]0,T^*[;H^{r-1}(\mathbb{R}^d)).$$

We can assume that for all $t \in [0,T^*[$,

$$\|\omega(t)\|_{L^\infty} > 0. \tag{33}$$

Otherwise there will exists $t_0 \in [0,T^*[$ such that $\|\omega(t_0)\|_\infty = 0$.
In the case of 2D QG equations (3), we get that $\omega(t_0) = 0$ and then $\nabla u(t_0) = 0$. Since $u(t_0,x)$ vanishes at infinity, then we will get $u(t_0) = 0$. Then by using inequality (21) concerning the local regularity, we deduce that $u(t) = 0$ for all $t \in [t_0,T^*[$ which allow us to conclude the proof.
In the case of Navier-Stokes or Euler equations (1), by following step by step

the proof of Lemma 4 given in [30] but keeping the term $\|u(t)\|_{L^2(\mathbb{R}^3)}$ after using the Cauchy-Schwarz inequality, we obtain for all $t \in [0, T^*[$,

$$\begin{aligned} \|u(t)\|_\infty &\lesssim \|u(t)\|_2^{\frac{2}{5}} \|\omega(t)\|_\infty^{\frac{3}{5}} \\ &\leq \|u_0\|_2^{\frac{2}{5}} \|\omega(t)\|_\infty^{\frac{3}{5}}, \end{aligned}$$

where we have used (15) for the last inequality. Then, we will obtain that $\|u(t_0)\|_\infty = 0$ which will imply that $u(t_0) = 0$. Then by using the inequality (18) of local regularity, we will deduce $u(t) = 0$ for all $t \in [t_0, T^*[$ and this will allows us to conclude the proof.

Let $0 < T < T^*$. We want to apply Lemma 5.1 to the function ω, then we check that the hypotheses of the Lemma are satisfied.
Since $u \in C([0,T]; H^r(\mathbb{R}^d))$, then we get $\omega \in C([0,T]; H^{r-1}(\mathbb{R}^d))$. Thanks to the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow BC^2(\mathbb{R}^d)$ for $r > \dfrac{d}{2} + 3$, we deduce that $\omega \in C([0,T]; BC^2(\mathbb{R}^d))$. Then thanks to (33), we infer that $\inf\limits_{t\in[0,T]} \|\omega(t)\|_\infty > 0$. Moreover, since $\omega \in C([0,T]; H^{r-1}(\mathbb{R}^d))$ with $r > \dfrac{d}{2} + 3$, we have for any $t \in [0,T]$, $|\omega(x,t)| \to 0$ as $|x| \to +\infty$, the proof follows immediately by using the density of $C_0^\infty(\mathbb{R}^d)$ in $H^{r-1}(\mathbb{R}^d)$ and the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $r > \dfrac{d}{2} + 3$.

Then thanks to Lemma 5.1, there exists $R > 0$ such that for all $t \in [0,T]$, $\|\omega(t)\|_\infty = \sup\limits_{x\in B_R} |\omega(x,t)|$. Then for all $t \in [0,T]$, the set $\Omega(t)$ given by (32) can be rewritten as follows:

$$\Omega(t) = \{x \in B_R; |\omega(x,t)| = \|\omega(t)\|_\infty\}.$$

We introduce the direction of the vorticity $\xi = \dfrac{\omega}{|\omega|}$ defined on the non-empty open set $\mathcal{O} = \{(x,t) \in \mathbb{R}^d \times [0,T]; |\omega(x,t)| > 0\}$.

We set $\boldsymbol{v} = u$ in the case of Navier-Stokes or Euler equations and $\boldsymbol{v} = -R^\perp u$ with $\nu = 0$ in the case of 2D QG equation.

Then by multiplying (11) or (13) by ξ, we get that for all $(x,t) \in \mathcal{O}$,

$$\begin{aligned} \frac{\partial|\omega|}{\partial t}(x,t) + \boldsymbol{v}(x,t)\cdot\nabla|\omega|(x,t) \quad & -(\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot\xi(x,t) \\ & -\nu\Delta|\omega|(x,t) + \nu|\omega(x,t)||\nabla\xi(x,t)|^2 = 0. \end{aligned} \tag{34}$$

We introduce the function φ defined for all $t \in [0,T]$ by $\varphi(t) \overset{\text{def}}{=} \sup\limits_{x\in B_R} |\omega(x,t)|$ and we search the expression of its derivative. For this, we will use the main Theorem obtained in [93] or Theorem 1 in [4] and then we have to check that the hypotheses of the Theorem are satisfied.

Since $\omega \in C([0,T];BC^2(\mathbb{R}^d))$, then from (11) and (13), we get $\dfrac{\partial \omega}{\partial t} \in C([0,T];BC(\mathbb{R}^d))$. Therefore, we infer that $|\omega| \in BC(\mathcal{O})$ and $\dfrac{\partial |\omega|}{\partial t} \in BC(\mathcal{O})$. Since for any $t \in [0,T]$, $\Omega(t) \subset \mathcal{O}$, then, thanks to the results obtained in [93] (see also Theorem 1 in [4]), we obtain the expression of the derivative of φ given for any $t \in [0,T]$ by,

$$\varphi'(t) = \sup_{x \in \Omega(t)} \frac{\partial |\omega|}{\partial t}(x,t). \tag{35}$$

Further for all $x \in \Omega(t) \subset B_R$, we have $|\omega(x,t)| = \varphi(t) = \|\omega(t)\|_\infty$, we thus infer that

$$\nabla |\omega|(x,t) = 0 \text{ and } \Delta |\omega|(x,t) \leq 0. \tag{36}$$

Therefore, we have for all $x \in \Omega(t)$,

$$\begin{aligned} \frac{\partial |\omega|}{\partial t}(x,t) &= \frac{\partial |\omega|}{\partial t}(x,t) + \boldsymbol{v}(x,t) \cdot \nabla |\omega|(x,t) \\ &= (\omega(x,t) \cdot \nabla)\boldsymbol{v}(x,t) \cdot \xi(x,t) + \nu \Delta |\omega|(x,t) - \nu |\omega(x,t)||\nabla \xi(x,t)|^2 \\ &\leq (\omega(x,t) \cdot \nabla)\boldsymbol{v}(x,t) \cdot \xi(x,t), \end{aligned} \tag{37}$$

where we have used (34) for the second equality and (36) for the last inequality. We can notice that we get equality for (37) in the case of Euler or 2D QG equations, since for these equations we have not the terms $\nu \Delta |\omega|(x,t)$ and $\nu |\omega(x,t)||\nabla \xi(x,t)|^2$.

Then using (37), from (35), we obtain,

$$\varphi'(t) \leq \sup_{x \in \Omega(t)} (\omega(x,t) \cdot \nabla)\boldsymbol{v}(x,t) \cdot \xi(x,t),$$

which means that

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \sup_{x \in \Omega(t)} (\omega(x,t) \cdot \nabla)\boldsymbol{v}(x,t) \cdot \xi(x,t), \tag{38}$$

where equality holds in the case of Euler or 2D QG equations.
We use now the function α introduced in [19] for the Navier-Stokes or Euler equations and in [17] for the 2D QG equation, defined for all $(x,t) \in \mathcal{O}$ by,

$$\alpha(x,t) = c_d P.V. \int_{\mathbb{R}^d} D(\hat{y}, \xi(x+y,t), \xi(x,t)) \, |\omega(x+y,t)| \, \frac{dy}{|y|^d}, \tag{39}$$

where $\hat{y} = \dfrac{y}{|y|}$ and
In the case of Navier-Stokes or Euler equations c_d and D are given by

$$c_d = \frac{3}{4\pi},$$

and

$$D(a_1,a_2,a_3) = (a_1 \cdot a_3)\,\mathrm{Det}(a_1,a_2,a_3).$$

The Det in D is the determinant of the matrix whose columns are the three unit column vectors a_1, a_2, a_3. We observe that $\mathrm{Det}(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$, then, we get

$$D(a_1, a_2, a_3) = (a_1 \cdot a_3)\ a_1 \cdot (a_2 \times a_3). \tag{40}$$

In the case of 2D QG equations, c_d and D are given by

$$c_d = 1,$$

and

$$D(a_1, a_2, a_3) = (a_1 \cdot a_3^{\perp})\,(a_2 \cdot a_3^{\perp}), \tag{41}$$

where for any $z = (z_1, z_2) \in \mathbb{R}^2$, $z^{\perp} = (-z_2, z_1)$.

520 We can notice that D is linear from its second variable. From [19] and [17], we get that for all $x \in \mathcal{O}$

$$(\omega(x,t) \cdot \nabla)\boldsymbol{\upsilon}(x,t)\ \cdot \xi(x,t) = \alpha(x,t)|\omega(x,t)|.$$

Therefore, from (38), we deduce,

$$\begin{aligned} \frac{d}{dt}\|\omega(t)\|_{\infty} &\leq \sup_{x\in\Omega(t)} \alpha(x,t)|\omega(x,t)| \\ &= \left(\sup_{x\in\Omega(t)} \alpha(x,t)\right)\|\omega(t)\|_{\infty}, \end{aligned} \tag{42}$$

where we have used the fact that for all $x \in \Omega(t)$, $|\omega(x,t)| = \|\omega(t)\|_{\infty}$.

We take a smooth cutoff function $\chi : [0, +\infty[\longmapsto [0,1]$ such that $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. We introduce the function γ defined for all $t \in [0, T^*[$ by $\gamma(t) = \dfrac{1}{4}\min\left(\dfrac{1}{\|\omega(t)\|_{\infty}}, \rho(t)\right)$.
Let us estimate now $|\alpha(x,t)|$ for any $t \in [0,T]$ and $x \in \Omega(t)$. We bound $|\alpha(x,t)|$ by the sum of three terms

$$|\alpha(x,t)| \leq |I_1| + |I_2| + |I_3| \tag{43}$$

with,

$$I_1 = c_d \int_{\mathbb{R}^d} \chi\left(\frac{|y|}{\gamma(t)}\right) D(\hat{y}, \xi(x+y,t), \xi(x,t))|\omega(x+y,t)|\,\frac{dy}{|y|^d}, \tag{44}$$

$$I_2 = c_d \int_{\mathbb{R}^d} \left(1 - \chi\left(\frac{|y|}{\gamma(t)}\right)\right) \chi\left(\frac{8|y|}{\rho(t)}\right) D(\hat{y}, \xi(x+y,t), \xi(x,t))|\omega(x+y,t)|\,\frac{dy}{|y|^d} \tag{45}$$

and

$$I_3 = c_d \int_{\mathbb{R}^d} \left(1 - \chi\left(\frac{|y|}{\gamma(t)}\right)\right)\left(1 - \chi\left(\frac{8|y|}{\rho(t)}\right)\right) D(\hat{y}, \xi(x+y,t), \xi(x,t))|\omega(x+y,t)|\,\frac{dy}{|y|^d}. \tag{46}$$

Then, we estimate each term I_1, I_2, I_3. From (40) and (41), we observe that for all $y \in B\left(0, \dfrac{\rho(t)}{2}\right)$

$$|D(\hat{y}, \xi(x+y,t), \xi(x,t))| \leq |\xi(x+y,t) - \xi(x,t)| \leq \|\nabla \xi(t)\|_{L^\infty(B(x, \frac{\rho(t)}{2}))} |y|. \quad (47)$$

Since $B(0, 2\gamma(t)) \subset B(0, \frac{\rho(t)}{2})$, then using (47), for the first term I_1, we obtain,

$$\begin{array}{rl} |I_1| & \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x, \frac{\rho(t)}{2}))} \|\omega(t)\|_\infty \gamma(t) \\ & \leq \|\nabla \xi(t)\|_{L^\infty(B(x, \rho(t)))}. \end{array} \quad (48)$$

Furthermore, we have $|\omega(x+y,t)| = \omega(x+y,t) \cdot \xi(x+y,t)$ and $\omega(x+y,t) = \operatorname{curl}_y u(x+y,t)$. Then the second term I_2 can be rewritten under the form

$$I_2 = c_d \int_{\mathbb{R}^d} \psi(x,y,t) \cdot \operatorname{curl}_y u(x+y,t)\, dy$$

where

$$\psi(x,y,t) = \frac{1}{|y|^d} \left(1 - \chi\left(\frac{|y|}{\gamma(t)}\right)\right) \chi\left(\frac{8|y|}{\rho(t)}\right) D(\hat{y}, \xi(x+y,t), \xi(x,t))\, \xi(x+y,t).$$

To get a fine estimate of I_2, we introduce a dyadic unity partition of Littlewood-Paley decomposition. Let $\Psi : [0, +\infty[\mapsto [0,1]$ be a smooth function such that $\Psi \equiv 1$ on $[0,1]$, $\Psi \equiv 0$ on $[2, +\infty[$ and Ψ is strictly decreasing on $[1,2]$. For $r \geq 0$, let $\Phi(r) := \Psi(r) - \Psi(2r)$, so that Φ is a smooth bump function supported on $[1/2, 2]$ satisfying for any $r > 0$, $\displaystyle\sum_{j \in \mathbb{Z}} \Phi(2^{-j} r) = 1$. We thus introduce the radial maps $\phi : \mathbb{R}^d \mapsto [0,1]$ defined for all $y \in \mathbb{R}^d$ by $\phi(y) = \Phi(|y|)$. Then ϕ is an non-negative smooth radial function compactly supported in $\{y \in \mathbb{R}^d; 1/2 \leq |y| \leq 2\}$ such that for all $y \in \mathbb{R}^d \backslash \{0\}$, $\displaystyle\sum_{j \in \mathbb{Z}} \phi(2^{-j} y) = 1$.

For any $j \in \mathbb{Z}$, we introduce the annular ring $\mathcal{C}_j = \{y \in \mathbb{R}^d; 2^{j-1} \leq |y| \leq 2^{j+1}\}$ and we set $\phi_j = \phi(2^{-j} \cdot)$. We notice that ϕ_j is supported on $\mathcal{C}_j$. We introduce also the annular ring $\mathcal{C}(t) = \left\{y \in \mathbb{R}^d; \gamma(t) \leq |y| \leq \dfrac{\rho(t)}{4}\right\}$. We can notice that

$\psi(x,\cdot,t)$ is supported on $\mathcal{C}(t)$. Then, we observe,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\phi_j(y)|\psi(x,y,t)|\,|\mathrm{curl}_y\,u(x+y,t)|\,dy &\leq \|\omega(t)\|_\infty\sum_{j\in\mathbb{Z}}\int_{\mathcal{C}(t)}\phi_j(y)\frac{dy}{|y|^d}\\ &\leq \frac{\|\omega(t)\|_\infty}{\gamma(t)^d}\sum_{j\in\mathbb{Z}}\int_{\mathcal{C}(t)}\phi_j(y)\,dy\\ &= \frac{\|\omega(t)\|_\infty}{\gamma(t)^d}\int_{\mathcal{C}(t)}\sum_{j\in\mathbb{Z}}\phi_j(y)\,dy\\ &= \frac{\|\omega(t)\|_\infty}{\gamma(t)^d}\int_{\mathcal{C}(t)}dy\\ &\leq \|\omega(t)\|_\infty\left(\frac{\rho(t)}{4\gamma(t)}\right)^d|B(0,1)|.\end{aligned}$$

For any $t\in[0,T]$, we infer $\displaystyle\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\phi_j(y)|\psi(x,y,t)|\,|\mathrm{curl}_y\,u(x+y,t)|\,dy<+\infty$. Hence, we can write I_2 as follows

$$I_2=c_d\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\phi_j(y)\psi(x,y,t)\cdot\mathrm{curl}_y\,u(x+y,t)\,dy.$$

For any $j\in\mathbb{Z}$, we introduce $u_j(x,t)=\dfrac{1}{|\mathcal{C}_j|}\displaystyle\int_{\mathcal{C}_j}u(x+y,t)$. Then I_2 can be rewritten as follows,

$$I_2=c_d\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\phi_j(y)\psi(x,y,t)\cdot\mathrm{curl}_y\left(u(x+y,t)-u_j(x,t)\right)dy.$$

Then, by using an integration by part and after elementary computations where we use the fact that $\chi'(r)$ is compactly supported in $[1,2]$, we deduce

$$|I_2|\lesssim\sum_{j\in\mathbb{Z}}\int_{\mathcal{A}_j(t)}f_j(x,y,t)\,|u(x+y,t)-u_j(x,t)|dy,\tag{49}$$

where,

$$f_j(x,y,t)=\left(\frac{2^{-j}}{|y|^d}+\frac{1}{|y|^{d+1}}\right)\mathcal{P}(\xi(x+y,t),\xi(x,t))+\frac{|\nabla\xi(x+y,t)|}{|y|^d},\tag{50}$$

$\mathcal{A}_j(t)=\mathcal{C}_j\cap\mathcal{C}(t)$ and $\mathcal{P}(\xi(x+y,t),\xi(x,t))=|\xi(x+y,t)\times\xi(x,t)|$ in the case of Navier-Stokes or Euler equations and $\mathcal{P}(\xi(x+y,t),\xi(x,t))=|\xi(x+y,t)\cdot\xi^{\perp}(x,t)|$ in the case of 2D QG equation. Further, we get for all $y\in B(0,\frac{\rho(t)}{4})$

$$\mathcal{P}(\xi(x+y,t),\xi(x,t))\leq|\xi(x+y,t)-\xi(x,t)|\leq\|\nabla\xi(t)\|_{L^\infty(B(x,\frac{\rho(t)}{4}))}\,|y|.\tag{51}$$

Then, owing to (51), from (50), we get for all $y \in B(0, \frac{\rho(t)}{4})$,

$$f_j(x,y,t) \leq \left(\frac{2^{-j}}{|y|^{d-1}} + \frac{2}{|y|^d} \right) \|\nabla \xi(t)\|_{L^\infty(B(x,\frac{\rho(t)}{4}))}. \tag{52}$$

Since for all $y \in \mathcal{C}_j$, we have $\dfrac{1}{|y|} \leq 2^{-(j-1)}$, then owing to (52), from 49 we deduce

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x,\rho(t)))} \sum_{j \in \mathbb{Z}} 2^{-jd} \int_{\mathcal{A}_j(t)} |u(x+y,t) - u_j(x,t)| dy. \tag{53}$$

However the set $\mathcal{A}_j(t)$ is empty if $\dfrac{\rho(t)}{4} < 2^{j-1}$ or if $\gamma(t) > 2^{j+1}$, which means that the set $\mathcal{A}_j(t)$ is empty if $\dfrac{\log(\frac{\rho(t)}{4})}{\log 2} < j-1$ or if $\dfrac{\log(\gamma(t))}{\log 2} > j+1$. Therefore, if the set $\mathcal{A}_j(t)$ is non-empty then we get $\dfrac{\log(\gamma(t))}{\log 2} - 1 \leq j \leq 1 + \dfrac{\log(\frac{\rho(t)}{4})}{\log 2}$. Then by introducing the interval $I(t) = \left[\dfrac{\log(\gamma(t))}{\log 2} - 1, \dfrac{\log(\frac{\rho(t)}{4})}{\log 2} + 1 \right]$, from (53) we obtain,

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x,\rho(t)))} \sum_{j \in I(t)} 2^{-jd} \int_{\mathcal{C}_j} |u(x+y,t) - u_j(x,t)| dy, \tag{54}$$

where we have used also the fact that $\mathcal{A}_j(t) \subset \mathcal{C}_j$. On one hand, for any $j \in I(t)$, we have $\mathcal{C}_j \subset B(0, \rho(t))$, then we get

$$2^{-jd} \int_{\mathcal{C}_j} |u(x+y,t) - u_j(x,t)| dy \leq \|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t),\rho(t))}. \tag{55}$$

On the other hand, thanks to Poincare-Wirtinger inequality, we get,

$$\begin{aligned} 2^{-jd} \int_{\mathcal{C}_j} |u(x+y,t) - u_j(x,t)| dy &\lesssim 2^j \left(2^{-jd} \int_{\mathcal{C}_j} |\nabla u(x+y,t)| dy \right) \\ &\lesssim 2^j \|\nabla u(t)\|_\infty. \end{aligned} \tag{56}$$

On one hand, thanks to (55), from (54) we have,

$$|I_2| \lesssim \|\nabla \xi(t)\|_{L^\infty(B(x,\rho(t)))} \|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t),\rho(t))} \sum_{j \in I(t)} 1, \tag{57}$$

moreover we notice that

$$\begin{aligned} \sum_{j \in I(t)} 1 &\leq 3 + \frac{\log(\frac{\rho(t)}{4\gamma(t)})}{\log 2} \\ &\leq \max\left(3, \frac{1}{\log 2}\right) (1 + \log^+(\rho(t)\|\omega(t)\|_\infty)). \end{aligned} \tag{58}$$

On the other hand, thanks to (56), from (54) we have

$$|I_2| \lesssim \|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))}\|\nabla u(t)\|_\infty \sum_{j\in I(t)} 2^j, \tag{59}$$

moreover we notice that

$$\begin{aligned}\sum_{j\in I(t)} 2^j &\leq \sum_{j\leq \frac{\log(\frac{\rho(t)}{4})}{\log 2}+1} 2^j \\ &= \sum_{\ell\geq -\frac{\log(\frac{\rho(t)}{4})}{\log 2}-1} 2^{-\ell} \\ &\leq 2\times 2^{\frac{\log(\frac{\rho(t)}{4})}{\log 2}+1} = \rho(t).\end{aligned} \tag{60}$$

Then, from (57)-(60) we obtain,

$$|I_2| \lesssim \|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))}J_0(t), \tag{61}$$

with $J_0(t) = \min(\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t),\rho(t))}(1+\log^+(\rho(t)\|\omega(t)\|_\infty), \rho(t)\|\nabla u(t)\|_\infty)$. Since $0<\rho(t)\leq\rho_0$ with $\rho_0 = \|\rho\|_{L^\infty([0,T^*])}$, then from (61), we deduce that,

$$|I_2| \lesssim \|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))}J_1(t), \tag{62}$$

with $J_1(t) = \min(\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t),\rho(t))}(1+\log^+\|\omega(t)\|_\infty), \rho(t)\|\nabla u(t)\|_\infty)$, where ρ_0 have been fixed as a constant.
For the last term I_3, we use the fact that $|\omega(x+y,t)|\xi(x+y,t) = \omega(x+y,t)$ and the fact that D is linear in comparison with its second variable, then we rewrite I_3 as follows

$$I_3 = c_d\int_{\mathbb{R}^d}\Lambda(x,y,t)\,dy,$$

where $\Lambda(x,y,t) = \left(1-\chi\left(\frac{|y|}{\gamma(t)}\right)\right)\left(1-\chi\left(\frac{8|y|}{\rho(t)}\right)\right)D(\hat{y},\omega(x+y,t),\xi(x,t))\,\frac{1}{|y|^d}$.

We observe that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\phi_j(y)\,|\Lambda(x,y,t)|\,dy &\leq \sum_{j\in\mathbb{Z}}\int_{|y|\geq\frac{\rho(t)}{8}}\phi_j(y)\,\frac{|\omega(x+y,t)|}{|y|^d}\,dy \\ &= \int_{|y|\geq\frac{\rho(t)}{8}}\sum_{j\in\mathbb{Z}}\phi_j(y)\,\frac{|\omega(x+y,t)|}{|y|^d}\,dy \\ &= \int_{|y|\geq\frac{\rho(t)}{8}}\frac{|\omega(x+y,t)|}{|y|^d}\,dy \\ &\lesssim \frac{\|\omega(t)\|_2}{\rho(t)^{\frac{d}{2}}},\end{aligned}$$

where we have used Cauchy-Schwarz inequality to obtain the last inequality. Then, we infer that for any $t \in [0,T]$, $\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y)\,|\Lambda(x,y,t)|\,dy < +\infty$.

Then, we can write I_3 as follows

$$I_3 = c_d \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \phi_j(y)\Big(1-\chi\left(\frac{|y|}{\gamma(t)}\right)\Big)\Big(1-\chi\left(\frac{8|y|}{\rho(t)}\right)\Big) D(\hat{y},\omega(x+y,t),\xi(x,t))\frac{dy}{|y|^d}. \tag{63}$$

However $\omega(x+y,t) = \mathrm{curl}_y\, u(x+y,t) = \mathrm{curl}_y\,(u(x+y,t)-u_j(x,t))$, then from (63) by using an integration by part and after elementary computations where we use the fact that $\chi'(r)$ is compactly supported in $[1,2]$, we deduce

$$|I_3| \lesssim \sum_{j\in\mathbb{Z}} \int_{\mathcal{B}_j(t)} \left(\frac{2^{-j}}{|y|^d} + \frac{1}{|y|^{d+1}}\right)|u(x+y,t)-u_j(x,t)|dy, \tag{64}$$

where $\mathcal{B}_j(t) = \{y \in \mathcal{C}_j; |y| \geq \frac{\rho(t)}{8}\}$. Then from (64), we deduce

$$\begin{aligned} |I_3| &\lesssim \sum_{j\in\mathbb{Z}, 2^{j-1}\geq\frac{\rho(t)}{8}} 2^{-j}\, 2^{-dj} \int_{\mathcal{C}_j} |u(x+y,t)-u_j(x,t)|dy \\ &\leq \|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t))} \sum_{j\in\mathbb{Z}, 2^{j-1}\geq\frac{\rho(t)}{8}} 2^{-j} \\ &\leq \frac{8\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t))}}{\rho(t)}. \end{aligned} \tag{65}$$

We set

$$\begin{aligned} a(t) &\stackrel{\mathrm{def}}{=} \sup_{x\in\Omega(t)} \|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))}, \\ b(t) &\stackrel{\mathrm{def}}{=} \min\left(\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t),\rho(t))}, \frac{\rho(t)\|\nabla u(t)\|_\infty}{1+\log^+\|\omega(t)\|_\infty}\right) \\ c(t) &\stackrel{\mathrm{def}}{=} \frac{\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t))}}{(1+\log^+\|\omega(t)\|_\infty)\rho(t)}. \end{aligned} \tag{66}$$

Then, owing to (48), (62) and (65), from (43) we deduce that there exists a constant $C_1 > 0$ such that for any $t \in [0,T]$ and $x \in \Omega(t)$,

$$|\alpha(x,t)| \leq C_1(a(t)(1+b(t))+c(t))(1+\log^+\|\omega(t)\|_\infty). \tag{67}$$

Then from (42), using (67), we deduce that for all $t \in [0,T]$,

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq C_1(a(t)+a(t)b(t)+c(t))(1+\log^+\|\omega(t)\|_\infty)\|\omega(t)\|_\infty. \tag{68}$$

After applying Gronwall inequality to (68), we obtain that for all $t \in [0,T]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp\left(C_1\int_0^t (a(\tau)+a(\tau)b(\tau)+c(\tau))(1+\log^+\|\omega(\tau)\|_\infty)\,d\tau\right). \tag{69}$$

We take the Logarithm of inequality (69), to obtain

$$\log \|\omega(t)\|_\infty \leq \log \|\omega_0\|_\infty + C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau))(1 + \log^+ \|\omega(\tau)\|_\infty)\, d\tau. \tag{70}$$

Then, from (70) we deduce that for all $t \in [0, T]$,

$$\begin{aligned} 1 + \log^+ \|\omega(t)\|_\infty \quad &\leq 1 + \log^+ \|\omega_0\|_\infty \\ &+ C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau))(1 + \log^+ \|\omega(\tau)\|_\infty)\, d\tau. \end{aligned} \tag{71}$$

Then thanks to Gronwall inequality, we deduce that for all $t \in [0, T]$,

$$1 + \log^+ \|\omega(t)\|_\infty \leq (1 + \log^+ \|\omega_0\|_\infty) e^{C_1 \int_0^t (a(\tau) + a(\tau)b(\tau) + c(\tau))\, d\tau}. \tag{72}$$

Plugging (72) into (69), we obtain

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp\left((1 + \log^+ \|\omega_0\|_\infty) C_1 \int_0^t \psi(\tau) e^{C_1 \int_0^\tau \psi(s)\, ds}\, d\tau \right). \tag{73}$$

where for all $t \in [0, T]$,

$$\psi(t) = a(t) + a(t)b(t) + c(t). \tag{74}$$

Further, we have

$$C_1 \int_0^t \psi(\tau) e^{C_1 \int_0^\tau \psi(s)\, ds}\, d\tau = e^{C_1 \int_0^t \psi(s)\, ds} - 1. \tag{75}$$

By plugging (75) into (73), we deduce that for all $t \in [0, T]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{-(1 + \log^+ \|\omega_0\|_\infty)} \exp\left((1 + \log^+ \|\omega_0\|_\infty) e^{C_1 \int_0^t \psi(s)\, ds} \right). \tag{76}$$

We notice that $\|\omega_0\|_\infty e^{-(1 + \log^+ \|\omega_0\|_\infty)} \leq 1$. Then from (76), we get for all $t \in [0, T]$,

$$\|\omega(t)\|_\infty \leq \exp\left((1 + \log^+ \|\omega_0\|_\infty) e^{C_1 \int_0^t \psi(s)\, ds} \right), \tag{77}$$

which is valid for all $0 < T < T^*$. Since by assumption $\displaystyle\int_0^{T^*} c(t)\, dt < +\infty$, then owing to (74) and (77), we deduce that there exists a real $C_2 > 0$ such that for all $t \in [0, T^*[$

$$\|\omega(t)\|_\infty \leq \exp\left(C_2 (1 + \log^+ \|\omega_0\|_\infty) e^{C_2 \int_0^t (a(s) + a(s)b(s))\, ds} \right),$$

which concludes the proof. □

Now, let us deal with the geometric properties for non-blow-up of 2D QG, Navier-Stokes and Euler equations obtained as an immediate consequence of Theorem 5.1 combined with the BKM blow-up criteria (16) and (20).

5.2. Geometric property for non blow-up of 2D QG equation

In this subsection, we give a geometric property for non blow-up of 2D QG equation based on the regularity of the direction of the vorticity. Then in the remark 5.1, we revisit the hyperbolic saddle scenario for the finite time blow-up of 2D QG equation introduced in [17].

We want to use Theorem 5.1 for the 2D QG equation. Then, let $u_0 \in H^r(\mathbb{R}^2)$ with $r > 4$ and $T^* > 0$ be such that there exists $u \in C([0, T^*[; H^r(\mathbb{R}^2))$ solution to the 2D QG equation (3)-(4). Thanks to (19) used with $p = \infty$, we get that for all $t \in [0, T^*[$,

$$\|u(t)\|_\infty = \|u_0\|_\infty. \tag{78}$$

Let us assume that u blows up at the finite time T^*. Then, we get that for all $t \in [0, T^*[$, $\|\omega(t)\|_\infty > 0$ otherwise by using the arguments of (33), we will get that there exists $t_0 \in [0, T^*[$ such that for all $t \in [t_0, T^*[$, $u(t) = 0$ and then no blow up will be possible at T^*. Then thanks to Lemma 4.2, we infer that for all $t \in [0, T^*[$, $\|\omega(t)\|_\infty \geq \dfrac{c_0}{T^* - t}$, with $c_0 > 0$ a real. Then, we infer for all $t \in [0, T^*[$, $1 + \log^+ \|\omega(t)\|_\infty \geq 1 + \log^+ \left(\dfrac{c_0}{T - t}\right)$. We observe that for all $x \geq 0$ and $\alpha > 0$, $\frac{1+\alpha}{\alpha}(1 + \log^+ x) \geq 1 + \log(1 + \alpha x)$, then with $x = \frac{c_0}{T-t}$ and $\alpha = \frac{1}{c_0}$, we infer that $1 + \log^+ \left(\frac{c_0}{T-t}\right) \geq \frac{1}{1+c_0} \left(1 + \log\left(1 + \frac{1}{T^*-t}\right)\right)$. Then we get for all $t \in [0, T^*[$,

$$1 + \log^+ \|\omega(t)\|_\infty \geq \frac{1}{1 + c_0} \left(1 + \log\left(1 + \frac{1}{T^* - t}\right)\right). \tag{79}$$

Now, let us take a continuous function ρ from $[0, T^*]$ to $[0, +\infty[$ such that ρ positive on $[0, T^*[$ and $\displaystyle\int_0^{T^*} \frac{1}{(1 + \log(1 + \frac{1}{T^*-t}))\rho(t)}\, dt < +\infty$. Then, owing to (78) and (79), we deduce that

$$\begin{aligned} \int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t))}}{(1 + \log^+ \|\omega(t)\|_\infty)\rho(t)}\, dt &\leq 2 \int_0^{T^*} \frac{\|u(t)\|_\infty}{(1 + \log^+ \|\omega(t)\|_\infty)\rho(t)}\, dt \\ &\leq \int_0^{T^*} \frac{2\|u_0\|_\infty}{(1 + \log(1 + \frac{1}{T^*-t}))\rho(t)}\, dt < +\infty. \end{aligned} \tag{80}$$

Let b be the function introduced in (32), then owing (78), we get for all $t \in [0, T^*[$, $b(t) \leq 2\|u_0\|_\infty$. Then, owing to (80), thanks to Theorem 5.1 combined with the BKM blow-up criterion (20), we obtain the following Corollary

Corollary 5.1. *Let $u_0 \in H^r(\mathbb{R}^2)$ with $r > 4$. Let $T^* > 0$ be such that there exists $u \in C([0, T^*[; H^r(\mathbb{R}^2))$ solution to the 2D QG equation (3)-(4). Let ρ be a continuous function from $[0, T^*]$ to $[0, +\infty[$ such that ρ positive on $[0, T^*[$ and $\displaystyle\int_0^{T^*} \frac{dt}{(1 + \log(1 + \frac{1}{T^*-t}))\rho(t)} < +\infty$. If we have*

$$\int_0^{T^*} a(t)\, dt < +\infty, \tag{81}$$

then the solution u cannot blow up at the finite time T^* which means that $u \in C([0,T^*];H^r(\mathbb{R}^2))$, where for all $t \in [0,T^*[$

$$a(t) \stackrel{\text{def}}{=} \sup_{x\in\Omega(t)} \|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))} \tag{82}$$

and $\Omega(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2; |\omega(x,t)| = \|\omega(t)\|_\infty\}$. Moreover we have for all $t \in [0,T^*]$

$$\|\omega(t)\|_\infty \leq \exp\left(c(1+\log^+\|\omega_0\|_\infty)e^{c(1+\|u_0\|_\infty)\int_0^t a(s)\,ds}\right), \tag{83}$$

where $c > 0$ is a constant.

Remark 5.1. *For Corollary 5.1, we can take $\rho(t) = (T^*-t)\log\left(1+\frac{1}{T^*-t}\right)^\alpha$ with $\alpha > 0$. Then, we can observe that our non blow-up criterion (81) for the 2D QG equation is an improvement of the one given in [17] and also an improvement of the one given in [31]. Further, the numerical experiments performed in [31] on the hyperbolic saddle test case (a candidate for finite-time blow-up suggested in [17]), indicate that the region of large vorticity ω in magnitude and the region of large $\nabla\xi$ in magnitude are disjoints (see Fig.2-Fig.5 in [31]) and shrink to zero as the time tends to the alleged time of singularity. This suggests that $\nabla\xi$ remains bounded in regions where $|\omega|$ achieves its maximum and these regions shrink to zero as the time tends to the alleged time of singularity T^*. Thus, if these regions are contained in balls of radius of type $O(\rho(t))$ then we obtain that $a(t)$ remains bounded up to the alleged time of singularity T^* and therefore from Corollary 5.1, we will obtain that the solution u cannot blow up at the time T^*. From (83), we will infer also that the maximum of the vorticity $\|\omega(t)\|_\infty$ is bounded by double exponential, which is consistent with the recent numerical experiments [88, 21, 22, 25].*

5.3. Geometric property for non blow-up of Navier-Stokes and Euler equations

In this subsection, we give a geometric property for non blow-up of Navier-Stokes and Euler equations based on the regularity of the velocity and the direction of the vorticity.

We want to use Theorem 5.1 for the Navier-Stokes and Euler equations. Then, let $u_0 \in H^r_\sigma(\mathbb{R}^3)$ with $r > \dfrac{9}{2}$ and $T^* > 0$ be such that there exists $u \in C([0,T^*[;H^r_\sigma(\mathbb{R}^3))$ solution to the Navier-Stokes equations and Euler (1)-(2). Let us assume that u blows up at the finite time T^*. Then, we notice that (79) holds also for Navier-Stokes or Euler equations and then we infer for all $t \in [0,T^*[$,

$$\frac{1}{1+\log^+\|\omega(t)\|_\infty} \lesssim \frac{1}{1+\log\left(1+\frac{1}{T^*-t}\right)}. \tag{84}$$

Then for any continuous function ρ from $[0,T^*]$ to $[0,+\infty[$ such that ρ positive

on $[0,T^*[$ and $\displaystyle\int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t))}}{(1+\log(1+\frac{1}{T^*-t}))\rho(t)}\,dt<+\infty$, we have

$$\int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t))}}{(1+\log^+\|\omega(t)\|_\infty)\rho(t)}\,dt \lesssim \int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t))}}{(1+\log(1+\frac{1}{T^*-t}))\rho(t)}\,dt<+\infty. \tag{85}$$

Then, owing to (85), thanks to Theorem 5.1 and the BKM criterion 16, we obtain the following Corollary.

Corollary 5.2. *Let $u_0\in H^r_\sigma(\mathbb{R}^3)$ with $r>\dfrac{9}{2}$. Let $T^*>0$ be such that there exists $u\in C([0,T^*[;H^r_\sigma(\mathbb{R}^3))$ solution to the Navier-Stokes and Euler equations (1)-(2). Let ρ be a continuous function from $[0,T^*]$ to $[0,+\infty[$ such that ρ positive on $[0,T^*[$ and $\displaystyle\int_0^{T^*} \frac{\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t))}}{(1+\log(1+\frac{1}{T^*-t}))\rho(t)}\,dt<+\infty$. If we have*

$$\int_0^{T^*} a(t)(1+b(t))\,dt<+\infty,$$

then the solution u cannot blow up at the finite time T^* which means that $u\in C([0,T^*];H^r_\sigma(\mathbb{R}^3)^3)$, where for all $t\in[0,T^*[$,

$$\begin{array}{ll} \Omega(t) & \stackrel{\rm def}{=}\{x\in\mathbb{R}^3;|\omega(x,t)|=\|\omega(t)\|_\infty\},\\ a(t) & \stackrel{\rm def}{=}\displaystyle\sup_{x\in\Omega(t)}\|\nabla\xi(t)\|_{L^\infty(B(x,\rho(t)))}\\ b(t) & \stackrel{\rm def}{=}\min\left(\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t),\rho(t))},\dfrac{\rho(t)\|\nabla u(t)\|_\infty}{1+\log^+\|\omega(t)\|_\infty}\right).\end{array} \tag{86}$$

Moreover, we have for all $t\in[0,T^*]$

$$\|\omega(t)\|_\infty\le\exp\left(c_0(1+\log^+\|\omega_0\|_\infty)e^{c_0\int_0^t a(s)(1+b(s))\,ds}\right),$$

where $c_0>0$ is a constant.

Remark 5.2. *Most of numerical experiments show that if u blows up at a finite time T^* then we get the following blow-up rates $\|\omega(t)\|_\infty\sim O(T^*-t)^{-1}$, $\|\nabla u(t)\|_\infty\sim O(T^*-t)^{-1}$, $\|\nabla\xi(t)\|_\infty\sim O(T^*-t)^{-\frac12}$, $\|u(t)\|_\infty\sim O(T^*-t)^{-\frac12}$ (see for instance [73]). Owing to these blow-up rates, we expect that the blow-up rate of $\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t))}$ be slightly smaller than the one of $\|u(t)\|_\infty$ such that we get*

$$\|u(t)\|_{\mathcal{BMO}_{\rm d}(\Omega(t))}\lesssim\frac{(T-t)^{-\frac12}}{(1+\log(1+\log(1+\frac{1}{T^*-t})))^{2\beta}}\ \text{with}\ \beta>1. \tag{87}$$

Indeed from the sharp version of logarithmic inequality (2.5) of [87] and thanks also to Bernstein's inequality, we infer that

$$\|u(t)\|_\infty\lesssim 1+\|u(t)\|_{\rm BMO(\mathbb{R}^3)}(\log^+(\|\omega(t)\|_\infty+\|u(t)\|_2))^{\frac12},$$

where the exponent $\frac{1}{2}$ *is optimal. By using the energy inequality (15), we get* $\|u(t)\|_\infty \lesssim 1 + \|u(t)\|_{\mathrm{BMO}(\mathbb{R}^3)}(\log^+(\|\omega(t)\|_\infty + \|u_0\|_2))^{\frac{1}{2}}$ *which implies that* $\|u(t)\|_\infty \lesssim (1+\log^+\|u_0\|_2)(1+\|u(t)\|_{\mathrm{BMO}(\mathbb{R}^3)}(1+\log^+\|\omega(t)\|_\infty)^{\frac{1}{2}})$. *Then we get that there exists a constant* $c_0 > 0$ *such that*

$$\frac{\|u(t)\|_\infty}{(1+\log^+\|\omega(t)\|_\infty)^{\frac{1}{2}}} \leq c_0(1+\|u(t)\|_{\mathrm{BMO}(\mathbb{R}^3)}). \tag{88}$$

Thanks to (84), we get also

$$\begin{array}{rcl} \dfrac{\|u(t)\|_\infty}{(1+\log^+\|\omega(t)\|_\infty)^{\frac{1}{2}}} & \lesssim & \dfrac{\|u(t)\|_\infty}{(1+\log(1+\frac{1}{T^*-t}))^{\frac{1}{2}}} \\ & \lesssim & \dfrac{(T^*-t)^{-\frac{1}{2}}}{(1+\log(1+\frac{1}{T^*-t}))^{\frac{1}{2}}}. \end{array} \tag{89}$$

Moreover $\mathrm{BMO}(\mathbb{R}^3) \hookrightarrow \mathcal{BMO}_{\mathrm{d}}(\mathbb{R}^3) \hookrightarrow \mathcal{BMO}_{\mathrm{d}}(\Omega(t))$. *Since the exponent* $\frac{1}{2}$ *is optimal in inequality (88) and thanks to Proposition 5.1 in [86] then we could expect that*

$$\|u(t)\|_{\mathcal{BMO}_{\mathrm{d}}(\Omega(t))} \lesssim \frac{(T^*-t)^{-\frac{1}{2}}}{(1+\log(1+\frac{1}{T^*-t}))^{\frac{1}{2}}}.$$

But, here we require only that (87) holds.

Let us assume that u *blows up at the finite time* T^* *with these blow-up rates which correspond to the Kerr's critical case studied by Deng et al. in [30, 32]. Then, owing to (87), from Corollary 5.2 we can take the function* ρ *as* $\rho(t) = \dfrac{(T^*-t)^{\frac{1}{2}}}{(1+\log(1+\log(1+\frac{1}{T^*-t})))^{\beta}}$. *From (86) of Corollary 5.2, we get* $a(t) \lesssim (T^*-t)^{-\frac{1}{2}}$ *and*

$$\begin{array}{rcl} b(t) & \leq & \dfrac{\rho(t)\|\nabla u(t)\|_\infty}{1+\log^+\|\omega(t)\|_\infty} \\ & \lesssim & \dfrac{(T^*-t)^{-\frac{1}{2}}}{(1+\log(1+\frac{1}{T^*-t}))(1+\log(1+\log(1+\frac{1}{T^*-t})))^{\beta}}. \end{array}$$

Since $\beta > 1$, *we thus infer that* $\displaystyle\int_0^{T^*} a(t)(1+b(t))\,dt < +\infty$ *and thanks to Corollary 5.2, we deduce that in fact* u *cannot blow up at the time* T^*.

Then, we have shown with our criterion that for the Kerr's critical case, we obtain the non blow-up of the solutions of Navier-Stokes or Euler equations provided that the very weak assumption (87) holds.

Remark 5.3. *From Proposition 5.1 in [86], we notice that our criterion given in Corollary 5.2 can reveal the non blow-up of solutions of Navier-Stokes or Euler equations even in some cases of possible blow-up of type II.*

6. Geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations evaluated only on the positions of maximum vorticity

In this section, we improve the geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations obtained in our Corollaries 5.1 and 5.2, but also the ones obtained in [30, 32]. Moreover, our geometric properties for non blow-up is given in an Eulerian setting.

Up to now, most of recent geometric constraints for non blow-up criteria of Navier-Stokes, Euler and 2D QG equations required the local regularity of the velocity and the direction of vorticity in all a region containing the position where maximum of vorticity is reached, this latter could shrink to zero as time increases. Therefore, these non blow-up criteria are limited due to the geometric regularity required in all a given region $\mathcal{R}(t)$ containing the positions where the maximum of vorticity is reached. Indeed it could happen that in the region $\mathcal{R}(t)$ we have not enough regularity for the velocity or the direction of vorticity to conclude non blow-up of the solutions of Navier-Stokes, Euler or 2D QG equations, but sufficiently regularity in a region strictly included in $\mathcal{R}(t)$ to prevent the non blow-up of these solutions.

Moreover, the recent geometric constraints for non blow-up criteria of Navier-Stokes, Euler and 2D QG equations based on the local geometric regularity of the Lagrangian vortex filaments [30, 32, 31] make the assumption that the position where the maximum of vorticity is reached, is advected with the flow, however it is not always the case, as described in [74] (see also section 5.4.5 of [49]).

Then, in our Theorem 6.1, we obtain again some improvements of the non blow-up criteria for Navier-Stokes, Euler and 2D QG equations. Indeed we obtain geometric properties for non blow-up of Navier-Stokes, Euler and 2D QG equations evaluated only on the positions where the maximum of vorticity are reached.

Moreover, in the case of Euler and 2D QG equations, the non blow-up criterion obtained is a necessary and sufficient condition in preventing finite time blow-up of the solutions.

Later, from our Theorem 6.1, we derive the Corollary 6.1 where we improve our non blow-up criterion logarithmically.

Theorem 6.1. *Let $d \in \{2, 3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \dfrac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1(]0, T^*[; H^{r-1}(\mathbb{R}^d)).$$

Moreover for all $t \in [0, T^*[$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s)\, ds},$$

where equality holds in the case of Euler and 2D QG equations and for all $t \in [0, T^*[$,

$$\begin{aligned} \Phi(t) &= \sup_{x\in\Omega(t)} -(\ \nabla\cdot\boldsymbol{v}_{\xi^\perp} + \boldsymbol{v}_{\xi^\perp}\cdot(\xi\cdot\nabla)\xi\ )(x,t), \\ \boldsymbol{v}_{\xi^\perp}(x,t) &= (\boldsymbol{v} - (\boldsymbol{v}\cdot\xi)\xi\,)(x,t) \\ \Omega(t) &= \{x\in\mathbb{R}^d; |\omega(x,t)| = \|\omega(t)\|_\infty\}, \end{aligned}$$

with $\boldsymbol{v} \equiv u$ *in the case of Navier-Stokes or Euler equations, and* $\boldsymbol{v} \equiv -R^\perp u$ *in the case of 2D QG equation.*

If we have,

$$\limsup_{t\to T^*} \int_0^t \Phi(s)\, ds < +\infty$$

then the solution u *cannot blow up at the finite time* T^*.

Proof. Thanks to the section 3, we have effectively that there exists a time of existence strictly positive $T^* > 0$ such that there exists a unique strong solution u to the Navier-Stokes, Euler or 2D QG equations (3)-(4) in the class

$$u \in C([0,T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1(]0,T^*[; H^{r-1}(\mathbb{R}^d)).$$

Similarly as for the proof of Theorem 5.1, we assume that for all $t \in [0, T^*[$, $\|\omega(t)\|_\infty > 0$.
We introduce the non-empty set $\mathcal{O} \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^d \times [0,T^*[; |\omega(x,t)| > 0\}$. We introduce also the direction of the vorticity $\xi = \dfrac{\omega}{|\omega|}$ defined on $\mathcal{O}$.

Let $0 < T < T^*$. From (38), we get that for all $t \in [0,T]$

$$\frac{d}{dt}\|\omega(t)\|_\infty \le \sup_{x\in\Omega(t)} (\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot\xi(x,t), \tag{90}$$

where equality holds in the case of Euler and 2D QG equations.
Since for any $(x,t) \in \mathcal{O}$, $(\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot\xi(x,t) = (\xi(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot \xi(x,t)|\omega(x,t)|$, then, from (90), we deduce that for all $t \in [0,T]$

$$\begin{aligned} \frac{d}{dt}\|\omega(t)\|_\infty &\le \sup_{x\in\Omega(t)} (\xi(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot\xi(x,t)\ |\omega(x,t)| \\ &= \left(\sup_{x\in\Omega(t)} (\xi(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot\xi(x,t)\right)\|\omega(t)\|_\infty, \end{aligned} \tag{91}$$

where we have used the fact that for all $x \in \Omega(t)$, $|\omega(x,t)| = \|\omega(t)\|_\infty$. We notice,

$$(\xi\cdot\nabla)\boldsymbol{v}\ \cdot\xi = \xi\cdot\nabla(\boldsymbol{v}\cdot\xi) - \boldsymbol{v}\cdot((\xi\cdot\nabla)\xi). \tag{92}$$

Since $\nabla\cdot\omega = 0$ and $\omega = |\omega|\xi$, then we get

$$|\omega|\nabla\cdot\xi + \xi\cdot\nabla|\omega| = 0. \tag{93}$$

However, for all $x \in \Omega(t)$, $|\omega(x,t)| = \|\omega(t)\|_\infty > 0$ and similarly as (36), we have $\nabla|\omega(x,t)| = 0$. Therefore, from (93), we deduce that for all $t \in [0,T]$ and $x \in \Omega(t)$,

$$\nabla \cdot \xi(x,t) = 0. \tag{94}$$

Owing to (94), we get that for all $t \in [0,T]$ and $x \in \Omega(t)$,

$$\xi(x,t) \cdot \nabla(\boldsymbol{v} \cdot \xi)(x,t) = \nabla \cdot ((\boldsymbol{v} \cdot \xi)\xi)(x,t). \tag{95}$$

We introduce $\boldsymbol{v}_{\xi^\perp}$ the normal component to ξ of the velocity $\boldsymbol{v}$

$$\boldsymbol{v}_{\xi^\perp} = \boldsymbol{v} - (\boldsymbol{v} \cdot \xi)\xi. \tag{96}$$

Notice that $\boldsymbol{v}_{\xi^\perp} \cdot \xi = 0$, which means that $\boldsymbol{v}_{\xi^\perp}$ belongs to $\xi^\perp$ the plan orthogonal to ξ. Since $\nabla \cdot \boldsymbol{v} = 0$, then, we deduce that $\nabla \cdot \boldsymbol{v}_{\xi^\perp} = -\nabla \cdot ((\boldsymbol{v} \cdot \xi)\xi)$. Therefore from (95), we get that for all $t \in [0,T]$ and $x \in \Omega(t)$,

$$\xi(x,t) \cdot \nabla(\boldsymbol{v} \cdot \xi)(x,t) = -\nabla \cdot \boldsymbol{v}_{\xi^\perp}(x,t). \tag{97}$$

Then using (92) and (97), we infer that for all $t \in [0,T]$ and $x \in \Omega(t)$,

$$(\xi(x,t) \cdot \nabla)\boldsymbol{v}(x,t) \cdot \xi(x,t) = -\nabla \cdot \boldsymbol{v}_{\xi^\perp}(x,t) - \boldsymbol{v}(x,t) \cdot ((\xi(x,t) \cdot \nabla)\xi(x,t)). \tag{98}$$

Since $\boldsymbol{v} = (\boldsymbol{v} \cdot \xi)\xi + \boldsymbol{v}_{\xi^\perp}$, then we observe

$$\begin{aligned} \boldsymbol{v} \cdot ((\xi \cdot \nabla)\xi) &= (\boldsymbol{v} \cdot \xi)\ (\xi \cdot (\xi \cdot \nabla)\xi) + \boldsymbol{v}_{\xi^\perp} \cdot ((\xi \cdot \nabla)\xi) \\ &= (\boldsymbol{v} \cdot \xi)\ \xi \cdot \nabla|\xi|^2 + \boldsymbol{v}_{\xi^\perp} \cdot ((\xi \cdot \nabla)\xi) \\ &= \boldsymbol{v}_{\xi^\perp} \cdot ((\xi \cdot \nabla)\xi), \end{aligned}$$

where for the last equality, we have used the fact that $\nabla|\xi|^2 = 0$ since $|\xi| = 1$. Then, from (98), we deduce that for all $t \in [0,T]$ and $x \in \Omega(t)$,

$$(\xi(x,t) \cdot \nabla)\boldsymbol{v}(x,t) \cdot \xi(x,t) = -\nabla \cdot \boldsymbol{v}_{\xi^\perp}(x,t) - \boldsymbol{v}_{\xi^\perp}(x,t) \cdot ((\xi(x,t) \cdot \nabla)\xi(x,t)). \tag{99}$$

Therefore using (99), from (91), we have for all $t \in [0,T]$,

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \left(\sup_{x \in \Omega(t)} -(\nabla \cdot \boldsymbol{v}_{\xi^\perp} + \boldsymbol{v}_{\xi^\perp} \cdot (\xi \cdot \nabla)\xi\ )(x,t) \right) \|\omega(t)\|_\infty. \tag{100}$$

Thanks to Gronwall inequality used in its differential form, we get for all $t \in [0,T]$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s)\, ds}, \tag{101}$$

where $\Phi(s) = \sup_{x \in \Omega(s)} -(\nabla \cdot \boldsymbol{v}_{\xi^\perp} + \boldsymbol{v}_{\xi^\perp} \cdot (\xi \cdot \nabla)\xi\ )(x,s)$. Notice that for (101) equality holds in the case of Euler and 2D QG equations. Since inequality (101) is valid for all $0 < T < T^*$, then we infer that for all $t \in [0,T^*[$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s)\, ds}, \tag{102}$$

which concludes the first part of the proof.

We assume now that $\limsup_{t\to T^*} \int_0^t \Phi(s)\,ds < +\infty$, then from (102), we get

$$\limsup_{t\to T^*} \|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp\left(\limsup_{t\to T^*} \int_0^t \Phi(s)\,ds\right) < +\infty.$$

However if u blows up at the time T^* then thanks to (16) and (20) we have $\int_0^{T^*} \|\omega(t)\|_\infty\,dt = +\infty$ which leads to a contradiction with the fact that $\limsup_{t\to T^*} \|\omega(t)\|_\infty < +\infty$. Then, we deduce that u cannot blow up at the time T^* which completes the proof.

□

From Theorem 6.1, we obtain the following Corollary,

Corollary 6.1. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the Navier-Stokes, Euler equations (1)-(2) or 2D QG equation (3)-(4) in the class*

$$u \in C([0,T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1(]0,T^*[; H^{r-1}(\mathbb{R}^d)).$$

Moreover for all $t \in [0,T^*[$,

$$\|\omega(t)\|_\infty \leq \exp\left((1+\log^+\|\omega_0\|_\infty)\exp\left(\int_0^t \frac{|\Phi(s)|}{1+\log^+\|\omega(s)\|_\infty}\,ds\right)\right),$$

where for all $t \in [0,T^*[$,

$$\begin{aligned}
\Phi(t) &= \sup_{x\in\Omega(t)} -(\ \nabla\cdot \boldsymbol{v}_{\xi^\perp} + \boldsymbol{v}_{\xi^\perp}\cdot(\xi\cdot\nabla)\xi\ )(x,t),\\
\boldsymbol{v}_{\xi^\perp}(x,t) &= (\boldsymbol{v} - (\boldsymbol{v}\cdot\xi)\xi\,)(x,t)\\
\Omega(t) &= \{x\in\mathbb{R}^d; |\omega(x,t)| = \|\omega(t)\|_\infty\},
\end{aligned}$$

with $\boldsymbol{v} \equiv u$ in the case of Navier-Stokes or Euler equations, and $\boldsymbol{v} \equiv -R^\perp u$ in the case of 2D QG equation.

Furthermore, if we have,

$$\int_0^{T^*} \frac{|\Phi(s)|}{1+\log^+\|\omega(s)\|_\infty}\,ds < +\infty$$

then the solution u cannot blow up at the finite time T^*.

Proof. Indeed thanks to Theorem 6.1, we get that for all $t \in [0,T^*[$,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \Phi(s)\,ds},$$

which implies that

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t |\Phi(s)|\, ds}. \tag{103}$$

We observe that we can rewrite inequality (103) as follows

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{\int_0^t \psi(s)(1+\log^+ \|\omega(s)\|_\infty)\, ds}. \tag{104}$$

where $\psi(s) = \frac{|\Phi(s)|}{1+\log^+ \|\omega(s)\|_\infty}$. We take the logarithm of inequality (104) to obtain,

$$\log(\|\omega(t)\|_\infty) \leq \log(\|\omega_0\|_\infty) + \int_0^t \psi(s)(1 + \log^+(\|\omega(s)\|_\infty))\, ds,$$

which yields to

$$1 + \log^+ \|\omega(t)\|_\infty \leq 1 + \log^+ \|\omega_0\|_\infty + \int_0^t \psi(s)(1 + \log^+ \|\omega(s)\|_\infty)\, ds. \tag{105}$$

Then thanks to Gronwall inequality used in its integral form, we deduce that for all $t \in [0, T^*[$,

$$1 + \log^+ \|\omega(t)\|_\infty \leq (1 + \log^+ \|\omega_0\|_\infty) e^{\int_0^t \psi(s)\, ds}. \tag{106}$$

By plugging inequality (106) into (104), we obtain,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{(1+\log^+ \|\omega_0\|_\infty) \int_0^t \psi(s) e^{\int_0^s \psi(\tau)\, d\tau}\, ds}. \tag{107}$$

However $\displaystyle\int_0^t \psi(s) e^{\int_0^s \psi(\tau)\, d\tau}\, ds = e^{\int_0^t \psi(\tau)\, d\tau} - 1$. Hence, from (107), we obtain,

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty e^{-(1+\log^+ \|\omega_0\|_\infty)} e^{(1+\log^+ \|\omega_0\|_\infty) e^{\int_0^t \psi(\tau)\, d\tau}\, ds}. \tag{108}$$

We notice that $\|\omega_0\|_\infty e^{-(1+\log^+ \|\omega_0\|_\infty)} \leq 1$, then from (108), we get

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega_0\|_\infty) e^{\int_0^t \psi(\tau)\, d\tau}\, ds}, \tag{109}$$

which concludes the first part of the proof. We assume now that $\displaystyle\int_0^{T^*} \psi(s)\, ds < +\infty$, then from (109), we get

$$\limsup_{t \to T^*} \|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega_0\|_\infty) e^{\int_0^{T^*} \psi(\tau)\, d\tau}\, ds} < +\infty.$$

However if u blows up at the time T^* then thanks to (16) and (20) we have $\displaystyle\int_0^{T^*} \|\omega(t)\|_\infty\, dt = +\infty$ which leads to a contradiction with the fact that $\limsup_{t \to T^*} \|\omega(t)\|_\infty < +\infty$. Then, we deduce that u cannot blow up at the time T^* which completes the proof.

□

7. Some arguments in favor of a double-exponential in time growth of the maximum vorticity

In this section, we give some heuristic arguments combined with Corollary (6.1) to show some evidence in favor of a double-exponential in time growth of the maximum vorticity in the cases of Euler and 2D QG equations. For this, we take T^* as the maximal time of existence of solutions. We assume that $T^* < +\infty$, which means that the solutions blow up at the time T^*, then thanks to Lemma 4.2, we infer that

$$\liminf_{t \to T^*} \|\omega(t)\|_\infty = +\infty. \tag{110}$$

We introduce $\kappa \overset{\text{def}}{=} |(\xi \cdot \nabla)\xi|$ which seen as the curvature of vortex lines (see [30, 31]).

We highlight now the stabilizing effect of the convection term by using the Lagrangian formulation of the vorticity equation (see [60][Section 2]). Indeed, when we consider the convection term together with the vortex stretching term of the vorticity equation, we preserve the Lagrangian structure of the solution $\omega(X(\alpha,\tau,t),t) = \nabla_\alpha X(\alpha,\tau,t)\omega(\alpha,\tau)$ [84, 15, 31] where $X(\alpha,\tau,t)$ is the particle path that passes α at time τ and defined by $\dfrac{\partial}{\partial t}X(\alpha,\tau,t) = \boldsymbol{v}(X(\alpha,\tau,t),t)$, $X(\alpha,\tau,\tau) = \alpha$. Therefore, we observe that the vorticity can increase in time only through the dynamic deformation of the Lagrangian flow map X. Due to the divergence-free property of the velocity field $\boldsymbol{v}$, the flow map is volume-preserving which means that $\mathrm{Det}(\nabla_\alpha X(\alpha,\tau,t)) \equiv 1$ (see [84]), thus as vorticity increases dynamically, the hyper-rectangle spanned by the d vectors, $\dfrac{\partial X}{\partial \alpha_i}$,$1 \leq i \leq d$ will experience severe deformation and become flattened dynamically.

Therefore at the region of high vorticity, a vortex tube will experience tremendous core deformation and become severely flattened which is in agreement with the numerical experiments [60, 57, 74], then thanks to (110) this suggests that the curvature $\kappa(x,t)$ for all $x \in \Omega(t)$ and $t \in [t_0, T^*[$ may remain uniformly bounded up to T^*, for some t_0 sufficiently close to T^*. The time $t_0 \in [0, T^*[$ is chosen also such that for all $t \in [t_0, T^*[$, $\|\omega(t)\|_\infty \geq \|\omega(t_0)\|_\infty$. Then, we expect that for all $t \in [t_0, T^*[$,

$$\sup_{x \in \Omega(t)} \kappa(x,t) \leq C_0, \tag{111}$$

where $C_0 > 0$ is a constant.

7.1. Heuristic arguments for Euler equations in favor of a double-exponential in time growth of the maximum vorticity $\|\omega(t)\|_\infty$

Let $x \in \Omega(t)$ and $t \in]t_0, T^*[$. Since the flow map X is a C^1-diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ (see [84]), there exists an unique $x_0 \in \mathbb{R}^3$ such that $x_0 =$

$X(x_0,t_0,t_0)$ and $x = X(x_0,t_0,t)$. Let $R(t) \overset{\text{def}}{=} O(\kappa^{-1}(x,t)) \ll \kappa^{-1}(x,t)$ (R depends also on x but for simplicity in the notation, we keep only t), $\kappa^{-1}(x,t)$ is in fact the radius of curvature of the vortex filament passing through x at the time t. Let $\mathcal{C}_{R(t)}(t_0)$ be a vortex tube at the time t_0 containing x_0 of length ϵ such that any of its cross sections is assumed to be similar to a disk of radius $O(R(t))$. For any $s \in [t_0,t]$, we consider the vortex tube $\mathcal{C}_{R(t)}(s) \overset{\text{def}}{=} X(\mathcal{C}_{R(t)}(t_0),t_0,s)$ with its length denoted $\ell(s)$ and whose any of its cross sections is assumed to be similar to a disk of radius $O(\delta(s))$ for some $\delta(s) > 0$. We observe that $x \in \mathcal{C}_{R(t)}(t)$ and due to the vortex stretching, we can assume that $\delta(t) \leq R(t)$.

Following numerical experiments (see [73]), the blow-up of the solution takes place progressively in a anisotropic collapsing region characterized at the time t by two length scales similar to $\delta(t)$ and $R(t)$. This region at the time t is identified to the vortex tube $\mathcal{C}_R(t) \cap B(x,R(t))$. Then, since the very high vorticities at the time t are concentrated in the vortex tube $\mathcal{C}_{R(t)}(t) \cap B(x,R(t))$, we make the assumption that the velocity at the time t in the vortex tube $\mathcal{C}_{R(t)}(t) \cap B(x,R(t))$ is mainly induced by itself.

Then, by using the arguments involved in section 13.2 pp. 322 in [37], from (13.25) of [37] with $\epsilon = R(t)$, we get that for all $y \in \mathcal{C}_{R(t)}(t) \cap B(x,R(t))$

$$|v_{\xi^\perp}(y,t)| \lesssim |\Gamma(t)|\,\kappa(x,t)(1+\log(R(t)/\delta(t))), \tag{112}$$

where $\Gamma(t)$ is the Kelvin's circulation obtained from any boundary of a cross section of the vortex tube $\mathcal{C}_{R(t)}(t)$. Let us give an estimate of $\dfrac{R(t)}{\delta(t)}$. Thanks to the conservation of the volume of the vortex tubes $\mathcal{C}_{R(t)}(s)$, $s \in [t_0,t]$ during the transport by the flow map X, we get $|\mathcal{C}_{R(t)}(t_0)| = |\mathcal{C}_{R(t)}(t)|$ and therefore $R(t)^3 \lesssim \delta(t)^2\ell(t)$. Moreover from [84, 15], we assume that the stretching of the vortex tube $\mathcal{C}_{R(t)}(t_0)$ experienced during the flow gives us $\dfrac{\ell(t)}{R(t)} \lesssim \dfrac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}$. Then we deduce that $R(t)/\delta(t) \lesssim \left(\dfrac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}\right)^{\frac{1}{2}}$. Then from (112), we get for all $y \in \mathcal{C}_{R(t)}(t) \cap B(x,R(t))$

$$|v_{\xi^\perp}(y,t)| \lesssim |\Gamma(t)|\,\kappa(x,t)\left(1+\log\left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}\right)\right). \tag{113}$$

Thanks to Kelvin's circulation Theorem, we get $|\Gamma(t)| = |\Gamma(t_0)| \lesssim R(t)\|u(t_0)\|_\infty$. Then, from (113), we get for all $y \in \mathcal{C}_{R(t)}(t) \cap B(x,R(t))$

$$|v_{\xi^\perp}(y,t)| \lesssim R(t)\|u(t_0)\|_\infty\,\kappa(x,t)\left(1+\log\left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}\right)\right). \tag{114}$$

Since $R(t) = O(\kappa^{-1}(x,t))$, then from (114) we deduce for all $y \in \mathcal{C}_{R(t)}(t) \cap B(x,R(t))$

$$|v_{\xi^\perp}(x,t)| \lesssim \|u(t_0)\|_\infty\left(1+\log\left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}\right)\right). \tag{115}$$

By assuming that,

$$|\nabla \cdot v_{\xi^\perp}(x,t)| \lesssim \left| \frac{1}{|B(x,R(t))|} \int_{B(x,R(t))} \nabla \cdot v_{\xi^\perp}(y,t)\, dy \right|,$$

we deduce thanks to Stokes Theorem that

$$\begin{aligned} |\nabla \cdot v_{\xi^\perp}(x,t)| &\lesssim \frac{|S(x,R(t))|}{|B(x,R(t))|} \|v_{\xi^\perp}(t)\|_{L^\infty(B(x,R(t)))} \\ &= \frac{3}{R(t)} \|v_{\xi^\perp}(t)\|_{L^\infty(B(x,R(t)))} \end{aligned}$$

where $S(x,R(t))$ is the sphere of center x and radius $R(t)$. Since the very high vorticities at the time t are concentrated in the vortex tube $\mathcal{C}_{R(t)}(t) \cap B(x,R(t))$, we expect that $\|v_{\xi^\perp}(t)\|_{L^\infty(B(x,R(t)))} \lesssim \|v_{\xi^\perp}(t)\|_{L^\infty(\mathcal{T}(t))}$, where $\mathcal{T}(t) = \mathcal{C}_{R(t)}(t) \cap B(x,R(t))$. Then using again (114), we obtain

$$|\nabla \cdot v_{\xi^\perp}(x,t)| \lesssim \|u(t_0)\|_\infty\, \kappa(x,t) \left(1 + \log\left(\frac{\|\omega(t)\|_\infty}{\|\omega(t_0)\|_\infty}\right)\right). \tag{116}$$

Then, owing to (115) and (116), from the function Φ involved in Corollary 6.1, we get

$$\int_{t_0}^t \frac{|\Phi(s)|}{1+\log^+ \|\omega(s)\|_\infty}\, ds \lesssim \|u(t_0)\|_\infty (1 + |\log(\|\omega(t_0)\|_\infty)|) \int_{t_0}^t \sup_{x\in\Omega(s)} k(x,s)\, ds.$$

Then thanks to Corollary 6.1 used with the initial data $u(t_0)$, we deduce that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+(\|\omega(t_0)\|_\infty)) e^{\int_{t_0}^t f(s)\, ds}}, \tag{117}$$

where $f(s) = C\|u(t_0)\|_\infty (1 + |\log(\|\omega(t_0)\|_\infty)|) \sup_{x\in\Omega(s)} k(x,s)$ with $C > 0$ a constant.

Owing to (111), from (117), we deduce that there exists a real $C > 0$ depending only on $u(t_0)$ such that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+ \|\omega(t_0)\|_\infty) e^{C(t-t_0)}},$$

which indicates that we can expect a double-exponential in time growth of the maximum vorticity and then no blow-up is possible at T^*.

7.2. Heuristic arguments for 2D QG equations in favor of a double-exponential in time growth of the maximum vorticity $\|\omega(t)\|_\infty$

Let us assume that for all $t \in [t_0, T^*[$,

$$|\nabla \cdot \boldsymbol{v}_{\xi^\perp}(x,t)| \lesssim \left| \frac{1}{|B(x,\rho(t))|} \int_{B(x,\rho(t))} \nabla \cdot \boldsymbol{v}_{\xi^\perp}(y,t)\, dy \right|, \tag{118}$$

where ρ is a continuous function on $[0,T^*]$ such that

$$\int_0^{T^*} \frac{dt}{(1+\log(1+\frac{1}{T^*-t}))\rho(t)} < +\infty.$$

Then thanks to Stokes's Theorem, we deduce that for all $t \in [t_0, T^*[$

$$\begin{aligned}\left|\frac{1}{|B(x,\rho(t))|}\int_{B(x,\rho(t))} \nabla\cdot \boldsymbol{v}_{\xi^\perp}(y,t)\,dy\right| &\leq \frac{|S(x,\rho(t))|}{|B(x,\rho(t))|}\|\boldsymbol{v}(t)\|_\infty \\ &\lesssim \frac{1}{\rho(t)}\|\boldsymbol{v}(t)\|_\infty.\end{aligned}$$

Then from (118), we infer

$$|\nabla\cdot \boldsymbol{v}_{\xi^\perp}(x,t)| \lesssim \frac{1}{\rho(t)}\|\boldsymbol{v}(t)\|_\infty. \tag{119}$$

Thanks to the result obtained in [26], we get that there exists $C(u_0) > 0$ depending continuously only on u_0 such that

$$\|\boldsymbol{v}(t)\|_\infty \leq C(u_0)(1+\log^+\|\omega(t)\|_\infty). \tag{120}$$

Then thanks to (119), (120) and (79) for all $t \in [t_0, T^*[$, we infer

$$\int_{t_0}^t \frac{\sup\limits_{x\in\Omega(s)}|\nabla\cdot \boldsymbol{v}_{\xi^\perp}(x,s)|}{1+\log^+\|\omega(s)\|_\infty}\,ds \lesssim C(u_0)\int_{t_0}^{T^*}\frac{ds}{(1+\log(1+\frac{1}{T^*-s}))\rho(s)}\,ds < +\infty. \tag{121}$$

Then, owing to (121) and (120), from the function Φ involved in Corollary 6.1, we get

$$\int_{t_0}^t \frac{|\Phi(s)|}{1+\log^+\|\omega(s)\|_\infty}\,ds \lesssim C(u_0)\left(1+\int_{t_0}^t \sup_{x\in\Omega(s)} k(x,s)\,ds\right).$$

Then thanks to Corollary 6.1 used with the initial data $u(t_0)$, we deduce that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+\|\omega(t_0)\|_\infty)e^{CC(u_0)+\int_{t_0}^t f(s)\,ds}}, \tag{122}$$

where $f(s) = CC(u_0)\sup\limits_{x\in\Omega(s)} k(x,s)$ with $C>0$ a constant.

Owing to (111), from (122), we deduce that there exists a real $C_1(u_0) > 0$ depending only on u_0 such that for all $t \in [t_0, T^*[$,

$$\|\omega(t)\|_\infty \leq e^{(1+\log^+\|\omega(t_0)\|_\infty)e^{CC(u_0)+C_1(u_0)(t-t_0)}},$$

which indicates that we can expect a double-exponential in time growth of the maximum vorticity and then no blow-up is possible at T^*.

References

[1] Agélas, L.: Global regularity for logarithmically critical 2D MHD equations with zero viscosity, to appear in Monatsh. Math., (2016), (can be retrieved on my website: Léo Agélas - Homepage).

[2] Bardos, C. and Titi, E. S.: Euler equations for an ideal incompressible fluid, Uspekhi Mat. Nauk, 62(3), 5-46, (2007).

[3] Beale, J. T., Kato, T. and Majda, A.: Remarks on the Breakdown of Smooth Solutions for the 3-D Euler Equations, Comm. Math. Phys. 94(1), 61-66, (1984).

[4] Borisenko, O. F. and Minchenko, L. I.: Directional derivatives of the maximum function, Cybernet. Systems Anal., 28(2), 309-312, (1992).

[5] Bourguignon, J.P and Brezis, H.: Remarks on the Euler equation, J. of Func Analysis, 15, 341-363, (1974).

[6] Bustamante, M. D. and Kerr, R. M.: 3D Euler about a 2D symmetry plane, Phys. D, 23, pp. 1912-1920, (2008).

[7] Chae, D. and Choe, H-J.: Regularity of solutions to the Navier-Stokes equations, Electron. J. Differential Equations, 5, 1-7, (1999).

[8] Chae, D.: On the well-posedness of the Euler equation in the Triebel-Lizorkin spaces, Comm. Pure. Appl. Math., 5, 654-678, (2002).

[9] Chae, D.: The quasi-geostrophic equation in the Triebel-Lizorkin spaces, Nonlinearity 16, 479-495, (2003).

[10] Chae, D.: Remarks on the blow-up criterion of the three-dimensional Euler equations, Nonlinearity. 18(3), (2005).

[11] Chae, D., K. Kang and J. Lee: Notes on the asymptotically self-similar singularities in the Euler and the Navier-Stokes equations, Discrete contin. Dyn. Syst., 25, 1181-1193, (2009).

[12] Chae, D.: On the generalized self-similar singularities for the Euler and the Navier-Stokes equations, J. of Func. Analysis, 258, 2865-2883, (2010).

[13] Chae, D., Constantin, P. and Wu, J.: Deformation and Symmetry in the Inviscid SQG and the 3D Euler Equations, J. Nonlinear Sci., 22 (5), 665-688, (2012).

[14] Chemin, J.-Y.: Perfect Incompressible Fluids, Clarendon Press, Oxford, (1998).

[15] Chorin, A. J. and Marsden, J. E.: A Mathematical introduction to Fluid Mechanics , 3rd ed,New York, Springer-Verlag, (1993).

[16] Coifman, R., Meyer, Y.: Au delà des opérateurs pseudo-différentiels, Société Mathématique de France, Astérisque 57 (1978).

[17] Constantin, P., Majda, A. and Tabak, E.: Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar, Nonlinearity, 7, 1495-1533, (1994).

[18] Constantin, P.: Geometric statistic in turbulence, SIAM Rev. 36(1), 73-98, (1994).

[19] Constantin, P. and Fefferman, C.: Direction of Vorticity and the Problem of Global Regularity for the Navier-Stokes equations, Indiana Univ. Math. J. 42, 775-789, (1994).

[20] Constantin, P., Fefferman, C. and Majda, A. J.: Geometric constraints on potentially singular solutions for the 3-D Euler equation, Comm. Partial Differential Equations, 21, 559-571, (1996).

[21] Constantin, P., Nie, Q. and Schörghofer, N.: Nonsingular surface quasi-geostrophic flow, Phys. Lett. A, 241(3), 168-172, (1998).

[22] Constantin, P., Nie, Q. and Schörghofer, N.: Front formation in an active scalar equation, Phys. Rev. E, 60(3), 2858-2863, (1999).

[23] Constantin, P.: An Eulerian Lagrangian Approach to the Navier Stokes Equations, Commun. Math. Phys, 216(3), 663-686, (2000).

[24] Constantin, P.: On the Euler Equations of incompressible fluids, Bull. Amer. Math. Soc., 44, 603-621, (2007).

[25] Constantin, P., Lai, M-C., Sharma, R., Tseng, Y-H., Wu, J.: New Numerical Results for the Surface Quasi-Geostrophic Equation, J. Sci. Comput., 50(1)-28, (2012).

[26] Córdoba, D.: Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, Ann. of Math. 148, 1135-1152, (1998).

[27] Cordoba, D., Fefferman, C.: On the collapse of tubes carried by 3D incompressible flows, Comm. Math. Phys. 222, 293-298, (2001).

[28] Córdoba, D. and Ch. Fefferman, C.: Growth of solutions for QG and 2D Euler equations, J. Amer. Math. Soc. 15, 665-670, (2002).

[29] Chen, T. and Pavlović, N.: A Lower Bound on Blowup Rates for the 3D Incompressible Euler Equation and a Single Exponential Beale-Kato-Majda Type Estimate, Communications in Mathematical Physics, 314(1), 265-280, (2012).

[30] Deng, J., Hou, T. Y. and Yu, X.: Geometric Properties and Non-blow-up of 3-D Incompressible Euler Flow. Comm. Partial Differential Equations, 30(1), 225-243, (2005).

[31] Deng, J., Hou, T. Y., Li, R. and Yu, X.: Level set dynamics and the non-blow-up of the 2D Quasi-Geostrophic equation, Methods Appl. Anal., 13(2), 157-180, (2006).

[32] Deng, J., Hou, T. Y. and Yu, X.: Improved geometric conditions for non-blow-up of 3D incompressible Euler equation, Comm. Partial Differential Equations, 31, 293-306, (2006).

[33] De Lellis, C. Camillo and Székelyhidi, Jr. L.: The Euler equations as a differential inclusion, Ann. of Math., 170(3), 1417-1436, (2009).

[34] De Lellis, C. Camillo and Székelyhidi, Jr. L.: On admissibility criteria for weak solutions of the Euler equations, Arch. Ration. Mech. Anal. 195(1), 225-260, (2010).

[35] De Lellis, C. Camillo and Székelyhidi, Jr. L.: The h-principle and the equations of fluid dynamics, Bull. Amer. Math. Soc., 49(3), 347-375, (2012).

[36] DiPerna, R. J. and Majda, A. J.: Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys. 108(4),667-689, (1987).

[37] Eggers, J. and Fontelos, M. A.: Singularities: Formation, Structure and Propagation, Cambridge University Press, (2015).

[38] Fabes, E. B., Jones, B. F. and Rivière, N. M.: The initial Value Problem for the Navier-Stokes equations with data in L^p, Arch. Ration. Mech. Anal., 45, 222-240, (1972).

[39] Ferrari, A. B.: On the blow-up of solutions of the 3-D Euler equations in a bounded domain, Comm. Math. Phys., 155, 277-294, (1993).

[40] Fujita, H., Kato, T.: On the Navier-Stokes initial value problem 1. Arch. Ration. Mech. Anal. 16, 269-315 (1964).

[41] Furioli, G., Lemarié-Rieusset, P. G. and Terraneo, E.: Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces fonctionnels limites pour Navier-Stokes, Rev. Mat. Iberoam., 16(3), (2000).

[42] Galdi, G. P. and Maremonti, P.: Monotonic decreasing and asymptotic behaviour of the kinetic energy for weak solutions of the Navier-Stokes equations in exterior domains, Arch. Ration. Mech. Anal., 94, 253-66, (1986).

[43] Galdi, G. P.: An introduction to the Navier-Stokes Initial-Boundary Value Problem ,Fundamental Directions in Mathematical Fluids Mechanics, ed. G. P. Galdi et al., eds, (Birkhäuser, Basel), 1-70, (2000).

[44] Gallagher, I. and Planchon, F.: On global infinite energy solutions to the Navier-Stokes equations in two dimensions, Arch. Ration. Mech. Anal., 161, 307-337, (2002).

[45] Gallagher, I., Paicu, M.: Remarks on the blow-up of solutions to a toy model for the Navier-Stokes equations, Proc. Amer. Math. Soc., 137(6), 2075-2083, (2009).

[46] Gibbon, J. D. and Titi, E. S.: The 3D incompressible Euler equations with a passive scalar: A road to blow-up ?, J. Nonlinear Sci., 23, 993-1000, (2013).

[47] Giga, Y.: Weak and Strong Solutions of the Navier-Stokes Initial Value Problem, RIMS, Kyoto Univ, 19, 887-910, (1983).

[48] Giga, Y.: solutions for semilinear parabolic Equations in L^q and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations, 62 , 186-212, (1986).

[49] Grafke, T.: Finite-time Euler singularities: A Lagrangian perspective, PhD Thesis in der Fakultät für Physik und Astronomie der Ruhr-Universität Bochum, (2012).

[50] Grafke, T. and Grauer, R.: Finite-time Euler singularities: A Lagrangian perspective, Appl. Math. Lett., 26, 500-505, (2013).

[51] Grauer, R. and Sideris, T. C.: Numerical computation of 3D incompressible ideal fluids with swirl, Phys. Rev. Lett., 67, 3511-3514, (1991).

[52] He, C.: Regularity for solutions to the Navier-Stokes Equations with one velocity component regular, Electron. J. Differential Equations, 29, 1-13, (2002).

[53] Held, I.M., Pierrehumbert, R.T., Harner, S.T. and Swanson, K.L.: Surface quasi-geostrophic dynamics, J. Fluid Mech. 282, 1-20, (1995).

[54] Heywood, J. G.: The Navier-Stokes equations, on the existence, regularity and decay of solutions, Indiana Univ. Math. J. 29, 639-81, (1980).

[55] Heywood, J. G.: Epochs of regularity for weak solutions of the Navier-Stokes equations in unbounded domains, Tôhoku Math. J., 40, 293-313, (1988).

[56] Hopf, E.: Über die Anfangwertaufgabe für die hydrohynamischen Grundgleichungen, Math. Nachr, 4, 213-231, (1951).

[57] Hou, T. Y. and Li, R.: Dynamic depletion of vortex stretching and non-blow-up of the 3-D incompressible Euler equations, J. Nonlinear Sci., 16, 639-664, (2006).

[58] Hou, T. Y. and Li, C.: Dynamic stability of the 3D axi-symmetric Navier-Stokes equations with swirl, Commun. Pure Appl. Math., 61, 661-697, (2008).

[59] Hou, T. Y. and Li, R.: blow-up or no blow-up? The interplay between theory and numerics, Phys. D, 237, 1937-1944, (2008).

[60] Hou, T. Y.: Blow-up or no blow-up ? A unified computational and analytic approach to 3D incompressible Euler and Navier-Stokes equations, Acta Numer., 18, 1-70, (2009).

[61] Hou, T. Y. and Lei, Z.: On the stabilizing effect of convection for three dimensional incompressible flows, Commun. Pure Appl. Math., 62, 501-564, (2009).

[62] Hou, T. Y. and Shi., Z.: Dynamic Growth Estimates of Maximum Vorticity for 3D Incompressible Euler Equations and the SQG Model, Discrete Contin. Dyn. Syst. - A, 32(5), 1449-1463, (2012).

[63] Iskauriaza, L., Serëgin, G. A., Shverak, V.: $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness, Uspekhi Mat. Nauk, 58:2(350), 3-44, (2003).

[64] Kato, T.: Strong L^q solutions of the Navier-Stokes equations in $\mathbb{R}^m$, with application to weak solutions, Math. Z., 187, 471-80, (1986).

[65] Kato, T. and Ponce, G.: Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbb{R}^2)$, Rev. Math. Iberoam., 2, 73-88, (1986).

[66] Kato, T. and Ponce, G. : Commutator Estimates and the Euler and Navier-Stokes Equa- tions, Comm. Pure. Applied. Math, 41(7), 891-907, (1988).

[67] Kato, T.: Liapunov functions and monotonicity in the Navier-Stokes equations, Lecture Notes in Math., 1450, 53-63, (1990).

[68] Kerr, R. M.: Evidence for a singularity of the three-dimensional incompressible Euler equations, Phys. Fluids A, 5, 1725-1746, (1993).

[69] Kerr, R. M.: The role of singularities in Euler. In Small-Scale Structure in Hydro and Magnetohydrodynamic Turbulence Pouquet, A., Sulem, P. L., eds. Lecture Notes. Springer-Verlag, (1995).

[70] Kerr, R. M.: Cover illustration: vortex structure of Euler collapse. Nonlinearity, 9, 271-272, (1996).

[71] Kerr, R. M.: Euler singularities and turbulence. In 19th ICTAM Kyoto '96. Tatsumi, T., Watanabe, E., Kambe, T., eds. Elsevier Science, (1997).

[72] Kerr, R. M.: The outer regions in singular Euler. In Fundamental Problematic Issues in Turbulence. Tsinober and Gyr, eds. Boston: Birkhäuser, (1998).

[73] Kerr, R. M.: Velocity and scaling of collapsing Euler vortices, Phys. Fluids 17, 075103, (2005).

[74] Kerr, R. M. and Bustamante, M. D.: Exploring symmetry plane conditions in numerical Euler solutions, published in Mathematical Aspects of Fluid Mechanics, edited by Robinson, J. C., JRodrigo, J. L. and Sadowski, W. Cambridge University Press (2012).

[75] Kerr, R. M.: Bounds for Euler from vorticity moments and line divergence, J. Fluid Mech.,729 (2013).

[76] Kozono, H., Taniuchi, Y.: Bilinear estimates and critical Sobolev inequality in BMO, with applications to the Navier-Stokes and the Euler equations, RIMS Kokyuroku, 1146, 39-52, (2000).

[77] Ladyzhenskaya, O.: On the unique global solvability of the Cauchy problem for the Navier- Stokes equations in the presence of the axial symmetry, Zap. Nauch. Sem. LOMI, 7, 155-177, (1968) (in Russian).

[78] Ladyzhenskaya, O.: The Mathematical Theory of Viscous Incompressible Flows (2nd edition), Gordon and Breach, (1969).

[79] Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math., 63, 193-248, (1934).

[80] Lions, J. L. and Prodi, G.: Un théorème d'existence et d'unicité dans les Équations de Navier-Stokes en dimension 2, C. R. Acad. Sci. Paris, 248, 3519-3521, (1959).

[81] Lions, J. L.: Sur la régularité et l'unicité des solutions turbulentes des Équations de Navier-Stokes, *Rendiconti del Seminario Matematico della Università di Padova*, 30, 16-23, (1960).

[82] Lions, J. L.: Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod, Paris, (1969).

[83] Lions, P. L.: Mathematical topics in fluid mechanics. Vol.1. The Clarendon Press Oxford University Press, NY, (1996). Incompressible models, Oxford Science Publications.

[84] Majda, A. and Bertozzi, A.: Vorticity and Incompressible Flow, Cambridge Univ. Press. (2002).

[85] Monniaux, S.: Unicité dans L^d des solutions du système de Navier-Stokes : cas des domaines lipschitziens, Ann. Math. Blaise Pascal, 10, 107-116, (2000).

[86] Nakai, E. and Yoneda, T.: Bilinear estimates in dyadic BMO and the Navier-Stokes equations, J. Math. Soc. Japan, 64(2), 399-422, (2012).

[87] Ogawa, T.: Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow, Siam J. Math. Anal., 34(6), 1318-1330, (2003).

[88] Ohkitani, K. and Yamada, M.: Inviscid and inviscid-limit behavior of a surface quasi- geostrophic flow, Phys. Fluids, 9:4, 876-882, (1997).

[89] Pedlosky, J.: Geophysical Fluid Dynamics, Springer, New York, 1987.

[90] Pelz, R. B.: Locally self-similar, finite-time collapse in a high-symmetry vortex filament model. Phys. Rev. E 55(2), 1617-1620, (1997).

[91] Pelz, R. B.: Symmetry and the hydrodynamic blow-up problem. J. Fluid Mech. 444, 299-320, (2001).

[92] Ponce, G.: Remarks on a paper by J. T. Beale, T. Kato, and A. Majda, Comm. Math. Phys., 98, 349-353, (1985).

[93] Pshenichny, B. N.: Necessary Conditions for an Extremum, Marcel Dekker, Inc., New York, NY, 1971 (Translated from Russian 1969).

[94] Pumir, A. and Siggia, E. D.: Development of singular solutions to the axisymmetric Euler equations, Phys. Fluids A, 4, 1472-1491, (1992).

[95] Scheffer, V.: An inviscid flow with compact support in space-time, J. Geom. Anal. 3(4), 343-401, (1993).

[96] Serrin, J.: On the interior regularity of weak solutions of the Navier-Stokes equations, Arch. Ration. Mech. Anal. 9, 187-191, (1962).

[97] Serrin, J.: The initial value problem for the Navier-Stokes equations 1963 Nonlinear Problems, Proc. Sympos., Madison, Wis. 69-98 Univ. of Wisconsin Press, Madison, Wis. 1963.

[98] Shirota, T. and Yanagisawa, T.: A continuation principle for the 3-D Euler equations for incompressible fluids in a bounded domain, Proc. Japan Acad. Ser. A Math. Sci., 69, 77-82, (1993).

[99] Shnirelman, A.: On the nonuniqueness of the weak solution of the Euler equation, Comm. Pure Appl. Math. 50, 1261-1286, (1997).

[100] Shu, W. E and C.-W.: Small-scale structures in Boussinesq convection, Phys. Fluids, 6, 49-58, (1994).

[101] Silvestre, L., Vicol, V., Zlatoš, A.: On the Loss of Continuity for Super-Critical Drift-Diffusion Equations. Arch. Ration. Mech. Anal., 207(3), 845-877, (2013).

[102] Solonnikov, V. A.: Estimates of the solutions of a nonstationary linearized system of Navier-Stokes eauations, Amer. Math. Soc. Transl. Ser. 2, 75, 1-116, (1968).

[103] Sohr, H., v. Wahl, W.: On the singular set and the uniqueness of weak solutions of the Navier-Stokes equations, Manuscripta math., 49, 27-59, (1984).

[104] Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton NJ: Princeton University Press, (1970).

[105] Székelyhidi, Jr. L. and Wiedemann, E.: Young measures generated by ideal incompressible fluid flows. Arch. Ration. Mech. Anal., 206(1), 333-366, (2012).

[106] Tao, T.: Finite time blow-up for an averaged three-dimensional Navier-Stokes equations, J. Amer. Math. Soc (2015), DOI: http://dx.doi.org/10.1090/jams/838.

[107] Temam, R.: Navier-Stokes equations, North-Holland, Amsterdam, 1977.

[108] Uchovskii, M.R. and Yudovich, B.I.: Axially symmetric flows of an ideal and viscous fluid in the whole space (in Russian, also J. Appl. Math. Mech, 32, 1968, 52-61), Prikl. Mat. Mekh., 32, 59-69, (1968).

[109] Beirão da Veiga, H.: A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. Ser. B, 16(4): 407-412, (1995).

[110] Villani, C.: Paradoxe de Scheffer-Shnirelman revu sous l'angle de l'intégration convexe [d'après C. De Lellis et L. Székelyhidi], Séminaire Bourbaki 61^e année, no 1001, (2008-2009).

[111] Vishik, M.: Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type, Ann. Sci. Éc. Norm. Supér. (4), 32(6), 769-812, (1999).

[112] Von Wahl, W.: Regularity of weak solutions of the Navier-Stokes equations, Proceedings of the 1983 Summer Institute on Nonlinear Functional Analysis and Applications, Proc. Symposia in Pure Mathematics 45, Providence Rhode Island : Amer. Math. Soc, 497-503, (1986).

[113] Wiedemann, E.: Existence of weak solutions for the incompressible Euler equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 28(5), 727-730, (2011).

[114] Yudovich, V. I.: Non-stationary flows of an ideal incompressible fluid. Z. Vychisl. Mat. i Mat. Fiz., 6(3), 1032-1066, (1963) (Russian).

[115] Yudovich, V. I.: Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. Math. Res. Lett., 2(1), 27-38, (1995).

[116] Zhou, Y.: A new regularity criterion for the Navier-Stokes equations in terms of the gradient of one velocity component. Methods Appl. Anal., 9(4), 563-578, (2002).